The boolean Pythagorean Triples problem

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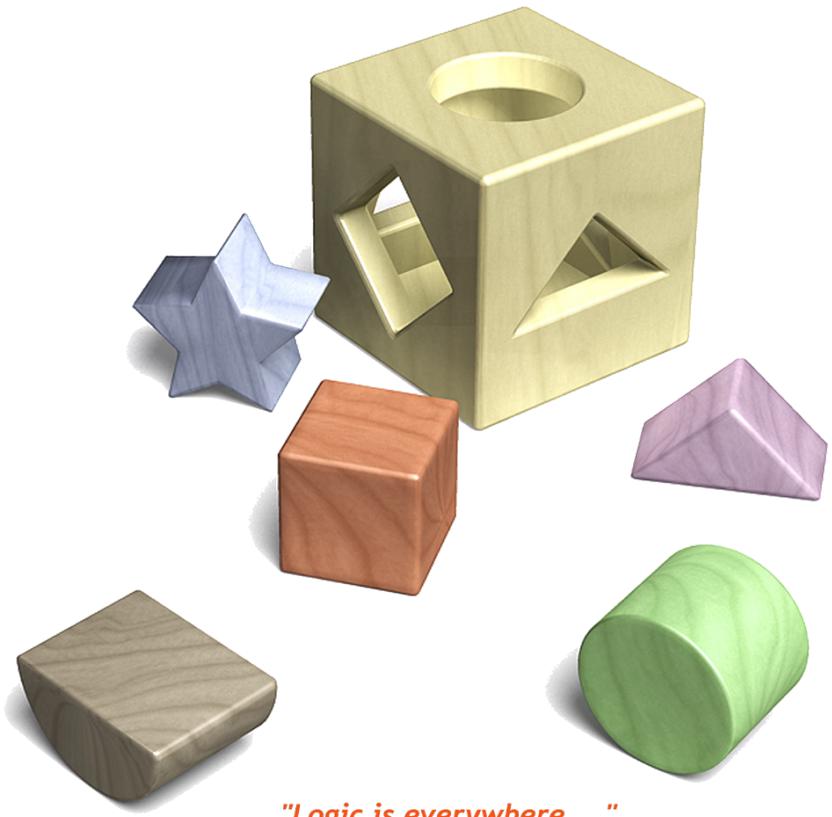
- Introduction
- History of the problem
- The framework
- Encoding
- Transformation
- Heuristic solving



"Logic is everywhere ..."

Introduction

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- Can the set $\mathbb{N} = \{1, 2, 3, \ldots\}$ be divided in two parts such that no part contains a triple (a, b, c) with $a^2 + b^2 = c^2$

Example

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- **Triples:**

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- **Partition:** $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}, \{5, 10, 13, 15\}$

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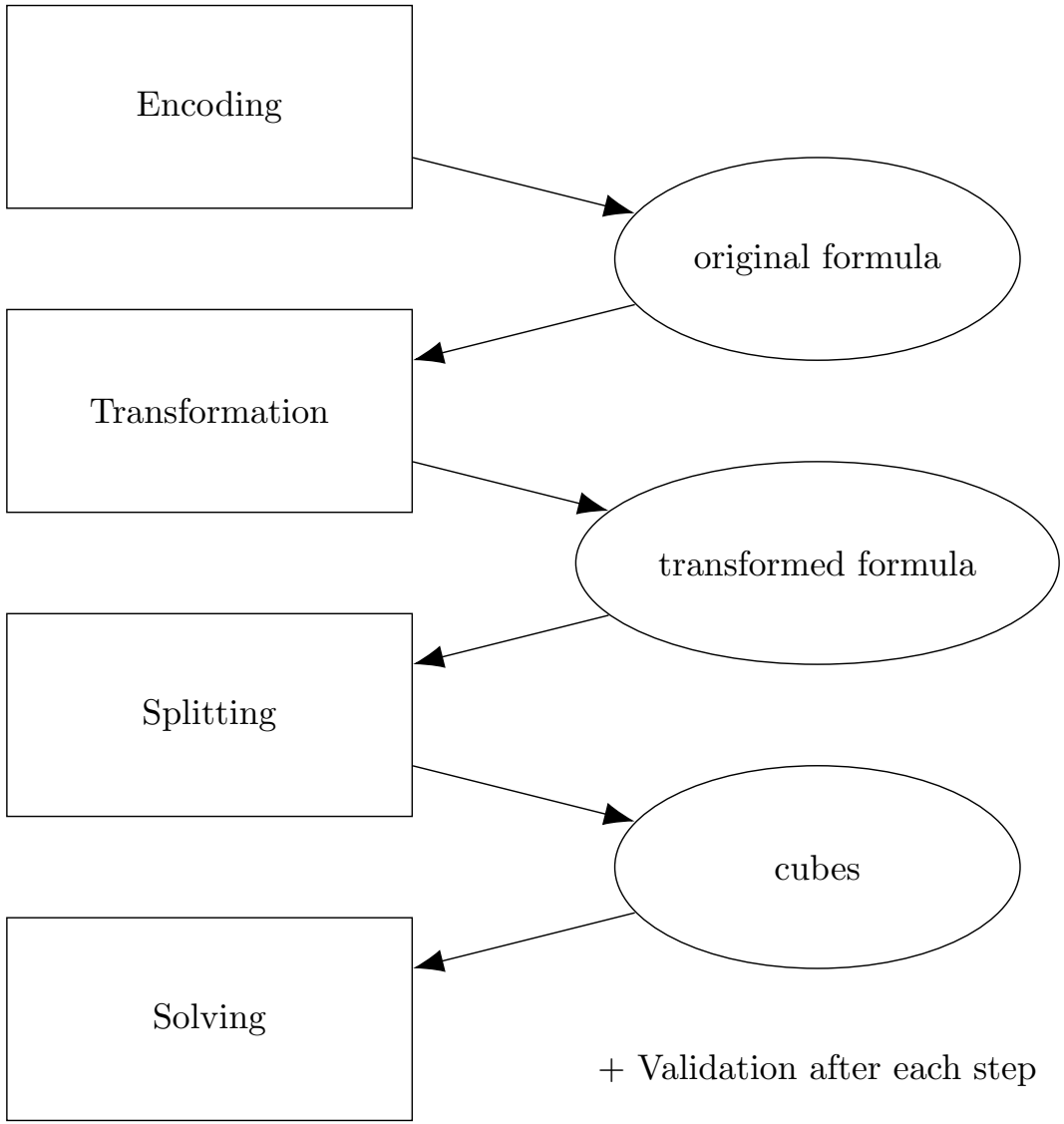
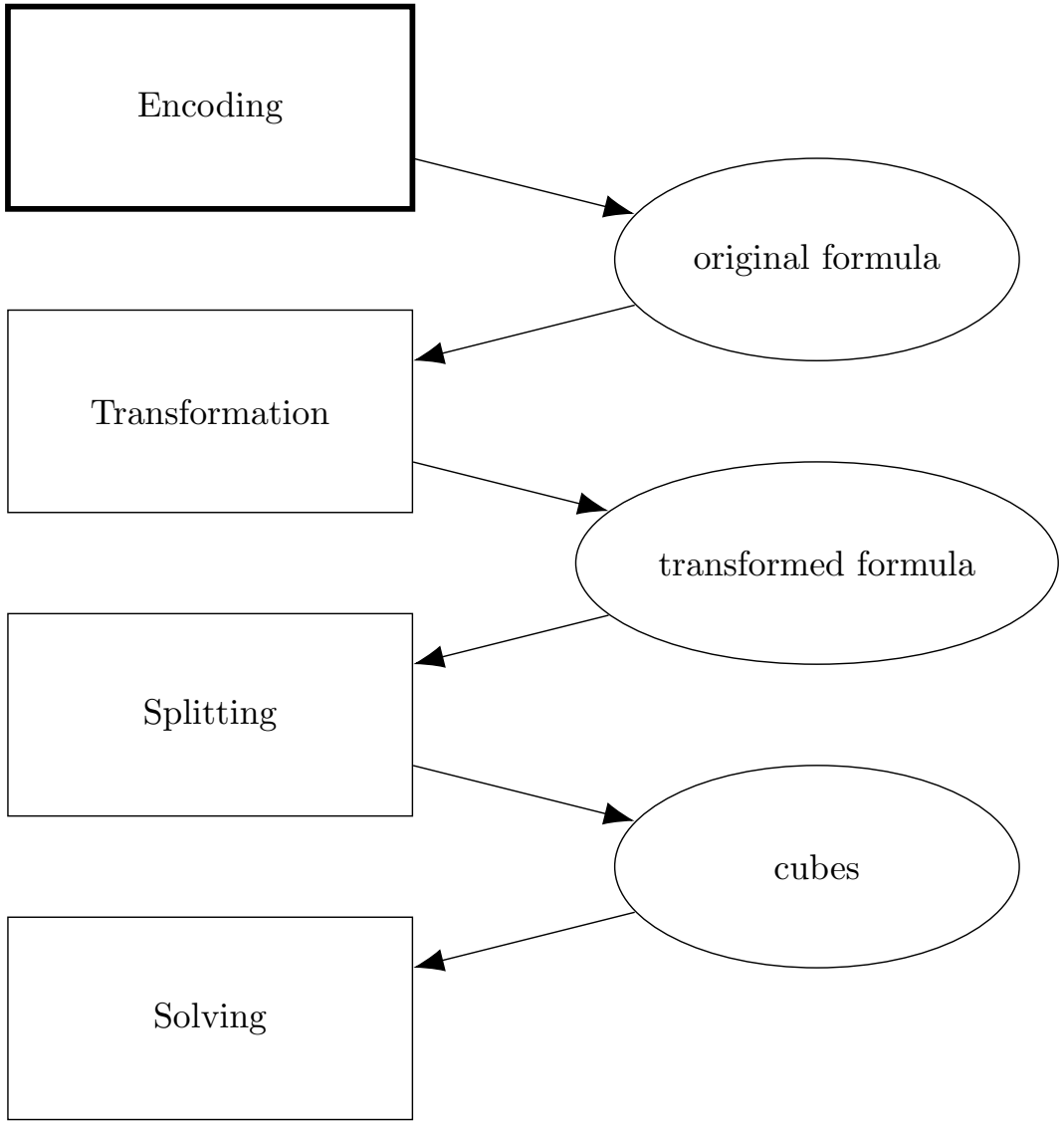
- We prove this theorem by considering two SAT problems:
  1. showing that $\{1, \ldots, 7824\}$ can be partitioned in two different parts.
  2. showing that any partition of $\{1, \ldots, 7825\}$ contains a Pythagorean triple.

Framework


Encoding
original formula
Transformation
transformed formula
Splitting
cubes
Solving
+ Validation after each step

Encoding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Encoding - Intuition

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- **one constraint clause for each Pythagorean triple**

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Binary Pythagorean triple problem with ***n*** numbers

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Set of variables $V = \{p_k \mid 1 \leq k \leq n\}$

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For an interpretation $I \subseteq V$ with $I \models F$, the resulting partition is:

- $P_1 = \{x \mid p_x \in I\}$
- $P_2 = \{x \mid p_x \notin I\}$

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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

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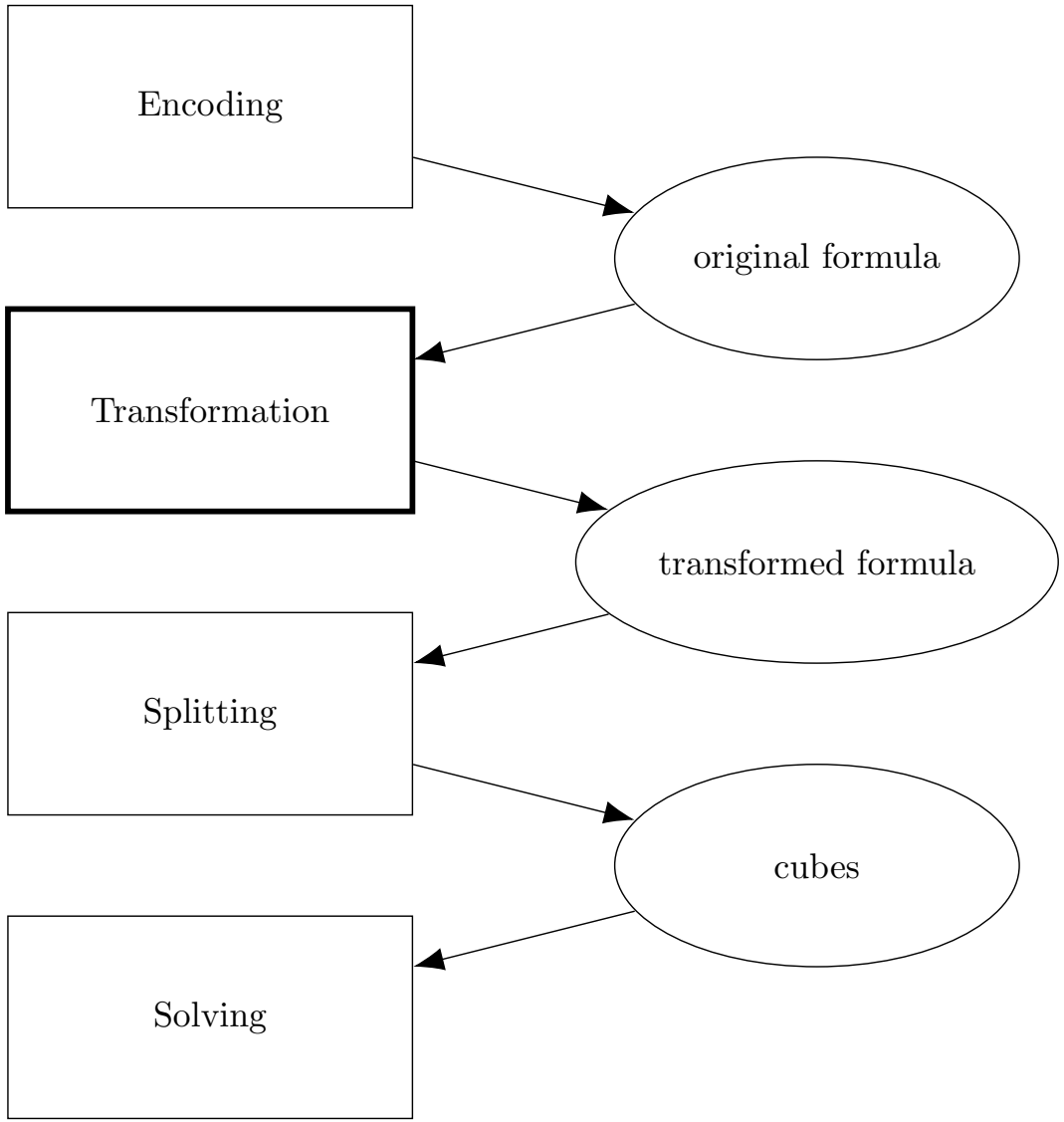
Resulting partition:

- $P_1 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}$
- $P_2 = \{5, 10, 13, 15\}$

Transformatio



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Approaches:

- **eliminate some particular clauses**
- **break partition symmetry**

Clause elimination

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F \quad = \quad & \mathit{NotEqual}(p_3, p_4, p_5) \\
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- **$\rightarrow$ remember deleted clauses and modify interpretation of F' accordingly**

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$I \models F'$ but $I \not\models F$

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$I' = \{p_3, p_6, p_{12}\}$

$I' \models F$

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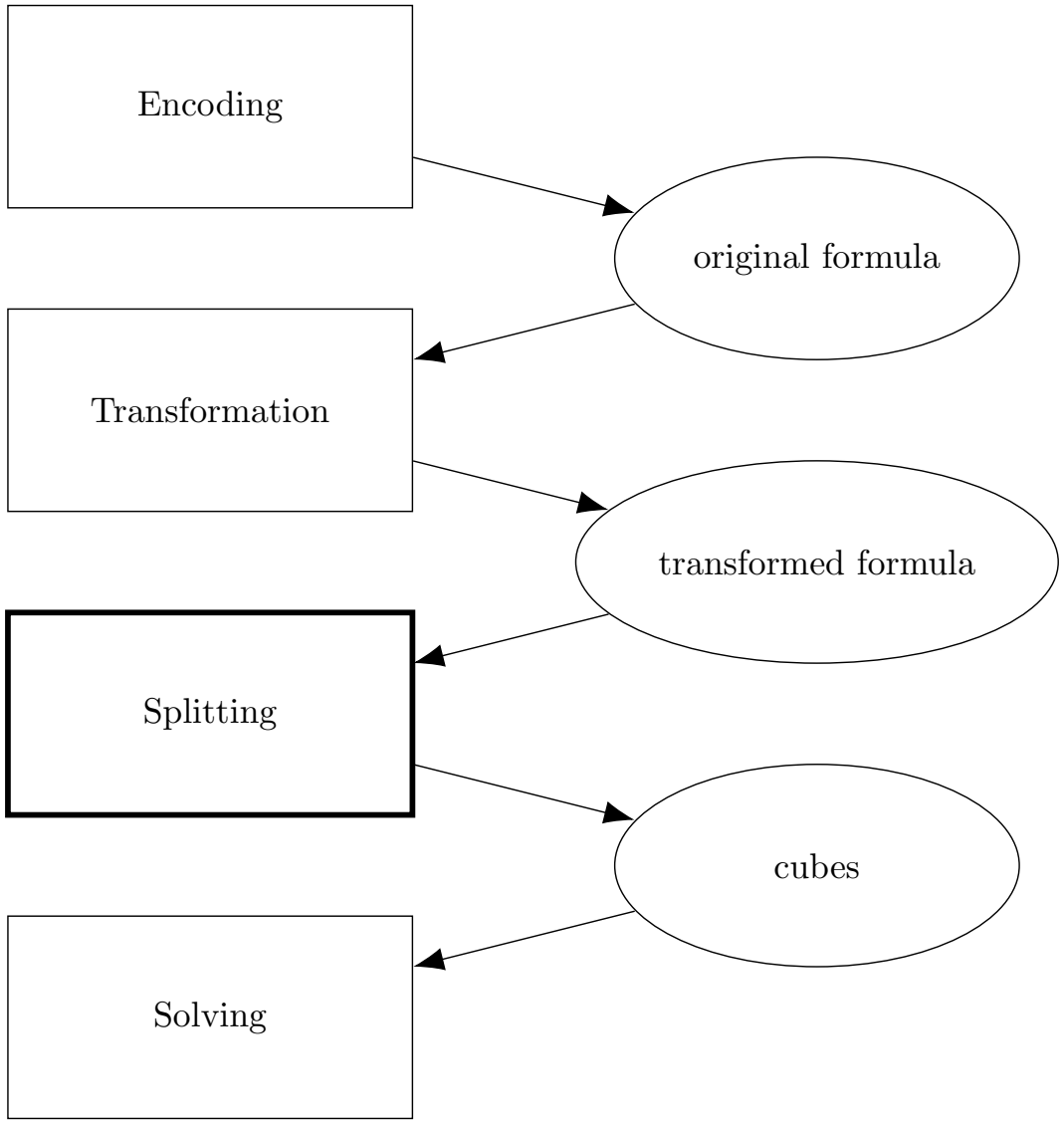
- **if $I \models F'$ then $V \setminus I \models F'$**
- **$\rightarrow$ introduce unit clause for variable p_x occurring in F'**
- **$F'' = F' \wedge p_x$**

Note that every model for F'' is a model for F' and the transformation is satisfiability preserving

Splitting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Cube-and-conquer solving

- **Problem: solving with conflict-driven clause learning (CDCL) is too slow**

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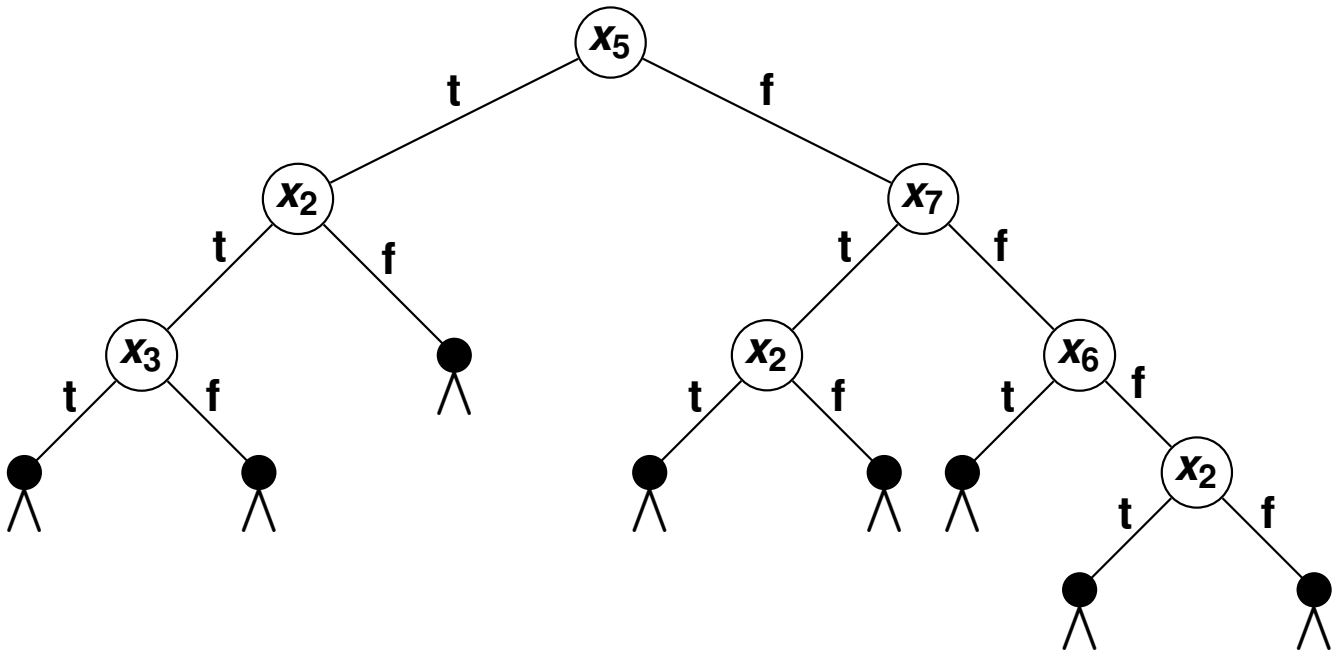

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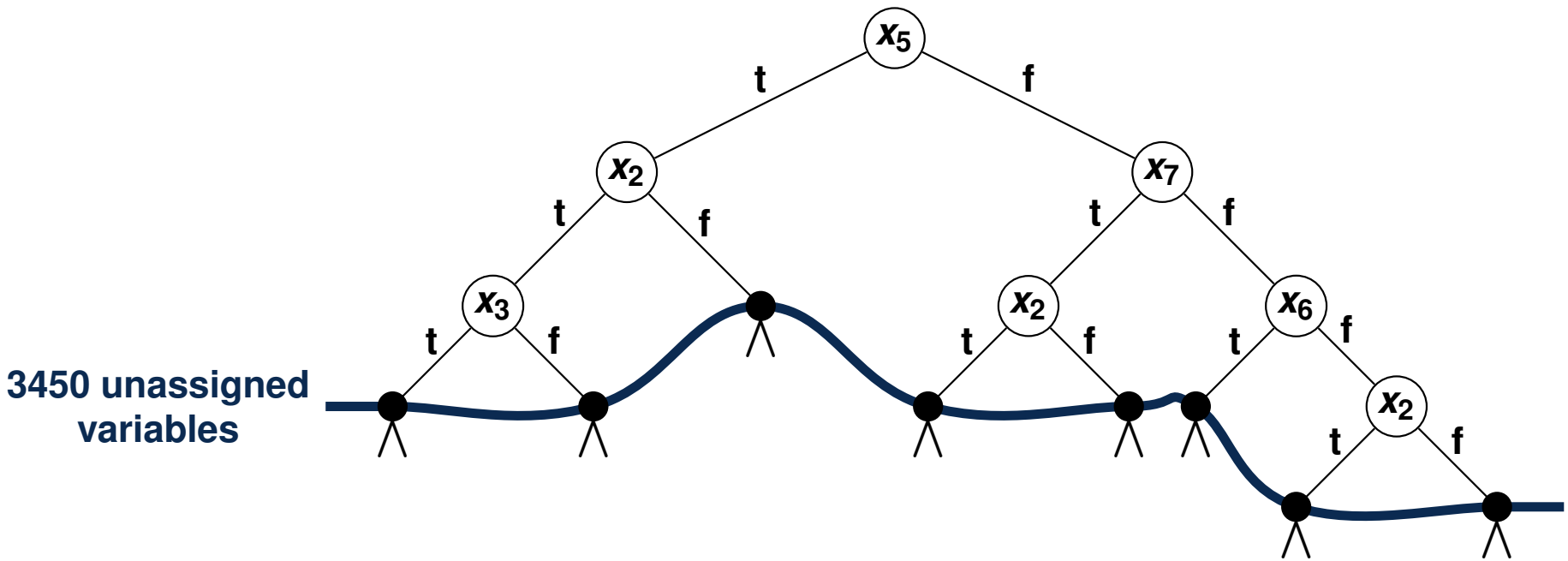

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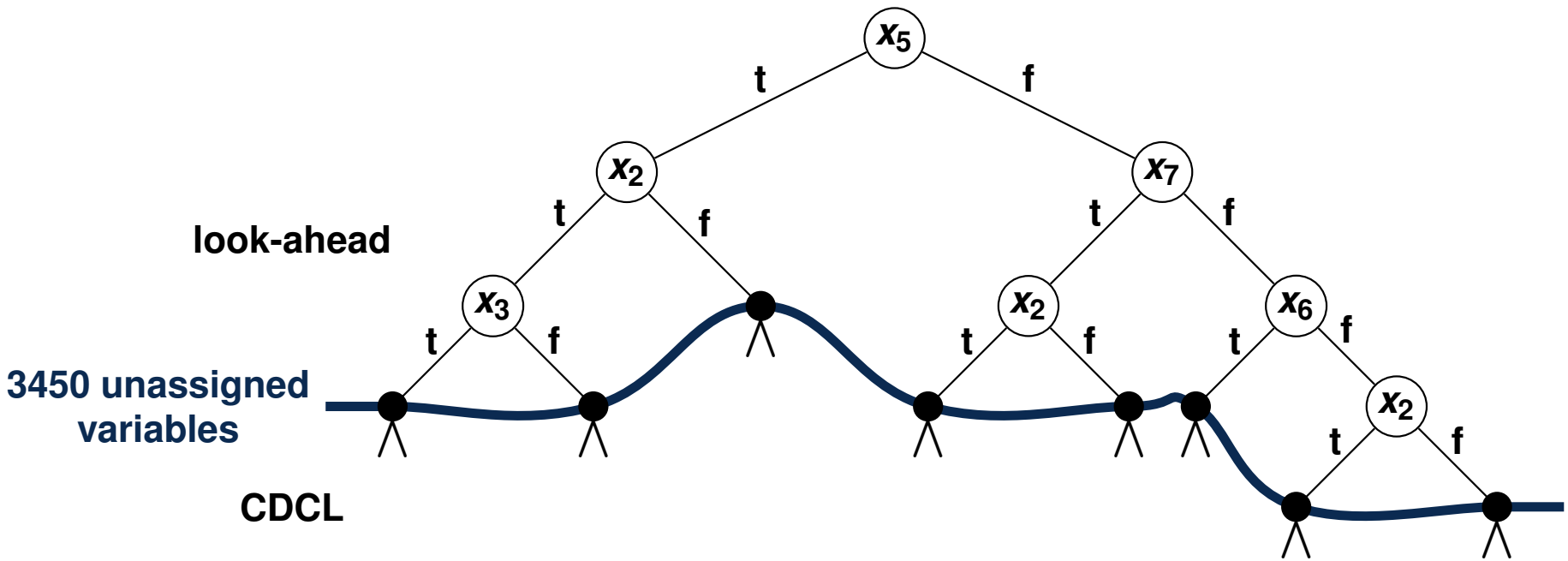

Cube-and-conquer solving

- **Problem: solving with conflict-driven clause learning (CDCL) is too slow**
- **Solution:**
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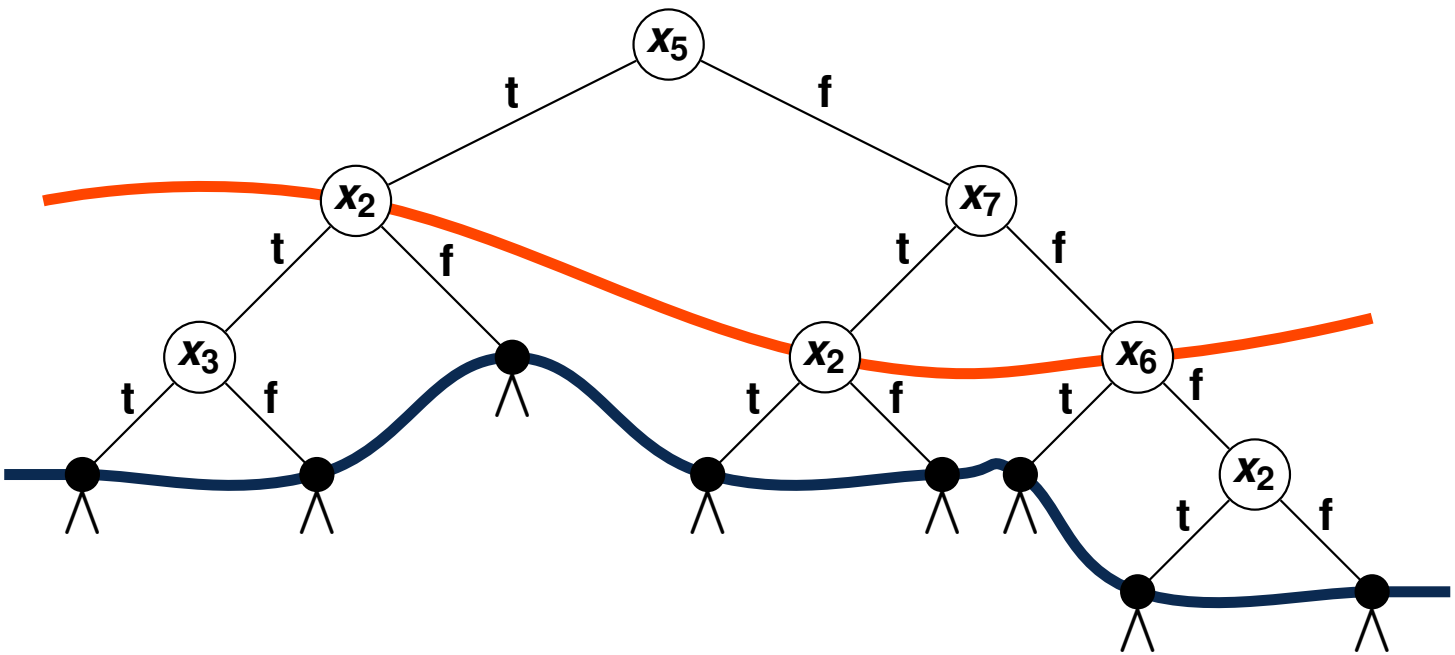
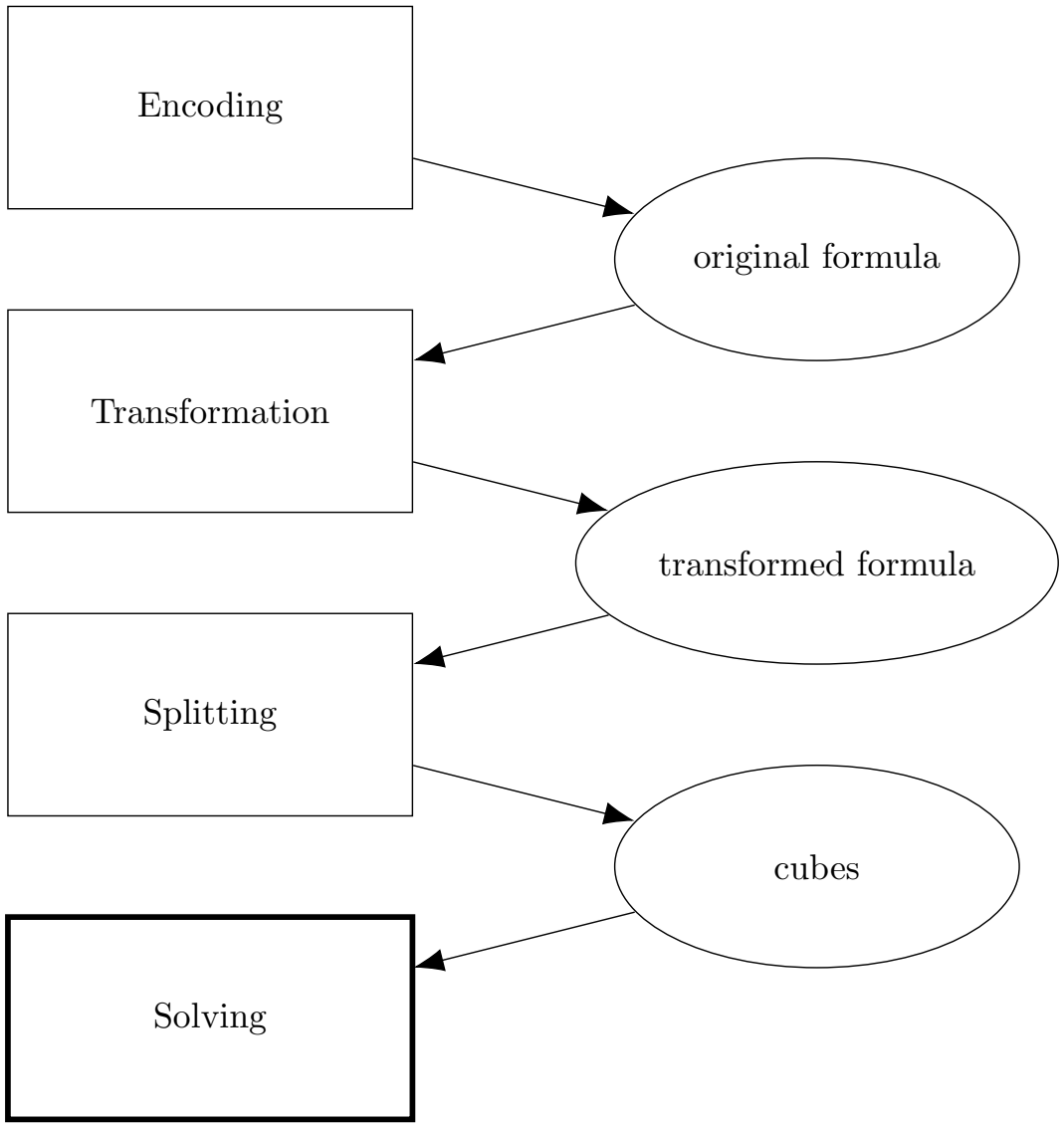
  - **use parallelization (800 cores)**

≈ 3000 binary clauses

3450 unassigned variables



Solving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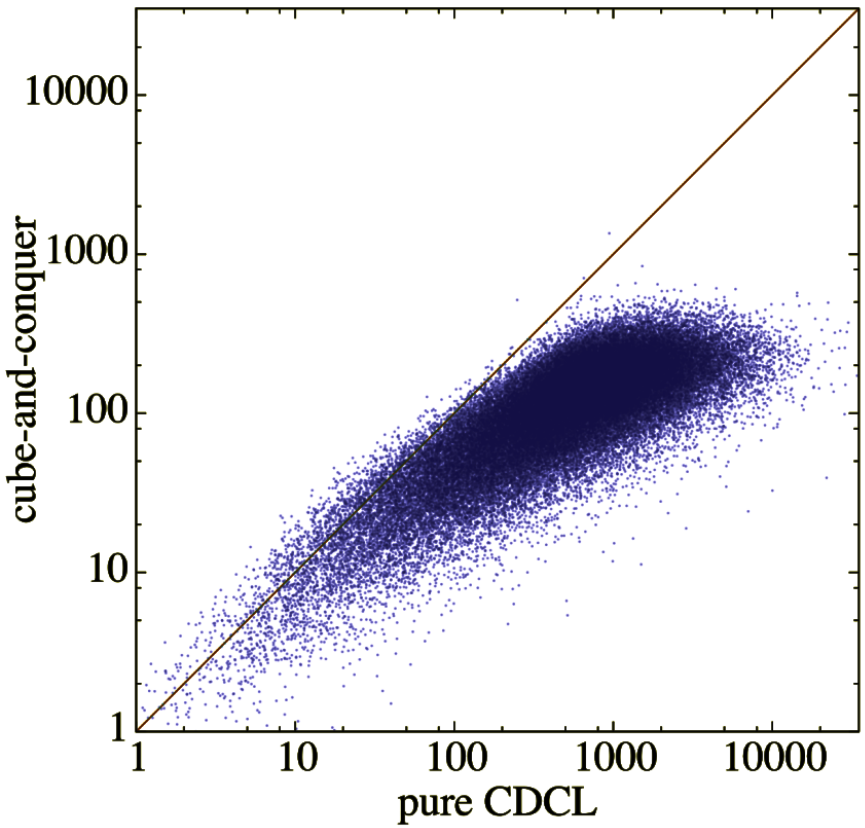

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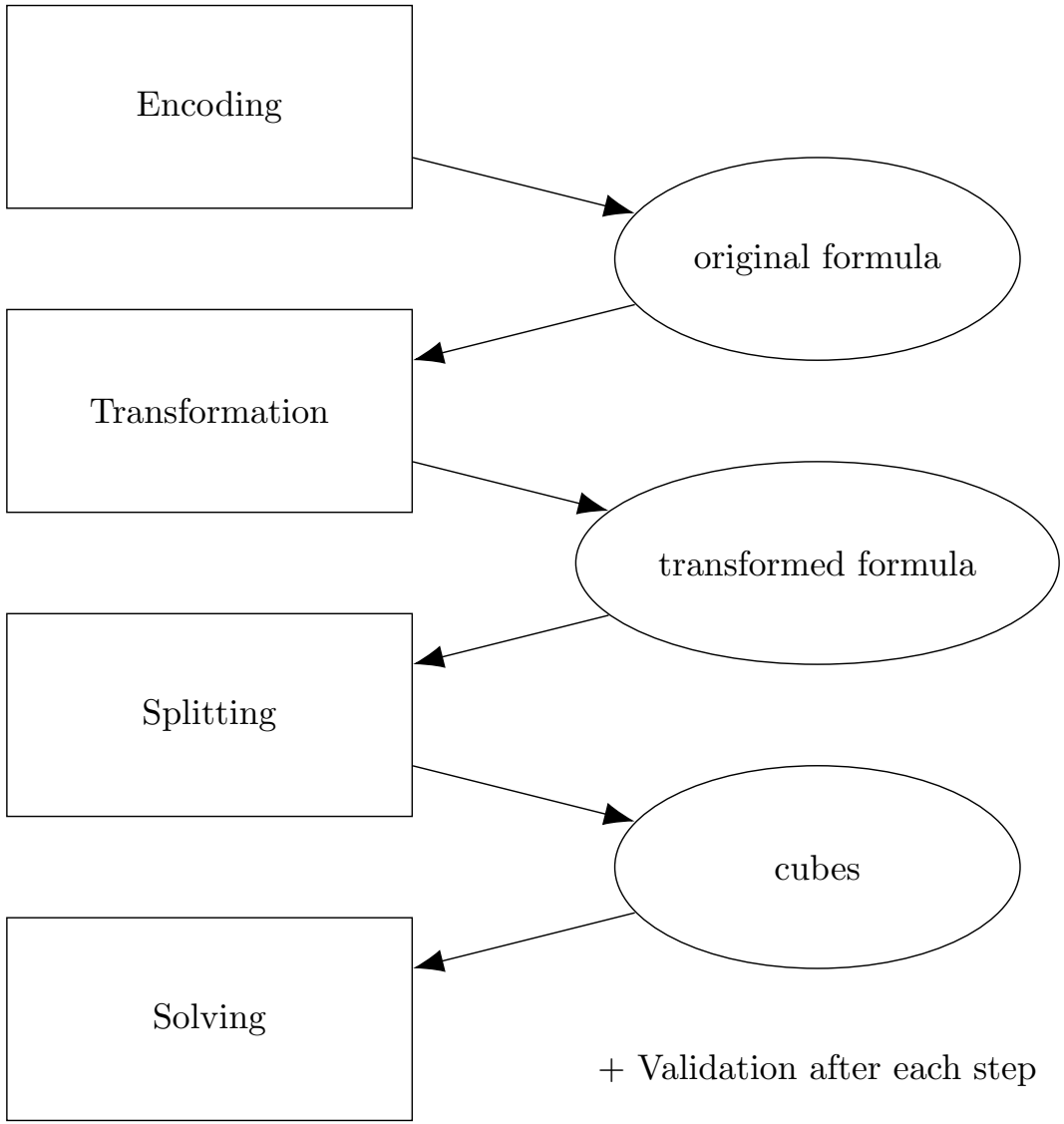
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