Black Swans framework:

1. Log transformation for S&P500 returns:

$P = (p_1, p_2, p_3, \dots, p_m)$ (Price levels S&P 500 Data)

$t \in [2, m]$

$$r_t = \log(p_t) - \log(p_{t-1})$$

$$r_t = \log\left(\frac{p_t}{p_{t-1}}\right)$$

$$r_t = \log\left(1 + \frac{p_t}{p_{t-1}} - 1\right)$$

$$r_t = \log\left(1 + \frac{p_t - p_{t-1}}{p_{t-1}}\right)$$

$$\Rightarrow r_t \simeq \frac{p_t - p_{t-1}}{p_{t-1}}$$

$$R = (r_1, r_2, r_3, \dots, r_{m-1})$$

2. Proof for expected number of days till black swan

$$R = (r_1, r_2, r_3, \dots, r_{n-1})$$

$$\Rightarrow R' = (r_{(1)} \leq r_{(2)} \leq r_{(3)} \leq \dots \leq r_{(T)})$$

$$\Rightarrow \mathcal{E}^- := \{r_{(1)}, r_{(2)}, r_{(3)}, r_{(4)}, r_{(5)}\}$$

$$\Rightarrow \mathcal{E}^+ := \{r_{(T)}, r_{(T-1)}, r_{(T-2)}, r_{(T-3)}, r_{(T-4)}\}$$

$$B := \mathcal{E}^- \cup \mathcal{E}^+$$

$$A_c^- := \{R \leq c\} \quad | \, c \in \mathcal{E}^- \land R : \Omega \to \mathbb{R}$$

$$\Rightarrow p_c^- := P(R \leq c) = F_R(c)$$

$$A_c^+ := \{R \geq c\} \quad | \, c \in \mathcal{E}^+$$

$$\Rightarrow p_c^+ := P(R \geq c) = 1 - P(R < c) = 1 - F_R(c)$$

$$P(T = k) = (1-p)^{k-1} p$$

$$\Rightarrow E[T] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p = \frac{1}{p} \quad | \, p \in p_c^- \cup p_c^+$$

3. Normal - and Lévy distribution fitting:

$$r_r \sim N(\mu, \sigma^2) \wedge r_r \sim L(\alpha, \beta, \mu, \gamma)$$

$$f_N(r \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right)$$

$$\varphi(q) = \begin{cases} \exp\left(i\mu q - \gamma |q|^\alpha \left[1 - i\beta \frac{q}{|q|} \tan\left(\frac{\pi}{2}\alpha\right)\right]\right) & , \alpha \neq 1 \\ \exp\left(i\mu q - \gamma |q|^\alpha \left[1 + i\beta \frac{q}{|q|} \frac{2}{\pi} \ln(|q|)\right]\right) & , \alpha = 1 \end{cases}$$

$$\mathcal{L}(\theta) = \prod_{t=1}^{n-1} f(r_r \mid \theta)$$

$$\ell(\theta) = \sum_{t=1}^{n-1} \log(f(r_r \mid \theta))$$

$$\Rightarrow \hat{\theta}_{MLE} = \arg\max_{\theta} \ell(\theta)$$