Bayesian_HW_3_Whitson

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4/29/2020

Part 1.1: Data

4 out of 14 rats developed a tumor:

```
(Data <- c(s=14, k=4))
```

```
##  s  k
## 14  4
```

Part 1.2: Model

Based on historical mean and variance of beta-distributed parameter theta, determine the parameters alpha and beta of the beta distribution.

Using definition of mean and variance of a beta distribution and solving algebraically for alpha and beta, we have:

```
Sigma <- 0.1034
Mu <- 0.136
(a_prior <- (1/Sigma^2)*(Mu^2 - Mu^3) - Mu)
```

```
## [1] 1.358688
```

```
(b_prior <- a_prior*(1/Mu -1))
```

```
## [1] 8.631663
```

Part 1.3: Posterior Distribution by Formulas:

Assuming theta = probabilty of success, and assuming that theta has a beta-distributed prior distribution with parameters alpha and beta, after sampling n items with y successes, the posterior distribution will be a beta distribution with parameters (alpha + y) and (beta + y - n):

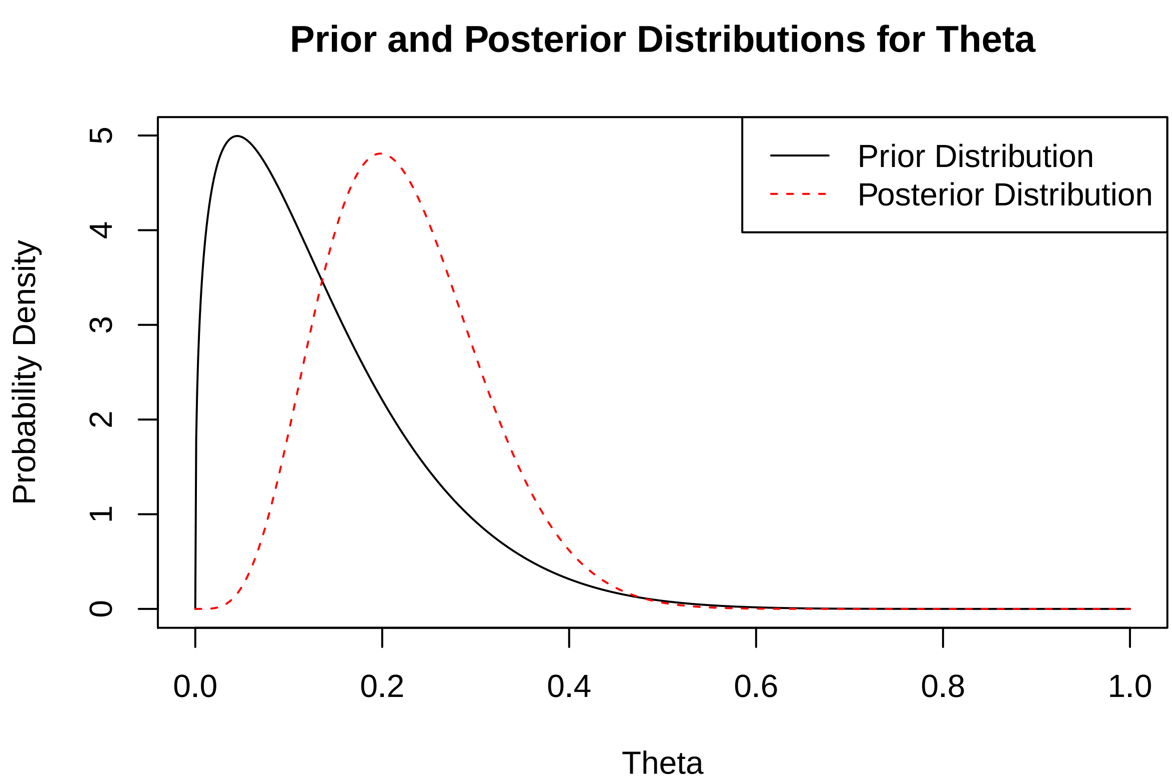
```
#Posterior Value of Alpha:
a_post <- a_prior + as.numeric(Data["k"])
#Posterior Value of Beta:
b_post <- b_prior + as.numeric(Data["s"]) - as.numeric(Data["k"])
(cbind("Alpha" = a_post,  "Beta" = b_post))
```

```
##         Alpha     Beta
## [1,] 5.358688 18.63166
```

Part 1.4 Comparison:

Compare prior and posterior distributions:

```
#define values for theta from 0 to 1:
Theta <- seq(0, 1, length.out = 1001)
#calculate probability density for prior and posterior distributions:
Prior1 <- dbeta(Theta, a_prior, b_prior)
Posterior1 <- dbeta(Theta, a_post, b_post)
#Plot data
matplot(x=Theta, y=cbind(Prior1, Posterior1), type = "l", main = "Prior and Posterior Distributions for
legend("topright", legend = c("Prior Distribution", "Posterior Distribution"), col = 1:2, lty = 1:2)
```



Summarize mean and mode of prior and posterior:

```
print(paste("Mode of Prior:", Theta[which.max(Prior1)]))
```

```
## [1] "Mode of Prior: 0.045"
```

```
print(paste("Mode of Posterior:", Theta[which.max(Posterior1)]))
```

```
## [1] "Mode of Posterior: 0.198"
```

```
print(paste("Mean of Prior:", a_prior/(a_prior+b_prior)))
```

```
## [1] "Mean of Prior: 0.136"
```

```
print(paste("Mean of Posterior:", a_post/(a_post+b_post)))
```

```
## [1] "Mean of Posterior: 0.223368458119222"
```

```
print(paste("StDev of Prior:", sqrt((a_prior+b_prior)/((a_prior+b_prior)^2*(a_prior+b_prior+1)))))
```

```
## [1] "StDev of Prior: 0.0954341553387098"
```

```
print(paste("StDev of Posterior:", sqrt((a_post*b_post)/((a_post+b_post)^2*(a_post+b_post+1)))))
```

```
## [1] "StDev of Posterior: 0.0833167374388394"
```

Discussion:

The prior distribution has a mean of 0.136 and a mode of approximately 0.045.

The observed data had 4 out of 14 “successes” (tumors), for a rate of approximately 0.286

The mean and mode of the posterior distribution are“pulled toward” the mean of the observed data, with a mean of 0.223 and a mode of 0.198

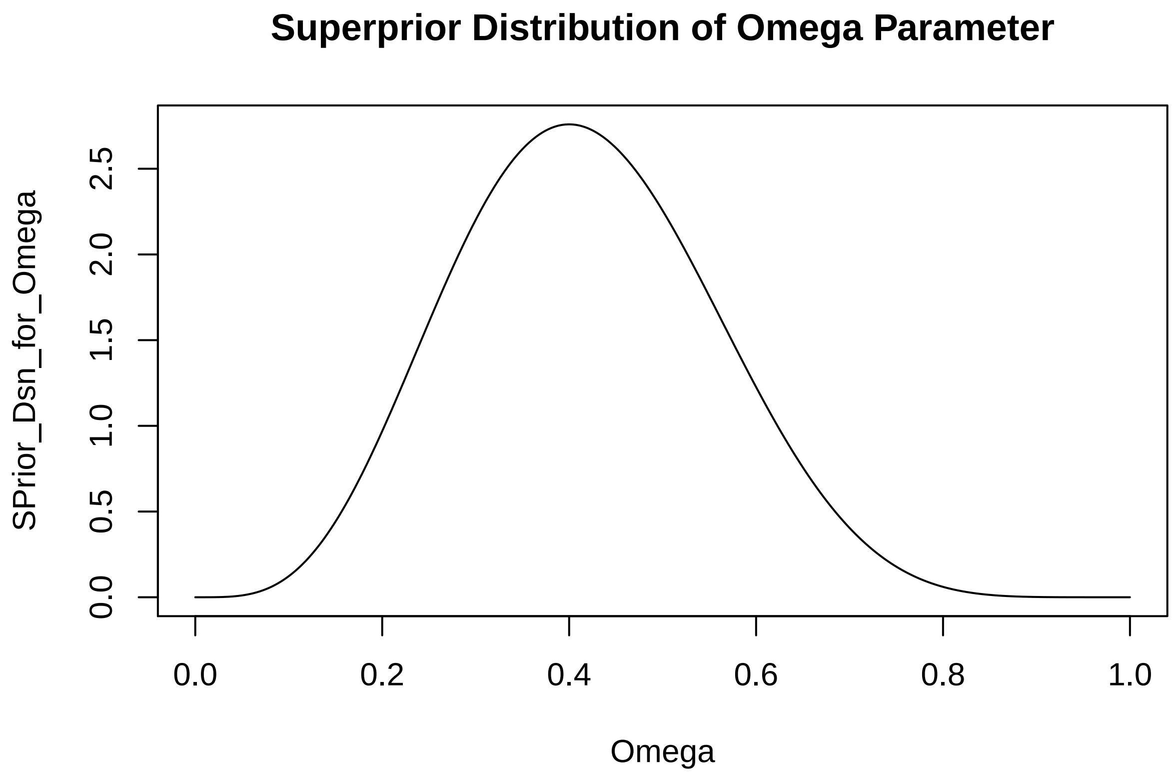
The standard deviation of the posterior distribution is 0.083, smaller than that of the prior distribution, reflecting the fact that there is less uncertainty about the parameter theta after observing the data.

Part 1.5: Grid approximation

Superprior distribution for Omega:

Omega is believed to be beta-distributed with intensity kappa = 12 and mode (omega) = 0.4.

```
Omega <- seq(0, 1, length.out = 1001)
SPrior_kappa <- 12
SPrior_omega <- 0.4
A_omega <-  SPrior_omega*(SPrior_kappa-2) + 1
B_omega <- (1-SPrior_omega)*(SPrior_kappa-2) + 1
#Superprior distribution for omega, the parameter for the prior distribution of theta:
SPrior_Dsn_for_Omega <- dbeta(Omega, A_omega, B_omega)
plot(Omega, SPrior_Dsn_for_Omega, type = "l", main = "Superprior Distribution of Omega Parameter")
```



Prior Distribution for Theta

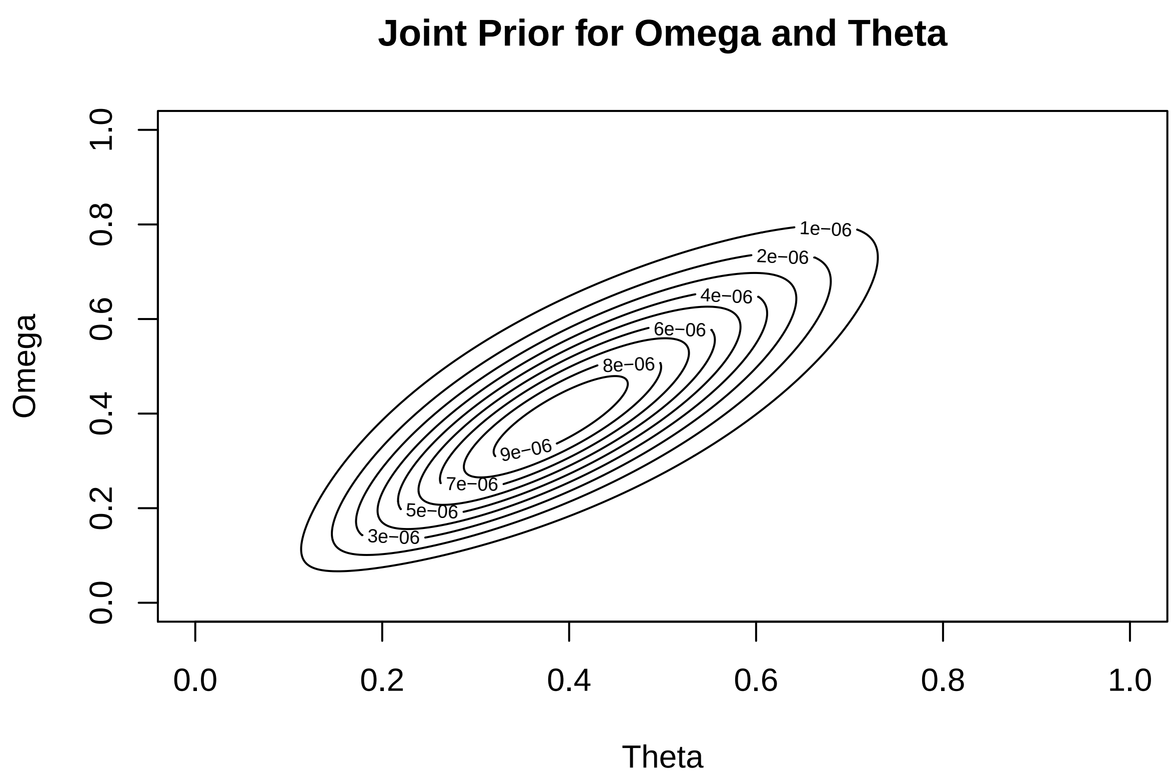
Assume theta is has a beta-distributed prior with a mode of omega (unknown, characterized by the superprior distribution above), and an intensity of kappa = 20

```
Prior_kappa <- 20

#Create Joint Prior for Theta and Omega

#initialize matrix
JointPrior <- matrix(nrow=1001, ncol = 1001, dimnames = list(paste("Omega=", Omega), paste("Theta=", The

#Calculate initial densities for joint prior:
#For each value of theta, calculate the probability of theta for all values
#of omega (using the prior distribution of theta given omega),
#times the probability of each value of omega
#(using the superprior distribution for omega)

for (i in 1:length(Theta)){
    JointPrior[,i] = SPrior_Dsn_for_Omega *
      dbeta(Theta[i], shape1 = Omega*(Prior_kappa - 2) + 1, shape2 = (1-Omega)*(Prior_kappa -2) + 1)
}

#Normalize joint prior to 1:
```

```r
JointPrior <- JointPrior / sum(JointPrior)
#Display first few rows and columns of joint prior:
JointPrior[1:5,1:5]
```

```
##              Theta= 0 Theta= 0.001 Theta= 0.002 Theta= 0.003 Theta= 0.004
## Omega= 0            0 0.000000e+00 0.000000e+00 0.000000e+00 0.000000e+00
## Omega= 0.001        0 4.028551e-14 4.006326e-14 3.963573e-14 3.912901e-14
## Omega= 0.002        0 6.020571e-13 6.062634e-13 6.041982e-13 5.995814e-13
## Omega= 0.003        0 2.845387e-12 2.901292e-12 2.912641e-12 2.905444e-12
## Omega= 0.004        0 8.390933e-12 8.663367e-12 8.761122e-12 8.785003e-12
```

```r
#Plot joint prior distribution of theta and omega:
JointPriorGraph <- contour(x = Theta, y = Omega,z = JointPrior,
                           main = "Joint Prior for Omega and Theta",
                           xlab = "Theta", ylab = "Omega")
```



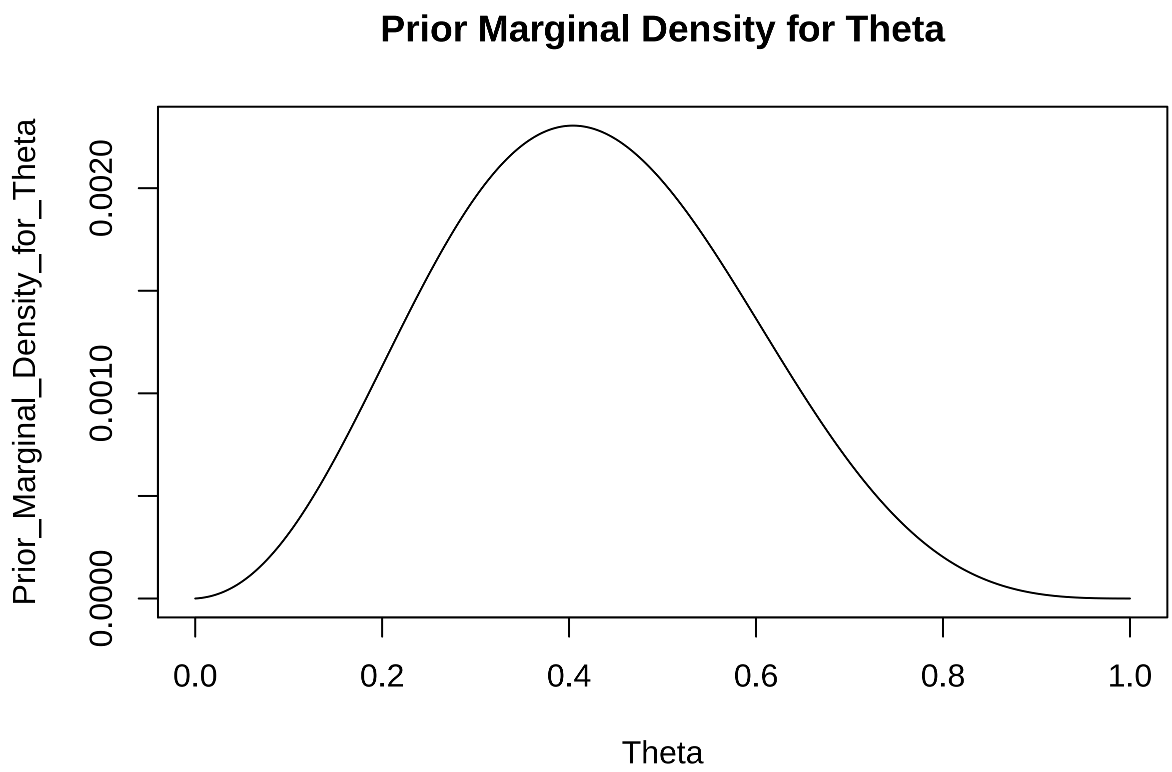
The joint prior indicates a positive correlation between the value of omega and the value of theta; if omega is large, we would expect theta to be large as well.

This is reasonable, as omega represents the mode of the marginal prior distribution for theta. So if omega is high (i.e., close to 1), we would expect theta to also be close to 1, because that is where the mode of theta's prior distribution will be.

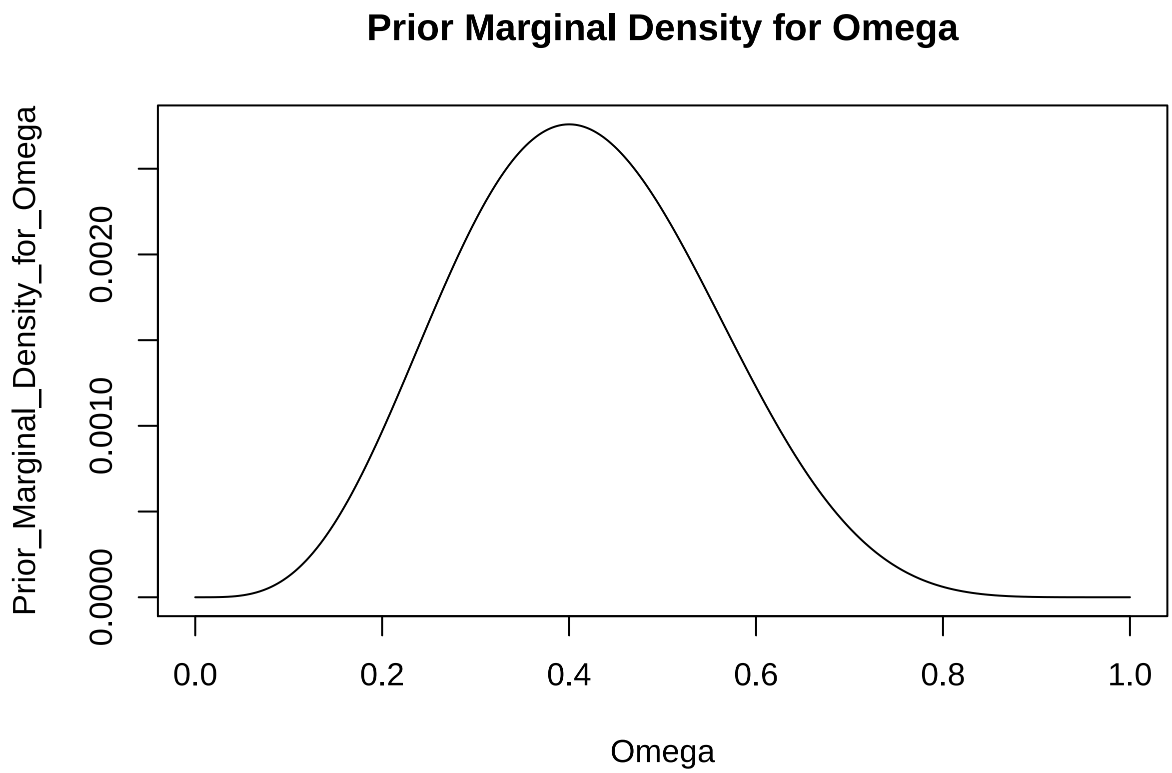
The prior distribution for omega showed a mode at 0.4; the joint distribution shows that the mode occurs about where both theta and omega = 0.4

Calculate marginal prior distributions for Theta and Omega:

```
#Marginal Density for Theta:  Sum joint prior over all values of omega for each value of theta:
Prior_Marginal_Density_for_Theta <- apply(JointPrior, MARGIN = 2, FUN = sum)
Prior_Marginal_Density_for_Theta <- Prior_Marginal_Density_for_Theta/sum(Prior_Marginal_Density_for_The
plot(x=Theta, y=Prior_Marginal_Density_for_Theta, main = "Prior Marginal Density for Theta", type = "l")
```




```
#Reconstruct prior marginal density for Omega:
Prior_Marginal_Density_for_Omega <- apply(JointPrior, MARGIN = 1, FUN = sum)
Prior_Marginal_Density_for_Omega <- Prior_Marginal_Density_for_Omega/sum(Prior_Marginal_Density_for_Omeg
plot(x=Omega, y=Prior_Marginal_Density_for_Omega, main = "Prior Marginal Density for Omega", type = "l")
```



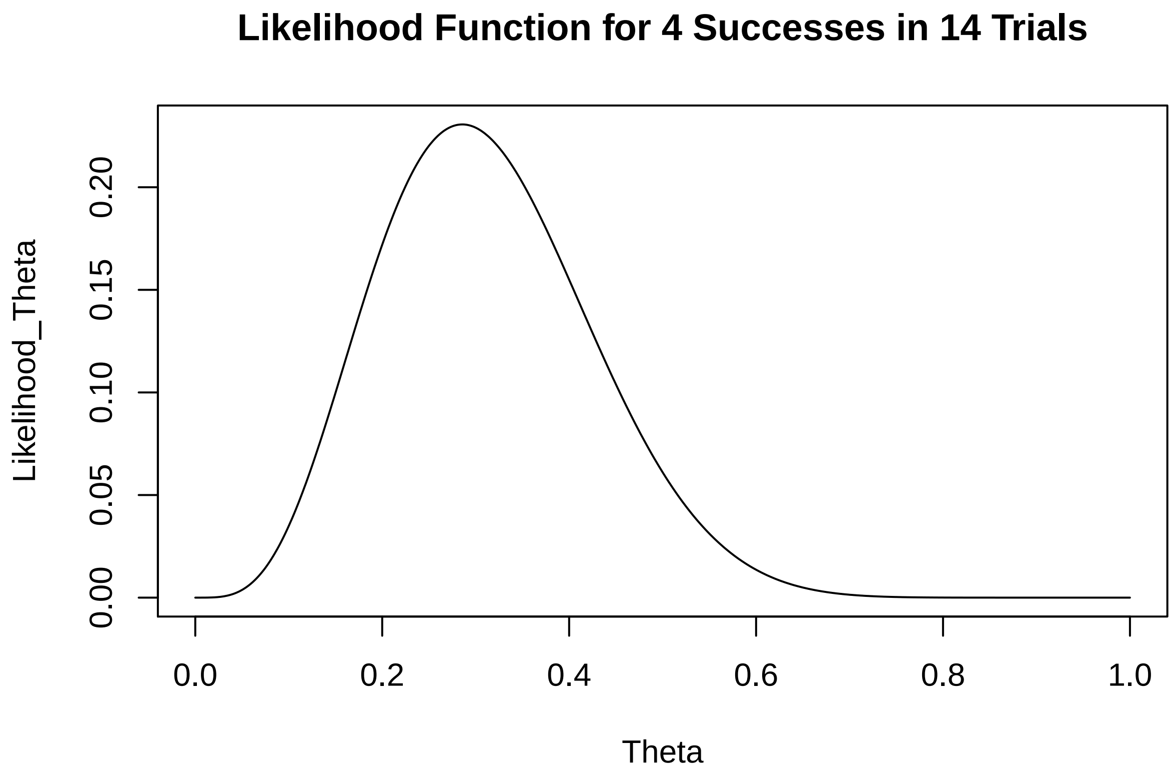
Dependence of Omega and Theta Based on Prior Distributions:

Because the superprior distribution for omega has its own omega parameter value of 0.4, we expect omega to have a mode at 0.4. Because omega and theta are correlated as shown in the joint prior, the prior distribution for theta also has a mode at 0.4.

Find likelihood function for observed data.

Likelihood function depends directly only upon theta, not upon omega.

```
Likelihood_Theta <- dbinom(x=4, size=14, prob=Theta)
plot(x=Theta, y=Likelihood_Theta,
     main = "Likelihood Function for 4 Successes in 14 Trials", type = "l")
```




```
#Find mode:
print(paste("Mode of Likelihood Function:", Theta[which.max(Likelihood_Theta)]))
```

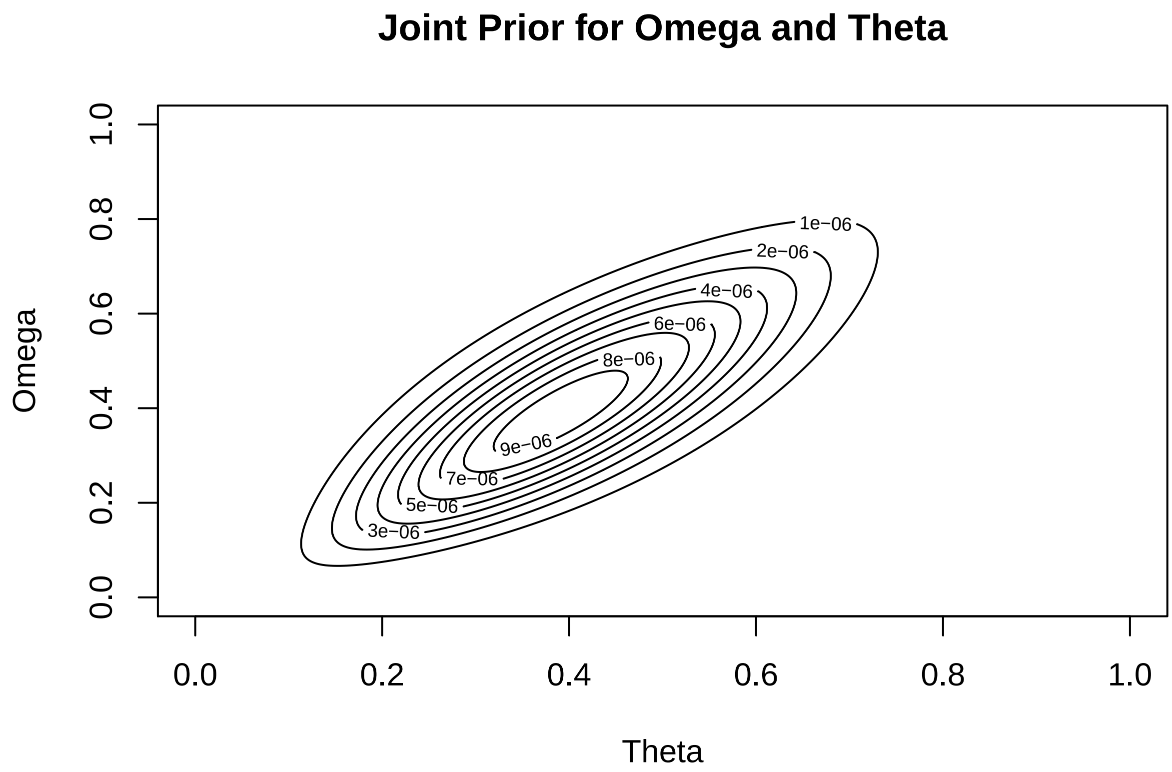
```
## [1] "Mode of Likelihood Function: 0.286"
```

Note that the mode of the likelihood function occurs at the value observed in the data: $4/14 = 0.286$.

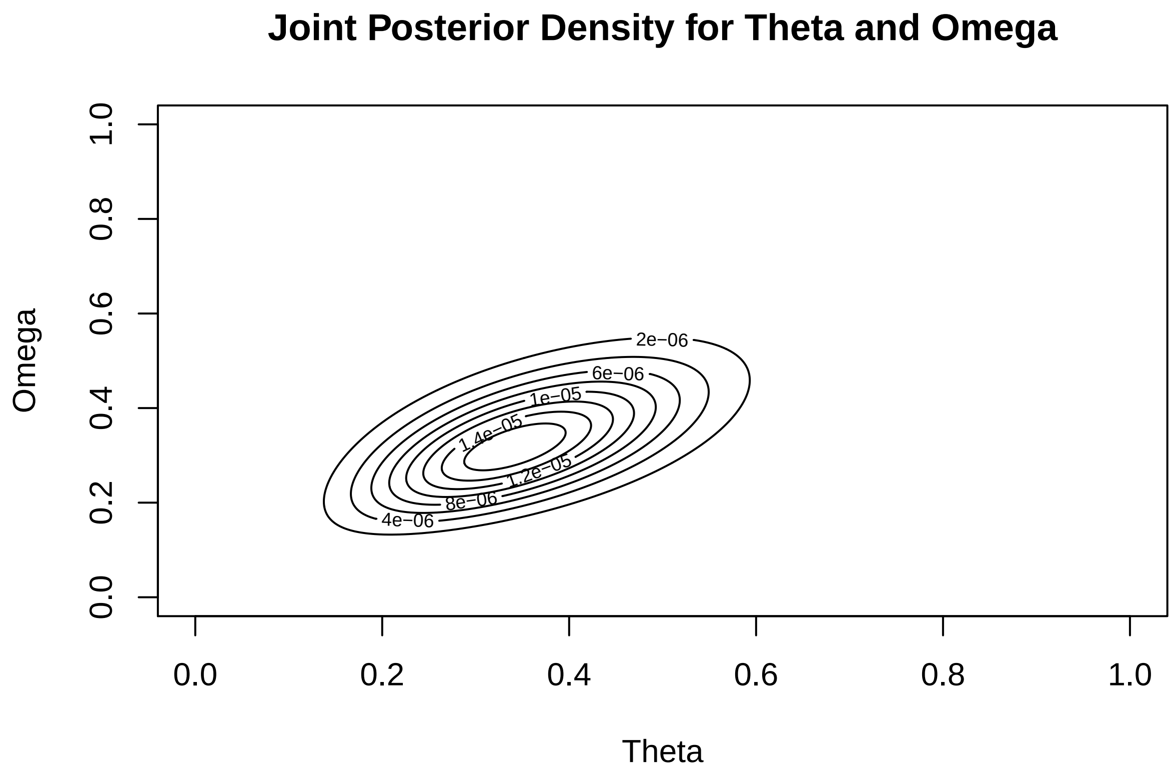
Calculate Joint Posterior for Omega and Theta

```
#Initialize matrix
JointPosterior <- matrix(nrow=1001, ncol = 1001, dimnames = list(paste("Omega=", Omega), paste("Theta="

#Calculate initial densities for joint posterior:
for (i in 1:length(Omega)){
    JointPosterior[i,] = Likelihood_Theta * JointPrior[i,]
}

#Normalize joint posterior:
JointPosterior <- JointPosterior / sum(JointPosterior)

#Plot joint prior and joint posterior distributions:
JointPriorGraph <- contour(x = Omega, y = Theta, z = JointPrior,
```

```
                  main = "Joint Prior for Omega and Theta",
                  xlab = "Theta", ylab = "Omega")
```




```
JointPosteriorGraph <- contour(x = Omega, y = Theta, z = JointPosterior,
                               main = "Joint Posterior Density for Theta and Omega",
                               xlab = "Theta", ylab = "Omega")
```




```r
#Find mode of joint posterior distribution:
Theta[which((JointPosterior == max(JointPosterior)), arr.ind = TRUE)[2]]
```

```
## [1] 0.317
```

```r
Omega[which((JointPosterior == max(JointPosterior)), arr.ind = TRUE)[1]]
```

```
## [1] 0.341
```

Note that the posterior joint distribution appears tighter / more concentrated, as we would expect given that we have collected data and reduced uncertainty about the values of theta and omega.

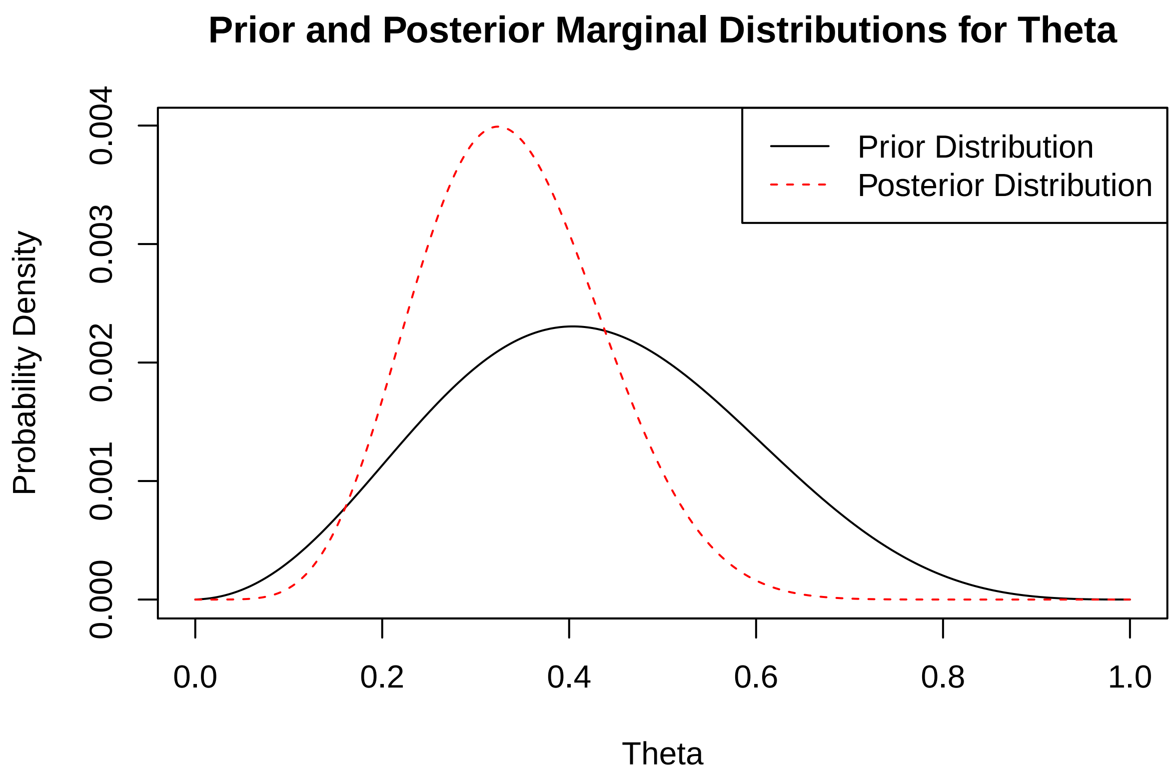
Center of highest density (mode) has shifted from being at Omega = Theta = 0.4 in the prior distribution to being at about Omega = 0.341 and Theta = 0.317 in the posterior distribution.

This is reasonable, as the data value was $4/14 = 0.286$, the mode of the posterior distributions are a compromise between the original values of 0.4 and the data value of 0.286

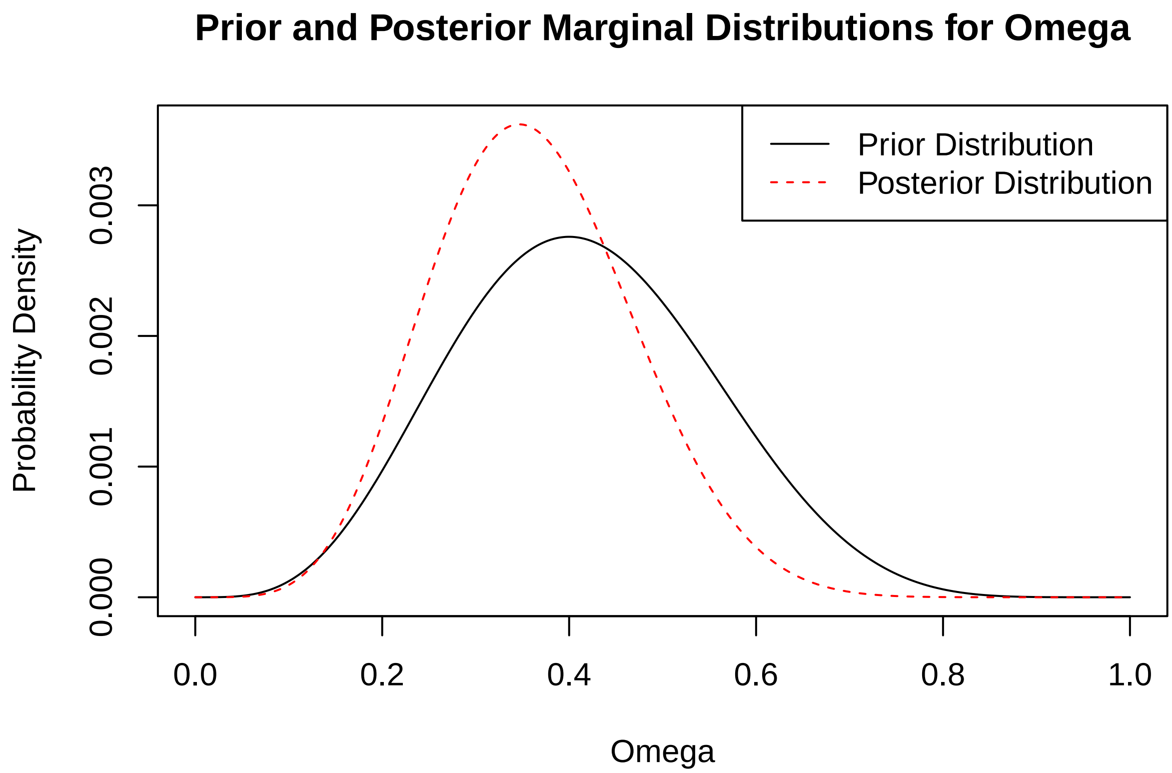
Compute marginal posterior distribution for Theta and Omega

```r
normalize <- function (x) x / sum(x)
```

```r
Posterior_Marginal_Theta <- normalize(apply(JointPosterior, MARGIN = 2, FUN = sum))

matplot(x=Theta, y=cbind(Prior_Marginal_Density_for_Theta, Posterior_Marginal_Theta), type = "l",
        main = "Prior and Posterior Marginal Distributions for Theta", ylab = "Probability Density")
legend("topright", legend = c("Prior Distribution", "Posterior Distribution"), col = 1:2, lty = 1:2)
```




```r
Posterior_Marginal_Omega <- normalize(apply(JointPosterior, MARGIN = 1, FUN = sum))

matplot(x=Omega, y=cbind(Prior_Marginal_Density_for_Omega, Posterior_Marginal_Omega), type = "l",
        main = "Prior and Posterior Marginal Distributions for Omega", ylab = "Probability Density")
legend("topright", legend = c("Prior Distribution", "Posterior Distribution"), col = 1:2, lty = 1:2)
```




```r
paste("Omega Posterior Mean: ", (sum(Posterior_Marginal_Omega*Omega)))
```

```
## [1] "Omega Posterior Mean:  0.362106855183688"
```

```r
paste("Omega Posterior Mode: ", Omega[which.max(Posterior_Marginal_Omega)])
```

```
## [1] "Omega Posterior Mode:  0.347"
```

```r
paste("Theta Posterior Mean: ", (sum(Posterior_Marginal_Theta*Theta)))
```

```
## [1] "Theta Posterior Mean:  0.338762452744305"
```

```r
paste("Theta Posterior Mode: ", Theta[which.max(Posterior_Marginal_Theta)])
```

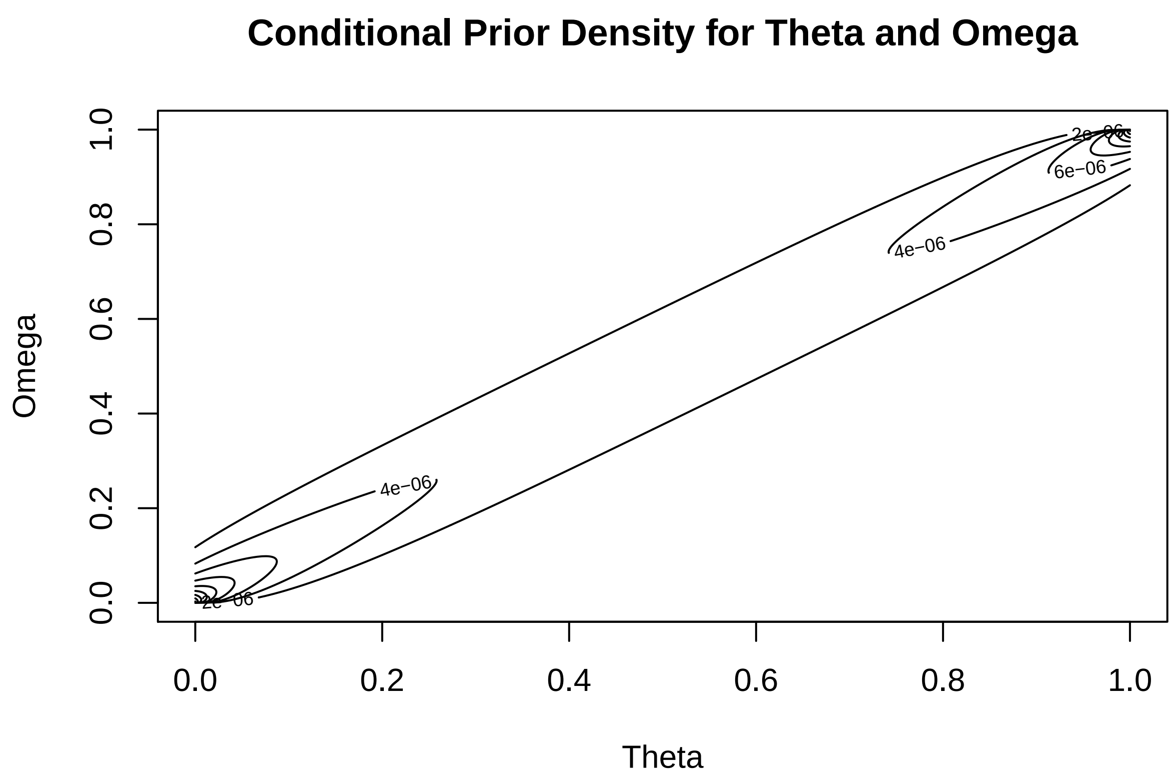
```
## [1] "Theta Posterior Mode:  0.324"
```

The posterior distributions of both omega and theta have been pulled toward the observed data value of 0.29.

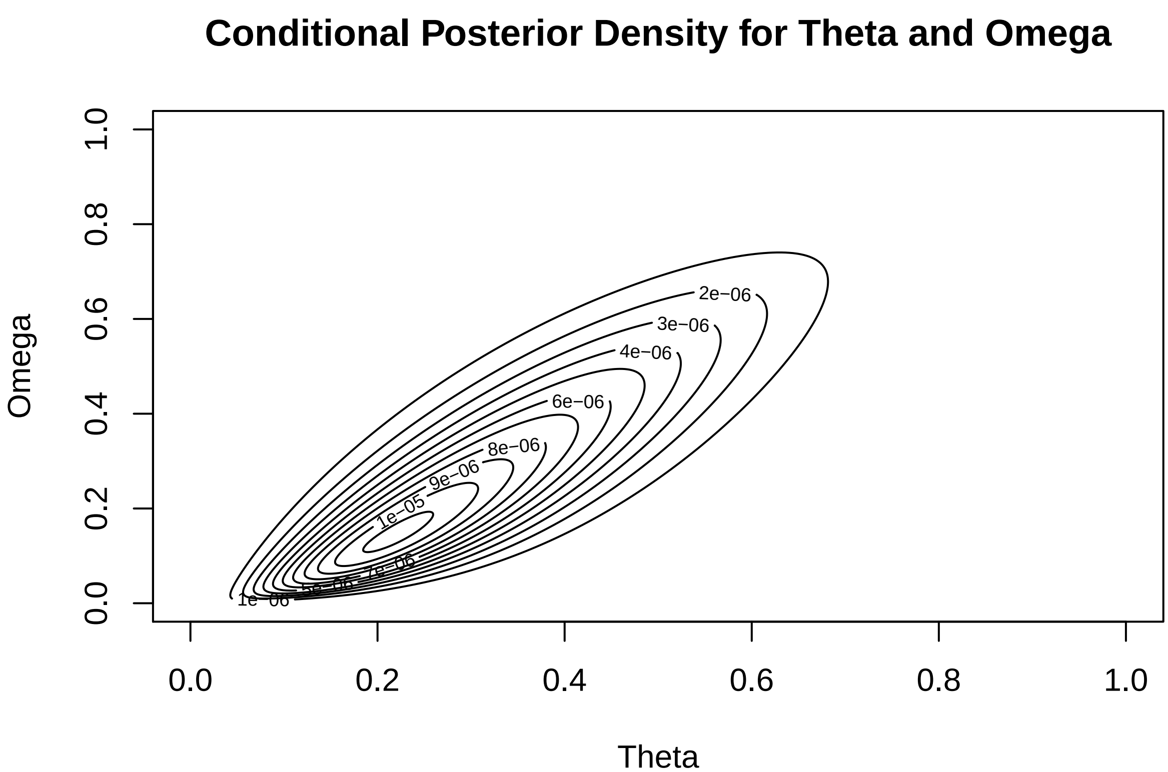
Theta's parameters moved more, and are closer to the observed value of 0.29. This makes sense, as theta is more directly affected by the data, whereas omega is affected only indirectly, through theta.

Compute and compare conditional priors and conditional posteriors:

```
ConditionalPrior <- matrix(nrow=1001, ncol = 1001,
                           dimnames = list(paste("Omega=", Omega), paste("Theta=", Theta)))
#Calculate densities of the conditional prior:
for (i in 1:length(Theta)){
    ConditionalPrior[,i] = dbeta(Theta[i], shape1 = Omega*(Prior_kappa - 2) + 1,
                                 shape2 = (1-Omega)*(Prior_kappa -2) + 1)
}
ConditionalPrior <- normalize(ConditionalPrior)

# calculate densities of conditional posterior:
ConditionalPosterior <- matrix(nrow=1001, ncol = 1001,
                               dimnames = list(paste("Omega=", Omega), paste("Theta=", Theta)))
for (i in 2:(length(Theta)-1)) {
    ConditionalPosterior[,i] = JointPosterior[,i]/Posterior_Marginal_Omega[i]
}
ConditionalPosterior[,2:1000] <- normalize(ConditionalPosterior[,2:1000])

ConditionalPriorGraph <- contour(x = Omega, y = Theta, z = ConditionalPrior,
                                 main = "Conditional Prior Density for Theta and Omega",
                                 xlab = "Theta", ylab = "Omega")
```



```
ConditionalPosteriorGraph <- contour(x = Omega, y = Theta[2:1000], z = ConditionalPosterior[,2:1000],
                                     main = "Conditional Posterior Density for Theta and Omega",
                                     xlab = "Theta", ylab = "Omega")
```



In the conditional posterior distribution, theta is less strongly dependent upon omega compared to in the prior distribution. This is because initially theta was determined only by the prior distribution of omega. However, after observing data, theta is influenced more strongly by the data than by the the prior distribution of omega.