

Accurate Predictions of Ionization and Atomization Energies without the Born-Oppenheimer Approximation

Yubo Yang,¹ Ilkka Kylänpää,¹ Norm M. Tubman,¹ Jaron T. Krogel,² Sharon Hammes-Schiffer,³ and David M. Ceperley¹

¹*Department of Physics, University of Illinois, Urbana, Illinois 61801 USA*

²*Materials Science & Technology Division, Oak Ridge National Laboratory, Oak Ridge, TN 37831*

³*Department of Chemistry, University of Illinois, Urbana, Illinois 61801 USA*

(Dated: January 28, 2015)

We obtained ground-state energies of a range of small atoms and molecules with Fixed-Node Diffusion Monte Carlo (FN-DMC) to an accuracy of 0.01mHa (0.27meV), treating both electrons and ions as quantum particles. We used "dragged node approximation" developed by Tubman et. al to construct the trial wavefunction without the adiabatic assumption. For each system, we optimized an all-electron trial wavefunction generated by quantum chemistry methods and used it to construct an all-electron-ion wavefunction. It is then used in FN-DMC to obtain the ground-state energy of said system. We found the ionization energies of first row atoms to be identical with or without the adiabatic assumption, whereas the atomization energies of simple hydrides to change by as much as 6.2%. The non-adiabatic results are in better agreement with the best available quantum chemistry literature in all tested hydrides.

INTRODUCTION

Under the Born-Oppenheimer (BO) approximation [1], the full molecular wavefunction considered to be a product of an electron component and an ion component. The electron problem is solved by "clamping" the ions to their equilibrium positions. Then the ion problem is solved using the equilibrium electron distribution as an effective potential energy surface. This assumption is based on the fact that protons are roughly 1836 times as heavy as electrons therefore ions, being even heavier than protons, move much more slowly than electrons. This allows the electrons to relax instantaneously and adiabatically to their equilibrium states as the ions move around. The BO approximation is excellent in most cases, but recent advances suggest one might be missing important physics by adopting this approximation in certain cases, particularly when a conical intersection is present [2]. For instance, the prediction of ground state structure of atomic hydrogen [3–5], the dissociation of hydrogen at finite temperature [6] and photon coupled electron transfer (PCET) in phenoxyl-phenol [7] were only possible due to the inclusion of non-adiabatic effects. We acknowledge the subtle differences between the Born-Oppenheimer approximation and the adiabatic assumption pointed out by Worth and Cederbaum [2], namely the former includes the kinetic energy of nuclei on a single electronic potential surface whereas the latter doesn't. However, we will use the phrases non-Born-Oppenheimer and non-adiabatic interchangeably since they both mean using the exact molecular Hamiltonian in the QMC community.

While there have been many cases in the literature where non-adiabatic effects were included for atomic and molecular systems, they tend to require a great deal of computational as well as human effort. One approach is to add corrections to a basic coupled cluster theory to produce highly accurate ground state energies for atoms and molecules[8]. In such an approach, a frozen-core coupled cluster singles and doubles (FC-CCSD) [9] calculation serves as a first approximation for the ground-state energies. Then a series of corrections including core/valance, spin-orbit coupling, higher order correlation, zero point motion, diagonal Born-Oppenheimer and scalar relativistic effects are introduced. Each correction involves a non-trivial and often costly calculation. In addition, extrapolation formulae are sometimes needed to take the result to the complete basis set limit. This method produces highly accurate expectation values, but contains uncontrolled errors that have to be estimated either empirically or analyzed on a molecule-by-molecule basis [10]. The addition of uncertainties in each correction also contributes to a rather large error bar, making it difficult to obtain a small confidence interval with this method. Another approach in a more recent development is to use the explicitly correlated Gaussian (ECG) basis [11, 12]. This approach is capable of calculating the ground-state energies of small molecules to an incredibly high accuracy. Unfortunately, the current implementation of ECG cannot be applied to moderately-sized systems due to factorial scaling with the number of identical particles [13]. Other methods with less aggressive scaling include nuclear-electronic orbital (NEO) Hatree-Fock (HF) [14], NEO explicitly correlated HF (NEO-ECHF) [15–17], path integral Monte Carlo [18–20] and multicomponent density functional theory [21–26]. While these methods can be applied to larger systems, it would be difficult to deliver the same kind of accuracy that ECG offers without significant development. In this paper, we would like to demonstrate

that, with simple modifications [27], the fixed-node diffusion Monte Carlo (FN-DMC) algorithm is capable of producing results on par with and even better than the most sophisticated quantum chemistry methods for systems as small as BH.

METHOD

Fixed-Node Diffusion Monte Carlo (FN-DMC)

Diffusion Monte Carlo is a projector method that evolves a trial wavefunction with the exact Hamiltonian in imaginary time and projects out the ground-state wavefunction in the infinite time limit. The trial wavefunction is often produced by some other mean field method such as Hartree-Fock (HF) or Density Functional Theory (DFT). The more popular variant (FN-DMC) introduces the fixed-node approximation to overcome the sign-problem suffered in fermion problems due to the presence of nodes in the ground-state wavefunction. FN-DMC is a simple but powerful method, since its accuracy is only limited by the quality of the nodal surface of the trial wavefunction and the finite length of the calculation. The uncertainty in the obtained ground-state energy is statistically controlled and may be shrunk arbitrarily by increasing computation time. If the trial wavefunction has the same nodal surface as the exact ground-state wavefunction, the final ground-state energy will be exact with zero variance. It should be noted that even with an approximate nodal surface, FN-DMC will still produce an excellent approximation of the exact ground-state energy, albeit with rather large variance if the quality of the trial function is low. In addition, since the exact Hamiltonian is used, the FN-DMC method is variational, that is, even when the nodes of the trial wavefunction are not exact the result will be a rigorous upper bound to the exact ground-state energy.

Due to the introduction of linear optimization method by Nightingale et. al.[28] and Umrigar et. al.[29], one can systematically improve the wave function ansatz generated by quantum chemistry calculations to obtain high quality wave functions for atoms and molecules as was done by Brown[30] and Toulouse[31]. However, in these benchmark studies, the authors always worked within the adiabatic assumption, i.e. only the electron Hamiltonian is used in evolving the trial wavefunction in imaginary time, while the ions are "clamped" to their equilibrium positions. Such an assumption is not fundamentally required by FN-DMC. It's inclusion is mostly due to a lack of mean field theories that include non-adiabatic effects. Although there is significant effort in the quantum chemistry community to develop such methodology as mentioned in the introduction, until a standardized package is developed we will have to rely on modification of the QMC algorithm itself to include non-adiabatic effects. To this end, we will use the technique developed by Tubman et. al. [27] to construct a high quality all-electron-ion wavefunction from an optimized all-electron wavefunction.

Electron Wavefunction and Optimization

We followed the basic strategies of Umrigar et. al.[31, 32] and Needs et. al. [30, 33] in generating our all-electron wavefunctions. The initial guess for the wavefunction is generated from Complete Active Space Self-Consistent Field (CASSCF) [34, 35] calculation using the quantum chemistry package GAMESS-US[36]. The optimized orbitals are then used in a Second Order Configuration Interaction (SOCF) calculation to generate a series of Configuration State Functions (CSF). This process is described in more detail in [37]. The multi-CSF expansion of the wavefunction generated by GAMESS can be expressed in the following form

$$\Psi_{SOCF}(\vec{r}) = \sum_{i=1}^{N_{CSF}} \alpha_i \phi_i(\vec{r}) \quad (1)$$

where \vec{r} refers to the spacial coordinates of all the electrons. $\phi_i(\vec{r})$ are the CSF generated from SOCF. We used the cc-pV5Z basis for all the atomic systems but switched to Roos Augmented Triple Zeta ANO basis for molecular systems due to GAMESS's limited ability to handle a large number of basis elements. Both basis sets are taken from Basis Set Exchange [38].

A Jastrow factor $J(\vec{r}, \vec{\beta})$, in the form of a B-spline with values $\vec{\beta}$ on a linear grid, is then added to the wave function to correlate electron motion and smooth out the divergence in the local energy near the ions by imposing the cusp condition [39]. Our Jastrow factor contains electron-electron, electron-nucleus and electron-electron-nucleus terms.

The actually wave function being optimized is then

$$\Psi_e(\vec{r}) = e^{J(\vec{r}, \vec{\beta})} \sum_{i=1}^{N_{CSF}} \alpha_i \phi_i(\vec{r}) \quad (2)$$

We optimized the CSF and Jastrow coefficients $\vec{\alpha}, \vec{\beta}$ simultaneously with QMCPACK[40].

Electron-Ion Wavefunction

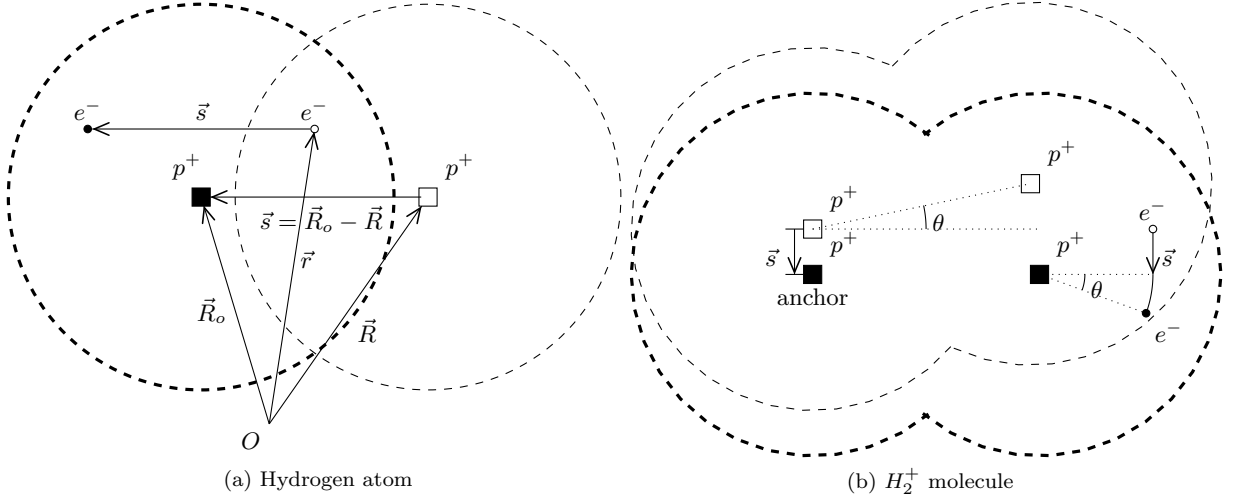


FIG. 1: **Dragged Node Approximation** (a) For hydrogen atom, we assume the entire wavefunction shifts with the ion. This process can be visualized by following a contour of the wavefunction. The thick dashed circle represents a contour of the electron wavefunction when the proton is at its reference position \vec{R}_o and the thin dashed circle represents the same contour when the proton has moved to a new position \vec{R} . To evaluate the ion-dependent electron wavefunction $\bar{\psi}_e(\vec{r}, \vec{R})$, we simply map the electron to its proper place in the reference wavefunction. That is $\bar{\psi}_e(\vec{r}, \vec{R}) = \bar{\psi}_e(\vec{r} + \vec{s}, \vec{R}_o) = \psi_e(\vec{r} + \vec{s})$ where \vec{s} is the shift required to put the the proton back to its reference position. (b) For H_2^+ , we pick one of the protons as an "anchor" and approximate the new wavefunction by dragging the reference wavefunction with the "anchor" proton. We also rotate the wavefunction to align its axis of symmetry with the orientation of the two protons.

Once a satisfactory electron wave function has been obtained, we construct the electron-ion wave function using the following ansatz suggested by Tubman et. al.[27]

$$\Psi_{ei}(\vec{r}, \vec{R}) = \psi_I(\vec{R}) \bar{\psi}_e(\vec{r}, \vec{R}) \quad (3)$$

where \vec{R} includes spatial coordinates of all ions. The ion wave function consists of simple products of Gaussian wave functions over each nuclei pair.

$$\psi_I(\vec{R}) \propto \prod_{i,j} e^{-a(|\vec{R}_i - \vec{R}_j| - b_{ij})^2} \quad (4)$$

where a is a contraction coefficient for the ion wave function that we optimize for each system and b_{ij} are taken to be the equilibrium distances between the nuclei in the adiabatic limit. Notice the new electron wavefunction $\bar{\psi}_e$ depends on both the electron and the ion positions. In general $\bar{\psi}_e(\vec{r}, \vec{R}) \neq \psi_e(\vec{r})$, but we do have $\bar{\psi}_e(\vec{r}, \vec{R}_o) = \psi_e(\vec{r})$, where \vec{R}_o are the ion positions used in the creation of the $\psi_e(\vec{r})$. The most straight-forward way to obtain $\bar{\psi}_e(\vec{r}, \vec{R})$ is to repeat the process described in the previous section for every new ion positions \vec{R} . However, such an approach would be horrendously expensive. To alleviate the computational cost, Tubman et. al. proposed a "Dragged Node Approximation" [27], where the contours (including the nodal surface) of $\psi_e(\vec{r})$ are dragged along the ions \vec{R} to

create $\bar{\psi}(\vec{r})$. Figure 1 demonstrates this strategy for the simple cases of a hydrogen atom and a H_2^+ molecule. For atoms, this "dragged-node" process is equivalent to re-running a quantum chemistry calculation and re-optimizing the wavefunction at each new ion position. However, for diatomic molecules, since the distance between ions fluctuates, the two processes will produce slightly different wavefunctions. It is also important to note that the trial wavefunction (3) is still in Born-Oppenheimer form for each set of ion coordinates, that is we are essentially using the Born-Oppenheimer wavefunction as the starting point of FN-QMC. Nevertheless, without modification to the Hamiltonian, FN-DMC will automatically include non-adiabatic effects not present in the trial wavefunction and the process remains variational. Therefore the ground-state energy we calculate will still serve as a rigorous upper bound to the exact ground-state energy of the full electron-ion problem.

Although the dragged-node technique is developed with atomic and diatomic systems in mind, it is not difficult to generalize it for use in larger systems or even apply to parts of a bigger system, treating light ions as quantum particles and heavy ions as "clamped". For a system of more than 3 particles, a general fitting procedure [41] can be done to determine the transformation needed to put the ions back to their reference positions. This is similar to the process of protein structure alignment implemented in some well-known biophysics software such as VMD [42]. Once the transformation is determined, we can map each electron individually and evaluate the wavefunction with moved ions using the reference wavefunction as was done for atoms and diatomics. One complication occurs when the ions become degenerate (when there are more than two protons with the same spin for example). In this case one has to explicitly anti-symmetrize the ion wavefunction in a manner similar to what's done for the electron wavefunction (Slater determinant).

RESULTS AND DISCUSSION

Ground State Energies

Ground state energies were calculated for first row atoms and ions with and without the adiabatic assumption (Table I). The first row of Table I lists the level of CASSCF calculation we used to generate the all-electron wavefunction guess. We first performed a CAS(m,n) calculation, meaning that we distribute of m electrons into n active orbitals, with the ground-state equilibrium geometries taken from [43]. The MCSCF optimized orbitals are then used in a SOCI calculation that includes single and double excitations of the m electrons into all of the available valance orbitals provided by the basis. The SOCI ground state CSF $\phi_0(\vec{r})$ always dominates the expansion (with $\alpha_0 > .95$). Nevertheless, we include all CSFs with coefficients bigger than some cutoff ϵ to lend reasonable flexibility to the wavefunction during optimization. The choice of ϵ is somewhat arbitrary. We wish to included as many CSFs as possible to maximize the flexibility of the wavefunction. However, the inclusion of too many CSFs with small expansion coefficients introduces unnecessary noise into the system and the optimization requires a large number of samples to reach our desired accuracy. Therefore, we have chosen ϵ to restrict the number of CSFs in the wave function to be ~ 1000 in all systems to maintain a balance between the flexibility and the cost of optimization. In all of the atoms and molecules tested, this criteria results in an ϵ of $0.001 \sim 0.0001$ and the sum of coefficients squared of the included CSFs $\sum_i a_i^2 > 0.999$ in all cases. Optimization was performed with 6×10^6 statistically independent samples and we chose a cost function consisting of equal parts average local energy and reweighted variance. We found this choice of cost function to produce slightly better wavefunctions than a highly biased one, albeit the differences are small and are most likely insignificant at the DMC level. The adiabatic ground state energies of atoms are in perfect agreement with the most recent QMC benchmark study [33]. The non-adiabatic ground-state energies for Be and B ($-14.66643(2)\text{Ha}$ and $-24.65244(3)\text{Ha}$) are in good agreement with ECG results (-14.66643544Ha [44] and -24.652598Ha [13]).

It is interesting to note that even though we used massive multi-determinant expansions (~ 1000 CSF) in our study whereas Seth et. al. [33] used moderately-sized multi-determinant expansions (~ 100 CSF) with a backflow transformation, meaning that we have wildly different wavefunctions, the final DMC energies we obtained were in perfect agreement.

Similar calculations were performed for diatomic systems (Table II) and the results are in excellent agreement with the sophisticated coupled cluster study [8]. It is important to note that with our method, the only included non-adiabatic effects are the zero point motion of the nuclei and any correlation that may exist among the quantum particles. We do not take into account spin-orbit coupling or relativistic effects. However, our method is much simpler. There's no need for separate computation for each correction factor, thus no addition of uncertainties. Further, the uncertainties in our calculations are statistically controlled and may be reduced by increasing computation time.

Ionization Energies

The ionization energies are listed at the end of Table I and they agree well with experimental results. Notice that even though ground state energies change significantly with the inclusion of non-adiabatic effects, the ionization energies match perfectly with or without the adiabatic assumption. This suggests that for atomic systems, the gradient coupling between electron and ion motions is indeed negligible. The difference in ground state energies can be entirely attributed to the zero point motion of the nucleus. Physically, this means that for all first row atoms, the outer most electron is nearly perfectly screened from the nucleus and all of the energy required for its removal can be attributed to its interaction with the rest of the electrons in the atom.

The ground state energy of LiH^- was also calculated to obtain the electron affinity of LiH . The energy obtained in the adiabatic limit at the DMC level is $-8.08220(2)\text{Ha}$ and $-8.07811(3)\text{Ha}$ with non-adiabatic effects included. Our non-adiabatic result is in good agreement with a previous ECG study [45] which reported a value of -8.07856887Ha , giving an electron affinity of $0.01191(4)\text{Ha}$ which is very close to the ECG prediction of $0.012132(2)\text{Ha}$ and agrees with experiment $0.0126(4)\text{Ha}$ up to the mHa level. It should be noted that the authors of [45] mislabeled the columns of LiH^- and LiD resulting in a miss-citation in [12].

Atom	Li(^2S)	Be(^1S)	B(^2P)	C(^3P)	N(^4S)	O(^3P)	F(^2P)
Guess	CAS(3,23)	CAS(4,20)	CAS(5,40)	CAS(6,16)	CAS(7,20)	CAS(8,15)	CAS(9,15)
FN-DMCe	-7.478056(4)	-14.66732(1)	-24.65377(1)	-37.84449(2)	-54.58858(3)	-75.06576(4)	-99.7316(1)
E_{ref}	-7.478067(5)	-14.667306(7)	-24.65379(3)	-37.84446(6)	-54.58867(8)	-75.0654(1)	-99.7318(1)
FN-DMCei	-7.47742(1)	-14.66643(2)	-24.65244(3)	-37.84277(6)	-54.58655(8)	-75.0631(1)	-99.7290(4)
Ion	Li $^+$ (^1S)	Be $^+$ (^2S)	B $^+$ (^1S)	C $^+$ (^2P)	N $^+$ (^3P)	O $^+$ (^4S)	F $^+$ (^3P)
Guess	CAS(2,5)	CAS(3,18)	CAS(4,40)	CAS(5,16)	CAS(6,20)	CAS(5,15)	CAS(6,30)
FN-DMCe	-7.279919(4)	-14.324753(6)	-24.34884(1)	-37.43075(2)	-54.05376(3)	-74.56588(4)	-99.0913(1)
E_{ref}	-7.279914(3)	-14.324761(3)	-24.34887(2)	-37.43073(4)	-54.05383(7)	-74.56662(7)	-99.0911(2)
FN-DMCei	-7.2793(1)	-14.32386(2)	-24.34750(3)	-37.42904(4)	-54.05182(9)	-74.56336(8)	-99.0885(3)
IP_e	0.19814(1)	0.34256(2)	0.3049(3)	0.4138(5)	0.53475(8)	0.500(1)	0.640(1)
IP_{ei}	0.1981(1)	0.34257(2)	0.3049(3)	0.4137(5)	0.53473(8)	0.500(1)	0.640(1)
IP_{exp}	0.19808	0.3425	0.30502	0.413797	0.533967	0.500526	0.640173

TABLE I: **Ionization Energies** Fixed-Node DMC was performed with and without the adiabatic assumption and the energies for each atom and ion is reported in units of Hartree. FN-DMCe denotes that only electrons are treated quantum mechanically while the ions are clamped in their equilibrium positions. FN-DMCei denotes that both electrons and ions are treated quantum mechanically. The reference energies are taken from [33]

Atomization Energies

We also calculated ground state energies for LiH , BeH , BH , OH and HF (see Table II). The energies calculated in the adiabatic limit are on par and sometimes better than the best available quantum chemistry results [46–48] and the energies calculated without the adiabatic assumption are in excellent agreement with state-of-the-art quantum chemistry calculations performed with ECG where available (LiH , BeH and BH). In the case of simple hydrides, non-adiabatic effects do make a noticeable contribution to the atomization energy. This is possibly due to the presence of the light weighted proton.

CONCLUSION

We calculated the ground-state energies of first row atoms, ions and a number of simple hydrides to an accuracy of 0.1mHa both with and without the adiabatic assumption. The ionization energies of first row atoms was independent of the adiabatic assumption, suggesting that either the energy difference between the adiabatic and non-adiabatic ground states is entirely due to the zero point motion of the nuclei or the coupling between the nucleus and electrons is not important in the ionization process. The atomization energies of simple hydrides were significantly different

Molecule	LiH	BeH	BH	CH	OH	HF
Guess	CAS(4,43)	CAS(5,44)	CAS(4,14)	CAS(7,20)	CAS(5,20)	CAS(6,20)
FN-DMCe	-8.070521(7)	-15.24793(1)	-25.28868(2)	-38.4781(1)	-75.7352(1)	-100.44941(6)
E_{Ref}	-8.07045	-15.247846	-25.287650	-38.478048(DFT)	-75.720	-100.442
FN-DMCei	-8.06620(2)	-15.24196(7)	-25.28103(4)	-38.4725(1)	-75.7260(4)	-100.4391(4)
ECG	-8.0664371(15)	15.24203(10)	-25.2803(10)	N/A	N/A	N/A
E_e^{at}	0.092465(8)	0.08062(1)	0.13491(2)	0.13361(2)	0.16944(4)	0.21781(6)
E_{ei}^{at}	0.08905(2)	0.0758(7)	0.12886(5)	0.13000(6)	0.16317(4)	0.21037(5)
$E_{\text{ref}}^{\text{at}}$	0.08940(5)	0.0761(4)	0.1299(2)	0.1276(2)	0.1622(2)	0.2166(3)
$E_{\text{exp}}^{\text{at}}$	0.08874(38)	0.0826(11)	0.1281(37)	0.1275(5)	0.1622(1)	0.2158(3)

TABLE II: **Atomization Energies** The adiabatic reference energies for OH and HF are taken from [49]. E_e^{at} is the atomization energy in the adiabatic limit, whereas E_{ei}^{at} is obtained without the adiabatic assumption. $E_{\text{ref}}^{\text{at}}$ is taken from [8] processed to excluded scalar relativistic and spin-orbit coupling corrections. $E_{\text{exp}}^{\text{at}}$ is obtained from [43]

in the adiabatic than in the non-adiabatic limit, possibly due to the presence of a light nucleus (the proton) in the molecule. We showed that it is necessary to include non-adiabatic effects to accurately predict the experimental values of atomization energies for these simple hydrides.

These calculations also verified the validity of the "dragged-node" approximation, namely it does indeed produce a high quality electron-ion trial wavefunction from a good electron wavefunction. This technique also has the potential to solve more interesting problems due to its ease of implementation as well as the polynomial scaling in computational time with respect to the number of electrons. As mentioned at end of Method section, this technique can be generalized quite easily to deal with larger systems. In addition, we are able to offer similar levels of accuracy compared to the most sophisticated quantum chemistry methods (coupled cluster and ECG) while maintaining a reasonable level of computational and human cost.

ACKNOWLEDGMENT

The authors would like to thank Mike Pak for useful discussions. This work was supported by the U.S. Department of Energy grant No. 1-485267-244000-191100 as part of the Scientific Discovery through Advanced Computing (SciDAC) program. We used the Extreme Science and Engineering Discovery Environment (XSEDE), which is supported by the National Science Foundation Grant No. OCI-1053575 and resources of the Oak Ridge Leadership Computing Facility (OLCF) at the Oak Ridge National Laboratory, which is supported by the Office of Science of the U.S. Department of Energy under Contract No. DE-AC05-00OR22725.

-
- [1] M. Born and R. Oppenheimer. Zur quantentheorie der molekeln. *Annalen der Physik*, 389(20):457–484, 1927.
 - [2] Graham A. Worth and Lorenz S. Cederbaum. Beyond born-oppenheimer: Molecular dynamics through a conical intersection. *Annual Review of Physical Chemistry*, 55(1):127–158, 2004. PMID: 15117250.
 - [3] D. Ceperley and B. Alder. Ground state of solid hydrogen at high pressures. *Phys. Rev. B*, 36:2092–2106, Aug 1987.
 - [4] V. Natoli, Richard Martin, and D. Ceperley. Crystal structure of atomic hydrogen. *Phys. Rev. Lett.*, 70:1952–1955, Mar 1993.
 - [5] Vincent Natoli, Richard Martin, and David Ceperley. Crystal structure of molecular hydrogen at high pressure. *Phys. Rev. Lett.*, 74:1601–1604, Feb 1995.
 - [6] Guglielmo Mazzola, Andrea Zen, and Sandro Sorella. Finite-temperature electronic simulations without the born-oppenheimer constraint. *The Journal of Chemical Physics*, 137(13):–, 2012.
 - [7] Andrew Sirjoosingh and Sharon Hammes-Schiffer. Proton-coupled electron transfer versus hydrogen atom transfer: Generation of charge-localized diabatic states. *The Journal of Physical Chemistry A*, 115(11):2367–2377, 2011. PMID: 21351757.
 - [8] David Feller, Kirk A. Peterson, and David A. Dixon. A survey of factors contributing to accurate theoretical predictions of atomization energies and molecular structures. *The Journal of Chemical Physics*, 129(20):–, 2008.
 - [9] George D. Purvis and Rodney J. Bartlett. A full coupledcluster singles and doubles model: The inclusion of disconnected triples. *The Journal of Chemical Physics*, 76(4), 1982.
 - [10] David Feller, Kirk A. Peterson, and T. Daniel Crawford. Sources of error in electronic structure calculations on small chemical systems. *The Journal of Chemical Physics*, 124(5):–, 2006.
 - [11] Donald Kinghorn and Ludwik Adamowicz. Improved nonadiabatic ground-state energy upper bound for dihydrogen. *Phys. Rev. Lett.*, 83:2541–2543, Sep 1999.
 - [12] Jim Mitroy, Sergiy Bubín, Wataru Horiuchi, Yasuyuki Suzuki, Ludwik Adamowicz, Wojciech Cencek, Krzysztof Szalewicz, Jacek Komasa, D. Blume, and Kálmán Varga. Theory and application of explicitly correlated gaussians. *Rev. Mod. Phys.*, 85:693–749, May 2013.
 - [13] Sergiy Bubín, Monika Stanke, and Ludwik Adamowicz. Non-bornoppenheimer calculations of the bh molecule. *The Journal of Chemical Physics*, 131(4):–, 2009.
 - [14] Simon P. Webb, Tzvetelin Iordanov, and Sharon Hammes-Schiffer. Multiconfigurational nuclear-electronic orbital approach: Incorporation of nuclear quantum effects in electronic structure calculations. *The Journal of Chemical Physics*, 117(9), 2002.
 - [15] Chet Swalina, Michael V. Pak, Arindam Chakraborty, and Sharon Hammes-Schiffer. Explicit dynamical electronproton correlation in the nuclearelectronic orbital framework. *The Journal of Physical Chemistry A*, 110(33):9983–9987, 2006. PMID: 16913669.
 - [16] Arindam Chakraborty, Michael V. Pak, and Sharon Hammes-Schiffer. Inclusion of explicit electron-proton correlation in the nuclear-electronic orbital approach using gaussian-type geminal functions. *The Journal of Chemical Physics*, 129(1):–, 2008.
 - [17] Andrew Sirjoosingh, Michael V. Pak, Chet Swalina, and Sharon Hammes-Schiffer. Reduced explicitly correlated hartree-fock approach within the nuclear-electronic orbital framework: Theoretical formulation. *The Journal of Chemical Physics*, 139(3):–, 2013.
 - [18] Ilkka Kylänpää, Tapio Rantala, and David Ceperley. Few-body reference data for multicomponent formalisms: Light-nuclei molecules. *Phys. Rev. A*, 86:052506, Nov 2012.
 - [19] Ilkka Kylnp and Tapio T. Rantala. First-principles simulation of molecular dissociationrecombination equilibrium. *The Journal of Chemical Physics*, 135(10):–, 2011.
 - [20] Ilkka Kylnp and Tapio T. Rantala. Finite temperature quantum statistics of h3+ molecular ion. *The Journal of Chemical Physics*, 133(4):–, 2010.
 - [21] Arindam Chakraborty, Michael Pak, and Sharon Hammes-Schiffer. Development of electron-proton density functionals for multicomponent density functional theory. *Phys. Rev. Lett.*, 101:153001, Oct 2008.
 - [22] Arindam Chakraborty, Michael V. Pak, and Sharon Hammes-Schiffer. Properties of the exact universal functional in multicomponent density functional theory. *The Journal of Chemical Physics*, 131(12):–, 2009.
 - [23] Andrew Sirjoosingh, Michael V. Pak, and Sharon Hammes-Schiffer. Multicomponent density functional theory study of the interplay between electron-electron and electron-proton correlation. *The Journal of Chemical Physics*, 136(17):–, 2012.
 - [24] Andrew Sirjoosingh, Michael V. Pak, and Sharon Hammes-Schiffer. Derivation of an electronproton correlation functional for multicomponent density functional theory within the nuclearelectronic orbital approach. *Journal of Chemical Theory and Computation*, 7(9):2689–2693, 2011.
 - [25] T. Kreibich and E. Gross. Multicomponent density-functional theory for electrons and nuclei. *Phys. Rev. Lett.*, 86:2984–2987, Apr 2001.
 - [26] Thomas Kreibich, Robert van Leeuwen, and E. Gross. Multicomponent density-functional theory for electrons and nuclei. *Phys. Rev. A*, 78:022501, Aug 2008.
 - [27] Norm M. Tubman, Ilkka Kylänpää, Sharon Hammes-Schiffer, and David M. Ceperley. Beyond the born-oppenheimer approximation with quantum monte carlo methods. *Phys. Rev. A*, 90:042507, Oct 2014.
 - [28] M. Nightingale and Vilen Melik-Alaverdian. Optimization of ground- and excited-state wave functions and van der waals clusters. *Phys. Rev. Lett.*, 87:043401, Jul 2001.
 - [29] C. Umrigar and Claudia Filippi. Energy and variance optimization of many-body wave functions. *Phys. Rev. Lett.*,

94:150201, Apr 2005.

- [30] M. D. Brown, J. R. Trail, P. Lpez Ros, and R. J. Needs. Energies of the first row atoms from quantum monte carlo. *The Journal of Chemical Physics*, 126(22):–, 2007.
- [31] Julien Toulouse and C. J. Umrigar. Full optimization of jastrowslater wave functions with application to the first-row atoms and homonuclear diatomic molecules. *The Journal of Chemical Physics*, 128(17):–, 2008.
- [32] C. J. Umrigar, Julien Toulouse, Claudia Filippi, S. Sorella, and R. G. Hennig. Alleviation of the fermion-sign problem by optimization of many-body wave functions. *Phys. Rev. Lett.*, 98:110201, Mar 2007.
- [33] P. Seth, P. Lpez Ros, and R. J. Needs. Quantum monte carlo study of the first-row atoms and ions. *The Journal of Chemical Physics*, 134(8):–, 2011.
- [34] G. Chaban, M.W. Schmidt, and M.S. Gordon. Approximate second order method for orbital optimization of scf and mscf wavefunctions. *Theoretical Chemistry Accounts*, 97(1-4):88–95, 1997. cited By 46.
- [35] Attila Szabo and Neil S. Ostlund. *Modern Quantum Chemistry*. McGraw-Hill, Inc., 1989.
- [36] Michael W. Schmidt, Kim K. Baldridge, Jerry A. Boatz, Steven T. Elbert, Mark S. Gordon, Jan H. Jensen, Shiro Koseki, Nikita Matsunaga, Kiet A. Nguyen, Shujun Su, Theresa L. Windus, Michel Dupuis, and John A. Montgomery. General atomic and molecular electronic structure system. *Journal of Computational Chemistry*, 14(11):1347–1363, 1993.
- [37] Bryan K. Clark, Miguel A. Morales, Jeremy McMinis, Jeongnim Kim, and Gustavo E. Scuseria. Computing the energy of a water molecule using multideterminants: A simple, efficient algorithm. *The Journal of Chemical Physics*, 135(24):–, 2011.
- [38] Karen L. Schuchardt, Brett T. Didier, Todd Elsethagen, Lisong Sun, Vidhya Gurumoorthi, Jared Chase, Jun Li, and Theresa L. Windus. Basis set exchange: a community database for computational sciences. *Journal of Chemical Information and Modeling*, 47(3):1045–1052, 2007. PMID: 17428029.
- [39] Tosio Kato. On the eigenfunctions of many-particle systems in quantum mechanics. *Communications on Pure and Applied Mathematics*, 10(2):151–177, 1957.
- [40] QMCPACK. www.qmcpack.org.
- [41] W. Kabsch. A solution for the best rotation to relate two sets of vectors. *Acta Crystallographica Section A*, 32(5):922–923, Sep 1976.
- [42] William Humphrey, Andrew Dalke, and Klaus Schulten. VMD – Visual Molecular Dynamics. *Journal of Molecular Graphics*, 14:33–38, 1996.
- [43] Nist computational chemistry comparison and benchmark database, nist standard reference database number 101, release 16a, august 2013. editor: Russell d. johnson iii, <http://cccbdb.nist.gov/>.
- [44] Sergiy Bubin and Ludwik Adamowicz. Calculations of the ground states of beh and beh+ without the born-oppenheimer approximation. *The Journal of Chemical Physics*, 126(21):–, 2007.
- [45] Sergiy Bubin, Ludwik Adamowicz, and Marcin Molski. An accurate non-bornoppenheimer calculation of the first purely vibrational transition in lih molecule. *The Journal of Chemical Physics*, 123(13):–, 2005.
- [46] Wei-Cheng Tung, Michele Pavanello, and Ludwik Adamowicz. Very accurate potential energy curve of the lih molecule. *The Journal of Chemical Physics*, 134(6):–, 2011.
- [47] Jacek Koput. Ab initio ground-state potential energy functions of beryllium monohydride ions: Beh+ and beh. *The Journal of Chemical Physics*, 139(10):–, 2013.
- [48] Evangelos Miliordos and Aristides Mavridis. Ab initio investigation of the electronic structure and bonding of bh, bh, and hbbh molecules. *The Journal of Chemical Physics*, 128(14):–, 2008.
- [49] Myung Won Lee, Massimo Mella, and Andrew M. Rappe. Electronic quantum monte carlo calculations of atomic forces, vibrations, and anharmonicities. *The Journal of Chemical Physics*, 122(24):–, 2005.