Borel cardinals and Ramsey ultrafilters (draft of Sections 1-3)

Richard Ketchersid Paul Larson Jindřich Zapletal

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Abstract

This is a draft of the first three sections of a paper in preparation in which we study cardinality of Borel quotients in extensions of determinacy models by the forcing $\mathcal{P}(\omega)/\mathrm{Fin}$.

1 Introduction

In this paper we revisit the study of models of the form $M[U]$, where M is an inner model of ZF satisfying certain regularity properties inconsistent with the Axiom of Choice, and U is a Ramsey ultrafilter on the integers. Such extensions have been studied by several authors, notably Henle, Mathias and Woodin [6] and Di Prisco and Todorcevic [2, 3], where the model M is variously taken to be a Solovay model or an inner model of Determinacy in the presence of large cardinals.

We let Fin denote the ideal of finite subsets of $\omega = \{0, 1, 2, \ldots\}$. It is easy to see that for any $\subseteq$-decreasing sequence $\langle x_n : n < \omega\rangle$ consisting of infinite subsets of ω, there is an infinite $y \subseteq \omega$ such that $y \setminus x_n \in \mathrm{Fin}$ for all n. It follows that forcing with the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$ does not add countable subsets of the ground model. Forcing with this Boolean algebra over a model M of ZF then produces a model $M[U]$, where the generic filter is naturally interpreted as a nonprincipal ultrafilter U on ω. In fact, the ultrafilter U is a selective (or Ramsey) ultrafilter, as defined below. under assumptions to the effect that the inner model M satisfies certain regularity properties inconsistent with the Axiom of Choice. In this paper, we take as our base theory ZFC plus the existence of a proper class of Woodin cardinals, which implies that there exist inner models (such as $L(\mathbb{R})$) containing the reals and satisfying the hypotheses of these papers. A theorem of Todorcevic (see [4]) implies that in this context any Ramsey ultrafilter is generic over the types of inner models we are considering. We give a proof of this theorem in Section 2. It follows that if we assume the existence of such an ultrafilter U (which is not a theorem of ZFC), we have that inner models $M[U]$ exist as above. Moreover, the models M that we consider are canonical, in that (among other things), their theories are the same in all forcing extensions. Since the theory of $M[U]$ can be computed inside M, this means that the models $M[U]$ are similarly canonical.

We were motivated originally by the following question, which is still unsolved.

1.1 Question. Does the existence of a nonprincipal ultrafilter on ω allow one to carry out paradoxical decompositions along the lines of the Hausdorff-Banach-Tarski Paradox? (see [18])

The models we consider are natural contexts in which to address this question. Moreover, many basic questions remain open. For instance:

1.2 Question. Do there exist sets X and Y in $L(\mathbb{R})$ for which there is no injection from X to Y in $L(\mathbb{R})$ but there is one in $L(\mathbb{R})[U]$?

There are specific sets X and Y for which a positive answer to Question 1.2 would mean that paradoxical decompositions can be carried out in $L(\mathbb{R})[U]$ (and for which we don't know the answer to Question 1.2).

An ultrafilter on ω is said to be *Ramsey* if for any collection $\{X_n : n \in \omega\} \subset U$ there is a set $\{i_n : n \in \omega\}$ (listed in increasing order) in U such that each $i_n \in X_n$. Ramsey ultrafilters exist if the Continuum Hypothesis holds, and their existence follows from weaker statements such as $cov(\mathcal{M}) = \mathfrak{c}$, where $cov(\mathcal{M})$ is the least cardinality of a collection of meager sets of reals whose union is the entire real line, and $\mathfrak{c}$ denotes the cardinality of the continuum (see Theorem 4.5.6 of [1]). Kunen has shown that consistently there are no Ramsey ultrafilters, see [**?**]. The ultrafilter added by $\mathcal{P}(\omega)/\text{Fin}$ is easily seen to be Ramsey, without any assumptions on $L(\mathbb{R})$ other than ZF + DC, which it inherits from being an inner model of a model of ZFC (see Lemma 4.5.5 of [1]).

Woodin has shown that under the assumption of a proper class of Woodin cardinals, the theory of $L(\mathbb{R})$ is invariant under set forcing (see [14]). Since $\mathcal{P}(\omega)/\text{Fin}$ is homogeneous, the theory of $L(\mathbb{R})[U]$ is also invariant under set forcing. This statement can be interpreted as a statement about the forcing language for $\mathcal{P}(\omega)/\text{Fin}$ in $L(\mathbb{R})$. By Todorcevic's theorem, it is also a statement about any inner model of the form $L(\mathbb{R})[U]$, where U is a Ramsey ultrafilter on ω. Under the same large cardinal assumption, the theory of $L(\mathbb{R})[U]$ is decided in Ω-logic (see [19]). This should mean that large cardinals give as detailed a theory for $L(\mathbb{R})[U]$ (i.e., answering most natural questions) as they do for the inner model $L(\mathbb{R})$. It remains to be seen whether this is the case. At the present moment the natural open questions about this model vastly outnumber the answers.

1.1 Notation

Given an infinite set $a \subseteq \omega$, we let $[a]^{\omega}$ denote the set of infinite subsets of a, and we let $[a]^{<\omega}$ denote the set of finite subsets of a. Given $s \in [\omega]^{<\omega}$ (= Fin) and $a \subseteq \omega$, we let $s \sqsubseteq a$ denote that s is an initial segment of a, i.e., that $s \subseteq a$ and that, if s is nonempty, for every $i < \max(s)$ not in s, $i \notin a$. We let $s \sqsubset a$

denote that s is proper initial segment of a. Given $s \in [\omega]^{<\omega}$ and $a \in [\omega]^{\omega}$, we let $[s, a]$ denote the set of infinite subsets of $s \cup a$ with s as an initial segment.

Given a finite set $s \subset \omega$ and a set $a \subseteq \omega$, a/s denotes a in the case that s is the emptyset, and $a \setminus (\max(s) + 1)$ otherwise. In a severe abuse of notation, we let a/n denote $a/\{n\}$ (i.e., $a/(n + 1)$) when $n \in \omega$.

We write $x \subseteq^* y$ for $|x \setminus y| < \aleph_0$. When x and y are subsets of ω and $x \subseteq^* y$, we write $CP(x, y)$ (for *containment point*) for the least $n \in \omega$ such that $x \setminus n \subseteq y$. When x and y are subsets of ω and $x \triangle y$ is finite, we let $AP(x, y)$ (for *agreement point*) be the least $n \in \omega$ such that $x \setminus n = y \setminus n$. Similarly, when f and g are functions with domain ω and $\{n \mid f(n) \neq g(n)\}$ is finite, we let $AP(f, g)$ be the least $n \in \omega$ such that $f(m) = g(m)$ for all $m \geq n$.

1.2 Selective coideals and Ramsey ultrafilters

A *coideal* C on a set X is a subset of $\mathcal{P}(X)$ such that $\mathcal{P}(X) \setminus C$ is an ideal. Given $a \in C$, we let $C \restriction a$ denote $\{b \in C \mid b \subseteq a\}$. A coideal C on ω is *selective* if it contains all cofinite sets, and if for all $\subseteq$-decreasing sequences $\langle a_n : n \in \omega \rangle$ contained in C, there is a set $\{k_i : i \in \omega\}$ (listed in increasing order) in C such that each $k_i \in a_i$. As defined above, a Ramsey ultrafilter is a selective ultrafilter on ω.

The following is part of Theorem 4.5.2 of [1].

Theorem 1.3. *A nonprincipal ultrafilter U on ω is Ramsey if and only if either of the following two statements holds.*

- *For every partition $\{y_n : n \in \omega\}$ of ω, either some $y_n \in U$ or there exists an $x \in U$ such that $|x \cap y_n| \leq 1$ for all $n \in \omega$.*
- *For all $a \subseteq [\omega]^2$, there is an $x \in U$ such that $[x]^2 \subseteq a$ or $[x]^2 \cap a = \emptyset$.*

Given $A \subseteq [\omega]^{\omega}$ and a coideal C on ω, we say that A is *C-Ramsey* (or has the *C-Ramsey property*) if there exists a $b \in C$ such that either $A \cap [b]^{\omega} = \emptyset$ or $[b]^{\omega} \subseteq A$. We say that A is *completely C-Ramsey* if for every finite $s \subseteq \omega$ and every $b \in C$, there exists a $d \in C \restriction b$ such that either $A \cap [s, d] = \emptyset$ or $[s, d] \subseteq A$. We drop the prefix C- when C is the coideal of infinite subsets of ω. It follows easily from the definitions that if every set of reals in an inner model M of ZF is C-Ramsey, then every set of reals in M is completely C-Ramsey, even if C is not a member of M. The axioms $\mathrm{AD}_{\mathbb{R}}$ and $\mathrm{AD} + V = L(\mathbb{R})$ each imply that every subset of $[\omega]^{\omega}$ is completely Ramsey; weakly homogeneously Suslin sets of reals are also completely Ramsey (see pages 382 and 458 of [12]).

Given a coideal C on ω, a set $A \subseteq [\omega]^{\omega}$ is said to be C-Baire if for every $s \in [\omega]^{<\omega}$ and $b \in C$ there exist $t \in [\omega]^{<\omega}$ and $d \in C \restriction b$ such that $[t, d] \subseteq [s, b]$ and $[t, d] \subseteq A$ or $[t, d] \cap A = \emptyset$. The following is Corollary 7.14 of [17] (a corollary to Theorem 2.6 below).

Theorem 1.4. *If C is a selective coideal on ω, then every C-Baire subset of $[\omega]^{\omega}$ is C-Ramsey.*

1.3 Forcing over Choice-less models

For the most part, the basic theory of forcing does not require Choice. For a given partial order P, the class of P-names is defined in the same recursive manner in models with or without Choice. In either case, the forcing extension is the class of realizations of these names. There are two important differences to keep in mind. When we force with a partial order P over a model of Choice, we have that if $p \in P$, τ is a P-name, ϕ is a binary formula and $p \Vdash \exists x\, \phi(x, \tau)$, then there is a P-name σ such that $p \Vdash \phi(\sigma, \tau)$. In the Choice-less context, one has only that for densely many $p' \leq p$ there is a P-name σ such that $p' \Vdash \phi(\sigma, \tau)$. The other important difference is that P may not have infinite maximal antichains. Indeed, in the context we are interested in $\mathcal{P}(\omega)/\mathrm{Fin}$ does not have infinite maximal antichains, by a theorem of Mathias [10] (see Corollary 2.9). Instead of working with maximal antichains, we use the fact that a generic filter is one that meets all predense sets.

1.4 Universally Baire sets

The following definition comes from page 206 of [5].

1.5 Definition. Given a set $A \subseteq^\omega \omega$ and an infinite cardinal λ, A is λ-universally Baire if for every topological space X with a regular open base of cardinality at most λ, and for every continuous function $f \colon X \to^\omega \omega$, $f^{-1}[A]$ has the property of Baire in X.

Given a tree $S \subseteq \omega \times Z$, for some set Z, $p[S]$ is the set of $x \in^\omega \omega$ such that for some $z \in {}^\omega Z$ the pair (x, z) makes an infinite path through S. The following is part of Theorem 2.1 of [5].

Theorem 1.6. *Given a set $A \subseteq^\omega \omega$ and an infinite cardinal λ, A is λ-universally Baire if and only if there are trees S and T on $\omega \times \lambda$ such that $A = p[S]$ and $p[S] =^\omega \omega \setminus p[T]$ in every forcing extension by a partial order of cardinality at most λ.*

2 Ramsey ultrafilters are generic

In this section we give a proof of the following theorem of Todorcevic, adapted from [4]. Many of the ideas in this section have their origin in [10].

Theorem 2.1 (Todorcevic)**.** *If there exist infinitely many Woodin cardinals below a measurable cardinal, then every Ramsey ultrafilter is $L(\mathbb{R})$-generic for $\mathcal{P}(\omega)/\mathrm{Fin}$.*

Theorem 2.7 is the main theorem of this section, and it applies to the class of models considered in this paper.

The following definition is due to Feng, Magidor and Woodin [5].

2.2 Definition. Given a cardinal κ, a set $A \subseteq 2^\omega$ is *κ-universally Baire* if for every Hausdorff topological space X with a base consisting of regular open sets such that the regular open algebra of X has cardinality at most κ, for every continuous $f\colon X \to 2^\omega$, $f^{-1}[A]$ has the property of Baire in X.

The following theorem follows from combining arguments given in [5] and [14].

Theorem 2.3 (Woodin)**.** *If δ is a limit of Woodin cardinals below a measurable cardinal, all subsets of 2^ω in $L(\mathbb{R})$ are $<\delta$-universally Baire.*

Given a nonprincipal ultrafilter U on ω, the *U-exponential* (or *U-Vietoris* or *U-Ellentuck*) topology on $[\omega]^\omega$ has a base consisting of all sets of the form $[s, a]$, where s is a finite subset of ω and $a \in U$. Note that these sets are regular, as each set of the form $[s, a]$ is clopen.

Let $\pi\colon [\omega]^\omega \to \omega^\omega$ be the function that sends each infinite subset of ω to its increasing enumeration. Letting U be a nonprincipal ultrafilter, π is continuous when its domain is given the U-exponential topology and its range is given the usual product topology. It follows that $A \subseteq [\omega]^\omega$ has the property of Baire in the U-exponential topology whenever $\pi[A]$ is $2^{\mathfrak{c}}$-universally Baire, where $\mathfrak{c}$ denotes $2^{\aleph_0}$, the cardinality of the continuum.

The following lemma follows from Theorem 1.4 above.

Lemma 2.4. *If D is a dense open set in the U-exponential topology, and $[s, a]$ is a basic open set, there exists a set $a' \subseteq a$ in U such that $[s, a'] \subseteq D$.*

Our proof will also use the following two facts, the second of which is Lemma 7.12 of [17].

Theorem 2.5. *If U is a Ramsey ultrafilter on ω, and for each finite $s \subseteq \omega$, A_s is a member of U, then there is a set $B \in U$ such that for all $n \in B$, $B/n \subseteq \bigcap_{s \subseteq n} A_{s \cup \{n\}}$.*

Proof. Let E be the set of pairs $i < j$ from ω such that $j \in A_{s\cup\{i\}}$ for all $s \subseteq i$, and let $B \in U$ be such that $[B]^2$ is contained in or disjoint from E. Since U is a filter, fixing $i \in B$ there must be $j \in B/i$ such that $\{i, j\} \in E$, so $[B]^2$ is not disjoint from E. □

Theorem 2.6. *[Selective Galvin lemma] If $F \subseteq [\omega]^{<\omega}$ and C is a selective coideal, then there exists an $a \in C$ such that $F \cap [a]^{<\omega}$ is either empty or contains an initial segment of every infinite subset of C.*

Todorcevic's theorem follows from the following more general fact.

Theorem 2.7. *If U is a Ramsey ultrafilter, $I \subseteq [\omega]^\omega$ is $\supseteq$-dense and I has the property of Baire in the U-exponential topology, then $U \cap I \neq \emptyset$.*

Proof. Let us note first that for any dense open set D in the U-exponential topology, and any $i \in \omega$, D contains a dense open set $D[i]$ which is closed under changes below i. To see this, we check that for all $t \subseteq i$, the set D_i^t consisting of those $b \in [\omega]^\omega$ such that $(b \setminus i) \cup t \in D$ is dense open. Fix t, and note that if $[t_0, c]$ is a basic open set, with $\max(t_0) > i$, then, letting $t_1 = (t_0 \setminus i) \cup t$, there exists a $t_2 \subseteq c/t_1$ and a set $c' \in U$ such that $[t_1 \cup t_2, c']$ is contained in $[t_1, c] \cap D$. Then for every $b \in [t_0 \cup t_2, c']$, $(b \setminus i) \cup t$ is in D. Now, let $D[i]$ be the dense open set formed by taking the intersection of all D_i^t, for $t \subseteq i$.

Let $I \subseteq [\omega]^\omega$ be $\supseteq$-dense, and suppose that I has the property of Baire in the U-exponential topology. There exists an open set O such that $O \triangle I$ is meager.

Let us see that O is dense. Fix a basic open set $[s, a]$ and dense open sets D_i $(i \in \omega)$ such that $(O \triangle I) \cap \bigcap_{i \in \omega} D_i = \emptyset$. We may assume that $D_{i+1} \subset D_i$ for each $i \in \omega$. Let $[s_i, a_i]$ $(i \in \omega)$ be such that

- $[s_0, a_0] \subseteq [s, a]$;
- $[t \cup \{\max(s_i)\}, a_i] \subseteq D_i$, for all $t \subseteq \max(s_i)$ and $i \in \omega$ (here we use Lemma2.4);
- each $[s_{i+1}, a_{i+1}] \subseteq [s_i, a_i]$;
- each s_{i+1} is a proper extension of s_i.

Then $\{\max(s_i) : i \in \omega\}$ is infinite and every infinite subset of it is in each D_i. It has an infinite subset in I, and therefore in O.

We have then that O is dense open in the U-exponential topology, so by adding its complement to our collection of dense sets if necessary, we may assume that $O = [\emptyset, \omega]$, and fix dense open sets D_i $(i \in \omega)$ such that $\bigcap_{i \in \omega} D_i \subseteq I$. We will be done with the proof once we establish the following claim.

Claim. $U \cap \bigcap_{i \in \omega} D_i \neq \emptyset$.

Replacing each D_i with $D_i[i+1]$ as above, we have that for all $b \in D_i$ and all $t \subseteq (i+1)$, $(b/i) \cup t \in D_i$. We may assume also that $D_j \subseteq D_i$ for all $i < j$ in ω.

For each $i \in \omega$ and each finite $t \subset \omega$, let $a_t^i \subseteq \omega/t$ be an element of U such that $[t, a_t^i] \subseteq D_i$, if such a set exists, otherwise, let $a_t^i = \omega/t$.

For each $i \in \omega$, let

- b_i be an element of U such that for all $n \in b_i$, $b_i/n \subseteq \bigcap\{a^i_{t \cup \{n\}} \mid t \subseteq n\}$ (here we use Theorem 2.5);
- S_i be the set of nonempty finite $t \subset \omega$ such that $[t, a_t^i] \subseteq D_i$;
- c_i be an element of U such that $S_i \cap [c_i]^{<\omega}$ contains an initial segment of every infinite subset of c_i (here we use the Selective Galvin lemma; note that the empty case cannot hold, since D_i is dense open).

Applying Theorem 2.5 again, let $e \in U$ be such that $e/i \subseteq b_i \cap c_i$ for all $i \in e$. We claim then that $e \in D_i$ for all $i \in \omega$. Since the D_i's are shrinking, and e is infinite, it suffices to consider $i \in e$. For each such i, it suffices to see that $e/i \in D_i$. This in turn follows from the fact that $e/i \subseteq c_i$, so some nonempty initial segment s_0 of e/i is in S_i, so $[s_0, a^i_{s_0}] \subseteq D_i$. Since $e/i \subseteq b_i$ and $b_i/s_0 \subseteq a^i_{s_0}$, we have that $e/s_0 \subseteq b_i$ and thus that $e/i \in [s_0, a^i_{s_0}]$. □

As a corollary, we get Mathias's result (in this context) that selective coideals in $L(\mathbb{R})$ must be equal to Fin densely often.

Corollary 2.8. *Suppose that M is an inner model containing the reals, and that every set of reals in M is $2^{\mathfrak{c}}$-universally Baire in every forcing extension of V by an (ω, ∞)-distributive partial order of cardinal $\mathfrak{c}$. Then for every selective coideal C on ω in M, and every $a \in [\omega]^{\omega}$, there is a $b \in [a]^{\omega}$ such that $[b]^{\omega} \subseteq C$.*

Proof. Let $I = \mathcal{P}(\omega) \setminus C$. Since C is selective, a V-generic filter for $\mathcal{P}(a)/I$ gives a Ramsey ultrafilter U which does not intersect I. This ultrafilter U is also M-generic for $\mathcal{P}(a)/\mathrm{Fin}$, which means that there must be a $b \in [a]^{\omega} \cap U$ such that $[b]^{\omega} \cap I = \emptyset$. □

This of course implies that there are no infinite maximal antichains in $\mathcal{P}(\omega)/\mathrm{Fin}$.

Corollary 2.9. *If M is an inner model containing the reals, and every set of reals in M is $2^{\mathfrak{c}}$-universally Baire in every forcing extension of V by an (ω, ∞)-distributive partial order of cardinal $\mathfrak{c}$, then the partial order $\mathcal{P}(\omega)/\mathrm{Fin}$ contains no infinite maximal antichains in M.*

Proof. If A were such an antichain, let I be the ideal of subsets of ω which are contained mod-finite in a union of finitely many members of A, and let C be the corresponding coideal. Then C is selective, and nowhere equal to Fin. □

3 Countable-to-one Uniformization

Given sets A, B, $a \in A$ and $X \subseteq A \times B$, we let X_a denote the set of $b \in B$ such that $(a, b) \in X$. *Uniformization* is the statement that for every $X \subseteq \mathbb{R} \times \mathbb{R}$ there is a function $f \subseteq X$ whose domain is the set of $a \in \mathbb{R}$ such that $X_a \neq \emptyset$. *Countable-to-one Uniformization* is Uniformization restricted to the case where each set X_a is countable (in this case we say that X has *countable cross sections*). In this section we first discuss Countable-to-one Uniformization in models of Determinacy, and we then show that it holds in the $\mathcal{P}(\omega)/\mathrm{Fin}$ extension.

3.1 ...in models of Determinacy

It is easy to see that Uniformization is equivalent to determinacy for one-round real games, which of course follows from $\mathrm{AD}_{\mathbb{R}}$. It is also well known that Uniformization fails in models of the form $L(A)$, for A a set of reals. In this subsection we present an unpublished theorem of Woodin that Countable-to-One Uniformization holds in $L(\mathbb{R})$ and other models of AD.

3.1 Definition. A set of ordinals S is an ∞-*Borel* code for a set of reals A if for some binary formula ϕ, $A = \{x \in \mathbb{R} \mid L[S, x] \models \phi(S, x)\}$.

The statement that every set of reals has an ∞-Borel code is one of the three statements that make up the axiom AD^{+}. Recall that for a model M of ZF and sets $x_1, \ldots, x_n$ in M, $\mathrm{HOD}^M_{x_1,\ldots,x_n}$ is the class HOD as defined in M, allowing $x_1, \ldots, x_n$ as parameters. This is always a model of ZFC, and has a natural definable wellordering.

Theorem 3.2 (Woodin)**.** *Countable-to-one Uniformization is a consequence of AD + $DC_{\mathbb{R}}$ + "every set of reals has an ∞-Borel code."*

Before beginning the proof, we note that AD can be replaced by the following consequences.

- (Martin) Every set of Turing degrees either contains or is disjoint from a cone.
- (Mycielski) There is no ω_1-sequence of distinct reals.
- (Woodin, Theorems 5.4 and 5.9 of [9]) For each real x there is a Turing cone of reals y such that
  - $\omega_2^{L[S,x,y]}$ is strongly inaccessible in $\mathrm{HOD}^{L[S,x,y]}_{S,x}$, and
  - CH holds in $L[S, x, y]$.

Proof of Theorem 3.2. Since all sets of reals are ∞-Borel, it suffices to fix a set of ordinals S and a formula ϕ and show that the set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid L[S, x, y] \models \phi(S, x, y)\}$$

can be uniformized, under the assumption that all of its cross sections are countable. For any set of ordinals T, let A_T denote the set

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid L[T, x, y] \models \phi(T, x, y)\}.$$

We will show that for each $x \in \mathbb{R}$, $(A_S)_x = \{y \in \mathbb{R} \mid (x, y) \in A_S\} \subseteq \mathrm{HOD}_{S,x}$. From this it follows, using the natural wellordering of $\mathrm{HOD}_{S,x}$, that A_S can be uniformized. Fix a real x_0. For each $z \in \mathbb{R}$, set

$$H_z = \mathrm{HOD}^{L[S,x_0,z]}_{S,x_0} \qquad \text{and} \qquad \delta_z = \omega_2^{L[S,x_0,z]}$$

Claim. *It suffices to show that $(A_S)_{x_0} \subseteq H_z$ on a cone of z.*

To see why this suffices suppose it is true and for each z in this cone let $\langle x^z_\alpha \mid \alpha < \gamma_z\rangle$ be the enumeration of $(A_S)_{x_0}$ in H_z via the natural wellordering of H_z. For each fixed $\alpha < \omega_1$, we get that on a cone of z, x^z_α is a fixed real x^∞_α. The ordinal γ_z must also be the same for a cone of z, or else we could construct an ω_1-sequence of distinct reals. So on a cone of z, $\langle x^z_\alpha \mid \alpha < \gamma_z\rangle = \langle x^\infty_\alpha \mid \alpha < \gamma_\infty\rangle$. Clearly $x^\infty_\alpha \in \mathrm{HOD}_{S,x_0}$ for all $\alpha < \gamma_\infty$. This finishes the proof of the claim.

We finish by proving the claim. Since $(A_S)_{x_0}$ is countable, it is a subset of $L[S, x_0, z]$ for a cone of z. Fix any z in this cone. Following Definition 2.3 of [7] (but changing the notation), we let $\mathbb{B}_0$ be the collection of sets of reals in $L[S, x_0, z]$ which are ordinal definable in $L[S, x_0, z]$ from S and x_0. Given a filter $G \subset \mathbb{B}_0$, let $y(G)$ be the set of $n \in \omega$ such that $\{y \subseteq \omega \mid n \in y\} \in G$. Then by Vopěnka's Theorem (Theorem 2.4 of [7]), there exist a Boolean algebra $\mathbb{B}_1$ in H_z, a $\mathbb{B}_1$-name $\dot{y} \in H_z$ and an isomorphism $h\colon \mathbb{B}_0 \to \mathbb{B}_1$ such that

1. for every real $y \in L[S, x_0, z]$, $G(y) = h[\{A \in \mathbb{B}_0 \mid y \in A\}]$ is H_z-generic for $\mathbb{B}_1$;
2. if $H \subseteq \mathbb{B}_1$ is H_z-generic and $G = h^{-1}[H]$, then $y(G) = \dot{y}_H$ and, for every binary formula ψ and every ordinal α,

   $$L_\alpha[S, x_0, y(G)] \models \psi(S, x_0, y(G)) \Leftrightarrow \{y \subseteq \omega \mid L_\alpha[S, x_0, y] \models \psi(S, x_0, y)\} \in G.$$

By (2), densely many conditions in $\mathbb{B}_1$ below

$$\{y \subseteq \omega \mid L[S, x_0, y] \models \phi(S, x_0, y)\}$$

must decide all of $\dot{y}$, since otherwise one can easily construct a real $y(G)$ distinct from all members of the countable set $(A_S)_{x_0}$ (here we use the fact that $\mathcal{P}(\mathbb{B}_1)^{H_z}$ is countable, which follows from the fact that there is no ω_1-sequence of distinct reals). By (1), and the assumption that $(A_S)_{x_0} \subseteq L[S, x_0, z]$, every member of $(A_S)_{x_0}$ is one of these completely determined values of $\dot{y}$, which means that $(A_S)_{x_0} \subseteq H_z$. □

3.3 Remark. A slight modification of the argument just given works just assuming ZF + DC + "there is a fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$"; this holds in the Solovay model for Levy collapsing a measurable cardinal to be ω_1.

3.4 Remark. The argument just given shows that under the assumption AD$^+$ + $V = \mathrm{HOD}\mathcal{P}(\mathbb{R})$, one can uniformize subsets of $\mathcal{P}(Ord) \times \mathbb{R}$ which have countable cross-sections.

3.2 ...in the $\mathcal{P}(\omega)/\mathrm{Fin}$ extension

Before proving Theorem 3.6 below, we separate out the following lemma, a variation of the results of Section 6 of [10].

Lemma 3.5. *Suppose that every set of reals is completely Ramsey, and let $f\colon [\omega]^\omega \to 2^\omega$ be a partial function whose domain is closed under subsets. Then for all $x_0 \in [\omega]^\omega$ there exists $x^* \in [x_0]^\omega$ such that either x^* is not in the domain of f or there exists a collection*

$$\{\tau^n_\sigma : n \in \omega \setminus \{0\}, \sigma \subseteq n\}$$

such that each $\tau^n_\sigma \in 2^n$ and such that for all infinite $x \subseteq \omega$ and all $m \in \omega$, if $x/m \subseteq x^*$, then $f = \bigcup\{\tau^n_{x' \cap n} : n \in [m, \omega) \cap x'\}$.

Proof. Fix x_0 such that $[x_0] \in dom(f)$. Find x_i ($i < \omega$) such that

- each $x_{i+1} \in [x_i]^\omega$;

- for each $i \in \omega$ and $s \subseteq i$, $f \restriction i$ is the same fixed set τ^i_s for all $x \in [s \cup \{i\}, x_i]$ (here we use the complete Ramsey property).

Then let x^* be an infinite subset of x_0 such that, for each $i \in x^*$, $x^*/i \subseteq x_i$. Then if $x' \subseteq^* x^*$ is infinite and $m \in \omega$ is such that $x'/m \subseteq x^*$, then for all $i \in x' \setminus m$,

$$x'/i \subseteq x^*/i \subseteq x_i,$$

so $f(x') \restriction i = \tau^i_{x' \cap i}$. Then $f(x') = \bigcup \{\tau^n_{x' \cap n} : n \in [m, \omega) \cap x'\}$. □

Theorem 3.6. *Suppose that every set of reals is completely Ramsey, and that Countable-to-one Uniformization holds. Then Countable-to-one Uniformization holds in the* $\mathcal{P}(\omega)/\mathrm{Fin}$ *extension.*

Proof. Let ρ be a $\mathcal{P}(\omega)/\mathrm{Fin}$-name for a subset of $2^\omega \times 2^\omega$. Let T be the set of triples (x, y, z) such that $[x]$ forces that (y, z) is in the realization of ρ. By refining T, we may suppose that for each pair (x, y),

$$\{z \in (x, y, z) \in T\} = \{z \mid \exists x' \in [x]^\omega \, (x', y, z)\}$$

whenever the first of these two sets is nonempty (note that it is always countable). To see this, note that since ρ is a name for a set with countable cross-sections, for each y, for densely many $[x]$ there is a sequence of reals that $[x]$ forces to be an enumeration of the cross section of ρ at y, and we may restrict T to triples starting with such pairs (x, y). Let P_0 be the set of pairs (x, y) for which there exists a z with $(x, y, z) \in T$. Applying Countable-to-one Uniformization, fix a function $Z \colon P_0 \to 2^\omega$ such that for each $(x, y) \in P_0$, $(x, y, Z(x, y)) \in T$.

Let P_1 be the set of pairs (x, y) such that there exists a collection

$$\{\tau^n_\sigma : n \in \omega \setminus \{0\}, \sigma \subseteq n\}$$

such that each $\tau^n_\sigma \in 2^n$ and such that for all infinite $x' \subseteq \omega$ and all $m \in \omega$, if $x'/m \subseteq x$, then

$$Z(x', y) = \bigcup \{\tau^n_{x' \cap n} : n \in [m, \omega) \cap x'\}.$$

Applying Lemma 3.5 to the function $Z(x, y)$ (with y fixed), we get the following.

Claim. *For each* $y \in {}^\omega 2$ *there are densely many classes* $[x] \in \mathcal{P}(\omega)/\mathrm{Fin}$ *such that either* $(x, y) \notin P_0$ *or* $(x, y) \in P_1$.

For each pair $[x]$, y, let $\Sigma^{[x]}_y$ be the emptyset if $(x, y) \notin P_0$, and otherwise let it be the set of finite $\sigma \subset \omega$ for which there exists an $x' \in [x]$ such that $Z(x'', y)$ is the same for all $x'' \in [\sigma, x']$. Noting that this constant value must be the same for all such x', we denote it by $Z^*([x], y, \sigma)$. Note that $[x_0] \leq [x_1]$ implies $\Sigma^{[x_1]}_y \subseteq \Sigma^{[x_0]}_y$, so for each y, $\Sigma^{[x]}_y$ is constant below densely many conditions $[x]$.

Claim. *If* $(x, y) \in P_1$*, then* $\Sigma^{[x]}_y \neq \emptyset$.

To prove the claim, fix $\{\tau^n_\sigma : n \in \omega \setminus \{0\}, \sigma \subseteq n\}$ witnessing that $(x, y) \in P_1$. For each $n \in \omega$ and $s \subset n$ such that $s \cup \{n\} \subseteq x$, try to find $t \cup \{m\}$ and $r \cup \{p\}$, subsets of x end-extending $s \cup \{n\}$, such that τ^m_t and τ^p_r are incompatible (necessarily proper) extensions of τ^n_s. If there always exists such a pair, then there is a perfect set Q consisting of infinite subsets of x' such that the values of $Z(x', y)$ for $x' \in Q$ are all distinct. This is impossible, by our refinement of T. This proves the claim.

Fixing some enumeration of $[\omega]^{<\omega}$, for each $[x] \in \mathcal{P}(\omega)/\mathrm{Fin}$, let $\sigma_{[x],z}$ denote the least element of $\Sigma^{[x]}_y$, whenever this set is nonempty (and be undefined otherwise). For each $y \in 2^\omega$, for densely many $[x]$, either $[x]$ forces that y is not in the first coordinate projection of the realization of ρ, or $\sigma_{[x],y}$ is defined and $\sigma_{[x']} = \sigma_{[x]}$ for all $[x'] \leq [x]$. Call this dense set D_y.

Now, suppose that $[a]$ and $[b]$ are two compatible conditions in D_y such that $\sigma_{[a],y}$ and $\sigma_{[b],y}$ are defined. Then for any $[c]$ below both $[a]$ and $[b]$, $\sigma_{[c],y}$ is equal to both $\sigma_{[a],y}$ and $\sigma_{[b],y}$. Call this set σ. Then if $d \in [a]$ and $e \in [b]$ are such that $Z(f, y)$ is the same for all $f \subseteq^* d$ with $f \cap |\sigma| = \sigma$, and $Z(g, y)$ is the same for all $g \subseteq^* e$ with $e \cap |\sigma| = \sigma$, then these two constant values are the same, since these two sets are not disjoint. We have then that

- for all $(x, y) \in P_0$, if $[x] \in D_y$, then $(x, y, Z^*([x], y, \sigma_{[x],y})) \in T$;
- for all $(a, y), (b, y) \in P_0$, if $[a], [b] \in D_y$ and $[a]$, $[b]$ are compatible, then $Z^*([a], y, \sigma_{[a],y}) = Z^*([b], y, \sigma_{[b],y})$.

It follows that the set of $(x, y, \sigma_{[x],y})$ for $(x, y) \in P_0$ and $[x] \in D_y$ gives rise to a name for function uniformizing the realization of ρ. $\square$

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Department of Mathematical Sciences
University of Texas at Dallas
Richardson, TX 75080
USA
ketchero@muohio.edu

Department of Mathematics
Miami University
Oxford, Ohio 45056
USA
larsonpb@muohio.edu

Department of Mathematics
University of Florida
Gainesville, Florida 32611-8105
USA
zapletal@math.ufl.edu