Lévy distribution file framework:

1. Log transformation for S&P500 returns:

$P = (p_1, p_2, p_3, \dots, p_m)$ (Price levels S&P 500 Data)

$t \in [2, m]$

$$r_t = \log(p_t) - \log(p_{t-1})$$

$$r_t = \log\left(\frac{p_t}{p_{t-1}}\right)$$

$$r_t = \log\left(1 + \frac{p_t}{p_{t-1}} - 1\right)$$

$$r_t = \log\left(1 + \frac{p_t - p_{t-1}}{p_{t-1}}\right)$$

$$\Rightarrow r_t \simeq \frac{p_t - p_{t-1}}{p_{t-1}}$$

$$R = (r_1, r_2, r_3, \dots, r_{m-1})$$

2. Descriptive statistics:

$$\mu = \frac{1}{|R|} \cdot \sum_{t=1}^{m-1} r_t \quad \text{(Mean)}$$

$$\sigma = \sqrt{\frac{1}{|R|} \cdot \sum_{t=1}^{m-1} (r_t - \mu)^2} \quad \text{(Standartdeviation)}$$

$$\mu_3 = \frac{1}{|R|} \cdot \sum_{t=1}^{m-1} \left(\frac{r_t - \mu}{\sigma}\right)^3 \quad \text{(Skewness)}$$

$$\mu_4 = \frac{1}{|R|} \cdot \sum_{t=1}^{m-1} \left(\frac{r_t - \mu}{\sigma}\right)^4 \quad \text{(Kurtosis)}$$

3. Normal - and Lévy distribution fitting:

$$r_t \sim N(\mu, \sigma^2) \wedge r_t \sim L(\alpha, \beta, \mu, \gamma)$$

$$f_N(r \mid \mu, \sigma) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right)$$

$$\varphi(q) = \begin{cases} \exp\left(i\mu q - \gamma |q|^\alpha \left[1 - i\beta \frac{q}{|q|} \tan\left(\frac{\pi}{2}\alpha\right)\right]\right) & , \alpha \neq 1 \\ \exp\left(i\mu q - \gamma |q|^\alpha \left[1 + i\beta \frac{q}{|q|} \frac{2}{\pi} \ln(|q|)\right]\right) & , \alpha = 1 \end{cases}$$

$$\mathcal{L}(\theta) = \prod_{t=1}^{m-1} f(r_t|\theta)$$

$$\ell(\theta) = \sum_{t=1}^{m-1} \log\left(f(r_t|\theta)\right)$$

$$\Rightarrow \hat{\theta}_{MLE} = \arg\max_{\theta} \ell(\theta)$$

4. Proof for Power-Law behavior of Lévy-stable distributions:

$$\varphi(q) = e^{i\mu q - \gamma|q|^{\alpha}\left[1 - i\beta\frac{q}{|q|}\tan\left(\frac{\pi}{2}\alpha\right)\right]} \qquad |\beta \overset{!}{=} \mu \overset{!}{=} 0$$

$$\varphi(q) = e^{-\gamma|q|^{\alpha}}$$

$$\Rightarrow P_L(r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \varphi(q)\, e^{-iqr}\, dq$$

$$P_L(r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\gamma|q|^{\alpha}} e^{-iqr}\, dr \qquad |e^{-iqr} \overset{!}{=} \cos(qr) - i\sin(qr)$$

$$P_L(r) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\gamma|q|^{\alpha}} \left(\cos(qx) - i\sin(qx)\right) dq \qquad |e^{-\gamma|q|^{\alpha}} \text{ is even}$$

$$\Rightarrow P_L(r) = \frac{1}{\pi}\int_{0}^{+\infty} e^{-\gamma|q|^{\alpha}} \cos(qx)\, dq \qquad |\gamma \overset{!}{=} 1$$

$\Rightarrow$ After series expansion:

$$P_L(|r|) = \frac{1}{\pi}\sum_{k=1}^{m} \frac{(-1)^k}{k!} \frac{\Gamma(\alpha k + 1)}{|x|^{\alpha k + 1}} \sin\left(\frac{k\pi\alpha}{2}\right) + \mathcal{O}\left(|x|^{\alpha(m+1)-1}\right)$$

$$\Rightarrow P_L(|r|) \sim \frac{\Gamma(1+\alpha)\sin\left(\frac{\pi\alpha}{2}\right)}{\pi|r|^{1+\alpha}} \sim |r|^{-(1+\alpha)} \quad \text{(Power law)}$$

5. Proof for exponential behavior of normal distributions:

$$f_N(r|\mu,\sigma) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\exp\left(-\frac{(r-\mu)^2}{2\sigma^2}\right) \qquad |\mu \overset{!}{=} 0 \wedge \sigma \overset{!}{=} 1$$

$$\Rightarrow \quad f(r) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{r^2}{2}\right)$$

$$\Rightarrow \quad P(R > r) = \int_r^{\infty} f(t)\, dt = \int_r^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$$

$$r \to +\infty \quad \int_r^{\infty} e^{-\frac{t^2}{2}}\, dt \sim \frac{e^{-\frac{r^2}{2}}}{r} \sim e^{-\frac{r^2}{2}} \quad \text{(Exponential law)}$$

$$\Rightarrow e^{-\frac{r^2}{2}} \ll r^{-(1+\alpha)}$$