Cycling in Circles

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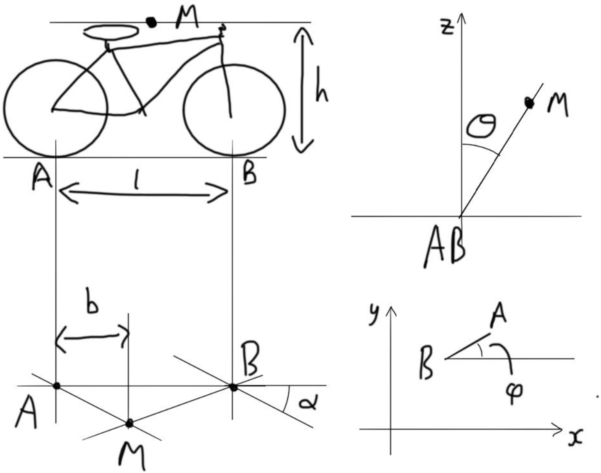
Studying the motion

Diagram



Constructing a bicycle

With the angles as we have defined in the previous slide, we define:
Coordinates (Cartesian):
Front wheel - $(x_f, y_f, 0)$
Back wheel - $(x_b, y_b, 0)$
Centre of Mass - (ξ, η, ζ)

Constants:
Height - h
Mass - m
Length of bike - l
Distance of rear wheel to centre of mass - b
Gravity - g $= 9.81 m/s^2$

Constructing a bicycle

How do we model the wheels?

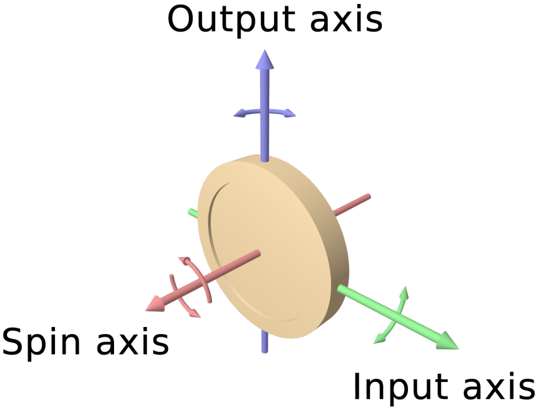
For simplicity:

- we consider them as points of contact of the bike with the ground.
- we add a non-slip condition
- define v to be the speed of the rear wheel

What about gyroscopic effects?

Gyroscopic forces

Reminder:
Gyroscopic percussion is a force which reacts perpendicular to an initial force due to the inertia in the spinning mass.



Gyroscopic forces

Things to discuss

- Coins rolling
- Bike with fixed handlebars
- Bike moving freely - what are we controlling?
- Stability of tilt and rotation of the front wheel

Motion of the bicycle

If alpha is predetermined then the path that the bicycle takes is geometrically predetermined but the motion of the center of mass depends on the physics of the problem.

We will first look at the motion of the wheels and then after consider the change in the tilt of the center of mass as it moves along this path.

The Path of the Wheels

Clearly:

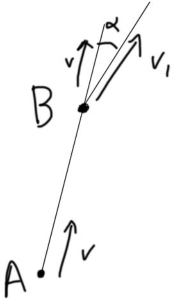
$$
\begin{aligned}
& \dot{\vec{x}}_b = v \frac{(\vec{x}_f - \vec{x}_b)}{|\vec{x}_f - \vec{x}_b|} && (1) \\
\Longrightarrow \quad & \dot{x}_b = \frac{v}{l} (x_f - x_b) && (2) \\
\text{and} \quad & \dot{y}_b = \frac{v}{l} (y_f - y_b) && (3) \\
\text{Now} \quad & \dot{x}_f = v_1 \cos(\alpha + \phi) && (4) \\
\text{and} \quad & \dot{y}_f = v_1 \sin(\alpha + \phi) && (5)
\end{aligned}
$$

where v_1 is to be found

The Path of the Wheels

From the diagram opposite it is clear that:

$$v_1 \cos(\alpha) = v \qquad (6)$$



The Path of the Wheels

We can expand $\dot{x}_f$ and $\dot{y}_f$

$$\dot{x}_f = v_1 \cos(\alpha) \cos(\phi) - v_1 \sin(\alpha) \sin(\phi) \tag{7}$$

$$\dot{y}_f = v_1 \sin(\alpha) \cos(\phi) + v_1 \cos(\alpha) \sin(\phi) \tag{8}$$

$$\dot{x}_f = v \cos(\phi) - v \tan(\alpha) \sin(\phi) \tag{9}$$

$$\dot{y}_f = v \tan(\alpha) \cos(\phi) + v \sin(\phi) \tag{10}$$

$$\text{Note :} \quad \cos(\phi) = (x_f - x_b)/l, \quad \sin(\phi) = (y_f - y_b)/l \tag{11}$$

$$\dot{x}_f = v(x_f - x_b)/l - v \tan(\alpha)(y_f - y_b)/l \tag{12}$$

$$\dot{y}_f = v \tan(\alpha)(x_f - x_b)/l + v(y_f - y_b)/l \tag{13}$$

$$\tag{14}$$

The Path of the Wheels

So we now have by substituting what we know:

$$\dot{x}_b = \frac{v}{l}\left(x_f - x_b\right) \tag{15}$$

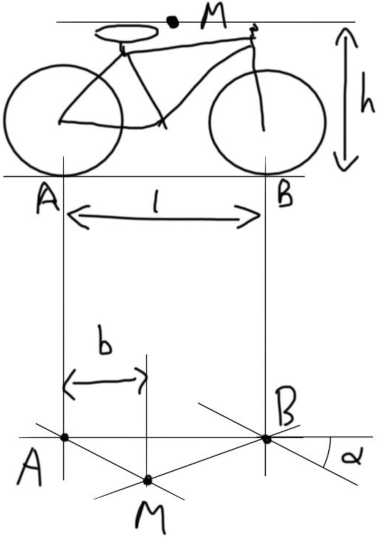
$$\dot{y}_b = \frac{v}{l}\left(y_f - y_b\right) \tag{16}$$

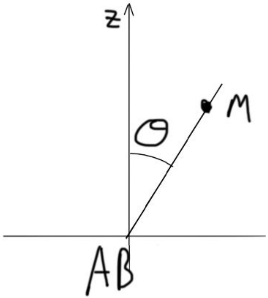
$$\dot{x}_f = \frac{v}{l}\left(\left(x_f - x_b\right) - \tan\left(\alpha\right)\left(y_f - y_b\right)\right) \tag{17}$$

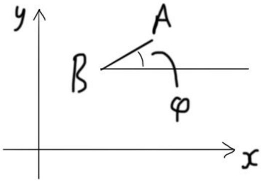
$$\dot{y}_b = \frac{v}{l}\left(\tan\left(\alpha\right)\left(x_f - x_b\right) + \left(y_f - y_b\right)\right) \tag{18}$$

We can now used this to generate the path that the wheels take given an alpha which could depend on time.

Motion of the centre of mass







So the coordinates of the centre of mass are:

$$\xi = x_b + b\cos(\phi) + h\sin(\theta)\sin(\phi) \tag{19}$$

$$\eta = y_b + b\sin(\phi) + h\sin(\theta)\cos(\phi) \tag{20}$$

$$\zeta = h\cos(\theta) \tag{21}$$

Thus the velocity and angular velocity are:

$$\vec{v} = \begin{pmatrix} \frac{d\xi}{dt} \\ \frac{d\eta}{dt} \\ \frac{d\zeta}{dt} \end{pmatrix} \tag{22}$$

$$\vec{\omega} = \vec{v} \times \vec{r}/|\vec{r}|^2 \tag{23}$$

$$\vec{\omega} = \begin{pmatrix} \frac{d\xi}{dt} \\ \frac{d\eta}{dt} \\ \frac{d\zeta}{dt} \end{pmatrix} \times \begin{pmatrix} 0 \\ h\sin(\theta) \\ h\cos(\theta) \end{pmatrix} /h^2 = \begin{pmatrix} \frac{d\eta}{dt}\cos(\theta) - \frac{d\zeta}{dt}\sin(\theta) \\ -\frac{d\xi}{dt}\cos(\theta) \\ \frac{d\xi}{dt}\sin(\theta) \end{pmatrix} /h \tag{24}$$

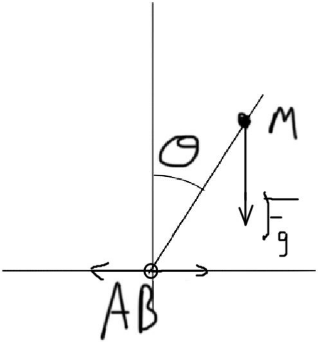
Angular Momentum

If we consider moments and momentum about the line AB

$$I = mh^2 \tag{25}$$

$$\frac{d}{dt}\left(\vec{\omega} I\right) = mgh\sin\left(\theta\right) \implies \tag{26}$$

$$\frac{d^2\eta}{dt^2}\cos\left(\theta\right) - \frac{d^2\zeta}{dt^2}\sin\left(\theta\right) = g\sin\left(\theta\right) \tag{27}$$



This is the equation of motion for the centre of mass. We would like to get it in terms of only θ.

Angular Momentum

So first we expand $\frac{d^2\zeta}{dt^2}$

$$\zeta = h\cos(\theta) \tag{28}$$

$$\frac{d\zeta}{dt} = -h\sin(\theta)\frac{d\theta}{dt} \tag{29}$$

$$\frac{d^2\zeta}{dt^2} = -h\cos(\theta)\left(\frac{d\theta}{dt}\right)^2 - h\sin(\theta)\frac{d\theta^2}{dt^2} \implies \tag{30}$$

$$\frac{d^2\eta}{dt^2}\cos(\theta) + h\cos(\theta)\left(\frac{d\theta}{dt}\right)^2\sin(\theta) + h\sin^2(\theta)\frac{d\theta^2}{dt^2} = g\sin(\theta) \tag{31}$$

Now we need to expand $\frac{d^2\eta}{dt^2}$

Angular Momentum

$$\frac{d^2\eta}{dt^2} = \frac{d^2y_b}{dt^2} - b\sin(\phi)\left(\frac{d\phi}{dt}\right)^2 + b\cos(\phi)\frac{d^2\phi}{dt^2}$$
$$- h\sin(\theta)\left(\frac{d\theta}{dt}\right)^2\cos(\phi) + h\cos(\theta)\frac{d^2\theta}{dt^2}\cos(\phi)$$
$$- 2h\cos(\theta)\frac{d\theta}{dt}\sin(\phi)\frac{d\phi}{dt} - h\sin(\theta)\cos(\phi)\left(\frac{d\phi}{dt}\right)^2$$
$$- h\sin(\theta)\sin(\phi)\frac{d^2\phi}{dt^2}$$

Maintaining a tilt

Well this is getting complicated.
So lets consider having a constant alpha, so bike is going in circles, and try to determine if there is a tilt which is constant along this path.
Then it is probably a good idea to introduce radius of curvature as this will be constant and equal to the radius of the path.

$$\left(\frac{d\phi}{ds}\right)^2 = \frac{1}{R^2} \implies \left(\frac{d\phi}{dt}\right)^2 = \frac{v^2}{R^2} \tag{32}$$

$$\text{and} \quad \frac{d^2\phi}{dt^2} = \pm\left(\frac{dv}{dt}\frac{1}{R} + \frac{d}{dt}\left(\frac{v}{R}\right)\right) = 0 \tag{33}$$

$$\text{Also} \quad \left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 = \frac{v^4}{R^2} \tag{34}$$

Maintaining a tilt

As we are moving round in a circle at some point ϕ will be zero. When this happens:

$$\frac{d^2x}{dt} = 0 \implies \left(\frac{d^2y}{dt^2}\right)^2 = \frac{v^4}{R^2} \tag{35}$$

$$\frac{d^2\eta}{dt^2} = \frac{v^2}{R} + b\frac{d}{dt}\left(\frac{v}{R}\right) - h\sin(\theta)\left(\frac{d\theta}{dt}\right)^2 + h\cos(\theta)\frac{d^2\theta}{dt^2} - h\sin(\theta)\frac{v^2}{R^2} \tag{36}$$

Which goes into this equation

$$\frac{d^2\eta}{dt^2}\cos(\theta) + h\cos(\theta)\left(\frac{d\theta}{dt}\right)^2\sin(\theta) + h\sin^2(\theta)\frac{d\theta^2}{dt^2} = g\sin(\theta) \tag{37}$$

Linearise

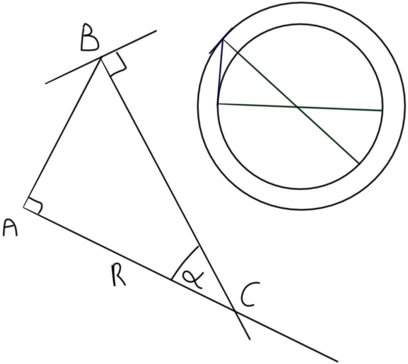
Now we consider a small tilt. In which case:

$$\cos(\theta) = 1, \quad \sin(\theta) = \theta, \quad h\sin(\theta)/R = 0 \tag{38}$$

$$\frac{d^2\eta}{dt^2} = \frac{v^2}{R} + b\frac{d}{dt}\left(\frac{v}{R}\right) + h\frac{d^2\theta}{dt^2} - h\theta\left(\frac{d\theta}{dt}\right)^2 \tag{39}$$

$$\frac{d^2\theta}{dt^2} + \frac{b}{h}\frac{d}{dt}\left(\frac{v}{R}\right) = \frac{g\theta}{h} - \frac{v^2}{hR} \tag{40}$$

Radius of Curvature



$$R \tan(\alpha) = l \quad (41)$$

Linearise

Now the equation has the form:

$$\frac{d^2\theta}{dt^2} + \frac{b}{h}\frac{d}{dt}\left(\frac{v\tan(\alpha)}{l}\right) = \frac{g\theta}{h} - \frac{v^2\tan(\alpha)}{hl} \tag{42}$$

If α and θ are constant then

$$g\theta = \frac{v^2}{R} \tag{43}$$

$$g\theta = \frac{v^2\tan(\alpha)}{l} \tag{44}$$

$$\alpha = \tan^{-1}\left(\frac{lg\theta}{v^2}\right) \tag{45}$$

Linearise

How ever if α is small and the speed is almost constant with an average v_a, then:

$$\frac{d^2\theta}{dt^2} + \frac{bv_a}{lh}\frac{d\alpha}{dt} = \frac{g\theta}{h} - \frac{v_a^2\alpha}{hl} \tag{46}$$

So to keep θ constant

$$\frac{d\alpha}{dt} = \frac{gl\theta}{v_a b} - \frac{v_a\alpha}{b} \tag{47}$$

$$\implies \quad \alpha = \frac{lg\theta}{v_a^2} + Ce^{-\frac{v_a t}{b}} \tag{48}$$

Solving the full system

We can use what we have learnt about radius of curvature and α to produce the following equations:

$$\frac{d^2\eta}{dt^2}\cos(\theta) + h\cos(\theta)\left(\frac{d\theta}{dt}\right)^2\sin(\theta) + h\sin^2(\theta)\frac{d\theta^2}{dt^2} = g\sin(\theta)$$

$$\frac{d^2\eta}{dt^2} = \frac{v^2\tan(\alpha)}{l} + b\frac{d\alpha}{dt}\frac{v\sec(\alpha)^2}{l} - h\sin(\theta)\left(\frac{d\theta}{dt}\right)^2$$
$$+ h\cos(\theta)\frac{d^2\theta}{dt^2} - h\sin(\theta)\frac{v^2\tan^2(\alpha)}{l^2}$$

Solving the full system

Now if we want θ to be constant, the derivatives must be zero

$$\frac{d^2\eta}{dt^2}\cos(\theta) = g\sin(\theta) \tag{49}$$

$$\frac{d^2\eta}{dt^2} = \frac{v^2\tan(\alpha)}{l} + b\frac{d\alpha}{dt}\frac{v\sec(\alpha)^2}{l} - h\sin(\theta)\frac{v^2\tan^2(\alpha)}{l^2} \tag{50}$$

$$g\tan(\theta) = \frac{v^2\tan(\alpha)}{l} + b\frac{d\alpha}{dt}\frac{v\sec(\alpha)^2}{l} - h\sin(\theta)\frac{v^2\tan^2(\alpha)}{l^2} \tag{51}$$

Solving the full system

$$\frac{d\alpha}{dt} = \frac{l}{v}\tan\left(\theta\right)\cos^2\left(\alpha\right) - v\sin\left(\alpha\right)\cos\left(\alpha\right) + \frac{vh}{l}\sin\left(\theta\right)\sin^2\left(\alpha\right) \tag{52}$$

which we can solve numerically:

Motion of the Centre of Mass

We can also use the system to find the tilt,

$$\frac{d^2\eta}{dt^2}\cos(\theta) + h\cos(\theta)\left(\frac{d\theta}{dt}\right)^2\sin(\theta) + h\sin^2(\theta)\frac{d\theta^2}{dt^2} = g\sin(\theta)$$

$$\frac{d^2\eta}{dt^2} = \frac{v^2\tan(\alpha)}{l} + b\frac{d\alpha}{dt}\frac{v\sec(\alpha)^2}{l} - h\sin(\theta)\left(\frac{d\theta}{dt}\right)^2 + h\cos(\theta)\frac{d^2\theta}{dt^2} - h\sin(\theta)\frac{v^2\tan^2(\alpha)}{l^2}$$

So if α and $\dot{\alpha}$ are known then we have an ordinary differential equation for θ.

Modelling the motion

Numerical Methods

For numerical integration we first turn the set of equations into a system of linear equations.

$$Y = \begin{pmatrix} \theta \\ \dot{\theta} \\ \alpha \end{pmatrix} \tag{53}$$

$$DY = \begin{pmatrix} Y(2) \\ -v^2 \cos(Y(1)) \frac{\tan(Y(3))}{hl} + \ldots \\ f(t) \end{pmatrix} \tag{54}$$

Then we can use the inbuilt ODE solvers in MATLAB

Putting the Code together

The motion of the wheels depend on α.
The motion of the centre of mass depends on $\dot{\alpha}$.

So we define $\dot{\alpha}$ for all time and integrate to deterimine α.

We can even put all the equations into one large system as solve in one swoop.

Demonstration of maintaining a tilt

We can now demonstrate the solution attained by the linearised equations.

However the solution is an unstable equilribium so when using numerical methods small error can be introduced.

This knock the system of balence and so the bike will fall over.

Changing tilt

Suppose we know α^* the alpha which maintains the current tilt.
Then an α closer to zero will increase the tilt
and a larger α will decrease the tilt

However there is also a dependence on the rate at which the tilt is changing.

Machine Learning

Gradient Descent

Suppose we have an unknown $f(x) = y$

Training set

Input	Output
x_1	y_1
x_2	y_2
x_3	y_3
etc.	

Hypothesis Function, $h_\lambda(x) \approx f(x)$ where λ is a vector of parameters and h_λ has a known form.

Gradient Descent

Cost Function,

$$J(\lambda) = \frac{1}{2m} \sum_{i=1}^{m} \left(h_{\lambda}(x_i) - y_i \right)^2$$

We would like to minimise this to attain an accurate estimate for the function given the data we have.

Gradient Descent

Idea:

1. Start with initial guess for λ
2. Change θ to minimise J

The best way to change λ is to look at the gradient of J.

Gradient Descent

Algorithm:

$$\lambda_j = \lambda_j - \alpha \frac{\partial}{\partial \lambda_j} J\left(\lambda\right) \tag{55}$$

Repeat until tolerance meet

$$\frac{\partial}{\partial \lambda_j} J\left(\lambda\right) = \frac{1}{m} \sum_{i=1}^{m} \left(h_\lambda\left(x_i\right) - y_i\right) \frac{\partial h_\lambda}{\partial \lambda_j} \tag{56}$$

Gradient Descent

For a bicycle we could use: θ, α, $\dot{\theta}$, $\dot{\alpha}$ as inputs
Time before the bike crashes as an output.

And use the h_λ to find the $\dot{\alpha}$ for a given θ, α, $\dot{\theta}$ which maximises the time taken before a crash.

Thanks for listening