Model description

February 2022

In accordance with the Unobserved Components approach (I refer to Harvey (1990)), a series y_t is broken down into:

$$y_t = \mu_t + \gamma_t \tag{1}$$

where μ_t gives a trend and γ_t a seasonal component.

Probabilities for the unobserved regimes $S_t = 0$ and $S_t = 1$ are governed by

$$S_t^* = \beta_0 + \beta_1 S_{t-1} + \eta_t \qquad \eta_t \sim i.i.d.N(0,1). \tag{2}$$

The Probit model discriminates between regimes whether S_t^* is non-negative. This mirrors a first order stationary Markov switching process (see Hamilton (1994, ch. 22)) and gives the identities:

$$\begin{aligned} S_t = 0 \;\; \forall \;\; \eta_t < -\beta_0 - \beta_1 S_{t-1} \\ S_t = 1 \;\; \forall \;\; \eta_t \geq -\beta_0 - \beta_1 S_{t-1}. \end{aligned} \tag{3}$$

The trend component in 1 is assumed to satisfy:

$$\mu_t = \mu_{t-1} + \nu_{S_t} + \xi_t \qquad \xi_t \sim i.i.d.N(0, \sigma^2_{\xi,S_t}) \tag{4}$$

with

$$\nu_{S_t} = \nu_0 + S_t \nu_1, \tag{5}$$

$$\sigma^2_{\xi,S_t} = \sigma_{\xi,0}(1 - S_t) + \sigma^2_{\xi,1} S_t. \tag{6}$$

Thus, the trend is modelled as a RW with both innovation variance and drift depending on the current regime. In order to uniquely identify the system it is required to impose $\nu_1 < 0$.

γ_t nets out seven-day (-period) seasonality:

$$\gamma_t = \sum_{j=1}^{6} \gamma_{t-j} + \omega_t \qquad \omega_t \sim i.i.d.N(0, \sigma^2_{\omega_t}). \tag{7}$$

The stochastic component ω_t allows the seasonal coefficients to vary and assumes measurement errors. By their nature, ξ_t and ω_s are uncorrelated for all t and s.

Endogeneity between the trend and regime processes arises from the joint distribution function of all innovations in the system:

$$\begin{bmatrix} \xi_{t,0} \\ \xi_{t,1} \\ \omega_t \\ \eta_t \end{bmatrix} \sim i.i.d.N(0, \Sigma), \qquad \Sigma = \begin{bmatrix} \sigma^2_{\xi,0} & 0 & 0 & \rho_0 \\ 0 & \sigma^2_{\xi,1} & 0 & \rho_1 \\ 0 & 0 & \sigma^2_\omega & 0 \\ \rho_0 & \rho_1 & 0 & 1 \end{bmatrix}. \tag{8}$$

Note, that the covariance ρ_{S_t} between innovations to the regime process and trend innovations is itself regime dependent. This is a function of the regime induced heteroskedasticity in 6. If a certain regime innovation triggers a switch from $S_{t-1} = 0$ to $S_t = 1$, $\sigma_{\xi,0}$ follows suit and changes to $\sigma_{\xi,1}$ in t. Since η_t has a constant variance by definition, ρ_0 is forced to change in proportion to $\sigma_{\xi,1}$ and becomes ρ_1. However, the correlation between ξ_{t,S_t} and η_t stays constant and is thus uniquely defined as

$$\varrho = \frac{\rho_0}{\sigma_{\xi,0}} = \frac{\rho_1}{\sigma_{\xi,1}}, \qquad \sigma_{\xi,0}, \sigma_{\xi,1} \neq 0. \tag{9}$$

I originally employed the approach to model log COVID-19 infections for my Bachelor thesis. An extension of the Kim filter (Kim, 1994) as proposed in Kim et al. (2008) is exploited to feasibly estimate the UC components, regime probabilities and all covariances.

Similar models were used in Chang et al. (2017); Kim et al. (2008); Sinclair (2009); Luginbuhl and de Vos (2003) and many more.

In state space form the measurement equation is defined as:

$$\underset{1\times 1}{y_t} = \underset{1\times 8}{Z}\alpha_t \tag{10}$$

with the transition equation:

$$\underset{8\times 1}{\alpha_{t+1}} = \begin{bmatrix} \mu_{t+1} \\ \nu_0 \\ \gamma_{t+1} \\ \vdots \\ \gamma_{t+1-5} \end{bmatrix} = \underset{8\times 8}{T}\alpha_t + \underset{8\times 1}{\lambda_{S_t}} + \underset{8\times 3}{R}\underset{3\times 1}{u_{t+1}} \qquad u_t \sim N(0, \underset{3\times 3}{Q_{S_t}}) \tag{11}$$

where

$$Z = \begin{bmatrix} 1 & 0 & 1 & 0 & \cdots & 0 \end{bmatrix},$$

$$T = diag\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \right),$$

$$\lambda_{S_t} = \begin{bmatrix} S_t \nu_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_{S_t} = \begin{bmatrix} \sigma^2_{\xi, S_t} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma^2_{\omega} \end{bmatrix}.$$

References

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