

Small Steps Semantics for tiny arithmetic expression grammar

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1 Grammar

We'll define the set of arithmetic expressions as the expressions that can be built from the following grammar:

$$E_A ::= n(\in \mathbb{Z}) \mid v(\in \mathbf{V}) \mid E_A + E_A \mid E_A \times E_A \mid E_A - E_A$$

With \mathbf{V} the set of variables.

2 Small Steps Semantic

Let $\sigma \in \mathcal{V}[\mathbb{Z}]$ ($= \mathcal{F}(\mathbf{V}, \mathbb{Z})$) a valuation.

$$(A_1) \frac{}{\langle n_1 \text{ op } n_2, \sigma \rangle \hookrightarrow \langle n, \sigma \rangle} (n_1, n_2 \in \mathbb{Z}, \text{op} \in \{+, -, \times\}, n = n_1 \text{ op } n_2)$$

$$(A_2) \frac{}{\langle v, \sigma \rangle \hookrightarrow \langle \sigma(v), \sigma \rangle} (v \in \mathbf{V})$$

$$(A_3) \frac{\langle e_1, \sigma \rangle \hookrightarrow \langle e'_1, \sigma \rangle}{\langle e_1 \text{ op } e_2, \sigma \rangle \hookrightarrow \langle e'_1 \text{ op } e_2, \sigma \rangle} (\text{op} \in \{+, -, \times\})$$

$$(A_4) \frac{\langle e_2, \sigma \rangle \hookrightarrow \langle e'_2, \sigma \rangle}{\langle n \text{ op } e_2, \sigma \rangle \hookrightarrow \langle n \text{ op } e'_2, \sigma \rangle} \text{ (op } \in \{+, -, \times\}, n \in \mathbb{Z})$$

$$(A_5) \frac{\langle e, \sigma \rangle \hookrightarrow \langle e', \sigma \rangle}{\langle \text{if } e \text{ then } a \text{ else } b, \sigma \rangle \hookrightarrow \langle \text{if } e' \text{ then } a \text{ else } b, \sigma \rangle} (a, b \in E_A, e \in E_B)$$

$$(A_6) \frac{}{\langle \text{if true then } a \text{ else } b, \sigma \rangle \hookrightarrow \langle a, \sigma \rangle} (a, b \in E_A)$$

$$(A_7) \frac{}{\langle \text{if false then } a \text{ else } b, \sigma \rangle \hookrightarrow \langle b, \sigma \rangle} (a, b \in E_A)$$

$$(B_1) \frac{}{\langle \text{not}(\text{false}), \sigma \rangle \hookrightarrow \langle \text{true}, \sigma \rangle}$$

$$(B_2) \frac{}{\langle \text{not}(\text{true}), \sigma \rangle \hookrightarrow \langle \text{false}, \sigma \rangle}$$

$$(B_3) \frac{}{\langle \text{true} \wedge e, \sigma \rangle \hookrightarrow \langle e, \sigma \rangle} (e \in E_B)$$

$$(B_4) \frac{}{\langle \text{false} \vee e, \sigma \rangle \hookrightarrow \langle e, \sigma \rangle} (e \in E_B)$$

$$(B_5) \frac{}{\langle \text{false} \wedge e, \sigma \rangle \hookrightarrow \langle \text{false}, \sigma \rangle} (e \in E_B)$$

$$(B_6) \frac{}{\langle \text{true} \vee e, \sigma \rangle \hookrightarrow \langle \text{true}, \sigma \rangle} (e \in E_B)$$

$$(B_7) \frac{n_1 < n_2}{\langle n_1 < n_2, \sigma \rangle \hookrightarrow \langle \text{true}, \sigma \rangle} (n_1, n_2 \in \mathbb{Z})$$

$$(B_8) \frac{n_1 \geq n_2}{\langle n_1 < n_2, \sigma \rangle \hookrightarrow \langle \text{false}, \sigma \rangle} (n_1, n_2 \in \mathbb{Z})$$

$$(B_9) \frac{n_1 = n_2}{\langle n_1 = n_2, \sigma \rangle \hookrightarrow \langle \text{true}, \sigma \rangle} (n_1, n_2 \in \mathbb{Z})$$

$$(B_{10}) \frac{n_1 \neq n_2}{\langle n_1 = n_2, \sigma \rangle \hookrightarrow \langle \text{false}, \sigma \rangle} (n_1, n_2 \in \mathbb{Z})$$

$$(B_{11}) \frac{\langle e_1, \sigma \rangle \hookrightarrow \langle e'_1, \sigma \rangle}{\langle e_1 < e_2, \sigma \rangle \hookrightarrow \langle e'_1 < e_2, \sigma \rangle} (e_1, e'_1, e_2 \in E_A)$$

$$(B_{12}) \frac{\langle e_1, \sigma \rangle \hookrightarrow \langle e'_1, \sigma \rangle}{\langle e_1 = e_2, \sigma \rangle \hookrightarrow \langle e'_1 = e_2, \sigma \rangle} (e_1, e'_1, e_2 \in E_A)$$

$$(B_{13}) \frac{\langle e_2, \sigma \rangle \hookrightarrow \langle e'_2, \sigma \rangle}{\langle e_1 < e_2, \sigma \rangle \hookrightarrow \langle e_1 < e'_2, \sigma \rangle} (e_1, e_2, e'_2 \in E_A)$$

$$(B_{14}) \frac{\langle e_2, \sigma \rangle \hookrightarrow \langle e'_2, \sigma \rangle}{\langle e_1 = e_2, \sigma \rangle \hookrightarrow \langle e_1 = e'_2, \sigma \rangle} (e_1, e_2, e'_2 \in E_A)$$

$$(B_{15}) \frac{\langle e_1, \sigma \rangle \hookrightarrow \langle e'_1, \sigma \rangle}{\langle e_1 \wedge e_2, \sigma \rangle \hookrightarrow \langle e'_1 \wedge e_2, \sigma \rangle} (e_1, e'_1, e_2 \in E_B)$$

$$(B_{16}) \frac{\langle e_1, \sigma \rangle \hookrightarrow \langle e'_1, \sigma \rangle}{\langle e_1 \vee e_2, \sigma \rangle \hookrightarrow \langle e'_1 \vee e_2, \sigma \rangle} (e_1, e'_1, e_2 \in E_B)$$

$$(B_{17}) \frac{\langle e, \sigma \rangle \hookrightarrow \langle e', \sigma \rangle}{\langle \text{not}(e), \sigma \rangle \hookrightarrow \langle \text{not}(e'), \sigma \rangle} (e, e' \in E_B)$$