This document contains the reorganized notes with Esteve's meeting at 10-02-26.

Definition of the Bare-bones Simulation.

We are going to define the bare-bones simulation. The objective of this is to build an engine which does not take into account *real data* but only all the concepts that are going to interact and test if the theoretical idea makes sense.

Notation

To not lose hours and sanity, I will specify the notation to be used in all the document here. If it's in anyway unorthodox, it's because I asked to an LLM to give me an idea.

- User from the set of users: $u \in U$, and $|U| = N$.
- Post (item) from the set of posts (item): $i \in I$, and $|I| = M$.

A user has several actions to perform in the simulation. We denote the set of actions as $\mathcal{A} = \{\varnothing, \text{like}, \text{comment}, \text{repost}, \text{quote}\} = \{\varnothing, l, c, r, q\}$, and we connect the user and the item with the following notation:

$$a_{u,i}^{(t)} \in \mathcal{A}$$

As a user can perform more than one option over a post, we must denote the action as a vector:

$$\boldsymbol{y}_{u,i} \in \{0,1\}^{|\mathcal{A}|\ -1}$$

where each index corresponds to an action type with the order Like, Comment Repost and Quote. E.g, $y = [1, 0, 1, 0]$ would be a like and a repost.

How, we have to distinguish between several traces, one for what the user has seen, another for what the user has done, and the last one for what has happened to the post.

- Impression history of a user: $H_u^{\text{imp}}(t) = \varepsilon_u(t) = (i_1, i_2, ..., i_k)$ where the item i_k is the last item smaller than
- User historic activity: $H_u^{\text{act}}(t) = \mathcal{H}_u(t) = \{(i, a, \tau) : a \neq \varnothing, \tau < t\}$
- Item trajectory: $T_i(t) = \{(u, \text{repost}, \tau) \mid \tau < t\}$. That is a list with all the users who have reposted at given time time τ.

Axioms

This lists what the simulations assumes (several simplifications) in order to simplify the implementations. This are not immutable, some of them will be torn down in more advanced versions of the simulation.

1. User/Agent Homogeneity: every user is indistinguishable from the other users, they behave absolutely the same, that is, they have the exact same decision policy π.

$$\forall u_i, u_j \in U : \pi_{u_i} = \pi_{u_j} = \pi$$

2. Action Independence: The agent (user) is memoryless regarding past actions:

$$\mathbb{P}\big(a_{u,i}^t \mid \mathcal{H}_u\big) = \mathbb{P}\big(a_{u,i}^t\big)$$

3. Stable User Population: no new users are added in the simulation duration.
4. Stable Post Population: no new posts are going to be created during the simulation duration.

5. Algorithm: chronologically followers recommendation.

Algorithm *Pseudo* implementation

There must be a list with all the posts, and then each user has a mean-heap with a index (or a pointer) to the post the user has to see (the oldest one).

Each user must have both which posts has he written, which posts has he interacted and what action did the user performed (well, that's the trace)

Regarding the time between actions of the user, we will assume a exponential distribution, such as an interarrival time.

Due to axiom 1, user will have all the same weights π, but which action is performed is a weighted probability (uniform from zero to one and in which interval falls)

Algorithm evaluation using Markov properties

Social networks are path-dependent (user action depends on past actions performed in different posts), we can formalize the Simulation Engine as a Markov Chain to prove convergence.

Markov chains are usually characterized as *memoryless* process, meaning that the probability of going to the next state does not depend on previous states:

$$\mathbb{P}(s_j \mid s_1, ..., s_{j-1}, s_j) = \mathbb{P}(s_j \mid s_{j-1})$$

This does not behave at all as a social network nor is not the statement of the first axiom. The tricky fact is that we can characterize the user history of impressions $\mathcal{H}_u$ and the item states $\mathcal{C}_i$ as a Markov chain with the memoryless property by defining the state as S:

$$\mathcal{S}_t = \left\{\mathcal{H}^t_{u_1}, ..., \mathcal{H}^t_{u_N}\right\} \cup \left\{\mathcal{C}^t_{i_1}, ..., \mathcal{C}^t_{i_M}\right\}$$

Given n the number of posts seen until now, we can consider all the posts until that point, so we can express the memoryless like this:

$$\mathbb{P}(\mathcal{S}_n \mid \mathcal{S}_{n-1}, ..., \mathcal{S}_1) = \mathbb{P}(\mathcal{S}_n \mid \mathcal{S}_{n-1})$$

Considering this fact gives us all the nice Markov chain properties, such as, giving an explicit probability distribution of changing between states, our simulation will converge to the stationary distribution π of the Markov Chain. Therefore, if the code produces a distribution of interactions that matches π then the code is bug-free. Let's narrow it down with an example.

A user i has it's policy defined by π_u, which is essentially a probability distribution which has to add up to 1.

$$\pi_u = (\pi_\varnothing, p_l, p_r, p_q, p_c); \sum \pi_j = 1$$

We can do that by axiom 1, which tells us that every user has the same policy.

Let's invoke the other axioms to guarantee that this is a markov chain. Axiom 2 makes π_u not change with t (what a user has seen depends on the time it has been seen) which makes the system time-homogenous. Axioms of stability (3 and 4) also makes it homogenous. Axiom 5 ensures that all users see all posts of the users they follow, so defines the state transitions, the topology of the chain. A more sophisticated recommender would change that, and it would not be a markov chain anymore. The definition of memoryless given in this section is the final nail in the coffin, the system can be modelized as a markov chain, therefore it will converge to it's analitical distribution.

As a final note, the User Homogeneity axiom is *not needed* for the system to be modelized as a Markov chain, but it's needed to find an analytical expression to test the code against.

As a final final note, axiom 5 *i think* could be rephrased. That is, imagine that user i sees posts $\{1, 2, 3\}$, a simple markov chain. Now, assume user j reposts post 3, and that's why user i is seeing that. Despite this not seeming markovian, i think it still is? State of user i depends on user j, that means that the path (current state $\mathcal{S}_t$) depends on the state of the user j, so it is still a markov chain with a bigger state which encompasses all users of the social network.