The boolean Pythagorean Triples problem

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Example

- **Set of Integers: $\{1, \ldots, 15\}$**

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- **Triples:**

$$3^2 + 4^2 = 5^2$$
$$6^2 + 8^2 = 10^2$$
$$9^2 + 12^2 = 15^2$$
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- Triples:

$$3^2 + 4^2 = 5^2$$
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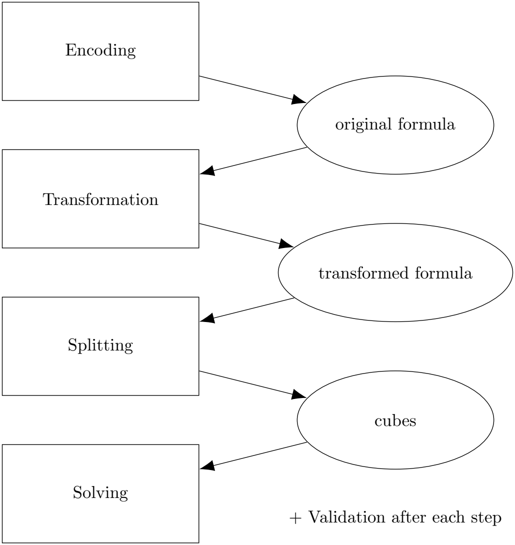
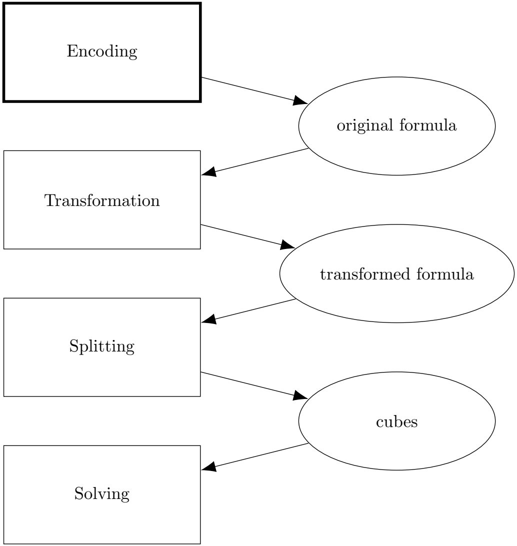
- Partition: $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}, \{5, 10, 13, 15\}$

Framework



+ Validation after each step

Encoding


Encoding
original formula
Transformation
transformed formula
Splitting
cubes
Solving

Encoding - Intuition

Idea:

- **one variable for each number**

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- **interpretation gives partition**

Encoding - Intuition

Idea:

- **one variable for each number**
- **interpretation gives partition**
- **one constraint clause for each Pythagorean triple**

Encoding

Binary Pythagorean triple problem with ***n*** numbers

Encoding

Binary Pythagorean triple problem with $\boldsymbol{n}$ numbers

Set of variables $\boldsymbol{V = \{p_k \mid 1 \leq k \leq n\}}$

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Binary Pythagorean triple problem with n numbers

Set of variables $V = \{p_k \mid 1 \leq k \leq n\}$

Constraint for non-equality: $\mathit{NotEqual}(x, y, z) = (x \lor y \lor z) \land (\lnot x \lor \lnot y \lor \lnot z)$

Encoding

Binary Pythagorean triple problem with n numbers

Set of variables $V = \{p_k \mid 1 \leq k \leq n\}$

Constraint for non-equality: $\mathit{NotEqual}(x, y, z) = (x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$

Constraint for all Pythagorean triples: $F = \bigwedge_{x^2+y^2=z^2} \mathit{NotEqual}(p_x, p_y, p_z)$

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For an interpretation $I \subseteq V$ with $I \models F$, the resulting partition is:

- $P_1 = \{x \mid p_x \in I\}$
- $P_2 = \{x \mid p_x \notin I\}$

Encoding - Example

As in beginning example, $\boldsymbol{n = 15}$

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$V = \{p_1, p_2, \ldots, p_{15}\}$

$$
\begin{aligned}
F \quad = \quad & (p_3 \vee p_4 \vee p_5) \wedge (\neg p_3 \vee \neg p_4 \vee \neg p_5) \\
\wedge \quad & (p_6 \vee p_8 \vee p_{10}) \wedge (\neg p_6 \vee \neg p_8 \vee \neg p_{10}) \\
\wedge \quad & (p_9 \vee p_{12} \vee p_{15}) \wedge (\neg p_9 \vee \neg p_{12} \vee \neg p_{15}) \\
\wedge \quad & (p_5 \vee p_{12} \vee p_{13}) \wedge (\neg p_5 \vee \neg p_{12} \vee \neg p_{13})
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Encoding - Example

As in beginning example, $n = 15$

$V = \{p_1, p_2, \ldots, p_{15}\}$

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F \quad = \quad & (p_3 \lor p_4 \lor p_5) \land (\neg p_3 \lor \neg p_4 \lor \neg p_5) \\
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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

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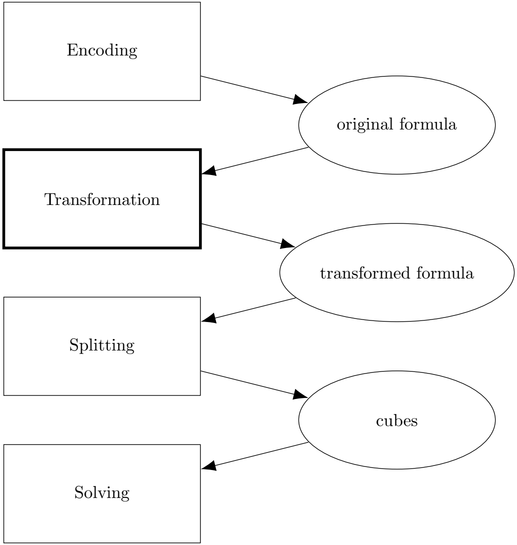
Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

Resulting partition:

- $P_1 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}$
- $P_2 = \{5, 10, 13, 15\}$

Transformatio



Transformation

Goal: from $\boldsymbol{F}$, find formula $\boldsymbol{F'}$ which

- **is easier to solve**

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Approaches:

- **eliminate some particular clauses**

Transformation

Goal: from F, find formula F' which

- **is easier to solve**
- **preserves satisfiability**
- **has models that can be easily transformed into models for F**

Approaches:

- **eliminate some particular clauses**
- **break partition symmetry**

Clause elimination

$$
\begin{aligned}
F \quad = \quad & \mathit{NotEqual}(p_3, p_4, p_5) \\
\wedge \quad & \mathit{NotEqual}(p_6, p_9, p_{12}) \\
\wedge \quad & \mathit{NotEqual}(p_9, p_{12}, p_{15}) \\
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Note p_3 only occurs in $\mathit{NotEqual}(p_3, p_4, p_5)$ and thus does not affect any other clauses

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- **if $p_4^I \neq p_5^I$ then $NotEqual(p_3, p_4, p_5)$ is satisfied**

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- interpretation of p_3 is important but not represented in F'
- $\rightarrow$ remember deleted clauses and modify interpretation of F' accordingly

Clause elimination - Example

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$I \models F'$ but $I \not\models F$

$\rightarrow$ account for deleted constraint $\mathit{NotEqual}(p_3, p_4, p_5)$

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$I' = \{p_3, p_6, p_{12}\}$

$I' \models F$

Breaking Symmetry

- **formulas F and F' are symmetric**

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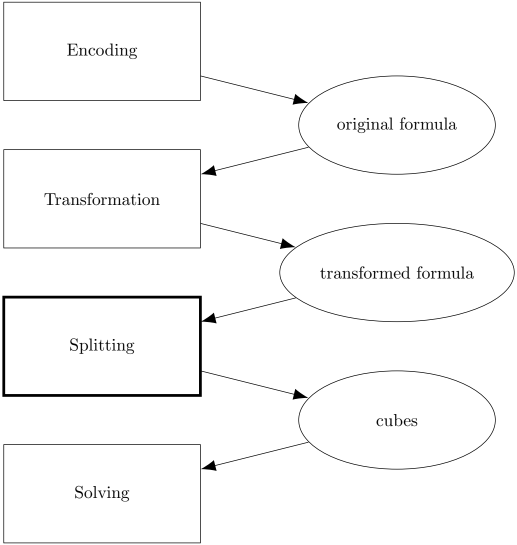
- $\rightarrow$ introduce unit clause for variable p_x occurring in F'
- $F'' = F' \wedge p_x$

Breaking Symmetry

- **formulas F and F' are symmetric**
- **if $I \models F'$ then $V \setminus I \models F'$**
- **$\rightarrow$ introduce unit clause for variable p_x occurring in F'**
- **$F'' = F' \wedge p_x$**

Note that every model for F'' is a model for F' and the transformation is satisfiability preserving

Splitting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Cube-and-conquer solving

- **Problem: solving with conflict-driven clause learning (CDCL) is too slow**

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- Solution:
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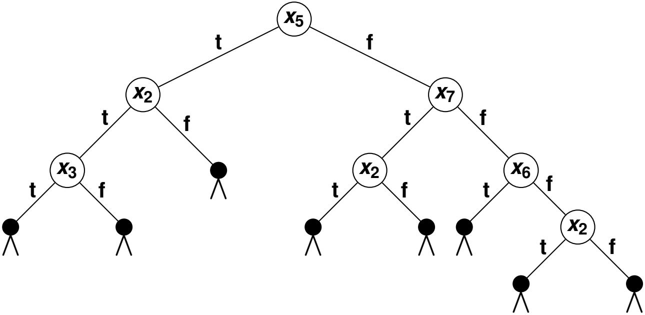

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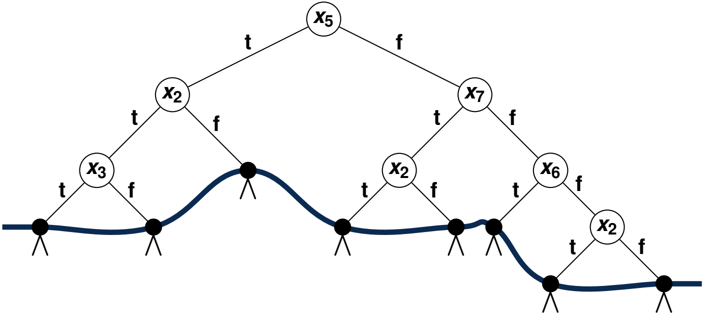
3450 unassigned variables



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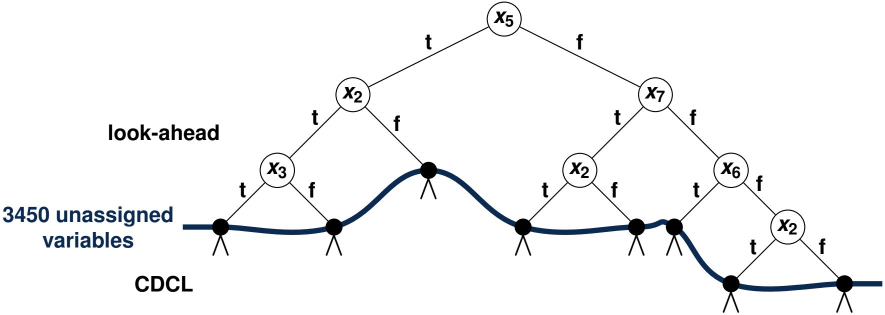

Cube-and-conquer solving

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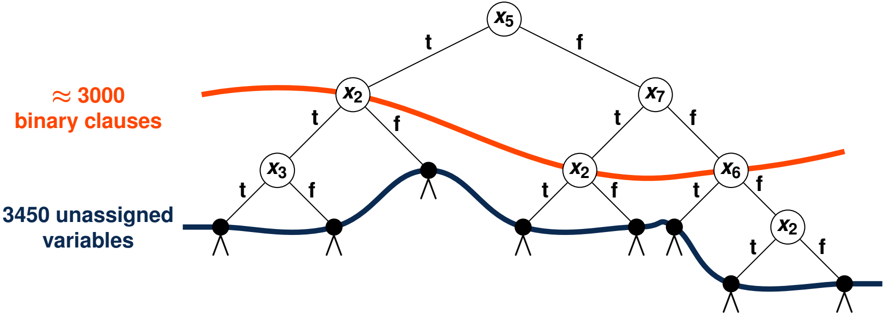
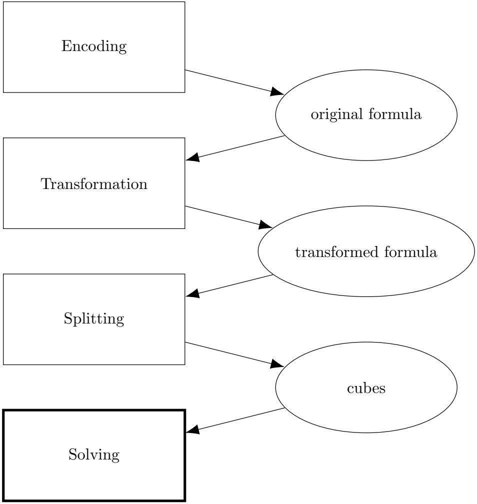
- Solution:
  - use different heuristics ⇒ cube-and-conquer solver (C&C)
  - use parallelization (800 cores)



Solving

- $n = 7824$

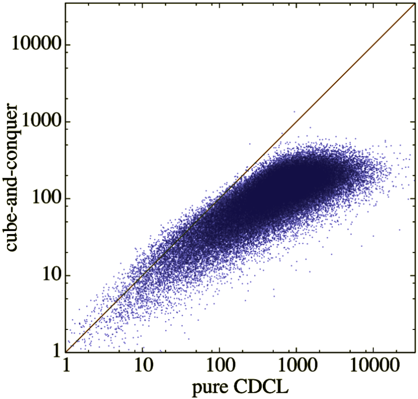


Runtime

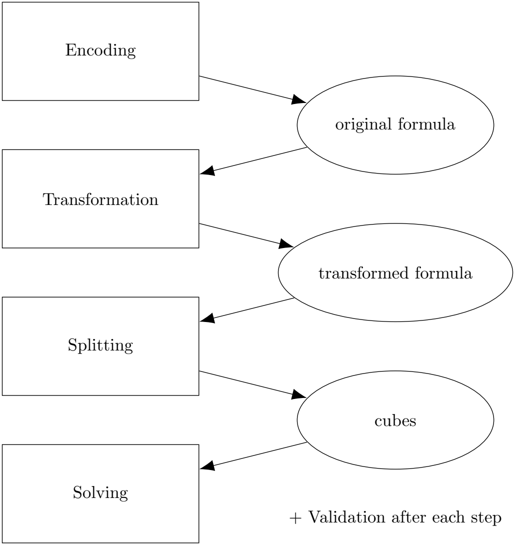
- **Splitting: 22000 CPU hours**
- **Solving: 13000 CPU hours**
- **Validation: 16000 CPU hours**
- **sums up to $\approx$ 5.8 CPU years**

Runtime

- Splitting: 22000 CPU hours
- Solving: 13000 CPU hours
- Validation: 16000 CPU hours
- sums up to ≈ 5.8 CPU years



Solution



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