ON THE HEREDITARY PARACOMPACTNESS OF LOCALLY COMPACT, HEREDITARILY NORMAL SPACES

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ABSTRACT. Assuming the consistency of a supercompact cardinal, it is shown consistent that if X is a locally compact, hereditarily normal space which includes no perfect pre-image of ω_1, then X is hereditarily paracompact and indeed is the topological sum of σ-compact, hereditarily Lindelöf, hereditarily separable spaces.

This is the fifth in a series of papers ([LTo], [To], [LT], [T] being the previous ones) that establish powerful topological consequences in models of set theory obtained by starting with a particular kind of Souslin tree S, iterating partial orders that don't destroy S, and then forcing with S. The particular case of the theorem stated in the abstract when X is perfectly normal (and hence has no perfect pre-image of ω_1) was proved in [LT], using essentially that locally compact perfectly normal spaces are locally hereditarily Lindelöf and first countable. Here we avoid these two last properties by combining the methods of [B_2] and [T]. To apply [B_2], we establish the new set-theoretic result that Fleissner's "Axiom R" [F] holds in a model of the form "PFA^{++}(S)[S]". This notation is explained below; the model is a prima facie strengthening of those used in the previous four papers.

It is easy to find locally compact, hereditarily normal spaces which are not paracompact – ω_1 is one such. Non-trivial perfect pre-images of ω_1 may also be hereditarily normal, but are not paracompact. Our result says that consistently, any example must in fact include such a canonical example.

Theorem 1. *If it is consistent that there is a supercompact cardinal, it's consistent that every locally compact, hereditarily normal space that does not include a perfect pre-image of ω_1 is (hereditarily) paracompact.*

This is not a ZFC result, since there are many consistent examples of locally compact, perfectly normal spaces which are not paracompact.

Let us state some axioms we will be using.

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PFA^{++}: *Suppose P is a proper partial order, $\{D_\alpha\}_{\alpha<\omega_1}$ is a collection of dense subsets of P, and $\{\dot{S}_\alpha : \alpha < \omega_1\}$ is a sequence of terms such that $(\forall\alpha < \omega_1) \Vdash \dot{S}_\alpha$ is stationary in ω_1. Then there is a filter $G \subseteq P$ such that*

(i) $(\forall\alpha < \omega_1)\ G \cap D_\alpha \neq 0$,

and (ii) $(\forall\alpha < \omega_1)\ S_\alpha(G) = \{\xi < \omega_1 : (\exists p \in G) p \Vdash \xi \in \dot{S}_\alpha\}$ *is stationary in* ω_1.

Baumgartner [Ba] introduced this axiom and called it "PFA$^+$". Since then, others have called this "PFA^{++}", using "PFA$^+$" for the weaker one-term version. As Baumgartner observed, the usual consistency proof for PFA, which uses a supercompact cardinal, yields a model for what we are calling PFA^{++}.

Definition. $\Gamma \subseteq [X]^{<\kappa}$ *is* **tight** *if whenever $\{C_\alpha : \alpha < \delta\}$ is an increasing sequence from Γ, and $\omega < cf\delta < \kappa$, then $\bigcup\{C_\alpha : \alpha < \delta\} \in \Gamma$.* **Axiom R:** *if $\Sigma \subseteq [X]^\omega$ is stationary and $\Gamma \subseteq [X]^{<\omega_2}$ is tight and unbounded, then there is a $Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma$ is stationary in $[Y]^\omega$.* **Axiom R^{++}:** *if $\Sigma_\alpha(\alpha < \omega_1)$ are stationary subsets of $[X]^\omega$ and $\Gamma \subseteq [X]^{<\omega_2}$ is tight and unbounded, then there is a $Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma_\alpha$ is stationary in $[Y]^\omega$ for each $\alpha < \omega_1$.*

Fleissner introduced Axiom R in [F] and showed it held in the usual model for PFA.

Σ^+: *Suppose X is a countably tight compact space, $\mathcal{L} = \{L_\alpha\}_{\alpha<\omega_1}$ a collection of disjoint compact sets such that each L_α has a neighborhood that meets only countably many L_β's, and $\mathcal{V}$ is a family of $\leq \aleph_1$ open subsets of X such that:*

a) $\bigcup\mathcal{L} \subseteq \bigcup\mathcal{V}$

b) For every $V \in \mathcal{V}$ there is an open U_V such that $\overline{V} \subseteq U_V$ and U_V meets only countably many members of $\mathcal{L}$.

Then $\mathcal{L} = \bigcup\limits_{n<\omega} \mathcal{L}_n$, *where each $\mathcal{L}_n$ is a discrete collection in $\bigcup\mathcal{V}$.*

Balogh [B$_1$] proved that MA$_{\omega_1}$ implies the restricted version of Σ^+ in which we take the L_α's to be points. The same proof establishes Σ^+.

Definition. *A space is (strongly) κ-collectionwise Hausdorff if for each closed discrete subspace $\{x_d\}_{d\in D}$, $|D| \leq \kappa$, there is a disjoint (discrete) family of open sets $\{U_d\}_{d\in D}$ with $d \in U_d$. A space is (strongly) collectionwise Hausdorff if it is (strongly) κ-collectionwise Hausdorff for all κ.*

It is easy to see that normal ($\kappa-$)collectionwise Hausdorff spaces are strongly ($\kappa-$)collectionwise Hausdorff.

Balogh [B$_2$] proved:

Lemma 2. *MA_{ω_1} + Axiom R implies locally compact hereditarily strongly $\aleph_1$-collectionwise Hausdorff spaces which do not include a perfect pre-image of ω_1 are paracompact.*

The consequences of MA_{ω_1} he used are first, that Σ^+ – indeed only the version in which the compact sets are points – and Szentmiklóssy's result [S] that *compact spaces with no uncountable discrete subspaces are hereditarily Lindelöf.* Our plan is to find a model in which these two consequences and Axiom R hold, as well as normality implying (strongly) $\aleph_1$-collectionwise Hausdorffness for the spaces under consideration. The model we will consider is of the same genre as those in [LTo], [To], [LT], and [T]. One starts off with a particular kind of Souslin tree S, a *coherent* one, which is obtainable from $\diamondsuit$ or by adding a Cohen real. One then iterates in standard fashion as in establishing MA_{ω_1} or PFA, but omitting partial orders that adjoin uncountable antichains to S. In the PFA case for example, this will establish *PFA(S)*, which is like PFA except restricted to partial orders that don't kill S. In fact it will also establish $\mathrm{PFA}^{++}(\mathrm{S})$, which is the corresponding modification of PFA^{++}. We then force with S. We use *$PFA^{++}(S)[S]$* and similar notation to stand for the intersection of the theories of all models formed in this fashion. For more information on such models, see [L].

In [T] it is established that:

Lemma 3. *PFA(S)[S] implies that locally compact normal spaces which do not include a perfect pre-image of ω_1 are (strongly) $\aleph_1$-collectionwise Hausdorff.*

By doing some preliminary forcing (see [LT]), one can actually get full collectionwise Hausdorffness, but we won't need that here.

We will assume all spaces are Hausdorff, and use "X^*" to refer to the one-point compactification of a locally compact space X.

There is a bit of a gap in Balogh's proof of Lemma 2. Balogh asserted that:

Lemma 4. *If X is locally compact and does not include a perfect pre-image of ω_1, then X^* is countably tight.*

and referred to [B_1] for the proof. However in [B_1], he only proved this for the case in which X is countably tight. It is not obvious that that hypothesis can be omitted, but in fact it can. We need a definition and lemma.

Definition. *A space Y is* $\boldsymbol{\omega}$**-bounded** *if each separable subspace of Y has compact closure.*

Lemma 5. [G], [Bu]. *If Y is ω-bounded and does not include a perfect pre-image of ω_1, then Y is compact.*

We then can establish Lemma 4 as follows.

Proof. By Lemma 5, every ω-bounded subspace of X is compact. By [B$_1$], it suffices to show X is countably tight. Suppose, on the contrary, that there is a $Y \subseteq X$ which is not closed, but is such that for all countable $Z \subseteq Y$, $\overline{Z} \subseteq Y$. Since X is a k-space, there is a compact K such that $K \cap Y$ is not closed. Then $K \cap Y$ is not ω-bounded, so there is a countable $Z \subseteq K \cap Y$ such that $\overline{Z} \cap K \cap Y$ is not compact. But $\overline{Z} \subseteq Y$, so $\overline{Z} \cap K \cap Y = \overline{Z} \cap K$, which is compact, contradiction.

Lemma 3 takes care of the hereditary strong $\aleph_1$-collectionwise Hausdorffness we need, since if open subspaces are $\aleph_1$-collectionwise Hausdorff, all subspaces are, and open subspaces of locally compact spaces are locally compact. The proposition

Σ: *in a compact countably tight space, locally countable subspaces of size $\aleph_1$ are σ-discrete.*

was established from PFA(S)[S] in [To]; the same proof works for Σ^+.

Also in [To], it was established that:

Corollary 6. *PFA(S)[S] implies that a subspace of a compact countably tight space is hereditarily separable if and only if it is hereditarily Lindelöf.*

It is standard to get from this conclusion to the result of Szentmiklóssy quoted earlier. It suffices to show that compact spaces with no uncountable discrete subspaces are separable, for then their closed subspaces are separable. They are also countably tight by an old result of Arhangel'skiǐ. (see e.g. [H]) and thence hereditarily separable. By Corollary 6, they are then hereditarily Lindelöf.

Thus all we have to do is prove that PFA^{++}(S)[S] implies Axiom R. Before doing that, however, we shall establish a topological corollary of our main result.

Corollary 7. *PFA^{++}(S)[S] implies every locally compact, hereditarily normal space which includes no perfect pre-image of ω_1 is the topological sum of σ-compact, hereditarily Lindelöf, hereditarily separable spaces. In particular, it is perfectly normal.*

Proof. Since we have that the space is locally compact and paracompact, it is standard that it breaks up into σ-compact clopen pieces. Take such a piece Y. Again, since open subspaces of locally compact spaces are locally compact, we apply our theorem to open subspaces of Y to conclude that they too are paracompact. But they can break up into only countably many σ-compact clopen pieces, since Y is σ-compact. We conclude that they too are σ-compact. It follows that Y is hereditarily Lindelöf, since its open subspaces are Lindelöf. It is shown in [LTo] that MA$_{\omega_1}$(S)[S] implies first countable hereditarily Lindelöf spaces are hereditarily separable. Since locally compact, locally hereditarily Lindelöf spaces are first countable, we conclude Y is hereditarily separable. Finally, it is easy to see that if X is the union of hereditarily Lindelöf, regular and hence perfectly normal clopen subspaces, then X too is perfectly normal.

In order to prove that PFA^{++}(S)[S] implies Axiom R, we first note that a straightforward argument using the forcing *Coll* (ω_1, X) (whose conditions are countable partial functions from ω_1 to X, ordered by inclusion) shows that PFA^{++}(S) inplies Axiom R^{++}.

It then suffices to prove:

Lemma 8. *If Axiom R^{++} holds and S is a Souslin tree, then Axiom R^{++} still holds after forcing with S.*

Proof. First note that if X is a set, P is a c.c.c. forcing and τ is a P-name for a tight unbounded subset of $[X]^{<\omega_2}$, then the set of $a \in [X]^{<\omega_2}$ such that every condition in P forces that a is in the realization of τ is itself tight and unbounded. The tightness of this set is immediate. To see that it is unbounded, let b_0 be any set in $[X]^{<\omega_2}$. Define sets b_α $(\alpha \leq \omega_1)$ and σ_α $(\alpha < \omega_1)$ recursively by letting σ_α be a P-name for a member of the realization of τ containing b_α and letting $b_{\alpha+1}$ be the set of members of X which are forced by some condition in P to be in σ_α. For limit ordinals $\alpha \leq \omega_1$, let b_α be the union of the b_β $(\beta < \alpha)$. Then b_{ω_1} is forced by every condition in P to be in τ.

Since we are assuming that the Axiom of Choice holds, Axiom R^{++} does not change if we require X to be an ordinal. Fix an ordinal γ and let $\rho_\alpha(\alpha < \omega_1)$ be S-names for stationary subsets of $[\gamma]^\omega$. Let T be a tight unbounded subset of $[\gamma]^{<\omega_2}$. For each countable ordinal α and each node $s \in S$, let $\tau_{s,\alpha}$ be the set of countable subsets a of γ such that some condition in S extending s forces that α is in the realization of ρ_α. Applying Axiom R^{++}, we have a set $Y \in [\tau]^{<\omega_2}$ such that each $\mathcal{P}(Y) \cap \tau_{s,\alpha}$ is stationary in $[Y]^\omega$.

Since S is c.c.c., every club subset of $[Y]^\omega$ that exists after forcing with S includes a club subset of $[\gamma]^\omega$ existing in the ground model. Letting $\rho_{\alpha G}$ (for each $\alpha < \omega_1$) be the realization of ρ_α, we have by genericity then that after forcing with S, each $\mathcal{P}(Y) \cap \rho_{\alpha G}$ will be stationary in $[Y]^\omega$.

This completes the proof of Theorem 1.

We do not know the answer to the following question; a postive answer would likely enable us to dispense with Axiom R, and possibly with the supercompact cardinal.

Problem. Does MA$_{\omega_1}$ imply every locally compact, hereditarily strongly collectionwise Hausdorff space which does not include a perfect pre-image of ω_1 is paracompact?

We also do not know whether in our main result, we can replace "perfect pre-image of ω_1" by "copy of ω_1".

Even for compact spaces, Theorem 1 is very strong:

Corollary 9. *PFA^{++}(S)[S] implies every compact, hereditarily normal countably tight space is hereditarily Lindelöf and hereditarily separable.*

Even for first countable (and hence countably tight) spaces, this is surprising.

Proof. Our only use of the condition that a locally compact space not include a perfect pre-image of ω_1 was to get its one-point compactification to be countably tight. Here we have X compact and countably tight already. It only remains to observe that *compact hereditarily paracompact spaces are hereditarily Lindelöf.*

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