DISTRIBUTIONS OF ULAM WORDS UP TO LENGTH 30

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ABSTRACT. We further explore the notion of Ulam words considered by Bade, Cui, Labelle, and Li. We find that when interpreted as integers in a natural way, Ulam words appear to follow a new, unexplained distribution. Gaps between words and words of special type also reveal remarkable structure. By substantially increasing the number of computed terms, we are also able to sharpen some of the conjectures made by Bade et al.

1. Introduction:

In their 2020 paper [1], Bade, Cui, Labelle, and Li introduced the notion of Ulam words, defined as follows. Consider the free semigroup $S[\{0,1\}]$ on two generators 0 and 1. We say that 0 and 1 are *Ulam* and then define all other Ulam words inductively: a word $w \neq 0, 1$ is Ulam if and only if there exist exactly one pair of Ulam words $u_1 \neq u_2$ such that $w = u_1 \cap u_2$. (Here, \cap denotes concatenation.) We shall denote the entire set of Ulam words as \mathcal{U} , and Ulam words of length n by \mathcal{U}_n . It is easy to check that:

$$\mathcal{U}_1 = \{0, 1\}$$
 $\mathcal{U}_3 = \{001, 011, 100, 110\}$ $\mathcal{U}_2 = \{01, 10\}$ $\mathcal{U}_4 = \{0001, 0010, 0100, 0111, 1000, 1011, 1101, 1110\}.$

Bade et al. computed all Ulam words up to length 24—using an algorithm described in Section 6, we were able to compute up to length 30. While this might appear as a small improvement at first glance, because the number of Ulam words of length n (almost) doubles on each iteration, in reality, this represents nearly 60 times as much data. One consequence of this additional data is that we can better project the growth rate of the density

$$\rho(n) := \frac{\#\mathscr{U}_n}{2^n}$$

as a function of n—see Section 4 for details, and Figure 1 for an illustration. It has also provided us with substantial numerical evidence for a new class of conjectures regarding the distribution of Ulam words, which we shall describe below.

This notion of Ulam words was built on the earlier notion of Ulam sets due to Kravitz and Steinerberger [8], which was itself a generalization of Ulam's eponymous integer sequence, also defined recursively [13]: the (classical) Ulam sequence begins with 1, 2, and then every subsequent term is the next smallest integer that can be written as the sum of two distinct prior terms in exactly one way. Generalizations of Ulam's classic sequence have become an increasingly popular object of study: in 1972, Queneau did some preliminary work studying generalizations where the initial two terms of the integer sequence are varied [10]; in the 1990s, Cassaigne, Finch, Shmerl, and Spiegel determined some of the families of such sequences such that the consecutive differences are eventually periodic [2–5,11];

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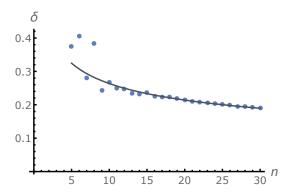


FIGURE 1. A plot of the densities $\rho(n)$ for $4 \le n \le 30$, together with a plot of $f(n) = 0.526n^{-3/10}$.

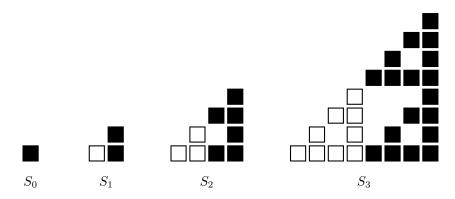


FIGURE 2. Visual of the first 4 steps of constructing discrete Sierpiński triangle.

in 2017, Kravitz and Steinerberger considered generalizing the Ulam condition for abelian groups [8]; in 2020, Bade et al. gave the aforementioned notion of Ulam words with some preliminary results [1]; and in 2021, Sheydvasser showed that there is an analogous notion of Ulam sets for integer polynomials [12] by building off earlier work of Hinman, Kuca, Schlesinger, and Sheydvasser [6,7].

Earlier work around Ulam words has largely centered around giving simple criteria for when words of some special type are Ulam—for example, Bade et al. showed that a word of the form $0^a 10^b$ is Ulam if and only if $\binom{a+b}{a}$ is odd [1]. Similarly, Mandelshtam considered Ulam words of the form $0^a 10^b 10^c$ and demonstrated a connection to the Sierpiński gasket [9]. We also prove a few such results, such as the following.

Theorem 1.1. Consider the set of points $(x,y) \in \mathbb{Z}^2_{\geq 1}$ such that $1^y 0^{x-y} \in \mathcal{U}$. This is the discrete Sierpiński triangle, union a point.

We will discuss this construction more precisely in Section 3, but, briefly, the discrete Sierpiński triangle is an approximation to the standard Sierpiński triangle. It can be constructed either iteratively (as in Figure 2) or by coloring Pascal's triangle by parity.

On the other hand, we also have a novel way of considering Ulam words, by interpreting them as integers. Observe that there exists a natural map $\pi: S(\{0,1\}) \to \mathbb{Z}_{\geq 0}$ via

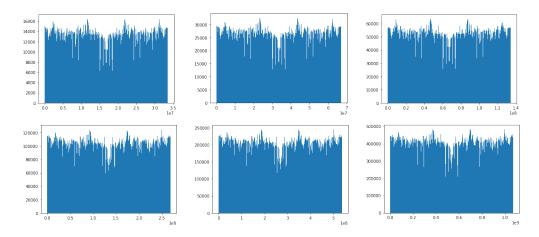


FIGURE 3. From left to right, top to bottom: histograms of words of length 25, 26, 27, 28, 29, 30.

interpreting a word as the binary representation of an integer. In general, this map is not injective—for example, $\pi(0) = \pi(00) = \pi(000) = 0$. However, if we restrict it to words of a fixed length, then it is. In particular, the restrictions $\pi : \mathcal{U}_n \to \mathbb{Z} \cap [0, 2^{n-1}]$ are injective maps. This gives a natural ordering on \mathcal{U}_n and allows us to ask questions about equidistribution. For instance: do Ulam words equidistribute modulo N?

Conjecture 1.1. For any integer N > 1 and $a \in \mathbb{Z}/N\mathbb{Z}$, define the relative density

$$\rho_{a,N}(n) := \frac{\# \{ w \in \mathcal{U}_n | \pi(w) = a \mod N \}}{\# \mathcal{U}_n}.$$

Then $\lim_{n\to\infty} \rho_{a,N}(n) = 1/N$.

Remark 1.1. As we discuss in Section 5, while this conjecture is consistent with the available data, it is somewhat surprising. For one thing, $\rho_{5,6}(1) = \rho_{5,6}(2) = \rho_{5,6}(3) = 0$, and it takes some time before it appears to start to converge to 1/6.

This conjecture can be viewed as saying that Ulam words equidistribute with respect to the non-archimedean valuations (essentially, modulo arbitrarily large prime powers). We might also ask whether they equidistribute with respect to the archimedean valuation, which we choose to measure as follows. Consider normalizing each \mathcal{U}_n such that it is within the interval [0,1]—i.e. we consider instead $2^{-n}\mathcal{U}_n$. Then we can apply the usual notion of the equidistribution of a sequence, which is that in the limit the proportion of points in an interval is the length of the interval. That is, is it true that for all $0 \le c < d \le 1$,

$$\lim_{n\to\infty} \frac{\#\left\{w\in\mathcal{U}_n|2^{-n}w\in[c,d]\right\}}{\#\mathcal{U}_n} = d-c?$$

It would appear not, as Figure 3 illustrates! It is clear that if $c, d \approx 1/2$ then there will be significantly fewer points in the interval than expected. Nevertheless, it appears that the limit does approach *some* distribution, even if cannot determine what it is precisely. To be concrete, we tentatively make the following conjecture.

Conjecture 1.2. Given a positive integer n, define a probability measure $\delta_{\mathscr{U}_n}:[0,1] \to [0,\infty)$ by

$$\delta_{\mathcal{U}_n}(S) = \frac{\# (2^{-n} \mathcal{U}_n \cap S)}{\# \mathcal{U}_n}.$$

As $n \to \infty$, $\delta_{\mathscr{U}_n}$ converges weakly to some probability distribution $\delta_{\mathscr{U}} : [0,1] \to [0,\infty)$.

There is a related conjecture that we can make with greater confidence, which is that if we look at the distribution of the gaps between consecutive Ulam words, this converges as $n \to \infty$.

Conjecture 1.3. Let $u_1 < u_2 < \ldots < u_{k_n}$ be the (ordered) elements of $\pi(\mathcal{U}_n)$. Define

$$p_n: \mathbb{Z}_{\geqslant 1} \to [0, \infty)$$

$$g \mapsto \frac{\# \{i | u_{i+1} - u_i = g\}}{k_n - 1}.$$

This has a natural interpretation as a probability measure. As $n \to \infty$, the functions p_n converge pointwise to a probability measure $p: \mathbb{Z}_{\geq 1} \to [0, \infty)$.

Illustrations of the probability measures p_n and some discussion about their properties can be found in Section 4.

2. Definitions and Visualizations

We start with some basic definitions and constructions. Given a word $w \in S[0,1]$, we define its *complement* \hat{w} to be the word with every instance of 0 replaced with a 1, and vice versa. We also define the *reverse* \overline{w} , which is the word obtained by reversing the order of the letters. It was shown by Bade et al. [1] that $w \in \mathcal{U}$ if and only if $\hat{w} \in \mathcal{U}$, if and only if $\overline{w} \in \mathcal{U}$.

In order to better visualize the set \mathscr{U} , we made use of heat maps, which depict each Ulam word as a colored bar and stacks all of the words vertically—that is, for a given word, a 0 corresponds to a rectangle of one color, and a 1 corresponds to a rectangle of a second color. An example is provided in Figure 4. In general, we abridge such diagrams: we created figures only using all the Ulam words that started with zero, since Ulam words are closed under complements.

Patterns in Ulam words become apparent in the heat maps if we impose the ordering discussed in the introduction: we say that $w \leq w'$ if and only if $\pi(w) \leq \pi(w')$. See Figure 5. Using this way of visualizing Ulam words allows us to easily see that there is both a clear binary tree structure that governs the existence of Ulam words, as well as a chaotic element to the set where the binary tree breaks down.

We can be more specific about our meaning regarding this breakdown: since Ulam words are preserved under the reverse map, this is equivalent to saying that for any n there exists $n > \ell_n > 0$ such that all possible subwords of length ℓ_n occur as the final ℓ_n characters of words in \mathcal{U}_n . In turn, that is equivalent to saying that the quotient map $\pi(\mathcal{U}_n) \to \mathbb{Z}/2^{\ell_n}\mathbb{Z}$ is surjective. Our observation is that ℓ_n appears to increase as a function of n, albeit not very quickly—see Figure 6.

Assuming that Conjecture 1.1 is true, then it would follow immediately that $\ell_n \to \infty$ simply by considering the case where $N=2^{\ell_n}$ —indeed, the heat maps were the original impetus for our equidistribution conjectures. On the other hand, the "chaotic" latter half of the heat map is more of a mystery.

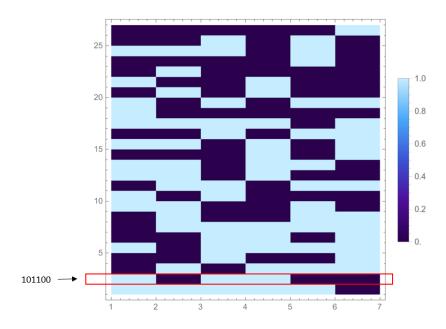


FIGURE 4. Example of how an Ulam word appears in a heatmap. Heat map shows all Ulam words of length 6 without forced ordering.

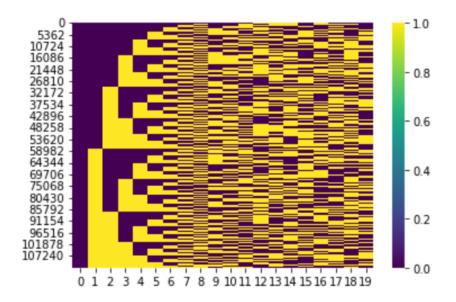


FIGURE 5. All words of length 20 beginning with a zero.

n	ℓ_n	n	ℓ_n	n	ℓ_n	n	ℓ_n	n	ℓ_n
1	1	7	4	13	5	19	9	25	11
2	1	8	4	14	5	20	9	26	11
3	1	9	4	15	6	21	9	27	12
4	1	10	4	16	7	22	10	28	12
5	3	11	4	17	7	23	10	29	13
6	2	12	5	18	8	24	11	30	13

FIGURE 6. Tables of n versus ℓ_n , where ℓ_n is the largest integer such that $\mathcal{U}_n \to \mathbb{Z}/2^{\ell_n}\mathbb{Z}$ is surjective.

3. Unconditional Results

Bade et al. proved various results regarding Ulam words containing certain patterns [1]. We offer a similar collection of results. We begin with a couple of intermediate results that to allow us to prove Theorem 1.1. The first two are due to Bade et al.

Theorem 3.1 (Theorem 3.4 in [1]). A word of the form $0^a 10^b$ is Ulam if and only if $\binom{a+b}{a} = 1 \mod 2$.

Theorem 3.2 (Theorem 3.5 in [1]). A word of the form $0^a 1^2 0^b$ is Ulam if and only if the length of the word is odd (that is, $a + b = 1 \pmod{2}$).

Remark 3.1. Since $w \in \mathcal{U}$ if and only if $\overline{w} \in \mathcal{U}$, if and only if $\hat{w} \in \mathcal{U}$, we get analogous results with 1's replaced with 0's and the order of the letters reversed. This is true for all the results that we prove here.

Lemma 3.3. For any $n \ge 3$, $1^30^{n-3} \in \mathcal{U}$ if and only if $n = 0 \pmod{4}$ or $n = 1 \pmod{4}$.

Proof. We will use proof by induction, where the base cases n = 3, 4, 5, 6 can all be verified by direct computation. Assume the statement holds for all words of length strictly less than n, and consider the word $u = 1^30^{n-3}$. The only two possible representations for u are

- (1) $1^{n-1} 1^2 0^{n-3}$ and
- (2) 1^30^{n-4} (2)

since neither 1^a nor 0^a are Ulam words for any a > 1. By Theorem 3.2, $1^20^{n-3} \in \mathcal{U}$ if and only if $n = 0 \pmod{2}$; thus, representation (1) is valid if and only if $n = 0, 2 \pmod{4}$. On the other hand, by the inductive hypothesis, representation (2) is valid if and only if $n = 1, 2 \pmod{4}$. Ergo, exactly one of the representations is valid if and only if $n = 0, 1 \pmod{4}$.

Lemma 3.4. Let $u \in S[\{0,1\}]$ be a word of length $n \ge 5$ with exactly four 1's such that the 1's are consecutive. The word $u \in \mathcal{U}$ if and only if $n = 1 \pmod{4}$.

Proof. We will use proof by induction, where the base cases n = 5, 6, 7, 8 can be verified directly. Assume the statement holds for all words of length strictly less than n, and consider the word $u = 0^k 1^4 0^l$ of length n, where since $n \ge 9$ at least one of $k \ge 3$ or $l \ge 3$. By applying the reverse map if necessary, we may assume that $k \ge 3$.

Case 1: l = 0.

The only possible representations are $0^{n-1}1^4$ and $0^k1^{3}1$. By the inductive hypothesis, the first is valid if and only if $n=2 \pmod{4}$. By Lemma 3.3, the second

is valid if and only if $n = 1, 2 \pmod{4}$. Thus, exactly one of these representations is valid if and only if $n = 1 \pmod{4}$.

Case 2: $l \ge 1$.

There are five potential representations:

- (1) $0^{-0^{k-1}}1^40^l$,
- (2) $0^k 1^{-13} 0^l$,
- (3) $0^k 1^2 \cap 1^2 0^l$,
- (4) $0^k 1^3 \cap 10^l$, and
- (5) $0^k 1^4 0^{l-1} \ 0$.

Observe that by the inductive hypothesis, representations (1) and (5) are valid if and only if $n = 2 \pmod{4}$, which is to say that $k + l = 2 \pmod{4}$. By Theorem 3.1 and Lemma 3.3, representation (2) is valid if and only if $l = 0, 3 \pmod{4}$; similarly, representation (4) is valid if and only if $k = 0, 3 \pmod{4}$. Finally, by Theorem 3.2, representation (3) is valid if and only if $k = l = 1 \pmod{2}$. This allows us to count the number of valid representations in terms of the congruence classes of k and k modulo 4.

In particular, there is a unique representation if and only if $n = 1 \pmod{4}$.

The proof of Lemma 3.4 illustrates how the modular length restrictions for $1^{a+1}0^b$ can easily be found using the length restrictions for 1^a0^b , which, in turn, means we could generate countless additional theorems, providing length restrictions for 1^40^b , 1^50^b , and so forth. However, this would quickly prove tedious. Instead, we will demonstrate the unifying pattern between all words of the form 1^a0^b . Recursively define

$$S_0 := (2,1)$$

$$S_{n+1} := S_n \cup (S_n + (2^n, 0)) \cup (S_n + (2^n, 2^n))$$

$$S := \bigcup_{n=0}^{\infty} S_n.$$

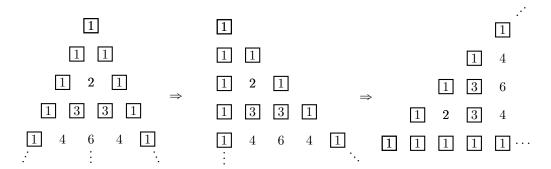
The set S is sometimes referred to as the discrete Sierpiński triangle. The reason for this is that if one considers a suitable limit of the sets $2^{-n}S$, the result is the usual Sierpiński triangle. Remarkably, this is exactly the correct construction to determine whether a word 1^a0^b is Ulam or not.

Theorem 3.5.

$$\left\{ (x,y) \in \mathbb{Z}_{\geqslant 1}^2 \middle| 1^y 0^{x-y} \in \mathscr{U} \right\} = (1,1) \cup \mathcal{S}.$$

Remark 3.2. Observe that this is Theorem 1.1, but stated more precisely.

Proof. It is easy to check that the only elements in both sets with y = 1 are (1,1) and (2,1). Our goal is to show that the iterative process for producing points in S with larger y values is the same as for the first set. To do so, we will use a modified version of Pascal's triangle, achieved by aligning all entries to the left and then rotating the resulting image counterclockwise 90 degrees.



In the modified version, if we let p(x,y) be the entry in row y and column x, we have

$$p(x,y) = p(x-1,y) + p(x-1,y-1).$$

We make two additional modifications:

- (1) we align Pascal's triangle such that the bottom left entry occurs at (2,1) and
- (2) we shade all odd entries.

On the one hand, this is well-known to be the discrete Sierpiński triangle S. On the other hand, because only odd entries are shaded, an entry (x, y) is shaded if and only if exactly one of (x - 1, y) or (x - 1, y - 1) is shaded.

Now, observe that there are only two potential representations for 1^x0^{x-y} , those being

- (1) $1^y 0^{x-y-1} \ 0$ and
- (2) $1^{y-1}0^{x-y}$.

Thus, (x, y) is in the desired set if and only if exactly one of (x - 1, y), (x - 1, y - 1) is in the set.

Remark 3.3. This is not the first time that Pascal's triangle and the Sierpiński triangle has showed up in the study of Ulam words. As mentioned previously, Mandelshtam found an analogous pattern for words $0^x 10^y 1^z$ [9]; even before that, Bade et al. proved that the number of words of length n with one 1 is the n-th term in Gould's sequence [1]—that is, the number of elements in the n-th row of Pascal's triangle.

There are many consequences of this result. First, it means that determining the set of Ulam words of the form 1^a0^b is quite simple: it is a straightforward iterative procedure. Counting the number of elements of such form is also simple. For example, consider the following simple corollary.

Corollary 3.6. Fix $x \in \mathbb{Z}_{\geq 2}$; let L_x be the set of $y \in \mathbb{Z}_{\geq 1}$ such that $0^y 1^{x-y} \in \mathcal{U}$. Then $\#L_x$ is the (x-1)-th term of Gould's sequence.

Proof. The k-th term of Gould's sequence is the number of odd entries in the k-th row of Pascal's triangle. After rotating and shifting, the k-th row corresponds to L_{k+1} .

There are certainly more symmetries lurking within S that would lead to more theorems and patterns. In particular, we believe that it might be possible to demonstrate the following.

Conjecture 3.1. Let u be a word of length n of the form $0^a 1^{2^k} 0^b$ for $a, b, k \in \mathbb{Z}_{\geq 1}$. The word u is in \mathscr{U} if and only if $n = 1 \mod 2^k$.

We close by giving one more result, which is analogous to the following theorem due to Bade et al. **Theorem 3.7** (Theorem 3.6 in [1]). For any $a, b \in \mathbb{Z}$ such that $a + b \ge 2$, $0^a 1010^b \in \mathcal{U}$ if and only if $a + b = 1 \mod 2$.

Theorem 3.8. For any $a, b \in \mathbb{Z}$ such that $a + b \ge 1$, $0^a 101010^b \in \mathcal{U}$ if and only if $a, b \in 2\mathbb{Z}$ and one is zero.

Proof. We induct on a + b. The base cases a + b = 1, 2 are easily established by pure computation. First, assume that both $a, b \neq 0$. Then there are six possible representations:

- (1) $0 \cap 0^{a-1} 101010^b$.
- (2) $0^a 1^{-} 01010^b$,
- (3) $0^a 10^a 1010^b$.
- $(4) 0^a 101^{\circ} 010^b$,
- (5) $0^a 1010^{\hat{}} 10^b$, and
- (6) $0^a 101010^{b-1} \ 0$.

Applying the inductive hypothesis and Theorem 3.7, we find that in all cases either none of these are valid representations, or multiple are simultaneously.

This leaves the case where either a = 0 or b = 0—without loss of generality, we assume that a=0, since we can always apply the reverse map if needed. We have three possible representations:

- $(1) 1^{\circ}01010^{b},$
- (2) $10^{\hat{}}1010^{b}$, and (3) 101010^{b-1} 0.

If b is odd, then (2) and (3) are valid representations due to Theorem 3.7; therefore, this word is not Ulam. If b is even, then (1) is a valid representation, but (2) and (3) are not; therefore, this word is Ulam. П

4. Density and the Distribution of the Gaps

In their paper, Bade et al. conjectured that the density $\rho(n)$ of Ulam words of length n—as defined in Section 1—converges to some real number 0 < r < 1 as $n \to \infty$ [1]. With our enlarged data set, we conjecture that $\rho(n) \to 0$; specifically, $\rho(n) = \Theta(n^{-3/10})$. We originally conjectured this due to illustrations such as Figure 1, but there is an even more striking way to see this, in terms of the average gap.

Let $u_1 < u_2 < \ldots < u_{k_n}$ be the (ordered) elements of $\pi(\mathcal{U}_n)$ —that is, the integers that are images of the Ulam words of length n. The gaps between them are the consecutive differences $g_1 = u_2 - u_1$, $g_2 = u_3 - u_2$, and so on. It is easy to see that $u_1 = \pi(0000...01) = 1$ and $u_{k_n} = \pi(1111...10) = 2^n - 2$. Ergo, the average gap between Ulam words of length n

$$\mu_g(n) := \frac{g_1 + g_2 + \ldots + g_{k_n - 1}}{k_n - 1} = \frac{u_{k_n} - u_1}{k_n - 1} = \frac{2^n - 3}{k_n - 1}$$
$$= \frac{1}{\rho(n)} \frac{k_n}{k_n - 1} - \frac{3}{k_n + 1},$$

from which we conclude that $\rho(n)^{-1} \simeq \mu_q(n)$. So, we must conclude that if $\rho(n)$ converges to a non-zero constant, then $\mu_q(n)$ must be bounded. But this does not appear to be the case, as evidenced by Figures 7 and 8. Instead, we conjecture the following.

Conjecture 4.1. As $n \to \infty$, $\mu_g(n) = \Theta(n^{3/10})$ —indeed, it may be that there is a constant $c \approx 1.9$ such that $\mu_a(n) = cn^{3/10} + o(1)$. Consequently, $\rho(n) = \Theta(n^{-3/10})$.

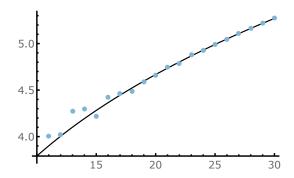


FIGURE 7. A plot of $\mu_g(n)$ for $11 \le n \le 30$ and $f(n) = 1.9n^{3/10}$.

n	Relative Error	n	Relative Error	n	Relative Error
13	4.08765%	19	0.196991%	25	0.0447467%
14	2.43579%	20	0.222403%	26	0.0981692%
15	1.54247%	21	0.24375%	27	0.028047%
16	1.27957%	22	0.332677%	28	0.0353147%
17	0.402629%	23	0.33529%	29	0.0365254%
18	0.87545%	24	0.104678%	30	0.00663021%

FIGURE 8. The relative error between the actual $\mu_g(n)$ and the estimate $1.9n^{3/10}$.

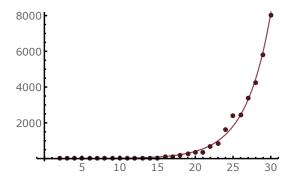


FIGURE 9. A plot of the maximal gap between words of length n for $2 \le n \le 30$, and $f(n) = 1.35^n$.

It is obvious that any gap is at least 1 (in fact, it is an easy exercise to show that a gap of 1 is always attained); on the other hand, numerical evidence suggests that the maximal gap grows exponentially—specifically, it is $O(r^n)$ for some constant $r \approx 1.35$ (see Figure 9). We can also study the distribution of the gaps more generally, and a very curious phenomenon emerges: it appears that these distributions converge to some limiting distribution as $n \to \infty$, as described in Conjecture 1.3. A large number of these distributions are illustrated in Figure 1.3.

There are some curious details to this apparent distribution. The first is that there is a clear bias against gaps that are either 2 or 4 modulo 6. This is very odd, considering that we conjecture that Ulam words are equidistributed modulo 6 in the limit. (See Section 5 for

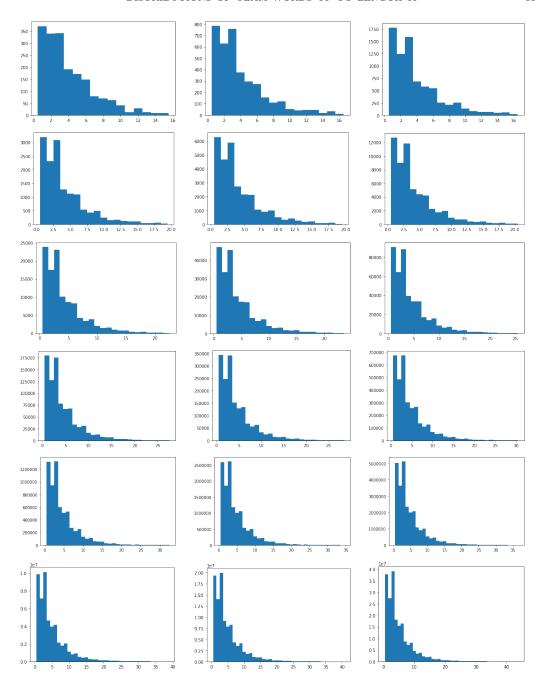


FIGURE 10. From left to right, top to bottom: bar graphs showing the frequency of gaps of various sizes between consecutive words in \mathcal{U}_n for $n = 13, \ldots, 30$, shown out to 4 standard deviations.

more about this.) The second is that just as the average gap appears to grow without bound, so does the standard deviation. However, the standard deviations are quite small—here is

a table of the last few we were able to compute:

n	Standard Deviation
21	6.090434611011227
22	6.5739141205506035
23	6.951985360977966
24	7.48652894820725
25	7.954513792250833
26	8.410261056967098
27	8.838421071953993
28	9.345660478229291
29	9.940553024054223
30	10.500749743720565

This means that the distribution is very tightly clustered toward the smaller side. However, it has extreme outliers: the maximal gap between words of length 30 is 8030, which is more than 764 standard deviations away from the mean! Somehow, this should be typical: as we noted already, the size of the maximal gap appears to grow exponentially, but the same is not true of either the average gap or the standard deviation.

5. Modular Distribution

Let us now consider the relative density of Ulam words. As we mentioned earlier, the set of Ulam words is preserved under the complement map. This forces a symmetry on congruence classes.

Theorem 5.1. If $w \in \mathcal{U}_n$ then $\pi^{-1}(2^{n+1} - 1 - \pi(w)) \in \mathcal{U}_n$. Consequently, for any positive integer N and $a \in \mathbb{Z}/N\mathbb{Z}$,

$$\rho_{a,N}(n) = \rho_{2^{n+1}-1-a,N}(a).$$

Proof. Given $w \in \mathcal{U}_n$, write $x = \pi(w) = a_n 2^n + \ldots + a_0$ in binary. Then

$$\pi\left(\hat{w}\right) = (1 - a_n)2^n + \ldots + (1 - a_0) = 2^{n+1} - 1 - x.$$

But $\hat{w} \in \mathcal{U}_n$. Now, observe that if $\pi(w) = a \mod N$, then $2^{n+1} - 1 - \pi(w) = 2^{n+1} - 1 - a \mod N$, which forces the equality of the relative densities.

For N=2,3, this is particularly simple.

Corollary 5.2. For any positive integer n, $\rho_{0,2}(n) = \rho_{1,2}(n)$. Furthermore, for any $a \in \mathbb{Z}/3\mathbb{Z}$,

$$\rho_{a,3}(n) = \begin{cases} \rho_{1-a,3}(n) & \text{if } n = 0 \mod 2\\ \rho_{-a,3}(n) & \text{if } n = 1 \mod 2. \end{cases}$$

Proof. Observe that $2^{n+1} - 1 - x = x + 1 \mod 2$, from which $\rho_{0,2}(n) = \rho_{1,2}(n)$ follows immediately. For the second part, observe that

$$2^{n+1} - 1 - x \mod 3 = \begin{cases} 1 - x & \text{if } n = 0 \mod 2 \\ -x & \text{if } n = 1 \mod 2. \end{cases}$$

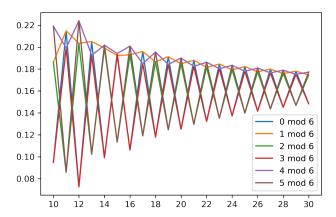


FIGURE 11. A plot of the relative densities modulo 6 as functions of the word length.

While there must always exist for any n two congruence classes $a, b \in \mathbb{Z}/3\mathbb{Z}$ such that $\rho_{a,3}(n) = \rho_{b,3}(n)$, there is no reason why the last congruence class c should be roughly equal. Indeed, for $n \leq 5$, $\rho_{c,3}(n) = 0$. However, as Figure 11 shows, for larger n, it does appear to be the case that $\rho_{c,3}(n) \to \rho_{a,3}(n) = \rho_{b,3}(n)$.

To help measure the extent to which words are equidistributing modulo N, we define the modular discrepancy.

Definition 5.3. For any positive integers n, N, the modular discrepancy is

$$d_N(n) := \max_{a,b \in \mathbb{Z}/N\mathbb{Z}} |\rho_{a,N}(n) - \rho_{b,N}(n)|.$$

Trivially, saying that Ulam words equidistribute modulo N is equivalent to saying that $d_N(n) \to 0$ as $n \to \infty$. Moreover, by appealing to the Chinese remainder theorem, proving that $d_N(n) \to 0$ for all N is reducible to prove that $d_{p^k}(n) \to 0$ for all prime powers p^k . To investigate Conjecture 1.1, we computed $d_{p^k}(n)$ for all prime powers $p^k < 30$ —as near as we can tell, $d_{p^k}(n)$ decays exponentially as a function of n (see Figure 12.)

6. Description of the Algorithm

Our code for generating Ulam words and a list of all Ulam words up to length 20 are available on GitHub¹. Below, we describe the algorithm, the current roadblocks, and possible theoretical improvements for the benefit of the reader.

One of the key components to the fast performance of the algorithm is the implementation of hash functions. Specifically, we made use of Python's 'set' data type, which uses hash tables to make searching through the set for an element take $\Theta(1)$ operations on average. We use this as follows: all Ulam words that have been computed during previous iterations are stored in a dictionary UlamSet such that for any integer k, UlamSet(k) returns the images of Ulam words of length k under the map π , stored as a set. Ulam words are stored in terms of their associated integers to reduce memory requirements.

Aside from the dictionary UlamSet, we will also need a command we shall call Split—given a word w and an integer k, it splits w into two subwords by taking the first k letters

¹https://github.com/asheydva/Ulam-Words.git

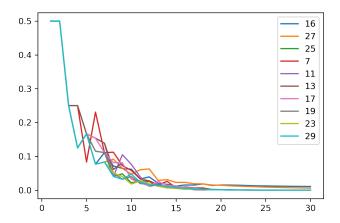


FIGURE 12. A plot of the modular discrepancies for prime power moduli $p^k < 30$.

of w as one word and the remaining letters as the second word. The specifics of how to implement this function are, of course, language-dependent, but it should never take more than O(n) operations, where n is the length of w. With this, we can define a function to test whether words of length n are Ulam or not, assuming we have already determined the Ulam words of length $1 \le k \le n-1$.

Algorithm 6.1. On an input of a word w of length n, outputs whether it is Ulam or not.

```
1: procedure ULAMWORDQ(w)
2:
        num\_reps \leftarrow 0
        for 1 \le k \le n - 1 do
3:
             w_1, w_2 \leftarrow \text{Split}(w, k)
4:
             m_1 \leftarrow \pi(w_1)
5:
             m_2 \leftarrow \pi(w_2)
6:
             if m_1 \in \text{UlamSet}(k) and m_2 \in \text{UlamSet}(n-k) and m_1 \neq m_2 then
 7:
                 num\_reps \leftarrow num\_reps + 1
8:
             if num\_reps > 1 then
9:
                 break
10:
11:
        return num\_reps = 1
```

Remark 6.1. Here, the command 'break' functions as it does in Python: if it is ever activated, the algorithm exits the for-loop that it is currently executing.

Proof of Correctness. The idea is that num_reps counts the number of ways that w can be written as $w_1 \cap w_2$, where $w_1 \neq w_2 \in \mathcal{U}$. It does this by looking at all ways to split w into two subwords and then checking whether those subwords are Ulam by using UlamSet. By definition, w is Ulam if and only if num_reps = 1.

Theoretically, the worst-case running time for UlamWordsQ is $O(n^2 \# \mathcal{U}_{n-1})$ and thus, conjecturally, $O(n^{17/10}2^n)$. However, in practice, the actual worst case seems to be $O(n^2)$,

which occurs whenever w is Ulam or when it has no representations at all; the latter case appears to be rare. This is sufficiently good for our purposes.

With this, we just need one extra function: Binarize(k, n), which returns the binary representation of k, padded with zeros on the left such that it has length n. Observe that listing all integers between 1 and $2^n - 1$ is then equivalent to listing all words of length n (other than 0^n and 1^n , which are never Ulam words).

Algorithm 6.2. On an input of an integer n, prints all Ulam words of length n.

```
1: procedure ULAMWORDS(n)

2: for 1 \le k \le 2^n - 2 do

3: w \leftarrow Binarize(k)

4: if UlamWordQ(w) then

5: print w
```

Remark 6.2. In practice, rather than printing, we instead write to a file. Moreover, we do not store w but $k = \pi(w)$ to reduce memory requirements.

Proof of Correctness. This is immediate.

The running time of this algorithm should be something like $O(n^2 2^n)$, which is consistent with our observations. Since there are, conjecturally, $\approx n^{-3/10} 2^n$ Ulam words of length n, there does not appear to be much room for improvement. The larger issue is memory. Because the algorithm requires storing the dictionary UlamSet in RAM, there is a hardware-dependent limit on the length of Ulam words that can be computed. On the personal computer that we used, we hit this limit at n=30—for context, the produced text file of all Ulam words of length ≤ 30 was 4.11 gigabytes.

The problem is imminently parallelizable: it is possible to test each word of length n for Ulam-ness separately. Running it on a supercomputer with significantly more RAM would allow pushing past the limit that we encountered. Unfortunately, because of the exponentially growing time and memory requirements, even pushing to n = 40 would pose substantial difficulties².

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²To give some perspective, we would expect that simply storing all Ulam words up to that size would require over 16 terabytes.

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