Model description

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A series y_t is decomposed into:

$$y_t = \mu_t + \gamma_t \tag{1}$$

where μ_t gives a trend and γ_t a seasonal component.

Probabilities for states $S_t = 0$ and $S_t = 1$ are governed by

$$S_t^* = \beta_0 + \beta_1 S_{t-1} + \eta_t \qquad \eta_t \sim i.i.d.N(0,1). \tag{2}$$

The Probit model discriminates between regimes whether S_t^* is non-negative. This gives the identities:

$$\begin{aligned} S_t = 0 \ \forall \ \eta_t < -\beta_0 - \beta_1 \\ S_t = 1 \ \forall \ \eta_t \geq -\beta_0 - \beta_1. \end{aligned} \tag{3}$$

The components themselves in 1 are assumed to satisfy:

$$\mu_t = \mu_{t-1} + \nu_{S_t} + \xi_t \qquad \xi_t \sim i.i.d.N(0, \sigma^2_{\xi,S_t}) \tag{4}$$

with

$$\nu_{S_t} = \nu_0 + S_t \nu_1, \tag{5}$$

$$\sigma^2_{\xi,S_t} = \sigma_{\xi,0}(1 - S_t) + \sigma^2_{\xi,1} S_t. \tag{6}$$

γ_t nets out seven-day (-period) seasonality:

$$\gamma_t = \sum_{j=1}^{6} \gamma_{t-j} + \omega_t \qquad \omega_t \sim i.i.d.N(0, \sigma^2_{\omega_t}). \tag{7}$$

The stochastic component ω_t allows the seasonal coefficients to vary and assumes measurement errors. By their nature, ξ_t and ω_s are uncorrelated for all t and s.

Endogeneity between the trend and regime processes arises from the joint density function of all innovations in the system:

$$\begin{bmatrix}\xi_{t,0}\\ \xi_{t,1}\\ \omega_t\\ \eta_t\end{bmatrix} \sim i.i.d.N(0,\Sigma), \qquad \Sigma = \begin{bmatrix}\sigma^2_{\xi,0} & 0 & 0 & \rho_0\\ 0 & \sigma^2_{\xi,1} & 0 & \rho_1\\ 0 & 0 & \sigma^2_\omega & 0\\ \rho_0 & \rho_1 & 0 & 1\end{bmatrix}. \tag{8}$$

Note, that the covariance ρ_{S_t} between innovations to the regime process and trend innovations is itself regime dependent. This is a function of the regime induced heteroskedasticity in 6. If a certain regime innovation triggers a switch from $S_{t-1} = 0$ to $S_t = 1$, $\sigma_{\xi,0}$ follows suit and changes to $\sigma_{\xi,1}$ in t. Since η_t has a constant variance by definition, ρ_0 is forced to change in proportion to $\sigma_{\xi,1}$ and becomes ρ_1. However, the correlation between ξ_{t,S_t} and η_t stays constant and is thus uniquely defined as

$$\varrho = \frac{\rho_0}{\sigma_{\xi,0}} = \frac{\rho_1}{\sigma_{\xi,1}}, \qquad \sigma_{\xi,0}, \sigma_{\xi,1} \neq 0. \tag{9}$$

I originally employed the approach to model log COVID-19 infections for my Bachelor thesis. Similar models were used in Chang, Choi, and Park 2017; Kim, Piger, and Startz 2008; Sinclair 2009; Luginbuhl and Vos 2003 and many more.

In state space form the measurement equation is defined as:

$$\underset{1\times1}{y_t} = \underset{1\times8}{Z}\alpha_t \tag{10}$$

with the transition equation:

$$\underset{8\times1}{\alpha_{t+1}} = \begin{bmatrix}\mu_{t+1}\\ \nu_0\\ \gamma_{t+1}\\ \vdots\\ \gamma_{t+1-5}\end{bmatrix} = \underset{8\times8}{T}\alpha_t + \underset{8\times1}{\lambda_{S_t}} + \underset{8\times3}{R}\underset{3\times1}{u_{t+1}} \qquad u_t \sim N(0, \underset{3\times3}{Q_{S_t}}) \tag{11}$$

where

$$Z = \begin{bmatrix}1 & 0 & 1 & 0 & \cdots & 0\end{bmatrix},$$

$$T = diag\left(\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \begin{bmatrix}-1 & -1 & \cdots & -1 & -1\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & 1 & 0\end{bmatrix}\right),$$

$$\lambda_{S_t} = \begin{bmatrix} S_t \nu_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_{S_t} = \begin{bmatrix} \sigma^2_{\xi, S_t} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma^2_{\omega} \end{bmatrix}.$$

References

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