Supplementary Materials for
"Distribution Analysis for Diagnostics and Therapeutics of Motor Actions"

I. Idealized Distributions Describe Human Actions (continued…)

In the 1920s, Paul Levy took this idea of kurtosis to the extreme, theorizing the ***Levy*** distribution where the kurtosis is infinity [1]. Therefore, it can be applied to datasets with highly variable data distant from the median, as Benoit Mandelbrot, the father of fractals, proposed in star clustering, turbulence, and stock market variations in 1982 [2] [3]. Neurotypical individuals showcase this distribution in a voluntary balance task. It was revealed that after practice of balancing a stick on a fingertip, transverse speeds consisting of both minor adjustments and large corrections converged to Levy [4] [5].

While Pearson and other mathematicians were developing measures to capture deviations from the Gaussian, others had started to discover that certain skewed distributions could be transformed into a Gaussian form using a logarithm scale, termed ***lognormal*** distributions [6]. These describe continuous, non-negative, positively skewed data. Early examples include the distribution of silver particle sizes in photographic emulsions and in drug sensitivity among animals of the same species [7], [8]. In motor learning, the *Kinematic Theory of Rapid Movement* leverages lognormality to profile motor velocities in voluntary actions. Handwriting follows this pattern, as lognormal distributions can reflect proficiency and age [9], [10], and are often seen in the sub-movements of larger actions [10]. It is also relevant for the firing rates of intrinsic hand muscles [9] [11]. Interestingly, the tongue's movement range during word pronunciation was found to be lognormal in neurotypical people, but deviations from the distribution were found in dysarthric individuals [12].

The ***Weibull*** distribution, described by Waloddi Weibull in 1935, unites many earlier statistical properties into one flexible curve that can adapt to a wide range of problems [13], [14]. First applied to particle size modeling in 1933, it has since become a foundation of quality control engineering. It has been employed in motor learning to model response times during inferred tasks to uncover the function of specific parts of the brain [15]. It may also be repurposed to model patient compliance with at-home therapy [16] and the inter-spike intervals in a single motor unit [17].

A few decades later, Ronald Fisher saw that there was no method to model data distributed over the surface of a sphere, something he wanted to study the magnetism of rocks and lava flows. In 1953, he extended the von Mises distribution, originally defined for data on a circle, to create the ***von Mises-Fisher*** distribution, a spherical analogue for analyzing directional data [18]. In motion tracking, it can estimate uncertainties in quaternion readings from inertial measurement units (IMUs), enhancing the accuracy of angular data [19]. In motor learning, the angular trajectories of reaching movements have been modeled with von Mises-Fisher to show faster learning with consistent lead-in movements prior to the influence of curl fields [20].

Until this point, distributions assumed unbounded variables. James Tobin, in 1965, introduced ***Censored*** distributions while studying household spending on durable goods, where the observed spending values were limited by survey constraints. In his study, the survey period caused many households to report $0 spending, creating a spike at the lower bound due to censoring rather than true zero demand [21]. Sensorimotor synchronization demonstrates these properties in rhythm coordination tasks: when participants tap their fingers to a metronome, they anticipate beats at shorter intervals [22], but as intervals randomly lengthen, the upper tail of the distribution becomes "censored" by the metronome, triggering reactive taps [23].

Some specialized probability distributions can only be used with *discrete* events, such as the opening of membrane channels, release of some number of neurotransmitter vesicles, or the number of spikes observed in a given timeframe. In these cases, abstract events are first mapped to a numeric space (such as 1 for when an action potential occurs, and 0 otherwise).

Among these, the ***Binomial*** probability distribution models the outcomes of a fixed number of binary Bernoulli trials (experiments with two possible outcomes, such as flipping a coin). In 1837, Siméon-Denis Poisson first described the likelihood of errors in criminal justice convictions, such as wrongful convictions or acquittals [24]. It models neural responses and estimate motor unit activation and has been shown to perform better in patients with severe motor unit loss [25], [26]. For example, a presynaptic terminal probability of k vesicles being released is $\frac{n!\,p^k(1-p)^{n-k}}{k!(n-k)!}$ where p is the probability of a single vesicle and n is the number of available vesicles.

An extension of Binomial, for when n is large and p is small, is the ***Poisson*** distribution, which is used to estimate the number of events. The simpler formula is $e^{-\lambda} \cdot \frac{\lambda^k}{k!}$. These distributions have been used to track injury frequency in professional athletes [27], fall rates in patients with Parkinson's [28], and to describe neural spiking events [26], [29], [30]. Multivariate Poisson distribution can model multi-class events such as injury severity [31] and motor vehicular crash types [32], [33]. Multivariate Poisson regression captures the interdependencies between events, such as the impact of training programs on injury frequency [34].

II. Metrics for Multidimensional Distribution Analysis (continued…)

Finally, beyond distance metrics, probabilistic modeling approaches have been used to compare distributions, identifying the latent structure of patterns across individuals or groups. **Gaussian Mixture Models** (Fig 2f) can be utilized for clustering data and determining the probability of each data point belonging to each cluster. It assumes that a complex multivariate dataset can be modeled as a mixture of Gaussian distributions [35]. Thus, the probability density function is the sum of each Gaussian distribution that describes a portion of the data. The number of Gaussians can be chosen by comparing models of varying component counts with criteria such as the Akaike Information Criterion or Bayesian Information Criterion (BIC). Both model the fit against complexity, though the BIC tends to favor simpler models. Validation of the fit can be later performed by examining the consistency of the results after changing the random initial parameters or visualizing the model and data. Gaussian mixture models have been used in gait to analyze distributions of symmetry, speed, and variability in neurotypical participants and individuals post-stroke. Three clusters formed from the distribution analysis: one for neurotypical individuals, one for intermediate impairment and one for worse impairment. The posterior probability assigned to each participant that predicts the likelihood of their membership in a specific group was proposed as a potential metric during rehabilitation [36]. Another group modeled elbow kinematic data of individuals post-stroke and neurotypical participants with Gaussian mixture models before using KL divergence to differentiate distributions of individuals post-stroke from that of control [37]. The Gaussian mixture models were also explored as the first step in clustering IMU accelerometer data in individuals' post-stroke as they performed daily activities for 7 days. In modeling the bilateral data in this way, the group uncovered patterns in upper extremity movement that could assess function continuously and outside the clinical setting [38].

Probabilistic graphical models (Fig 2g) represent relationships between the probability distributions of many variables via nodes and edges that capture dependencies [39]. With sufficient data, they can identify movement deficits, predict mechanisms, and remain robust to noise. Two main forms exist: Bayesian Networks, which model causal links between nodes, and Markov Random Fields, which model correlations [40], [41]. While both can be used in movement analysis, Bayesian Networks are more often used to identify deficits and guide therapy with good success. One study, for example, created a Bayesian Network to model dependencies in muscle activation of the biceps, triceps, deltoid, and latissimus dorsi during reaching tasks. In comparing individuals post-stroke and neurotypical participants, they found that individuals post-stroke had selectively recruited a subnetwork within the muscles, evidence supportive of the synergy hypothesis [42]. One subtype of Bayesian Networks, Hidden Markov Model (HMM), is better when sequential data is evaluated. In 2004, a team modeled the trajectories of neurotypical individuals as they navigated a labyrinth with HMMs. The HMM then guided assistive forces of a haptic interface to improve upper limb coordination and reduce tremors in two cerebral palsy participants [43].

III. References

[1] C. Tsallis, "Lévy distributions," *Phys. World*, vol. 10, no. 7, p. 42, July 1997, doi: 10.1088/2058-7058/10/7/32.

[2] T. Duquesne, O. Reichmann, K. Sato, and C. Schwab, *Lévy Matters I: Recent Progress in Theory and Applications: Foundations, Trees and Numerical Issues in Finance*. Springer Science & Business Media, 2010.

[3] B. B. Mandelbrot, "Fractals and an Art for the Sake of Science," *Leonardo. Supplemental Issue*, vol. 2, pp. 21–24, 1989, doi: 10.2307/1557938.

[4] J. L. Cabrera and J. G. Milton, "Human stick balancing: Tuning Lèvy flights to improve balance control," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 14, no. 3, pp. 691–698, Sept. 2004, doi: 10.1063/1.1785453.

[5] T. Cluff and R. Balasubramaniam, "Motor Learning Characterized by Changing Lévy Distributions," *PLOS ONE*, vol. 4, no. 6, p. e5998, June 2009, doi: 10.1371/journal.pone.0005998.

[6] J. H. Gaddum, "Lognormal Distributions," *Nature*, vol. 156, no. 3964, pp. 463–466, Oct. 1945, doi: 10.1038/156463a0.

[7] E. P. Wightman, A. P. H. Trivelli, and S. E. Sheppard, "The Size-Frequency Distribution of Grains of Silver Halide in Photographic,Emulsions its Relation to Sensitometric Characteristics.,VI," *The Journal of Physical Chemistry*, May 2002, doi: 10.1021/j150240a001.

[8] C. I. Bliss and M. K. Cattell, "Biological Assay," *Annual Review of Physiology*, vol. 5, no. Volume 5, 1943, pp. 479–539, Mar. 1943, doi: 10.1146/annurev.ph.05.030143.002403.

[9] R. Plamondon, C. O'Reilly, C. Rémi, and T. Duval, "The lognormal handwriter: learning, performing, and declining," *Front. Psychol.*, vol. 4, Dec. 2013, doi: 10.3389/fpsyg.2013.00945.

[10] T. Duval, C. Rémi, R. Plamondon, J. Vaillant, and C. O'Reilly, "Combining sigma-lognormal modeling and classical features for analyzing graphomotor performances in kindergarten children," *Hum Mov Sci*, vol. 43, pp. 183–200, Oct. 2015, doi: 10.1016/j.humov.2015.04.005.

[11] V. I. Rupasov, M. A. Lebedev, J. S. Erlichman, and M. Linderman, "Neuronal Variability during Handwriting: Lognormal Distribution," *PLoS One*, vol. 7, no. 4, p. e34759, Apr. 2012, doi: 10.1371/journal.pone.0034759.

[12] S. Duan, X. Zhang, M. Yan, and J. Zhang, "Statistical Distribution Exploration of Tongue Movement for Pathological Articulation on Word/Sentence Level," *IEEE Access*, vol. 8, pp. 91057–91069, 2020, doi: 10.1109/ACCESS.2020.2993856.

[13] W. Weibull, "A Statistical Distribution Function of Wide Applicability," *Journal of Applied Mechanics*, 1951, Accessed: Aug. 13, 2025. [Online]. Available: https://hal.science/hal-03112318

[14] R. B. Abernethy, *The New Weibull Handbook*. R.B. Abernethy, 1996.

[15] P. Rice and A. Stocco, "The Role of Dorsal Premotor Cortex in Resolving Abstract Motor Rules: Converging Evidence From Transcranial Magnetic Stimulation and Cognitive Modeling," *Top Cogn Sci*, vol. 11, no. 1, pp. 240–260, Jan. 2019, doi: 10.1111/tops.12408.

[16] Q. Sanders, V. Chan, R. Augsburger, S. C. Cramer, D. J. Reinkensmeyer, and A. H. Do, "Feasibility of Wearable Sensing for In-Home Finger Rehabilitation Early After Stroke," *IEEE Transactions on Neural Systems and Rehabilitation Engineering*, vol. 28, no. 6, pp. 1363–1372, June 2020, doi: 10.1109/TNSRE.2020.2988177.

[17] C. J. De Luca and W. J. Forrest, "Some properties of motor unit action potential trains recorded during constant force isometric contractions in man," *Kybernetik*, vol. 12, no. 3, pp. 160–168, Mar. 1973, doi: 10.1007/BF00289169.

[18] K. V. Mardia, "Fisher's legacy of directional statistics, and beyond to statistics on manifolds," *Journal of Multivariate Analysis*, vol. 207, p. 105404, May 2025, doi: 10.1016/j.jmva.2024.105404.

[19] G. To and M. R. Mahfouz, "Quaternionic Attitude Estimation for Robotic and Human Motion Tracking Using Sequential Monte Carlo Methods With von Mises-Fisher and Nonuniform Densities Simulations," *IEEE Transactions on Biomedical Engineering*, vol. 60, no. 11, pp. 3046–3059, Nov. 2013, doi: 10.1109/TBME.2013.2262636.

[20] I. S. Howard, C. Ford, A. Cangelosi, and D. W. Franklin, "Active lead-in variability affects motor memory formation and slows motor learning," *Sci Rep*, vol. 7, no. 1, p. 7806, Aug. 2017, doi: 10.1038/s41598-017-05697-z.

[21] J. Tobin, "Estimation of Relationships for Limited Dependent Variables," *Econometrica*, vol. 26, no. 1, pp. 24–36, 1958, doi: 10.2307/1907382.

[22] B. H. Repp and R. Doggett, "Tapping to a Very Slow Beat: A Comparison of Musicians and Nonmusicians," *Music Perception: An Interdisciplinary Journal*, vol. 24, no. 4, pp. 367–376, 2007, doi: 10.1525/mp.2007.24.4.367.

[23] R. Bååth, "Estimating the distribution of sensorimotor synchronization data: A Bayesian hierarchical modeling approach," *Behavior Research Methods*, vol. 48, no. 2, pp. 463–474, 2016, doi: 10.3758/s13428-015-0591-2.

[24] S. M. Stigler, "Poisson on the poisson distribution," *Statistics & Probability Letters*, vol. 1, no. 1, pp. 33–35, July 1982, doi: 10.1016/0167-7152(82)90010-4.

[25] J. H. Blok, G. H. Visser, S. de Graaf, M. J. Zwarts, and D. F. Stegeman, "Statistical motor number estimation assuming a binomial distribution," *Muscle Nerve*, vol. 31, no. 2, pp. 182–191, Feb. 2005, doi: 10.1002/mus.20256.

[26] A. Ghanbari, C. M. Lee, H. L. Read, and I. H. Stevenson, “Modeling stimulus-dependent variability improves decoding of population neural responses,” *J Neural Eng*, vol. 16, no. 6, p. 066018, Oct. 2019, doi: 10.1088/1741-2552/ab3a68.

[27] H. Nobari, S. M. Khalili, A. D. Zamorano, T. G. Bowman, and U. Granacher, “Workload is associated with the occurrence of non-contact injuries in professional male soccer players: A pilot study,” *Front Psychol*, vol. 13, p. 925722, 2022, doi: 10.3389/fpsyg.2022.925722.

[28] L. R. S. Almeida *et al.*, “Predicting falls in people with Parkinson’s disease: impact of methodological approaches on predictors identified,” *Aging Clin Exp Res*, vol. 32, no. 6, pp. 1057–1066, June 2020, doi: 10.1007/s40520-019-01281-9.

[29] S. Kuroda, K. Yamamoto, H. Miyamoto, K. Doya, and M. Kawato, “Statistical characteristics of climbing fiber spikes necessary for efficient cerebellar learning,” *Biol Cybern*, vol. 84, no. 3, pp. 183–192, Feb. 2001, doi: 10.1007/s004220000206.

[30] C. Y. Song and M. M. Shanechi, “Unsupervised learning of stationary and switching dynamical system models from Poisson observations,” *J Neural Eng*, vol. 20, no. 6, p. 066029, Dec. 2023, doi: 10.1088/1741-2552/ad038d.

[31] J. Ma and K. M. Kockelman, “Bayesian Multivariate Poisson Regression for Models of Injury Count, by Severity,” *Transportation Research Record*, vol. 1950, no. 1, pp. 24–34, Jan. 2006, doi: 10.1177/0361198106195000104.

[32] A. Faden, M. Abdel-Aty, N. Mahmoud, T. Hasan, and H. Rim, “Multivariate Poisson-Lognormal Models for Predicting Peak-Period Crash Frequency of Joint On-Ramp and Merge Segments on Freeways,” *Transportation Research Record*, vol. 2678, no. 3, pp. 133–147, Mar. 2024, doi: 10.1177/03611981231178797.

[33] Y.-C. Chiou and C. Fu, “Modeling crash frequency and severity using multinomial-generalized Poisson model with error components,” *Accid Anal Prev*, vol. 50, pp. 73–82, Jan. 2013, doi: 10.1016/j.aap.2012.03.030.

[34] O. B. A. Owoeye, L. M. Palacios-Derflingher, and C. A. Emery, “Prevention of Ankle Sprain Injuries in Youth Soccer and Basketball: Effectiveness of a Neuromuscular Training Program and Examining Risk Factors,” *Clin J Sport Med*, vol. 28, no. 4, pp. 325–331, July 2018, doi: 10.1097/JSM.0000000000000462.

[35] D. A. Reynolds, “Gaussian mixture models,” *Encyclopedia of biometrics*, vol. 741, pp. 659–663, 2009.

[36] E. Dolatabadi, A. Mansfield, K. K. Patterson, B. Taati, and A. Mihailidis, “Mixture-Model Clustering of Pathological Gait Patterns,” *IEEE Journal of Biomedical and Health Informatics*, vol. 21, no. 5, pp. 1297–1305, Sept. 2017, doi: 10.1109/JBHI.2016.2633000.

[37] I. Davidowitz *et al.*, “Relationship Between Spasticity and Upper-Limb Movement Disorders in Individuals With Subacute Stroke Using Stochastic Spatiotemporal Modeling,” *Neurorehabil Neural Repair*, vol. 33, no. 2, pp. 141–152, Feb. 2019, doi: 10.1177/1545968319826050.

[38] L. Tang, S. Halloran, J. Q. Shi, Y. Guan, C. Cao, and J. Eyre, “Evaluating upper limb function after stroke using the free-living accelerometer data,” *Stat Methods Med Res*, vol. 29, no. 11, pp. 3249–3264, Nov. 2020, doi: 10.1177/0962280220922259.

[39] D. Koller and N. Friedman, *Probabilistic graphical models: principles and techniques*, Nachdr. in Adaptive computation and machine learning. Cambridge, Mass.: MIT Press, 2010.

[40] N. Friedman, D. Geiger, and M. Goldszmidt, “Bayesian Network Classifiers,” *Machine Learning*, vol. 29, no. 2, pp. 131–163, Nov. 1997, doi: 10.1023/A:1007465528199.

[41] P. Clifford, *Disorder in Physical Systems: A Volume in Honour of John M. Hammersley*. Oxford: Oxford University Press, 1990.

[42] J. Li, Z. J. Wang, J. J. Eng, and M. J. McKeown, “Bayesian Network Modeling for Discovering ‘Dependent Synergies’ Among Muscles in Reaching Movements,” *IEEE Transactions on Biomedical Engineering*, vol. 55, no. 1, pp. 298–310, Jan. 2008, doi: 10.1109/TBME.2007.897811.

[43] W. Yu, R. Dubey, and N. Pernalete, “Robotic therapy for persons with disabilities using Hidden Markov Model based skill learning,” in *IEEE International Conference on Robotics and Automation, 2004. Proceedings. ICRA ’04. 2004*, Apr. 2004, pp. 2074-2079 Vol.2. doi: 10.1109/ROBOT.2004.1308129.