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TECHNICAL NOTE

UNSTEADY WALL SHEAR STRESS IN A DISTENSIBLE TUBE

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Abstract—An asymptotic expression of the wall shear stress (WSS) in an elastic tube is deduced for small values of the Womersley parameter. In the case of a rigid tube this asymptotic expression is shown to compare better with the exact solution than Poiseuille's or Lambossy's approximations. Its integration in a one-dimensional model of the internal carotid artery blood flow predicts more marked systolic and less marked diastolic WSS than those predicted by the commonly used Poiseuille's approximation.

INTRODUCTION

Numerical simulations of the unsteady flow of blood in the cardiovascular system are usually based on the one-dimensional set of equations (continuity, momentum and a tube law). A flow dependent wall shear stress (WSS) expression is required to solve this set of equations. To establish this additional relation, the local velocity profile is often anticipated and is usually assumed to be parabolic, the WSS being therefore deduced from Poiseuille's law (Rockwell, 1969; Stettler *et al.*, 1981). However, in an unsteady flow the velocity profile is not parabolic throughout the cycle except for quasi-steady flows. The WSS always has a phase lead on the mean and center line velocities in an unsteady flow. This feather is ignored when assuming a parabolic profile, and as a consequence the WSS amplitude is underestimated.

Several attempts have been made to improve the WSS evaluation in the one-dimensional modeling of pulsed blood flow in distensible tubes. Streeter *et al.* (1964) used a turbulent type of friction expression. In doing so, they achieved a better adjustment of the WSS amplitude than when the steady laminar flow approximation was used. This only shows that Poiseuille's approximation underestimates the WSS amplitude in an unsteady flow irrespective of the flow being turbulent or not. Olsen and Shapiro (1967) used a WSS expression deduced from the exact solution of a single harmonic periodic flow in a rigid tube to compute the WSS in an elastic tube, but in this expression they neglected the phase lag and the variations of the cross-sectional area. They obtained reasonably good wave propagation results for very low or very high values of the Womersley parameter, but the agreement was not so good for the intermediate values which are observed in the cardiovascular system (Pedley, 1980). Buthaud (1976) and Atabek (1980) combined the solution of one-dimensional and two-dimensional sets of equations. However, this numerical method requires a time-consuming iterative process. Kufahl and Clark (1985) evaluated the WSS using the slope of a Karman-Polhausen velocity profile at the vessel boundary. This method requires the solution of four additional equations at each time step.

Theoretical attempts to derive an analytical expression of the WSS, taking into account the unsteady nature of the flow, have only focused on the flow of a Newtonian fluid in a uniform rigid tube. Lambossy (1952) obtained an analytical expression of the viscous drag for a single harmonic periodic flow and deduced asymptotic approximations for low and high values of the frequency parameter. Zielke (1968) extended this approach to a more general periodic flow and established a one-dimensional expression of the WSS in the form of a convolution product. These last two approaches involved the use of Bessel functions.

In this study, we used a formal process to establish a one-dimensional incremental expression that relates the WSS to the instantaneous flow for the unsteady flow of a Newtonian fluid in an axisymmetric elastic tube. An asymptotic relation was deduced for a low Womersley parameter. This expression of the WSS was validated for rigid tubes and then was integrated into a one-dimensional mathematical model of blood flow in the internal carotid artery (ICA). Numerical results are given and compared with those obtained when using a parabolic velocity profile assumption to assess WSS.

MATHEMATICAL FORMULATION

An axisymmetric elastic tube of radius $R^*(z^*, t^*)$ and axis z^* was considered. The following dimensionless variables were defined:

$$r=\frac{r^*}{R_c} \quad z=\frac{z^*}{L_c} \quad t=\frac{R_c t^*}{L_c T_c} \quad u=\frac{u^*}{U_c}$$

$$v=\frac{L_c v^*}{R_c U_c} \quad p=\frac{p^*}{P_c} \quad \mathbb{P}=\frac{L_c \mathbb{P}^*}{R_c P_c} \quad A=\frac{A^*}{\pi R_c^2}$$

where u is the longitudinal velocity, v the radial velocity, p the intraluminal pressure, A the cross-section and $\mathbb{P}$ the extra stress tensor defining the rheological properties of the fluid. R_c, L_c, U_c, P_c and T_c are characteristic quantities of the problem under consideration. Dimensionless numbers were introduced:

$$\varepsilon=\frac{R_c}{L_c} \quad Re=\frac{\rho U_c^2}{P_c} \quad \beta=\frac{\rho R_c U_c}{P_c T_c}$$

where ρ is the density of the fluid, β characterizes the unsteadiness of the flow (for a periodical pressure signal, $\sqrt{(2\pi\beta)}$ is the well known Womersley parameter), Re is the Reynolds number and ε is a shape parameter.

The momentum equation for one-dimensional flow ($\varepsilon \ll 1$) in dimensionless form is:

$$\beta u_t + Re(uu_z + vu_r) = -p_z + r^{-1}(r\tau)_r, \tag{1}$$

where the subscript denotes a partial derivative; $\tau = \mathbb{P}rz$ is the shear stress. The associated boundary conditions expressing

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the no-slip condition at the wall and axial symmetry are:

$$u(r=R)=0 \quad v(r=R)=\frac{\beta}{Re}R_t \quad u_r(r=0)=0 \quad v(r=0)=0.$$

In the case of low Re or quasi-rigid uniform tubes, (1) reduces to:

$$\beta u_t=-p_z+r^{-1}(r\tau)_r \tag{2}$$

with the corresponding boundary conditions. Integration of (2) on a tube section area A yields:

$$G=Ap_z+\beta Q_t \tag{3}$$

where $G=2R\tau(R)$ is the dimensionless WSS over the circumference of the tube and Q is the dimensionless instantaneous flow rate. For a Newtonian fluid $\tau=u_r$, and (2) yields:

$$u(r,z,t)=-\left(\frac{R^2-r^2}{4}\right)p_z+\beta\int_R^r\frac{1}{x}\int_0^x yu_t\,\mathrm{d}y\,\mathrm{d}x. \tag{4}$$

The flow rate $Q(z,t)$ is given by:

$$Q(z,t)=2\int_0^R ru\,\mathrm{d}r \tag{5}$$

$$=-\frac{A^2}{8}p_z+2\beta\int_0^R r\int_R^r\frac{1}{x}\int_R^r yu_t\,\mathrm{d}y\,\mathrm{d}x \tag{6a}$$

where $A=R^2$. The last term in this expression is developed using successively the following properties (successive integration by parts):

$$\int_0^R r^n\int_R^r f(x,t)\,\mathrm{d}x\,\mathrm{d}r=-\int_0^R\frac{x^{n+1}}{n+1}f(x,t)\,\mathrm{d}x \tag{6b}$$

$$\int_0^R x^n\int_0^x g(r,t)\,\mathrm{d}r\,\mathrm{d}x=\int_0^R\frac{R^{n+1}-r^{n+1}}{n+1}g(r,t)\,\mathrm{d}r. \tag{6c}$$

Thus

$$Q(z,t)=-\frac{A^2}{8}p_z-\frac{\beta}{4}AQ_t+\frac{\beta}{2}\left(\int_0^R r^3u\,\mathrm{d}r\right)_t. \tag{7}$$

The last term can be expanded by substituting 4 for u. This iteration when repeated twice yields, by the aims of (6b) and (6c):

$$Q(z,t)=-\frac{A^2}{8}p_z-\frac{\beta}{4}AQ_t-\frac{\beta}{96}(A^3p_z)_t-\frac{\beta^2}{64}(A^2Q_t)_t$$
$$-\frac{\beta^2}{3072}(A^4p_z)_{tt}-\frac{\beta^3}{1152}\left(\int_0^R(R^6-y^6)yu_t\,\mathrm{d}y\right)_{tt}. \tag{8}$$

By substituting (3) for p_z in (8) one obtains:

$$Q+\frac{\beta}{8}AQ_t+\frac{\beta^2}{192}(A^2Q_t)_t-\frac{\beta^3}{3072}(A^3Q_t)_{tt}=-\frac{AG}{8}-\frac{\beta}{96}(A^2G)_t$$
$$-\frac{\beta^2}{3072}(A^3G)_{tt}-\frac{\beta^3}{1152}\left(\int_0^R(R^6-y^6)yu_t\,\mathrm{d}y\right)_{tt}. \tag{9}$$

This equation relates the averaged WSS to the instantaneous flow rate and cross-section and their successive derivatives. Thus it takes into account the unsteady nature of the flow as well as wall motion. The same iterative process leads by recurence to a more general equation (Bretteville *et al.*, 1986).

WSS ASYMPTOTIC APPROXIMATIONS

The last term of the second member in (9) is a function of the unknown instantaneous velocity profile $u(r,z,t)$. However this term can be neglected for small values of β, $\beta\ll 1$. In this case neglecting all the $O(\beta^3)$ terms in (9) yields:

$$AG+\frac{\beta}{12}(A^2G)_t+\frac{\beta^2}{384}(A^3G)_{tt}=$$
$$-8Q-\beta AQ_t-\frac{\beta^2}{24}(A^2Q_t)_t+O(\beta^3). \tag{10}$$

In the same manner one obtains for $O(\beta^2)$ order:

$$AG+\frac{\beta}{12}(A^2G)_t=-8Q-\beta AQ_t+O(\beta^2) \tag{11}$$

and for $O(\beta)$ order:

$$AG=-8Q+O(\beta). \tag{12}$$

Equation (12) is the well known Poiseuille approximation which is commonly used in one-dimensional models to compute the viscous terms in the momentum equation. This relation neglects both the flow unsteadiness and the tube distensibility in the evaluation of the WSS. Equation (11) is a distensible equivalent expression to Lambossy's rigid one. In fact, if $A_t\sim O(\beta)$, (quasi-rigid tubes or small perturbations of the tube wall), then $\beta A_t\sim O(\beta^2)$, and hence the term involving A_t in (11) can be replaced using the derivative of (12). One finally obtains:

$$AG=-8Q-\frac{\beta}{3}AQ_t+O(\beta^2) \tag{13}$$

which is the same relation as that obtained by Lambossy (1952) for small values of β in the case of a single harmonic periodic flow in a uniform rigid tube. (Note that it was obtained here for an unspecified unsteady flow.) Equation (10) is valid up to $O(\beta^3)$ and is therefore more accurate than the two preceding approximations. However one can obtain a more compact $O(\beta^3)$ expression by proceeding as follows. Equations (11) and (12) can be written:

$$a\beta AG_t+\frac{a\beta^2}{12}(A^2G)_{tt}=-8a\beta Q_t-a\beta^2(AQ_t)_t+O(\beta^3)$$
$$b\beta^2(AG)_{tt}=-8b\beta^2Q_{tt}+O(\beta^3)$$

where a and b are arbitrary constants. Adding these last two equations to equation (10) yields the linear combination:

$$\gamma_1G+\gamma_2G_t+\gamma_3G_{tt}=-8Q-\gamma_4Q_t-\gamma_5Q_{tt}+O(\beta^3) \tag{14}$$

where:

$$\gamma_1=A+a\beta A_t+\frac{a\beta^2A_t^2}{6}+\frac{\beta AA_t}{6}+\beta^2AA_t^2+\left(b+\frac{aA}{6}+\frac{A^2}{128}\right)A_{tt}$$

$$\gamma_2=\beta\left(aA+2b\beta A_t+\frac{a\beta}{3}AA_t+\frac{A^2}{12}+\frac{\beta}{64}A^2\right)$$

$$\gamma_3=\beta^2A\left(b+\frac{a}{12}A+\frac{A^2}{384}\right)$$

$$\gamma_4=\beta\left(A+8a+a\beta A_t+\frac{\beta}{12}AA_t\right)$$

$$\gamma_5=\beta^2\left(8b+aA+\frac{A^2}{24}\right).$$

Equation (14) has an infinite set of solutions depending on the choice of a and b. One convenient solution for (14) is obtained for $a=-\frac{A}{16}$ and $b=\frac{A^2}{384}$ which cancel all the X_{tt}

terms including A_{tt}. Hence (14) reduces to:

$$\left(1+\frac{5\beta}{48}A_t+\frac{\beta^2}{192}A_t^2\right)AG+\frac{\beta}{48}A^2G_t= -8Q-\frac{\beta}{2}\left(1+\frac{\beta}{24}A_t\right)AQ_t+O(\beta^3). \quad (15)$$

In the case where $A_t \sim \beta$, one gets from (15):

$$\left(1+\frac{5\beta}{48}A_t\right)AG+\frac{\beta}{48}A^2G_t=-8Q-\frac{\beta}{2}AQ_t+O(\beta^3). \quad (16)$$

For rigid tubes $A_t=0$, and for very small tube distensibility, i.e. $A_t \ll \beta$, which is the case of the cerebral vasculature, one obtains:

$$AG+\frac{\beta}{48}A^2G_t=-8Q-\frac{\beta}{2}AQ_t+O(\beta^3). \quad (17)$$

It is noteworthy that, although the β^2 terms do not appear, equation (17) is valid up to $O(\beta^3)$.

VALIDATION IN A RIGID TUBE

In Figs 1 and 2 the WSS values predicted by each of three approximations [Poiseuille (12), Lambossy (13) and equation (17)] are compared with the exact solution of a periodic pressure gradient flow in a uniform rigid tube for two values of β. As can be seen, equation (17) gives the same results as the Womersley solution for $\beta=2$, and is in good agreement with the exact solution, even for a value of β up to six, whereas the Lambossy and Poiseuille approximations lose their accuracy both in terms of amplitude and phase lag as β increases.

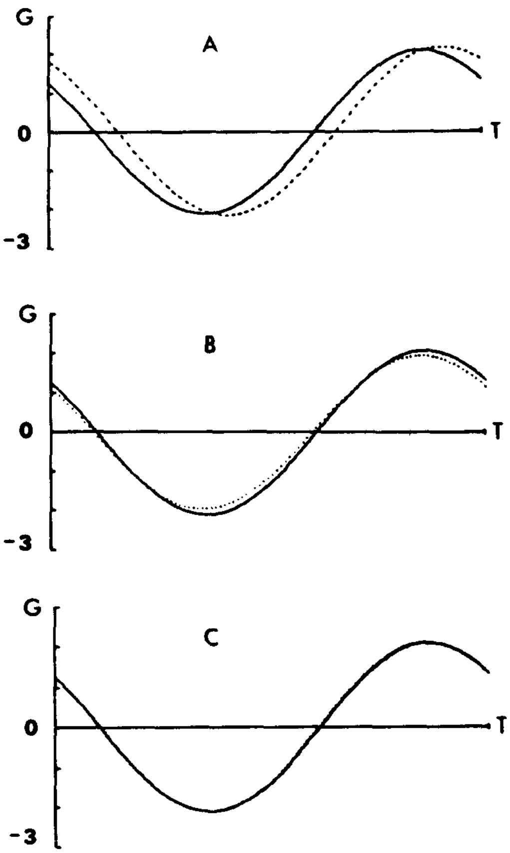

Fig. 1. Wall shear stress G as predicted by Poiseuille's approximation (A), Lambossy's (B) and equation (17) (C) compared with the exact solution (——) of a periodic flow in a rigid tube. $\beta=2$.

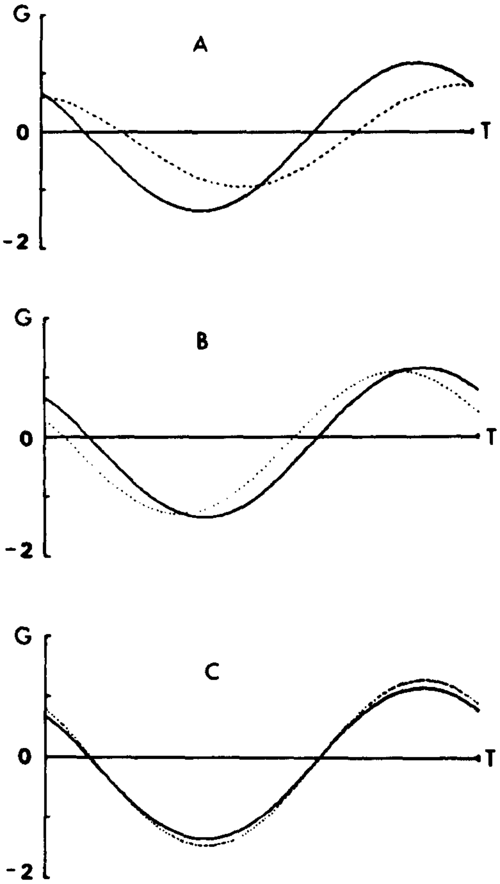

Fig. 2. Same as Fig. 1, except $\beta=6$.

APPLICATION TO ICA

Blood flow in the internal carotid artery is simulated using the following one-dimensional set of equations:

$$\beta A_t+ReQ_z=0$$

$$\beta Q_t+Re\left(\frac{Q^2}{A}\right)_z=-Ap_z+G \quad (18)$$

$$p=\delta(A-1) \quad \text{with} \quad \delta=\frac{2}{3}\frac{Eh}{P_cR_0}$$

E being Young's modulus, h the wall thickness and R_0 the radius at zero transmural pressure. P_c is a characteristic pressure, e.g. 100 mm Hg. The first two equations in (18) are respectively the continuity and momentm equations averaged over the artery cross-sectional area, A. The last equation is a tube law deduced from the linear elasticity theory. We have three equations for four unknown quantities: A, Q, P and G. A flow dependent WSS equation is required to solve (18). For this purpose, the local velocity profile is usually assumed to be parabolic, i.e. equation (12) is used (Stettler *et al.*, 1981; Zagzoule and Marc-Vergnes, 1986). In Fig. 3 we present numerical results of the local wall shear stress, $\tau_w=\frac{G}{2R}$, when using equation (17) to close the system (18) compared with when equation (12) was used. Whatever

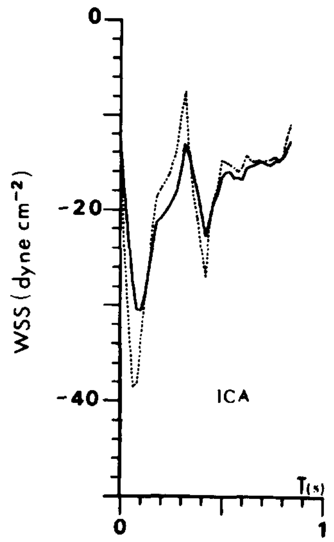

Fig. 3. Computed wall shear stress τ_w signals as predicted by the non-linear one-dimensional model (18) in the ICA when using Poiseuille's approximation (——) or equation (17) ($\cdots$).

expression was used for G, system (18) was solved numerically using the two-step Lax-Wendroff scheme (Richtmeyer and Morton, 1967). Numerical processing details are given in Zagzoule (1987).

DISCUSSION

An expression, equation (9), relating the averaged WSS G to the flow rate Q, the cross-section A and their successive derivatives, was established from the local linearized momentum equation. It thus takes into account the unsteady nature of the flow as well as wall motion. Its validity is limited to uniform elastic tubes in which the contribution of the convective terms can be neglected in the computation of the WSS. For small values of β^3, the term including explicitly the unknown instantaneous velocity profile was neglected and asymptotic relations were deduced. Accuracy of the different asymptotic relations, as far as unsteadiness was concerned, was investigated in the case of a periodic flow in a straight rigid tube for which an exact solution (i.e. the instantaneous velocity profile) could be obtained analytically (Lambossy, 1952; Womersely, 1957). As can be seen in Figs 1 and 2, equation (17) gives good results up to $\beta = 6$ despite the fact that it was deduced for small values of β, while Lambossy [equation (13)] and Poiseuille [equation (12)] approximations lose their accuracy both in terms of amplitude and phase lag as β increases. To illustrate the effect of flow unsteadiness on the WSS in arteries, we chose to simulate the blood flow in the internal carotid artery. This simulation is based on the one-dimensional set of equations, system (18). Either equations (12), (13), (16) or (17) could be used to complete system (18), depending on β magnitude and the value of A_t relative to β. Equation (12) neglects both unsteadiness and wall motion. Equation (13), which has been used by Shaaf (1971) and by Sikaskie *et al.* (1984), is shown in Figs 1 and 2 to be less accurate than equation (17). Equation (16) is the more appropriate for an unsteady flow in a distensible tube. However, an estimation of A_t in the ICA shows that $A_t \ll \beta$, and therefore equation (17) is expected to be sufficiently accurate. This can be shown as follows: according to the tube law, A_t is proportional to $\delta^{-1} p_t$, and δ lies in the range of 4–8 for the ICA (Hayashi *et al.*, 1980; Hudetz *et al.*, 1981). Taking 100 mm Hg as the mean blood pressure, 80 and 120 mm Hg as the diastolic and systolic pressures, respectively, yields $A_t \leqslant 0.1$. This is several times smaller than β, which is almost equal to 2 in the ICA. Thus equation (17), obtained for $A_t \ll \beta$, is well adapted to the evaluation of the WSS in the ICA. The second reason for using equation (17) instead of (16) in this application is that the former has been validated in the case of a rigid tube. Note that the use of equation (16) adds no difficulties to the numerical resolution of the one-dimensional set of equations. As can be seen in Fig. 3, closing the one-dimensional model by a time dependent expression of the WSS induced marked differences in the local wall shear stress τ_w amplitude compared with when a steady approximation was used. However, the mean τ_w of the local WSS over one cardiac cycle remained unchanged. The widely used Poiseuille approximation is seen to underestimate the τ_w systolic peak in the ICA of about 50% relative to the mean τ_w. On the other hand, it does predict a diastolic peak which is twice the one obtained when using a time dependent relation for the WSS. These results in ICA extend to all the major cerebral arteries and do have some interesting implications for the cerebral circulation. This will be presented in a separate paper dealing with the whole cerebral circulation.

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