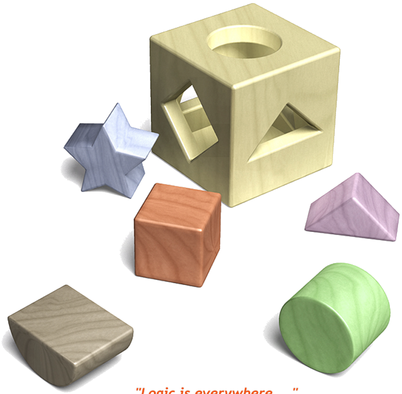
TECHNISCHE UNIVERSITÄT DRESDEN

The boolean Pythagorean Triples problem

Tobias John, Aldo Kurmeta, Patrick Wienhöft
International Center for Computational Logic
Technische Universität Dresden
Germany

- **Introduction**
- **History of the problem**
- **The framework**
- **Encoding**
- **Transformation**
- **Heuristic solving**



"Logic is everywhere ..."

Introduction

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- **Can the set $\mathbb{N}$ = $\{1, 2, 3, \dots\}$ be divided in two parts such that no part contains a triple (a, b, c) with $a^2 + b^2 = c^2$**

Example

- **Set of Integers: $\{1, \ldots, 15\}$**

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- Triples:

$$3^2 + 4^2 = 5^2$$
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$$5^2 + 12^2 = 13^2$$

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- **Triples:**

$$\begin{aligned} 3^2 + 4^2 &= 5^2 \\ 6^2 + 8^2 &= 10^2 \\ 9^2 + 12^2 &= 15^2 \\ 5^2 + 12^2 &= 13^2 \end{aligned}$$

- **Partition:** $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}, \{5, 10, 13, 15\}$

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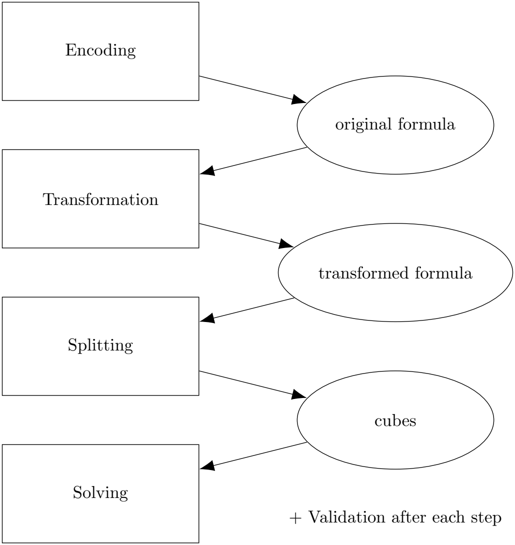
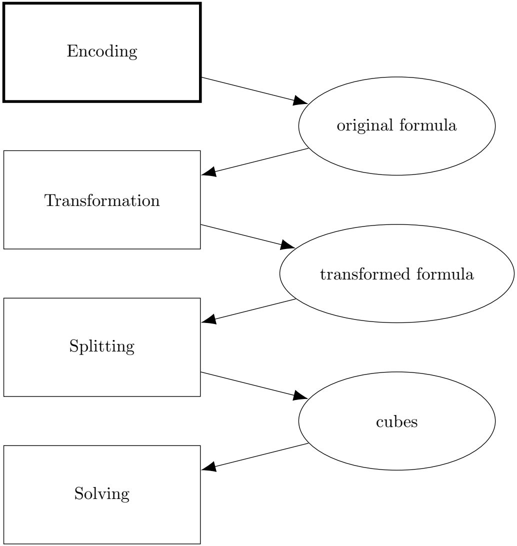
- We prove this theorem by considering two SAT problems:
  1. showing that $\{1, \ldots, 7824\}$ can be partitioned in two different parts.
  2. showing that any partition of $\{1, \ldots, 7825\}$ contains a Pythagorean triple.

Framework



+ Validation after each step

Encoding


Encoding
original formula
Transformation
transformed formula
Splitting
cubes
Solving

Encoding - Intuition

Idea:

- **one variable for each number**

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- **one variable for each number**
- **interpretation gives partition**
- **one constraint clause for each Pythagorean triple**

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Binary Pythagorean triple problem with n numbers

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For an interpretation $I \subseteq V$ with $I \models F$, the resulting partition is:

- $P_1 = \{x \mid p_x \in I\}$
- $P_2 = \{x \mid p_x \notin I\}$

Encoding - Example

As in beginning example, $\boldsymbol{n = 15}$

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F \quad = \quad & (p_3 \lor p_4 \lor p_5) \land (\neg p_3 \lor \neg p_4 \lor \neg p_5) \\
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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

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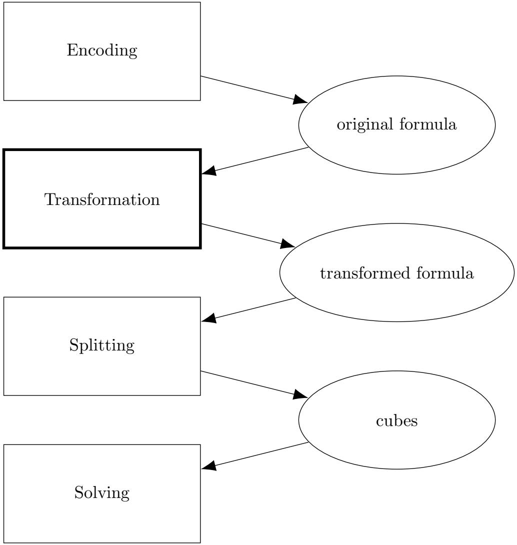
Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

Resulting partition:

- $P_1 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}$
- $P_2 = \{5, 10, 13, 15\}$

Transformatio



Transformation

Goal: from F, find formula F' which

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Approaches:

- **eliminate some particular clauses**

Transformation

Goal: from F, find formula F' which

- **is easier to solve**
- **preserves satisfiability**
- **has models that can be easily transformed into models for F**

Approaches:

- **eliminate some particular clauses**
- **break partition symmetry**

Clause elimination

$$
\begin{aligned}
F \quad = \quad & NotEqual(p_3, p_4, p_5) \\
\wedge \quad & NotEqual(p_6, p_9, p_{12}) \\
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Note p_3 only occurs in $NotEqual(p_3, p_4, p_5)$ and thus does not affect any other clauses

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- **$\rightarrow$ remember deleted clauses and modify interpretation of F' accordingly**

Clause elimination - Example

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$I \models F'$ but $I \not\models F$

$\longrightarrow$ account for deleted constraint $NotEqual(p_3, p_4, p_5)$

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$I' = \{p_3, p_6, p_{12}\}$

$I' \models F$

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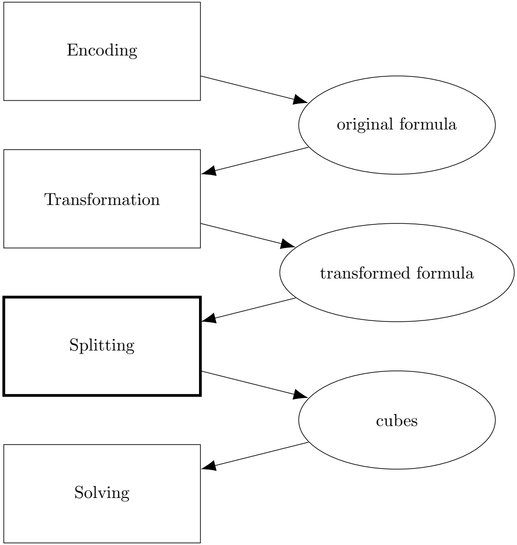
- **formulas F and F' are symmetric**
- **if $I \models F'$ then $V \setminus I \models F'$**
- **$\rightarrow$ introduce unit clause for variable p_x occurring in F'**
- **$F'' = F' \wedge p_x$**

Note that every model for F'' is a model for F' and the transformation is satisfiability preserving

Splitting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Cube-and-conquer solving

- **Problem: solving with conflict-driven clause learning (CDCL) is too slow**

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  - **use different heuristics $\Rightarrow$ cube-and-conquer solver (C&C)**

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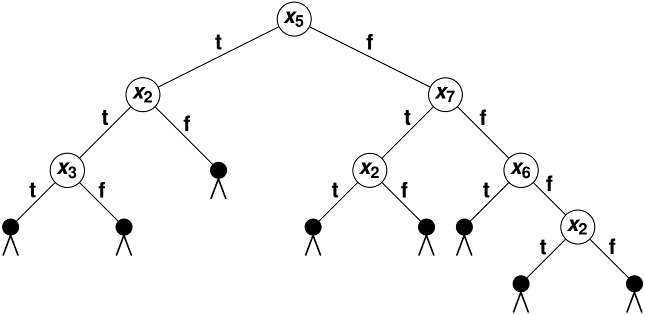

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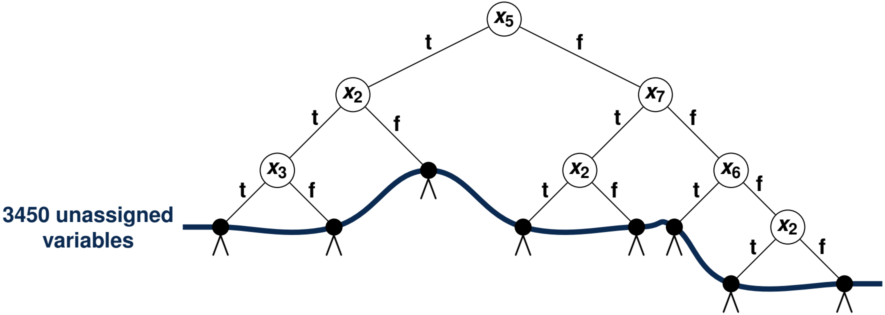

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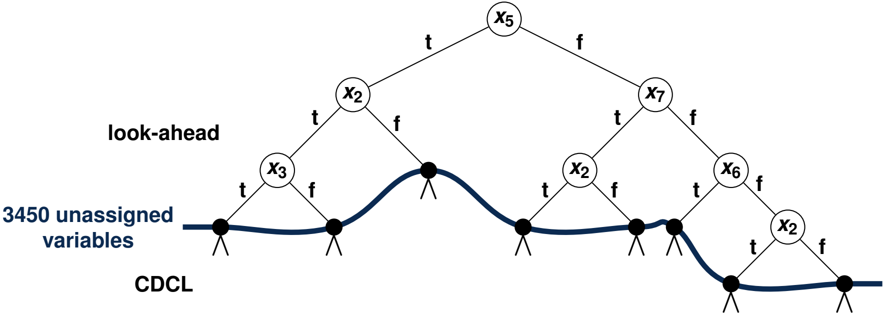

Cube-and-conquer solving

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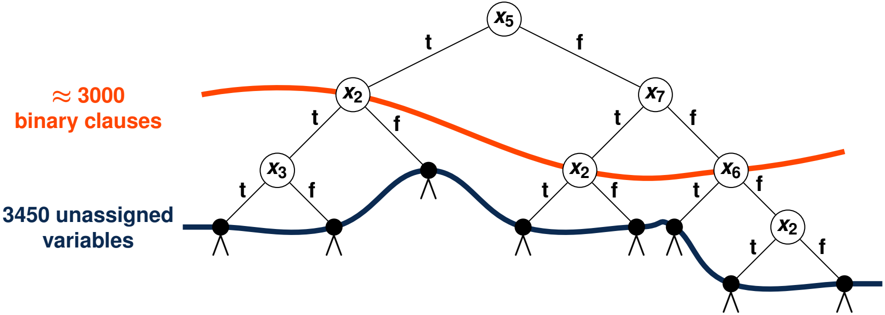
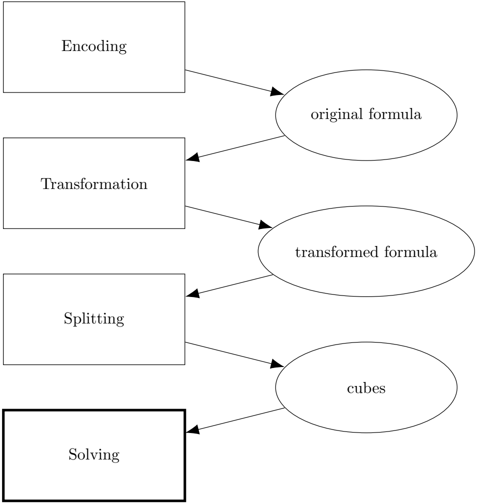
  - use parallelization (800 cores)



Solving

- $n = 7824$



Runtime

- **Splitting: 22000 CPU hours**
- **Solving: 13000 CPU hours**
- **Validation: 16000 CPU hours**
- **sums up to $\approx$ 5.8 CPU years**

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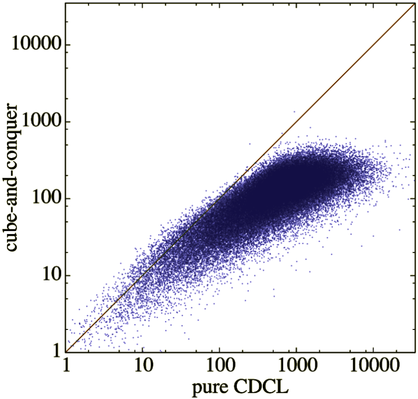
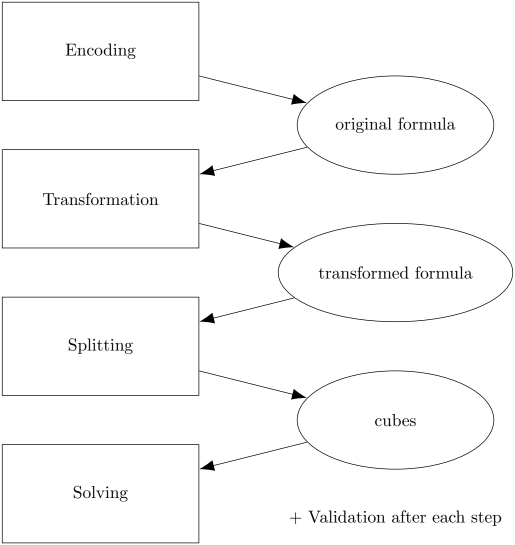


Solution



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