The boolean Pythagorean Triples problem

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Example

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- Triples:

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\end{aligned}
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- **Triples:**

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- **Partition:** $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}, \{5, 10, 13, 15\}$

Encoding - Intuition

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- **one variable for each number**

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- **one variable for each number**
- **interpretation gives partition**
- **one constraint clause for each Pythagorean triple**

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Binary Pythagorean triple problem with n numbers

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Set of variables $V = \{p_k \mid 1 \leq k \leq n\}$

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Constraint for non-equality: $\mathit{NotEqual}(x, y, z) = (x \lor y \lor z) \land (\neg x \lor \neg y \lor \neg z)$

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Constraint for all Pythagorean triples: $F = \bigwedge_{x^2+y^2=z^2} \mathit{NotEqual}(p_x, p_y, p_z)$

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Binary Pythagorean triple problem with $\boldsymbol{n}$ numbers

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For an interpretation $I \subseteq V$ with $I \models F$, the resulting partition is:

- $P_1 = \{x \mid p_x \in I\}$
- $P_2 = \{x \mid p_x \notin I\}$

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As in beginning example, $\boldsymbol{n} = \mathbf{15}$

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\begin{aligned}
F \quad = \quad & (p_3 \lor p_4 \lor p_5) \land (\neg p_3 \lor \neg p_4 \lor \neg p_5) \\
\land \quad & (p_6 \lor p_8 \lor p_{10}) \land (\neg p_6 \lor \neg p_8 \lor \neg p_{10}) \\
\land \quad & (p_9 \lor p_{12} \lor p_{15}) \land (\neg p_9 \lor \neg p_{12} \lor \neg p_{15}) \\
\land \quad & (p_5 \lor p_{12} \lor p_{13}) \land (\neg p_5 \lor \neg p_{12} \lor \neg p_{13})
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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

Resulting partition:

- $P_1 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}$
- $P_2 = \{5, 10, 13, 15\}$

Transformation

Goal: from $\boldsymbol{F}$, find formula $\boldsymbol{F'}$ which

- **is easier to solve**

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Approaches:

- **eliminate some particular clauses**
- **break partition symmetry**

Clause elimination

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\begin{aligned}
F \quad = \quad & NotEqual(p_3, p_4, p_5) \\
\wedge \quad & NotEqual(p_6, p_9, p_{12}) \\
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Note p_3 only occurs in $NotEqual(p_3, p_4, p_5)$ and thus does not affect any other clauses

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- **interpretation of p_3 is important but not represented in F'**
- **$\rightarrow$ remember deleted clauses and modify interpretation of F' accordingly**

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$I \models F'$ but $I \not\models F$

$\rightarrow$ account for deleted constraint $\mathit{NotEqual}(p_3, p_4, p_5)$

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$I' \models F$

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- **formulas F and F' are symmetric**
- **if $I \models F'$ then $V \setminus I \models F'$**
- **$\rightarrow$ introduce unit clause for variable p_x occurring in F'**
- **$F'' = F' \wedge p_x$**

Note that every model for F'' is a model for F' and the transformation is satisfiability preserving

Cube-and-conquer solving

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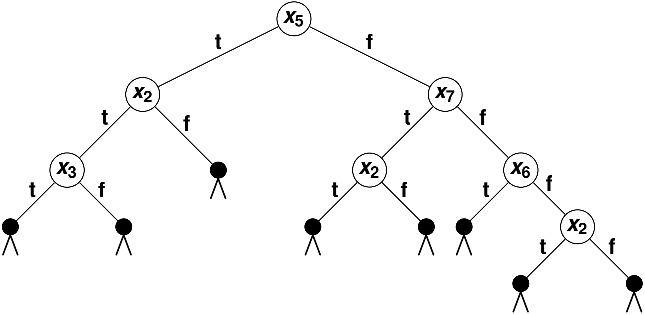

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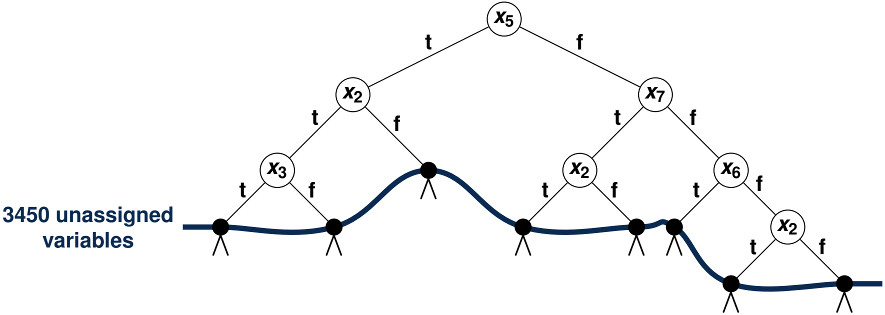

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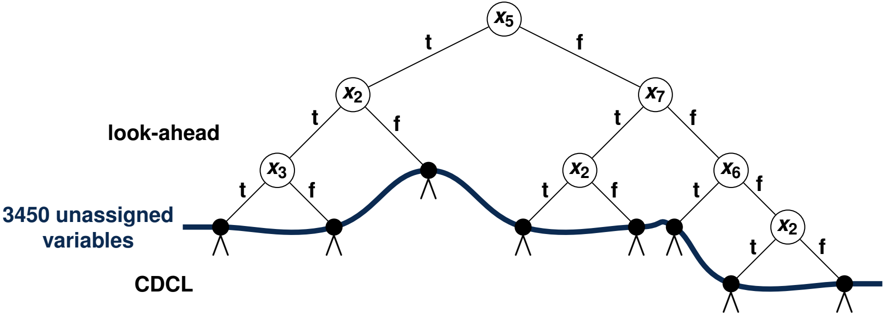

Cube-and-conquer solving

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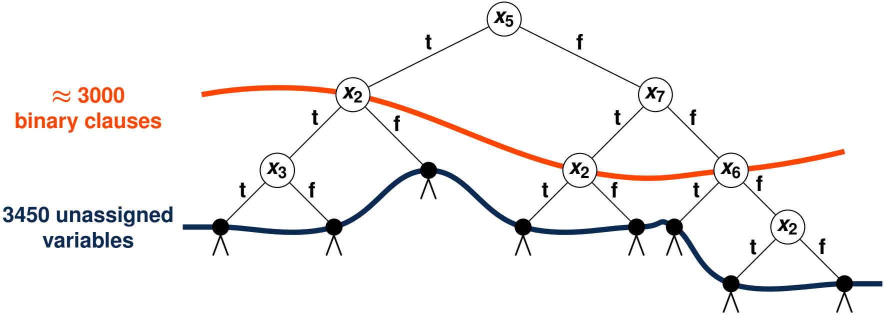
- **Solution:**
  - **use different heuristics ⇒ cube-and-conquer solver (C&C)**
  - **use parallelization (800 cores)**



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- Solving: 13000 CPU hours
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- sums up to $\approx$ 5.8 CPU years

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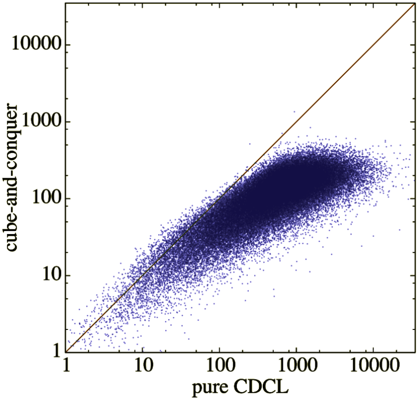

Framework

Picture of the Framework