# Local Class Field Theory

Kevin Buzzard, Imperial College London

Imperial, 24th Oct 2025

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I'll also be livestreaming on the Xena Discord.

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By the joys of infinite Galois theory, this is the same as proving facts about the *maximal abelian extension* $K^{ab}$ of $K$ (defined for example as the union of all the finite abelian extensions of $K$ within a fixed algebraic closure), and trying to figure out what either what $K^{ab}$ is or what $Gal(K^{ab}/K)$ is.

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The second question is easier than the first, it turns out; the first (Hilbert’s 12th problem) is still largely open for fields like $\mathbb{Q}(2^{1/3})$ where Shimura variety techniques do not apply, but the second was answered 100 years ago.

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Class field theory can be explained and reconceptualised via the Langlands program but that doesn't make the proofs any easier.

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## Theorem

*Let $K$ be a finite field. Then for each positive integer $n$ there is (up to non-unique isomorphism if $n > 1$) exactly one extension $L/K$ of degree $n$; it is Galois, with cyclic Galois group of order $n$, generated by the arithmetic Frobenius element $x \mapsto x^{|K|}$.*

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Proof omitted.

Put another way: if $K$ is finite then $\mathrm{Gal}(K^{ab}/K) = \mathrm{Gal}(K^{sep}/K)$ is canonically isomorphic to $\widehat{\mathbb{Z}}$, the profinite completion of $\mathbb{Z}$, a.k.a. the projective limit $\lim_{n\geq 1} \mathbb{Z}/n\mathbb{Z}$.

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There is no "best" choice of isomorphism; in practice the best choice depends on what you're doing.

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Indeed, historically the next developments in the study of class field theory involved figuring out $\mathrm{Gal}(K^{ab}/K)$ for these fields $K$, with $K = \mathbb{Q}$ going first (the Kronecker-Weber theorem: $\mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) = \widehat{\mathbb{Z}}^\times$, and even the description of $\mathbb{Q}^{ab}$ as $\bigcup_n \mathbb{Q}(\zeta_n)$).

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However, when it comes to absolute Galois groups, $\mathbb{R}$ is much simpler than $\mathbb{Q}$ ($\mathrm{Gal}(\mathbb{R}^{sep}/\mathbb{R})$ has size 2, $\mathrm{Gal}(\mathbb{Q}^{sep}/\mathbb{Q})$ is incomprehensible), and after the discovery of the $p$-adic numbers in 1897 it was realised that their Galois theory was also easier than that of global fields.

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The $\mathbb{R}$ case is easy: $\mathbb{C}^{ab} = \mathbb{C}$, $\mathrm{Gal}(\mathbb{C}^{ab}/\mathbb{C})$ is trivial, $\mathbb{R}^{ab} \cong \mathbb{C}$, and $\mathrm{Gal}(\mathbb{R}^{ab}/\mathbb{R})$ is cyclic of size 2 (note that we do not need to fix a preferred square root of $-1$ in this statement).

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So we'll spend the rest of our time thinking about the nonarchimedean case where $K$ is a finite extension of either $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$.

# Remarks we won't ever need

But just before we leave the $\mathbb{R}$ case let me say some things about why the definition ("finite extension of $\mathbb{Q}_p$ or of $\mathbb{F}_p((t))$ or of $\mathbb{R}$") is so weird.

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Other random facts: $\mathbb{R}$ and $\mathbb{Q}_p$ are rigid; they have no nontrivial field automorphisms (because the algebra determines the topology and $\mathbb{Q}$ is dense). However $\mathbb{F}_p((t))$ has uncountably many (e.g. $t \mapsto t + a_2t^2 + a_3t^3 + \cdots$).

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Every extension of $\mathbb{F}_p((t))$ is isomorphic to $k((t))$ for $k$ a finite field.

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Although it is far less obvious, in the local field case, for any finite abelian extension $L/K$ there's a canonical surjection $K^\times \to \mathrm{Gal}(L/K)$, with kernel $N_{L/K}L^\times$ (the norm map from algebra), and $\mathrm{Gal}(K^{ab}/K)$ is canonically isomorphic to the profinite completion of $K^\times$.

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Like in the finite field case, nobody can decide whether the Artin map or its inverse (precompose with $x \mapsto x^{-1}$ on $K^\times$) is the canonicalest.

Exercise: If $K$ is an *archimedean* local field and $L/K$ is finite Galois, there is also a canonial surjection $K^\times \to \mathrm{Gal}(L/K)$ with kernel $N_{L/K}(L^\times)$.

# Overview of group cohomology

If $G$ is a (finite for us, but it's not necessary) group, acting by group automorphisms on a (typically not finite) abelian group $A$ (we say "$A$ is a $G$-module"), then there are (typically not finite) abelian groups $H^n(G, A)$ and $H_n(G, A)$ associated to this set-up.

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For $n = 0$ they have a direct definition ($H^0(G, A) = A^G$, the maximal $G$-invariant subgroup, and $H_0(G, A) = A_G$, the maximal $G$-invariant quotient) and for $n > 0$ they can either be defined via derived functor nonsense (giving cheap theorems but not way to calculate) or explicitly as $n$-(co)cycles over $n$-(co)boundaries (giving easy calculations but now you have to work to prove the theorems).

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(If $G$ is not finitely-generated then group cohomology is probably "the wrong object", $G$ might well have a topology and one should use continuous cycles etc.)

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Example: a 1-cocycle is $f : G \to A$ such that $f(gh) = f(g) + g \bullet f(h)$ and a 1-coboundary is $f : G \to A$ of the form $g \mapsto g \bullet a - a$ (exercise: check a 1-coboundary is a 1-cocycle; you just proved that $d^2 = 0$).

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Example: a 2-cocycle is $\sigma : G^2 \to A$ such that $g \bullet \sigma(h, k) - \sigma(gh, k) + \sigma(g, hk) - \sigma(g, h) = 0$, so every element of $H^2(G, A)$ can be represented by such a function.

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Example: every $f : G \to_0 A$ is a 1-cocycle if $G$ acts trivially on $A$.

If $G$ acts trivially on $A$ then there's a natural surjective map from the 1-cycles to $G^{ab} \otimes_{\mathbb{Z}} A$, sending `single g a` to $g \otimes a$ and general $f$ to $\sum_{g \in G}(g \otimes f(g))$.

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The kernel is precisely the 1-boundaries.

# Category-theoretic approach

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This nonsense gives you that a short exact sequence $0 \to A \to B \to C \to 0$ of $G$-modules gives rise to long exact sequences $0 \to H^0(G, A) \to H^0(G, B) \to H^0(G, C) \to H^1(G, A) \to H^1(G, B) \to \cdots$ and $\cdots \to H_1(G, C) \to H_0(G, A) \to H_0(G, B) \to H_0(G, C) \to 0$.

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It also gives you the Hochschild–Serre spectral sequence $E_2^{i,j} := H^i(G/N, H^j(N, M)) \implies H^{i+j}(G, M)$ (see Grothendieck Tohoku).

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Idea: there's a map $M_G = H_0(G,M) \to M^G = H^0(G,M)$ sending $m$ to $\sum_{g\in G}(g \bullet m)$; $H^{-1}_{Tate}(G,M)$ is the kernel of this map, and $H^0_{Tate}(G,M)$ is the cokernel.

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Note: it only works for $G$ finite (but we only care about this case anyway).

# The Artin map

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The Artin map $K^\times \to G$ when $G$ is abelian, comes as a consequence, as we'll see today.

# Fundamental classes

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Example: $GL_1$ corresponds to $L = K$ and $G$ trivial and $d = 1$.

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However in June in Oxford, Richard Hill showed me a way to define the Artin map without defining cup products at all.

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To shift the degree by 2, we need to apply this twice.

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We're half way there.

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We can use this data to construct a "splitting module" $split(\tilde{\sigma})$, which is an action of $G$ on $M \times aug(G)$ with $g \bullet (m, f) = (g \bullet m + \sum_{\gamma \in G} f(\gamma)\tilde{\sigma}(g, \gamma), g \bullet f)$.

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The 2-cocycle equation shows that this is an action.
There's a short exact sequence $0 \to M \to split(\tilde{\sigma}) \to aug(G) \to 0$.

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So *if* $split(\sigma)$ has no Tate cohomology, we get an induced isomorphism $H^n_{Tate}(G, aug(G)) \to H^{n+1}_{Tate}(G, M)$ and hence an isomorphism $H^n_{Tate}(G, \mathbb{Z}) = H^{n+2}_{Tate}(G, M)$.

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Of course this will not be true in general – this is general $G$ and $M$.

# The story so far

So far: If we have some cohomology class $\sigma \in H^2(G, M)$ lifting to a 2-cocycle $\tilde{\sigma}$ such that $split(\tilde{\sigma})$ has no Tate cohomology, then for all $n$ we have isomorphisms $H^n_{Tate}(G, \mathbb{Z}) = H^{n+2}_{Tate}(G, M)$.

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A theorem we proved (in the notes, not in Lean) in the Oxford workshop in June is that if it's true that for all finite degree $d$ Galois extensions $L/K$ of nonarchimedean local fields, $H^2(Gal(L/K), L^\times)$ is finite cyclic of order $d$ generated by $\sigma$ which lifts to the 2-cocycle $\tilde{\sigma}$, then $split(\tilde{\sigma})$ has no Tate cohomology.

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Corollary: $H^{-2}_{Tate}(G, \mathbb{Z}) = H^0_{Tate}(G, L^\times)$ if $L/K$ is a finite Galois extension of local fields, and thus $G^{ab} = K^\times / N(L^\times)$, which gives us the Artin map.

# What's left?

A lot of the basics of this story are already formalized in Lean.
We do *not* have the hard theorem that $H^2(Gal(L/K), L^\times)$ is cyclic of size $d = [L : K]$ when $K$ is nonarch local, and we need to work on this (and I didn't even tell you the proof yet).

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But before we start on this, I want to formalize in Lean the construction of the Artin map, assuming that $split(\tilde{\sigma})$ has no Tate cohomology.

# What's left?

A lot of the basics of this story are already formalized in Lean.
We do *not* have the hard theorem that $H^2(Gal(L/K), L^\times)$ is cyclic of size $d = [L : K]$ when $K$ is nonarch local, and we need to work on this (and I didn't even tell you the proof yet).

But before we start on this, I want to formalize in Lean the construction of the Artin map, assuming that $split(\tilde{\sigma})$ has no Tate cohomology.

So let's start there.