Teoria del Senyal

Class Notes

Autor: JP Alumni

UPC

February 8, 2026

Legend and Visual Guide

This document uses colored boxes to highlight the nature of each concept or result.

Properties

Used for key **definitions**, **relations**, and **mathematical properties**. They contain rigorous results or fundamental equations.

Examples

Used for solved **examples** or **illustrative cases**. They help visualize the meaning of a concept and the application to the problem.

Special Phenomena

Used for important effects such as **aliasing**, **leakage**, or **quantization noise**. These sections highlight critical insights.

Whenever you see a colored box, its color indicates the type of information: property, example, or phenomenon.

Contents

0 Prelude **4**
- 0.1 Basic Signals . . . 4
- 0.2 Energy and Mean Power of Signals . . . 5
- 0.3 Fundamental Transformations in Signal Processing . . . 6

1 Classification of Systems **9**
- 1.1 Linear vs. Non-linear Systems . . . 9
- 1.2 Time-Invariant vs. Time-Variant Systems . . . 9
- 1.3 Static (Memoryless) vs. Dynamic (With Memory) Systems . . . 9
- 1.4 Causal vs. Non-causal Systems . . . 9
- 1.5 Stable vs. Unstable Systems . . . 10
- 1.6 Invertible vs. Non-invertible Systems . . . 10

2 Laplace Transform **11**

3 Convolution of Two Signals **13**
- 3.1 Interpretation . . . 13
- 3.2 Properties of Convolution . . . 13
- 3.3 Connection with Transforms . . . 13

4 Trigonometric Fourier Series **15**
- 4.1 Complex Fourier Series . . . 15
- 4.2 Dirichlet Conditions for Fourier Series Convergence . . . 16

5 Fourier Transformation **18**
- 5.1 Definition . . . 18
- 5.2 Properties of the Fourier Transform . . . 19

6 Bandpass Communication **21**
- 6.1 DSB Transmission (Double Sideband) . . . 21
- 6.2 Coherent Receiver (DSB) . . . 21
- 6.3 Non-Coherent Receiver (DSB) . . . 21

7 Z-transform **23**
- 7.1 Properties of Z-transformation . . . 24
- 7.2 Inverse Z-Transform . . . 25
- 7.3 Calculate from the recursion formula, the transfer function . . . 26
- 7.4 System Stability in the Z-Domain . . . 27

8 Equalizer of $H(z)$ **29**

9 Discrete Fourier Transform (DFT) **31**
- 9.1 Properties . . . 31
- 9.2 Aliasing . . . 32
- 9.3 Interpretation . . . 32
- 9.4 Circular Delay Property . . . 33
- 9.5 Circular Convolution Property . . . 34

10 Sampling and alias generation: antialiasing filter **37**
10.0.1 Sampling of a generic signal 37
10.0.2 Analog Anti-Aliasing Filter 38
10.1 Signal reconstruction: ideal and zero-order-hold (ZOH) interpolators 39
10.1.1 Oversampling 41
10.2 Change of the sampling frequency: decimation and interpolation. 41
10.2.1 Decimation 41
10.2.2 Interpolation 42
10.2.3 Fractional change (combination of both previous) 43

11 Statistical characterization **45**

12 Strict-sense and wide-sense stationary processes. Cyclo-stationary processes. **46**
12.1 Autocorrelation for Non-Stationary Processes 47
12.2 Cross-Correlation for Jointly Stationary and Non-Stationary Processes 48

13 Auto-correlation and power spectral density. Wiener-Khinchin theorem. **49**
13.1 Filtering of random signals 53
13.2 Vector–Matrix Notation for Discrete-Time Processes. The Auto-Correlation Matrix 54

14 Tables of Transforms **56**

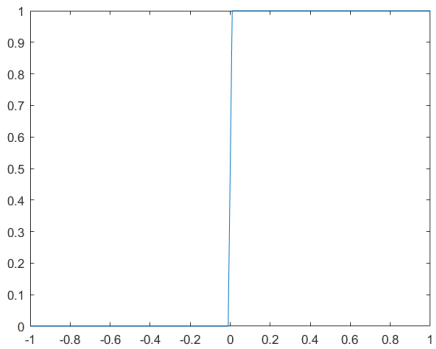
0 Prelude

0.1 Basic Signals

Unitary step

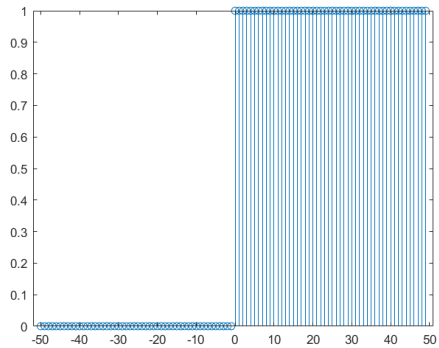
Analog:

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$



Discrete-time:

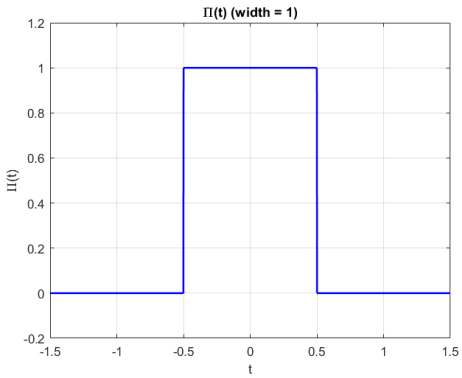
$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$



Rectangular Pulse

Analog:

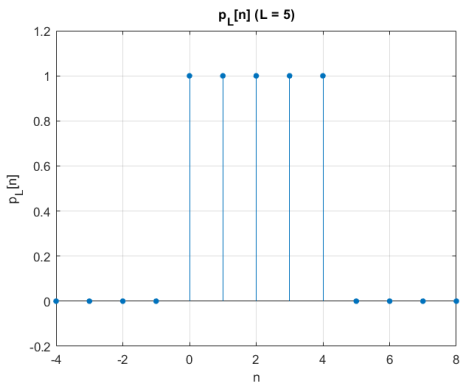
$$\Pi(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$



$$\Pi(t) = u\left(t + \tfrac{1}{2}\right) - u\left(t - \tfrac{1}{2}\right)$$

Discrete-time:

$$p_L[n] = \begin{cases} 1, & 0 \leq n < L \\ 0, & \text{otherwise} \end{cases}$$

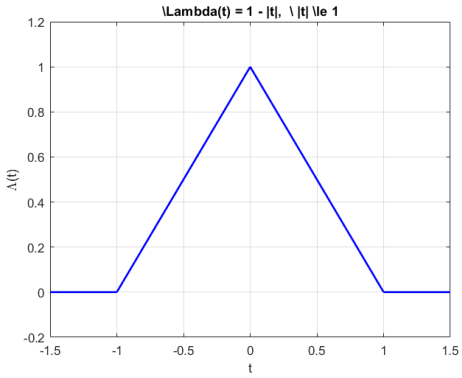


$$p_L[n] = u[n] - u[n - L], \quad L = 5$$

Triangular analog pulse & Discrete-time ramp

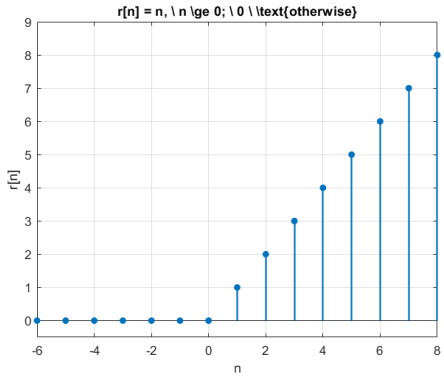
Analog triangular pulse:

$$\Lambda(t) = \begin{cases} 1 - |t|, & |t| \leq 1, \\ 0, & |t| > 1. \end{cases}$$



Discrete-time ramp:

$$r[n] = \begin{cases} n, & n \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$



0.2 Energy and Mean Power of Signals

	Energy	**Mean Power**
Analog	$E_x = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 \, dt$	$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \lvert x(t)\rvert^2 \, dt$
Discrete-time	$E_x = \sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$	$P_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \lvert x[n]\rvert^2$

0.3 Fundamental Transformations in Signal Processing

Laplace Transform

- **Definition:**

$$X(s) = \int_0^\infty x(t)e^{-st}\,dt, \quad s = \sigma + j\omega$$

- **Domain:** Continuous-time signals.
- **Purpose:** Transforms signals from time domain t to complex frequency domain s.
- **Key Concepts:**
  - Region of Convergence (ROC): defines values of s for which $X(s)$ exists.
  - Poles and Zeros: determine system stability and response.
  - Connection to Fourier Transform: when $\sigma = 0$, the Laplace Transform reduces to the Fourier Transform.
- **Applications:**
  - Solving differential equations.
  - Stability analysis of continuous-time systems.
  - Control systems and electrical circuit analysis.

Fourier Transform

- **Definition:**

$$X(\omega) = \int_{-\infty}^\infty x(t)e^{-j\omega t}\,dt$$

- **Domain:** Continuous-time signals (finite energy).
- **Purpose:** Represents a signal as a sum of sinusoidal frequency components.
- **Key Concepts:**
  - Spectrum $X(\omega)$ shows amplitude and phase at each frequency.
  - Inverse transform reconstructs the signal:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^\infty X(\omega)e^{j\omega t}\,d\omega$$

  - Fourier Series is related but only applies to periodic signals.
- **Applications:**
  - Frequency analysis of signals (e.g. audio, communications).
  - Filter design.
  - Image and sound processing.

Z-Transform

- **Definition:**

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \quad z = re^{j\omega}$$

- **Domain:** Discrete-time signals.
- **Purpose:** Generalizes the Discrete-Time Fourier Transform (DTFT) by adding a radial component r for convergence.
- **Key Concepts:**
  - Region of Convergence (ROC): determines existence and stability.
  - Poles and Zeros: graphical representation in the z-plane gives insight into filter properties.
  - Relationship: if $|z| = 1$, the Z-transform reduces to the DTFT.
- **Applications:**
  - Analysis and design of digital filters (FIR, IIR).
  - Solving linear difference equations.
  - Stability and frequency response of discrete-time systems.

Chapter 1

Signals and Systems in the Time Domain

1 Classification of Systems

Systems can be classified according to several criteria. Below we summarize the main types, their mathematical definitions, and one representative example of each.

1.1 Linear vs. Non-linear Systems

A system is **linear** if it satisfies both *superposition* and *homogeneity*:

$$x_1(t) \xrightarrow{\mathcal{S}} y_1(t), \quad x_2(t) \xrightarrow{\mathcal{S}} y_2(t)$$

$$\Rightarrow \ \alpha x_1(t) + \beta x_2(t) \xrightarrow{\mathcal{S}} \alpha y_1(t) + \beta y_2(t)$$

If this property does not hold, the system is **non-linear**.

Example:

- Linear: $y(t) = 3x(t) + 2\dot{x}(t)$
- Non-linear: $y(t) = [x(t)]^2$

1.2 Time-Invariant vs. Time-Variant Systems

A system is **time-invariant** if a time shift in the input causes an identical time shift in the output:

$$x(t) \xrightarrow{\mathcal{S}} y(t) \quad \Rightarrow \quad x(t - t_0) \xrightarrow{\mathcal{S}} y(t - t_0)$$

Otherwise, the system is **time-variant**.

Example:

- Time-invariant: $y(t) = x(t) + 5$
- Time-variant: $y(t) = t\,x(t)$

1.3 Static (Memoryless) vs. Dynamic (With Memory) Systems

A system is **static** (or **memoryless**) if its output at time t depends only on the input at that same instant:

$$y(t) = f(x(t))$$

It is **dynamic** if $y(t)$ depends on past or future values of $x(t)$.

Example:

- Static: $y(t) = 2x(t)$
- Dynamic: $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

1.4 Causal vs. Non-causal Systems

A system is **causal** if the output at time t depends only on the input for times $\tau \leq t$. If it depends on future inputs ($\tau > t$), it is **non-causal**.

Example:

- Causal: $y(t) = x(t - 1)$
- Non-causal: $y(t) = x(t + 1)$

1.5 Stable vs. Unstable Systems

A system is **BIBO stable** (Bounded Input–Bounded Output) if every bounded input produces a bounded output:

$$|x(t)| \leq M_x < \infty \;\Rightarrow\; |y(t)| \leq M_y < \infty$$

Otherwise, it is **unstable**.

Example:

- Stable: $y(t) = 0.5\,y(t-1) + x(t)$
- Unstable: $y(t) = 2\,y(t-1) + x(t)$

1.6 Invertible vs. Non-invertible Systems

A system is **invertible** if each input $x(t)$ produces a unique output $y(t)$ and there exists an inverse system $\mathcal{S}^{-1}$ such that:

$$\mathcal{S}^{-1}\{\mathcal{S}\{x(t)\}\} = x(t)$$

Otherwise, it is **non-invertible**.

Example:

- Invertible: $y(t) = 2x(t)$ (inverse: $x(t) = y(t)/2$)
- Non-invertible: $y(t) = [x(t)]^2$

2 Laplace Transform

The **Laplace Transform** is a mathematical operation that converts a signal from the time domain $x(t)$ to the complex frequency domain $X(s)$. It generalizes the Fourier Transform by allowing analysis of signals that may grow or decay exponentially, not just oscillate.

The bilateral Laplace Transform is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt, \qquad s = \sigma + j\omega$$

and its inverse transform is:

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

For causal physical systems (where $x(t) = 0$ for $t < 0$), the unilateral form is used:

$$X(s) = \int_{0^-}^{\infty} x(t)e^{-st}\,dt$$

Main Properties of the Laplace Transform

- **Linearity:**
$$\alpha x_1(t) + \beta x_2(t) \overset{\mathcal{L}}{\leftrightarrow} \alpha X_1(s) + \beta X_2(s)$$
- **Time Shifting:**
$$x(t-t_0)u(t-t_0) \overset{\mathcal{L}}{\leftrightarrow} e^{-st_0}X(s)$$
- **Frequency Shifting:**
$$e^{at}x(t) \overset{\mathcal{L}}{\leftrightarrow} X(s-a)$$
- **Differentiation in Time:**
$$\frac{dx(t)}{dt} \overset{\mathcal{L}}{\leftrightarrow} sX(s) - x(0^-)$$
- **Integration in Time:**
$$\int_0^t x(\tau)\,d\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s}X(s)$$
- **Convolution Theorem:**
$$(x_1 * x_2)(t) \overset{\mathcal{L}}{\leftrightarrow} X_1(s)X_2(s)$$
- **Initial Value Theorem:**
$$x(0^+) = \lim_{s\to\infty} sX(s)$$
- **Final Value Theorem:**
$$x(\infty) = \lim_{s\to 0} sX(s) \quad \text{(if all poles of } sX(s) \text{ are in } \Re(s) < 0)$$
- **Time Scaling:**
$$x(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{a}X\left(\frac{s}{a}\right), \quad a > 0$$

—

Signal $x(t)$	**Laplace Transform** $X(s)$	**ROC (Region of Convergence)**
$\delta(t)$	1	all s
$u(t)$	$\frac{1}{s}$	$\Re(s) > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\Re(s) > -a$
$t\,u(t)$	$\frac{1}{s^2}$	$\Re(s) > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\Re(s) > 0$
$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\Re(s) > 0$
$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\Re(s) > 0$
$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\Re(s) > -a$
$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\Re(s) > -a$

Table 1: Common Laplace Transform pairs and their regions of convergence.

3 Convolution of Two Signals

Definition

Convolution is an operation that expresses the output of a linear time-invariant (LTI) system when an input signal is applied. It combines two signals to produce a third one.

Continuous-Time Convolution

For two continuous signals $x(t)$ and $h(t)$:

$$y(t) = (x * h)(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Discrete-Time Convolution

For two discrete signals $x[n]$ and $h[n]$:

$$y[n] = (x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

3.1 Interpretation

- $h(t)$ or $h[n]$ usually represents the **impulse response** of a system.
- Convolution describes how the input signal x is "filtered" by the system.
- It can be seen as a "weighted overlap" of x and a shifted/reversed version of h.

3.2 Properties of Convolution

- **Commutative:** $x * h = h * x$
- **Associative:** $(x * h) * g = x * (h * g)$
- **Distributive:** $x * (h_1 + h_2) = x * h_1 + x * h_2$
- **Identity:** $x * \delta = x$ (where δ is the Dirac or Kronecker delta)

3.3 Connection with Transforms

- **Fourier Transform:**
$$\mathcal{F}\{x * h\} = X(\omega) \cdot H(\omega)$$
Convolution in time $\rightarrow$ multiplication in frequency.
- **Laplace Transform:**
$$\mathcal{L}\{x * h\} = X(s) \cdot H(s)$$
- **Z-Transform:**
$$\mathcal{Z}\{x * h\} = X(z) \cdot H(z)$$

Chapter 2

Signals and Systems in Transform domains

4 Trigonometric Fourier Series

A **periodic signal** $x(t)$ with period T can be expressed as a sum of **sinusoidal components** (sines and cosines) whose frequencies are integer multiples of the **fundamental frequency**:

$$\omega_0 = \frac{2\pi}{T}$$

The **Trigonometric Fourier Series (TFS)** representation of $x(t)$ is given by:

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)]$$

where:

- a_0 represents the **average (DC) component** of the signal.
- a_n and b_n are the **Fourier coefficients** corresponding to cosine and sine components.

These coefficients are obtained as:

$$a_0 = \frac{1}{T}\int_T x(t)\,dt, \qquad a_n = \frac{2}{T}\int_T x(t)\cos(n\omega_0 t)\,dt, \qquad b_n = \frac{2}{T}\int_T x(t)\sin(n\omega_0 t)\,dt$$

An alternative form of the TFS, often clearer for interpretation, is expressed in terms of **amplitude** and **phase**:

$$x(t) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

where:

$$A_n = \sqrt{a_n^2 + b_n^2}, \qquad \phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

This form directly shows that any periodic signal is composed of **sinusoids with distinct amplitudes and phases**, a perspective widely used in acoustics, vibration analysis, and electrical engineering.

4.1 Complex Fourier Series

The **Complex Fourier Series (CFS)** provides a more compact and elegant formulation using complex exponentials $e^{j\omega t}$. Using Euler's identity:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}, \qquad \sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

the TFS can be rewritten as:

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where:

- C_n are the **complex Fourier coefficients**.
- $e^{jn\omega_0 t}$ represents a complex exponential at the n-th harmonic frequency.

The coefficients are calculated as:

$$C_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} \, dt$$

For real-valued signals, the coefficients satisfy the **conjugate symmetry** property:

$$C_{-n} = C_n^*$$

ensuring that $x(t)$ remains real.

The relationship between trigonometric and complex coefficients is:

$$a_0 = C_0, \qquad a_n = C_n + C_{-n}, \qquad b_n = j(C_n - C_{-n})$$

Gibbs Phenomenon

When a discontinuous periodic signal (such as a square wave) is approximated using a finite number of Fourier series terms, the resulting partial sum exhibits oscillations near the points of discontinuity. This effect is known as the **Gibbs phenomenon**.
Even as the number of harmonics increases, the overshoot near the discontinuity does *not* disappear — it approaches a limiting value of approximately **9**% of the jump amplitude. However, the oscillations become more concentrated around the discontinuity as more terms are added, and the approximation improves everywhere else.
Mathematically, the phenomenon arises from the slow convergence of the Fourier series of a discontinuous function and reflects the inherent limitation of representing a sharp edge with smooth sinusoidal components.

4.2 Dirichlet Conditions for Fourier Series Convergence

For a periodic function $f(t)$ with period T, the Fourier series representation converges to the original signal (if certain mild requirements are met). These requirements, known as the **Dirichlet conditions**, specify when the Fourier series is guaranteed to converge.

Mathematical Conditions

The Fourier series of $f(t)$ will converge if, within any period of the function, the following conditions are satisfied:

1. **Single-valued and finite:** The function $f(t)$ must be single-valued and finite everywhere. That is, $f(t)$ cannot go to infinity or be multivalued.

2. **Finite number of discontinuities:** The function may have a finite number of finite discontinuities (no infinite jumps).

3. **Finite number of extrema:** The function must have a finite number of maxima and minima within one period.

4. **Absolute integrability:** The integral of the absolute value of the function over one period must be finite:

$$\int_{t_0}^{t_0+T} |f(t)| \, dt < \infty$$

ensuring that all Fourier coefficients exist.

Convergence Behavior

Under these conditions:

- At every point where $f(t)$ is continuous, the Fourier series converges to $f(t)$.
- At a discontinuity, it converges to the **average of the left and right limits**:

$$S_N(t) \to \frac{f(t^+) + f(t^-)}{2}$$

5 Fourier Transformation

The Fourier Transform (FT) is one of the most fundamental tools in signal analysis. It decomposes a time-domain signal into its constituent frequency components.

5.1 Definition

For a continuous-time signal $x(t)$, the Fourier Transform and its inverse are defined as:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Here:

- $x(t)$ is the time-domain representation.
- $X(\omega)$ is the frequency-domain representation (spectrum).
- ω is the angular frequency in radians per second.

5.2 Properties of the Fourier Transform

Main Properties of the Fourier Transform

- **Linearity:**

$$\alpha x_1(t) + \beta x_2(t) \overset{\mathcal{F}}{\leftrightarrow} \alpha X_1(\omega) + \beta X_2(\omega)$$

- **Time Shifting:**

$$x(t - t_0) \overset{\mathcal{F}}{\leftrightarrow} e^{-j\omega t_0} X(\omega)$$

- **Frequency Shifting:**

$$e^{j\omega_0 t} x(t) \overset{\mathcal{F}}{\leftrightarrow} X(\omega - \omega_0)$$

- **Time Scaling:**

$$x(at) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

- **Differentiation in Time:**

$$\frac{dx(t)}{dt} \overset{\mathcal{F}}{\leftrightarrow} j\omega X(\omega)$$

- **Integration in Time:**

$$\int_{-\infty}^{t} x(\tau)\, d\tau \overset{\mathcal{F}}{\leftrightarrow} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

- **Convolution Theorem:**

$$x_1(t) * x_2(t) \overset{\mathcal{F}}{\leftrightarrow} X_1(\omega)\, X_2(\omega)$$

- **Multiplication Theorem:**

$$x_1(t)\, x_2(t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi} X_1(\omega) * X_2(\omega)$$

- **Parseval's Theorem:**

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Signal $x(t)$	**Fourier Transform** $X(\omega)$	**Remarks**
$\delta(t)$	1	Impulse $\leftrightarrow$ Flat spectrum
1	$2\pi\delta(\omega)$	DC signal
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	Complex exponential
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	Real tone
$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}\left(\frac{\omega T}{2}\right)$	Pulse $\leftrightarrow$ sinc
$e^{-\alpha\lvert t\rvert}$	$\dfrac{2\alpha}{\alpha^2+\omega^2}$	Real exponential decay
$u(t)e^{-at}$	$\dfrac{1}{a+j\omega}$, $\Re(a)>0$	Causal exponential

Table 2: Common continuous-time Fourier Transform pairs.

6 Bandpass Communication

In many practical communication systems, transmitting a baseband (low frequency) signal directly is not feasible:

- Low frequency signals cannot be efficiently radiated by antennas.
- Noise and interference often dominate at low frequencies.

The solution is to perform **modulation**, which shifts the spectrum of the signal to a higher frequency band, more suitable for transmission over radio channels. At the receiver, the modulation must be undone (demodulation) in order to recover the original information.

6.1 DSB Transmission (Double Sideband)

In Double Sideband modulation, a baseband signal $f(t)$ is multiplied with a carrier $\cos(\omega_c t)$. The modulated signal is:

$$s(t) = f(t)\cos(\omega_c t).$$

In the frequency domain:

$$S(\omega) = \frac{1}{2}\left[F(\omega - \omega_c) + F(\omega + \omega_c)\right],$$

which means the spectrum of the baseband signal $F(\omega)$ is shifted both to $+\omega_c$ and $-\omega_c$, producing the two sidebands.

6.2 Coherent Receiver (DSB)

A coherent receiver assumes perfect knowledge of the carrier frequency and phase. The received signal is again multiplied by $\cos(\omega_c t)$:

$$r(t)\cos(\omega_c t) = \frac{1}{2}f(t) + \frac{1}{2}f(t)\cos(2\omega_c t).$$

The first term corresponds to the desired baseband signal, while the second is a high-frequency component around $2\omega_c$. By applying a **low-pass filter**, the high-frequency term is removed, and the receiver recovers:

$$\hat{f}(t) \approx \frac{1}{2}f(t).$$

The factor 1/2 is a scaling that can be compensated.

6.3 Non-Coherent Receiver (DSB)

If the receiver does not have exact carrier phase information, it uses an **envelope detector**. This method works for signals whose envelope carries the information (like in AM). The detector outputs an approximation of the original signal:

$$\hat{f}(t) \approx |r(t)|.$$

While simpler, non-coherent detection is less robust and can introduce distortion if the signal is not a pure amplitude modulation.

Note: In the case of pure DSB-SC modulation ($s(t) = f(t)\cos(\omega_c t)$), the envelope detector does *not* recover $f(t)$, but rather $|f(t)|$. If the message $f(t)$ changes sign, the information about the sign is lost.

To overcome this limitation, one of the following must be used:

- **Add a carrier (DC offset)**: transmit

$$s(t) = [A_c + f(t)]\cos(\omega_c t), \quad A_c > |f(t)|$$

  so that the envelope is always positive. The offset A_c can then be subtracted after detection.

- **Use coherent detection**: synchronize with the carrier and multiply, recovering $f(t)$ directly without requiring an offset.

7 Z-transform

$$X(z) = Z[x[n]] = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$$

with the **region of convergence (ROC)**:

$$\sum_{n=-\infty}^{+\infty} |x[n]z^{-n}| < +\infty$$

We can write any signal such as:

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

with the Laplace's transformation this becomes:

$$X_s(s) = L[x_s(t)] = L\left[\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)\right] = \sum_{n=-\infty}^{\infty} x[n]e^{-nsT} = X(z)|_{z=e^{sT}}$$

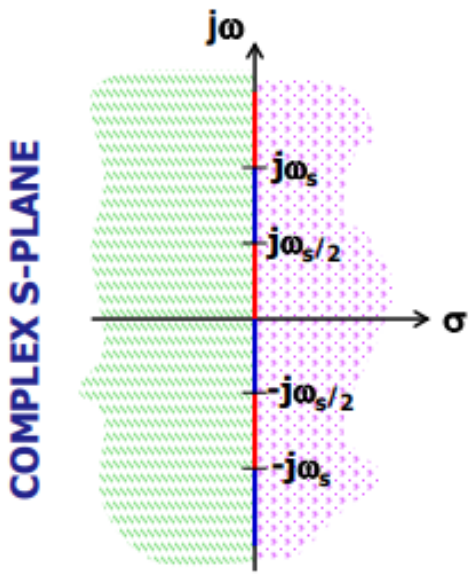
We can then split the exponential coefficient as:

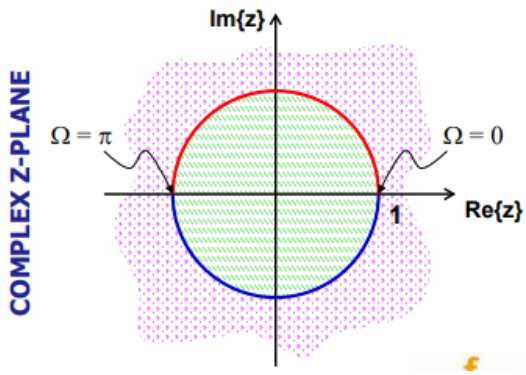
$$z = e^{sT} \doteq e^{(\sigma+j\omega)T} = e^{\sigma T}e^{j\omega T} = \rho e^{j\Omega}$$

where we have selected:

$$\boxed{\rho = e^{\sigma T} \quad \Omega = \omega T}$$

This leads us to plot both real and complex regions as:





7.1 Properties of Z-transformation

Main Properties of the Z-Transform

- **Linearity:**
$$\alpha x_1[n] + \beta x_2[n] \ \leftrightarrow \alpha X_1(z) + \beta X_2(z)$$

- **Temporal Shift (Delay):**
$$x[n - n_0] \ \leftrightarrow z^{-n_0} X(z)$$

- **Convolution:**
$$x_1[n] * x_2[n] \ \leftrightarrow X_1(z) X_2(z)$$

- **Derivation in Z:**
$$nx[n] \ \leftrightarrow -z \frac{dX(z)}{dz}$$

- **Initial Value Theorem:**
$$x[0] = \lim_{z \to \infty} X(z) \quad \text{if } x[n] = 0, \ n < 0$$
and the poles of $(z-1)X(z)$ are inside $|z| = 1$.

- **Final Value Theorem:**
$$x[\infty] = \lim_{z \to 1} X(z)(1 - z^{-1})$$

- **Multiplication by a Potential Sequence:**
$$a^n x[n] \ \leftrightarrow X\left(\frac{z}{a}\right)$$

- **Time Inversion:**
$$x[-n] \ \leftrightarrow X\left(\frac{1}{z}\right) = X(z^{-1})$$

Assumption

From now on, it is assumed that all sequences with which we are going to deal with are defined from a finite time instant (n_0). Hence, $f[n] = 0, \quad n < n_0$.
In this case, the ROC is the outside of the circle:

$$|z| > r$$

Table 3: Common sequences, their z–transforms, and regions of convergence (ROC).

Signal	**Waveform** $x[n]$	**Transform** $X(z)$	**ROC**
Impulse	$\delta[n]$	1	all z
Delayed impulse	$\delta[n-n_0]$	z^{-n_0}	$\lvert z\rvert > 0, n_0 > 0$ $\forall z, n_0 < 0$
Rectangular pulse	$p_N[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{1-z^{-N}}{1-z^{-1}}$	$\lvert z\rvert > 0$
Step function (causal)	$u[n]$	$\dfrac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
Causal real power	$a^n\, u[n]$	$\dfrac{1}{1-a\,z^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
Causal cosine	$\cos(\Omega_0 n)\, u[n]$	$\dfrac{1-\cos(\Omega_0)\, z^{-1}}{1-2\cos(\Omega_0)\, z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
Causal sine	$\sin(\Omega_0 n)\, u[n]$	$\dfrac{\sin(\Omega_0)\, z^{-1}}{1-2\cos(\Omega_0)\, z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
Causal damped cosine	$r^n \cos(\Omega_0 n)\, u[n]$	$\dfrac{1-r\cos(\Omega_0)\, z^{-1}}{1-2r\cos(\Omega_0)\, z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > \lvert r\rvert$

To perform the anti-transformation we compute in this way:

7.2 Inverse Z-Transform

General expression for the inverse Z-transform:

$$x[n] = \mathcal{Z}^{-1}[X(z)] = \frac{1}{2\pi j}\oint_C X(z) z^{n-1}\, dz$$

where C is a closed anti-clockwise contour path within the ROC and centered at the origin ($z = 0$).

Method based on partial fraction expansion:

$$X(z) = \frac{d_0 + d_1 z^{-1} + \cdots + d_M z^{-M}}{c_0 + c_1 z^{-1} + \cdots + c_N z^{-N}} = \frac{\displaystyle\sum_{k=0}^{M} d_k z^{-k}}{\displaystyle\sum_{k=0}^{N} c_k z^{-k}}$$

Let $p_1, p_2, \ldots, p_N$ be the roots of the denominator (we will assume simple poles). Then $X(z)$ can be written as:

$$X(z) = \sum_{r=0}^{M-N} B_r z^{-r} + \sum_{k=1}^{N} \frac{A_k}{1-p_k z^{-1}}$$

where A_k are the *residues*.

Hence, the inverse Z-transform is given by:

$$x[n] = \mathcal{Z}^{-1}[X(z)] = \sum_{r=0}^{M-N} B_r\, \delta[n-r] + \sum_{k=1}^{N} A_k\, p_k^n u[n]$$

7.3 Calculate from the recursion formula, the transfer function

This is a quite useful method to find the transfer function (gain) of the IIR system:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad \xrightarrow{\mathcal{Z}} \quad \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

$$\text{Zeros } (z_1, z_2, \ldots): \text{ roots of } \sum_{k=0}^{M} b_k z^{-k} \qquad \text{Poles } (p_1, p_2, \ldots): \text{ roots of } \sum_{k=0}^{N} a_k z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 \prod_{k=1}^{M} (1 - z_k z^{-1})}{a_0 \prod_{k=1}^{N} (1 - p_k z^{-1})}$$

$$\begin{cases} H(z_k) = 0, & \text{zeros} \\ H(p_k) = \infty, & \text{poles} \end{cases}$$

✓ If $M > N$, there are poles (order $M - N$) at the origin:

$$H(0) = \lim_{z \to 0} H(z) = \infty$$

✓ If $N > M$, there are zeros (order $N - M$) at the origin:

$$H(0) = \lim_{z \to 0} H(z) = 0$$

Note: There has to be, actually, the same order in the numerator and the denominator.

How to calculate output signals?

1. Given $x[n]$, calculate $X(z) = Z[x[n]]$
2. Calculate $Y(z) = X(z) \cdot H(z)$
3. Calculate $y[n] = Z^{-1}[Y(z)] = Z^{-1}[X(z) \cdot H(z)]$

Inverse Z-Transform Example

Given:

$$X(z) = \frac{2 - 3.5z^{-1} + 2.5z^{-2} - 0.5z^{-3}}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Step (i): Division until deg(numerator) < deg(denominator)

$$X(z) = -z^{-1} + 2 + \frac{0.5z^{-1}}{1 - 1.5z^{-1} + 0.5z^{-2}}$$

Step (ii): Calculation of poles and denominator decomposition

$$1 - 1.5z^{-1} + 0.5z^{-2} = 0 \quad \Rightarrow \quad z^{-2} - 1.5z^{-1} + 0.5 = 0 \quad \Rightarrow \quad p_1 = 1,\ p_2 = 0.5$$

$$X(z) = -z^{-1} + 2 + \frac{0.5z^{-1}}{(1 - z^{-1})(1 - 0.5z^{-1})}$$

Step (iii): Partial fraction expansion

$$X(z) = -z^{-1} + 2 + \frac{A_1}{1 - z^{-1}} + \frac{A_2}{1 - 0.5z^{-1}}$$

Step (iv): Residue calculation

$$A_1 = \frac{0.5z^{-1}}{(1 - 0.5z^{-1})}(1 - z^{-1})\Big|_{z=1} = 1 \qquad A_2 = \frac{0.5z^{-1}}{(1 - z^{-1})}(1 - 0.5z^{-1})\Big|_{z=0.5} = -1$$

Inverse Z-transform:

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} = -\delta[n-1] + 2\delta[n] + u[n] - 0.5^n u[n]$$

7.4 System Stability in the Z-Domain

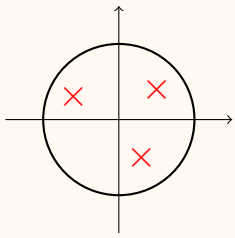
Stability Criteria in Discrete-Time Systems (Z-Domain)

- **Stable System:** A discrete-time system is **STABLE** if **all its poles are located within the unit circle** in the complex z-plane:

$$|z_p| < 1$$

Equivalently, the impulse response $h[n]$ satisfies:

$$\lim_{n\to\infty} h[n] = 0$$



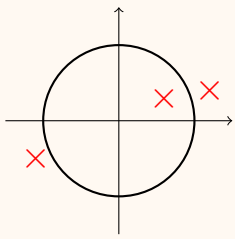
Poles inside unit circle

- **Unstable System:** A system is **UNSTABLE** if **at least one pole lies outside the unit circle:**

$$|z_p| > 1$$

In this case, the impulse response diverges:

$$\lim_{n\to\infty} |h[n]| = \infty$$



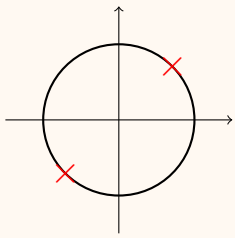
At least one pole outside unit circle

- **Marginally Stable System:** A system is **MARGINALLY STABLE** if it has **simple poles on the unit circle** and **no poles outside**:

$$|z_p| = 1 \quad \text{(simple poles only)}$$

Then the impulse response remains bounded but non-decaying:

$$\lim_{n\to\infty} |h[n]| = \text{constant}$$



Simple poles on unit circle

Ejercicio 4 — Parcial 22–23: Identificación de polos y ceros

Idea clave: En un sistema LTI discreto causal, los **polos** y **ceros** pueden deducirse comparando la **entrada** y la **salida** en el dominio temporal.

Si un componente aparece en $y[n]$ pero no en $x[n] \Rightarrow$ Polo
Si un componente aparece en $x[n]$ pero no en $y[n] \Rightarrow$ Cero

Ejemplo:

$$\begin{cases} x[n] = \cos(\pi n)\, u[n] + \cos\left(\dfrac{\pi n}{2}\right) u[n], \\ y[n] = \cos(\pi n)\, u[n] + A(-0.5)^n u[n] + B\,\delta[n]. \end{cases}$$

- La componente $\cos\left(\dfrac{\pi n}{2}\right)$ está en la entrada pero no en la salida $\rightarrow \Rightarrow$ **cero** en:

$$z = e^{\pm j\pi/2} = \pm j.$$

- La componente $A(-0.5)^n u[n]$ aparece sólo en la salida $\rightarrow \Rightarrow$ **polo** en:

$$z = -0.5.$$

- La componente $\cos(\pi n)$ aparece en ambas $\rightarrow$ transmitida $\rightarrow H(e^{j\pi}) \neq 0$.

Conclusión:

$$H(z) = K\,\frac{(1+z^{-2})}{1+0.5z^{-1}}, \quad \Rightarrow \begin{cases} \text{Ceros: } z = \pm j, \\ \text{Polo: } z = -0.5. \end{cases}$$

Interpretación física:

- Los ceros representan frecuencias que el sistema *bloquea*.
- Los polos representan modos naturales que el sistema *genera por sí mismo*.
- Si $|p| < 1$, el modo natural decae $\rightarrow$ sistema estable.

Observaciones:

- Un polo en $z = 1$ produciría un término constante $C\,u[n]$.
- Un polo en $z = -1$ generaría $C(-1)^n u[n]$ (oscilación alternante).
- Si no aparecen tales términos, esos polos no existen (a menos que estén cancelados por ceros).

8 Equalizer of $H(z)$

The goal of an **equalizer** is to compensate the frequency distortion introduced by a system $H(z)$. If a discrete-time system produces an output

$$y[n] = h[n] * x[n],$$

the equalizer $H_{eq}(z)$ is designed so that the overall system behaves as an ideal (flat) system:

$$h[n] * h_{eq}[n] = \delta[n]$$

Equalization Condition

$$h[n] * h_{eq}[n] = \delta[n] \quad \Longrightarrow \quad \begin{cases} H(z)\, H_{eq}(z) = 1 \\ H(e^{j\Omega})\, H_{eq}(e^{j\Omega}) = 1 \end{cases}$$

Therefore, the frequency response of the equalizer must satisfy:

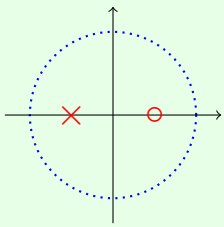
$$\boxed{|H_{eq}(e^{j\Omega})| = \frac{1}{|H(e^{j\Omega})|}}$$

Example

Consider a discrete-time system:

$$H(z) = \frac{1 - 0.5z^{-1}}{1 + 0.5z^{-1}} = \frac{z - 0.5}{z + 0.5}$$

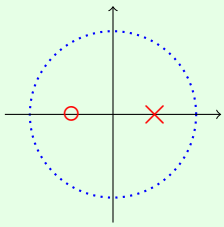
Its **zero** is at $z = 0.5$ and its **pole** at $z = -0.5$, both located on the real axis inside the unit circle.



$H(z)$: $z_0 = 0.5$, $p_0 = -0.5$

The equalizer is defined as the inverse system:

$$H_{eq}(z) = \frac{1}{H(z)} = \frac{1 + 0.5z^{-1}}{1 - 0.5z^{-1}} = \frac{z + 0.5}{z - 0.5}$$



$H_{eq}(z)$: $z_0 = -0.5$, $p_0 = 0.5$

Interpretation: The equalizer $H_{eq}(z)$ inverts the amplitude distortion of $H(z)$, so that the cascade $H(z)H_{eq}(z)$ acts as an identity system.

$$x[n] \xrightarrow{H(z)} y[n] \xrightarrow{H_{eq}(z)} x[n]$$

9 Discrete Fourier Transform (DFT)

The **Discrete Fourier Transform (DFT)** converts a discrete-time sequence of finite length N into a set of N complex frequency coefficients. It provides a frequency-domain representation suitable for numerical computation and digital signal analysis.

Definition

For a discrete-time signal $x[n]$ of length N:

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1$$

The inverse DFT reconstructs $x[n]$ from its spectral components:

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k]\, e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1$$

9.1 Properties

Main Properties of the DFT

- **Linearity:**

$$\alpha x[n] + \beta y[n] \overset{\text{DFT}}{\leftrightarrow} \alpha X[k] + \beta Y[k]$$

- **Time Shift:**

$$x[(n - n_0)_N] \overset{\text{DFT}}{\leftrightarrow} e^{-j\frac{2\pi kn_0}{N}} X[k]$$

- **Time Inversion:**

$$x[(-n)_N] \overset{\text{DFT}}{\leftrightarrow} X[-k]_N$$

- **Circular Convolution:**

$$x_1[n] \circledast x_2[n] \overset{\text{DFT}}{\leftrightarrow} X_1[k]\, X_2[k]$$

- **Windowing (Multiplication in Time):**

$$x[n]\, y[n] \overset{\text{DFT}}{\leftrightarrow} \frac{1}{N} \sum_{m=0}^{N-1} X[m]\, Y[(k-m)_N]$$

  (Product in time $\leftrightarrow$ circular convolution in frequency)

- **Frequency Shift:**

$$x[n]\, e^{j2\pi n_0 n/N} \overset{\text{DFT}}{\leftrightarrow} X[(k - n_0)_N]$$

- **Parseval's Theorem:**

$$\sum_{n=0}^{N-1} x[n] y^*[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] Y^*[k]$$

- **Symmetry for Real Sequences:**

$$x[n] \in \mathbb{R} \;\Rightarrow\; X[N-k] = X^*[k]$$

9.2 Aliasing

Aliasing in DFT

The DFT implicitly assumes that $x[n]$ is **periodic with period** N. If the original sequence is not zero outside the interval $[0, N-1]$, the reconstructed signal through the IDFT becomes a **periodic extension**:

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN] = x[n] * t_N[n], \qquad t_N[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

In this case, $\tilde{x}[n] \neq x[n]$ in the interval $[0, N-1]$, which produces **temporal aliasing**.

If $x[n] \neq 0$ outside $[0, N-1]$, then the periodic extension $\tilde{x}[n]$ is *not equal* to the original signal.

Sequence $x[n]$	**DFT** $X[k]$	**Remarks**
$\delta[n]$	1	Flat spectrum
1	$N\,\delta[k]$	DC component only
$\delta[n - n_0]$	$e^{-j2\pi k n_0/N}$	Phase shift
$e^{j2\pi m_0 n/N}$	$N\,\delta[k - m_0]$	Pure frequency tone
$\cos\left(\frac{2\pi m_0 n}{N}\right)$	$\frac{N}{2}\left[\delta[k - m_0] + \delta[k + m_0]\right]$	Real sinusoid
$u[n]$ (first N samples)	$\dfrac{1 - e^{-j2\pi k N/N}}{1 - e^{-j2\pi k/N}}$	Finite rectangular sequence

Table 4: Common Discrete Fourier Transform (DFT) pairs.

9.3 Interpretation

The DFT samples the Discrete-Time Fourier Transform (DTFT) at equally spaced frequencies:

$$\omega_k = \frac{2\pi k}{N}, \quad k = 0, 1, \ldots, N-1$$

Thus, the DFT provides a discrete spectral representation suitable for computation by the Fast Fourier Transform (FFT).

9.4 Circular Delay Property

If $x_N[n]$ is a sequence of length N and its DFT is $X_N[k]$, then a circular time shift of n_0 samples corresponds to a complex exponential modulation in the frequency domain:

$$x_N[(n-n_0)_N] \overset{\text{DFT}}{\longleftrightarrow} e^{-j\frac{2\pi}{N}kn_0} X_N[k]$$

where $(n-n_0)_N$ denotes the index module N:

$$(n-n_0)_N = (n-n_0) \bmod N$$

This means the sequence is shifted cyclically — samples that "exit" one end re-enter from the other. The exponential factor introduces a linear phase term in $X_N[k]$, just as in the continuous-time Fourier transform, but here the periodicity in N makes the delay *circular*.

Example:

$$x_N[n] = \{1, 2, 3, 4\}, \quad N = 4$$

Circularly delayed by $n_0 = 1$:

$$x_N[(n-1)_4] = \{4, 1, 2, 3\}$$

and in the frequency domain:

$$X_N[k] \to e^{-j\frac{2\pi}{4}k} X_N[k]$$

Properties of the Discrete Fourier Transform (DFT)

9.5 Circular Convolution Property

For two periodic sequences $x_N[n]$ and $h_N[n]$, both of length N, their **circular convolution** is defined as:

$$y_N[n] = (x_N[n] \circledast h_N[n]) = \sum_{m=0}^{N-1} x_N[m]\, h_N[(n-m)_N]$$

The DFT converts this operation into a simple multiplication in the frequency domain:

$$x_N[n] \circledast h_N[n] \overset{\text{DFT}}{\longleftrightarrow} X_N[k] \cdot H_N[k]$$

This is analogous to the linear convolution property of the standard Fourier transform, but with periodic boundary conditions. In practice, if you compute a DFT of two finite signals and multiply them, the inverse DFT produces their *circular* (not linear) convolution.

Relation with Linear Convolution:
If the original sequences $x[n]$ and $h[n]$ are of lengths L_x and L_h, their linear convolution has length $L_y = L_x + L_h - 1$. To obtain the correct (non-circular) convolution using DFTs, you must use a transform length $N \geq L_y$, typically implemented as:

$$y[n] = \text{IDFT}\{\text{DFT}_N\{x[n]\} \cdot \text{DFT}_N\{h[n]\}\}$$

Example:

$$x_N[n] = \{1, 2, 3, 4\}, \quad h_N[n] = \{1, 1, 1, 1\}$$

Then

$$x_N[n] \circledast h_N[n] = \{10, 10, 10, 10\}$$

since all samples wrap around after each period.

Example: Circular vs Linear Convolution

Let two discrete-time sequences be:

$$a[n] = \{2, 3, -1, -4\}, \qquad b[n] = \{1, 2, 3, 4\}$$

Their **linear convolution** has length

$$L_y = L_a + L_b - 1 = 4 + 4 - 1 = 7$$

and produces

$$y_{\text{lin}}[n] = \{2, 7, 11, 11, 1, -16, -16\}$$

—

Circular convolution of period N is defined as:

$$y[n] = a[n] \circledast_N b[n] = \sum_{m=0}^{N-1} a[m]\, b[(n-m)_N]$$

where $(n-m)_N$ denotes subtraction modulo N, i.e. the periodic extension of $b[n]$.

—

Case $N = 4$:

$$y[n] = a[n] \circledast_4 b[n] = \{3, -9, -5, 11\}$$

Since $N < 7$, part of the linear result wraps around — this is **aliasing in time**.

Case $N = 6$:

$$y[n] = a[n] \circledast_6 b[n] = \{-14, 7, 11, 11, 1, -16\}$$

Wrapping still occurs because $N < 7$, although less severe.

Case $N = 7$:

$$y[n] = a[n] \circledast_7 b[n] = \{2, 7, 11, 11, 1, -16, -16\} = a[n] * b[n]$$

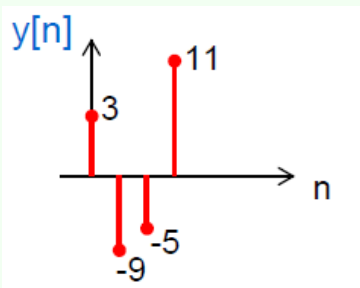
Here $N = L_a + L_b - 1$, so there is no overlap and:

$$\boxed{a[n] \circledast_N b[n] = a[n] * b[n] \quad \text{if } N \geq L_a + L_b - 1}$$

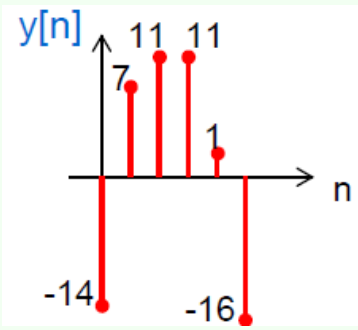
—

Interpretation: The value of N determines whether the periodic extension of the signals causes overlap. When N is large enough to contain all non-zero samples of the linear convolution ($N \geq L_a + L_b - 1$), the circular and linear convolutions are identical.

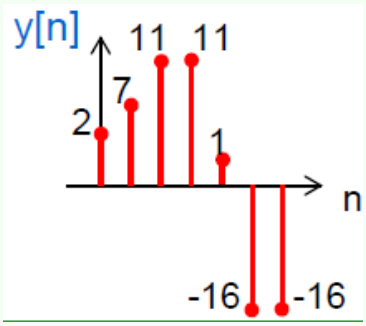
—



N = 4: Strong aliasing



N = 6: Partial overlap



N = 7: Linear and circular coincide

Chapter 3

A/D and D/A conversion

10 Sampling and alias generation: antialiasing filter

We make a sampling from an analog signal to obtain it digital. Then, the question will be: From this discrete signal, if we perform an interpolation, ¿What is the condition that should be satisfied to recover the analog signal?

To perfectly reconstruct a continuous-time signal $x(t)$ from its discrete-time samples

$$x[n] = x(nT_s),$$

the signal must satisfy the conditions of the **Nyquist–Shannon Sampling Theorem**.

If the original signal $x(t)$ is **band-limited**, i.e.

$$X(f) = 0 \quad \text{for} \quad |f| > f_{\max},$$

then perfect reconstruction by interpolation is possible **if and only if** the sampling frequency f_s fulfills:

Nyquist-Shannon Sampling Theorem

$$f_s \geq 2f_{\max}$$

where:

- $f_s = \frac{1}{T_s}$ is the sampling frequency,
- $f_{\max}$ is the maximum frequency present in $x(t)$.

When $f_s \geq 2f_{\max}$, the spectral alias of $X(f)$ generated by sampling do not overlap.

10.0.1 Sampling of a generic signal

The sampling of a continuous-time signal $x(t)$ can be modeled as the multiplication of the signal with a train of Dirac delta impulses $p(t)$:

$$x_p(t) = x(t) \cdot p(t)$$

where the impulse train is defined as:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Substituting, we obtain:

$$x_p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT) = x(nT)$$

Which corresponds to the original signal but evaluated on the multiples of T.

Hence, we could try to find how this affects to the frequency domain:

$$X_p(\Omega) = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)\right\} = \sum_{n=-\infty}^{\infty} x(nT)\,e^{-j\Omega nT} = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\Omega - k\frac{2\pi}{T}\right)$$

Thus, the spectrum of the sampled signal is a **periodic repetition** of the original continuous-time spectrum $X(\Omega)$ every $\frac{2\pi}{T}$ radians per second (i.e., every $f_s = \frac{1}{T}$ Hz).

We can express this relation equivalently in the discrete-frequency domain:

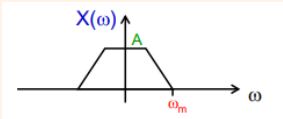
$$X(e^{j\omega}) = X_p\left(\frac{\Omega}{T}\right) \quad \text{with} \quad \Omega = \omega T, \quad \omega = \frac{\Omega}{T}$$

Alias

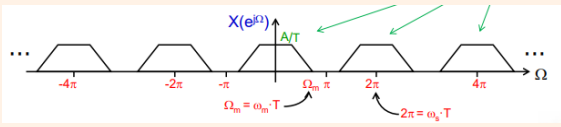
Each repetition of the original spectrum $X(\Omega)$ is an alias of the analog signal.

$$\text{Replicas appear at } \Omega = 0, \pm\frac{2\pi}{T}, \pm\frac{4\pi}{T}, \ldots$$

Then, let X(w) be the Fourier transform of the analog signal:



Then, the sampled Fourier transform is the following:



Notice what happens when $\Omega_m > \pi \rightarrow \omega_m T > \pi \rightarrow \omega_m \frac{2\pi}{\omega_s} > \pi \rightarrow \omega_s < 2\omega_m$ which does not fulfill Nyquist Theorem.

Summary relation:

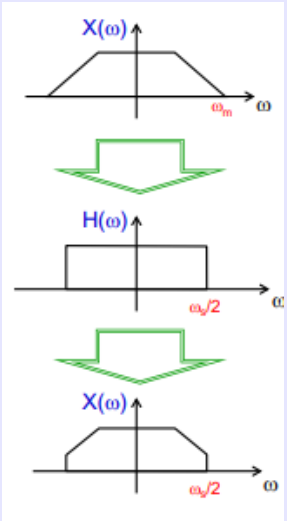
$$X_p(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\left(\Omega - k\frac{2\pi}{T}\right) \quad \Longleftrightarrow \quad X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} X(\omega - 2\pi k)$$

10.0.2 Analog Anti-Aliasing Filter

Before the sampling process, the analog signal $x(t)$ is passed through a **low-pass filter** known as the **anti-aliasing filter**. Its purpose is to remove all frequency components of $x(t)$ which **do not fulfill Nyquist Theorem**, ensuring that the spectral replicas do not overlap after sampling.

Anti-aliasing Filter

Anti-aliasing filter cutoff frequencies above $\omega_s/2$. See the example below. This allow the alias not to be superposed.



After filtering, the effective spectrum of the signal becomes limited to:

$$|\Omega| \leq \frac{\Omega_s}{2}$$

which guarantees that, when the sampling is performed, the repeated spectra in the discrete-time domain $X(e^{j\omega})$ remain separated and can be perfectly reconstructed.

10.1 Signal reconstruction: ideal and zero-order-hold (ZOH) interpolators

The clue idea in this section is to find if it exists an ideal interpolator such that we obtain the original function from the set of data.

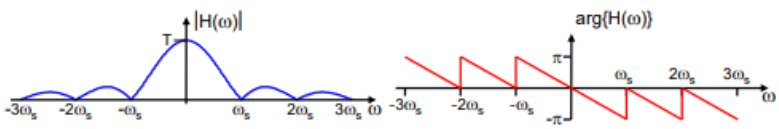
Ideal interpolator

Let's assume the ideal interpolator is our anti-aliasing filter (with the Nyquist condition), with response $H(w) = T, \quad -\omega_s/2 < \omega < \omega_s/2$ and impulse response $h(t) = sinc(\frac{t}{T})$. Then this filter will reconstruct the signal as:

$$\boxed{x_{reconstruct}(t) = x_p(t) * h(t) = \sum_{n=-\infty}^{+\infty} x(nT) sinc(\frac{t - nT}{T})}$$

ZOH interpolator

In a similar way, we can find the interpolation that corresponds to applying the ZOH filter, with impulse response $h(t) = 1, \quad 0 < t < T$ then frequency response corresponding to $H(\omega) = T sinc(\frac{\omega}{\omega_s}) e^{-j\omega\frac{T}{2}}$, thus the filter acts like:



$$\boxed{x_{reconstruct} = x_p(t) * h(t) = \sum_{n=-\infty}^{+\infty} x(nT)h(t-nT)}$$

Hence, notice this filter acts, in the frequency domain, as a low-pass filter distorting a lot high frequencies. Notice that this is such a bad interpolation that changes even the low-frequency term.

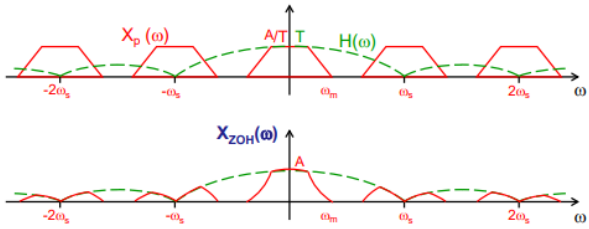
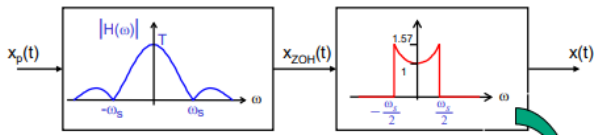


Figure 1: Aperture Distortortion Filter

Then the question is if we can fix what ZOH has distorted

To recover the original signal, we need to apply after the Aperture distorion filter, an analog-low pass filter:



Interpolation Type	Time-domain behavior	Frequency-domain effect	Advantages	Drawbacks
Zero-Order Hold (ZOH)	Each sample value is held constant until the next one, producing a staircase waveform.	Acts as multiplication by $\text{sinc}\left(\frac{\omega T}{2\pi}\right)$ $\rightarrow$ attenuates and distorts high frequencies.	Simple, widely used in DACs, easy hardware implementation.	High-frequency loss and phase distortion; not band-limited.
Linear Interpolation	Joins samples with straight lines, giving a continuous but non-smooth signal.	Equivalent to $\text{sinc}^2\left(\frac{\omega T}{2\pi}\right)$ $\rightarrow$ stronger high-frequency attenuation than ZOH.	Simple, smooth transition between samples, less distortion than ZOH.	Not exact reconstruction; still modifies the spectrum.
Ideal (Sinc) Interpolation	Each sample weighted by shifted $\text{sinc}\left(\frac{t-nT}{T}\right)$, achieving perfect reconstruction if band-limited.	Implements ideal brick-wall low-pass filter (cutoff $\omega_s/2$).	Exact recovery of $x(t)$ under Nyquist criterion; theoretical reference.	Requires infinite-length sinc; impractical for real-time systems.

Table 5: Comparison of interpolation methods: ZOH, Linear, and Ideal (Sinc).

10.1.1 Oversampling

Under the sampling over $\omega_s >> 2\omega_m$, it enables us to reduce phenomenon such as: Aliasing, Aperture distortion, quantization noise...

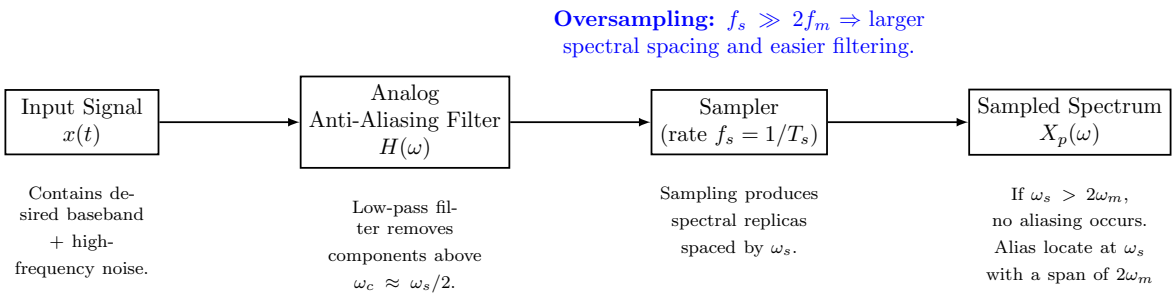


Figure 2: Oversampling and anti-aliasing filtering process. The low-pass filter removes high-frequency noise before sampling, avoiding aliasing. Oversampling increases the distance between replicated spectra, simplifying filter design.

10.2 Change of the sampling frequency: decimation and interpolation.

10.2.1 Decimation

$$x[n] \longrightarrow \boxed{\downarrow M} \longrightarrow y[n] = x[Mn], \quad M \in \mathbb{N}$$

We select only those multiple values of M. Hence, this reduces the sampling frequency to:

$$\boxed{f_s' = \frac{f_s}{M}}$$

Then, to analyse its frequency domain counterpart, we build the mechanism as it follows:

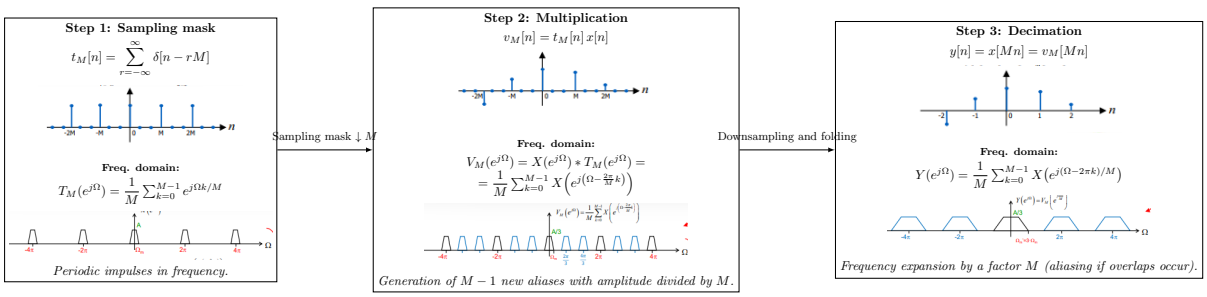


Figure 3: **Conceptual process of discrete-time decimation in frequency domain (example $M = 3$).** The first step defines the periodic sampling mask $t_M[n]$, whose frequency response $T_M(e^{j\Omega})$ consists of periodic impulses. Multiplying the input signal by this mask ($v_M[n] = t_M[n]x[n]$) replicates the spectrum, producing $M - 1$ additional aliases with amplitude scaled by $1/M$. Finally, decimation by M expands the frequency axis by a factor of M, which can cause overlapping between replicas (*aliasing*) if the original signal was not band-limited. In summary, decimation involves: (1) spectral replication, and (2) frequency expansion, as shown conceptually for $M = 3$.

Anti-aliasing

So, notice it could have aliasing iif:

$$\boxed{\Omega'_m = M\Omega_m > \pi}$$

What we will solve by applying an **anti-aliasing filter** with cutoff frequency of $\Omega_c = \frac{\pi}{M}$

$$H(e^{j\Omega}) = \begin{cases} 1, & |\Omega| \leq \dfrac{\pi}{M}, \\ 0, & \text{otherwise.} \end{cases}$$

10.2.2 Interpolation

$$x[n] \longrightarrow \boxed{\uparrow L} \longrightarrow \tilde{x}[n] = x[\tfrac{n}{L}], \quad n = kL, k \in \mathbb{Z}$$

We generate $L-1$ zeros between consecutive samples. Hence, this amplifies the frequency to:

$$\boxed{f'_s = Lf_s}$$

Notice, here there is **no possible aliasing**, since we are just compressing the frequency scale.

Ideal interpolator

As a result, the original spectrum $X(e^{j\Omega})$, which was initially periodic with period 2π, now becomes compressed and repeats L times within the interval $[-\pi, \pi]$. In other words, $L-1$ additional *spectral images* (or replicas) appear:

$$\tilde{X}(e^{j\Omega}) = \frac{1}{L}\sum_{k=0}^{L-1} X\left(e^{j(\Omega-2\pi k)/L}\right) = X(e^{jL\Omega})$$

To recover the original spectral shape and eliminate the undesired images, a **low-pass interpolation filter** is applied. This filter preserves only the main spectral lobe and suppresses all higher-frequency replicas. Its cutoff angular frequency is

$$\Omega_c = \frac{\pi}{L}.$$

In the discrete-time domain, the ideal interpolation filter has an impulse response

$$h[n] = \operatorname{sinc}\left(\frac{n}{L}\right),$$

whose frequency response is a rectangular function:

$$H(e^{j\Omega}) = \begin{cases} L, & |\Omega| \leq \dfrac{\pi}{L}, \\ 0, & \text{otherwise.} \end{cases}$$

After filtering, the resulting signal $y[n]$ exhibits a spectrum equivalent to that of the original continuous-time signal sampled at the higher rate $f'_s = L\,f_s$.

10.2.3 Fractional change (combination of both previous)

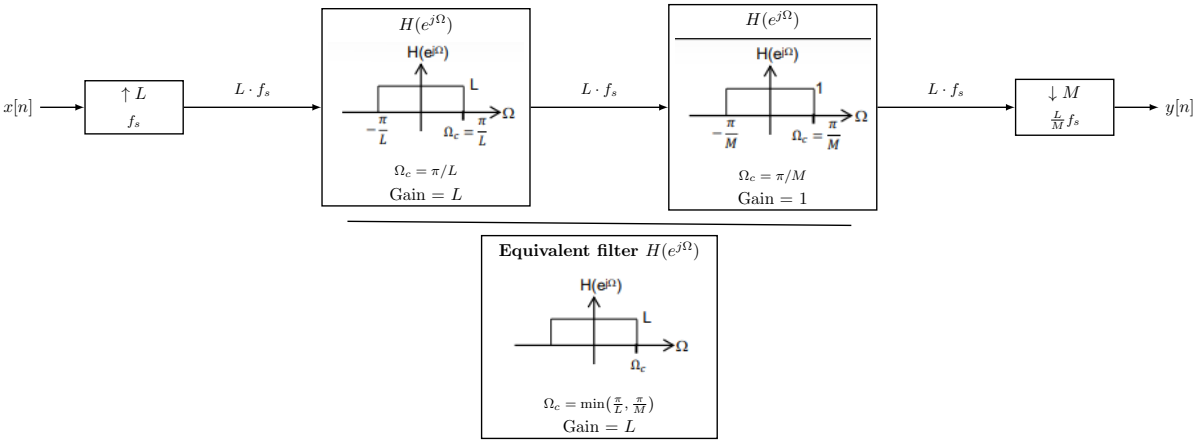


Figure 4: Conceptual diagram of interpolation and decimation in a multirate system. The overall equivalent filter has cutoff $\Omega_c = \min(\pi/L, \pi/M)$. Images `H_L.png`, `H_M.png`, and `H_combined.png` correspond to the respective filter responses.

Chapter 4

Stochastic processes – Random signals

11 Statistical characterization

Order-i Characterization	$t_1, t_2, \ldots, t_i \Rightarrow X(t_1), \ldots, X(t_i)$ are random variables.
CDF	$F_X(x_1, \ldots, x_i; t_1, \ldots, t_i) = P[X(t_1) \leq x_1, \ldots, X(t_i) \leq x_i]$
PDF	$f_X(x_1, \ldots, x_i; t_1, \ldots, t_i) = \dfrac{\partial^i F_X}{\partial x_1 \ldots \partial x_i}$
Extension	Applies to discrete-time. Lower-order pdf/cdf via marginalization.
Complete Characterization	A process is fully described if f_X or F_X is known for any order i, any instants $(t_1, \ldots, t_i)$ or $(n_1, \ldots, n_i)$, and any values $(x_1, \ldots, x_i)$.
Moments (1st & 2nd order)	
Mean (1st moment)	$m_X(t) = \mathbb{E}[X(t)] = \int x f_X(x;t)\, dx \qquad m_X[n] = \sum x f_X(x;n)$
Auto-correlation (2nd moment)	$r_X(t_1, t_2) = \mathbb{E}[X(t_1)X(t_2)] = \iint x_1 x_2 f_X(x_1, x_2; t_1, t_2)\, dx_1 dx_2$
Instantaneous Power	$P(t) = \mathbb{E}[\lvert X(t)\rvert^2] = r_X(t,t) \qquad P[n] = r_X[n,n]$
Second-Order Centered Moments	
Auto-covariance	$c_X(t_1, t_2) = \mathbb{E}[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))] = r_X(t_1, t_2) - m_X(t_1) m_X^*(t_2)$
Variance	$\sigma_X^2(t) = c_X(t,t) = r_X(t,t) - \lvert m_X(t)\rvert^2 = P_x(t) - \lvert m_X(t)\rvert^2$
Two Processes $X(t), Y(t)$	
Joint 1st-order CDF/PDF	$F_{XY}(x, y; t_x, t_y) = P[X(t_x) \leq x,\, Y(t_y) \leq y], \quad f_{XY}(x, y; t_x, t_y) = \frac{\partial^2 F_{XY}}{\partial x\, \partial y}$
Joint higher-order CDF/PDF	$F_{XY}(x_1, \ldots, y_j; t_{x1}, \ldots, t_{yj}), \quad f_{XY} = \dfrac{\partial^{i+j} F_{XY}}{\partial x_1 \ldots \partial y_j}$
Independence	$f_{XY}(x, y) = f_X(x)\, f_Y(y)$
Cross-correlation & Cross-covariance	$r_{XY}(t_1, t_2) = \mathbb{E}[X(t_1)Y^*(t_2)] = \iint x y^* f_{XY}(x, y; t_1, t_2)\, dx\, dy$ $c_{XY}(t_1, t_2) = \mathbb{E}[(X(t_1) - m_X(t_1))(Y^*(t_2) - m_Y^*(t_2))] = r_{XY}(t_1, t_2) - m_X(t_1) m_Y^*(t_2)$
Uncorrelated, Orthogonal, Independent	Two processes are **uncorrelated** ($\Rightarrow$ independent only if Gaussian processes) if $c_{XY}(t_1, t_2) = 0$, **orthogonal** if $r_{XY}(t_1, t_2) = 0$, and **independent** ($\Rightarrow$ uncorrelated) if $f_{XY}(x, y) = f_X(x) f_Y(y)$.

12 Strict-sense and wide-sense stationary processes. Cyclostationary processes.

Strict-Sense Stationary (SSS)	A stochastic process is **strict-sense stationary** if its joint PDFs are invariant under a time shift: $$f_X(x_1,\dots,x_i;t_1,\dots,t_i) = f_X(x_1,\dots,x_i;t_1+\Delta,\dots,t_i+\Delta), \quad \forall t_i, \forall \Delta$$
First-Order Stationarity	The **mean** does not depend on time: $$m_X(t) = \mathbb{E}[X(t)] = \text{constant}$$
Second-Order Stationarity	The **autocorrelation** depends only on the time difference: $$r_X(t_1,t_2) = \mathbb{E}[X(t_1)X^*(t_2)] = r_X(t_1-t_2) = r_X(\tau)$$ Hence, the average power is constant: $$P_X = \mathbb{E}[\lvert X(t)\rvert^2] = r_X(0)$$
Wide-Sense Stationary (WSS)	A process is **WSS** if it satisfies first- and second-order stationarity: $$\begin{cases} m_X(t) = \mathbb{E}[X(t)] = \text{constant} \\ r_X(t_1,t_2) = r_X(t_1-t_2) \end{cases}$$
Discrete-Time Case	$$m_X[n] = \mathbb{E}[X[n]] = \text{constant}$$ $$r_X[n_1,n_2] = \mathbb{E}[X[n_1]X^*[n_2]] = r_X[n_1-n_2] = r_X[m]$$ $$P_X = \mathbb{E}[\lvert X[n]\rvert^2] = r_X[0]$$
Relationship	**Second-order stationarity $\Rightarrow$ Wide-Sense Stationarity**

Sinusoid with a random phase

Let
$$X(t) = A\cos(\omega_0 t + \Theta), \quad \Theta \sim U(-\pi,\pi)$$
Then:
$$f_x(x;t) = f_x(x) = \frac{1}{\pi\sqrt{A^2-x^2}}, |x| \le A$$

Then, it is **First order stationary** (prove $m_x(t) = 0 = ctte$).
It is also **Second order stationary** since:

$$r_x(t_1,t_2) = E[X(t_1)X^*(t_2)] = E[A^2\cos(\omega_0 t_1+\Theta)\cos(\omega_0 t_2+\Theta)] = \frac{A^2}{2}\cos(\omega_0\tau)$$

Then, it is WSS. If we had randomized amplitude, we would have not obtained the WSS.

12.1 Autocorrelation for Non-Stationary Processes

General Definition

$$r_X(t, t+\tau) = \mathbb{E}[X(t)\, X^*(t+\tau)]\,, \qquad r_X[n, n+m] = \mathbb{E}[X[n]\, X^*[n+m]]\,. \tag{1}$$

Particular Cases

- **Wide-Sense Stationary (WSS) Processes:**

$$r_X(t, t+\tau) = r_X(\tau), \qquad r_X[n, n+m] = r_X[m].$$

  The correlation depends only on the time difference τ (or lag m).

Non-Stationary Processes

- $r_X(t, t+\tau)$ or $r_X[n, n+m]$ depend on both t and the lag (τ or m).
- The autocorrelation therefore varies over time and must be averaged to extract meaningful periodic or global behavior.

Time-Averaged Autocorrelation

$$\overline{r}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} r_X(t, t+\tau)\, dt, \qquad \overline{r}_X[m] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} r_X[n, n+m]. \tag{2}$$

Cyclostationary Processes

- $r_X(t, t+\tau)$, $m_X(t)$, $r_X[n, n+m]$, $m_X[n]$ are periodic in t (or n) with period T (or N).
- These signals exhibit statistical periodicity — the autocorrelation repeats every cycle.

Time-Averaged Autocorrelation for Cyclostationary Processes

$$\overline{r}_X(\tau) = \frac{1}{T_0} \int_{0}^{T_0} r_X(t, t+\tau)\, dt, \qquad \overline{r}_X[m] = \frac{1}{N} \sum_{n=0}^{N-1} r_X[n, n+m]. \tag{3}$$

Summary Schema

Process Type	**Dependence**	**Averaged Correlation Expression**
WSS	$r_X(\tau)$ or $r_X[m]$	Constant over time
Non-Stationary	$r_X(t, t+\tau)$ or $r_X[n, n+m]$	$\overline{r}_X(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} r_X(t, t+\tau) dt$
Cyclostationary	Periodic in t (or n)	$\overline{r}_X(\tau) = \frac{1}{T_0} \int_{0}^{T_0} r_X(t, t+\tau) dt$

12.2 Cross-Correlation for Jointly Stationary and Non-Stationary Processes

Definition

$$r_{XY}(t, t+\tau) = \mathbb{E}[X(t)\, Y^*(t+\tau)]\,, \qquad r_{XY}[n, n+m] = \mathbb{E}[X[n]\, Y^*[n+m]]\,. \tag{4}$$

Then, two processes $X(t)$ and $Y(t)$ are:

Joint Wide-Sense Stationarity (WSS)

- Both processes $X(t)$ and $Y(t)$ (or $X[n], Y[n]$) must be **WSS individually**.
- Their cross-correlation depends *only on the time difference* (τ or m):

$$r_{XY}(t, t+\tau) = r_{XY}(\tau), \qquad r_{XY}[n, n+m] = r_{XY}[m]. \tag{5}$$

- This implies:

$$r_{XY}(\tau) = \mathbb{E}[X(t)Y^*(t+\tau)] = \mathbb{E}[X(0)Y^*(\tau)]\,.$$

Non–Jointly Stationary Processes

- If $X(t)$ and $Y(t)$ are not jointly WSS, the cross-correlation depends on both time variables t and τ (or n and m).
- To obtain a meaningful measure of correlation, one defines the *time-averaged cross-correlation.*

Time-Averaged Cross-Correlation

$$\overline{r}_{XY}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} r_{XY}(t, t+\tau)\, dt, \qquad \overline{r}_{XY}[m] = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} r_{XY}[n, n+m]. \tag{6}$$

Summary Schema

Process Pair	**Condition**
Jointly WSS	Both $X(t)$, $Y(t)$ are WSS and correlation depends only on lag
Not Jointly WSS	Correlation depends on time origin t or sample index n

PROPERTIES OF THE CORRELATION FUNCTION

Assumption: The following properties are defined assuming **wide-sense stationarity (WSS)**. If the processes are not stationary, these properties remain valid for the *time-averaged* auto- and cross-correlation functions.

1. Properties of the Auto-Correlation

- **Hermiticity:**
$$r_X(\tau) = r_X^*(-\tau), \qquad r_X[m] = r_X^*[-m].$$
- **Maximum at the origin:**
$$|r_X(\tau)| \leq r_X(0) = P_X, \qquad |r_X[m]| \leq r_X[0] = P_X.$$

2. Properties of the Cross-Correlation

- **Hermiticity:**
$$r_{XY}(\tau) = r_{YX}^*(-\tau), \qquad r_{XY}[m] = r_{YX}^*[-m].$$
- **Inequality:**
$$|r_{XY}(\tau)| \leq \sqrt{r_X(0)\, r_Y(0)}, \qquad |r_{XY}[m]| \leq \sqrt{r_X[0]\, r_Y[0]}.$$
- **Sum of Processes:** For $Z(t) = X(t) + Y(t)$,
$$r_Z(\tau) = \mathbb{E}[Z(t+\tau)\, Z^*(t)] = r_X(\tau) + r_Y(\tau) + r_{XY}(\tau) + r_{YX}(\tau).$$

13 Auto-correlation and power spectral density. Wiener-Khinchin theorem.

Let S be a random variable ($S \in \Omega$), then we define the Fourier transform of a process as a new process:

1. Analog case:
$$X(t, S_i) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega, S_i) e^{j\omega t} d\omega$$
2. Discrete-time case:
$$X[n, S_i] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\Omega}, S_i) e^{j\Omega n} d\Omega$$

In a similar way, we could define the time-windowed FT:

1. Analog case:
$$X_T(t, S) = \begin{cases} X(t, S), & -T \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$
2. Discrete-time case:
$$X_N[n, S] = \begin{cases} X[n, S], & -N \leq n \leq N \\ 0, & \text{otherwise} \end{cases}$$

Which enables us to rewrite the energy of a 2T-length process as:

Analog Process (window of length $2T$)	
Energy	$$E_T(S) = \int_{-T}^{T} \lvert X_T(t,S)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X_T(\omega,S)\rvert^2 d\omega$$
Power	$$P_X(S) = \lim_{T\to\infty} \frac{E_T(S)}{2T} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{\lvert X_T(\omega,S)\rvert^2}{2T}\, d\omega$$
Discrete-Time Process (window of length $2N+1$)	
Energy	$$E_N(S) = \sum_{n=-N}^{N} \lvert X_N[n,S]\rvert^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X_N(e^{j\Omega},S)\rvert^2 d\Omega$$
Power	$$P_X(S) = \lim_{N\to\infty} \frac{E_N(S)}{2N+1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \lim_{N\to\infty} \frac{\lvert X_N(e^{j\Omega},S)\rvert^2}{2N+1} d\Omega$$

In general, we define the power of a process as:

$$P_X = E[P_X(S)] = \frac{1}{2\pi}\int_{-\infty \text{ or } (-\pi)}^{\infty \text{ or } (\pi)} S_X(\omega) d\omega$$

WIENER–KHINCHIN THEOREM

Definition: The *Power Spectral Density (PSD)* is the Fourier transform of the time-averaged auto-correlation function.

Analog process:

$$S_X(\omega) = \mathcal{F}\big[\overline{r_X(\tau)}\big] = \int_{-\infty}^{\infty} \overline{r_X(\tau)}\, e^{-j\omega\tau}\, d\tau$$

Discrete-time process:

$$S_X(e^{j\Omega}) = \mathcal{F}\big[\overline{r_X[m]}\big] = \sum_{m=-\infty}^{\infty} \overline{r_X[m]}\, e^{-j\Omega m}$$

If the process is WSS: The time average is not needed in the correlation function,

$$S_X(\omega) = \mathcal{F}\big[r_X(\tau)\big], \qquad S_X(e^{j\Omega}) = \mathcal{F}\big[r_X[m]\big].$$

Properties:

- The PSD is **real and non-negative:** $S_X(\omega) \geq 0, \quad S_X(e^{j\Omega}) \geq 0.$
- For **real-valued processes**, the PSD is **symmetric:**

$$S_X(\omega) = S_X(-\omega), \quad S_X(e^{j\Omega}) = S_X(e^{-j\Omega}).$$

- The **total power** of the process equals the area under the PSD:

$$r_X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)\, d\omega \quad \text{or} \quad r_X[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_X(e^{j\Omega})\, d\Omega.$$

- The **inverse relation** (from PSD to correlation) is:

$$r_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega) e^{j\omega\tau}\, d\omega.$$

Retaking the example of the sinuoid, we are now able to find the PSD as well as the power:

Recalling

$$r_X(\tau) = \frac{A^2}{2}\cos(\omega_0\tau)$$

Then:

1. PSD:

$$S_X(\omega) = F[\frac{A^2}{2}\cos(\omega_0\tau)] = \frac{A^2\pi}{2}\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]$$

2. Power:

$$P_X = r_X(0) = \frac{A^2}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega) d\omega$$

WHITE NOISE CONDITIONS

Definition: A *wide-sense stationary (WSS) white noise* process is a random process whose samples are uncorrelated in time and have a constant power spectral density (PSD) across all frequencies.

Zero mean:

$$m_X = E[X(t)] = 0, \qquad m_X = E[X[n]] = 0$$

Auto-correlation:

$$r_X(\tau) = \frac{N_0}{2}\,\delta(\tau), \qquad r_X[m] = \frac{N_0}{2}\,\delta[m]$$

where $N_0/2$ is the two-sided power spectral density level.

Power Spectral Density (PSD):

$$S_X(\omega) = \frac{N_0}{2}, \qquad S_X(e^{j\Omega}) = \frac{N_0}{2}$$

(Constant for all frequencies — the process has equal power at every frequency.)

Power:

$$P_X = \begin{cases} \infty, & \text{analog process (infinite bandwidth)} \\ \dfrac{N_0}{2}, & \text{discrete-time process} \end{cases}$$

Complementary Notes:

- Ideal white noise is not physically realizable because it would require infinite power.
- In practice, **band-limited white noise** is used, where $S_X(\omega)$ is constant within a finite bandwidth B.
- White noise has a flat PSD, implying a completely uncorrelated time-domain signal.
- The delta-shaped autocorrelation represents perfect time independence between samples.

13.1 Filtering of random signals

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t) = X(t) * h(t)$$

We already know that the following should fulfil:

$$Y(t, S_i) = X(t, S_i) * h(t) \qquad Y(\omega, S_i) = X(\omega, S_i) \cdot H(\omega)$$

Now, let's assume that $X(t)$ is WSS ($\implies E[X(t)] = m_X := cnst$)

1. Mean:

$$E[Y(t)] = E[X(t) * h(t)] = E\left[\int_{-\infty}^{\infty} h(\alpha) X(t-\alpha)\, d\alpha\right] = \int_{-\infty}^{\infty} h(\alpha) E[X(t-\alpha)]\, d\alpha$$

$$= \int_{-\infty}^{\infty} h(\alpha) m_X\, d\alpha = m_X \int_{-\infty}^{\infty} h(\alpha)\, d\alpha = m_X\, H(0)$$

2. Cross-relation:

$$r_{YX}(\tau) = E[Y(t+\tau) X^*(t)] = E\left[\int_{-\infty}^{\infty} h(\alpha) X(t+\tau-\alpha)\, d\alpha\; X^*(t)\right]$$

$$= \int_{-\infty}^{\infty} h(\alpha)\, E[X(t+\tau-\alpha) X^*(t)]\, d\alpha = \int_{-\infty}^{\infty} h(\alpha)\, r_X(\tau-\alpha)\, d\alpha = r_X(\tau) * h(\tau)$$

3. Auto-correlation of $Y(t)$:

$$r_Y(\tau) = E[Y(t+\tau) Y^*(t)] = E\left[\left(\int_{-\infty}^{\infty} h^*(\alpha) X^*(t-\alpha)\, d\alpha\right)\left(\int_{-\infty}^{\infty} h(\beta) X(t+\tau-\beta)\, d\beta\right)\right]$$

$$= \int_{-\infty}^{\infty} h^*(\alpha) E[Y(t+\tau) X^*(t-\alpha)]\, d\alpha = \int_{-\infty}^{\infty} h^*(\alpha) r_{YX}(\tau+\alpha)\, d\alpha$$

$$= \int_{-\infty}^{\infty} h^*(\alpha)\, r_{YX}(\tau-\alpha)\, d\alpha = r_{YX}(\tau) * h^*(-\tau) = r_X(\tau) * h(\tau) * h^*(-\tau)$$

Type of Process	**Continuous-Time:** $Y(t)$	**Discrete-Time:** $Y[n]$
Non-stationary process	$S_Y(\omega) = S_X(\omega)\, H(\omega)\, H^*(\omega)$	$S_Y(e^{j\Omega}) = S_X(e^{j\Omega})\, H(e^{j\Omega})\, H^*(e^{j\Omega})$
WSS process	$S_Y(\omega) = S_X(\omega)\, \lvert H(\omega)\rvert^2$	$S_Y(e^{j\Omega}) = S_X(e^{j\Omega})\, \lvert H(e^{j\Omega})\rvert^2$
Output power	$P_Y = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega)\, \lvert H(\omega)\rvert^2\, d\omega$	$P_Y = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_X(e^{j\Omega})\, \lvert H(e^{j\Omega})\rvert^2\, d\Omega$

13.2 Vector–Matrix Notation for Discrete-Time Processes. The Auto-Correlation Matrix

1. FIR Filtering Representation. A discrete-time process $X[n]$ filtered by a FIR system of L taps with impulse response $h[n]$ can be written as:

$$Y[n] = X[n] * h[n] = \mathbf{h}^H \mathbf{x}[n],$$

where

$$\mathbf{h} = \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{L-1} \end{bmatrix}, \qquad \mathbf{x}[n] = \begin{bmatrix} X[n] \\ X[n-1] \\ \vdots \\ X[n-L+1] \end{bmatrix}.$$

2. Output Mean. The mean of the output signal is given by:

$$m_Y[n] = E[Y[n]] = \mathbf{h}^H \boldsymbol{\mu}_X[n], \qquad \boldsymbol{\mu}_X[n] = E[\mathbf{x}[n]].$$

For a stationary process:

$$m_Y = m_X \sum_{i=0}^{L-1} h_i^*.$$

3. Output Power and Auto-Correlation Matrix. For a wide-sense stationary (WSS) input $X[n]$:

$$P_Y = E[|Y[n]|^2] = \mathbf{h}^H \mathbf{R}_X \mathbf{h},$$

where the **auto-correlation matrix** is defined as:

$$\mathbf{R}_X = E[\mathbf{x}[n]\mathbf{x}^H[n]] = \begin{bmatrix} r_X[0] & r_X[1] & \dots & r_X[L-1] \\ r_X[-1] & r_X[0] & \dots & r_X[L-2] \\ \vdots & \vdots & \ddots & \vdots \\ r_X[-L+1] & r_X[-L+2] & \dots & r_X[0] \end{bmatrix}.$$

4. Properties of $\mathbf{R}_X$.

- Hermitian: $\mathbf{R}_X = \mathbf{R}_X^H$ (real case: $\mathbf{R}_X = \mathbf{R}_X^T$)
- Toeplitz: elements along each diagonal are constant.
- Positive semidefinite: $\mathbf{h}^H \mathbf{R}_X \mathbf{h} \geq 0$ for all $\mathbf{h} \in \mathbb{C}^{L \times 1}$.

5. Spectral Decomposition Theorem. Since $\mathbf{R}_X$ is Hermitian, it can be diagonalized as:

$$\mathbf{R}_X = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^H, \qquad \boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_L), \qquad \mathbf{Q} = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_L],$$

with $\mathbf{Q}^H\mathbf{Q} = \mathbf{I}$ and $\lambda_i \geq 0$.

6. **Useful Results.**

- **Power at the output of a FIR filter:**

$$P_Y = \mathbf{h}^H \mathbf{R}_X \mathbf{h} = \mathbf{h}^H \mathbf{Q} \boldsymbol{\Lambda} \mathbf{Q}^H \mathbf{h} = \sum_{i=1}^{L} \lambda_i |\mathbf{h}^H \mathbf{q}_i|^2.$$

- **Trace and determinant relations:**

$$\mathrm{Tr}(\mathbf{R}_X) = \sum_{i=1}^{L} \lambda_i, \qquad |\mathbf{R}_X| = \prod_{i=1}^{L} \lambda_i.$$

- **Rayleigh quotients:**

$$\frac{\mathbf{h}^H \mathbf{R}_X \mathbf{h}}{\mathbf{h}^H \mathbf{h}} \in [\lambda_{\min}, \lambda_{\max}],$$

  with equality for $\mathbf{h}$ aligned with the eigenvectors $\mathbf{q}_{\min}$ or $\mathbf{q}_{\max}$.

14 Tables of Transforms

Table 6: Common signals and their continuous–time Fourier transforms.

Signal	$x(t)$	$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$	**Notes**
Impulse	$\delta(t)$	1	—
Shifted impulse	$\delta(t-t_0)$	$e^{-j\omega t_0}$	Time shift
Unit step (distribution)	$u(t)$	$\pi\delta(\omega) + \frac{1}{j\omega}$	Principal value
Rectangular pulse	$\mathrm{rect}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$	Width T
Triangular pulse	$\mathrm{tri}\left(\frac{t}{T}\right)$	$T\,\mathrm{sinc}^2\left(\frac{\omega T}{2\pi}\right)$	Convolution rect$*$rect
Exponential (causal)	$e^{-at}u(t),\ \Re\{a\} > 0$	$\dfrac{1}{a+j\omega}$	Stable pole
Two-sided exponential	$e^{-a\lvert t\rvert},\ a > 0$	$\dfrac{2a}{a^2+\omega^2}$	Even spectrum
Cosine	$\cos(\omega_0 t)$	$\pi[\delta(\omega \pm \omega_0)]$	Line spectrum
Sine	$\sin(\omega_0 t)$	$\dfrac{\pi}{j}[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)]$	Odd symmetry
Complex exponential	$e^{j\omega_0 t}$	$2\pi\,\delta(\omega-\omega_0)$	Frequency shift
Gaussian	$e^{-\alpha t^2}$	$\sqrt{\frac{\pi}{\alpha}}\,e^{-\omega^2/(4\alpha)}$	Self-Fourier
Derivative	$\dfrac{dx(t)}{dt}$	$j\omega X(\omega)$	Differentiation property
Time scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert}\,X\left(\frac{\omega}{a}\right)$	Compression/expansion
Convolution	$x(t) * h(t)$	$X(\omega)H(\omega)$	Conv. $\leftrightarrow$ product

Table 7: Common sequences and their discrete–time Fourier transforms (DTFT).

Signal	$x[n]$	$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$	**Notes**
Impulse	$\delta[n]$	1	Periodic in ω
Shifted impulse	$\delta[n-n_0]$	$e^{-j\omega n_0}$	Time shift
Unit step (distribution)	$u[n]$	$\pi\delta(\omega) + \dfrac{1}{1-e^{-j\omega}}$	Non-summable
Causal exponential	$a^n u[n],\ \lvert a\rvert < 1$	$\dfrac{1}{1-ae^{-j\omega}}$	Stable pole
Two-sided exponential	$\lvert a\rvert < 1$	$\dfrac{1-a^2}{1-2a\cos\omega + a^2}$	Even sequence
Rectangular window	$p_N[n]$	$\dfrac{\sin(\omega N/2)}{\sin(\omega/2)}\, e^{-j\omega(N-1)/2}$	Dirichlet kernel
Triangular window	$p_N * p_N$	$\left(\dfrac{\sin(\omega N/2)}{\sin(\omega/2)}\right)^2$	Bartlett window
Complex sinusoid	$e^{j\omega_0 n}$	$2\pi\sum_k \delta(\omega-\omega_0-2\pi k)$	Line spectrum
Cosine	$\cos(\omega_0 n)$	$\pi[\delta(\omega\pm\omega_0)]$	Even spectrum
Sine	$\sin(\omega_0 n)$	$\dfrac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$	Odd spectrum
First difference	$x[n]-x[n-1]$	$(1-e^{-j\omega})X(e^{j\omega})$	Discrete derivative
Time reversal	$x[-n]$	$X(e^{-j\omega})$	Spectral conjugation
Convolution	$(x*h)[n]$	$X(e^{j\omega})H(e^{j\omega})$	LTI property

Table 8: Common finite–length sequences and their N–point DFT.

Signal	$x[n],\ 0 \le n \le N-1$	$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$	**Notes**
Unit sample	$\delta[n]$	1	Flat spectrum
Constant	1	$N\,\delta[k]$	DC component
Ramp	n	$\frac{N}{1-e^{-j2\pi k/N}} - \frac{1}{(1-e^{-j2\pi k/N})^2}$	Polynomial pair
Bin-aligned complex sinusoid	$e^{j2\pi mn/N}$	$N\,\delta[k-m]$	Orthogonal basis
Bin-aligned cosine	$\cos(2\pi mn/N)$	$\frac{N}{2}[\delta[k \pm m]]$	Real symmetric
Rectangular window	$p_N[n]$	$\frac{\sin(\pi k)}{\sin(\pi k/N)}$	Dirichlet kernel
Exponential sequence	a^n	$\frac{1-a^N e^{-j2\pi k/N}}{1-ae^{-j2\pi k/N}}$	Geometric sum
Circular convolution	$(x \circledast h)[n]$	$X[k]\,H[k]$	DFT property
Time shift	$x[(n-n_0) \bmod N]$	$e^{-j2\pi kn_0/N}X[k]$	Circular shift
Frequency shift	$x[n]e^{j2\pi mn/N}$	$X[(k-m) \bmod N]$	Modulation
Zero padding (to $M > N$)	$[x[0], \ldots, x[N-1], 0 \ldots]$	Densifies spectrum samples	Interpolation in freq.