Compressed Sensing

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Algorithms interest group
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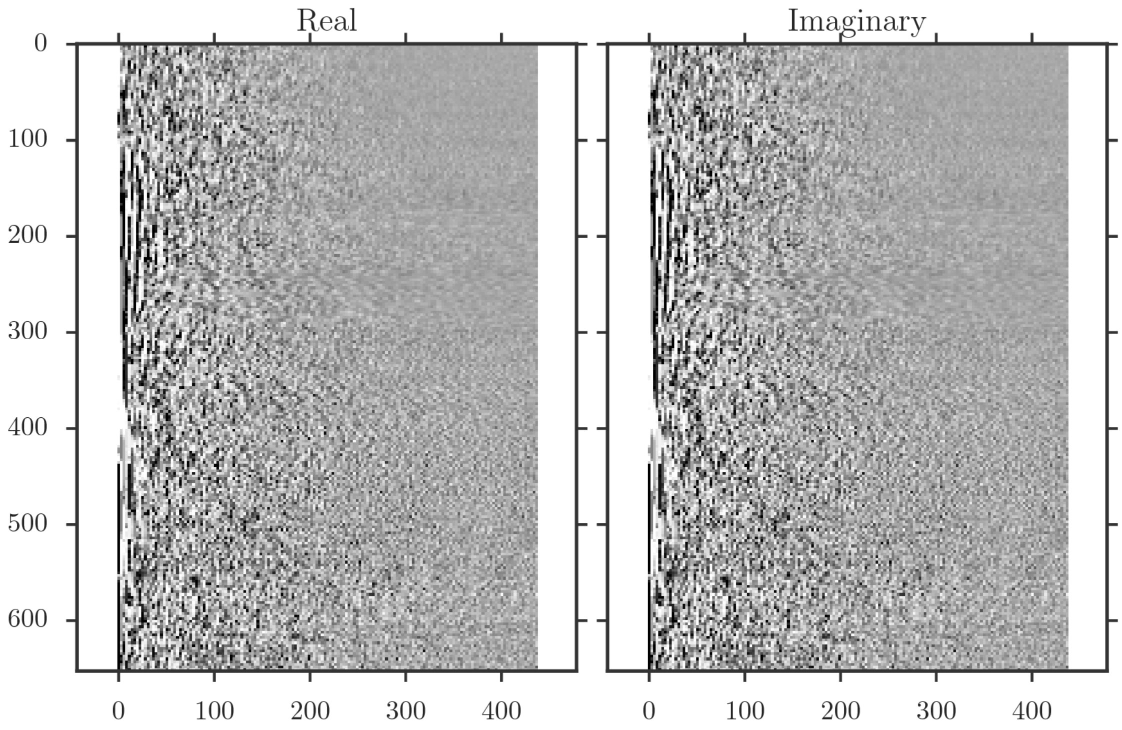
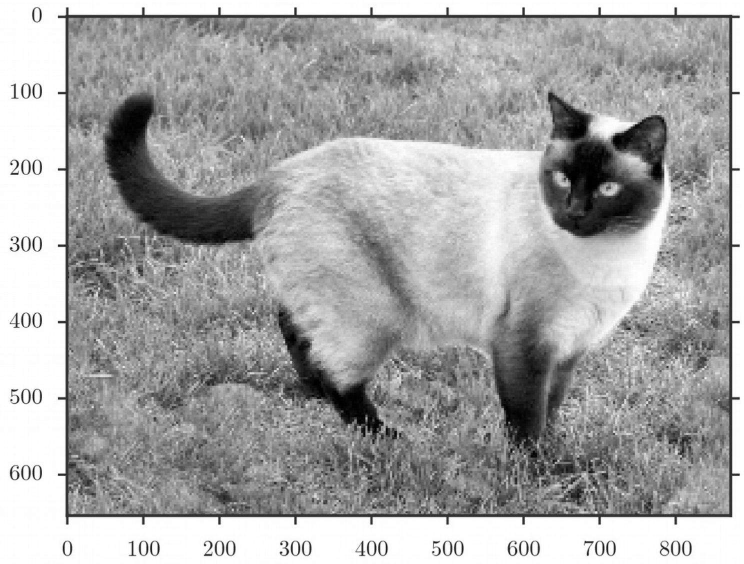
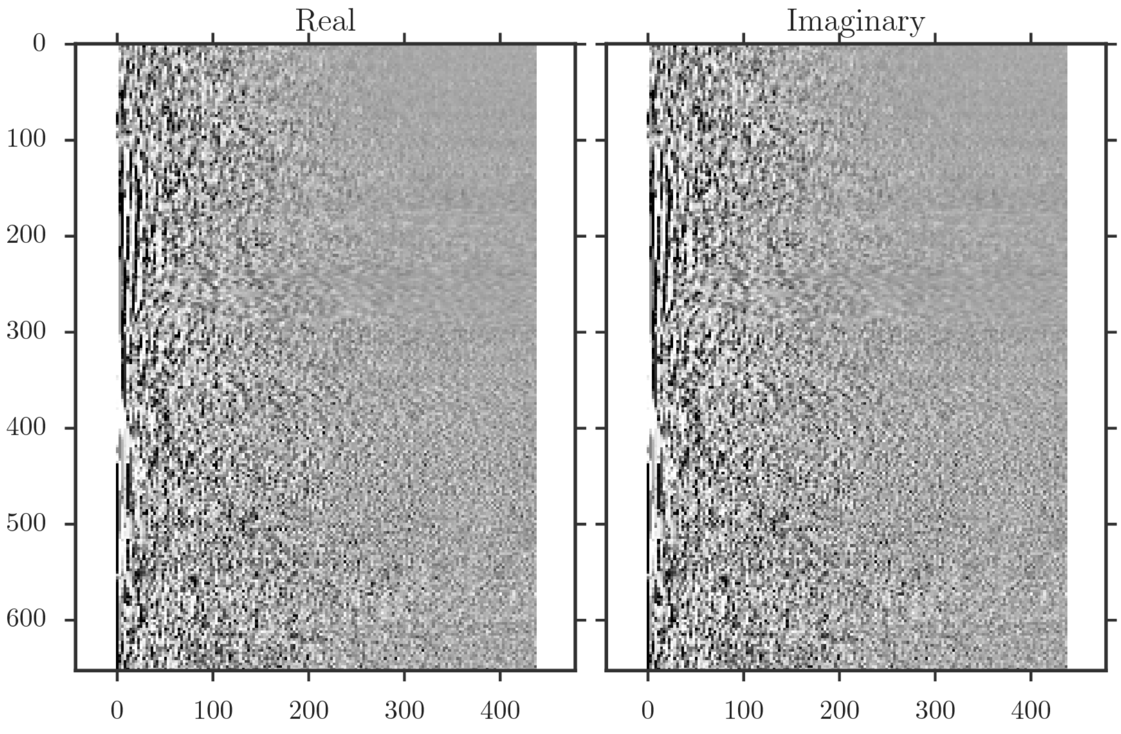


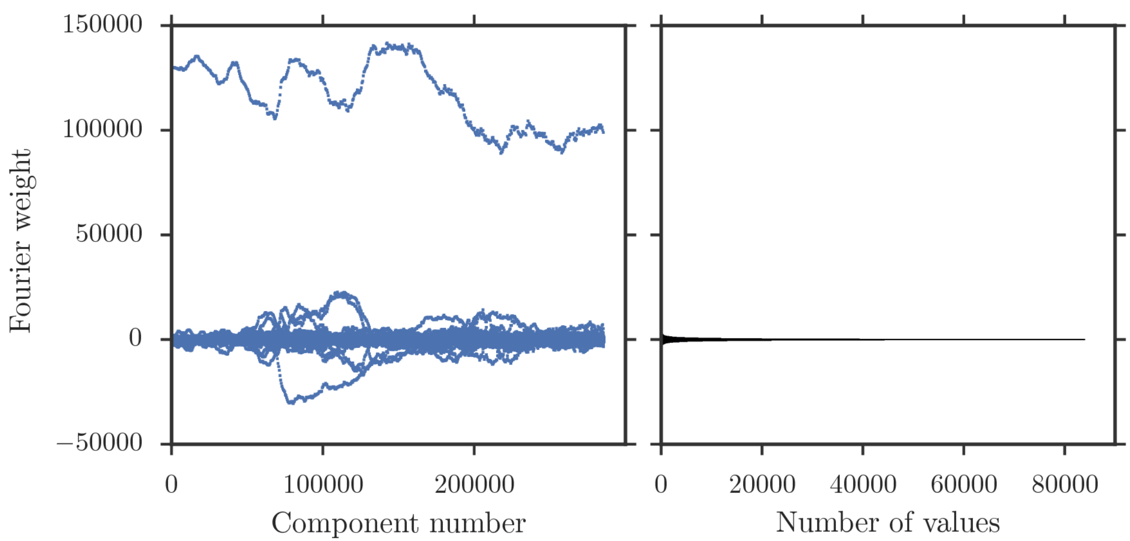
Compression algorithms often work really well.

Real space image:



Fourier space image:

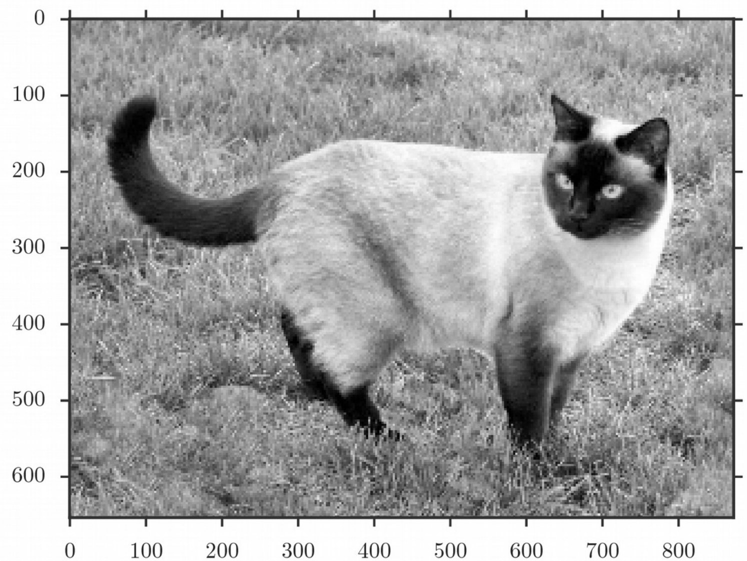




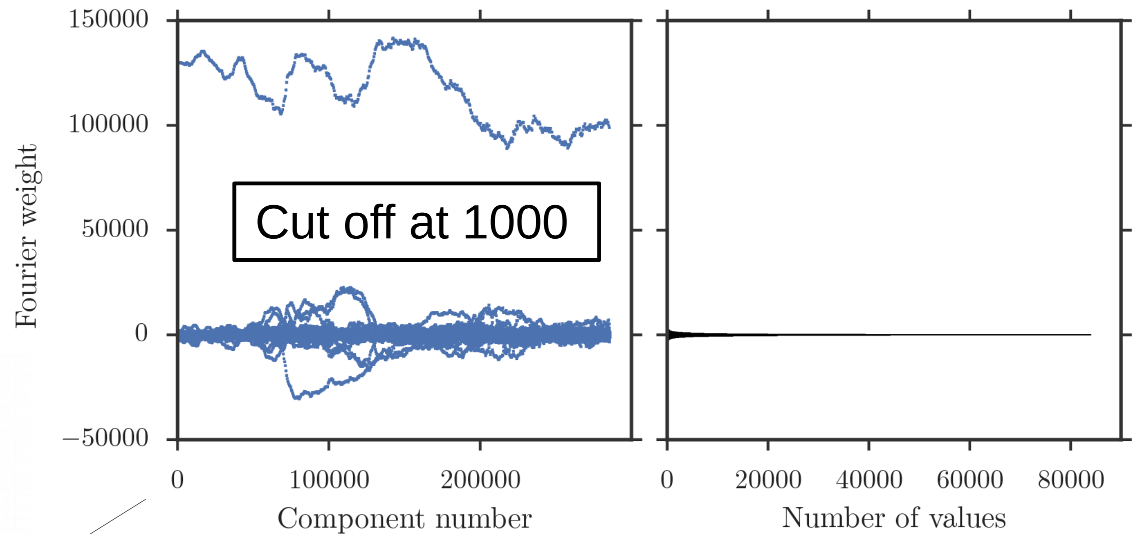
Almost all components are relatively small!

Compression algorithms often work really well.

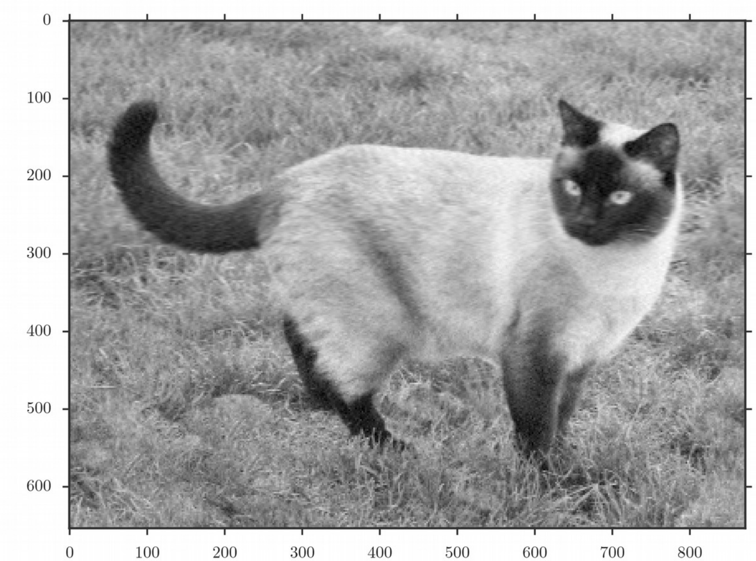
285,798 components



Fourier transform



Inverse transform



46,171 components

A “sparse” representation exists for most interesting signals

Do we *really* need to measure the whole of space?

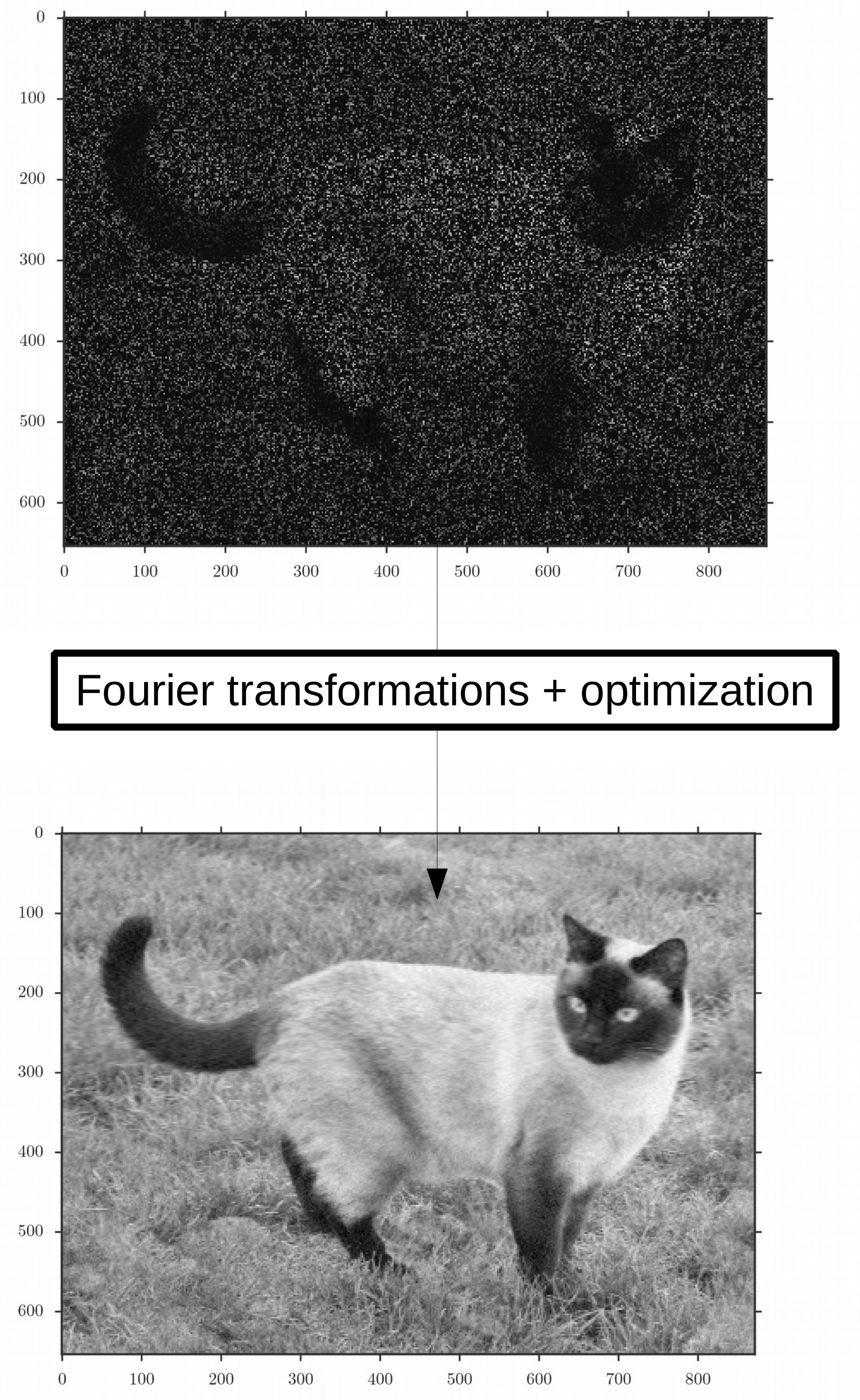
The idea.

1. Under-sample in non-sparse basis.

2. Underdetermined system of equations.

3. Impose additional constraint of sparseness

4. Minimize sparseness to determine unique solution.



The basics of the problem.

Fourier decompositions are examples of linear basis transformations.

$$\mathrm{Ax} = \mathrm{y}$$

Sparse (e.g. Fourier space) Nonsparse (e.g. real space)

If the length of y is less than the length of x, underdetermined: many x map to same y!

Least squares functions will allow you to minimize

$$||Ax - y||_2$$

over all possible x

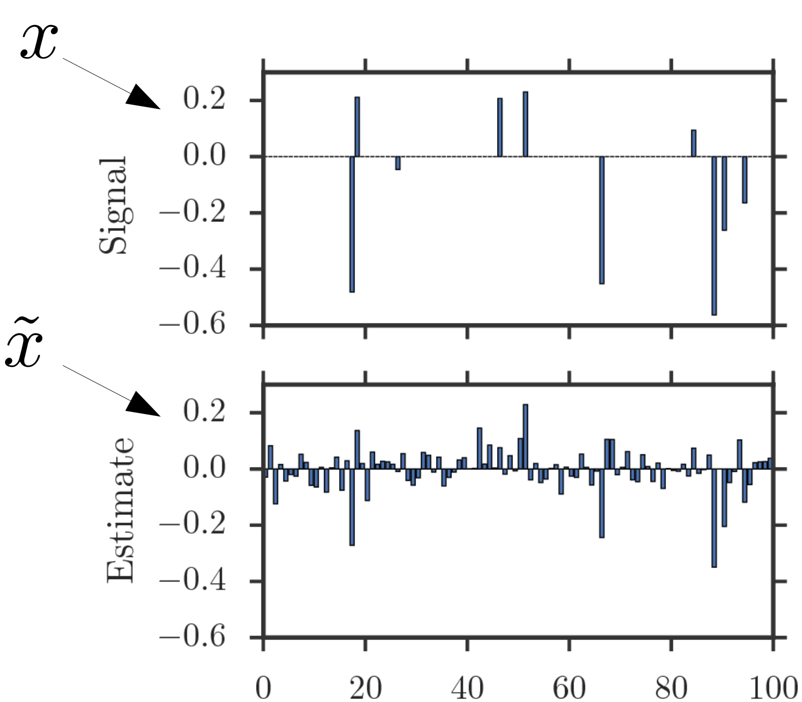
Test process.

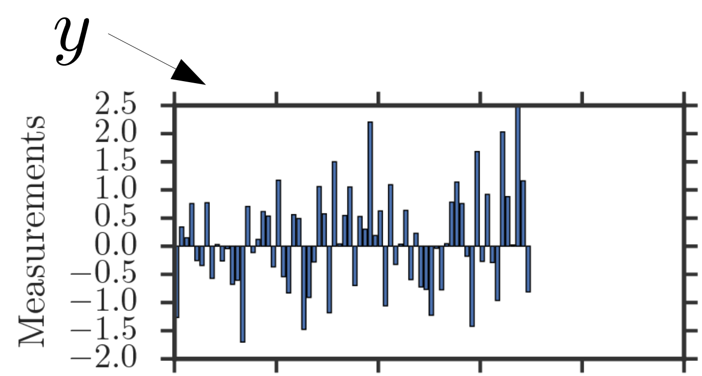
1. Create a sparse vector x (just make it mostly zeros).

2. Create a random basis changing matrix A.

3. Create a measurement vector y from $Ax = y$.

4. Create an estimate for x by minimizing $\|A\tilde{x} - y\|_2$ over $\tilde{x}$





Unlikely that we'll pick original signal from space of solutions!

Test process.

1. Create a sparse vector x (just make it mostly zeros).
2. Create a random basis changing matrix A.
3. Create a measurement vector y from $Ax = y$.
4. Create an estimate for x by minimizing $\|A\tilde{x} - y\|_2$ over $\tilde{x}$
5. Choose unique solution by minimizing number of components of $\tilde{x}$.

Prefer l_0 norm: number of nonzero entries (hard)

Instead, use l_1 norm, which we can use linear programming.

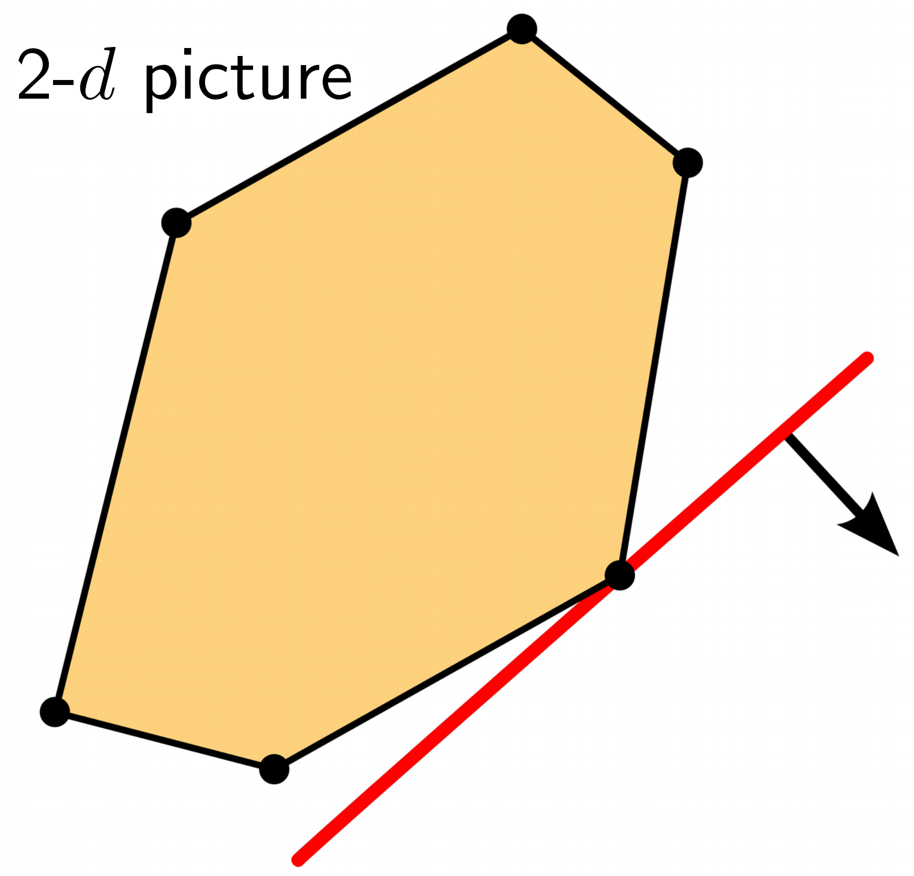
$$l_1 = \sum_i |x_i|$$

Linear programming.

Wide class of problems can be reduced to:

Minimize $c \cdot x$ subject to:

1. $M_1 x = b_1$
2. $M_2 x \leq b_2$



Our problem of minimizing components

Minimize $1 \cdot v$ subject to

$$\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ v \end{pmatrix} \leq 0 \qquad (1)$$

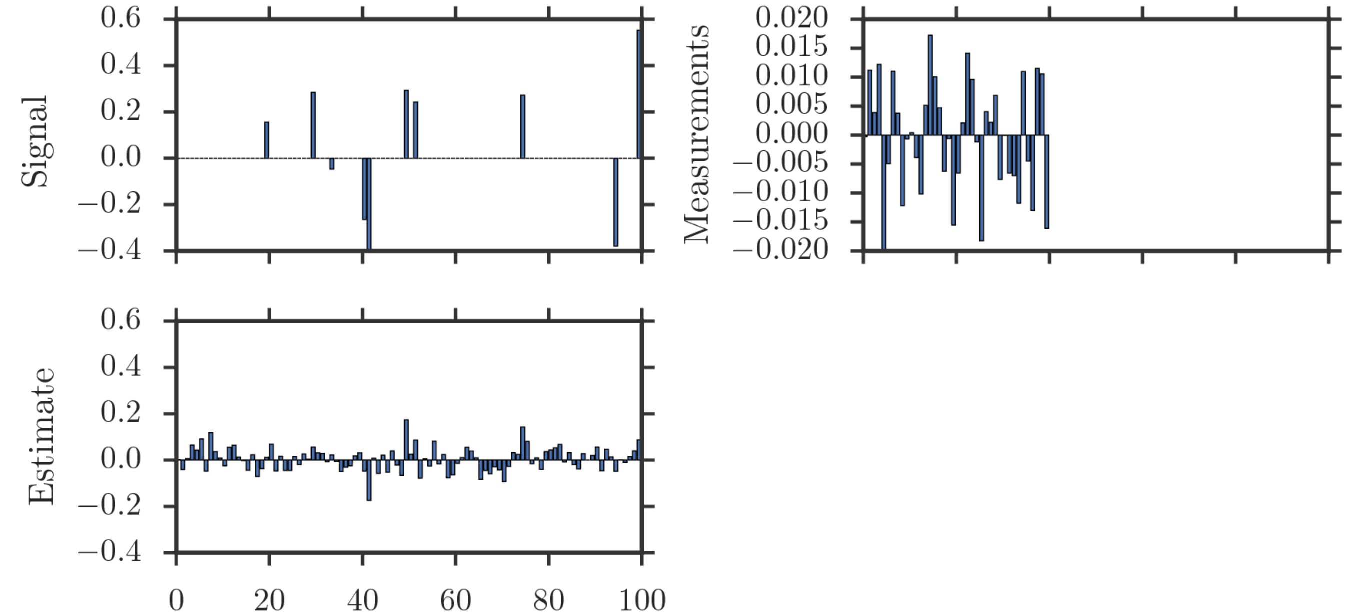
$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x} \\ v \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix} \qquad (2)$$

(1) is equivilent to $-v \leq \tilde{x} \leq v$, and minimizes $\|\tilde{x}\|_1$.

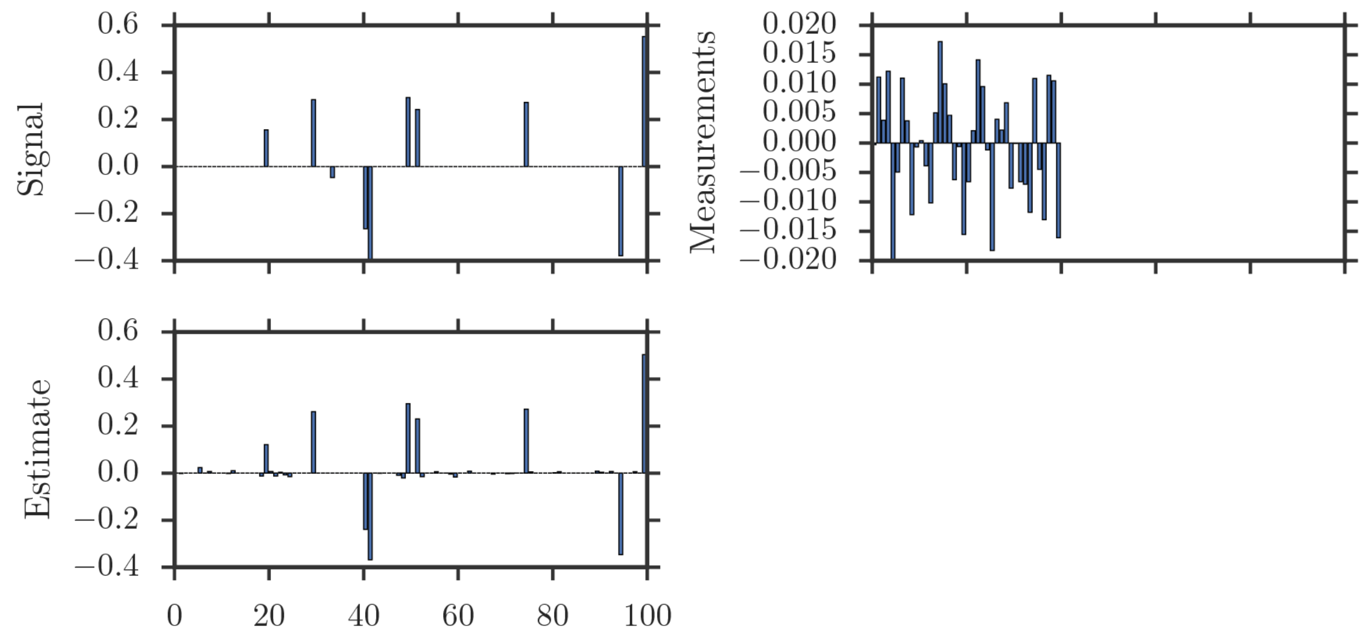
(2) constrains that $\tilde{x}$ would produce our measurements:

$$A\tilde{x} = y$$

Least square solution



Linear programming solution



Signal needs to be sparser then measurements.

K number of non zero components.

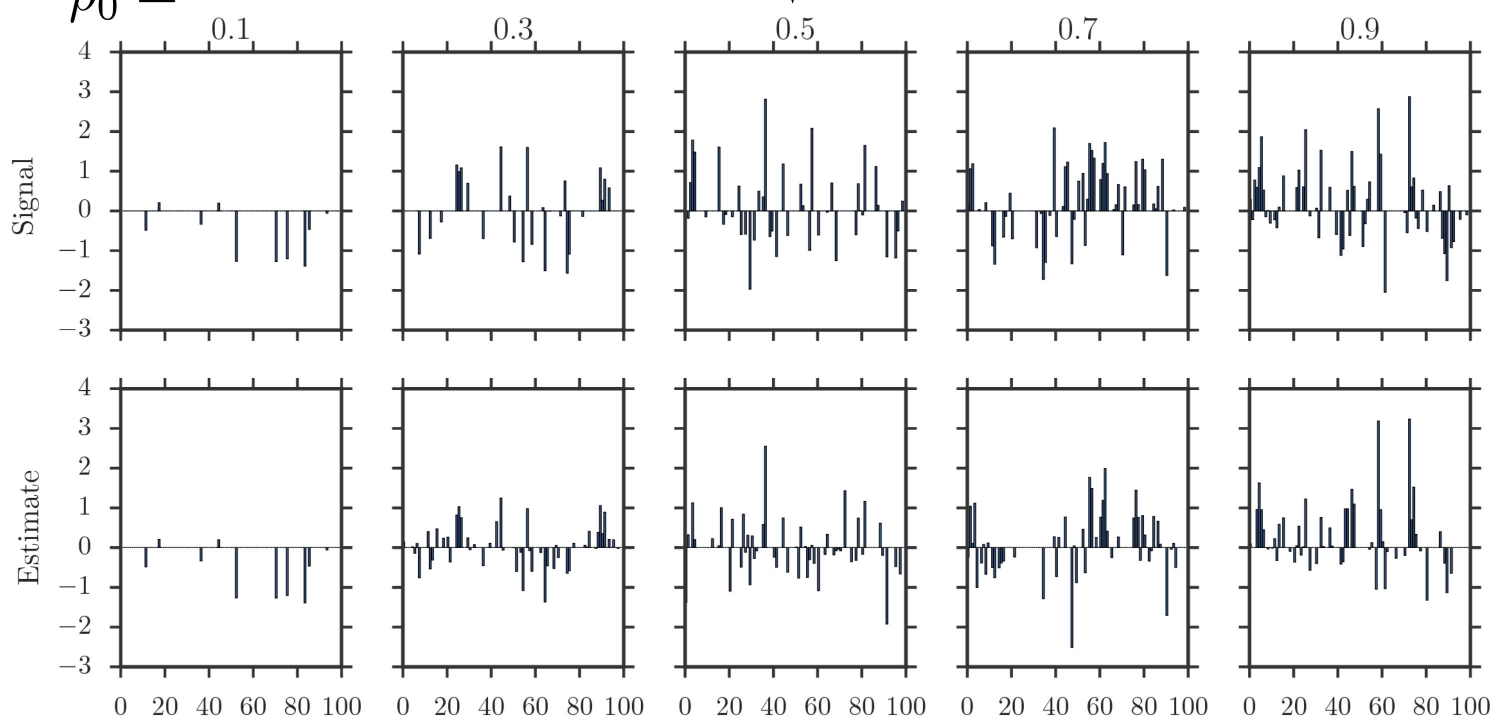
M number of measurements.

N number of vector components.

Before here, good match

$K \equiv \rho_0 N$ and $M \equiv \alpha N$

$\rho_0 =$



for $\alpha = 0.5$

As you accumulate measurements, signal stabilizes

K number of non zero components.

M number of measurements.

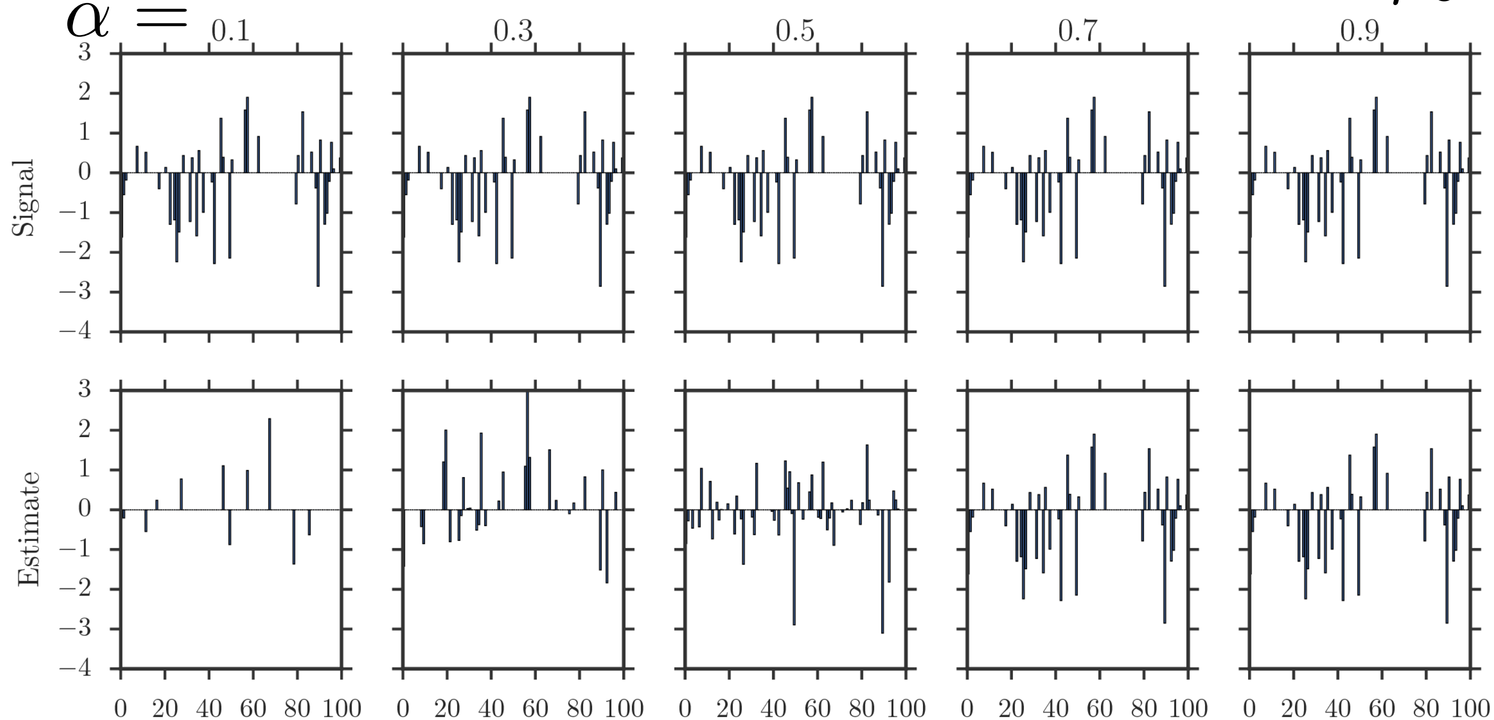
N number of vector components.

$K \equiv \rho_0 N$ and $M \equiv \alpha N$

After here, good match

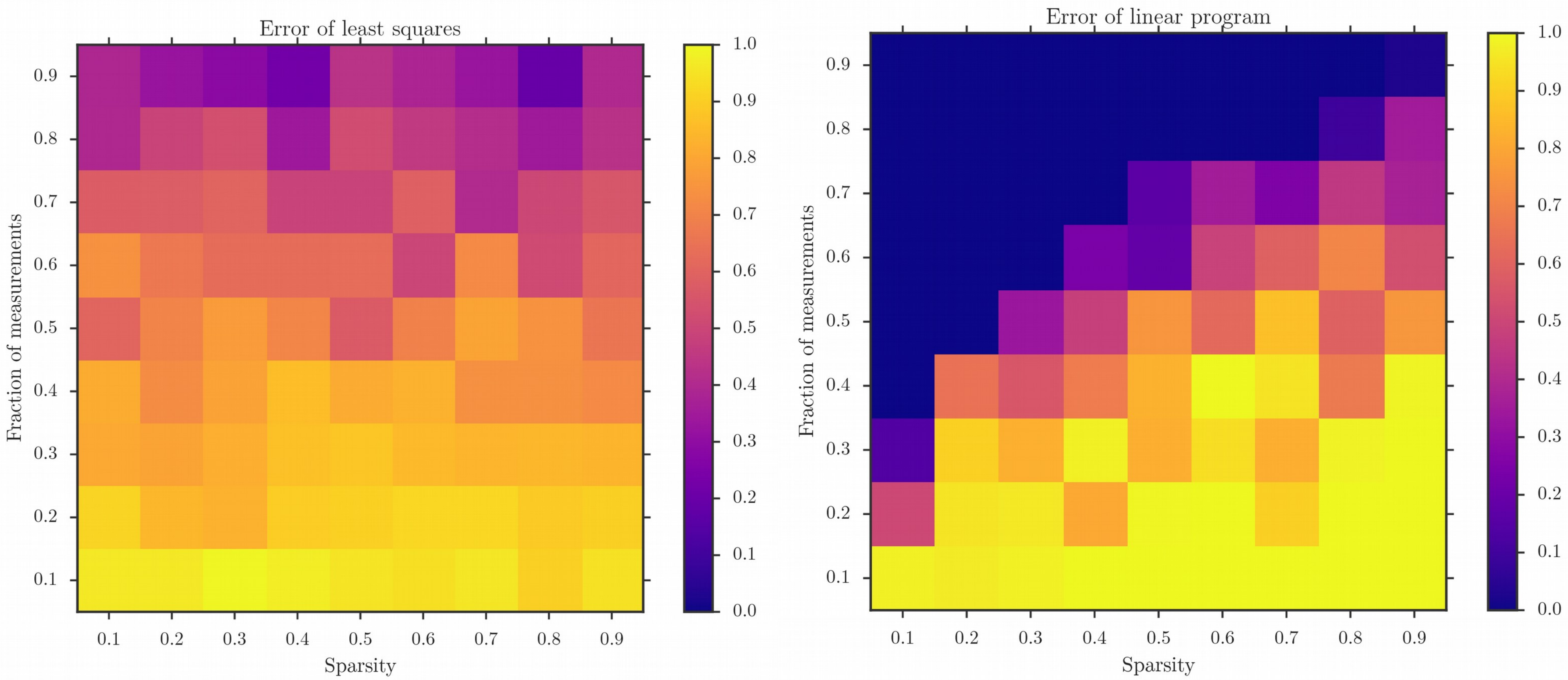
for $\rho_0 = 0.5$

$\alpha =$

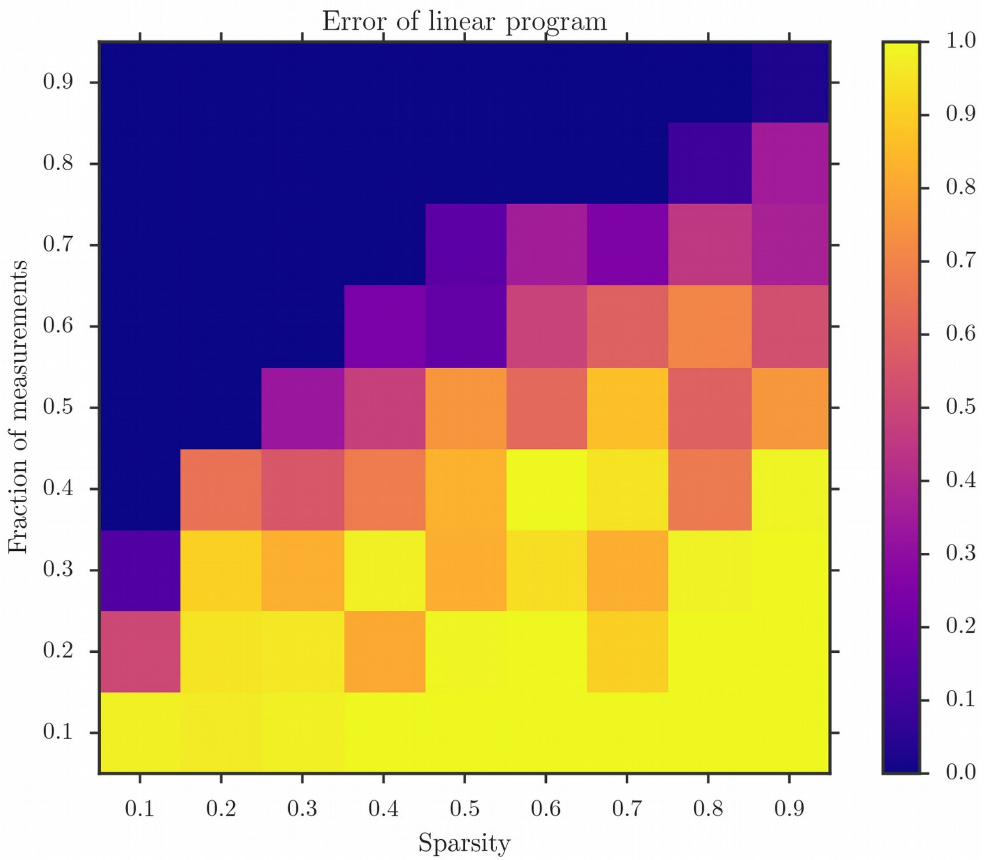


Mapping out the parameter space.

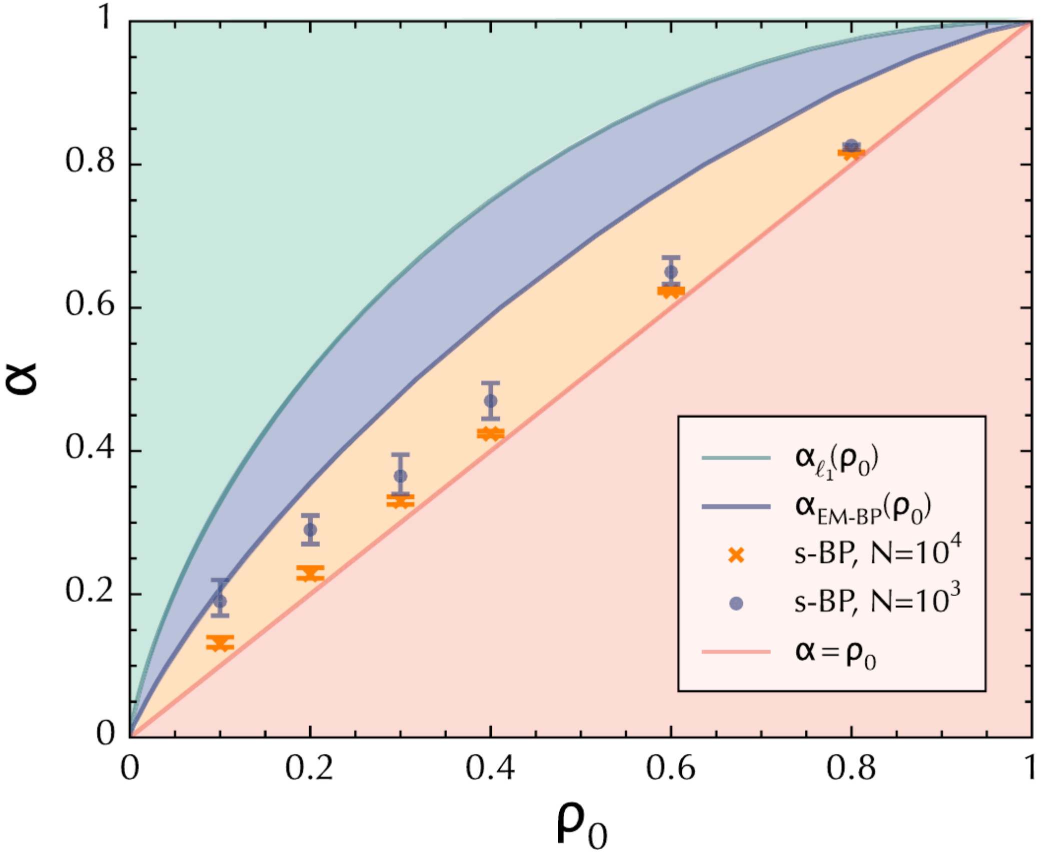
Comparison with straight least squares.



What is the state-of-the-art?



Krzakala *et. al*



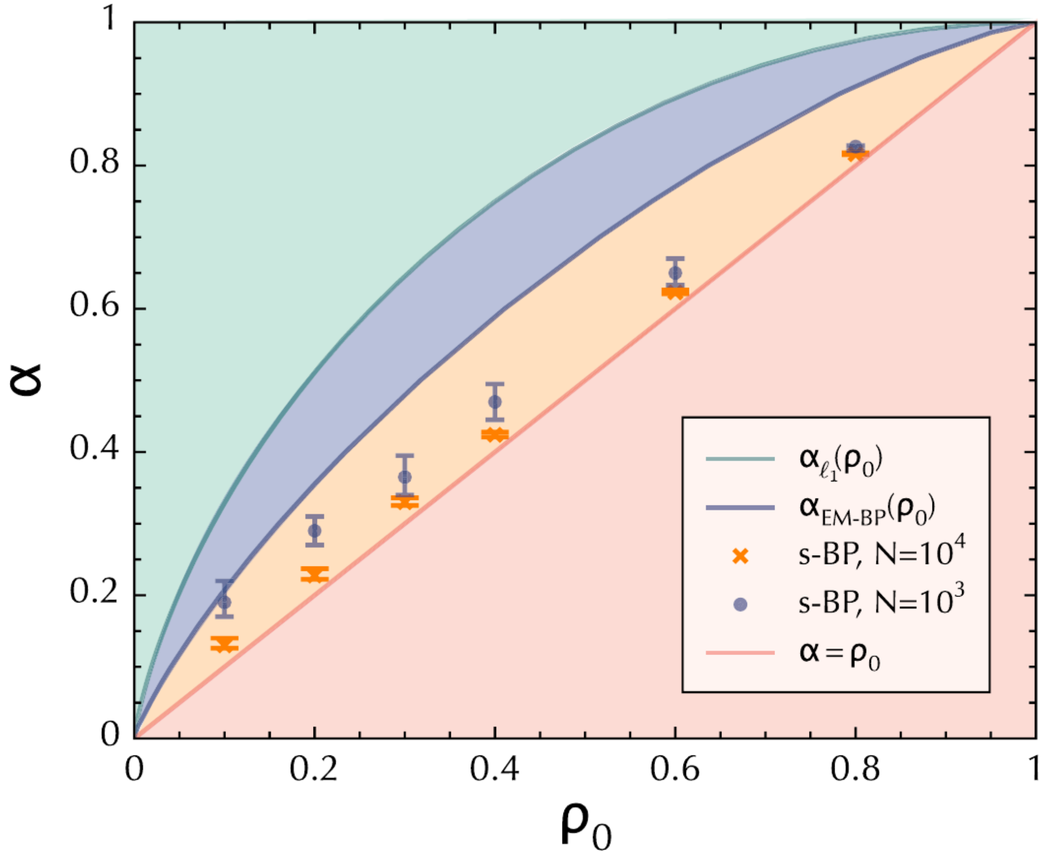
"Phase transition" in the limit of large signal sizes.

PHYS. REV. X **2,** 021005 (2012)

What is the state-of-the-art?

- Goal: Minimize l_0 norm through probabilistic method.
- Restrict movement to space where $Ax = y$.
- Message passing and belief propagation formalism.
- Analog with phase transitions
- "Seeding" crystallization.
- Also: "mean field" methods available.

Krzakala *et. al*



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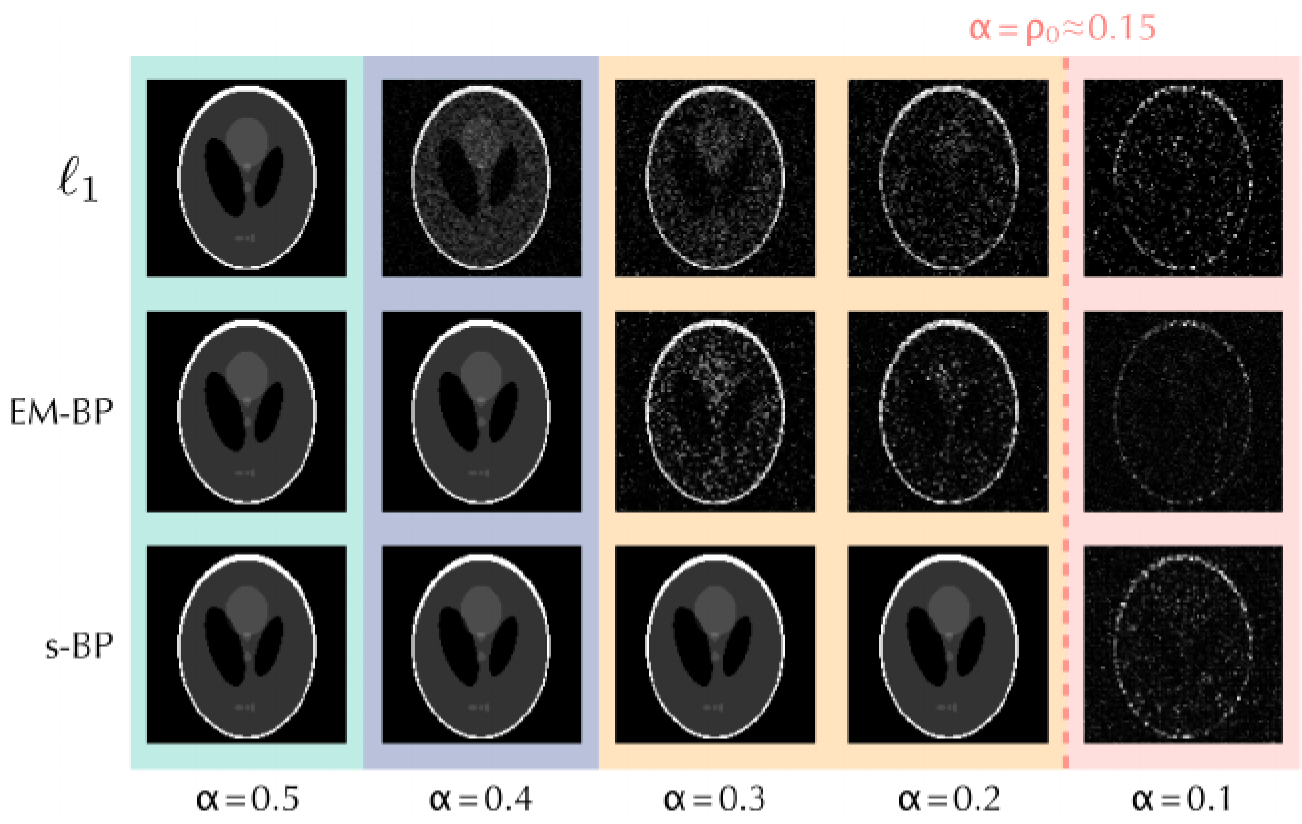
So what about the cat?

Sorry, the cat was too much for my code to handle.

Here's some examples from the paper.



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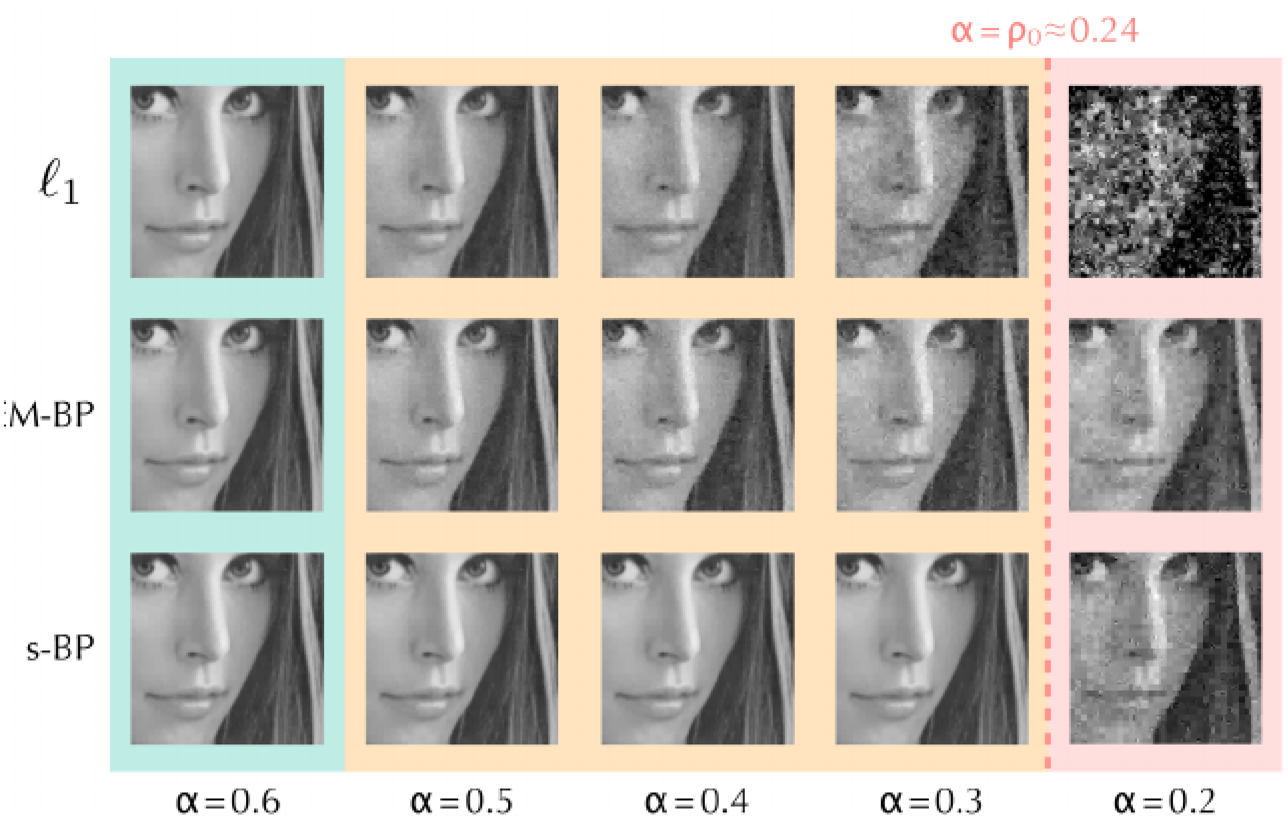
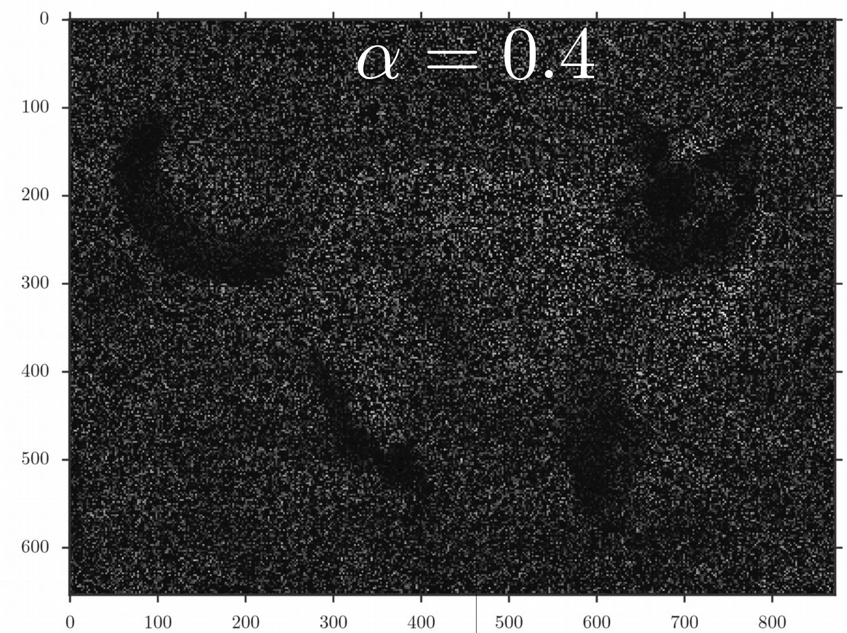
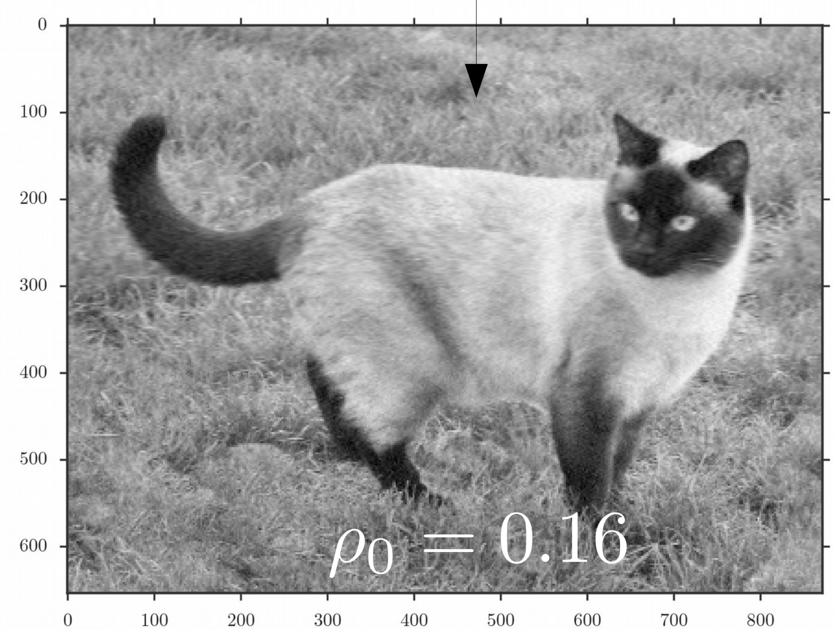


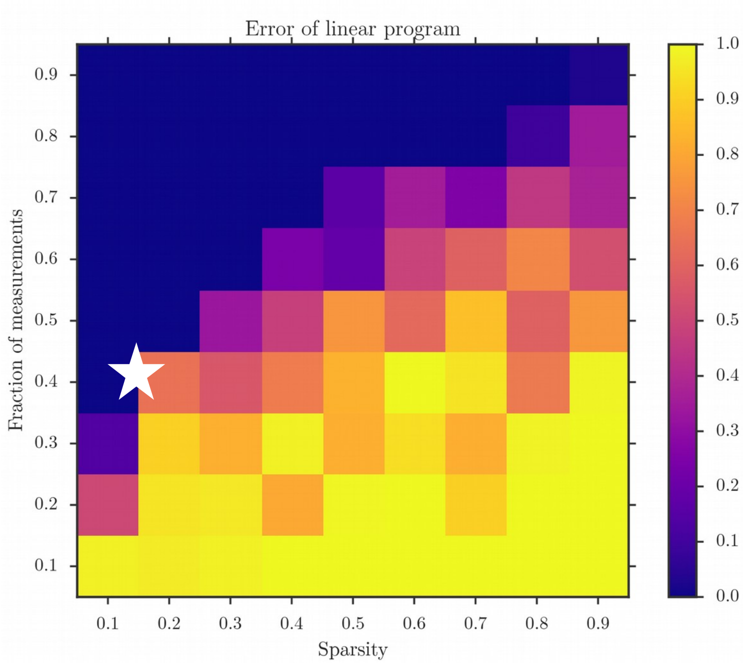
Summary

If a signal has a sparse representation, you can regenerate all of it with very few measurements!



Fourier transformations + optimization





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Simultaneous measurement:
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