The boolean Pythagorean Triples problem

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- **Triples:**

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- **Triples:**

$$3^2 + 4^2 = 5^2$$
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$$9^2 + 12^2 = 15^2$$
$$5^2 + 12^2 = 13^2$$

- **Partition:** $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}, \{5, 10, 13, 15\}$

Encoding - Intuition

Idea:

- **one variable for each number**

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- **one constraint clause for each Pythagorean triple**
- **interpretation gives partition**

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Binary Pythagorean triple problem with ***n*** numbers

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Constraint for all Pythagorean triples: $F = \bigwedge_{\substack{1 \leq x,y,z \leq n \\ x^2+y^2=z^2}} NotEqual(p_x, p_y, p_z)$

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For an interpretation $I \subseteq V$ with $I \models F$, the resulting partition is:

- $P_1 = \{x \mid p_x \in I\}$
- $P_2 = \{x \mid p_x \notin I\}$

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$$\begin{aligned} F \quad = \quad & (p_3 \lor p_4 \lor p_5) \land (\neg p_3 \lor \neg p_4 \lor \neg p_5) \\ \land \quad & (p_6 \lor p_8 \lor p_{10}) \land (\neg p_6 \lor \neg p_8 \lor \neg p_{10}) \\ \land \quad & (p_9 \lor p_{12} \lor p_{15}) \land (\neg p_9 \lor \neg p_{12} \lor \neg p_{15}) \\ \land \quad & (p_5 \lor p_{12} \lor p_{13}) \land (\neg p_5 \lor \neg p_{12} \lor \neg p_{13}) \end{aligned}$$

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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

Resulting partition:

- $P_1 = \{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}$
- $P_2 = \{5, 10, 13, 15\}$

Transformation

Goal: from $\boldsymbol{F}$, find formula $\boldsymbol{F'}$ which

- **is easier to solve**

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Approaches:

- **eliminate clauses with variables only occuring once**
- **break partition symmetry**

Clause elimination

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$\rightarrow$ clause $NotEqual(p_3, p_4, p_5)$ will not cause conflict

$$\begin{aligned} F' \quad = \quad & NotEqual(p_6, p_9, p_{12}) \\ \wedge \quad & NotEqual(p_9, p_{12}, p_{15}) \\ \wedge \quad & NotEqual(p_5, p_{12}, p_{13}) \end{aligned}$$

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- **deleted clause $NotEqual(p_3, p_4, p_5)$ because p_3 only occurred once**

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- **$I' \models F'$ and $\{p_3, p_4, p_5\} \subseteq I'$**

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- $\rightarrow$ remember deleted clauses and modify interpretation of F' accordingly

Example:

- deleted clause $\mathit{NotEqual}(p_3, p_4, p_5)$ because p_3 only occurred once
- $I' \models F'$ and $\{p_3, p_4, p_5\} \subseteq I'$
- $I = I' \setminus p_3$
- while $I' \not\models F$, we have $I \models F$

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- **formulas F and F' are symmetric**

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- **$\rightarrow$ introduce unit clause for variable occurring in F'**
- **$F'' = F' \wedge p_x$**

Note that every model for F'' is a model for F' and the transformation is satisfiability preserving

Cube-and-conquer solving

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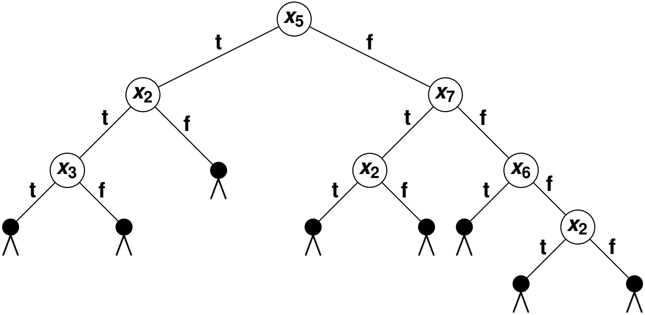

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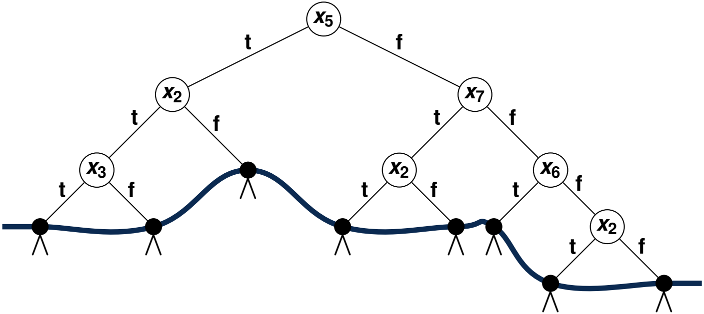
3450 unassigned variables



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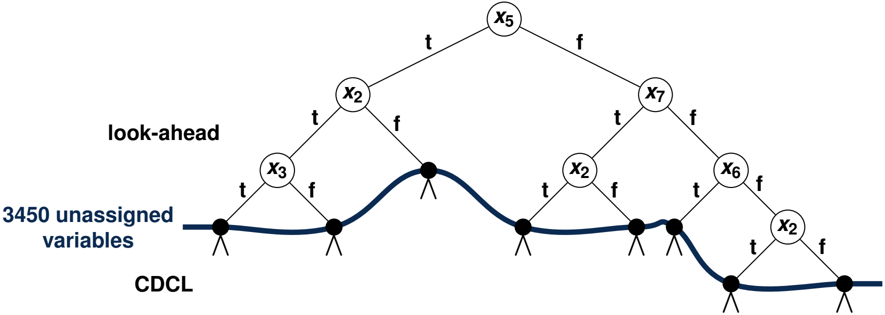

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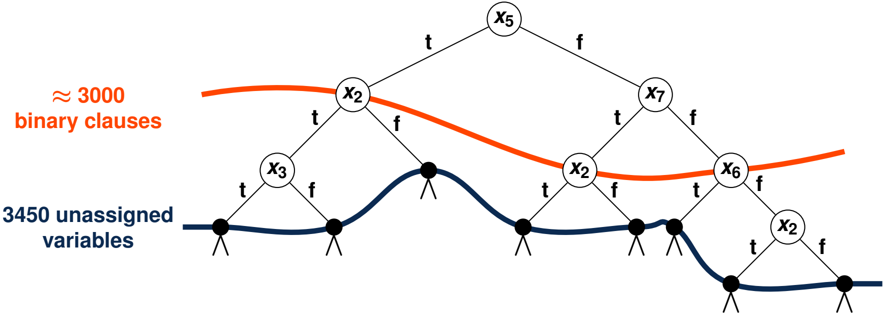
  - **use parallelization (800 cores)**



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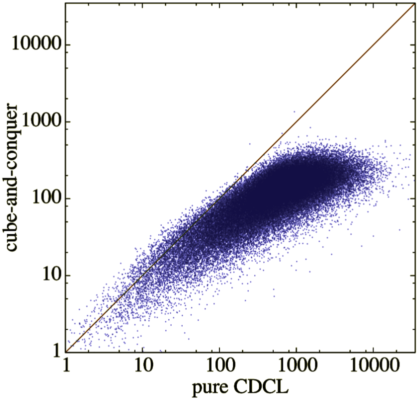

Framework

Picture of the Framework