The boolean Pythagorean Triples problem

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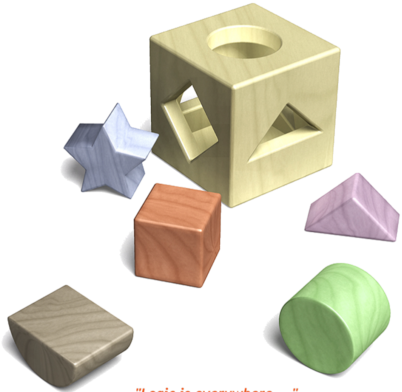
- **Introduction**
- **Framework**
- **Encoding**
- **Transformation**
- **Heuristic solving**



"Logic is everywhere ..."

Introduction

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- **Can the set $\mathbb{N}$ = {1, 2, 3, . . . } be divided in two parts such that no part contains a triple (a, b, c) with $a^2 + b^2 = c^2$**

Example

- **Set of Integers: $\{1, \ldots, 15\}$**

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- **Triples:**

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\begin{aligned}
3^2 + 4^2 &= 5^2 \\
6^2 + 8^2 &= 10^2 \\
9^2 + 12^2 &= 15^2 \\
5^2 + 12^2 &= 13^2
\end{aligned}
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- **Partition:** $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}, \{5, 10, 13, 15\}$

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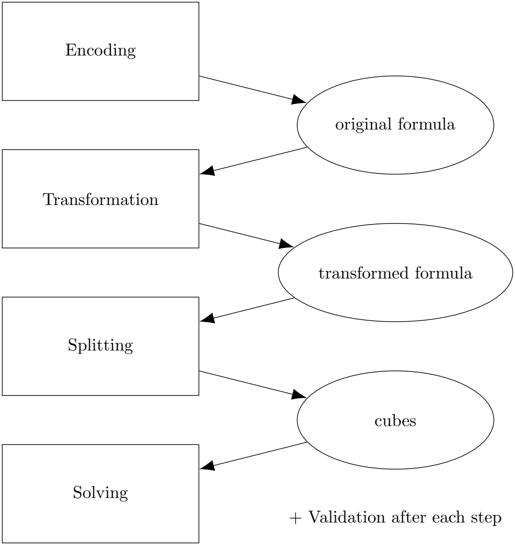
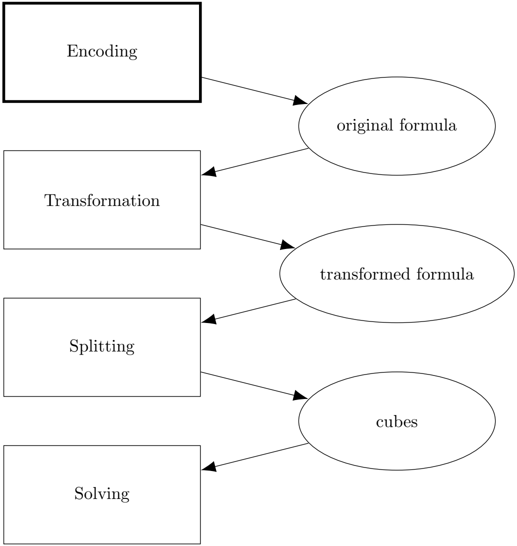
- We prove this theorem by considering two SAT problems:
  1. showing that $\{1, \ldots, 7824\}$ can be partitioned in two different parts.
  2. showing that any partition of $\{1, \ldots, 7825\}$ contains a Pythagorean triple.

Framework



+ Validation after each step

Encoding


Encoding
original formula
Transformation
transformed formula
Splitting
cubes
Solving

Encoding - Intuition

Idea:

- **one variable for each number**

Encoding - Intuition

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Encoding - Intuition

Idea:

- **one variable for each number**
- **interpretation gives partition**
- **one constraint clause for each Pythagorean triple**

Encoding

Binary Pythagorean triple problem with ***n*** numbers

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Set of variables $V = \{p_k \mid 1 \leq k \leq n\}$

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Constraint for all Pythagorean triples: $F = \bigwedge_{x^2+y^2=z^2} NotEqual(p_x, p_y, p_z)$

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Binary Pythagorean triple problem with n numbers

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Constraint for all Pythagorean triples: $F = \bigwedge_{x^2+y^2=z^2} \mathit{NotEqual}(p_x, p_y, p_z)$

For an interpretation $I \subseteq V$ with $I \models F$, the resulting partition is:

- $P_1 = \{x \mid p_x \in I\}$
- $P_2 = \{x \mid p_x \notin I\}$

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As in beginning example, $\boldsymbol{n = 15}$

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$$\begin{aligned} F \quad = \quad & (p_3 \vee p_4 \vee p_5) \wedge (\neg p_3 \vee \neg p_4 \vee \neg p_5) \\ \wedge \quad & (p_6 \vee p_8 \vee p_{10}) \wedge (\neg p_6 \vee \neg p_8 \vee \neg p_{10}) \\ \wedge \quad & (p_9 \vee p_{12} \vee p_{15}) \wedge (\neg p_9 \vee \neg p_{12} \vee \neg p_{15}) \\ \wedge \quad & (p_5 \vee p_{12} \vee p_{13}) \wedge (\neg p_5 \vee \neg p_{12} \vee \neg p_{13}) \end{aligned}$$

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\land \quad & (p_9 \lor p_{12} \lor p_{15}) \land (\neg p_9 \lor \neg p_{12} \lor \neg p_{15}) \\
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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

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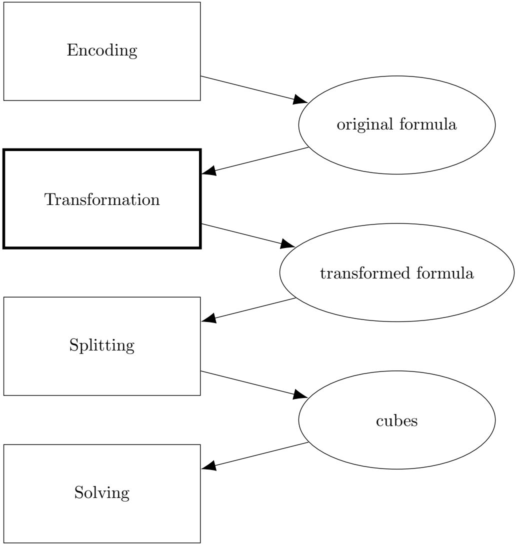
Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

Resulting partition:

- $P_1 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}$
- $P_2 = \{5, 10, 13, 15\}$

Transformation



Transformation

Goal: from F, find formula F' which

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Approaches:

- **eliminate some particular clauses**

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Goal: from F, find formula F' which

- **is easier to solve**
- **preserves satisfiability**
- **has models that can be easily transformed into models for F**

Approaches:

- **eliminate some particular clauses**
- **break partition symmetry**

Clause elimination

$$
\begin{aligned}
\boldsymbol{F} &= \boldsymbol{NotEqual}(\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5) \\
&\wedge \boldsymbol{NotEqual}(\boldsymbol{p}_6, \boldsymbol{p}_9, \boldsymbol{p}_{12}) \\
&\wedge \boldsymbol{NotEqual}(\boldsymbol{p}_9, \boldsymbol{p}_{12}, \boldsymbol{p}_{15}) \\
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Note $\boldsymbol{p}_3$ only occurs in $\boldsymbol{NotEqual}(\boldsymbol{p}_3, \boldsymbol{p}_4, \boldsymbol{p}_5)$ and thus does not affect any other clauses

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Note p_3 only occurs in $NotEqual(p_3, p_4, p_5)$ and thus does not affect any other clauses

- **if $p_4^I \neq p_5^I$ then $NotEqual(p_3, p_4, p_5)$ is satisfied**

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- **if $p_4^I \neq p_5^I$ then $NotEqual(p_3, p_4, p_5)$ is satisfied**
- **if $p_4^I = p_5^I = \top$ then $p_3^I = \bot$**

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- **remove $NotEqual(p_3, p_4, p_5)$ from F to obtain F'**
- **interpretation of p_3 is important but not represented in F'**
- **$\rightarrow$ remember deleted clauses and modify interpretation of F' accordingly**

Clause elimination - Example

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F' &= NotEqual(p_6, p_9, p_{12}) \\
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$I \models F'$ but $I \not\models F$

$\rightarrow$ account for deleted constraint $\mathit{NotEqual}(p_3, p_4, p_5)$

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$I' = \{p_3, p_6, p_{12}\}$

$I' \models F$

Breaking Symmetry

- **formulas F and F' are symmetric**

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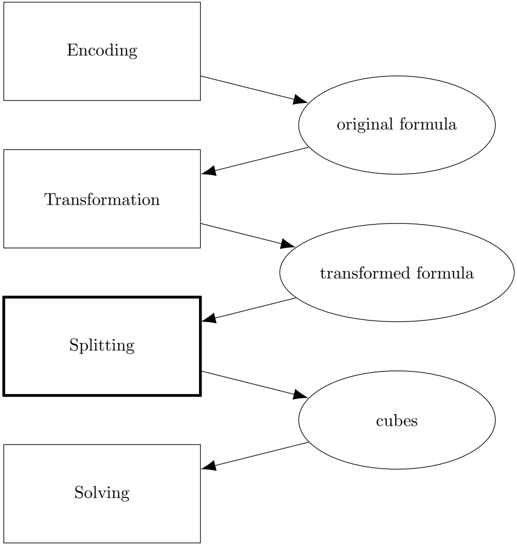
Breaking Symmetry

- **formulas F and F' are symmetric**
- **if $I \models F'$ then $V \setminus I \models F'$**
- **$\rightarrow$ introduce unit clause for variable p_x occurring in F'**
- **$F'' = F' \wedge p_x$**

Note that every model for F'' is a model for F' and the transformation is satisfiability preserving

Splitting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Cube-and-conquer solving

- **Problem: solving with conflict-driven clause learning (CDCL) is too slow**

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- **Solution:**
  - **use different heuristics $\Rightarrow$ cube-and-conquer solver**

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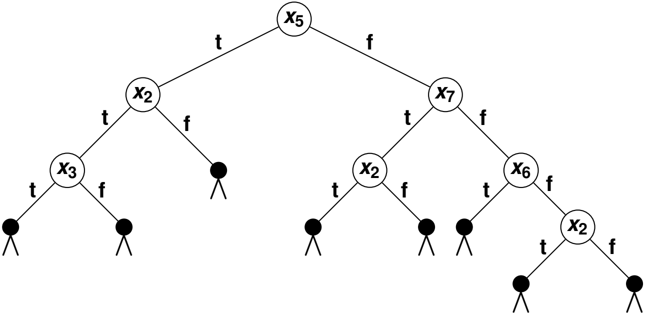

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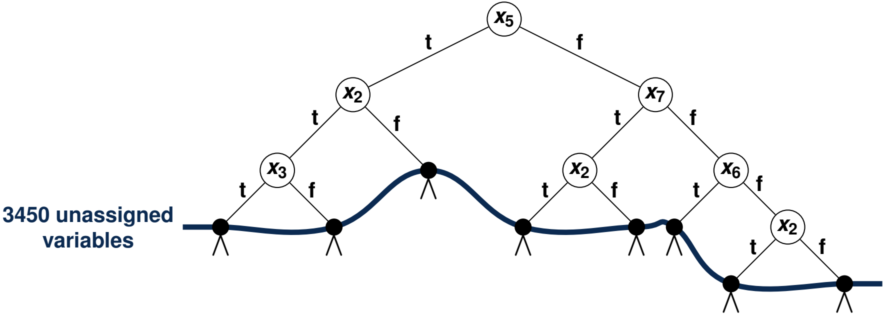

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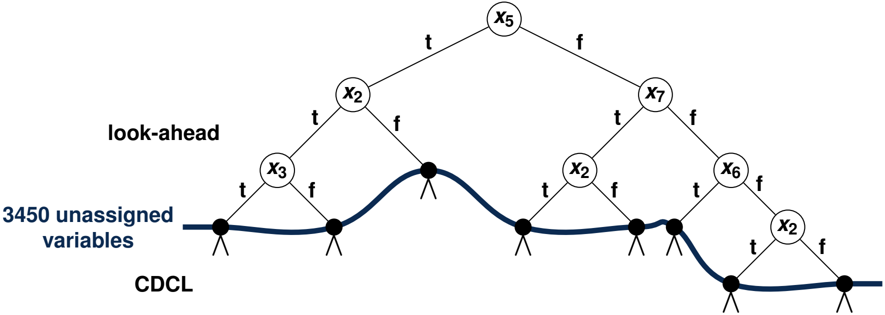

Cube-and-conquer solving

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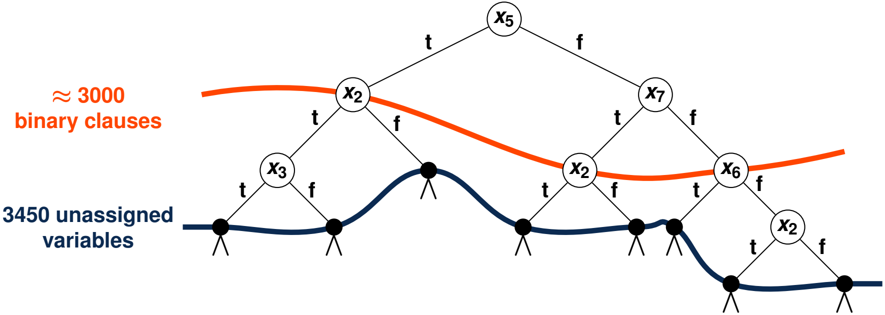
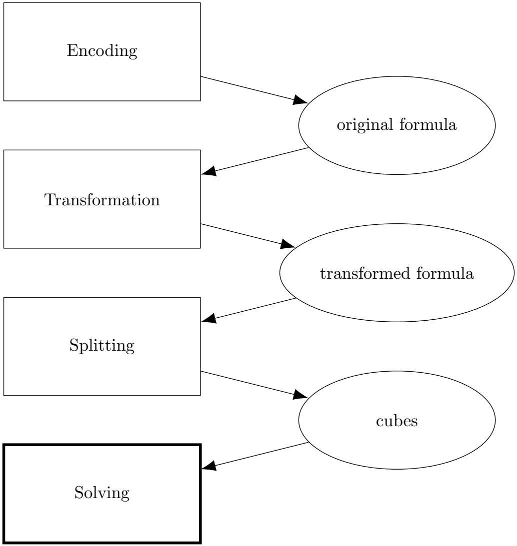
  - use parallelization (800 cores)



Solving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Runtime

- Splitting: 22000 CPU hours
- Solving: 13000 CPU hours
- Validation: 16000 CPU hours
- sums up to $\approx$ 5.8 CPU years

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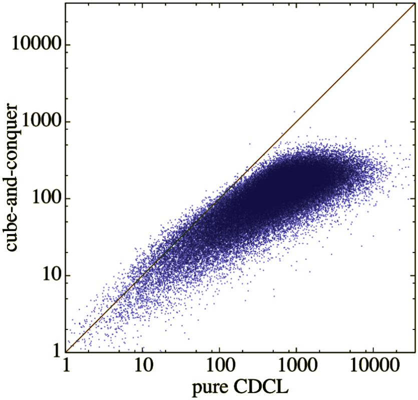
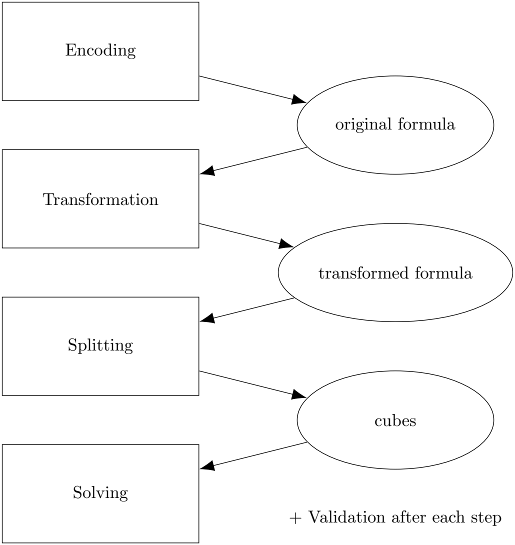


Solution



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