Finger formation at the base of ash clouds: A linear stability analysis

Paul Jarvis [1] Jonathon Lemus [1] Allan Fries [1]
Amanda Clarke [2] Jeremy Phillips [3] Costanza Bonadonna [1]

[1]Section of Earth and Environmental Sciences, University of Geneva

[2]School of Earth and Space Exploration, Arizona State University

[3]School of Earth Sciences, University of Bristol

UNIVERSITÉ DE GENÈVE

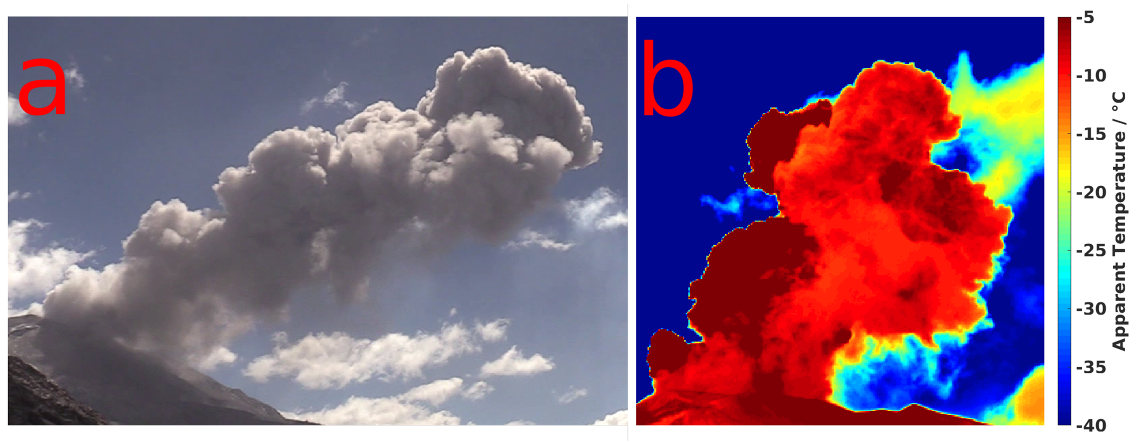

Figure 1. a) Visible and b) infrared images of ash clouds from Vulcanian explosions of Sabancaya volcano, Peru. Fingers can be seen in both images.

Introduction

Volcanic ash presents a hazard for infrastructure and human health. Understanding ash settling is therefore crucial for assessing the associated risk. Downard-propagating fingers(Figure 1), hypothesised to form from a gravitational instability at the base of the ash cloud (Figure 2) have been seen at many volcanoes, and in analogue experiments (**Allan Fries**) and numerical models (**Jonathon Lemus**). Here we present a linear stability analysis, that predicts the initial growth rate of the fingers.

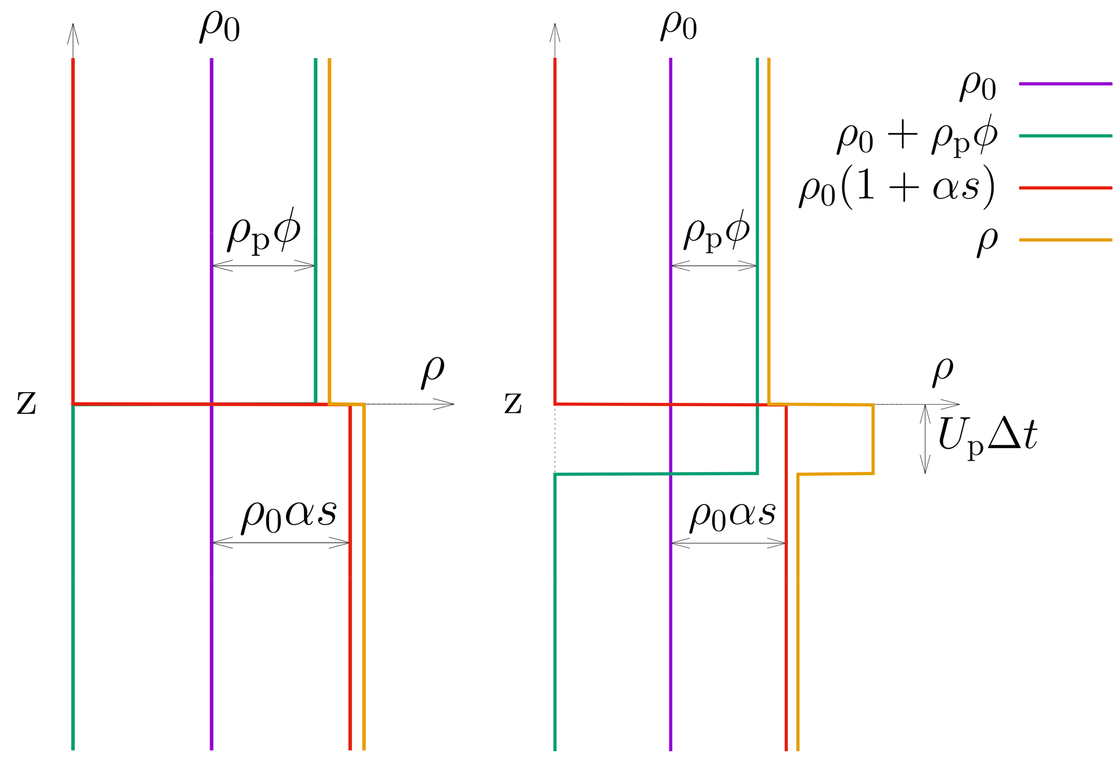

Figure 2. Despite an initial stable configuration, ash settling leads to the formation of a gravitationally unstable **particle boundary layer**.

Equations of motion

Expansion

		Expansion
Conservation of mass	$\nabla\cdot\mathbf{u}(\mathbf{x},t)$	$\mathbf{u}(\mathbf{x},t)=\hat{\mathbf{u}}(z)e^{i(k_x x+k_y y-\omega t)}$
Conservation of momentum	$\frac{\partial\mathbf{u}(\mathbf{x},t)}{\partial t}+[\mathbf{u}(\mathbf{x},t)]\cdot\nabla]\mathbf{u}(\mathbf{x},t)=\frac{\nabla P(\mathbf{x},t)}{\rho_0}+\nu\nabla^2\mathbf{u}(\mathbf{x},t)-g'\hat{\mathbf{z}}$	$P(\mathbf{x},t)=P^{(0)}(\mathbf{x},t)+\hat{P}(z)e^{i(kx-\omega t)}$
Conservation of particles	$\frac{\partial\phi(\mathbf{x},t)}{\partial t}+[\mathbf{u}(\mathbf{x},t)]-\mathbf{U}_p]\cdot\nabla\phi(\mathbf{x},t)=D_p\nabla^2\phi(\mathbf{x},t)$	$\phi(\mathbf{x},t)=\phi^{(0)}(\mathbf{x},t)+\hat{\phi}(z)e^{i(kx-\omega t)}$
Conservation of solute	$\frac{\partial s(\mathbf{x},t)}{\partial t}+\mathbf{u}(\mathbf{x},t)x\cdot\nabla s(\mathbf{x},t)=D_s\nabla^2 s(\mathbf{x},t)$	$s(\mathbf{x},t)=s^{(0)}(\mathbf{x},t)+\hat{s}(z)e^{i(kx-\omega t)}$
Reduced gravity	$g'=g\left([1+\alpha s(\mathbf{x},t)][1-\phi(\mathbf{x},t)]+\frac{\rho_p\phi(\mathbf{x},t)}{\rho_0}\right)$	$\hat{P}\ll P^{(0)},\hat{\phi}\ll\phi^{(0)},\hat{s}\ll s^{(0)}$

Unknowns		Parameters	
$\mathbf{u}(\mathbf{x},t)$	Fluid velocity	ρ_0	Reference fluid density
$P(\mathbf{x},t)$	Pressure	ν	Kinematic viscosity
$\phi(\mathbf{x},t)$	Particle volume	$\mathbf{U}_p$	Particle settling velocity
$s(\mathbf{x},t)$	Solute concentration	ρ_p	Particle density
Variables		D_p	Particle diffusivity
$\mathbf{x}$	Position vector	D_s	Solute diffusivity
t	time	g	Gravity
$\hat{\mathbf{z}}$	Vertical unit vector	α	Expansivity

Base states

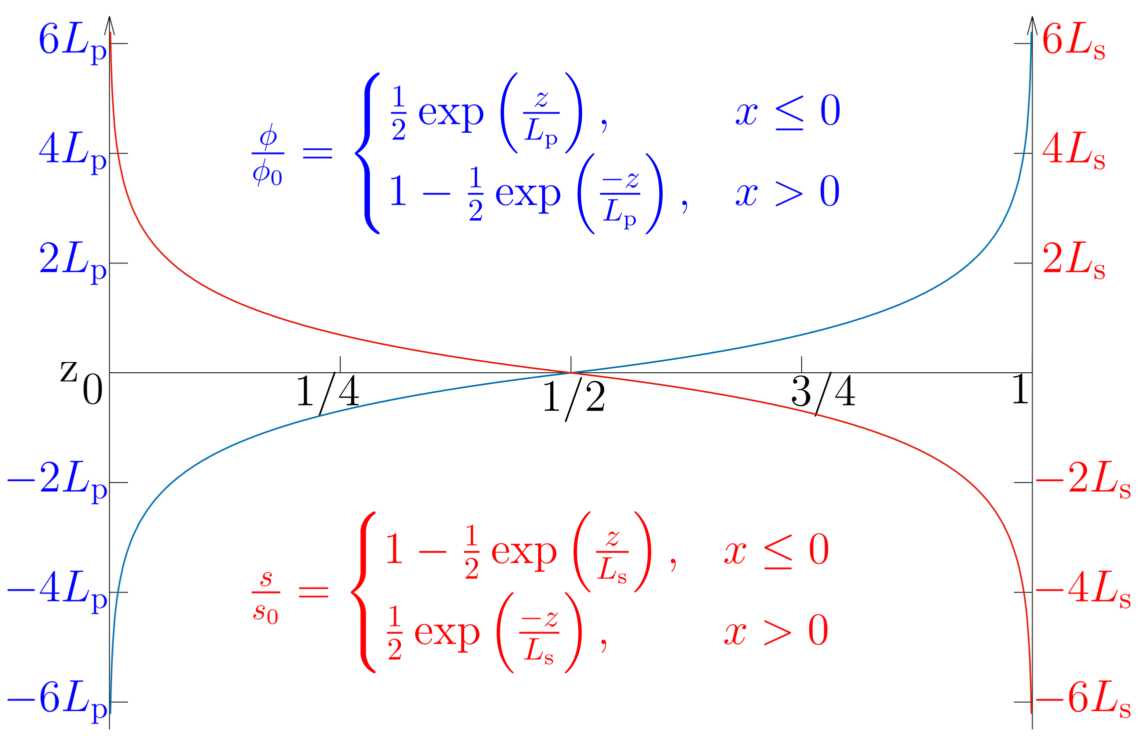

Figure 3. Choice of base states for the particle and solute concentrations. The transitions have length scales L_p and L_s respectively.

Dimensionless parameters

We nondimensionalise the system using the characteristic length and velocity scales

$$U_c=U_p \qquad L_c=\left(\frac{\nu U_p}{g}\right)^{1/2}$$

This gives six dimensionless parameters

$$Re=\frac{U_p^3}{g\nu}, \qquad L_p'=L_p\left(\frac{gU_p}{\nu}\right)^{1/2}$$

$$\beta=\alpha s_0, \qquad \gamma=\frac{\rho_p}{\rho_0}$$

$$Sc_s=\frac{\nu}{D_s}, \qquad Sc_p=\frac{\nu}{D_p}$$

Dimensionless equations

Can express problemas a generalised eigenvalue equation

$$\mathbf{M}\mathbf{e}=\omega\mathbf{Q}\mathbf{e},$$

where

$$\mathbf{M}=\begin{pmatrix}\frac{d^4}{dz'^4}-2k^2\frac{d^2}{dz'^2}+k^4 & k^2\beta & k^2\gamma\\ \frac{\partial s^{(0)}}{\partial z'} & \frac{1}{ReSc_s}\left(k^2-\frac{d^2}{dz'^2}\right) & 0\\ \frac{\partial\phi^{(0)}}{\partial z'} & 0 & \frac{1}{ReSc_p}\left(k^2-\frac{d^2}{dz'^2}\right)-U_p'\frac{d}{dz'}\end{pmatrix},$$

and

$$\mathbf{e}=\begin{pmatrix}\hat{u}_z\\ \hat{s}\\ \hat{\phi}\end{pmatrix}, \qquad \mathbf{Q}=i\omega\begin{pmatrix}\left(\frac{d^2}{k^2-dz'^2}\right) & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Special case: Rayleigh-Taylor instability