Electromagnetism

Class Notes

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Contents

0 Remember from physics 2...

0.1 Formulation

$$\bar{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \rho(\vec{r'})\frac{\vec{r}-\vec{r'}}{\mid \vec{r}-\vec{r'} \mid^3}$$

while,

$$\begin{cases} \int E \cdot da = Q_{inside}/\varepsilon_0 \\ \int E \cdot d\ell = 0 \end{cases}$$

$$\bar{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau' J(\vec{r'}) \times \frac{\vec{r}-\vec{r'}}{\mid \vec{r}-\vec{r'} \mid^3}$$

while,

$$\begin{cases} \int B \cdot da = 0 \\ \int \frac{\partial V_0}{\partial \Omega} \cdot d\ell = \mu_0 I \end{cases}$$

So we use the expressions in blue instead of using the other ones:

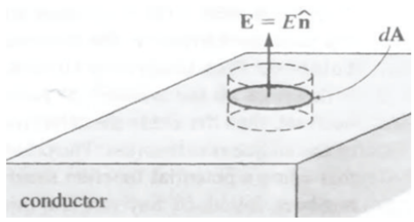
1. Because these expressions cannot be generalized to time-varying fields, while these can; In fact, the flux equations (Gauss law) are correct also for time-dependent fields.

2. Because field equations (flux, line integral) have actually a profound meaning.

3. They simplify the solution of certain problems:

   (a) high-symmetry charge distributions

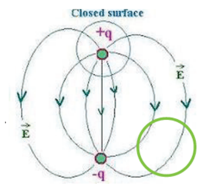
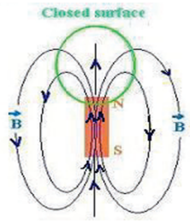
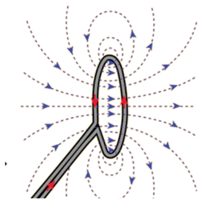
   (b) fields near interfaces ("boundary conditions")



$$E_n = \frac{\sigma}{\varepsilon_0}$$

4. Helmholtz theorem states that a field that only depends on r (and not time) is completely specified if both its curl and divergence are known. Hence we are not really settling for less!

0.2 Phyisical meaning

Integral Form	Differential Form	Explanation	Example/Diagram
$\oint_{\partial\text{Vol}} \vec{E} \cdot d\vec{a} = \frac{Q_{\text{inside}}}{\varepsilon_0}$	$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$	E-field lines start at positive charges and end at negative charges. This law explains how electric charges generate electric fields. **Example:** Electric dipole.	
$\oint_{\partial\text{Vol}} \vec{B} \cdot d\vec{a} = 0$	$\nabla \cdot \vec{B} = 0$	There are no magnetic monopoles (i.e., no isolated north or south poles exist). Magnetic field lines form closed loops, never starting or ending. **Example:** Bar magnet.	
$\oint_{\partial\text{area}} \vec{E} \cdot d\vec{l} = 0$	$\nabla \times \vec{E} = 0$	A changing magnetic field induces a circulating electric field. This is the principle behind electromagnetic induction. **Example:** Changing magnetic flux.	faraday_law.png
$\oint_{\partial\text{area}} \vec{B} \cdot d\vec{l} = \mu_0 I$	$\nabla \times \vec{B} = \mu_0 \vec{J}$	A circulating magnetic field is generated by an electric current and a changing electric field. This law unifies electricity and magnetism. **Example:** Current flowing in a ring.	

0.3 Mathematical reasoning for trespassing between different forms

Differential Form	Integral Form	Boundary Form
$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$	$\oint_{\partial\text{Vol}} \vec{E} \cdot d\vec{a} = \frac{Q_{\text{inside}}}{\varepsilon_0}$	$\vec{E}_1 \cdot \hat{n}_1 + \vec{E}_2 \cdot \hat{n}_2 = \frac{\sigma}{\varepsilon_0}$

0.3.1 From integral form to boundary form

Using the Gauss symmetry of a cylinder and the net flux we can easily determine the charge is only located at the boundary and the electric field does not depend on the surface.

1 Electrostatic of dipoles and metals

Given the formula of a set of point charges:

$$\vec{E}(\vec{r}) = \sum_{k=1}^{N} \frac{q_k}{4\pi\epsilon_0} \left| \frac{\vec{r}-\vec{r}_k}{\vec{r}-\vec{r}_k} \right|^3$$

And the ones given by a continuous distribution of charges (taking account the identity: $dq = \rho d\tau$)
Thus, the identity:

$$\frac{\vec{r}}{r^3} = -\vec{\nabla}(\frac{1}{r})$$

Leads us to define the general expression of a conservative field: $\vec{E} = -\vec{\nabla}V$ (Inherently, this allows us to determine the following property)

$$\nabla \times \vec{E} = -\nabla \times \nabla V = 0$$

so, we just define the **electrostatic potential**:

$$V(\vec{r}) = \sum_{k=1}^{N} \frac{q_k}{4\pi\epsilon_0} \left| \frac{1}{\vec{r}-\vec{r}_k} \right|$$

1.1 Dirac's delta effect

The Dirac's delta tool enables us to just rewrite all the formulas seen using this **distribution** since there is no difference between continuous and discrete distributions. The identity $\rho = q\delta^3(\vec{r}-\vec{r}_i)$ will be meaningful to determine the integral. This leads us to define:

$$\boxed{\vec{\nabla}(\frac{\vec{r}}{r^3}) = 4\pi\delta(\vec{r})}$$

Proof:

$$\int_{\text{sphere}} \left(\nabla \cdot \frac{\vec{r}}{r^3}\right) d\tau = \int_S \frac{\vec{r}}{r^3} \cdot d\vec{a} = \int_S \frac{\hat{r}}{r^2} \cdot r^2 \, d\Omega = \int_S d\Omega = 4\pi = \int_{\text{sphere}} 4\pi\delta(\vec{r}) \, d\tau$$

Just applying the divergence on the electric field expression we find out the so-called: **Poisson Law**, which is given by the laplacian such in the way:

$$\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$$

1.2 Strategy of resolution

If the charge density is given

1. For high density use **Gauss Law** (Gaussian surfaces)
$$\vec{\nabla}\vec{E} = \frac{\rho}{\epsilon_0}$$
2. For low density use **Integration**, either
$$\bar{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau'\rho(\vec{r}')\frac{\vec{r}-\vec{r}'}{\mid \vec{r}-\vec{r}' \mid^3}$$
or
$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int d\tau' \frac{\rho(\vec{r}')}{\mid \vec{r}-\vec{r}' \mid}$$
3. For sum of two or more simple charge distributions use **Superposition**

If the potential of the conductor is given

1. For high simmetry use either
$$\textbf{Laplace Law:}\nabla^2 V = 0$$
$$\textbf{Gauss Law:}\vec{\nabla}\vec{E} = \frac{\rho}{\epsilon_0}$$

For other cases try with **Image Charges**

1.3 Electric dipoles

We define the electric dipole $\vec{p}$ just the way:

$$\vec{p} = q\vec{d}$$

In units of Cm or eÅ
The total potential is:

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

where the vectors $\vec{r}_+$ and $\vec{r}_-$ are:

$$\frac{\vec{d}}{2} + \vec{r}_+ = \vec{r} \quad \rightarrow \quad \vec{r}_+ = \vec{r} + \frac{\vec{d}}{2}$$
$$\frac{\vec{d}}{2} + \vec{r} = \vec{r}_- \quad \rightarrow \quad \vec{r}_- = \vec{r} - \frac{\vec{d}}{2}$$

Since:
$$\frac{1}{r_+} = \frac{1}{\sqrt{\vec{r}_+\cdot\vec{r}_+}} = \frac{1}{\sqrt{\left(\vec{r}+\frac{\vec{d}}{2}\right)\cdot\left(\vec{r}+\frac{\vec{d}}{2}\right)}} = \frac{1}{\sqrt{r^2+\vec{r}\cdot\vec{d}+\frac{d^2}{4}}}$$

and similarly for r_-, we get:

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\left[\left(1 - \frac{\hat{r}\cdot\vec{d}}{r} + \frac{d^2}{4r^2}\right)^{-1/2} - \left(1 + \frac{\hat{r}\cdot\vec{d}}{r} + \frac{d^2}{4r^2}\right)^{-1/2}\right]$$

This result is exact, but we will use a simpler expression based on the Taylor expansion of V getting advantage of the property: $(1+\epsilon)^{-1/2} \approx 1 - \frac{\epsilon}{2}$

1.4 The Point dipole

As we have said in the previous chapter, just substituting the Taylor expansion in the expressions above we can easily get the answer:

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}\frac{\hat{r}\cdot\vec{d}}{r}$$

Since $\vec{p} = q\vec{d}$, we can rewrite the expression in terms of the dipole momentum.

$$V(\vec{r}) \approx \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$$

In particular, if $\vec{p}$ is parallel to the z axis:

$$V_{pointdipole}(\vec{r}) = \frac{p\vec{z}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{pcos\theta}{4\pi\varepsilon_0 r^2}$$

And by taking the gradient definition $\vec{E} = -\vec{\nabla}V$, the expression of the electric field:

$$\vec{E}(\vec{r}) = -\left[\frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0}\vec{\nabla}(\frac{1}{r^3}) + \frac{1}{4\pi\varepsilon_0 r^3}\vec{\nabla}(\vec{p}\cdot\vec{r})\right] = \frac{3\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0}\frac{\vec{r}}{r^5} - \frac{\vec{p}}{4\pi\varepsilon_0 r^3}$$

The same way, if $\vec{p}$ is parallel to z axis ($\hat{z} = cos\theta\hat{r} - sin\theta\hat{\theta}$) we get:

$$\vec{E}_{pointdipole} = \frac{p}{4\pi\varepsilon_0 r^3}(2cos\theta\hat{r} - sin\theta\hat{\theta})$$

1.5 Real dipoles

1. **Permanent dipoles:** polar molecules in which the center of positive charges does not coincide with that of the negative charges $(CO, H_2O, Na^+Cl^-, \dots)$

1.5.1 Polar atom in an external field

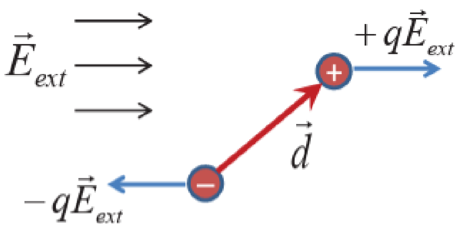


Figure 1: The permanent dipole moment of the atom/molecule orients parallel to the applied field

If the external field is **uniform**, thus:

$$\vec{F}_{total} = +q\vec{E}_{ext} - q\vec{E}_{ext} = 0$$

$$\vec{\Gamma} = q\vec{d}\times\vec{E}_{ext} = \vec{p}\times\vec{E}_{ext}$$

And its related energy expression is:

$$U^{\text{rigid}}_{\text{permanent dipole}} = +qV_{\text{ext}}(\vec{r}+\vec{d}) - qV_{\text{ext}}(\vec{r}) \approx +q\left[V_{\text{ext}}(\vec{r}) + \vec{\nabla}V_{\text{ext}}(\vec{r})\cdot\vec{d}\right] - qV_{\text{ext}}(\vec{r})$$

$$\approx q\vec{d}\cdot\vec{\nabla}V_{\text{ext}}(\vec{r}) = -\vec{p}\cdot\vec{E}_{\text{ext}}$$

$$\Longrightarrow \boxed{U_{\text{permanent point dipole}} = -\vec{p}\cdot\vec{E}_{\text{ext}}}$$

when $\vec{E}_{ext}$ is non-uniform, then the force is non-zero and we can calculate it in the way:

$$\vec{F}(\vec{r}) = -\vec{\nabla}U = \vec{\nabla}\left[\vec{p}\cdot\vec{E}(\vec{r})\right]$$

2. **Induced dipoles:** apolar atoms and molecules under and applied field become "polarized", that is, the center of positive charge (nucleus for an atom) moves away from the field and the center of negative charge (center of the electron cloud for an atom) moves towards the field: $\vec{p} = \alpha\vec{E}$, where α is the so-called **polarizability**.

1.5.2 Apolar atom in an external field

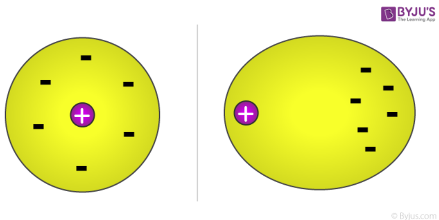


Figure 2: The electronic cloud is displaced from nucleus a total distance δ (induced polarization)

The net force on the nucleus is the sum of the force due to the external field and the force due to the electron cloud: at equilibrium $E_{cloud} = E_{ext}$ Assuming that the cloud is not deformed by the applied field, the nucleus is at a distance δ from the center of the cloud and the electronic force is calculated with Gauss' law:

$$-\vec{E}_{cloud}(s)4\pi\delta^2 = \frac{Q(\delta)}{\varepsilon_0} = \frac{\rho_{cloud}(4/3)\pi\delta^3}{\varepsilon_0} \Rightarrow \vec{E}_{cloud}(\vec{\delta}) = -\frac{\rho_{cloud}\vec{\delta}}{3\varepsilon_0} \equiv \vec{E}_{ext}$$

$$\Rightarrow \vec{\delta} = -\frac{3\varepsilon_0}{\rho}\vec{E}_0\frac{\mathcal{V}}{\mathcal{V}} = -\frac{3\varepsilon_0\mathcal{V}}{Ne}\vec{E}_0 \Rightarrow \vec{p} = \alpha\vec{E},$$

where $\vec{p} = -Ne\vec{\delta}$ and we find the polarizability value as $\alpha = 3\varepsilon_0 Vol$.
Note: N is the number of valence.

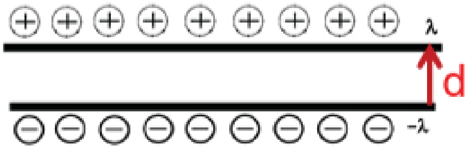
1.6 The line dipole

Line charge:

$$E(s)\cdot 2\pi s = \frac{\lambda h}{\varepsilon_0} \quad \Rightarrow \quad \vec{E}(s) = \frac{\lambda}{2\pi\varepsilon_0 s}\hat{s} \quad ; \quad V(s) = -\frac{\lambda}{2\pi\varepsilon_0}\ln s$$

(where s: radial coordinate)

Line dipole: We define:



$$\vec{\wp} = \lambda \vec{d} \quad \text{(where } \wp\text{: linear dipole density)}$$

$$[\wp] = \frac{C}{\mathrm{m}} = \mathrm{C} \quad (\text{since } \log(a^2) = 2\log(a))$$

The total potential is:

$$V_{\text{tot}}(\vec{s}) = V_{+\lambda}(\vec{s}_+) + V_{-\lambda}(\vec{s}_-) = -\frac{\lambda}{2\pi\varepsilon_0} \ln s_+ - \frac{\lambda}{2\pi\varepsilon_0} \ln s_- = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{s_-}{s_+}\right)$$

With:

$$\frac{\vec{d}}{2} + \vec{s}_+ = \vec{s} \quad \text{and} \quad \frac{\vec{d}}{2} + \vec{s} = \vec{s}_-,$$

we get:

$$V_{\text{tot}}(\vec{R}) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{s^2 + 2\vec{s}\cdot\frac{\vec{d}}{2} + \frac{d^2}{4}}{s^2 - 2\vec{s}\cdot\frac{\vec{d}}{2} + \frac{d^2}{4}}\right) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{1 + \frac{\hat{s}\cdot\vec{d}}{s} + \frac{d^2}{4s^2}}{1 - \frac{\hat{s}\cdot\vec{d}}{s} + \frac{d^2}{4s^2}}\right)$$

Using the Taylor expansion $\ln(1+\varepsilon) \approx \varepsilon$, we get:

$$V_{\text{tot}} = \frac{\lambda}{4\pi\varepsilon_0} \left[\ln(1+\varepsilon) - \ln(1+\varepsilon')\right] \approx \frac{\lambda}{4\pi\varepsilon_0}(\varepsilon - \varepsilon').$$

$$\Rightarrow V_{\text{tot}} = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{\hat{s}\cdot\vec{d}}{s} + \frac{d^2}{4s^2} - \left[-\frac{\hat{s}\cdot\vec{d}}{s} + \frac{d^2}{4s^2}\right]\right] = \frac{\lambda\hat{s}\cdot\vec{d}}{2\pi\varepsilon_0 s} = \frac{\lambda\vec{s}\cdot\vec{d}}{2\pi\varepsilon_0 s^2}.$$

$$\Longrightarrow V_{\text{line dipole}} = \frac{\vec{\wp}\cdot\vec{s}}{2\pi\varepsilon_0 s^2} = \frac{\vec{\wp}\cdot\hat{s}}{2\pi\varepsilon_0 s}$$

If $\vec{\wp} = \wp\hat{x}$, then:

$$V_{\text{line dipole}} = \frac{\vec{\wp}\cdot\hat{s}}{2\pi\varepsilon_0 s} = \frac{\wp\cos\theta}{2\pi\varepsilon_0 s}.$$

The last one is the expression of the line dipole potential in cylindrical coordinates.

1.7 Sphere dipole (plasma model of a point dipole)

$$\vec{E}(r)4\pi r^2 = \frac{\rho 4/3\pi r^3}{\varepsilon_0} \Rightarrow \vec{E}(r) = \frac{\rho\vec{r}}{3\varepsilon_0}$$

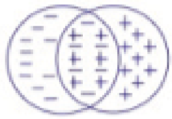


Figure 3: Two plasma spherical dipoles each of opposite charge are placed in contact each other.

$$\textbf{Field inside plasma: } \vec{E}_{total} = \vec{E}_{+} + \vec{E}_{-} = \frac{\rho\vec{r}_{+}}{3\varepsilon_0} + (\frac{-\rho\vec{r}_{-}}{3\varepsilon_0}) = \frac{\rho}{3\varepsilon_0}(\vec{r}_{+} - \vec{r}_{-}) = -\frac{\rho\vec{d}}{3\varepsilon_0}$$

Outsides, we use the same field created by a point dipole using superposition.

So, see the example of applying an external field to a simple metal sphere (just the combination of two plasmas of different sign). Then:

$$\vec{E}_{total} = \vec{E}_{ext} + \vec{E}_{response} = 0 \implies \vec{E}_{response} = -\vec{E}_{ext} = -\vec{E}_0 = -\frac{\rho\vec{d}}{3\varepsilon_0}$$

$$\implies \rho\vec{d} = 3\varepsilon_0\vec{E}_{ext} \implies \vec{p} = \rho\text{Vol}\vec{d} = 3\varepsilon_0\text{Vol}\vec{E}_{ext}$$

1.8 Line dipole (plasma model)

$$\vec{E}(s)2\pi sh = \frac{\rho\pi s^2 h}{\varepsilon_0} \Rightarrow \vec{E}(s) = \frac{\rho\vec{s}}{2\varepsilon_0}$$

$$\textbf{Field inside plasma: } \vec{E}_{total} = \vec{E}_{+}(\vec{s}_{+}) + \vec{E}_{(}\vec{s}_{-}) - = \frac{\rho\vec{s}_{+}}{2\varepsilon_0} + (\frac{-\rho\vec{s}_{-}}{2\varepsilon_0}) = \frac{\rho}{3\varepsilon_0}(\vec{s}_{+} - \vec{s}_{-}) = -\frac{\rho\vec{d}}{2\varepsilon_0}$$

where we have used $\vec{d} + \vec{s}_{+} = \vec{s}_{-}$.

Outsides, we use the same field created by a point dipole using superposition.

So, see the example of applying an external field to a simple metal wire (just the combination of two plasmas of different sign). Then:

$$\vec{E}_{total} = \vec{E}_{ext} + \vec{E}_{response} = \vec{E}_{ext} - \frac{\rho\vec{d}}{2\varepsilon_0} = 0 \implies \vec{E}_{ext} = \frac{\rho\vec{d}}{2\varepsilon_0} = \frac{\lambda\vec{d}}{2\varepsilon_0\text{Area}} = \frac{\vec{\wp}}{2\varepsilon_0\text{Area}}$$

$$\vec{\wp} = 2\varepsilon_0\text{Area}\vec{E}_{ext}$$

1.9 Uniqueness Theorem

If we have a set of conductors and for each of them we specify the potential V or the free charge density $\sigma_f = \varepsilon_0 E_n = -\varepsilon_0 \vec{\nabla} V \cdot \hat{n}$, then the solution to Laplace's equation $\nabla^2 V = 0$ in the region between the conductors is unique.

1.9.1 Application 1: Capacitance of a metal

Consider a metal far from anything else, charged to a voltage v_0. Outside the metal, $V(r)$ is a solution of Laplace's law:

$$\nabla^2 V = 0$$

with boundary conditions:

$$\begin{cases} V_{\text{metal surface}} = v_0 \\ V_\infty = 0 \end{cases}$$

Now change the voltage to v_1. The new potential $V'(r)$ must satisfy:

$$\nabla^2 V' = 0$$

with:

$$\begin{cases} V'_{\text{metal surface}} = v_1 \\ V'_\infty = 0 \end{cases}$$

Suppose we know the solution $V(\vec{r})$ to the first problem; for the second problem, we guess:

$$V'(\vec{r}) = \frac{v_1}{v_0} V(\vec{r})$$

This function has the right behavior $\Rightarrow$ unicity tells us it is **the** solution.
The new field is:

$$\vec{E}'(\vec{r}) = \frac{v_1}{v_0} \vec{E}(\vec{r})$$

and thus:

$$\sigma'_f(\vec{r}) = \frac{v_1}{v_0} \sigma_f(\vec{r}) \Rightarrow \quad Q'_f = \frac{v_1}{v_0} Q_f$$

Hence:

$$\boxed{\frac{Q'_f}{v_1} = \frac{Q_f}{v_0} = C = \text{const}}$$

Energy stored on a metal:

$$U = \frac{1}{2} V \int \sigma_f(\vec{r}) da = \frac{1}{2} V Q_f = \frac{1}{2} C V^2$$

The particular case for two pieces of metal leads to the same formula such as:

$$C = \frac{|Q_f|}{\Delta V} = \frac{Q_f}{V_A - V_B}, \text{ where } \Delta V > 0.$$

1.9.2 Application 2: Image charge Method

Summary of the Resolution Strategy

1) Find the charge distribution inside the metal

- The metal must maintain $V = 0$ on its surface.
- To achieve this, introduce *image charges* inside the metal.
- Image charges are fictitious but ensure that the total potential satisfies Laplace's equation $\nabla^2 V_{\text{TOT}} = 0$ everywhere except at the external charge locations.
- By the uniqueness theorem, the potential V_{TOT} generated by the external and image charges is the true solution outside the metal.

2) Calculate the total electric field $\vec{E}_{\text{TOT}}(\vec{r})$ outside the metal

- The total field is the sum of the fields produced by both external and image charges.
- Inside the metal, $\vec{E} = 0$ due to electrostatic shielding.

3) Determine the induced surface charge density σ_f

- The induced charge density on the metal surface is calculated using:

$$\sigma_f = \varepsilon_0 \vec{E}_{\text{TOT}}(\text{surface}) \cdot \hat{n}$$

- This follows from Gauss's law and the boundary conditions for conductors in electrostatics.

See some examples of configurations getting advantage of image charge method: For the

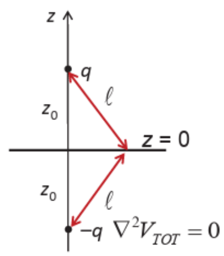


Figure 4: Point charge applying image method (imaginary charge inside the metal)

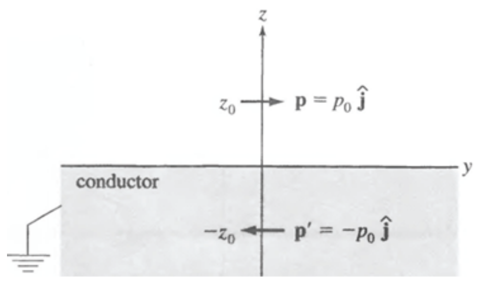


Figure 5: Point dipole applying image method (imaginary point dipole inside the metal)

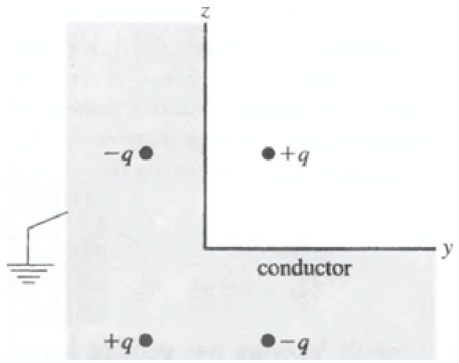


Figure 6: Point charge in L-shaped geometry applying image method (imaginary charges inside the metal)

case of a **metal sphere** in applied external field, something similar could happen, which leads us to apply image method. See it below:

1.9.3 Neutral metal sphere in applied external field

Potential

The external potential is just $V_{ext} = -E_0 \cdot z$ and if we take **the origin of potential in the center of the sphere** we can easily see how $z = R_0 \cos\theta$. So it gives us: $V_{ext} = -E_0 \cdot R_0 \cos\theta$ which is clearly non-uniform.
So that, ensuring the potential is constant and zero along the surface we can get the expression:

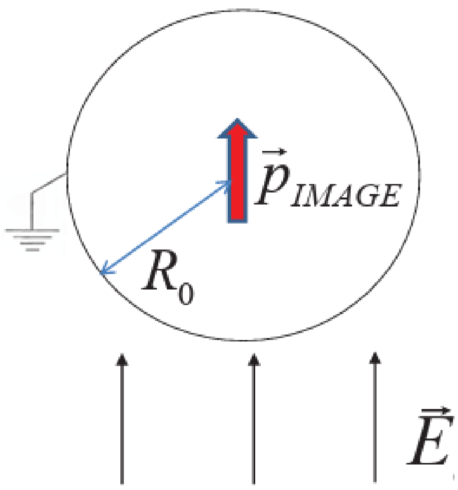
$$V_{TOT} = V_{ext} + V_{\text{IMAGE CHARGE}}$$

$$\implies V_{\text{TOT}}(r = R_0) = -E_0 R_0 \cos\theta + \frac{p\cos\theta}{4\pi\varepsilon_0 R_0^2}$$

Which leads to:

$$p_{\text{IMAGE}} = 4\pi\varepsilon_0 R_0^3 E_0 \,(= 3\varepsilon_0 Vol_{\text{sphere}} E_0)$$

which is just the same result of the plasma model of a point dipole.



Density

Thus, we need to calculate the induced charge density which is found by the boundary conditions:

$$\vec{E}_{TOT} \cdot \hat{n} = \frac{\sigma_f}{\varepsilon_0}$$

Since the total field is the sum of the external and the image dipole field we get:

$$\sigma_f = \varepsilon_0 E_0 \cos\theta + 2\varepsilon_0 E_0 \cos\theta = 3\varepsilon_0 E_0 \cos\theta$$

$$\sigma_f(r = R_0) = \sigma_0 \cos\theta, \text{ with } \sigma_0 = 3\varepsilon_0 E_0$$

Field

And the field has to be uniform and zero inside the metal:

$$\vec{E}_{TOT} = \vec{E}_0 + \vec{E}_{IMAGE} = 0 \implies \vec{E}_{IMAGE} = -\vec{E}_0 = -\frac{\sigma_0}{3\varepsilon_0}\hat{E}_0$$

And outsides, the field made by the point-dipole:

$$\vec{E}_{\vec{p}} = \frac{p}{4\pi\varepsilon_0 r^3}(2cos\theta\hat{r} - sin\theta\hat{\theta})$$

with $p = Vol_{sphere}\sigma_0$

1.9.4 Metal cylinder (wire) in applied external field

Potential

The same way we have done in the previous exercise, the total potential would be:

$$V_{TOT}(s = R_0) = -E_0 R_0 \cos\theta + \frac{\wp \cos\theta}{2\pi\varepsilon_0 R_0}$$

which cancels out if

$$\wp_{IMAGE} = 2\pi\varepsilon_0 R_0^2 E_0 = 2\varepsilon_0 \text{Area} E_0$$

Density

With the boundary conditions we can determine the charge density per unit area:

$$\sigma_f = \varepsilon_0(\vec{E}_0 \cdot \hat{s} + \vec{E}_{\vec{\wp}}(s = R_0) \cdot \hat{s}) = 2\varepsilon_0 E_0 \cos\theta$$

Field

Like in the previous case, the total field:

$$\vec{E}_{IMAGE} = -\vec{E}_0 = \frac{\sigma_0}{2\varepsilon_0}$$

And outsides, the field made by a line-dipole.

1.9.5 Point charge outside a metal sphere

The set-up for this kind of problems is a grounded metal sphere centered at the origin and a point charge along its z axis (see the figure below).

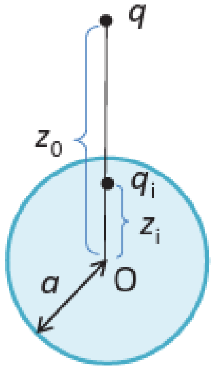


Figure 7: Point charge on the z axis at a distance z_0 induces a charge on the top-side of the sphere

We see how the image charge that makes the potential in the surface be zero has to be placed at a distance z_i from the center of the sphere. We will determine this value by:

$$V_{center} = \frac{1}{4\pi\varepsilon_0}\frac{q}{z_0} + \frac{1}{4\pi\varepsilon_0}\frac{Q_{induced}}{a} = 0 \implies Q_{induced} = -q\frac{a}{z_0}$$

This value means only the $Q_{induced}$ at the surface of the sphere.
Thus, with the image above, we can easily determine that the flux produced by q_i has to be the same that the ones produced by the $Q_{induced}$, so this leads to:

$$\Phi(\vec{E}) = \frac{Q_{induced}}{\varepsilon_0} = \frac{q_i}{\varepsilon_0} \to -\frac{q}{\varepsilon_0}\frac{a}{z_0}$$

$$\implies q_i = -q\frac{a}{z_0}$$

To find the position of q_i just define the set of vectors you see on the picture below and by using some trigonometric aspects you can find:

image.png

Figure 8: Vectorization of the set-up

$$V_{TOT} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R_+} + \frac{1}{4\pi\varepsilon_0}\frac{q_i}{R_-} = 0 \implies \frac{R_-}{R_+} = -\frac{q_i}{q} = \frac{a}{z_0}$$

This identity determines both triangles are **similar** by Tales, so we can define also the ratio:

$$\frac{a}{z_0} = \frac{z_i}{a} \implies z_i = \frac{a^2}{z_0}$$

1.10 Summary of E-field of Special Charge Distributions

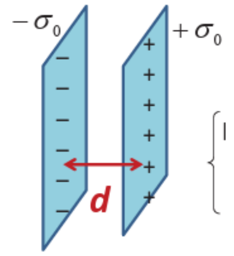
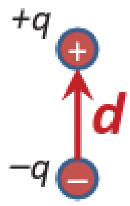
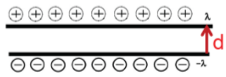
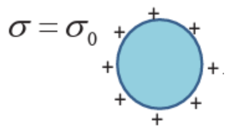
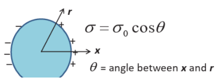
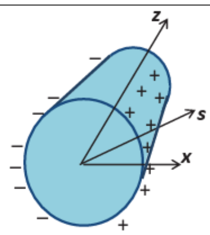
Charge Distribution	**Surface Charge Density**	**Electric Field**	**Diagram**
Two parallel planes (Uniform and opposite charge density, separated by $d \ll$ Area)	$\pm\sigma_0$	Between planes: $\vec{E} = \frac{\sigma_0}{\varepsilon_0}\hat{x}$ Outside: $\vec{E} = 0$	
Point dipole (Charge $+q$ and $-q$ separated by d, dipole moment $\vec{p} = q\vec{d}$)	$-q, +q$	$\vec{E}_{\text{point dipole}} = \frac{3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}}{4\pi\varepsilon_0 r^3}$	
Line dipole (Charge line density $\wp$, separated by d)	$\wp = \lambda d$	$\vec{E}_{\text{line dipole}} = \frac{2(\vec{\wp}\cdot\hat{s})\hat{s}-\vec{\wp}}{2\pi\varepsilon_0 s^2}$	
Spherical charge density (Uniform charge distribution on a sphere)	$\sigma = \sigma_0$	Inside: $\vec{E} = 0$, $V =$ constant $= V_{\text{center}}$ Outside: Point-charge field	
Spherical charge density (Varying with θ)	$\sigma = \sigma_0 \cos\theta$	Inside: $\vec{E} = -\frac{\sigma_0}{3\varepsilon_0}\hat{x}$ Outside: Point-dipole field	
Cylindrical charge density (Varying with θ)	$\sigma = \sigma_0 \cos\theta$	Inside: $\vec{E} = -\frac{\sigma_0}{2\varepsilon_0}\hat{x}$ Outside: Line-dipole field	

Table 1: Electric fields of various charge distributions with their respective diagrams

Summary of Section 1

Problems of electrostatics with conductors

Electric field and potential

From the relation of the potential energy of a point charge $U = qV$ we derive:

$$\vec{F} = -\nabla U = -q\nabla V = q\vec{E} \implies \vec{E} = -\nabla V (\implies \nabla \times \vec{E} = 0)$$

with

$$V = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r})\frac{d\tau'}{R}$$

$$\implies V_B - V_A = -\int_a^b \vec{E}\cdot d\vec{l}$$

Then by using $\nabla(\frac{1}{R}) = -\frac{\vec{R}}{R^3}$ we can write ($\vec{R} = \vec{r} - \vec{r}_{source}$):

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int_V \rho(\vec{r})\frac{\vec{R}}{R^3}d\tau'$$

Gauss' Law

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0} \to \oint_S \mathbf{E}\cdot\hat{\mathbf{n}}\,da = \frac{Q}{\varepsilon_0}$$

Point Charge	**Sphere (Uniform Q)**	**Plane (Uniform σ)**	**Line Charge λ**
$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$	$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0}\frac{\hat{r}}{r^2}$	$\mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$	$\mathbf{E} = \frac{\lambda}{2\pi\varepsilon_0}\frac{\hat{s}}{s}$
$V = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}$	$V = \frac{Q}{4\pi\varepsilon_0}\frac{1}{r}$		$V = -\frac{\lambda}{2\pi\varepsilon_0}\ln s$

Superposition

$$\rho = \rho_1 + \rho_2 \implies \begin{cases}\vec{E} = \vec{E}_1 + \vec{E}_2\\ V = V_1 + V_2\end{cases}$$

Conductors

$\vec{E}_{inside} = 0$ in electrostatic equilibrium

$$\implies \begin{cases}\rho = 0 \text{ Gauss' law}\\ V = \text{constant inside} = V|_S\end{cases}$$

Boundary conditions

- $\nabla\times\vec{E} = 0$

$$\implies \hat{n}_{out}\times\vec{E}_{outside}\Big|_S + \hat{n}_{in}\times\vec{E}_{inside}\Big|_S = 0 = \hat{n}_{out}\times\vec{E}_{outside}\Big|_S \implies E_{out_\perp} = 0$$

- $\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

$$\implies \hat{n}_{out}\cdot\vec{E}_{outside}\Big|_S + \hat{n}_{in}\cdot\vec{E}_{inside}\Big|_S = \frac{\sigma}{\varepsilon_0} = \hat{n}_{out}\cdot\vec{E}_{outside}\Big|_S \implies E_{out_\parallel} = \frac{\sigma}{\varepsilon_0}$$

Where σ is the surface charge density and is determined by: $\sigma = \varepsilon_0\,\hat{n}\cdot\vec{E}\Big|_S$

Energy and capacitance

Since $U = qV$ for a point charge, then for a system of charges (we use $(\nabla \cdot \vec{E})V = \vec{E}(\nabla V)$):

$$U = \frac{1}{2}\int_V \rho V \, d\tau = \frac{1}{2}\int_V \varepsilon_0 |\vec{E}|^2 d\tau$$

We define the capacitance as:

$$C = \frac{Q}{V} \implies U = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q^2}{C}$$

Electric infinitessimal dipole

Dipolar moment is defined as $\vec{p} = q\vec{d}$. Then, for huge distances:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

In external electric field $\vec{E}$:

$$\begin{cases} \vec{F} = (\vec{p}\cdot\nabla)\vec{E}_{\text{ext}} \\ \vec{\Gamma} = \vec{p}\times\vec{E}_{\text{ext}} \\ U_e = -\vec{p}\cdot\vec{E}_{\text{ext}} \end{cases}$$

Line dipole

$$\vec{p} = \lambda\vec{d}$$

Then for huge distances:

$$\begin{cases} V = \frac{1}{2\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{s}}{s} \\ \vec{E} = \frac{1}{2\pi\varepsilon_0 s^2}\left[2(\vec{p}\cdot\hat{s})\hat{s} - \vec{p}\right] \end{cases}$$

Sphere with $\sigma = \sigma_0\cos\theta$

$$\vec{E} = \begin{cases} -\frac{\sigma_0}{3\varepsilon_0}\hat{z}, & \text{for } r < a \quad \text{(inside)} \\ \\ \text{dipolar with } \vec{p} = \text{Vol}\,\sigma_0\,\hat{z} = \frac{4\pi}{3}a^3\sigma_0\,\hat{z}, & \text{for } r > a \quad \text{(outside)} \end{cases}$$

Infinite cylinder with $\sigma = \sigma_0\cos\theta$

$$\vec{E} = \begin{cases} -\frac{\sigma_0}{2\varepsilon_0}\hat{z}, & \text{for } r \leq a \quad \text{(inside)} \\ \\ \text{line-dipolar with } \vec{\mathcal{P}} = \text{Area}\,\sigma_0\,\hat{z} = \pi a^2\sigma_0\,\hat{z}, & \text{for } r > a \quad \text{(outside)} \end{cases}$$

Problems on image theory

Uniqueness Theorem

The field in a volume V is uniquely determined by the sources in V (ρ or q) and either $V|_S$ or $\vec{E}\cdot\hat{n}\Big|_S$ at $S = \partial V$

Image Theory

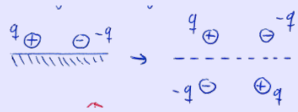


Figure 9: point images

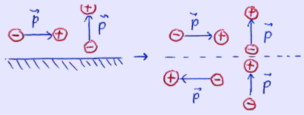


Figure 10: dipole images

In general terms, the formula to determine the position and the value of a charge of the image charge inside an sphere is:

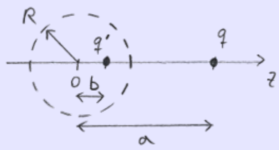


Figure 11: sphere images

with:

$$\begin{cases} q' = -q\frac{R}{a} \\ b = \frac{R^2}{a} \end{cases}$$

The sphericall shell of charge $\sigma = \sigma_0 \cos\theta$

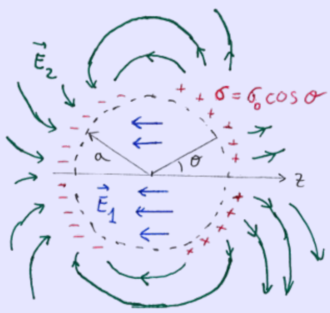


Figure 12: sphericall shell

This shell produces different field inside or outside:

$$\begin{cases} \vec{E}_{inside} = -E_0\hat{z} = -\frac{\sigma_0}{3\varepsilon_0}(\cos\theta\hat{r} - \sin\theta\hat{\theta}), \text{ uniform} \\ \vec{E}_{outisde} = \frac{3(\vec{p}\cdot\hat{r})\hat{r}-\vec{p}}{4\pi\varepsilon_0 r^3} = \frac{\sigma_0}{3\varepsilon_0}\frac{a^3}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta}), \text{ dipolar} \end{cases}$$

Notice that both fields fulfill $\nabla \times \vec{E} = 0$ and $\nabla \cdot \vec{E} = 0$ since there is no source.
But we have a discontinuity at the surface ($r = a$) where the field also fulfills the boundary conditions: $\hat{n}_{out} \cdot \vec{E}_{out} + \hat{n}_{in} \cdot \vec{E}_{in} = \frac{\sigma}{\varepsilon_0}$
So, by uniqueness: these field have to be created due to σ.

Potential:

Since $\vec{E} = -\nabla \cdot V$

$$\begin{cases} V_{inside} = E_0 z = \frac{\sigma_0}{3\varepsilon_0} r\cos\theta \implies V_{inside}|_{r=a} \frac{\sigma_0}{3\varepsilon_0} a\cos\theta \\ V_{outside} = \text{dipole potential} = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{a^3\sigma_0}{3\varepsilon_0 r^2}\cos\theta \implies V_{inside}|_{r=a} \frac{\sigma_0}{3\varepsilon_0} a\cos\theta \end{cases}$$

Uniqueness Theorem

Notice that both different field produces the same solution of the potential for the laplacian equation, which could seem it is violating Uniqueness theorem. But it is not like this since, when we state Uniqueness, we need the sources be inside a volume -V- where we calculate the E-field. In this case, for the region inside, the sources are in the surface of the sphere (outside the inner volume). The region outside has its source inside the sphere (dipole at the origin)

2 Dielectrics and polarized media

A dielectric may be represented by a collection of point dipoles.
We define, so, the **POLARIZATION**:

$$\vec{P} = \text{dipole moment per unit volume} = \frac{1}{\delta\tau}\sum_{i=1}^{\delta N}\vec{p}_i$$

Thus, it will be important to know the concept of **Boundary charges**

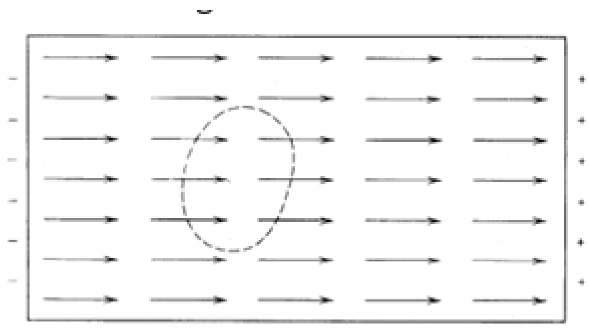


Figure 13: in both left and right side, there is a negative and positive, respectively, bound charge

2.1 Fundamental Theorem of polarized media

The most-commonly used definition for the polarization field is:

$$\vec{P}(\vec{r}) = \frac{d\vec{p}(\vec{r})}{d\tau} \rightarrow d\vec{p} = \vec{P}d\tau$$

For the electrostatic potential we use the point dipole adapted formula. In other words, while the electrostatic potential for a point dipole is

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$

Then for a continuous collection of dipoles:

$$V(\vec{r'}) = \frac{1}{4\pi\varepsilon_0}\int_{Vol} d\tau \frac{\vec{P}(\vec{r})\cdot(\vec{r'}-\vec{r})}{\left|\vec{r'}-\vec{r}\right|^3} = \int_{Vol} d\tau \vec{P}(\vec{r})\cdot\vec{\nabla}\left(\frac{1}{\left|\vec{r'}-\vec{r}\right|}\right)$$

which integrating by parts $\vec{\nabla}(f\vec{G}) = f\cdot\vec{\nabla}G + \vec{G}\cdot\vec{\nabla}f$ and then using the divergence theorem:

$$V(\vec{r'}) = \frac{1}{4\pi\varepsilon_0}\int_{\text{volume}} d\tau\left[\vec{\nabla}\cdot\left(\frac{\vec{P}}{|\vec{r'}-\vec{r}|^3}\right) - \frac{1}{|\vec{r'}-\vec{r}|}(\vec{\nabla}\cdot\vec{P})\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\oint_{\text{surface}}\frac{\vec{P}}{|\vec{r'}-\vec{r}|^3}\cdot\hat{n}da - \frac{1}{4\pi\varepsilon_0}\int_{\text{volume}} d\tau\frac{(\vec{\nabla}\cdot\vec{P})}{|\vec{r'}-\vec{r}|}.$$

Using the **bound charge densities** definitions ($\sigma_b = \vec{P}\cdot\hat{n}$ and $\rho_b = -\vec{\nabla}\cdot\vec{P}$):

$$V(\vec{r'}) = \frac{1}{4\pi\varepsilon_0}\oint_{\text{Area}}\frac{\sigma_b}{|\vec{r'}-\vec{r}|}da + \frac{1}{4\pi\varepsilon_0}\int_{\text{Vol}}\frac{\rho_b}{|\vec{r'}-\vec{r}|}d\tau$$

2.2 Uniformly Polarized Sphere

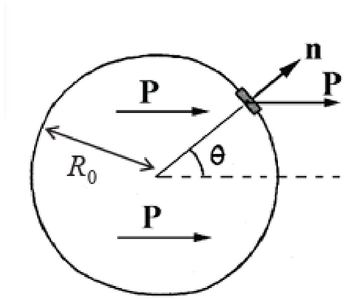


Figure 14: The electric field inside is uniform and the surface distribution is, actually, already calculated

Because of the uniformity inside of the electric field, and the dielectric media, we see the following distribution:

$$\begin{cases} \rho_b = -\vec{\nabla} \cdot \vec{P} = 0 \\ \sigma_b = \vec{P} \cdot \hat{n} = P\cos\theta \end{cases}$$

Then see that this uniform distribution insides generates an electric field opposite to the dipolar moment (which is called **depolarization field**):

$$\begin{cases} \vec{E}_{\text{inside}} = -\frac{\sigma_0}{3\varepsilon_0}\hat{x} = -\frac{|\vec{P}|\hat{x}}{3\varepsilon_0} = -\frac{\vec{P}}{3\varepsilon_0} \quad \text{with:} \quad \vec{p} = \vec{P} \cdot Vol_{\text{sphere}} = \frac{4}{3}\pi R_0^3 \vec{P} \\ \vec{E}_{\text{outside}} = \vec{E}_{\text{point dipole } \vec{p}} \end{cases}$$

where we have used:

$$\sigma = \sigma_0 \cos\theta = |\vec{P}| \cos\theta$$

2.3 Uniformly polarized cylinder

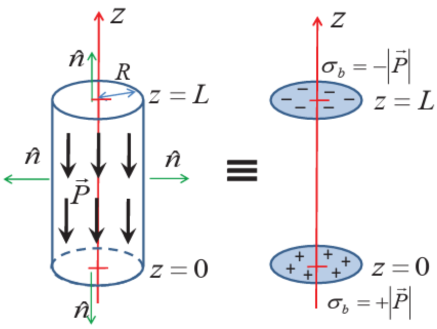


Figure 15: See the depolarized field phenomena due to the polarization

In this case is quite similar than the last one such as:

$$\begin{cases} \rho_b = -\vec{\nabla} \cdot \vec{P} = 0 \\ \sigma_b = \vec{P} \cdot \hat{n} = \begin{cases} 0 \text{ (side)} \\ \pm \left|\vec{P}\right| \text{ (bases)} \end{cases} \end{cases}$$

Potential

On the z axis, we have due to the influence of only one base:

$$V(z) = \int \frac{\sigma_0 da}{4\pi\varepsilon_0 l} = \frac{\sigma_0}{4\pi\varepsilon_0}\int_0^R 2\pi s ds(s^2+z^2)^{-1/2} = \frac{\sigma_0}{2\varepsilon_0}(\sqrt{R^2+z^2} - |z|)$$

Field

The field due to the bottom surface is easily calculated:

$$E_z(z) = -\vec{\nabla}\cdot V(z) = \frac{\sigma_0}{2\varepsilon_0}\left(sign(z) - \frac{2z}{2\sqrt{R^2+z^2}}\right)$$

By using a new coordinate centered at top base such as $z' = z - L$, then:

$$E_z^{tot} = \frac{+\sigma_b}{2\varepsilon_0}\left(sign(z) - \frac{z}{\sqrt{R^2+z^2}}\right) + \frac{-\sigma_b}{2\varepsilon_0}\left(sign(z-L) - \frac{z-L}{\sqrt{R^2+(z-L)^2}}\right)$$

So we obtain two important results:

$$\begin{cases} E_z^{inside} = \frac{\sigma_b}{2\varepsilon_0}\left[2 + \frac{z-L}{\sqrt{R^2+(z-L)^2}} - \frac{z}{\sqrt{R^2+z^2}}\right] \\ E_z^{outside} = \frac{\sigma_b}{2\varepsilon_0}\left[\frac{z-L}{\sqrt{R^2+(z-L)^2}} - \frac{z}{\sqrt{R^2+z^2}}\right] \end{cases}$$

2.4 Uniform vs Non-Uniform

For non-uniform cases we have:

$$\vec{P}(z) = (kz^2 + h)\hat{z}$$

which implies:

$$\rho_b = -\vec{\nabla}\cdot\vec{P}(\vec{r}) = -\frac{dP(z)}{dz} = -2kz \neq 0$$

And sorting the particular cases:

- **Lateral surface**

$$\sigma_b = \vec{P}\cdot\hat{n} = 0$$

- **Bottom base**

$$\sigma_b = \vec{P}(z=0)\cdot\hat{n} = -h$$

- **Top base**

$$\sigma_b = \vec{P}(z=L)\cdot\hat{n} = kL^2 + h$$

2.5 Linear Dielectrics

The effect of applying an external field in some dielectrics make huge differences depending on the geometry of it. We will see some examples, but first, is important to mention the relation with the polarization vector:

$$\vec{P} = \varepsilon_0\chi\vec{E}_{macro}$$

where χ is the susceptibility.

See the example of a thin disk or slab:

$$\vec{\nabla}\cdot\vec{P} = -\rho_b = 0$$

$$\vec{P} \cdot \hat{n} = \sigma_b = \pm |P|$$

In this case, the depolarizing field is:

$$E_{depolarizing} = \frac{\sigma}{\varepsilon_0} = \frac{|P|}{\varepsilon_0}$$

So applying the definition for dielectrics:

$$\vec{P} = \varepsilon_0 \chi \vec{E}_{TOT} = \varepsilon_0 \chi (\vec{E}_{ext} + \vec{E}_{depolarizing}) = \chi \varepsilon_0 \left(\vec{E}_{ext} - \frac{\vec{P}}{\varepsilon_0} \right)$$

$$\implies \vec{P} = \frac{\chi \varepsilon_0}{1 + \chi} \vec{E}_{ext} = \frac{(\varepsilon_r - 1)\varepsilon_0}{\varepsilon_r} \vec{E}_{ext}$$

2.6 Linear dielectric cylinder

If the external field is applied perpendicular to the z axis. Like in so many examples of this kind we have already done;

$$\sigma_b = \vec{P} \cdot \hat{n} = |P| \cos\theta$$

Therefore,

$$\vec{P} = \chi \varepsilon_0 (\vec{E}_{applied} + \vec{E}_{depolarizing}) = \chi \varepsilon_0 \left(\vec{E}_{applied} - \frac{\vec{P}}{2\varepsilon_0} \right)$$

$$\implies \vec{P} \left(1 + \frac{\chi}{2} \right) = \chi \varepsilon_0 \vec{E}_{applied} \implies \vec{P}_{induced} = \frac{\chi \varepsilon_0}{1 + \frac{\chi}{2}} \vec{E}_{applied}$$

2.7 Depolarizing factor

The depolarizing factor is defined as:

$$\gamma = \frac{E_{depolarizing} \sigma_b}{P/\varepsilon_0} = \varepsilon_0 \frac{|\vec{E_{in}}(\vec{P})|}{|\vec{P}|}$$

which we can see that depends only on the shape of the dielectric. So, we can get the following results:

	Slab	**Sphere**	**Needle** $E \perp$ axis	**Needle** $E \parallel$ axis	**Prolate Spheroid** $E \parallel$ symmetry axis	**Oblate Spheroid** $E \parallel$ symmetry axis
γ	1	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{r^2-1}\left(\frac{r}{\sqrt{r^2-1}} \ln(r + \sqrt{r^2-1}) - 1\right)$	$\frac{1}{1-r^2}\left(1 - \frac{r}{\sqrt{1-r^2}} \sin^{-1}(\sqrt{1-r^2})\right)$

Table 2: Values of γ for different geometries

2.8 Dielectrics without free charges

The depolarizing $\vec{E}$ field is just:

$$\vec{E}_{depolarizing} = -\gamma \frac{\vec{P}}{\varepsilon_0}, \text{ where } \gamma \text{ is described}$$

However, for a linear dielectric under the influence of an external field we have instead:

$$\vec{P} = \chi \varepsilon_0 \vec{E}_{tot} = \chi \varepsilon_0 \left(\vec{E}_{applied} + \vec{E}_{depolarizing} \right) = \chi \varepsilon_0 \left(\vec{E}_{applied} - \gamma \frac{\vec{P}}{\varepsilon_0} \right) = \chi \varepsilon_0 \vec{E}_{\text{applied}} - \chi \gamma \vec{P}, \quad \text{hence:}$$

$$\vec{P}_{induced} = \frac{\chi}{1 + \chi\gamma} \varepsilon_0 \vec{E}_{applied}$$

2.9 Force on a polarized dielectric

We could understand the force on the dielectric as the sum of the forces on each single dipole, which is just:

$$\vec{F}_{singledipole} = (\vec{p} \cdot \vec{\nabla})\vec{E}_o(\vec{r}) \implies \vec{F}_{dielectric} = \int d\tau (\vec{P}(\vec{r}) \cdot \vec{\nabla})\vec{E}_o(\vec{r})$$

since using integration at parts we get:

$$\vec{F}_{ondielectric} = \int d\tau \rho_b(\vec{r})\vec{E}_o(\vec{r}) + \int da \sigma_b(\vec{r})\vec{E}_o(\vec{r})$$

2.10 Free charges

An external field acting on a dielectric may be caused by a set of free charges in vacuum surrounding the dielectric. So, we define the so-called **Displacement field**: Given:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0} = \frac{\rho_f}{\varepsilon_0} + \frac{\rho_b}{\varepsilon_0}$$

now using $\rho_b = -\vec{\nabla} \cdot \vec{P}$, we can write all this as:

$$\vec{\nabla} \cdot (\varepsilon_0 \vec{E} + \vec{P}) = \rho_f \equiv \vec{\nabla} \cdot \vec{D} \ [\mathrm{C}/m^2]$$

then, we got the two fundamental identities:

$$\vec{\nabla} \cdot \vec{D} = \rho_f$$

$$\vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$$

2.11 D-field in linear dielectrics

In a linear dielectric the polarization is directly propportional to the macroscopic field inside:

$$\vec{P} = \chi_{el}\varepsilon_0\vec{E} \implies \vec{D} = (1+\chi)\varepsilon_0\vec{E} = \varepsilon_0\varepsilon_r\vec{E}$$

where ε_r is the relative permittivity

In a perfect metal:

$$\vec{P} = 0 \implies \vec{D} = \varepsilon_0\vec{E} \implies \varepsilon_r = 1$$

2.12 Boundary conditions and charge in dielectrics

Same as we have done with the E-field, we determine:

$$\vec{\nabla} \cdot \vec{D} = \rho_f \implies \vec{D}_1 \cdot \hat{n}_1 + \vec{D}_2 \cdot \hat{n}_2 = \sigma_f$$

Then for non-homogenius medium we can obtain:

$$\rho_f = \vec{\nabla} \cdot \vec{D} = \vec{\nabla} \cdot (\varepsilon_r\varepsilon_0\vec{E}) = \varepsilon_r\varepsilon_0\vec{\nabla} \cdot \vec{E} = \varepsilon_r\varepsilon_0\frac{\rho_{tot}}{\varepsilon_0} = \varepsilon_r\rho_{tot}$$

Then the capacitance:

$$C = \frac{Q_f}{V}$$

2.13 Electroestatic energy with linear dielectric

$$U = \frac{1}{2}\int_{volume} d\tau \rho_f V + \frac{1}{2}\int_{surface} dA\sigma_f V$$

where V is the total potential due to the bound and free charges.

$$u_{\text{el}} = \frac{1}{2}\rho_f V = \frac{1}{2}\vec{D}\cdot\vec{E} = \frac{1}{2}\varepsilon_r\varepsilon_0 E^2$$

$$\implies U_{el} = \frac{1}{2}Q_f V = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q_f^2}{C}$$

Then, we can determine:

$$\vec{F}_{el} = -\left.\frac{dU_{el}}{dx}\right|_{Q_f=cnst} = -\frac{1}{2}Q_f^2\frac{d}{dx}\left(\frac{1}{C}\right) = \frac{1}{2}\frac{Q_f^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$

2.14 Electrified dielectrics: dielectric screening

It is too easy to create free charges inside the dielectric (simply rubbing or bombarding with ions) that do not belong to any dipole moment. In essence, you can notice the relation between D and E fields by simply placing a positive free charge q_f inside a dielectric ε_r. Hence, by GL:

$$\Phi_D = D(r)4\pi r^2 = q_f \implies D(r) = \frac{q_f}{4\pi r^2}\hat{r}$$

Then, we could derive:

$$\vec{E} = \frac{\vec{D}}{\varepsilon_r\varepsilon_0} = \frac{q_f}{4\pi\varepsilon_0\varepsilon_r r^2}\hat{r} \iff \vec{P} = (\varepsilon_r - 1)\varepsilon_0\vec{E} = \frac{(\varepsilon_r - 1)}{\varepsilon_r}\vec{D} = \left(1 - \frac{1}{\varepsilon_r}\right)\frac{q_f}{4\pi r^2}\hat{r}$$

Notice the electric field is equal to the field generated by a charge $q_{TOT} = q_f/\varepsilon_r$.
Then, the polarization implies bound point charge q_b superposed to the free charges, see the picture:

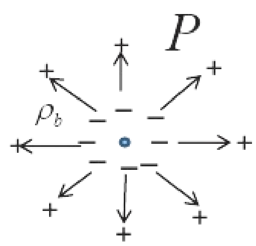


Figure 16: See how the polarization superposes the free charges

Then, we can determine the densities, since:

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\left(1 - \frac{1}{\varepsilon_r}\right)\frac{q_f}{4\pi}\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = -\left(1 - \frac{1}{\varepsilon_r}\right)\frac{q_f}{4\pi}4\pi\delta(\vec{r}) \implies q_b = -\left(1 - \frac{1}{\varepsilon_r}\right)q_f, \text{ placed at the origin}$$

And a similar procedure to the volume density, see the relation:

$$\rho_f = \vec{\nabla}\cdot\vec{D} = \vec{\nabla}\cdot(\varepsilon_0\varepsilon_r\vec{E}) = \varepsilon_0\varepsilon_r\vec{\nabla}\cdot\vec{E} = \varepsilon_0\varepsilon_r\frac{\rho_{TOT}}{\varepsilon_0} = \varepsilon_r\rho_{TOT}$$

2.15 Microscopic theory of ε_r for linear media

The field felt by an atom (E_{local})is just the sum of the external field (E_{ext}) and the induced one (E_{ind}). There is cases where the induced dipole moments are proportional to the this local field, then the microscopical polarization is n times each dipole moment. Then, we sort between this two cases:

- **Case 1: dilute apolar gas.**
  In a gas, the molecules are separated a large distance, which causes no interaction between them. Hence:
$$E_{local} = E_{ext}$$
  and
$$\vec{P} = n\vec{p} = n\alpha\vec{E} = \chi\varepsilon_0\vec{E} \implies n\alpha = \chi\varepsilon_0$$
$$\implies \alpha = \frac{\varepsilon_0}{n}(\varepsilon_r - 1) \iff \varepsilon_r = 1 + \frac{n\alpha}{\varepsilon_0}$$

- **Case 2: Dense apolar liquid/solid** In condensed matter E_{local} differs considerably from the applied external field and also from the macroscopic (average) field E_{macro} inside the dielectric. We use the following approximation: we take E_{local} to be equal to the macroscopic field inside the dielectric plus the field produced by the walls of a small spherical cavity carv the polarized dielectric, which is assumed to have a locally uniform polarization P. Hence:

$$\vec{P} = n\vec{p} = n\alpha\vec{E}_{local} = n\alpha\left(\vec{E}_{macro} + \frac{\vec{P}}{3\varepsilon_0}\right) \implies \vec{P}\left(1 - \frac{n\alpha}{3\varepsilon_0}\right) = n\alpha\vec{E} \Rightarrow \vec{P} = \left(\frac{n\alpha}{1 - n\alpha/3\varepsilon_0}\right)\vec{E}$$

$$\chi_{el} = \varepsilon_r - 1 = \frac{n\alpha}{\varepsilon_0 - n\alpha/3} \iff \alpha = \frac{\varepsilon_0}{n}\frac{\chi_{el}}{1 + \chi_{el}/3} = \frac{3\varepsilon_0}{n}\frac{\varepsilon_r - 1}{\varepsilon_r + 2}$$

Summary of Section 2

Problems with electrets and linear media

Polarization

Average dipole moment per unit volume: $\vec{P} = n\vec{p}$.
From here, we have two types of dielectrics:

- **Ferroelectrics:** those with a given numerical value of polarization.
- **Linear media:** those with polarization: $\vec{P} = \varepsilon_0 \chi_{el} \vec{E}$, with total macroscopic field .

`FUNDAMENTAL THEOREM:` The associated bound charges expressions are given by:

$$\begin{cases} \rho_b = -\nabla \cdot \vec{P} \\ \sigma_b = \hat{n} \cdot \vec{P}|_S \end{cases}$$

And produce the same field as the dipolar moments in the dielectric (E_{depol}). Hence:

$$\vec{E} = \vec{E}_{ext} + \vec{E}_{depol}$$

where:

$$\vec{E}_{depol} = -\gamma \frac{\vec{P}}{\varepsilon_0} \Longleftrightarrow \vec{P} = \frac{\chi}{1+\gamma\chi} \varepsilon_0 \vec{E}_{ext}$$

Displacement field

We define the displacement field as:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

In linear dielectrics:

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P} = \varepsilon_0(\chi_{el}+1)\vec{E} = \varepsilon_0 \varepsilon_r \vec{E}, \text{ with } \varepsilon_r = \chi_{el} + 1$$

We could easily see $\nabla \cdot \vec{D} = \rho_f$ where we have used $\varepsilon_0 \nabla \cdot \vec{E} = \rho = \rho_f + \rho_b$

Boundary conditions

$$\nabla \times \vec{E} = 0 \rightarrow \hat{n}_2 \times \vec{E}_2 + \hat{n}_1 \times \vec{E}_1 = 0 \rightarrow \hat{n}_2 \times (\vec{E}_2 - \vec{E}_1) = 0 \quad (E_{2\perp} = E_{1\perp})$$
$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \rightarrow \hat{n}_2 \cdot \vec{E}_2 + \hat{n}_1 \cdot \vec{E}_1 = \frac{\sigma}{\varepsilon_0} \rightarrow \hat{n}_2 \cdot (\vec{E}_2 - \vec{E}_1) = \frac{\sigma}{\varepsilon_0} \quad (E_{2\parallel} - E_{1\parallel} = \frac{\sigma}{\varepsilon_0})$$
$$\nabla \cdot \vec{D} = \rho_f \rightarrow \hat{n}_2 \cdot \vec{D}_2 + \hat{n}_1 \cdot \vec{D}_1 = \sigma_f \rightarrow \hat{n}_2 \cdot (\vec{D}_2 - \vec{D}_1) = \sigma_f \quad (D_{2\parallel} - D_{1\parallel} = \sigma_f)$$

Electrostatic energy and force

$$u_e = \frac{1}{2}\int_{V_{dielectric}} \rho_f V d\tau + \frac{1}{2}\int_{S_{conductor}} \sigma_f V da$$
$$u_e = \frac{1}{2}\int \vec{D}\cdot\vec{E} d\tau = \frac{1}{2}\int \varepsilon_0 \varepsilon_r |\vec{E}|^2 d\tau$$

where V is the potential due to both free and bound charges.
Capacitor:

$$u_e = \frac{1}{2} Q_f V = \frac{1}{2}\frac{Q_f^2}{C}$$

Force:

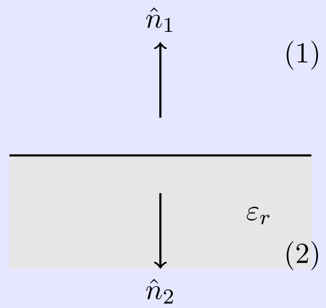
$$F_e = -\frac{d}{dx}U_e(\text{Energy conservation})$$

Plugging this two equations:

$$F_e = \frac{1}{2}\frac{Q_f^2}{C^2}\frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dx}$$

Capacitor filled with 1 dielectric

Connecting the capacitor to a battery makes the battery supply charge setting a different voltage to both plates. As we have seen in FIS2, an electric field is produced ($\vec{E}_{inside} = E_0\hat{z}$) so a depolarizing field is created inside the dielectric.



Then, once we know:

$$\nabla \cdot \vec{D} = \rho_f \implies \vec{D}_1 \cdot \hat{n}_1 + \vec{D}_2 \cdot \hat{n}_2 = (\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_2 = \sigma_f$$

Since we have the dielectric in surface (2) and nothing in surface (1), we can easily derive:

$$\vec{D} = -\sigma_f \hat{z} \implies \sigma_f = \varepsilon_0\varepsilon_r E_0$$

Then, by definition:

$$Q = \sigma_f \cdot S = \varepsilon_0\varepsilon_r E_0 S$$

and, since E is uniform inside:

$$E_0 = \frac{V}{d}$$

Capacitance

Then, we can compute the capacitance as:

$$C = \frac{Q}{V} = \varepsilon_0\varepsilon_r\frac{S}{d}$$

Polarization

$$\vec{D} = \varepsilon_0\varepsilon_r\vec{E} = \varepsilon_0\vec{E} + \vec{P} \implies \vec{P} = \varepsilon_0(\varepsilon_r - 1)\vec{E} = \frac{\varepsilon_r - 1}{\varepsilon_r}\vec{D} = -\frac{\varepsilon_r - 1}{\varepsilon_r}\sigma_f\hat{z}$$

Bound charge

As we know, we need firstly to compute the polarization in order to find bound parameters, like we have done in the previous section. Then:

$$\sigma_b = \vec{P} \cdot \hat{n}|_S = \begin{cases} \vec{P} \cdot (+\hat{z})|_{z=d} = -\frac{\varepsilon_r - 1}{\varepsilon_r}\sigma_f \\ \vec{P} \cdot (-\hat{z})|_{z=0} = +\frac{\varepsilon_r - 1}{\varepsilon_r}\sigma_f \end{cases}$$

Then, we distinguish between:

- **Empty capacitor:** $\sigma_b = 0 \Longleftrightarrow \sigma_f = \varepsilon_0 \vec{E} \cdot \hat{n}|_S = \varepsilon_0 \frac{V}{d}$
- **Filled capacitor**

$$\begin{cases} \sigma_f = \vec{D} \cdot \hat{n}|_S = \varepsilon_0 \varepsilon_r \frac{V}{d} \\ \sigma_b = \vec{P} \cdot \hat{n}|_S = -\varepsilon_0 (\varepsilon_r - 1) \frac{V}{d} \end{cases} \implies \varepsilon_0 \frac{V}{d} = \sigma_f + \sigma_b$$

Problems with 2 dielectrics capacitors

We will need to study the geometry of the problem (attending to the E-field direction) and distinguish if it is:

Parallel

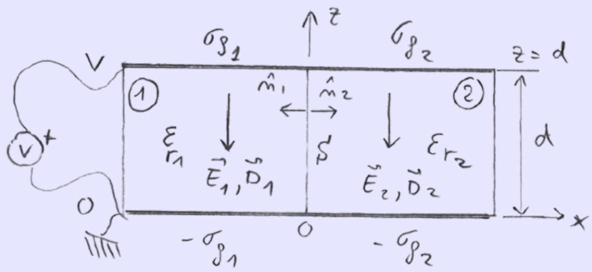


Figure 17: Parallel dielectrics inside a capacitor

Continuity

Since $\vec{E} \perp \hat{n}$, then the proper boundary condition selected will be $\nabla \times \vec{E} = 0$, which implies:

$$-E_1 \hat{z} \times \hat{n}_1 - E_2 \hat{z} \times \hat{n}_2 = 0 = E_1 - E_2 \implies E_1 = E_2$$

Distribution charge

Then total charge is $Q = \sigma_{f_1} + \sigma_{f_2}$ with this two distributions determined by computing $\sigma_f = \vec{D} \cdot \hat{n}|_z$.

Capacitance

So, we reach to the expression:

$$C = \frac{Q}{V} = \frac{S_1 \varepsilon_{r_1} \varepsilon_0}{d} + \frac{S_2 \varepsilon_{r_2} \varepsilon_0}{d} \equiv C_1 + C_2$$

Series

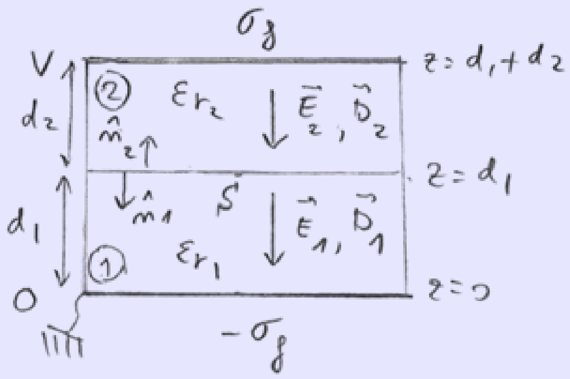


Figure 18: Series dielectrics inside a capacitor

Continuity

Since $\vec{D}$ is continuous across the boundary because $\nabla \cdot \vec{D} = 0$ then, this is the proper boundary condition which leads to the expression below:

$$\vec{D}_1 \cdot \hat{n}_1 + \vec{D}_2 \cdot \hat{n}_2 = 0 = (D_1\hat{z}) \cdot (-\hat{z}) + (D_2\hat{z}) \cdot (+\hat{z}) = D_2 - D_1 \implies D_1 = D_2$$

Distribution charge

Close to the top surface we will have a sigma distribution due to the free charges due to D-field:

$$\sigma_f = \vec{D} \cdot \hat{n} = (-D_0\hat{z}) \cdot (-\hat{z}) = D_0$$

E-field

with this result we can derive the E-field using the identity ($\vec{D} = \varepsilon\vec{E}$):

$$\begin{cases} \vec{E}_1 = \frac{\vec{D}_1}{\varepsilon_1} = -\frac{D_0\hat{z}}{\varepsilon_1} = -\frac{\sigma_f\hat{z}}{\varepsilon_1}, 0 < z < d_1 \\ \vec{E}_2 = \frac{\vec{D}_2}{\varepsilon_2} = -\frac{D_0\hat{z}}{\varepsilon_2} = -\frac{\sigma_f\hat{z}}{\varepsilon_2}, d_1 < z < d_2 \end{cases}$$

Potential

Then with this results we compute V:

$$V = -\int_0^{d_1+d_2} \vec{E} \cdot \vec{dl} = \int_0^{d_1+d_2} E_z dz = \frac{\sigma_f}{\varepsilon_1} d_1 + \frac{\sigma_f}{\varepsilon_2} d_2$$

Capacitance

So, we have everything to determine the capacity as:

$$C = \frac{Q}{V} = \frac{S\sigma_f}{\frac{\sigma_f}{\varepsilon_1}d_1 + \frac{\sigma_f}{\varepsilon_2}d_2} = \frac{S}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}} \equiv \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

Force and Energy

Introducing a dielectric inside a capacitor produces a force that tends to pull the dielectric into the capacitor. This effect is essentially produced by the curved E-field lines at the edges of the (finite) capacitor slabs which produces a force (due to the horizontal component of the E-field) to the free charges of the dielectric. In fact, the effect is also produced because the configuration of minimum energy is when the dielectric is completely inside the capacitor.

$$u_{el} = \frac{1}{2}\int_S \rho_f V da = \frac{1}{2}\int_V \vec{D}\cdot\vec{E} d\tau = \frac{1}{2}Q_f V$$

which can also be written using $C = \frac{Q}{V}$:

$$u_{el} = \frac{1}{2}Q_f V = \frac{1}{2}CV^2 = \frac{1}{2}\frac{Q_f^2}{C}$$

For the force we only need to use:

$$F_{el} = -\frac{du_{el}}{dx}$$

Then, when we connect a charge to a set potential, we ensure the charge is constant on the capacitor so, the expression:

$$\vec{F}_{el} = -\frac{d}{dx}(\frac{1}{2}\frac{Q_f^2}{C}) = \frac{1}{2}V^2\frac{dC}{dx}\hat{x}$$

3 Magnetic materials and magnets

Like we have done in electrostatics, we define a correlation between E-field and B-field, since:

$$\vec{F}_{el} = q\vec{E}_{ext} \Longleftrightarrow \vec{F}_{mag} = q_m\vec{B}_{ext}$$
$$\vec{p} = q\vec{d} \Longleftrightarrow \vec{m} = q_m\vec{d}$$
$$\vec{\Gamma} = \vec{p} \times \vec{E}_{ext} \Longleftrightarrow \vec{\Gamma} = \vec{m} \times \vec{B}_{ext}$$
$$U = -\vec{p} \cdot \vec{E}_{ext} \Longleftrightarrow U = -\vec{m} \cdot \vec{B}_{ext}$$

3.1 The magnetic dipole (spin) field

Just like the electric field produced by a point dipole, a **magnetic dipole (also called spin)** produces a B-field:

$$\vec{E}_{\text{point dipole}} = \frac{1}{4\pi\varepsilon_0}(3\frac{\vec{p}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{p}}{r^3}) \Longleftrightarrow \vec{B}_{dipole} = \frac{\mu_0}{4\pi}(3\frac{\vec{m}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{m}}{r^3})$$

This field is formally **conservative** so we can write:

$$\vec{B}_{dipole} = -\mu_0\vec{\nabla}\Xi_{dipole}, \text{ with } \Xi_{dipole} = \frac{\vec{m}\cdot\vec{r}}{4\pi r^3}$$

However, B-field is not conservative but solenoidal, but in this case, notice that the discontinuity in the origin (produced mainly by the non-existance of magnetic monopoles) makes the field not to be solenoidal (solenoidal $\equiv$ not closed loops).

3.2 Vector potential of the magnetic dipole

We want to find a vector potential (*'potencial vector'*) of the B-field, so we use the definition:

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

with

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3}$$

Then if we compute the cross product we find:

$$\vec{\nabla} \times \vec{A} = \mu_0\delta^3(\vec{r})\vec{m} - \mu_0\vec{\nabla}(\frac{\vec{m}\cdot\vec{r}}{4\pi r^3}) = \mu_0\vec{M} - \mu_0\vec{\nabla}\Xi$$

3.3 Magnetization media

Like we have done in the previous chapter, we define a Magnetization field:

$$\vec{M} = \frac{\sum_i^{\delta N}\vec{m}_i}{\delta\tau}$$

Then its respective densities like:

$$\begin{cases} \rho_m = -\vec{\nabla}\cdot\vec{M} \\ \sigma_m = \vec{M}\cdot\hat{n} \end{cases}$$

With the expression of $\vec{B} = \vec{\nabla} \times \vec{A}$ we have found the result:

$$\vec{B} = \mu_0\vec{M} - \mu_0\vec{\nabla}\Xi$$

where, integrating by parts we reach to the expression:

$$\Xi(\vec{r}) = \frac{1}{4\pi}\int d\tau \vec{M}(\vec{r})\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{1}{4\pi}\left(\int_S \frac{\sigma_m}{|\vec{r}-\vec{r}'|}da + \int_V \frac{\rho_m}{|\vec{r}-\vec{r}'|}d\tau\right)$$

So we will define the vectorial field:

$$\vec{H}(\vec{r}) = -\vec{\nabla}\Xi$$

which make the last expression llok like:

$$\vec{B} = \mu_0(\vec{M}+\vec{H})$$

3.4 Problems with magnetized media

Summary of the Resolution Strategy

1. From $\vec{M}$, find the densities:

$$\rho_m = -\vec{\nabla}\cdot\vec{M}$$

$$\sigma_m = \vec{M}\cdot\hat{n}$$

2. From the previous results, calculate $\vec{H}$ as you were calculating $\vec{E}$:

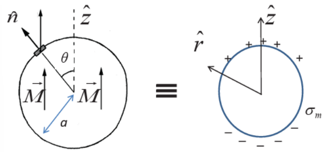
$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\left(\int_S \sigma_m\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}da + \int_V \rho_m\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|^3}d\tau\right)$$

3. Finally, compute $\vec{B}$ as:

$$\vec{B} = \mu_0(\vec{H}+\vec{M})$$

Here we will present two examples of two different geometries:

Spherical magnet uniformly magnetized



1. Since M is uniform:

$$\rho_m = -\nabla \cdot \vec{M} = 0$$

$$\rho_m = -\nabla \cdot \vec{M} = 0$$

$$\sigma_m = \vec{M} \cdot \hat{n} = M\hat{z} \cdot \hat{n} = M\cos\theta$$

2. As we have already seen for a charged cosinuidal sphere, we have almost the same:

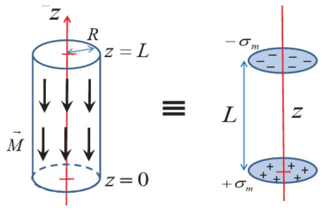
$$\vec{H}(\vec{r}) = \begin{cases} \vec{H}_{dipole} = \frac{1}{4\pi}(3\frac{\vec{m}\cdot\vec{r}}{r^5}\vec{r} - \frac{\vec{m}}{r^3}), r > a \ (\vec{m} = \vec{M}Vol) \\ \vec{H}_{in} = -\frac{\sigma_0}{3}\hat{z} = -\frac{\vec{M}}{3}, r < a \end{cases}$$

3. Compute finally B

$$\vec{B} = \mu_0(\vec{M} + \vec{H}) = \begin{cases} \texttt{INSIDE:}\ \vec{B} = \mu_0(\vec{M} + \vec{H}) = \mu_0(\vec{M} - \frac{\vec{M}}{3}) \\ \texttt{OUTSIDE:}\ \vec{B} = \mu_0\vec{H} \end{cases}$$

since outside there is no polarization.

Bar magnet uniformly magnetized



1. Since M is uniform:

$$\rho_m = -\nabla \cdot \vec{M} = 0$$
$$\rho_m = -\nabla \cdot \vec{M} = 0$$
$$\sigma_m = \vec{M} \cdot \hat{n} = \pm M$$

2. Like in the electric counterpart, H field is the field produced by two opposite charged disks. This H-field produced by a surface is the one calculated below:

By "Coulomb's law"

$$dH = \frac{1}{4\pi}\frac{dq_m}{l^2} \implies dH_z = \frac{1}{4\pi}\frac{dq_m}{l^2}\frac{1}{4\pi}\frac{dq_m}{l^2}\cos\theta = \frac{1}{4\pi}\frac{dq_m}{l^2}\frac{z}{l} = \frac{\sigma_0 z da}{4\pi l^3}$$

Then integrate and obtain:

$$H_z(z) = \frac{\sigma_0}{2}\left(\frac{z}{\sqrt{z^2}} - \frac{z}{\sqrt{R^2+z^2}}\right)$$

Then the total field in between both plates is:

$$\begin{cases} \texttt{INSIDE:}\ \ H_z^{TOT} = H_z^{+} + H_z^{-} = \frac{\sigma_0}{2}\left(2 + \frac{z-L}{\sqrt{R^2+(z-L)^2}} - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z} \\ \texttt{OUTSIDE:}\ \ H_z = \frac{\sigma_0}{2}\left(\frac{z-L}{\sqrt{(z-L)^2+R^2}} - \frac{z}{\sqrt{R^2+z^2}}\right) \end{cases}$$

3. Then, the B-field is determined by definition:

$$\vec{B} = \mu_0(\vec{M}+\vec{H}) = \begin{cases} \texttt{INSIDE:}\ \ \vec{B} = \mu_0(\vec{M}+\vec{H}) = \mu_0\frac{\sigma_0}{2}\left(\frac{z-L}{\sqrt{(z-L)^2+R^2}} - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z} \\ \texttt{OUTSIDE:}\ \ \vec{B} = \mu_0\vec{H} = \mu_0\frac{\sigma_0}{2}\left(\frac{z-L}{\sqrt{(z-L)^2+R^2}} - \frac{z}{\sqrt{R^2+z^2}}\right)\hat{z} \end{cases}$$

3.5 Special shape magnets: demagnetizing factor

In analogy with dielectrics, we define a demagnetizing factor:

$$\gamma = \frac{H_{in}(\sigma_{in})}{M} = \frac{|H_{in}(M)|}{|M|}$$

However, the demagnetizing field H is opposite to the polarizing one M.
Look at the chart for different geometries:

	Slab	Sphere	Needle $E \perp$ axis	Needle $E \parallel$ axis	Prolate Spheroid $E \parallel$ symmetry axis	Oblate Spheroid $E \parallel$ symmetry axis
γ	1	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{r^2-1}\left(\frac{r}{\sqrt{r^2-1}}\ln(r+\sqrt{r^2-1})-1\right)$	$\frac{1}{1-r^2}\left(1-\frac{r}{\sqrt{1-r^2}}\sin^{-1}(\sqrt{1-r^2})\right)$

Table 3: Values of γ for different geometries

3.6 Internal fields spherical and cylindrical magnets

The demagnetizing field ($\vec{H}$) is **opposite** to the "polarizing" one ($\vec{M}$) with proportionality:

$$\vec{H}_{demagnetizing} = -\gamma \vec{M}$$

Hence:

$$B_{magnetizing} = \mu_0(\vec{H}_{demagnetizing} + \vec{M}) = \mu_0(1-\gamma)\vec{M}$$

So, in the absence of external field, we have the solutions for spherical and cylindrical geometries:

$$\vec{H}_{\text{demagnetizing sphere}} = -\frac{\vec{M}}{3} \iff \vec{B} = \frac{2}{3}\mu_0 \vec{M}$$

$$\vec{H}_{\text{demagnetizing cylinder}} = -\frac{\vec{M}}{2} \iff \vec{B} = \frac{1}{2}\mu_0 \vec{M}$$

3.7 Boundary condition for B

Since B is solenoidal:

$$\nabla \cdot \vec{B} = 0$$

Then this forces B normal components to be continuous ($\vec{B}_{\parallel 1} = \vec{B}_{\parallel 2}$) Hence, the same happens with H tangential components since

$$\begin{cases} \nabla \times \vec{H} = 0 \implies H_{\perp 1} = H_{\perp 2} \\ \nabla \cdot \vec{H} = \rho_m \implies \vec{H}_1 \cdot \hat{n}_1 + \vec{H}_2 \cdot \hat{n}_2 = \sigma_m \end{cases}$$

3.8 Analogy with electric field

Notice both fields have so much concepts in common, which are, essentially, summarized in the following table:

Magnetic Quantity	Electric Quantity
$\frac{\vec{B}}{\mu_0}$	$\vec{D}$
$\vec{H}$	$\varepsilon_0 \vec{E}$
$\vec{M}$	$\vec{P}$
ρ_m	ρ_b
0	ρ_f

Table 4: Magnetic $\leftrightarrow$ Electric analogies

So, all this lead to the following classification (similar to the electric counterpart):

Magnetic Media

Ferromagnets
(and Antiferromagnets)
Spontaneous intrinsic magnetization

Linear Magnetic Media
(Induced magnetization)

Paramagnetic Media
Analogous to linear dielectrics
$\chi_m > 0, \mu_r > 1$

Diamagnetic Media
No dielectric analog exists
$\chi_m < 0, \mu_r < 1$

Summary of Section 3

Problems with magnets and linear media

Magnetization

Changes in the spin in the magnetic material lead to the production of current loops that are a problem equivalent to magnetic infinitesimal dipoles of moment $\vec{m}$ (the justification of this equivalence will be seen in topic 5) Then, we use the definition to compute, the magnetic dipole moment:

$$\vec{m} = q_m \vec{d}$$

Note: Remember magnetic charge (q_m) is fictitious.
And also the Magnetization:

$$\vec{M} = n\vec{m}$$

[average magnetic dipole moment per unit volume].
Other magnitudes such as the following ones can be computed just using the proper formula:

$$\vec{F} = q_m \vec{B}_{ext}$$

$$\vec{\Gamma} = \vec{m} \times \vec{B}_{ext}$$

$$u_m = -\vec{m} \cdot \vec{B}_{ext}$$

Vector potential $\vec{A}$

Such as we had done with electric counterpart, we can define a vector potential related to a magnitude in magnetic field (analogy: A↔V):

$$\nabla \cdot \vec{B} = 0 \implies \vec{B} = \nabla \times \vec{A} \implies \Phi_B = \iint_S \vec{B} \cdot \hat{n} da = \oint \vec{A} \cdot \vec{dl}$$

This is the magnetic flux.

Vector potential in a magnetic dipole

$$\vec{A} = \frac{\mu_o}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \to \vec{B} = \nabla \times \vec{A} = \mu_o \delta(\vec{r})\vec{m} + \frac{\mu_o}{4\pi r^3}[3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}]$$

Then, introducing the magnetization will become:

$$\vec{A} = \frac{\mu_o}{4\pi} \int_V \frac{\vec{M}(\vec{r}) \times \vec{R}}{R^3} d\tau', \quad R = ||\vec{r} - \vec{r'}||$$

Auxiliar field $\vec{H}$

`FUNDAMENTAL THEOREM:` The magnetization spins $\vec{M}$ produce $\vec{B} = \nabla \times \vec{A}$.
So, we will use the same method we used to explain the displacement field, and now:

$$\vec{B} = \mu_0(\vec{M} + \vec{H})$$

What let us define the proper density and surface densities such as:

$$\begin{cases} \rho_m = -\nabla \cdot \vec{M} = \nabla \cdot \vec{H} \\ \sigma_m = \hat{n} \cdot \vec{M}|_S \end{cases}$$

Linear media

$$\begin{cases} \vec{M} = \chi_m \vec{H} \\ \vec{B} = \mu_o(\vec{M} + \vec{H}) \end{cases} \implies \vec{B} = \mu_o(1+\chi_m)\vec{H} = \mu_o\mu_r\vec{H}, \quad \mu_r = 1 + \chi_m$$

With this result we can distinguish between:

- Paramagnetic:
$$\chi_m > 0, \mu_r > 1$$
- Diamagnetic:
$$\chi_m < 0, \mu_r < 1$$

Boundary conditions

$$\nabla \times \vec{H} = 0 \rightarrow \hat{n}_2(\vec{H}_2 - \vec{H}_1) = 0 \implies H_{2\perp} = H_{1\perp}$$

$$\nabla \cdot \vec{H} = \rho_m \rightarrow \hat{n}_2(\vec{H}_2 - \vec{H}_1) = \sigma_m \implies H_{2\parallel} - H_{1\parallel} = \sigma_m$$

$$\nabla \cdot \vec{B} = 0 \rightarrow \hat{n}_2(\vec{B}_2 - \vec{B}_1) = 0 \implies B_{2\parallel} = B_{1\parallel}$$

Magnetic energy

$$u_m = \frac{1}{2}\int \vec{B} \cdot \vec{H} d\tau = \frac{1}{2}\int \mu_o\mu_r|\vec{H}|^2 d\tau$$

Analogies

$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$	$\vec{B} = \mu_o(\vec{H} + \vec{M})$
$\nabla \cdot \vec{E} = \frac{\rho_b}{\varepsilon_0}$	$\nabla \cdot \vec{H} = \rho_m$
$\nabla \cdot \vec{P} = -\rho_b$	$\nabla \cdot \vec{M} = -\rho_m$
$\nabla \cdot \vec{D} = 0 \quad (\rho_f = 0)$	$\nabla \cdot \vec{B} = 0$
$\nabla \times \vec{E} = 0$	$\nabla \times \vec{H} = 0 \quad (\vec{J}_f = 0)$
$u_e = \frac{1}{2}\int \vec{D} \cdot \vec{E} d\tau$	$u_m = \frac{1}{2}\int \vec{B} \cdot \vec{H} d\tau$

Equivalent currents

$$\nabla \times \vec{B} = \mu_o(\nabla \times \vec{M} + \nabla \times \vec{H}) = \mu_o\vec{J}_e, \quad \nabla \times \vec{H} = 0 \quad (\text{ if } \vec{J}_f = 0)$$

Then:

$$\begin{cases} \vec{J}_e = \nabla \times \vec{M} \\ \vec{K}_e = \vec{M} \times \hat{n}|_S \end{cases}$$

4 Steady currents and magnetism

4.1 Current density

Like in the electric counterpart, we define a:

$$\textbf{Volume current density: } \vec{J} = \frac{\sum_i^{\delta N} q_i \vec{\nu_i}}{\delta \tau} \to \vec{J} = \rho \nu_{drift}$$

where ν_{drift} is called the average speed at the volume δN (for same q-charged particles).

$$\textbf{Surface current density: } \vec{K} = \cdots = \sigma \vec{\nu}_{drift}$$

4.2 Gauss law for current

We can also apply Gauss' law to the flux of J. This, by definition, equal to the **current**:

$$I = \frac{\Delta Q_{across}}{\Delta t} = \Phi_j = \oint_S \vec{J} \cdot d\vec{a}$$

4.3 Local charge conservation law

The flux of J through a closed surface is the total current through S. This means the charge crossing the surface r

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Hence, the free and bound charges problem is just studied the same way we have already done. However, it will be interesting to recall:

- **Bound charges:** that displacement due to the same atom/molecule (it creates a dipole that tend to orient to the external field...). Then, at equilibrium, no motion of this charges takes place:
$$\begin{cases} \vec{\nabla} \cdot \vec{J}_b + \frac{\partial \rho_b}{\partial t} = 0 \\ \partial_t \rho_b = 0 \end{cases} \implies \vec{J}_b = 0$$
- **Free charges:** This distribution is due to the free charges which are able to move freely across macroscopic distances.

So, all this results on:

$$\vec{J}_{TOT} = \vec{J}_f + \vec{J}_b \implies \vec{J}_{TOT} = \vec{J}_f$$

However, notice that there can be bound charge current when the polarization is being created or altered due to σ_b distributions on the surfaces of, for example, a tiny cylinder. Then:

$$\vec{J}_b = \frac{\partial \vec{P}}{\partial t}$$

4.4 Magnetostatic condition

Since in magnetostatic conditions there is no movement of charges nor densities, then this leads to the expression:

$$\vec{\nabla} \cdot \vec{J}_f = 0$$

where we can notice the magnetostatic field is solenoidal.

[1] This is **Global charge conservation**.

4.5 Local Ohm's Law

Then define, the free current density as "something" proportional to the external field.

$$\vec{J}_f = g\vec{E}_{macro}$$

where g is the **conductivity** of the material.

Note: Ohm's law is only valid for free charges.

4.6 Global's Ohm's law

Given all the mentioned before (local's Ohm's law and charge conservation), later we can derive:

$$\vec{\nabla} \cdot \vec{J} = g(\vec{\nabla} \cdot \vec{E}) = 0 \implies \vec{\nabla} \cdot \vec{E} = 0$$

Hence, we define a global law as:

$$\Delta V = IR, \quad I = G\Delta V$$

with $R = \frac{l}{gA} \leftrightarrow G = R^{-1}$ for a cylinder.

4.7 Non-homogeneous conductors

Non-homogeneous means $g \equiv g(r)$. This kind of problems cannot be solved as the homogeneous ones, but we can use continuity and boundary conditions just like the following procedure indicates:

1. in steady state the current through a cross-section of the conductor must equal the current through any other, or equivalently the NORMAL COMPONENT of J MUST BE CONTINUOUS.
2. The line integral of E, that is, the voltage, must display the same drop across all conductors, or equivalently, the TANGENTIAL COMPONENT of E MUST BE CONTINUOUS.

Hence, we can deal with two kind of problems depending on the symmetry of the conductances:

4.7.1 Series conductances

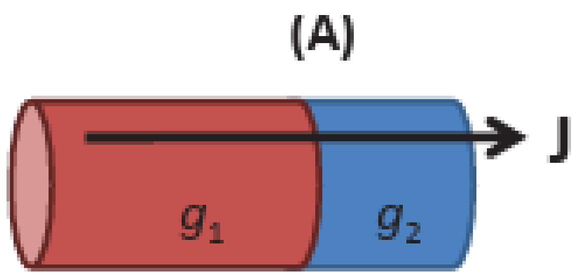


Figure 19: Series conductances

The proper bounder condition to use is:

$$\nabla \cdot J = 0 \implies J_1 = J_2$$

Also we know J has the form:

$$\begin{cases} \texttt{PLANAR SYMMETRY:}\ J = k \\ \texttt{SPHERICAL SYMMETRY:}\ J = k/r^2 \\ \texttt{CYLINDRICAL SYMMETRY:}\ J = k/s \end{cases}$$

Hence obtain E from Ohm's local law ($J = gE$), integrate it to obtain ΔV and obtain the resistance by computing $R = \Delta V/I$

4.7.2 Parallel conductances

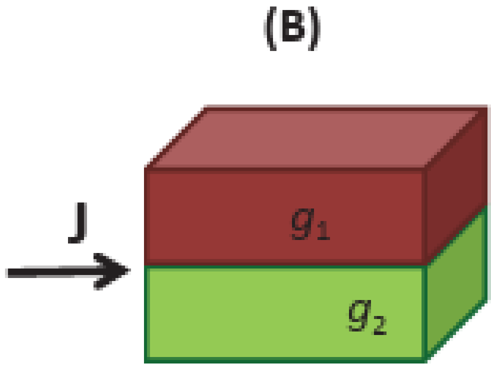


Figure 20: Parallel conductances

The proper bounder condition to use is:

$$\nabla \times E = 0 \implies E_1 = E_2$$

Also we know E has the form:

$$\begin{cases} \texttt{PLANAR SYMMETRY:}\ E = k \\ \texttt{SPHERICAL SYMMETRY:}\ E = k/r^2 \\ \texttt{CYLINDRICAL SYMMETRY:}\ E = k/s \end{cases}$$

Hence obtain ΔV and obtain the current I with Ohm's law to J. Finally, find resistance by computing $R = \Delta V/I$.
<u>**Note:**</u> Remember that the capacitance is $C = Q_f/V$.

4.8 B field generated by currents

Using Biot-Savart's law:

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I \cdot d\vec{l} \times \hat{r}}{r^2} \implies \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int I \cdot d\vec{l} \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

In analogous way, we define a potential scalar as:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int d\tau \frac{\vec{J}(\vec{r}')}{|\vec{r}-\vec{r}'|}$$

4.9 Ampère's law for B

Given Gauss' law for B:

$$\vec{B} = \vec{\nabla} \times \vec{A} \implies \nabla \cdot B = 0$$

Then, Ampère's law states that:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \Leftrightarrow \oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

4.9.1 Particular case: No magnetic media

In the case there is no magnetic media, this is; no magnets, no linear magnetic materials... Then, we have $\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0\vec{H}$. So, Biot-Savart's law gives us a simple result:

$$\vec{H}(\vec{r}) = \frac{1}{4\pi}\int I \cdot d\vec{l} \times \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}$$

and Ampère's law:

$$\vec{\nabla} \times \vec{H} = \vec{J} \Leftrightarrow \oint \vec{B} \cdot d\vec{l} = I$$

Hence, all of this is summarised in Maxwell's equations for magnetostatics:

$$\boxed{\begin{gathered}\vec{\nabla} \cdot \vec{B} = 0 \iff \Phi(\vec{B}) = 0 \\ \vec{\nabla} \times \vec{H} = \vec{J}_f \iff \oint \vec{H} \cdot d\vec{l} = I\end{gathered}}$$

Now, let's derive to the expressions of H and B of some important geometries:

Nonmagnetic infinite wire in vacuum

1. By symmetry, H cannot depend on z nor φ. This makes:

$$\vec{H} = \vec{H}(s)$$

2. By Biot-Savart's law, when $s \to \infty$, then $H \to 0$

3. One procedure consists on noticing how this H field behaves at switching the sign of current (J), this should also reverse H. Then, notice, in this case, reversing J does not affect essentially to H, what makes:

$$H_s = 0$$

4. Then, performing Ampère's law (Ampère's laces) both inside and outside makes:

$$H_z = 0$$

5. So, finally, the remaining component has to depend on φ

$$H_\varphi \neq 0$$

So, Gauss' law provides:

$$\begin{cases} \texttt{INSIDE:} \ \int \vec{H} \cdot d\vec{l} = J(\pi s^2) \implies H_\varphi = \frac{Js}{2} \implies B_\varphi = \mu_0 \frac{Js}{2} \\ \texttt{OUTSIDE:} \ \int \vec{H} \cdot d\vec{l} = I \implies H_\varphi = \frac{I}{2\pi a} \implies B_\varphi = \mu_0 \frac{I}{2\pi a} \end{cases}$$

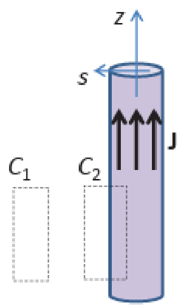


Figure 21: Nonmagnetic infinite wire in vacuum

Infinite solenoid in vacuum

1. By symmetry, $\vec{H}$ cannot depend on z nor φ:

$$\Rightarrow H = H(s)$$

2. From Biot–Savart's law, as $s \to \infty$, $H \to 0$.

3. Changing J_f direction implies reversal of H, so:

$$H_s = 0$$

4. Applying Ampère's law to the circles shown, we get that both outside and inside:

$$H_\varphi = 0$$

5. Apply Ampère's law:

   - Inside (C_2): $H \cdot \ell = NI \Rightarrow H = \frac{N}{\ell} I = nI$
   - Outside (C_1): $H_z = 0$ (top side $\to \infty$)

$$\Rightarrow \begin{cases} H_{\text{inside}} = nI, & B_{\text{inside}} = \mu_0 nI \\ H_{\text{outside}} = 0, & B_{\text{outside}} = 0 \end{cases}$$

where $n =$ number of turns per unit length.

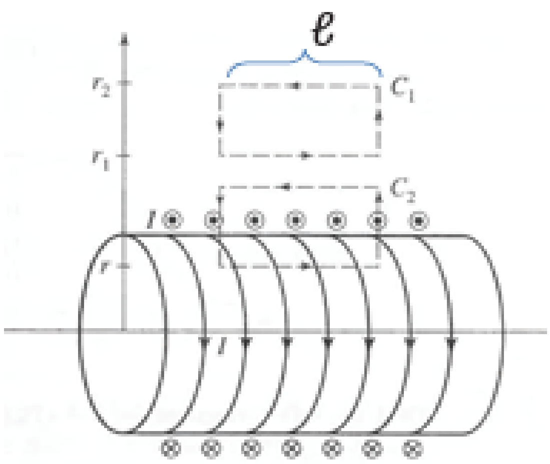


Figure 22: Infinite solenoid in vacuum

Hence, both two problems can be discussed with linear magnetic media. Then, proceed as it follows:

Nonmagnetic infinite wire in linear media

1. We found
$$\vec{H}_J = H_\varphi \hat{\varphi}$$
2. Then, we guess magnetisation should be the same way:
$$\vec{M} = M_\varphi \hat{\varphi}$$
3. Find pole densities everywhere in space, since:
$$\begin{cases} \rho_m = -\vec{\nabla} \cdot \vec{M} = 0 \\ \sigma_m = \vec{M} \cdot \hat{n} = 0 \end{cases}$$
4. Then, magnetic poles do not produce any kind of magnetic field, which lead to define the H-field as simply the one produced by the current:
$$\vec{M} = \chi_m \vec{H}_T = \chi_m \vec{H}_J$$
5. Then, same for B-field:
$$\vec{B} = \mu_r \mu_0 \vec{H}_T = \mu_r \mu_0 \vec{H}_J$$

Infinite solenoid in linear media

1. We found that H is parallel to z and nonzero only inside
$$\vec{H} = H_z \hat{z}$$
2. Then, we guess magnetisation should be axial inside and zero outside:
$$\vec{M} = M_z \hat{z}$$
3. Find pole densities everywhere in space. Then, we reach to the conclusion that there should be surface pole densities on both bases (which are at the infinite), so the field created by this " two plates located at infinite separation" will be zero.
4. Then, magnetic poles do not produce any kind of magnetic field, which lead to define the H-field as simply the one produced by the current:
$$\vec{M} = \chi_m \vec{H}_T = \chi_m \vec{H}_J$$
5. Then, same for B-field:
$$\vec{B} = \mu_r \mu_0 \vec{H}_T = \mu_r \mu_0 \vec{H}_J$$

And now, let's study what happens when a resistor element is placed along the infinite wire. Then, according to the picture below:

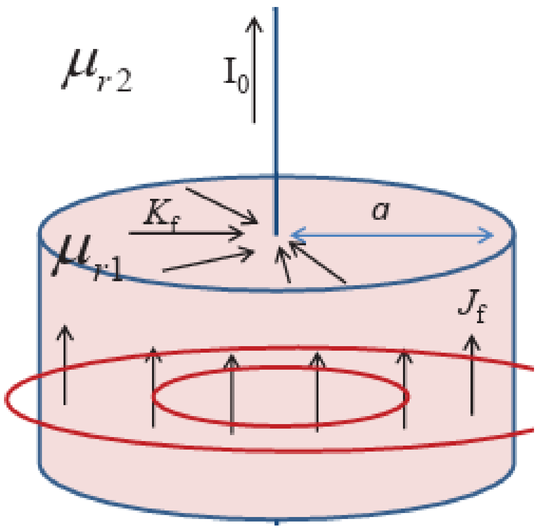


Figure 23: Infinite wire with a resistor

H-field

- We find $H_z = 0$ everywhere by BS law, since all current generators I_0 and J_f are parallel to z (thus, generate H field normal to z), while K_f on both surfaces compensate z and s contributions of dH generic point P in space.
- In a similar way, $H_s = 0$, since K_f does not contribute for simple geometric reasoning. And for current elements such as I_0 and J_0, remember thie rule that states, the contribution of this current elements should change sign by reversing the POV (Looking the resistor from the top or the bottom).
- Therefore, the remaining component of H is:

$$\vec{H} = H_\varphi \hat{\varphi}$$

- To compute this value we only need to use Ampère's law, as it follows:

$$\oint \vec{H} \cdot \vec{dl} = H_\varphi 2\pi s = I_0 \implies H_\varphi(s) = \frac{I_0}{2\pi s}$$

B-field

- Since the lines run in circles, there is nowhere any magnetic pole density. Then, the magnetic field:

$$B_\varphi(s) = \mu_{r2}\mu_0 H_\varphi(s) = \mu_{r2}\frac{\mu_0}{2\pi}\frac{I_0}{s}$$

outside the resistor.

- In the resistor,

$$\Phi_J = J_f \pi a^2 = I_0 \implies J_f = \frac{I_0}{\pi a^2}$$

- Then, the amount of current that flows through the integration loop (red lace) will be

$$\begin{cases} \int \vec{H} \cdot \vec{dl} = 2\pi s H_\varphi = J_f \pi s^2 \implies H_\varphi = \frac{J_f s}{2} = \frac{I_0}{2\pi a^2}s \implies B_\varphi = \mu_{r1}\mu_0 \frac{I_0}{2\pi a^2}s, \quad s < a. \\ \int \vec{H} \cdot \vec{dl} = \Phi_J = I_0 \implies H_\varphi = \frac{I_0}{2\pi s} \implies B_\varphi = \mu_{r2}\frac{\mu_0}{2\pi}\frac{I_0}{s}, \quad s > a. \end{cases}$$

4.10 Boundary conditions for magnetostatics

From Maxwell's equations for magnetostatics:

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{H} = \vec{J}_f \end{cases}$$

Then derive this expressions and obtain:

$$\vec{B}_1 \cdot \hat{n}_1 + \vec{B}_2 \cdot \hat{n}_2 = 0$$

$$\vec{H}_1 \times \hat{n}_1 + \vec{H}_2 \times \hat{n}_2 = \vec{K}$$

4.11 Ferromagnetic behavior when applying H-field

The experiment of applying an external H-field to a ferromagnet lead to two different behaviors depending on the material of the ferromagnet. Then, we call:

- **HARD ferromagnets:** to those cases which is <u>hard to change their magnetization</u>. Then the Histeresis phase is large enough which lead to define the PERMANENT MAGNETS (see the right graph)
- **SOFT ferromagnets:** to those cases which is <u>easy to change their magnetization</u>. Then the Histeresis phase is small enough leading to quasilinear response. Then, we define SHIELDING AND MAGNETIC CIRCUITS (see the left graph)

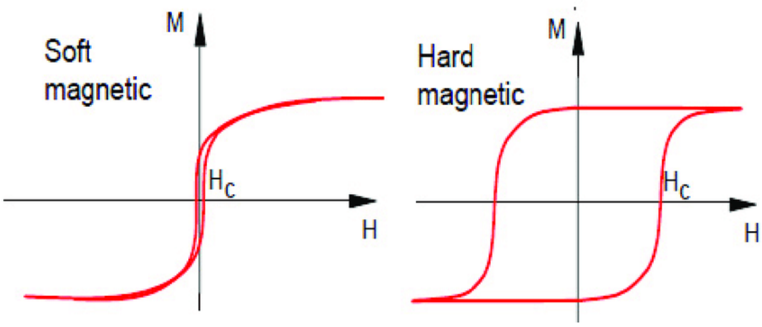


Figure 24: Hard and Soft ferromagnets graphs

Then, let's study some of the applications of this kind of ferromagnets.

4.11.1 Applications of SOFT ferromagnets

- **Amplification of B** Using the principle that $H = H_{\|}$ both inside and outside for the boundary condition. It would be easy to check that:

$$\begin{cases} \vec{B}_{out} = \mu_0 \vec{H} \\ \vec{B}_{in} = \mu_r \mu_0 \vec{H} \end{cases} \implies \frac{B_{in}}{B_{out}} = \mu_r \sim 10^3$$

- **Magnetic shield**
- **Magnetic circuits**

4.12 Magnetic circuits: Zero-Leakage approximation

Given the circuit in the figure below:

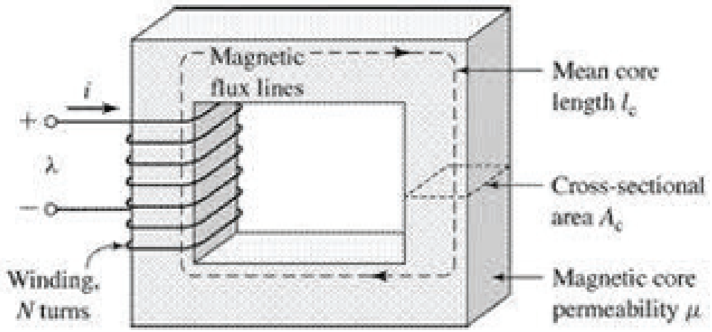


Figure 25: Magnetic circuit

Then, a simple analysis lead to:

1. $\vec{B} \parallel \vec{M} \parallel \vec{H}$ everywhere outside permanent magnets (linear soft ferromagnet constitutive equation)
2. $\Phi_B = cnst \implies B = cnst$ (Zero-leakage approximation)
3. $\oint \vec{H} \cdot \vec{dl} = H_{core} \cdot l_{core} + H_{gap} \cdot l_{gap} = NI$ (Ampère's law for H, assuming (2) holds)

Then, all this implies:

$$\frac{B}{\mu_{core}} \cdot l_{core} + \frac{B}{\mu_{gap}} \cdot l_{gap} = NI \implies B = \frac{\mu_o NI}{l_{gap} + \frac{l_{core}}{\mu_r}}$$

And this leads to the following law:

4.13 Hopkinson's law

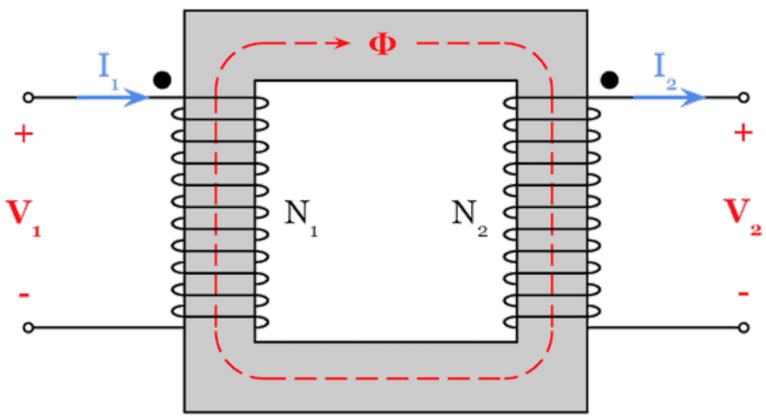


Figure 26: Hopkinson's circuit

Given the circuit in the figure:

$$\oint \vec{H} \cdot \vec{dl} = NI + NI = 2NI \implies H_q l_q + H_p l_p = NI = \frac{B_q l_q}{\mu_q} + \frac{B_p l_p}{\mu_p}$$

being ”q” the lateral side (where the solenoides) and ”p” the lower and top sides. Then, knowing:

$$\Phi = B_p S_p = B_q S_q \implies B_p = B_q \frac{A_q}{A_p}$$

So all this lead to the conclusion:

$$\Phi \frac{1}{A_q} \frac{l_q}{\mu_q} + \Phi \frac{1}{A_p} \frac{l_p}{\mu_p} = NI \equiv (\mathfrak{R}_q + \mathfrak{R}_p)\Phi = \mathfrak{R}\Phi$$

If we define $\mathfrak{M} = N_1 I_1 + N_2 I_2$ as the MAGNETOMOTIVE FORCE and $\mathfrak{R}$ as the RECLUCTANCE of the magnetic circuit, then, this summarizes in the analogy with electric circuits like:

$$\mathfrak{M} = \mathfrak{R}\Phi$$

4.14 Magnetic circuit with a permanent magnet

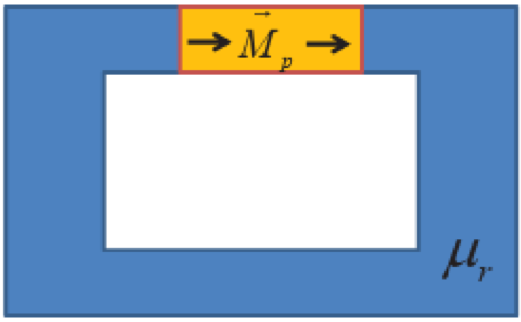


Figure 27: Magnetic circuit with a permanent magnet

Consider the magnetic circuit consisting of a permanent magnet of permanent magnetization M_p linked with a linear magnetic material. Ampère's law states:

$$\oint \vec{H} \cdot \vec{dl} = NI = 0 = H_p l_p + H_c l_c$$

being ”p” the yellow area and ”c” the blue area. The line integral of H is zero since H is opposite to M and B inside a permanent magnet.
In the linear medium we use the expression:

$$H_c = \frac{B}{\mu_r \mu_0}$$

valid for **lineal medium**.

In the magnet we use:

$$H_p = \frac{B}{\mu_0} - M_p$$

valid for **permanent magnets NOT linear**.

Plugging this two equations we obtain:

$$\left(\frac{B}{\mu_0} - M_p\right) l_p + \frac{B}{\mu_r \mu_0} l_c = 0 \implies M_p l_p = B\left(\frac{l_p}{\mu_o} + \frac{l_c}{\mu_o \mu_r}\right)$$

so all this lead too:

$$\begin{cases} B = \frac{\mu_o M_p l_p}{l_p + \frac{l_c}{\mu_r}} \\ \Phi(\mathfrak{R}_p + \mathfrak{R}_c) = \mathfrak{M} = M_p l_p \end{cases}$$

This enables to rewrite an equivalent simpler circuit such that the energy supplier will be $\mathfrak{M} = M_p l_p$ and two resistors placed at series of value $\mathfrak{R}_c$ and $\mathfrak{R}_p$.
Then, we could also compute:

$$M_c = (\mu_r - 1)H_c = (\mu_r - 1)\frac{B}{\mu_r \mu_o} \sim \mu_r H_c = \frac{B}{\mu_o} \implies \sigma_m = \frac{B}{\mu_0}$$

Then notice that the distribution of poles densities will be placed at the edges of the yellow area such that:

$$\sigma_T = \vec{M}_p \cdot \hat{n} + \vec{M}_c \cdot (-\hat{n}) = \pm(\sigma_p - \sigma_c) \sim \pm(M_p - \frac{B}{\mu_o}) = \pm M_p(1 - \frac{\mathfrak{R}_p}{\mathfrak{R}})$$

4.15 Ampère's equivalence theorem

We can use a quite tricky tool such as the integration by parts to integrate the expression of the scalar potential:

$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi} \int d\tau' \vec{M}(\vec{r}') \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \frac{\mu_0}{4\pi} \int d\tau' \vec{M}(\vec{r}') \times \vec{\nabla}(\frac{1}{|\vec{r} - \vec{r}'|})$$

After applying integration by parts we obtain:

$$\begin{cases} \vec{J}_{eq} = \vec{\nabla} \times \vec{M} \\ \vec{K}_{eq} = \vec{M} \times \hat{n} \end{cases}$$

All this enable us to study all the problems in such an easier way since, instead of using magnetic poles, we will use equivalent currents. So, the theorem that prevents this behavior:

Ampère's equivalence theorem

A magnetized body with magnetization $\vec{M}$ is equivalent to a set of volume and surface EQUIVALENT current densities:

$$\vec{J}_{eq} = \vec{\nabla} \times \vec{M}, \quad \vec{K}_{eq} = \vec{M} \times \hat{n}$$

We can just check some examples of this behavior:

Solenoid analogy with uniformly magnetized bar

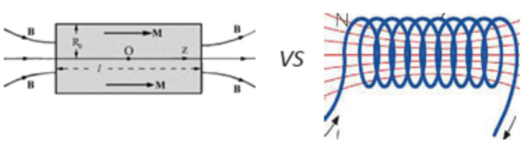


Figure 28: Solenoid ↔ uniformly magnetized bar

Under the assumption of infinite length...

then both magnetic poles are such far away that we consider $\vec{H} = 0$ and so on $\vec{B} = \mu_0 \vec{M}$. Notice how the respective equivalent problem holds Ampère's theorem:

$$\begin{cases} \vec{J}_{eq} = \vec{\nabla} \times \vec{M} = 0 \\ \vec{K}_{eq} = M\hat{z} \times \hat{s} = M\hat{\theta} \end{cases}$$

Considering a solenoid of length l...

Then use Ampère's law to calculate:

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I = \mu_0 K l$$

Since B is uniform inside the solenoid, we still get:

$$Bl = \mu_0 K l \implies B = \mu_0 K = \mu_0 M$$

which obviously holds Ampère's equivalence theorem as we have shown.

Loop analogy with uniformly magnetized thin disk

Spherical solenoid analogy with uniformly magnetized sphere

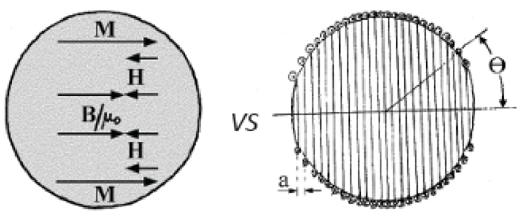


Figure 29: Spherical solenoid $\leftrightarrow$ uniformly magnetized sphere

Toroidal solenoid analogy with uniformly magnetized toroid (with[out] gap)

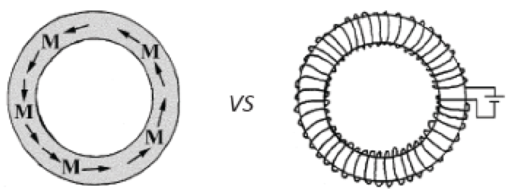


Figure 30: Toroidal solenoid $\leftrightarrow$ uniformly magnetized toroid

4.16 Superconductivity

Superconductivity effect is essentially characterized by:

- Zero Resistance ($R \to 0 \leftrightarrow g \to \infty$)
- Perfect diamagnetism ($B_{in} = 0$)

Also, properties of suèrconductivity implies there is some special conditions to be fulfilled. On the graph above:

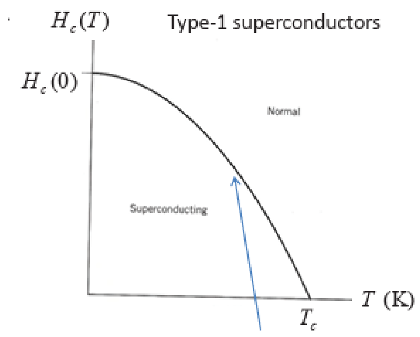


Figure 31: Critical values of superconductivity

- There is a critical temperature (T_c) above which superconductivity ceases.
- There is a critical field (H_c) above which superconductivity ceases.

For type-1, empirically:

$$H_c(T) \sim H_c(0)\left[1 - \left(\frac{T}{T_c}\right)^2\right]$$

4.17 Microscopic mechanism of superconductivity

The physical mechanism of superconductivity is the following:

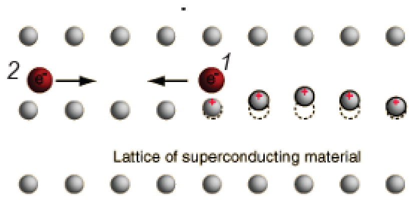


Figure 32: First step of mechanism

When an electron passes through the lattice, it dissorts it such in a way that lattice atoms create a quantized vibration or phonon.

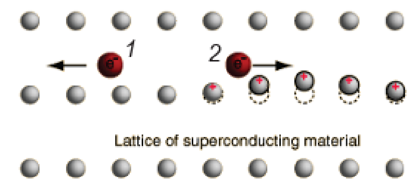


Figure 33: Second step of mechanism

This distorsion produced (also called phonon) creates an excess of positive charge that attracts a second electron (which absorbs the phonon).

This mechanism results in an effective interaction that brings 2 electrons closer enough, the so-called **Cooper pair**

4.18 Effects of the effective inter-electron attraction

A metal is characterized by a conduction band that is partially filled up to the so-called "Fermi energy", EF). When two electrons interact by exchange of a phonon to form a Cooper pair, the energy of the pair is lowered by an amount comparable to the phonon energies, of the order of some meV. This gives rise to a so-called superconducting gap.

- **Lack of resistance:** Due to the superconducting gap, there are no available energy levels for the electrons to scatter into.
- **Observation of superconductivity at low T** at normal temperatures, otherwise, thermal excitations provide enough energy for electrons to jump across the gap.
- **Diamagnetic behavior** The resulting coupling become stronger when the two electrons have opposite spins. The importance of this phenomenon is that the application of a magnetic field tensçds to align all electron spins in the same direction thus breaking the Cooper pairs.

This leads to state:
Superconductivity and magnetism are incompatible each other

4.19 Superconducting currents

Although a superconductor is nonmagnetic in nature, it responds as a **perfectly diamagnetic medium**.
This makes that we will study all the superconductors problems as a diamagnetic medium with $\chi_m = -1$, that is $\mu_r = 0 \implies B_{in} = 0$, one of the conditions of superconductivity.

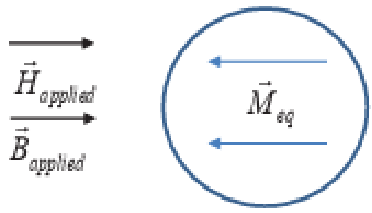


Figure 34: Superconducting sphere problem

$$\vec{M} = \chi_m \vec{H}_T = \chi_m(\vec{H}_0 + \vec{H}_{ind}) = \chi_m(\vec{H}_0 - \vec{M}/3)$$
$$\implies \vec{M} = -\frac{3}{2}\vec{H}_0$$

<u>Note:</u> We can easily verify that:

$$\vec{B}_T = \vec{B}_0 + \vec{B}_{ind} = 0$$

Then, something very important to see is that:

> *The induced magnetic field is not produced by a magnetization (since it is a superconductor), but rather by superconducting currents, according to Ampère's equivalence Theorem*

Then, we can compute this result by just:

$$\begin{cases} \vec{J}_{sc}(= \vec{J}_{eq}) = \vec{\nabla} \times \vec{M}_{eq} = 0 \\ \vec{K}_{sc}(= \vec{K}_{eq} = \vec{M}_{eq} \times \hat{n} = -|\vec{M}_{eq}|\hat{z} \times \hat{r} = -|\vec{M}_{eq}|sin\theta\hat{\varphi}) \end{cases}$$

Summary of section 4

Problems on electric current and conductivity

Current density

$$\vec{J} = \rho \vec{v}_{drift}$$

`Total current:`

$$I = \iint_S \vec{J} \cdot \hat{n} da$$

`Charge conservation:`

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \implies \oint \vec{J} \cdot \hat{n} da = -\frac{\partial Q_{inside}}{\partial t}$$

`Bound charges:`

$$\rho_b = -\nabla \cdot \vec{P}$$

`Bound currents:`

$$\nabla \cdot \vec{J}_b = -\frac{\partial \rho_b}{\partial t} = \nabla \cdot (\frac{\partial \vec{P}}{\partial t}) \implies \vec{J}_b = \frac{\partial \vec{P}}{\partial t}$$

`Total currents:`

$$\vec{J} = \vec{J}_f + \vec{J}_b$$

in steady state becomes:

$$\vec{J}_b = 0, \quad \vec{J} = \vec{J}_f, \quad \nabla \cdot \vec{J} = 0$$

Ohm's law

$$\vec{J}_f = g\vec{E}$$

Homogeneous medium in steady state

$$\vec{J}_f = g\vec{E}, \quad \nabla \cdot \vec{J}_f = 0 \implies \begin{cases} \nabla \cdot \vec{E} = 0 \implies \rho = 0 \\ \vec{D} = \varepsilon_0 \varepsilon_r \vec{E} \implies \nabla \cdot \vec{D} = 0 \implies \rho_f = 0 \end{cases} \implies \rho_b = 0$$

Also we have:

$$\nabla \times \vec{E} = 0$$

Then, we use this result to solve the uniqueness potential problem (this is solve the laplacian equation according to the symmetry of the problem):

$$\begin{cases} \nabla \cdot \vec{E} = 0 \\ \nabla \times \vec{E} = 0 \end{cases} \quad and \quad \nabla^2 V = 0$$

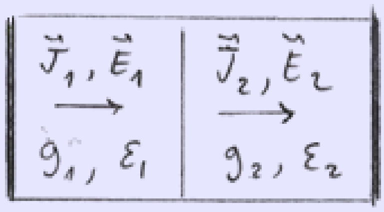
Then, we will find the magnitude: <u>Resistor</u>:

$$R = \frac{V}{I} \begin{cases} V = -\int \vec{E} \cdot d\vec{l} \\ I = \iint_S \vec{J} \cdot \hat{n} da = \iint_S g\vec{E} \cdot \hat{n} da \end{cases} \longleftrightarrow G = R^{-1} = \frac{I}{V}$$

Planar slab	**Cylinder**	**Sphere**
$R = \frac{l}{gA}$	$R = \rho \frac{L}{A}$	$R = \int_{R_1}^{R_2} \rho \frac{dr}{4\pi r^2}$

Piece-wise homogeneous medium in steady state

As we have already done in previous sections, we attend to two symmetries to study different kinds of problems:



We use the proper boundary condition:

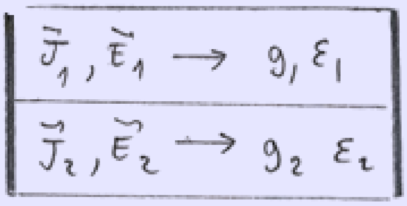
$$\nabla \cdot \vec{J} = 0 \rightarrow \vec{J}_1 = \vec{J}_2 = \vec{J}$$

Then, we use the definition $\vec{J} = g\vec{E}$ and find:

$$\begin{cases} \vec{E}_1 = \frac{\vec{J}}{g_1} \\ \vec{E}_2 = \frac{\vec{J}}{g_2} \end{cases}$$

Now, the target will be to find the resistor, so we will need to compute V and I separately:

$$\begin{cases} V = -\int \vec{E} \cdot \vec{dl} \\ I = \iint_S \vec{J} \cdot \hat{n} da \end{cases} \implies R = \frac{V}{I}$$



The proper boundary condition:

$$\nabla \times \vec{E} = 0 \rightarrow \vec{E}_1 = \vec{E}_2 = \vec{E}$$

And using the definition $\vec{J} = g\vec{E}$ we will find:

$$\begin{cases} \vec{J}_1 = g\vec{E}_1 \\ \vec{J}_2 = g\vec{E}_2 \end{cases}$$

Since we eventually want to compute the resistor we will need to find separately V and I:

$$\begin{cases} V = -\int \vec{E} \cdot \vec{dl} \\ I = \iint_S \vec{J} \cdot \hat{n} da \end{cases} \implies R = \frac{V}{I}$$

For non-linear medium ($\varepsilon_r \neq 1$)

Then we will have:

$$\vec{D}_i = \varepsilon_o \varepsilon_{r_i} \vec{E}_i \implies \sigma_f = \hat{n}_2 \cdot \vec{D}_2 + \hat{n}_1 \cdot \vec{D}_1 \quad (= 0 \text{ in second case})$$

So on, we proceed like before:

$$\vec{P}_i = \varepsilon_o \chi_e \vec{E} = \varepsilon_o(\varepsilon_r - 1)\vec{E}_i \implies \sigma_b = -(\hat{n}_2 \cdot \vec{P}_2 + \hat{n}_1 \cdot \vec{P}_1) \quad (= 0 \text{ in second case})$$

So, for the total expression:

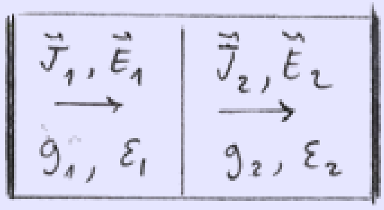
$$\sigma = \sigma_f + \sigma_b = \hat{n}_2 \cdot \vec{E}_2 + \hat{n}_1 \cdot \vec{E}_1 \quad (= 0 \text{ in second case})$$

Nonhomogenius media

We call nonhomogenius media to expressions like:

$$\vec{J} = g(\vec{r})\vec{E}$$

In this cases, we proceed just the same way we did with the electric case:



Fro the bounder condition:

$$\nabla \cdot \vec{J} = 0$$

Obtain J using Gauss' law:

$$I_f = J_f S$$

Then, use the definition, with the previous result:

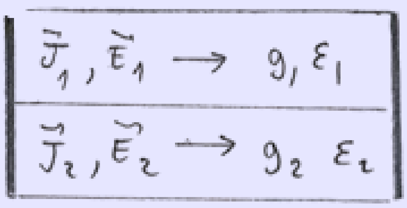
$$\vec{E} = \frac{\vec{J}}{g(r)}$$

Finally find ΔV by just integrating the electric field (the elimination of J's might help a lot!):

$$\Delta V = -\int \vec{E} \cdot \vec{dl}$$

<u>Note:</u> notice there is a discontinuity in E and D fields by just computing:

$$\begin{cases} \nabla \cdot \vec{E} = \rho \\ \nabla \cdot \vec{D} = \rho_f \end{cases}$$



The proper boundary condition is:

$$\nabla \times \vec{E} = 0$$

Then, write E ($\vec{J} = g(r)\vec{E}$) attending only on the symmetry of J.
Compute ΔV by performing the proper integral.
For I_f expression we only need to use Gauss' law, since:

$$I_f = J_f S = \iint \vec{J} \cdot \hat{n}(da)$$

Problems on B-field due to currents and magnets

This is only valid for the static case.

B-field and vector potential

Biot-Savart's law

$$\vec{B} = \frac{\mu_o}{4\pi}\int_V \vec{J} \times \frac{\vec{R}}{R^3} d\tau, \text{being } \vec{R} = \vec{r} - \vec{r}'$$

Then, from here we derive Ampère's law, since:

Ampère's law

$$\nabla \times \vec{B} = \mu_0 \vec{J} \implies \oint \vec{B} \cdot \vec{dl} = \mu_0 I_{enc}$$

And also we can derive the definition of scalar and vector potentials, since:

$$\nabla \cdot \vec{B} = 0 \implies \vec{B} = \nabla \times A \iff \nabla \cdot A = 0 \implies \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Then, from here we get:

$$\vec{A} = \frac{\mu_0}{4\pi}\int \vec{J} \cdot \frac{1}{R} d\tau$$

Note: See the analogy with magnetic field is present in this case with A (magnetic) and V (electric).

Magnets

From the general equation:

$$\vec{B} = \mu_o(\vec{H} + \vec{M}) \to \nabla \times B = \mu_0(\vec{J_f} + \vec{J_e})$$

where we have used the definition of current densities for free charges and magnetic ones (equivalent to bound ones).

This last current densities due to the M-field contributes at creating a B-field which could be determined by applying Biot-Savart to the currents.

$$\begin{cases} \vec{J_e} = \nabla \times \vec{M} \\ \vec{K_e} = \vec{M} \times \hat{n}|_S \end{cases}$$

Linear media

$$\vec{M} = \chi_m \vec{H} \to \vec{B} = \mu_0(1 + \chi_m)\vec{H} = \mu_0 \mu_r \vec{H}$$

Auxiliary H-field

We could have noticed that there will be, eventually, two kind of contributions to the H-field, the one produced by the same "magnetic charges" and another one due to the current densities produced, then:

$$\vec{H} = \vec{H}_{J_f} + \vec{H}_{\rho_m}$$

where:

$$\vec{H}_{J_f} = \frac{1}{\mu}\nabla \times \vec{A} \implies \nabla \cdot \vec{H}_{J_f} = 0 \text{ (solenoidal)}$$

$$\vec{H}_{\rho_m} = -\nabla V_m \implies \nabla \times \vec{H}_{\rho_m} = 0 \text{ irrotational}$$

This makes then, two kinds of discontinuities in H-field:

$$\begin{cases} \nabla \times \vec{H} = \vec{J}_f \\ \nabla \cdot \vec{H} = \rho_m \end{cases}$$

Useful trick

Since:

$$\nabla \cdot \vec{B} = 0 \implies \nabla \cdot \vec{H} = -\nabla \cdot \vec{M} = \rho_m$$

1. Obtain magnetic pole densities (volumetric and surface)
2. Obtain H due to ρ_m as it was due to ρ
3. Add, finally, $\vec{H}_{J_f}$ if $\vec{J}_f \neq 0$

Boundary condition

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{H} = \vec{K}_f$$

$$\nabla \times \vec{B} = \mu_0(\vec{J}_f + \vec{J}_e)$$

Magnetic dipole moment

Like we did in the electrical counterpart;

$$\vec{A} = \frac{\mu_0}{4\pi r^2}\vec{m} \times \hat{r}$$

where

$$\vec{m} = I \iint_S \vec{da} = IA\hat{n}$$

And this magnetic dipole moment produces a magnetic field:

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0}{4\pi r^3}[3(\vec{m}\cdot\hat{r})\hat{r} - \vec{m}]$$

or

$$\vec{B} = -\mu_0 \nabla V_{\text{mag}} \text{ with } V_{\text{mag}} = \frac{\vec{m}\cdot\hat{r}}{4\pi r^2} \tag{1}$$

And the, the proper magnitudes as: `Energy:`

$$u_m = -\vec{m}\cdot\vec{B}_{ext}$$

`Torque:`

$$\Gamma = \vec{m} \times \vec{B}_{ext}$$

`Force:`

$$\vec{F} = \vec{m}\cdot(\nabla \cdot \vec{B})$$

Magnetic circuits

Let a circuit be made of different segments (length (l), cross section (A), conductivity (g)).

Electric circuits	Magnetic circuits
EMF $\xi = \int \vec{E} \cdot \vec{dl} = -\dfrac{d\Phi}{dt}$	MMF $\mathcal{M} = \int \vec{H} \cdot \vec{dl} = NI = ML$ (NI coil, ML magnet)
CURRENT $I_i = \iint_S \vec{J_i} \cdot \vec{da}$, with $\vec{J_i} = g_i \vec{E_i}$	FLUX $\Phi_i = \iint_S \vec{B_i} \cdot \vec{da}$, with $\vec{B_i} = \mu_i \vec{H_i}$
Assuming $\vec{J}, \vec{E}, \vec{B}$ and $\vec{H}$ are uniform along the cross section	
$I_i = J_i A_i = g_i A_i E_i$	$\Phi_i = B_i A_i = \mu_i A_i H_i$
Zero-leakage approximation: I and Φ are constant along the circuit	
$E_i = \dfrac{I}{g_i A_i}$	$H_i = \dfrac{\Phi}{\mu_i A_i}$
$\xi = \int \vec{E} \cdot d\vec{\ell} = \sum_i E_i \ell_i = I \sum_i \dfrac{\ell_i}{g_i A_i}$	$\mathcal{M} = \int \vec{H} \cdot d\vec{\ell} = \sum_i H_i \ell_i = \Phi \sum_i \dfrac{\ell_i}{\mu_i A_i}$
Ohm's law: $\xi = RI$	Hopkinson's law: $\mathcal{M} = \mathcal{R}\Phi$
$R = \sum_i R_i$ with $R_i = \dfrac{\ell_i}{g_i A_i}$	$\mathcal{R} = \sum_i \mathcal{R}_i$ with $\mathcal{R}_i = \dfrac{\ell_i}{\mu_i A_i}$
	or in a magnet: $\mathcal{R}_m = \dfrac{\ell_m}{\mu_0 A_m}$

Table 5: Comparison between Electric and Magnetic circuits

Problems with superconductors

Superconductors are non-magnetic materials with the following properties:

- They are perfect conductors: $g \to \infty$
- They fulfill:

$$\vec{B}_{in} = 0 \quad (\text{since } \vec{M} = 0 \implies \vec{H}_{in} = 0)$$

under conditions of $T < T_c$ and $H < H_c$, with:

$$H_c(T) \approx H_c(0) \left[1 - \left(\frac{T}{T_c} \right)^2 \right]$$

Although they are non-magnetic, we will study them as they were <u>perfectly diamagnetic linear media</u>:

$$\vec{B} = \mu \vec{H} = 0 \implies \mu_r = 1 + \chi_m = 0 \implies \chi_m = -1$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = 0 \implies \vec{M} = \chi_m \vec{H} = -\vec{H}, \text{inside the equivalent diamagnetic}$$

And, obviously, a magnetic field implies the production of electric field, so <u>superconducting currents</u> are produced:

$$\vec{K}_{f_{sc}} = \frac{1}{\mu_0} \hat{n} \times \vec{B}|_S$$

which are free currents on the surface of the superconductor (however, volumetric density is zero since there is no magnetic field inside the superconductor).

Physical behavior under external field

When an external field is applied to a SC, then a free current sheet appears such that the induced field generated by the currents cancels out the external field, to obtain $\vec{B}_{in} = 0$.

$$\vec{K}_{f_{sc}} = \frac{1}{\mu_0} \hat{n} \times \vec{B}|_S = \hat{n} \times \vec{H}|_S = -\hat{n} \times \vec{M}|_S = \vec{K}_{eq}$$

<u>Note:</u>

- Inside the SC, we have $\vec{B} = 0$ and $\vec{H} = 0$
- Inside the equivalent perfectly diamagnetic medium, we have $\vec{B} = 0$ but $\vec{H} = -\vec{M} = 0$
- Oustide the both problems, we have $\vec{B}$ and $\vec{H}$ are the same in both problems.

5 Time-varying fields and Maxwell's equations

*A time varying magnetic field generates an electric field (**induction**) and a time-varying electric field generates a magnetic field (**displacement currents**)*

5.1 Lorentz force and emf force

Lorentz stated a formula for the magnetic force:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \Longleftrightarrow \vec{f} := \frac{\vec{F}}{q} = \vec{E} + \vec{v} \times \vec{B}$$

In a simmilar way, we can also define a density (volumetric):

$$\vec{f}(t,\vec{r}) = \rho(t,\vec{r})[\vec{E}(t,\vec{r}) + \vec{v}_{drift}(t,\vec{r}) \times \vec{B}(t,\vec{r})] = \rho\vec{E} + \vec{J} \times \vec{B}$$

Hence, we define the *electromotive force* as:

$$\mathcal{E} = \int \vec{f} \cdot \vec{dl} = \cdots = \mathcal{E}_{electric} + \mathcal{E}_{magnetic}$$

Then, it might makes us think there is two different ways to generate current as mentioned in the following two chapters.

5.2 Magnetic emf

Is usually generated by a generator, then:

$$\vec{F}_m = q\vec{v}_{charge} \times \vec{B} = q(\vec{v}_{drift} + \vec{v}_{wire}) \times \vec{B}$$

where $\vec{v}_{drift}$ is just the speed of charge with respect to the wire.

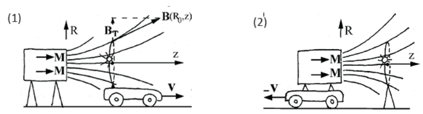
A very important result is the fact that the magnetic electromotive force is **non-conservative** (the line integral $\oint \vec{f}_m \cdot \vec{dl} \neq 0$).
Later on, to determine which way a magnetic emf goes we essentially think:

- Find out the direction of the magnetic force
- Recall Lenz's law:

Lenz's law:

The current generated by a magnetic emf always flows so as to oppose the change in external flux.

5.3 Faraday's law



This two situations are the same but in a different reference frame:

1. The charges in the loop are moving in a static B-field and thus subject to the Lorentz force, which gives a non-zero electromotive force.

2. There should be the same electromotive force as in the previous equivalent experiment.

This leads to modify the field equations for E to introduce this corrections:

$$\oint \vec{E} \cdot \vec{dl} = \int (\nabla \times \vec{E}) \cdot \vec{dl} = -\frac{d}{dt} \int \vec{B} \cdot \vec{da} = -\frac{d\Phi_B}{dt}$$

Then we can state Faraday's law (notice it works no matter the geometry):

Faraday's law:

$$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$$

A changing magnetic field produces an electric field.

Some implications of this law are shown in the following reasoning ($\vec{B} = \vec{\nabla} \times \vec{A}$):

$$\nabla \times \left(\vec{E} + \partial_t \vec{A}\right) = 0 \implies \nabla \times (-\nabla V) = 0$$

The last equality implies that that thing in brackets is a conservative field ($\nabla \times G = 0 \implies G$ is conservative)

- is no longer conservative, since:
$$\vec{E} = -\nabla V - \partial_t \vec{A}$$

- In the presence of time-variations, the energy U = q V associated with electrical interactions e.g. inside a battery, goes not only in the creation of electrical fields, but also of magnetic fields.

5.4 Self-inductance

This concept of Self-inductance ismathematically expressed as:

$$\Phi_B = \int \vec{B} \cdot \vec{da} = \frac{\mu_0 I}{4\pi} \int \left(\int \frac{\vec{dl} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \right) \cdot \vec{da} := LI$$

Then, it allows us to write:

$$\mathcal{E} = -L\frac{dI}{dt}$$

Application: AC current in a sollenoid

Given the setup above, we start finding which is the H-field produced in all points:

$$\vec{H} = \begin{cases} \vec{H}_{in} = \frac{NI(t)}{l} = \frac{NI_0}{l} \sin(wt)\hat{z} \\ \vec{H}_{out} = 0 \end{cases}$$

Then, under the infinite wire approximation, magnetic poles do not contribute in creating a B-field inside, so:

$$\vec{B} = \begin{cases} \vec{B}_{in} = \mu \frac{NI_0}{l} \sin(wt)\hat{z} \\ \vec{B}_{out} = 0 \end{cases}$$

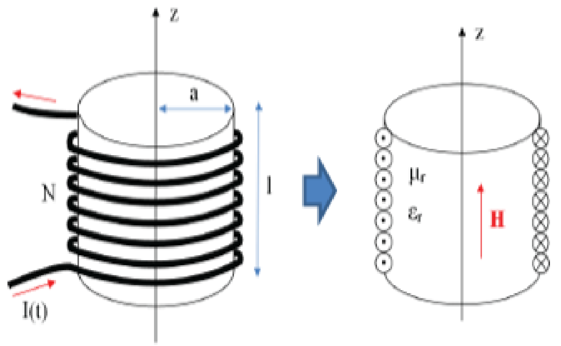


Figure 35: AC current in a sollenoid

Then, we could find the electric field produced by this magnetic field, since we notice it is a time-varying field. Using the analogy between Faraday's and Ampère's law, we notice E-field culs just like magnetic field curls around a current.

$$\vec{E} = \begin{cases} E_\phi^{in} \cdot (2\pi s) = \iint -\frac{\partial B}{\partial t} \cdot da = -\partial_t B \pi s^2 \\ \implies E_\phi = -\frac{\mu N I_0 \omega}{2l} \cos(\omega t) s \\ E_\phi^{out} \cdot (2\pi s) = \iint -\frac{\partial B}{\partial t} \cdot da = -\partial_t B \pi a^2 \\ \implies E_\phi = -\frac{\mu N I_0 \omega a^2}{2l} \frac{\cos(\omega t)}{s} \end{cases}$$

Then, for the case of the bounder ($s = a$), then we get what we were looking for:

$$\xi_{loop} = -\frac{d\Phi}{dt} = -\pi a^2 \mu \frac{N}{l} \frac{dI}{dt} = -L_{loop} \frac{dI}{dt}$$

5.5 Energy stored in magnetic fields

We start calculating the power supplied by a battery which is $\wp = -\Delta V \cdot I$. In fact, this can be written by considering current densities $J_f = IdS$ and an infinitessimal element of potential $dV = (\nabla V) \cdot dl$, then, the resulting expression will just be:

$$d\wp = -\nabla V \cdot J_f d\tau$$

Recalling Faraday's law:

$$-\nabla \vec{V} = \vec{E} + \partial_t \vec{A}$$

Then, we write the expression above as two different parts:

$$d\wp = \vec{E} \cdot \vec{J}_f d\tau + \partial_t \vec{A} \cdot \vec{J}_f d\tau \implies \int d\tau \vec{E} \cdot \vec{J}_f + \int d\tau \partial_t \vec{A} \cdot \vec{J}_f := \wp_{charges + \frac{dU_{magnetic}}{dt}}$$

1. First term is mechanical power absorbed by the charges, which is finally dissipated as Joule heat.

2. Second term is the magnetic energy stored in themagnetic field. In fact, we can get an easier expression by just:

$$\int d\tau \partial_t \vec{A} \cdot \vec{J}_f = \int (d\tau) \partial_t \vec{A} \cdot (\nabla \times \vec{H}) = \int (d\tau) \nabla \cdot (\vec{H} \times \partial_t \vec{A}) + \int (d\tau) \vec{H} \cdot (\vec{\nabla} \times \partial_t \vec{A}) = \int (d\tau) \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

since $\int (da) \cdot (\vec{H} \times \partial_t \vec{A}) = 0$. Hence, using the expression in leniar media: $\vec{B} = \mu \vec{H}$

$$\frac{dU_m}{dt} = \frac{d}{dt} \int (d\tau) \frac{B^2}{2\mu} := \frac{d}{dt} \int (d\tau) u_m \implies u_m = \frac{B^2}{2\mu_0 \mu_r} = \frac{1}{2} \vec{H} \cdot \vec{B}$$

With this last equation, we can rewrite the magnetic energy as the integral:

$$U_m = \frac{1}{2}\int (d\tau)\vec{H}\cdot\vec{B} = \cdots = \frac{1}{2}\int (d\tau)\vec{A}\cdot\vec{J}_f$$

5.5.1 Particular case: loop

For the particular case of a loop:

$$U_m = \frac{1}{2}\int (d\tau)\vec{J}_f\cdot\vec{A} = ... = \frac{1}{2}I\Phi_B$$

Other equivalent expressions built from the expression $\Phi_B = LI$:

$$U_m = \frac{1}{2}I\Phi_B = \frac{1}{2}LI^2 = \frac{\Phi_B^2}{2L}$$

Summary of section 5

Problems on Faraday's law and emf

Faraday's law

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} \rightarrow \oint \vec{E} \cdot \vec{dl} = -\iint_S \frac{d\vec{B}}{dt} \cdot \vec{da}$$

Then, from this definition, we define the electromotive force as:

$$\xi = \frac{1}{q} \oint \vec{F} \cdot \vec{dl}$$

if F is the Lorentz force, then this definition leads to:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

$$\xi = \oint \vec{E} \cdot \vec{dl} + \oint (\vec{v} \times \vec{B}) \cdot \vec{dl} = \cdots = -\iint_S \frac{d\vec{B}}{dt} \cdot \vec{da} - \oint \vec{B} \cdot \frac{\vec{da}}{dt}$$

where the first term corresponds to the **INDUCTION**.

$$\xi = -\frac{d}{dt}\Phi_B, \quad \Phi_B := \iint_S \vec{B} \cdot \vec{da}$$

The EMF is due to the magnetic flux (INDUCTION) across a loop changing with time, and generates, inherently, a current given by Ohm's law (assuming the loop has a resistivity), such that $\xi = IR$.

Lenz's law

The sense of I is such that $\vec{B}_{ind}$ due to I is just opposite to $\vec{B}$ if $\frac{d\Phi_B}{dt} > 0$ or as $\vec{B}$ if $\frac{d\Phi_B}{dt} < 0$

Note: The force that $\vec{B}$ exerts on moving I, $\vec{F} = \int (I \times \vec{B} dl)$ is always a **Breaking force**, meaning is opposite to the movement (friction-like force).

Problems on magnetic energy and force

Potentials V and A

We can define an scalar potential:

$$\begin{cases} \vec{E} = -\nabla V - \frac{d}{dt}\vec{A} \\ \vec{B} = \nabla \times \vec{A} \end{cases} \implies \Phi_B = \oint \vec{A} \cdot \vec{dl}$$

Magnetic energy

In a similar way than in the electrical counterpart:

$$u_m = \frac{1}{2}\int_V \vec{J}_f \cdot \vec{A} d\tau \rightarrow \frac{1}{2}\int_V \vec{H} \cdot \vec{B} d\tau = \frac{1}{2}\int_V \mu |\vec{H}|^2 d\tau$$

Inductance

Then, we define the self-inductance as "something" proportional to the current I:

$$\Phi_B = \iint_S \vec{B} \cdot \vec{da} = LI$$

which then, allows us to rewrite the magnetic energy as:

$$u_m = \cdots = \frac{1}{2} I \Phi_B = \frac{1}{2} L I^2 = \frac{1}{2} \frac{\Phi_B^2}{L}$$

Reluctance

Reluctance is an analogous to the resistance in electrical circuits so, a kind-of-Ohm's law is the following:

$$\mathcal{M} = \mathcal{R} \Phi_B$$

, where $\mathcal{M}$ is the magnetomotive force (MMF).

If there is only one coil:

$$L = \frac{\Phi_B}{I} = \frac{N\Phi_{core}}{I} = \frac{N\mathcal{M}}{I\mathcal{R}} = \frac{N(NI)}{I\mathcal{R}} = \frac{N^2}{\mathcal{R}}$$

Then, energy is rewritten as:

$$u_m = ... = \frac{1}{2} \frac{\mathcal{M}^2}{\mathcal{R}}$$

Magnetic force

Then, since we have a potential, we can define a force coming from this potential as:

$$\vec{F} = -\frac{du_m}{dx}|_{\Phi_B = cnst} = -\frac{1}{2} \Phi_{core}^2 \frac{d\mathcal{R}}{dx}$$

we can notice the force tends to decrease reluctance or decrease x so, this is close the gap.

Problems on displacement current and Maxwell's equations

Macroscopic Maxwell's equations

$$
\begin{aligned}
\nabla \times \vec{H} &= \vec{J}_f + \frac{\partial \vec{D}}{\partial t} := \vec{J}_f + \vec{J}_d && \text{(Ampère-Maxwell law)} \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} && \text{(Faraday's law)} \\
\nabla \cdot \vec{D} &= \rho_f && \text{(Gauss's law)} \\
\nabla \cdot \vec{B} &= 0 && \text{(No magnetic monopoles)}
\end{aligned}
$$

with

$$\vec{D} = \begin{cases} \varepsilon_0 \vec{E} + \vec{P} \\ \varepsilon \vec{E} \end{cases}$$

$$\vec{B} = \begin{cases} \mu_0 (\vec{H} + \vec{M}) \\ \mu \vec{H} \end{cases}$$

Displacement current density

$$\vec{J}_d = \frac{d\vec{D}}{dt} = \begin{cases} \varepsilon_0 \partial_t \vec{E} + \partial_t \vec{P} \\ \varepsilon \partial_t \vec{E} \end{cases}$$

Sources of B

$$\begin{cases} \nabla \times \vec{H} = \vec{J}_f + \varepsilon_0 \partial_t \vec{E} + \partial_t \vec{P} \\ \vec{B} = \mu_0 (\vec{H} + \vec{M}) \end{cases} \implies \nabla \times \vec{H} = \vec{J}_f + \vec{J}_e + \vec{J}_d + \vec{J}_b$$

Poynting's theorem

$$-\frac{d}{dt} \iiint_V \left(\frac{1}{2} \vec{E} \cdot \vec{D} + \frac{1}{2} \vec{H} \cdot \vec{B} \right) d\tau = \iiint_V (\vec{E} \cdot \vec{J})\, d\tau + \iint_S (\vec{E} \times \vec{H}) \cdot d\vec{a}$$

energy density *Joule dissipated* *Power exiting volume*

Poynting's vector

$$\vec{S} = \vec{E} \times \vec{H}$$

Poynting's vector indicates energy flux density ("power density") that exits a volume V.