Zapletal's u_2 argument

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This is an outline of the proof of the following theorem from [2].

Theorem 0.1. *Suppose that there exist a measurable cardinal and a stationary $S \subset \omega_1$ such that $NS_{\omega_1}\restriction S$ is saturated. Then there is a forcing preserving stationary subsets of S which increases u_2.*

If $u_2 = \omega_2$ then Namba forcing (or any forcing making ω_2^V have cofinality ω while preserving ω_1) increases u_2, so we will concentrate on the issue of making u_2 as large as the ω_2 of the ground model. We break the proof in this case up into the two following theorems.

Theorem 0.2. *Suppose that there is a stationary $S \subset \omega_1$ such that $NS_{\omega_1}\restriction S$ is saturated. Then there is a forcing preserving stationary subsets of S which adds a collection $\{C_n : n \in \omega\}$ of club subsets of ω_1 from the ground model such that $\bigcap_{n\in\omega} C_n \subset S$.*

Theorem 0.3. *Suppose that $V \subset W$ are models of ZFC which are correct about ω_1, and that κ is a measurable cardinal in both models as witnessed by a measure in W that restricts to a measure in V. Let $S \subset \omega_1$ be stationary such that $NS_{\omega_1}\restriction S$ is saturated in V, and let $\{C_n : n < \omega\} \in W$ be such that each C_n is a club subset of ω_1 in V, and $\bigcap_{n\in\omega} C_n \subset S$. Then $u_2^W \geq \omega_2^V$.*

We work on the proof of Theorem 0.2 first. Given a tree T and node $p \in T$, we let $|p|$ be the length (equivalently, the domain) of p. An *immediate successor* of p (in T) is a $q \in T$ such that $p \subset q$ and $|q| = |p| + 1$. For a node $p \in T$, let $\mathcal{S}_p$ be the set of X such that $p^\frown\langle X\rangle$ is in T. A *splitnode* in T is a node having at least two distinct immediate successors (equivalently, such that $\mathcal{S}_p$ has size at least 2). The *root* of a tree is the unique splitnode r of the tree (possibly the empty sequence) such that for all $p \in T$, $r \subset p$ or $p \subset r$.

Given collections $\mathcal{S}$, $\mathcal{N}$ of subsets of ω_1, we say that $\mathcal{S}$ is $\mathcal{N}$-*broad* (with $S \subset \omega_1$ as a suppressed parameter) if

1. $\bigcup\{X \mid X \in \mathcal{S}\} \supset (\omega_1 \setminus S)$;
2. For every $(A \in NS_{\omega_1}\restriction S) \cap \mathcal{N}$ there exists a $B \in \mathcal{S}$ such that $A \setminus B$ is countable.

We drop "$\mathcal{N}$" from "$\mathcal{N}$-broad" when $\mathcal{N} = \mathcal{P}(\omega_1)$. Note that since $NS_{\omega_1}\restriction S$ is normal, item (2) is equivalent to: there exists a $\gamma < \omega_1$ such that for all $A \in NS_{\omega_1}\restriction S$ there is a $B \in \mathcal{S}$ such that $A \subset \gamma \cup B$.

Fix S and let P be the set of trees T contained in $(NS_{\omega_1})^{<\omega}$ such that

- For every $p \in T$ there exists a splitnode $q \in T$ such that $p \subset q$.
- For every splitnode $p \in T$, $\mathcal{S}_p$ is broad.

Forcing with P adds an ω-sequence (the members of the roots of the members of the generic filter) of elements of NS_{ω_1} whose union is all of $\omega_1 \setminus S$ (by item (1) and genericity). It remains to see that P preserves ω_1. We show this in several steps.

Lemma 0.4. *Suppose that $NS_{\omega_1}\restriction S$ is saturated, for some $S \in NS^+_{\omega_1}$. Let $\mathcal{S}$ be a broad subset of NS_{ω_1}, and suppose that $j\colon V \to M$ is an elementary embedding derived from forcing with $\mathcal{P}(\omega_1)/(NS_{\omega_1}\restriction S)$ then $j[\mathcal{S}]$ is $\mathcal{P}(\omega_1)^M$-broad.*

Proof. Let $j\colon V \to M$ be a generic ultrapower embedding induced by a generic filter $G \subset (NS_{\omega_1}\restriction S)^+$. Noting that $j(f)(j(B)) = j(f(B)) < \omega_1^V$ for all $B \in \mathcal{S}$, we claim that ω_1^V works for $j(\mathcal{S})$ and $j(S)$ in M. First, let $h\colon S \to \omega_1$ be a function representing an element of $\omega_1^M \setminus j(S)$. If $[h]_G < \omega_1^V$, then since $\mathcal{S}$ is broad there is a $B \in \mathcal{S}$ such that $[h]_G \in B$. Since $\omega_1^V \in j(S)$, $\{\alpha < \omega_1 \mid h(\alpha) = \alpha\} \notin G$, so we may assume that $h(\alpha) \in \omega_1 \setminus (S \cup (\alpha + 1))$ for all $\alpha < \omega_1$. Then the range of h is in $NS_{\omega_1}\restriction S$, so there is a $B \in \mathcal{S}$ containing all but countably many elements of the range of h, which means that $[h]_G \in j(B)$. Now consider $h\colon \omega_1 \to NS_{\omega_1}\restriction S$. The diagonal union of the range of h is an element of $NS_{\omega_1}\restriction S$, and so there exists a $B \in \mathcal{S}$ containing all but countably many members of this diagonal union. Therefore, there is a $\gamma < \omega_1^V$ such that for all $\beta \in (\gamma, \omega_1^V)$, if there is an $\alpha < \beta$ such that $\beta \in h(\alpha)$, then β is in B. To see that $[h]_G \subset j(B) \cup (\omega_1^V + 1)$, consider a function $e\colon \omega_1 \to \omega_1$ such that $e(\alpha) \in h(\alpha) \setminus (\alpha + 1)$ for all $\alpha < \omega_1$. Then $e_\alpha \in B$ for all $\alpha > \gamma$. $\square$

Given a P-name τ for a function from ω_1 to ω_1, a bijection $\pi\colon \omega \to \beta$ for some countable ordinal β and a condition $T \in P$ we consider the game $G(\tau, \pi, T)$. For notational ease, let $T_{-1} = T$. In the ith round of the game, I plays either an element α of $\omega_1 \setminus S$ or an element A of $NS_{\omega_1}\restriction S$, and II plays a condition $T_i \leq T_{i-1}$ such that T_i forces $\tau(\pi(i))$ to be less than β, the root of T_i is longer than the root of T_{i-1}, and for B the $|root(T_{i-1})|$th member of the root of T_i, either $\alpha \in B$, if I played α, or $A \setminus B$ is countable, if I played A. If ever II cannot play meeting these conditions, she loses. Otherwise, if the game lasts for ω many rounds, she wins.

For each τ, π and T, this is a closed game, and thus determined, and a winning strategy for II defines a condition $T' \leq T$ which forces that $\tau[\beta] \subset \beta$. To show that P preserves stationary subsets of S, then, it suffices to show that for each pair τ, T and each stationary $S' \subset S$ there exist a $\beta \in S'$ and a bijection $\pi\colon \omega \to \beta$ such that II has a winning strategy in $G(\tau, \pi, T)$. To see that this is the case, fix τ, T and S', let $G \subset \mathcal{P}(\omega_1)/(NS_{\omega_1}\restriction S)$ be a generic filter containing S' and let $j\colon V \to M$ be the corresponding embedding. Fix a bijection $\pi\colon \omega \to \omega_1^V$ in M and a strategy Σ for I in $G(j(\tau), \pi, j(T))$. By Lemma 0.4, there is an infinite run of $G(j(\tau), \pi, j(T))$ where I plays by Σ and all of II's moves are in $j[P]$. Therefore, Σ is not a winning strategy for I, so II must have a winning strategy. By the elementarity of j, then we are done.

We now turn to the proof of Theorem 0.3. Let ZFC$^\circ$ denote the theory ZFC $-$ Powerset $-$ Replacement + "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the following scheme, which is a strengthening of ω_1-Replacement: every (possibly proper class) tree of height ω_1 definable from set parameters has a maximal branch

We note two facts from Section 3.1 of [1].

Theorem 0.5. *If M is countable transitive model of ZFC$^\circ$, I is a precipitous ideal on ω_1^M in M and M is a rank initial segment of a model containing ω_1, then the pair (M, I) is iterable.*

Theorem 0.6. *Suppose that M is countable transitive model of ZFC$^\circ$, I is a precipitous ideal on ω_1^M in M and the pair (M, I) is iterable, and let x be any real coding (M, I). If $f\colon \omega_1 \to \omega$ is a canonical function for an ordinal $\gamma < \omega_2$, and f appears in the last model of an iteration of (M, I) of length $\omega_1 + 1$, then the least x-indiscernible of x above ω_1 is bigger than γ.*

The following is a slight variation of a standard fact.

Lemma 0.7. *Suppose that $V \subset W$ are models of ZFC and that μ is a normal measure on κ in W such that $\mu \cap V$ is a normal measure on κ in V. Fix θ such that $\mu \cap V \in H(\theta)^V$, and let $X \in W$ be a countable elementary submodel of $H(\theta)^V$. Let $\gamma = \min(X \cap \mu)$. Then*

$$\{f(\gamma) \mid f\colon \kappa \to V \wedge f \in X\}$$

is an elementary submodel of $H(\theta)^V$ end-extending X below κ.

We now prove Theorem 0.3, which completes the proof of the main theorem.

Proof of Theorem 0.3. Fix $\gamma < \omega_2^V$ and let f be a canonical function for γ in V. Let $X \in W$ be a countable elementary substructure of $H(\theta)^V$ containing f, S and each C_n, where θ is as in Lemma 0.7. Let I denote $(NS_{\omega_1}{\restriction}s)^M$, where s is the image of S under the transitive collapse of X. By Lemma 0.7 and Theorem 0.5, the pair (M, I) is iterable. Recursively define X_α ($\alpha \leq \omega_1$) by letting $X_0 = X$, taking unions at limit stages, and letting

$$X_{\alpha+1} = \{g(X_\alpha \cap \omega_1) : g\colon \omega_1 \to H(\theta)^V \wedge g \in X_\alpha\}.$$

Letting M_α be the transitive collapse of X_α and I_α the image of $NS_{\omega_1}{\restriction}S$ under this collapse, it remains only to see that $\langle (M_\alpha, I_\alpha) : \alpha \leq \omega_1\rangle$ is an iteration of (M, I). This follows almost immediately from the fact that $NS_{\omega_1}{\restriction}S$ is saturated in V, noting that since each C_n is in X, each $X_\alpha \cap \omega_1$ is in S. $\square$

References

[1] W.H. Woodin, **The axiom of determinacy, forcing axioms, and the nonstationary ideal**, de Gruyter Series in Logic and its Applications, 1. Walter de Gruyter & Co., Berlin, 1999

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