Small Steps Semantics for tiny expressions grammar

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1 Grammar

We'll define the set of arithmetic and boolean expressions as the expressions that can be built from the following grammars:

$$
\begin{aligned}
E_A &::= n(\in \mathbb{Z}) \mid v(\in \mathbf{V}) \mid E_A + E_A \mid E_A \times E_A \mid E_A - E_A \mid \text{if } E_B \text{ then } E_A \text{ else } E_A \\
E_B &::= \text{true} \mid \text{false} \mid E_A = E_A \mid E_A < E_A \mid \text{not}(E_B) \mid E_B \wedge E_B \mid E_B \vee E_B
\end{aligned}
$$

with $\mathbf{V}$ the set of variables.

2 Small Steps Semantic

Let $\sigma \in \mathcal{V}[\mathbb{Z}]$ $(= \mathcal{F}(\mathbf{V}, \mathbb{Z}))$ a valuation.

2.1 Arithmetic Rules

$$(A_1)\ \frac{}{\langle n_1 \text{ op } n_2, \sigma \rangle \hookrightarrow \langle n, \sigma \rangle}\ (n_1, n_2 \in \mathbb{Z}, \text{op } \in \{+, -, \times\}, n = n_1 \text{ op } n_2)$$

$$(A_2)\ \frac{}{\langle v, \sigma \rangle \hookrightarrow \langle \sigma(v), \sigma \rangle}\ (v \in \mathbf{V})$$

$$(A_3)\ \frac{\langle e_1, \sigma \rangle \hookrightarrow \langle e_1', \sigma \rangle}{\langle e_1 \text{ op } e_2, \sigma \rangle \hookrightarrow \langle e_1' \text{ op } e_2, \sigma \rangle}\ (\text{op } \in \{+, -, \times\})$$

$$(A_4)\ \frac{\langle e_2, \sigma \rangle \hookrightarrow \langle e_2', \sigma \rangle}{\langle n \text{ op } e_2, \sigma \rangle \hookrightarrow \langle n \text{ op } e_2', \sigma \rangle}\ (\text{op } \in \{+, -, \times\}, n \in \mathbb{Z})$$

2.2 Conditional Rules

$$(A_5)\ \frac{\langle e, \sigma \rangle \hookrightarrow \langle e', \sigma \rangle}{\langle \text{if } e \text{ then } a \text{ else } b, \sigma \rangle \hookrightarrow \langle \text{if } e' \text{ then } a \text{ else } b, \sigma \rangle}\ (a, b \in E_A, e \in E_B)$$

$$(A_6)\ \frac{}{\langle \text{if true then } a \text{ else } b, \sigma \rangle \hookrightarrow \langle a, \sigma \rangle}\ (a, b \in E_A)$$

$$(A_7)\ \frac{}{\langle \text{if false then } a \text{ else } b, \sigma \rangle \hookrightarrow \langle b, \sigma \rangle}\ (a, b \in E_A)$$

2.3 Boolean Rules

Negation:

$$(B_1)\ \frac{}{\langle \mathrm{not}(\mathrm{false}), \sigma\rangle \hookrightarrow \langle \mathrm{true}, \sigma\rangle}$$

$$(B_2)\ \frac{}{\langle \mathrm{not}(\mathrm{true}), \sigma\rangle \hookrightarrow \langle \mathrm{false}, \sigma\rangle}$$

Conjunction and Disjunction:

$$(B_3)\ \frac{}{\langle \mathrm{true} \wedge e, \sigma\rangle \hookrightarrow \langle e, \sigma\rangle}\ (e \in E_B)$$

$$(B_4)\ \frac{}{\langle \mathrm{false} \vee e, \sigma\rangle \hookrightarrow \langle e, \sigma\rangle}\ (e \in E_B)$$

$$(B_5)\ \frac{}{\langle \mathrm{false} \wedge e, \sigma\rangle \hookrightarrow \langle \mathrm{false}, \sigma\rangle}\ (e \in E_B)$$

$$(B_6)\ \frac{}{\langle \mathrm{true} \vee e, \sigma\rangle \hookrightarrow \langle \mathrm{true}, \sigma\rangle}\ (e \in E_B)$$

Comparison:

$$(B_7)\ \frac{n_1 < n_2}{\langle n_1 < n_2, \sigma\rangle \hookrightarrow \langle \mathrm{true}, \sigma\rangle}\ (n_1, n_2 \in \mathbb{Z})$$

$$(B_8)\ \frac{n_1 \geq n_2}{\langle n_1 < n_2, \sigma\rangle \hookrightarrow \langle \mathrm{false}, \sigma\rangle}\ (n_1, n_2 \in \mathbb{Z})$$

$$(B_9)\ \frac{n_1 = n_2}{\langle n_1 = n_2, \sigma\rangle \hookrightarrow \langle \mathrm{true}, \sigma\rangle}\ (n_1, n_2 \in \mathbb{Z})$$

$$(B_{10})\ \frac{n_1 \neq n_2}{\langle n_1 = n_2, \sigma\rangle \hookrightarrow \langle \mathrm{false}, \sigma\rangle}\ (n_1, n_2 \in \mathbb{Z})$$

Structural Rules for Comparisons:

$$(B_{11})\ \frac{\langle e_1, \sigma\rangle \hookrightarrow \langle e_1', \sigma\rangle}{\langle e_1 < e_2, \sigma\rangle \hookrightarrow \langle e_1' < e_2, \sigma\rangle}\ (e_1, e_1', e_2 \in E_A)$$

$$(B_{13})\ \frac{\langle e_2, \sigma\rangle \hookrightarrow \langle e_2', \sigma\rangle}{\langle e_1 < e_2, \sigma\rangle \hookrightarrow \langle e_1 < e_2', \sigma\rangle}\ (e_1, e_2, e_2' \in E_A)$$

$$(B_{12})\ \frac{\langle e_1, \sigma\rangle \hookrightarrow \langle e_1', \sigma\rangle}{\langle e_1 = e_2, \sigma\rangle \hookrightarrow \langle e_1' = e_2, \sigma\rangle}\ (e_1, e_1', e_2 \in E_A)$$

$$(B_{14})\ \frac{\langle e_2, \sigma\rangle \hookrightarrow \langle e_2', \sigma\rangle}{\langle e_1 = e_2, \sigma\rangle \hookrightarrow \langle e_1 = e_2', \sigma\rangle}\ (e_1, e_2, e_2' \in E_A)$$

Structural Rules for Boolean Operators:

$$(B_{15})\ \frac{\langle e_1, \sigma\rangle \hookrightarrow \langle e_1', \sigma\rangle}{\langle e_1 \wedge e_2, \sigma\rangle \hookrightarrow \langle e_1' \wedge e_2, \sigma\rangle}\ (e_1, e_1', e_2 \in E_B)$$

$$(B_{16})\ \frac{\langle e_1, \sigma\rangle \hookrightarrow \langle e_1', \sigma\rangle}{\langle e_1 \vee e_2, \sigma\rangle \hookrightarrow \langle e_1' \vee e_2, \sigma\rangle}\ (e_1, e_1', e_2 \in E_B)$$

$$(B_{17})\ \frac{\langle e, \sigma\rangle \hookrightarrow \langle e', \sigma\rangle}{\langle \mathrm{not}(e), \sigma\rangle \hookrightarrow \langle \mathrm{not}(e'), \sigma\rangle}\ (e, e' \in E_B)$$