# The boolean Pythagorean Triples problem

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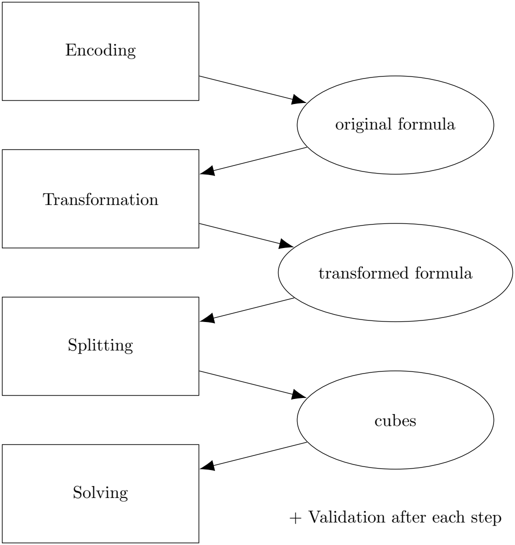
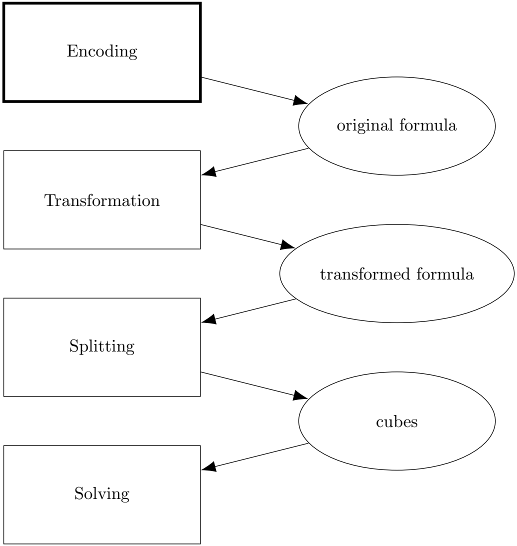
- **Partition:** $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 14\}, \{5, 10, 13, 15\}$

# Framework




+ Validation after each step

# Encoding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# Encoding - Intuition

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- **interpretation gives partition**
- **one constraint clause for each Pythagorean triple**

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Binary Pythagorean triple problem with ***n*** numbers

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Constraint for all Pythagorean triples: $F = \bigwedge_{x^2+y^2=z^2} NotEqual(p_x, p_y, p_z)$

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Constraint for all Pythagorean triples: $F = \bigwedge_{x^2+y^2=z^2} \mathit{NotEqual}(p_x, p_y, p_z)$

For an interpretation $I \subseteq V$ with $I \models F$, the resulting partition is:

- $P_1 = \{x \mid p_x \in I\}$
- $P_2 = \{x \mid p_x \notin I\}$

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$$\begin{aligned} F \quad = \quad & (p_3 \lor p_4 \lor p_5) \land (\neg p_3 \lor \neg p_4 \lor \neg p_5) \\ \land \quad & (p_6 \lor p_8 \lor p_{10}) \land (\neg p_6 \lor \neg p_8 \lor \neg p_{10}) \\ \land \quad & (p_9 \lor p_{12} \lor p_{15}) \land (\neg p_9 \lor \neg p_{12} \lor \neg p_{15}) \\ \land \quad & (p_5 \lor p_{12} \lor p_{13}) \land (\neg p_5 \lor \neg p_{12} \lor \neg p_{13}) \end{aligned}$$

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Possible interpretation: $I = \{p_5, p_{10}, p_{13}, p_{15}\}$

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Resulting partition:

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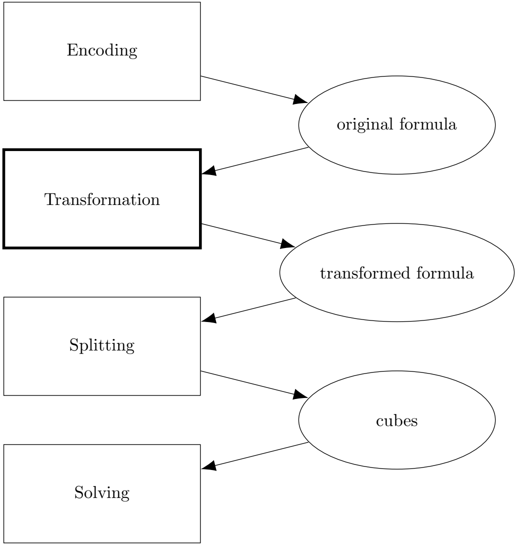
# Transformatio



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Approaches:

- **eliminate some particular clauses**
- **break partition symmetry**

# Clause elimination

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\begin{aligned}
F \quad = \quad & \mathit{NotEqual}(p_3, p_4, p_5) \\
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- remove $\mathit{NotEqual}(p_3, p_4, p_5)$ from $F$ to obtain $F'$
- interpretation of $p_3$ is important but not represented in $F'$
- $\rightarrow$ remember deleted clauses and modify interpretation of $F'$ accordingly

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$I' \models F$

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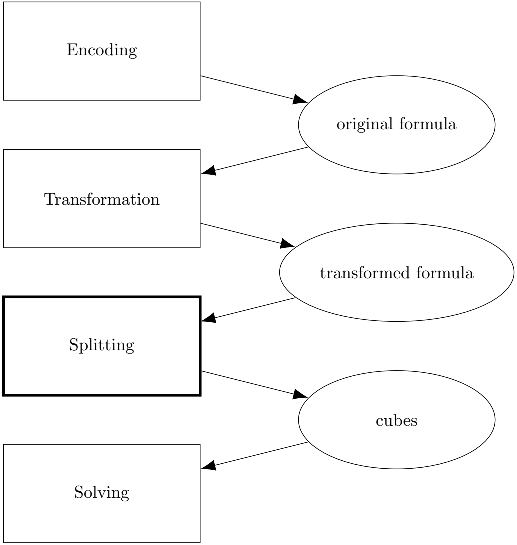
- **formulas $F$ and $F'$ are symmetric**
- **if $I \models F'$ then $V \setminus I \models F'$**
- **$\rightarrow$ introduce unit clause for variable $p_x$ occurring in $F'$**
- **$F'' = F' \wedge p_x$**

Note that every model for $F''$ is a model for $F'$ and the transformation is satisfiability preserving

# Splitting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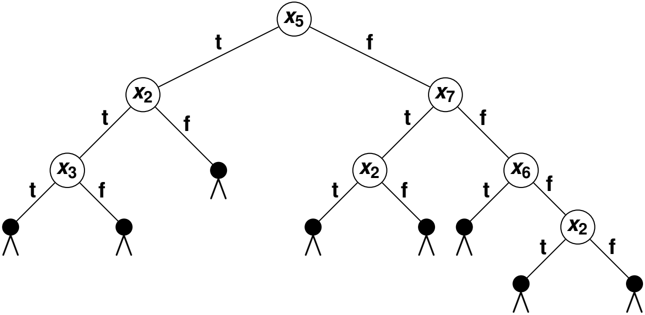

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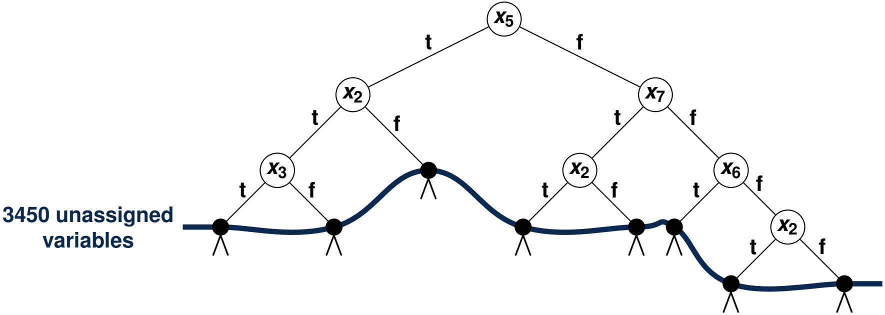

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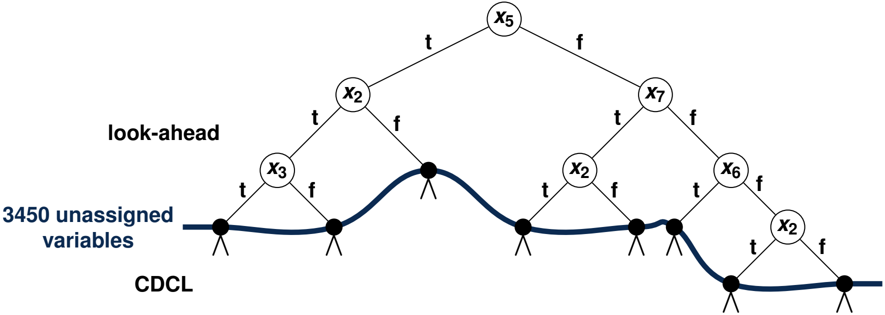

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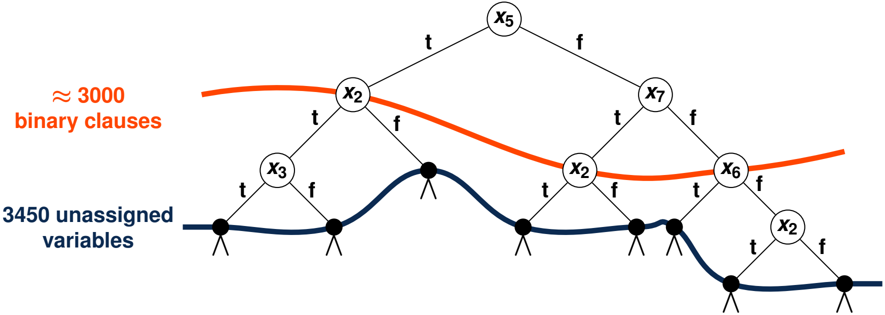
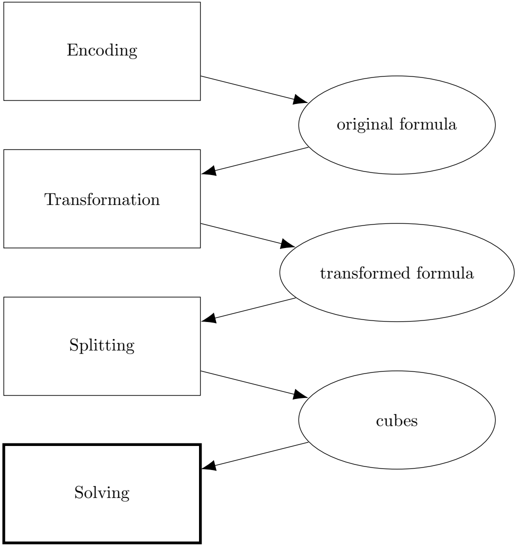
  - use parallelization (800 cores)



# Solving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