The boolean Pythagorean Triples problem

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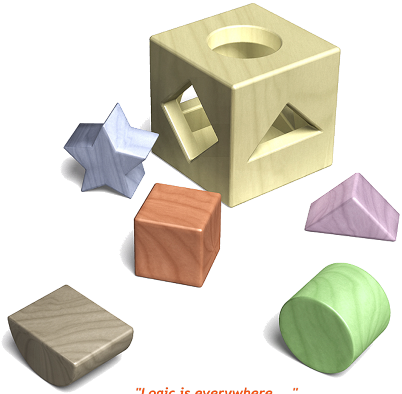
Germany

- Introduction
- History of the problem
- The framework
- Encoding
- Transformation
- Splitting
- Validation for each step



"Logic is everywhere ..."

Introduction

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- **Can the set $\mathbb{N}$ = {1, 2, 3, . . . } be divided in two parts such that no part contains a triple (a, b, c) with $a^2 + b^2 = c^2$**

Example

- **Set of Integers: $\{1, ..., 15\}$**

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- **Triples:**

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$$6^2 + 8^2 = 10^2$$
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- **Partition:** $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}, \{5, 10, 13, 15\}$

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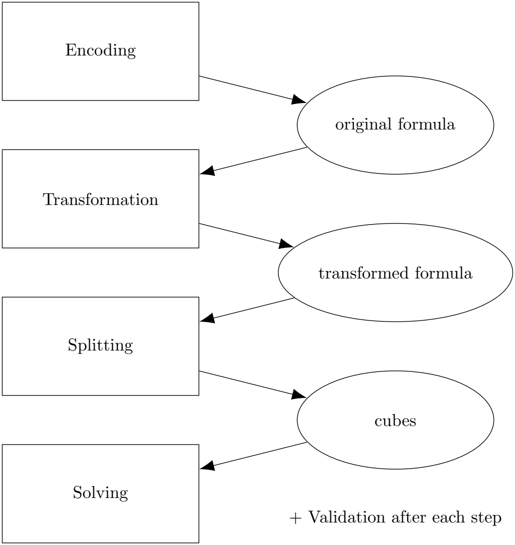
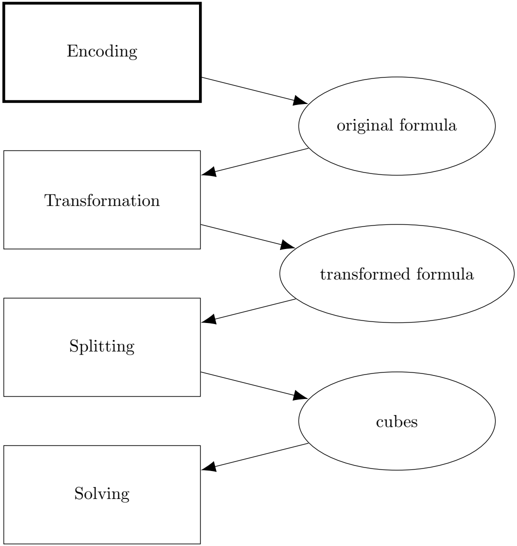
- We prove this theorem by considering two SAT problems:
  1. showing that $\{1, ..., 7824\}$ can be partitioned in two different parts.
  2. showing that any partition of $\{1, ..., 7825\}$ contains a Pythagorean triple.

Framework



+ Validation after each step

Encoding


Encoding
original formula
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Encoding - Intuition

Idea:

- **one variable for each number**

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- **interpretation gives partition**

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- **one variable for each number**
- **interpretation gives partition**
- **one constraint clause for each Pythagorean triple**

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Binary Pythagorean triple problem with n numbers

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Set of variables $V = \{p_k \mid 1 \leq k \leq n\}$

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Constraint for non-equality: $\mathit{NotEqual}(x, y, z) = (x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$

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Constraint for all Pythagorean triples: $F = \bigwedge_{x^2+y^2=z^2} NotEqual(p_x, p_y, p_z)$

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Binary Pythagorean triple problem with n numbers

Set of variables $V = \{p_k \mid 1 \leq k \leq n\}$

Constraint for non-equality: $NotEqual(x, y, z) = (x \lor y \lor z) \land (\neg x \lor \neg y \lor \neg z)$

Constraint for all Pythagorean triples: $F = \bigwedge_{x^2+y^2=z^2} NotEqual(p_x, p_y, p_z)$

For an interpretation $I \subseteq V$ with $I \models F$, the resulting partition is:

- $P_1 = \{x \mid p_x \in I\}$
- $P_2 = \{x \mid p_x \notin I\}$

Encoding - Example

As in beginning example, $n = 15$

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$$
\begin{aligned}
F \quad = \quad & (p_3 \vee p_4 \vee p_5) \wedge (\neg p_3 \vee \neg p_4 \vee \neg p_5) \\
\wedge \quad & (p_6 \vee p_8 \vee p_{10}) \wedge (\neg p_6 \vee \neg p_8 \vee \neg p_{10}) \\
\wedge \quad & (p_9 \vee p_{12} \vee p_{15}) \wedge (\neg p_9 \vee \neg p_{12} \vee \neg p_{15}) \\
\wedge \quad & (p_5 \vee p_{12} \vee p_{13}) \wedge (\neg p_5 \vee \neg p_{12} \vee \neg p_{13})
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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

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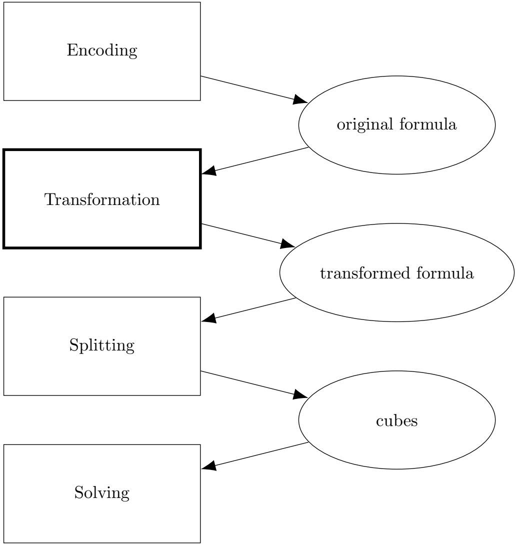
Resulting partition:

- $P_1 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}$
- $P_2 = \{5, 10, 13, 15\}$

Transformatio



Transformation

Goal: from F, find formula F' which

- **is easier to solve**

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Approaches:

- **eliminate some particular clauses**

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- **is easier to solve**
- **preserves satisfiability**
- **has models that can be easily transformed into models for *F***

Approaches:

- **eliminate some particular clauses**
- **break partition symmetry**

Clause elimination

$$\begin{aligned} F \quad = \quad & \mathit{NotEqual}(p_3, p_4, p_5) \\ \wedge \quad & \mathit{NotEqual}(p_6, p_9, p_{12}) \\ \wedge \quad & \mathit{NotEqual}(p_9, p_{12}, p_{15}) \\ \wedge \quad & \mathit{NotEqual}(p_5, p_{12}, p_{13}) \end{aligned}$$

Note p_3 only occurs in $\mathit{NotEqual}(p_3, p_4, p_5)$ and thus does not affect any other clauses

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- **remove $NotEqual(p_3, p_4, p_5)$ from F to obtain F'**

Clause elimination

$$F = NotEqual(p_3, p_4, p_5) \wedge NotEqual(p_6, p_9, p_{12}) \wedge NotEqual(p_9, p_{12}, p_{15}) \wedge NotEqual(p_5, p_{12}, p_{13})$$

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- **$\rightarrow$ remember deleted clauses and modify interpretation of F' accordingly**

Clause elimination - Example

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$I \models F'$ but $I \not\models F$

$\rightarrow$ account for deleted constraint $NotEqual(p_3, p_4, p_5)$

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$I' = \{p_3, p_6, p_{12}\}$

$I' \models F$

Breaking Symmetry

- **formulas F and F' are symmetric**

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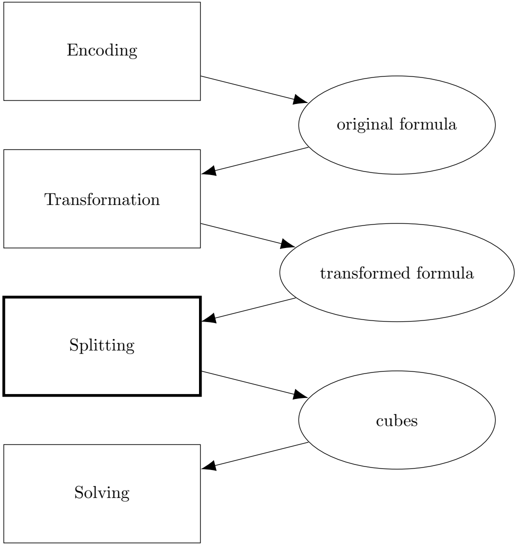
Breaking Symmetry

- **formulas F and F' are symmetric**
- **if $I \models F'$ then $V \setminus I \models F'$**
- **$\rightarrow$ introduce unit clause for variable p_x occurring in F'**
- **$F'' = F' \wedge p_x$**

Note that every model for F'' is a model for F' and the transformation is satisfiability preserving

Splitting


Encoding
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Solving

Cube-and-conquer solving

- **Problem: solving with conflict-driven clause learning (CDCL) is too slow**

Cube-and-conquer solving

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- **Solution:**
  - **use different heuristics $\Rightarrow$ cube-and-conquer solver (C&C)**

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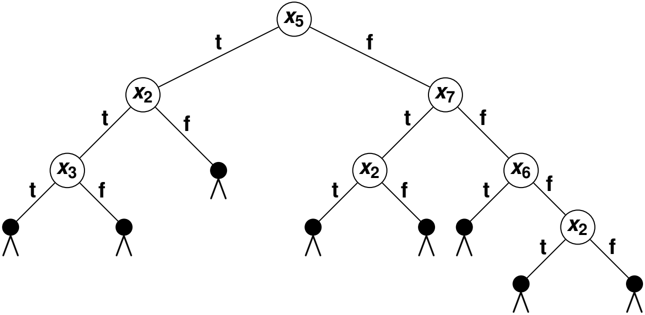

Cube-and-conquer solving

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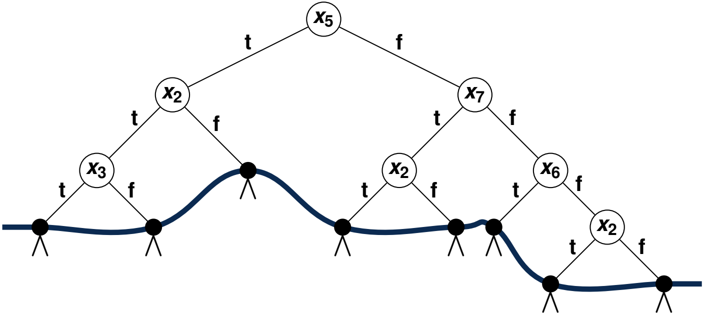
3450 unassigned variables



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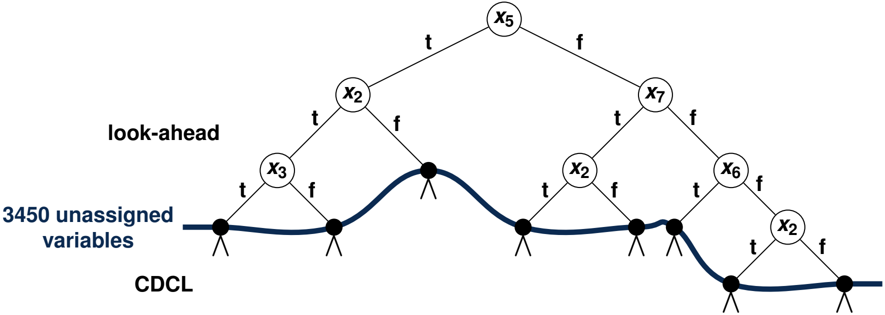

Cube-and-conquer solving

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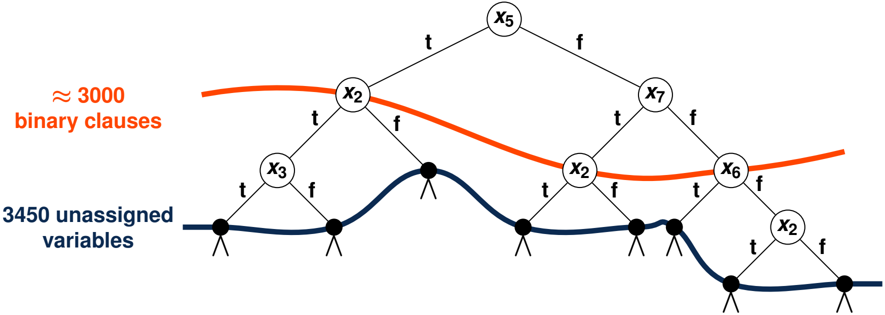
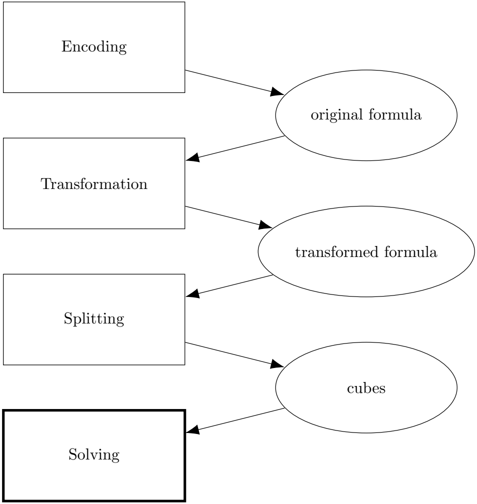
  - use different heuristics ⇒ cube-and-conquer solver (C&C)
  - use parallelization (800 cores)



Solving

- $n = 7824$



Runtime

- **Splitting: 22000 CPU hours**
- **Solving: 13000 CPU hours**
- **Validation: 16000 CPU hours**
- **sums up to $\approx$ 5.8 CPU years**

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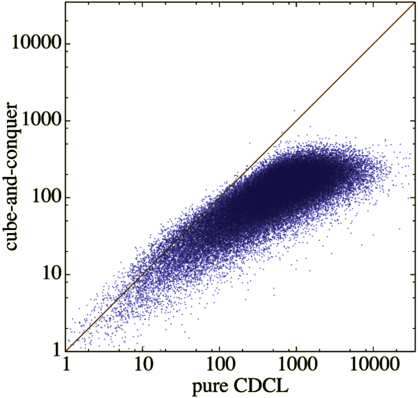
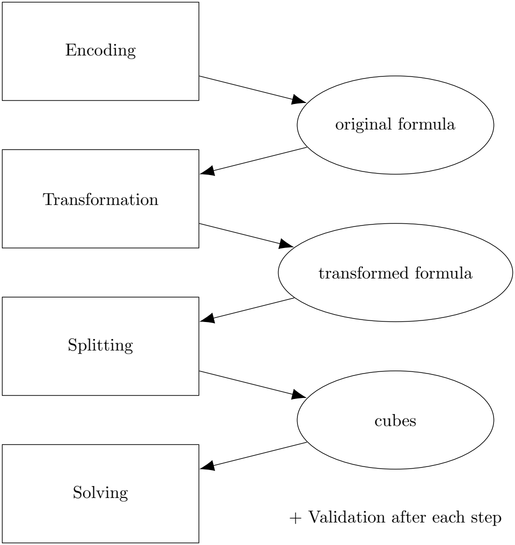


Solution



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