The boolean Pythagorean Triples problem

subtitle

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date of the presentation

Example

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- **Triples:**

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3^2 + 4^2 &= 5^2 \\
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- **Partition:** $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}, \{5, 10, 13, 15\}$

Encoding - Intuition

Idea:

- **one variable for each number**

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- **one constraint clause for each Pythagorean triple**
- **interpretation gives partition**

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Binary Pythagorean triple problem with n numbers

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Set of variables $V = \{p_k \mid 1 \leq k \leq n\}$

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Constraint for non-equality: $\boldsymbol{NotEqual(x, y, z) = (x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)}$

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Constraint for non-equality: $NotEqual(x, y, z) = (x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$

Constraint for all Pythagorean triples: $F = \bigwedge_{\substack{1 \leq x,y,z \leq n \\ x^2+y^2=z^2}} NotEqual(p_x, p_y, p_z)$

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For an interpretation $I \subseteq V$ with $I \models F$, the resulting partition is:

- $P_1 = \{x \mid p_x \in I\}$
- $P_2 = \{x \mid p_x \notin I\}$

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\begin{aligned}
F \quad = \quad & (p_3 \vee p_4 \vee p_5) \wedge (\neg p_3 \vee \neg p_4 \vee \neg p_5) \\
\wedge \quad & (p_6 \vee p_8 \vee p_{10}) \wedge (\neg p_6 \vee \neg p_8 \vee \neg p_{10}) \\
\wedge \quad & (p_9 \vee p_{12} \vee p_{15}) \wedge (\neg p_9 \vee \neg p_{12} \vee \neg p_{15}) \\
\wedge \quad & (p_5 \vee p_{12} \vee p_{13}) \wedge (\neg p_5 \vee \neg p_{12} \vee \neg p_{13})
\end{aligned}
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Encoding - Example

As in beginning example, $n = 15$

$V = \{p_1, p_2, \dots, p_{15}\}$

$$\begin{aligned} F \quad = \quad & (p_3 \lor p_4 \lor p_5) \land (\neg p_3 \lor \neg p_4 \lor \neg p_5) \\ \land \quad & (p_6 \lor p_8 \lor p_{10}) \land (\neg p_6 \lor \neg p_8 \lor \neg p_{10}) \\ \land \quad & (p_9 \lor p_{12} \lor p_{15}) \land (\neg p_9 \lor \neg p_{12} \lor \neg p_{15}) \\ \land \quad & (p_5 \lor p_{12} \lor p_{13}) \land (\neg p_5 \lor \neg p_{12} \lor \neg p_{13}) \end{aligned}$$

Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

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Resulting partition:

- $P_1 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}$
- $P_2 = \{5, 10, 13, 15\}$

Transformation

Goal: from F, find formula F' which

- **is easier to solve**

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- **eliminate clauses with variables only occuring once**

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Approaches:

- **eliminate clauses with variables only occuring once**
- **break partition symmetry**

Clause elimination

$$\begin{aligned} F \quad = \quad & \mathit{NotEqual}(p_3, p_4, p_5) \\ \wedge \quad & \mathit{NotEqual}(p_6, p_9, p_{12}) \\ \wedge \quad & \mathit{NotEqual}(p_9, p_{12}, p_{15}) \\ \wedge \quad & \mathit{NotEqual}(p_5, p_{12}, p_{13}) \end{aligned}$$

Note p_3 only occurs in $\mathit{NotEqual}(p_3, p_4, p_5)$ and thus does not affect any other clauses

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$\longrightarrow$ clause $NotEqual(p_3, p_4, p_5)$ will not cause conflict

$$\begin{aligned} F' = & \; NotEqual(p_6, p_9, p_{12}) \\ & \wedge \; NotEqual(p_9, p_{12}, p_{15}) \\ & \wedge \; NotEqual(p_5, p_{12}, p_{13}) \end{aligned}$$

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- **deleted clause $NotEqual(p_3, p_4, p_5)$ because p_3 only occurred once**

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- **$I' \models F'$ and $\{p_3, p_4, p_5\} \subseteq I'$**

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- $\rightarrow$ remember deleted clauses and modify interpretation of F' accordingly

Example:

- deleted clause $NotEqual(p_3, p_4, p_5)$ because p_3 only occurred once
- $I' \models F'$ and $\{p_3, p_4, p_5\} \subseteq I'$
- $I = I' \setminus p_3$
- while $I' \not\models F$, we have $I \models F$

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- **if $I \models F'$ then $V \setminus I \models F'$**
- **$\rightarrow$ introduce unit clause for variable occurring in F'**
- **$F'' = F' \wedge p_x$**

Note that every model for F'' is a model for F' and the transformation is satisfiability preserving

Cube-and-conquer solving

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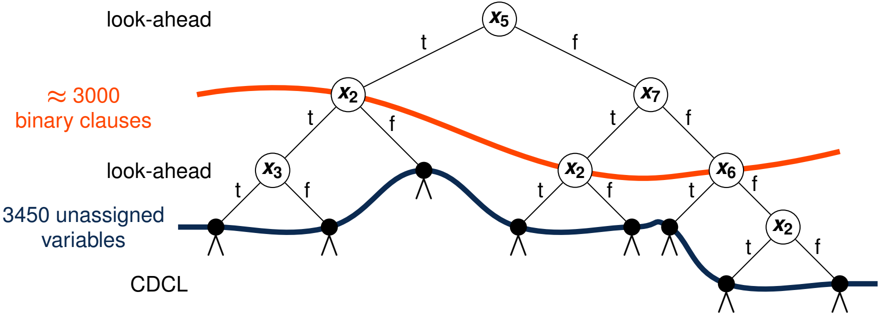
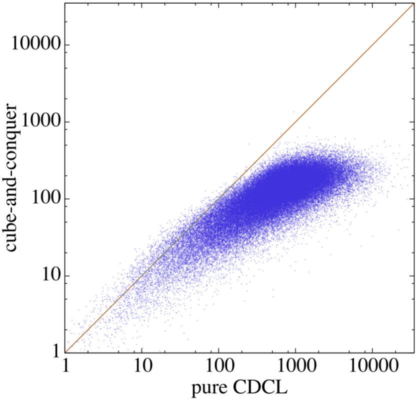


Runtime

- x time



Validation of the program