# Unilateral weighted shifts on $\ell^2$

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**Definition 1.** Given $w \in \ell^\infty$, define a bounded linear operator $B_w : \ell^2 \to \ell^2$ by

$$B_w(x)(i) = w(i) \cdot x(i+1).$$

Such a $B_w$ is called a *unilateral weighted shift.* A vector $x \in \ell^2$ is *hypercyclic* for $B_w$ iff the set

$$\{B_w^k(x) : k \in \omega\}$$

of forward iterates is dense in $\ell^2$. Let $\mathsf{HC}(w)$ denote the set of all hypercyclic vectors for $B_w$.

It is routine to check that $\mathsf{HC}(w)$ is a $G_\delta$ set for any $w \in \ell^\infty$. The question addressed in the present paper is how much the complexity of $\mathsf{HC}(w)$ can be increased by looking at those sequences which are hypercyclic for many $w$ simultaneously. Concretely, for $W \subseteq \ell^\infty$, let

$$X_W = \bigcap_{w \in W} \mathsf{HC}(w).$$

It turns out that $X_W$ can be made arbitrarily complicated by making $W$ sufficiently complex (Theorem 4). Even for a $G_\delta$ set $W$, however, the set $X_W$ can still be non-Borel (Theorem 5).

It is necessary to introduce a few preliminaries and some terminology before proceeding. Let $\|\cdot\|_2$ denote the usual $\ell^2$ norm. In what follows, this notation will be used for finite sequences as well, i.e., for $s \in \mathbb{R}^{<\omega}$,

$$\|s\|_2 = \sqrt{s(0)^2 + \ldots + s(n)^2}$$

assuming $s$ is of length $n+1$.

The notation $|s|$ will be used to denote both the length of a string (if $s \in 2^{<\omega}$) and the length of an interval (if $s \subseteq \omega$ is an interval). The notation $\|x\|_\infty$ will denote the $\ell^\infty$ or sup-norm of $x$. Again, this definition makes sense for any string $x$ – either finite or infinite. There is a relationship between the 2-norm and the sup-norm of a finite string which will be useful in what follows. Indeed, if $s$ is a finite string of real numbers, having length $n$, a computation shows that

$$\|s\|_2 \leq n^{1/2}\|s\|_\infty.$$

One of the key descriptive set theoretic concepts in this paper is that of a "pointclass". There are many variations on the definition of "pointclass". For the purposes of the present work, use the following definition of a pointclass $\Gamma$:

**Definition 2.** A *pointclass* $\Gamma$ is a collection of subsets of Polish (separable completely metrizable) spaces such that

- $\Gamma$ is closed under continuous preimages,
- $\Gamma$ is closed under finite unions and
- $\Gamma$ is closed under finite intersections.

Given a pointclass $\Gamma$, the *dual pointclass* $\bar{\Gamma}$ consists of those $Y$ contained in some Polish space $X$ such that $X \setminus Y \in \Gamma$. A pointclass is *non-self-dual* iff there is a Polish space $X$ and a set $Y \subseteq X$ such that $Y \in \Gamma$ but $Y \notin \bar{\Gamma}$ (equivalently, $X \setminus Y \notin \Gamma$).

To take a few examples, "closed" and "open" are dual pointclasses as are "$F_\sigma$" and "$G_\delta$". All four of these classes are non-self-dual.

**Proposition 3.** *For a Borel set $W \subseteq \ell^\infty$, the intersection $\bigcap_{w\in W} \mathsf{HC}(w)$ is co-analytic.*

*Proof.* To see this, observe that, for $y \in \ell^2$,

$$y \in \bigcap_{w\in W} \mathsf{HC}(w) \iff (\forall w \in \ell^\infty)(w \in W \implies y \in \mathsf{HC}(w)).$$

The key observation is that, although $\ell^\infty$ is not Polish, its Borel structure is the same as that inherited from $\mathbb{R}^\omega$ (which is Polish). Therefore, the claim that $\bigcap_{w\in W} \mathsf{HC}(w)$ is co-analytic follows by regarding $W$ and $\ell^\infty$ as subsets of $\mathbb{R}^\omega$ and using the fact that the relation

$$P(y, w) \iff y \in \mathsf{HC}(w)$$

is itself $G_\delta$. □

The next two theorems show that the upper bound from the last proposition cannot be improved.

**Theorem 4.** *Given a non-self-dual pointclass $\Gamma$ which contains the closed sets, there is a set $W \subseteq \ell^\infty$ such that $\bigcap_{w\in W} \mathsf{HC}(w)$ is not in $\Gamma$.*

**Theorem 5.** *There is a Borel set $W$ such that $\bigcap_{w\in W} \mathsf{HC}(w)$ is properly co-analytic, i.e., not analytic.*

The key to proving Theorems 4 and 5 lies with the next three lemmas.

**Lemma 6.** *If $s \in \mathbb{R}^n$ and $\|s\|_\infty < n^{-1/2}\varepsilon$, then $\|s\|_2 < \varepsilon$.*

*Proof.* Suppose that $s \in \mathbb{R}^n$ and $\|s\|_\infty < n^{-1/2}\varepsilon$, i.e., $|s(i)| < n^{-1/2}\varepsilon$ for all $i < n$. It follows that

$$\begin{aligned}\|s\|_2 &= \sqrt{s(0)^2 + \ldots + s(n-1)^2}\\ &< \sqrt{n \cdot (n^{-1/2}\varepsilon)^2}\\ &= \varepsilon\end{aligned}$$

This proves the lemma. $\square$

**Lemma 7.** *If $A$ is a countable set and $f : 2^A \to \ell^2$ is such that*

1. *$f$ is continuous with respect to the product topologies on $2^A$ and $\ell^2$ (inherited from $R^\omega$) and*

2. *there exists $y \in \ell^2$ such that $|f(x)(i)| \leq y(i)$ for all $x \in 2^A$ and $i \in \omega$,*

*then $f$ is continuous with respect to the norm-topology on $\ell^2$.*

*Proof.* Let $y \in \ell^2$ be as in the statement of the lemma. Towards the goal of showing that $f$ is $\ell^2$-continuous, fix $\varepsilon > 0$ and let $n$ be such that

$$\|y \restriction [n, \infty)\|_2 < \varepsilon/4.$$

Since $f$ is continuous into the product topology on $\ell^2$, let $F \subseteq A$ be finite and such that, for $x_1, x_2 \in 2^A$, if $x_1 \restriction F = x_2 \restriction F$, then

$$|f(x_1)(i) - f(x_2)(i)| < n^{-1/2}\varepsilon/2$$

for all $i < n$. In particular, $x_1 \restriction F = x_2 \restriction F$ guarantees

$$\|f(x_1) - f(x_2) \restriction n\|_2 < \varepsilon/2$$

by Lemma 6. It now follows that, whenever $x_1, x_2 \in 2^A$ and $x_1 \restriction F = x_2 \restriction F$,

$$\begin{aligned}\|f(x_1) - f(x_2)\|_2 &\leq \|f(x_1) - f(x_2) \restriction n\|_2 + \|f(x_1) \restriction [n,\infty)\|_2\\ &\qquad + \|f(x_2) \restriction [n,\infty)\|_2\\ &< \varepsilon/2 + 2\|y \restriction [n,\infty)\|_2\\ &< \varepsilon/2 + 2\varepsilon/4 = \varepsilon.\end{aligned}$$

Since $\varepsilon$ was arbitrary this completes the proof. Note that a stronger result was in fact proved: $f$ is uniformly continuous with respect to the standard ultrametric on $2^A$. $\square$

**Lemma 8.** *Given a countable set $A$. It is possible to assign to each $a \subseteq A$, sequences $y_a \in \ell^2$ and $w_a \in \{1,2\}^\omega$ such that*

$$y_a \in \mathsf{HC}(w_b) \iff a \not\supseteq b$$

*Moreover the maps $a \mapsto y_a$ and $a \mapsto w_a$ are homeomorphism between $2^A$ and their ranges.*

Before proving this lemma, it will be helpful to introduce an alternative topological basis for $\ell^2$. Given a finite string $q \in \mathbb{Q}^{<\omega}$ of rationals and a (rational) number $\varepsilon > 0$, let

$$U_{q,\varepsilon} = \{x \in \ell^2 : \|(x \restriction |q|) - q\|_\infty < \varepsilon |q|^{-1/2} \text{ and } \|x \restriction [|q|, \infty)\|_2 < \varepsilon\}$$

First note that each $U_{q,\varepsilon}$ is open. In order to check that the $U_{q,\varepsilon}$ form a basis for $\ell^2$, fix a basic open ball

$$V = \{x \in \ell^2 : \|x - x_0\|_2 < \varepsilon\}$$

where $x_0 \in \ell^2$ and $\varepsilon > 0$ are fixed. Let $n \in \omega$ be such that

$$\|x_0 \restriction [n, \infty)\|_2 < \varepsilon/4$$

and choose $q \in \mathbb{Q}^n$ such that

$$\|x_0 \restriction n - q\|_2 < \varepsilon/4.$$

First of all, it follows from the definition of $U_{q,\varepsilon}$ that $x_0 \in U_{q,\varepsilon/4}$. To see that $U_{q,\varepsilon/4} \subseteq V$, observe that if $x \in U_{q,\varepsilon/4}$,

$$\begin{aligned}
\|x - x_0\| &\le \|(x - x_0) \restriction n\|_2 + \|(x - x_0) \restriction [n, \infty)\|_2 \\
&\le n^{1/2} \|(x - x_0) \restriction n\|_\infty + \|x \restriction [n, \infty)\|_2 + \|x_0 \restriction [n, \infty)\|_2 \\
&< n^{1/2} (\|(x \restriction n) - q\|_\infty + \|(x_0 \restriction n) - q\|_\infty) + \varepsilon/4 + \varepsilon/4 \\
&< n^{1/2} ((\varepsilon/4) n^{-1/2} + (\varepsilon/4) n^{-1/2}) + \varepsilon/2 \\
&= \varepsilon
\end{aligned}$$

As $x \in U_{q,\varepsilon/4}$ was arbitrary, it follows that $U_{q,\varepsilon/4} \subseteq V$. Since $V$ was an arbitrary open ball, this shows that the $U_{q,\varepsilon}$ form a topological basis for $\ell^2$.

*Proof of Lemma 8.* Let $\pi \colon \omega \to \mathbb{Q}^{<\omega}$ be a surjection. Let $A$ be the fixed countable set from the statement of the lemma. for "coding" purposes, fix a bijection

$$\langle \cdot, \cdot, \cdot \rangle : \omega \times (\mathbb{Q} \cap (0, 1)) \times A \to \omega.$$

Given $n \in \omega$, let $p_n \in \omega$, $\varepsilon_n > 0$ and $i_n \in A$ be such that

$$n = \langle p_n, \varepsilon_n, i_n \rangle.$$

Finally, let
$$\rho_n = \min\{\varepsilon_r : r < n\}.$$

The first step of the proof is to choose a suitable partition

$$I_0, J_0, I_1, J_1, \ldots$$

of $\omega$ into consecutive intervals, i.e., such that $\min(J_n) = \max(I_n) + 1$ and $\min(I_{n+1}) = \max(J_n) + 1$. Each $J_n$ will be chosen with $|J_n| = |\pi(p_n)|$. The lengths of the $I_n$ will be chosen recursively and, for concreteness, of minimal length satisfying

1. $|I_n| \geq |I_{n-1}|$,

2. $|I_n| > \max(J_{n-1})$ and

3. $2^{-|I_n|} \cdot \|\pi(p_n)\|_2 \leq 2^{-n-1} \cdot \rho_n \cdot 2^{-\max(J_{n-1})} \cdot 2^{-|I_{n-1}|}$.

for $n > 1$. The length of $I_0$ is arbitrary – $I_0$ can even be the empty interval.

The next step is to define the desired $y_a$ and $w_a$ for each $a \subseteq A$. For $n = \langle p, \varepsilon, i\rangle$, define $y_a$ on each $I_n$ and $J_n$ by

1. $(\forall n)(y_a \upharpoonright I_n = \bar{0})$,

2. $(\forall n)(i \in a \implies y_a \upharpoonright J_n = \bar{0})$ and

3. $(\forall n)(i \notin a \implies y_a \upharpoonright J_n = 2^{-|I_n|} \cdot \pi(p)$.

The first important observation about the map $a \mapsto y_a$ is that it is continuous. To see this, first observe that every initial segment of $y_a$ is determined by an initial segment of $a$. This implies that $a \mapsto y_a$ is continuous into the product topology on $\ell^2$ (which it inherits from $\mathbb{R}^\omega$). Now invoke Lemma 7 and use the fact that $y_a$ is always termwise bounded by $y_\emptyset \in \ell^2$. It now follows that $a \mapsto y_a$ is in fact continuous with respect to the norm-topology on $\ell^2$.

It also follows from the definition of $y_a$ that the function $a \mapsto y_a$ is injective. As the domain of this map ($2^A$) is compact, $a \mapsto y_a$ must therefore be a homeomorphism with its range.

Now define $w_a \in \{1, 2\}^\omega$ (for $a \subseteq A$) by making sure that the restrictions $w_a \upharpoonright I_n \cup J_n$ satisfy

1. $(\forall n)(i_n \notin a \implies w_a \upharpoonright I_n \cup J_n = \bar{1})$,

2. $(\forall n)(i_n \in a \implies (\forall j \in J_n)(|\{i < j : w_a(i) = 2\}| = |I_n|)$ and

3. if $i, j \in I_n$ with $i < j$ and $w_a(j) = 2$, then $w_a(i) = 2$.

The continuity of $a \mapsto w_a$ follows from the fact that initial segments of $w_a$ are completely determined by initial segments of $a$.

The next three claims will complete the proof. The proofs of these three claims all follows similar arguments using the definitions of the $y_a$ and $w_a$.

*Claim.* Each $y_a$ is in $\ell^2$.

It suffices to show that the $\ell^2$ norm of $y_a$ is finite. Indeed, by the triangle inequality and the third part of the definition of $y_a$,

$$\begin{aligned}\|y_a\|_2 &\leq \sum_{n\in\omega} \|y_a \restriction J_n\|_2 \\ &\leq \sum_{n\in\omega} 2^{-|I_n|} \cdot \|\pi(p_n)\|_2 \\ &\leq \sum_{n\in\omega} 2^{-n-1} \cdot \rho_n \cdot 2^{-\max(J_{n-1})} \cdot 2^{-|I_{n-1}|} \\ &\leq \sum_{n\in\omega} 2^{-n-1} \\ &\leq 1\end{aligned}$$

This proves the claim.

*Claim.* If $a, b \subseteq A$ with $a \supseteq b$, then $y_a \notin \mathsf{HC}(w_b)$.

For this claim, it suffices to show that $\|B^k_{w_b}(y_a)\|_2 \leq 1$ or $B^k_{w_b}(y_a)(0) = 0$ for each $k \in \omega$. This will establish that there is no $k \in \omega$ such that $B^k_{w_b}(y_a)$ is in the open set

$$U = \{y \in \ell^2 : \|y\|_2 > 1 \text{ and } y(0) \neq 0\}.$$

To this end, fix $k \in \omega$ and let $n \in \omega$ be such that $k \in I_n \cup J_n$. First of all, if $i_n \in a$, then $y_a \restriction I_n \cup J_n = \bar{0}$ and hence

$$B^k_{w_b}(y_a)(0) = w_b(0) \cdot \ldots \cdot w_b(k-1) \cdot y_a(k) = 0.$$

On the other hand, if $i_n \notin a \supseteq b$, then $w_b \restriction I_n \cup J_n = \bar{1}$ and hence

$$|\{j < k : w_b(j) = 2\}| \leq \max(J_{n-1}).$$

To obtain an estimate on $\|B^k_{w_b}(y_a)\|_2$, a couple preliminary observations will be useful. Suppose $t \in \omega$ is such that $k + t \in I_r$ for some $r \in \omega$. In this case,

$$B^k_{w_b}(y_a)(t) = 0$$

since $y_a(k+t) = 0$. If $k + t \in J_n$ (where $k \in I_n \cup J_n$), then

$$|B^k_{w_b}(y_a)(t)| \leq 2^{\max(J_{n-1})} \cdot |y_a(k+t)|$$

since $w_b \upharpoonright I_n \cup J_n = \bar{1}$. Finally, if $k+t \in J_r$ for some $r > n$, then

$$\begin{aligned} |B^k_{w_b}(y_a)(t)| &\le 2^k \cdot |y_a(k+t)| \\ &\le 2^{\max(J_{r-1})} \end{aligned}$$

since $k \le \max(J_n) \le \max(J_{r-1})$. It now follows by the triangle inequality that

$$\begin{aligned} \|B^k_{w_b}(y_a)\|_2 &\le \sum_{r \ge n} 2^{\max(J_{r-1})} \cdot \|y_a \upharpoonright J_r\|_2 \\ &\le \sum_{r \ge n} 2^{\max(J_{r-1})} \cdot 2^{-r-1} \cdot \rho_r \cdot 2^{-\max(J_{r-1})} \cdot 2^{|I_{r-1}|} \\ &\le \sum_{r \ge n} 2^{-r-1} \\ &\le 1 \end{aligned}$$

This completes the proof of the claim.

*Claim.* If $a, b \subseteq A$ with $a \not\supseteq b$, then $y_a \in \mathsf{HC}(w_b)$.

For this final claim, it suffices to show that, for each $q \in \mathbb{Q}^{<\omega}$ and $\varepsilon > 0$, there is a $k \in \omega$ such that $B^k_{w_b}(y_a)$ is in the open set

$$U_{q,\varepsilon} = \{x \in \ell^2 : \|(x \upharpoonright |q|) - q\|_\infty < \varepsilon |q|^{-1/2} \text{ and } \|x \upharpoonright [|q|, \infty)\|_2 < \varepsilon\}$$

as these open sets form a topological basis for $\ell^2$ by remarks preceding the proof. Indeed, fix $q \in \mathbb{Q}^{<\omega}$ and let $p \in \omega$ be such that $\pi(p) = q$. Fix $i \in b \setminus a$ and let $n = \langle p, \varepsilon, i \rangle$. Since $i \in b$ and $i \notin a$, the second case in the definition of $w_b \upharpoonright I_n \cup J_n$ and the second case in the definition of $y_a \upharpoonright J_n$ are active. In particular, for each $j \in J_n$,

$$|\{t < j : w_b(t) = 2\}| = |I_n|.$$

It follows that

$$B^{\min(J_n)}_{w_b}(y_a) = \pi(p)^\frown y$$

for some $y \in \ell^2$. To show that $B^{\min(J_n)}_{w_b}(y_a) \in U_{q,\varepsilon}$ (for any given $\varepsilon > 0$), it now suffices to show that $\|y\|_2 < \varepsilon$, since $q \prec B^{\min(J_n)}_{w_b}(y_a)$ by choice of $n$. Indeed, observe that, again by the triangle inequality,

$$
\begin{aligned}
\|y\|_2 &\leq 2^{|I_n|} \cdot \sum_{r>n} \|y_a \restriction J_r\|_2 \\
&\leq 2^{|I_n|} \cdot \sum_{r>n} 2^{-|I_r|} \cdot \|\pi(p_r)\|_2 \\
&\leq 2^{|I_n|} \cdot \sum_{r>n} 2^{-r-1} \cdot \rho_r \cdot 2^{-\max(J_{r-1})} \cdot 2^{-|I_{r-1}|} \\
&\leq 2^{|I_n|} \cdot \sum_{r>n} 2^{-r-1} \cdot \rho_n \cdot 2^{-|I_n|} \\
&\leq \varepsilon \cdot \sum_{r>n} 2^{-r-1} \\
&< \varepsilon
\end{aligned}
$$

since $\rho_n \leq \varepsilon = \varepsilon_n$. This complete the proof of the claim and proves Lemma 8. $\square$

*Proof of Theorem 4.* Let $P \subseteq 2^\omega$ be a perfect set such that $a \not\supseteq b$ for any two distinct $a, b \in P$. The construction of such a set is a standard inductive argument (similar to the construction of a perfect independent set). Let $y_a$ and $w_a$ be as in the lemma for all $a \subseteq \omega$. It follows from the independence of $P$ that $y_a \in \mathsf{HC}(w_b)$ iff $a \neq b$ for all $a, b \in P$.

Given a non-self-dual pointclass $\Gamma$ which contains the closed sets, fix $Y \subseteq P$ with $Y \in \Gamma \setminus \bar{\Gamma}$. Since $P$ is closed, it follows that $P \setminus Y \in \bar{\Gamma} \setminus \Gamma$. Let

$$W = \{w_a : a \in Y\}.$$

Now consider the set

$$X_W = \bigcap_{w \in W} \mathsf{HC}(w).$$

For $a \in P$, notice that $y_a \in X_W$ iff $a \notin Y$. Hence,

$$X_W \cap \{y_a : a \in P\} = \{y_a : a \in P \text{ and } a \notin Y\} = \{y_a : a \in P \setminus Y\}$$

It follows that $X_W \notin \Gamma$ since $\{y_a : a \in P\}$ is closed and $\{y_a : a \in P \setminus Y\} \in \bar{\Gamma} \setminus \Gamma$ (because $a \mapsto y_a$ is a homeomorphism). This completes the proof of the theorem. $\square$

*Proof of Theorem 5.* The key to this proof is an application of Lemma 8 with the countable set $A$ taken to be $\omega^{<\omega}$. With this in mind, let

$$\mathsf{Wf} = \{T \subseteq \omega^{<\omega} : T \text{ is a well-founded subtree}\}$$

and

$$\mathsf{C} = \{p \subseteq \omega^{<\omega} : p \text{ is a maximal } \prec\text{-chain}\}.$$

In other words, $\mathsf{C}$ may be identified with the set of infinite branches through $\omega^{<\omega}$. The set $\mathsf{Wf}$ proper co-analytic while $\mathsf{C}$ is $G_\delta$. Let $W = \{w_p : p \in \mathsf{C}\}$ and notice that $W$ is also $G_\delta$ since $p \mapsto w_p$ is a homeomorphism by Lemma 8. To see that

$$X_W = \bigcap_{w \in W} \mathsf{HC}(w)$$

is not analytic, observe that, for any subtree $T \subseteq \omega^{<\omega}$,

$$\begin{aligned} [T] = \emptyset &\iff (\forall p \in \mathsf{C})(T \not\supseteq p) \\ &\iff (\forall p \in \mathsf{C})(y_T \in \mathsf{HC}(w_p)) && \text{(by Lemma 8)} \\ &\iff (y_T \in X_W). \end{aligned}$$

It follows that $\mathsf{Wf}$ is a continuous preimage of $X_W$ under the map $T \mapsto y_T$. In turn, this implies that $X_W$ cannot be analytic. $\square$

# References


[1] F. Bayart, E. Matheron, **Dynamics of Linear Operators**, Cambridge University Press, 2009