

The nonstationary ideal in the \mathbb{P}_{max} extension

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Abstract

The forcing construction \mathbb{P}_{max} , invented by W. Hugh Woodin, produces a model whose collection of subsets of ω_1 is in some sense maximal. In this paper we study the Boolean algebra induced by the nonstationary ideal on ω_1 in this model. Among other things we show that the induced quotient does not have a simply definable form. We also prove several results about saturation properties of the ideal in this extension.

1 Introduction

We let NS_{ω_1} denote the ideal of nonstationary subsets of ω_1 . Many basic properties of the Boolean algebra $\mathcal{P}(\omega_1)/NS_{\omega_1}$ have long been known to be independent of ZFC. One well-studied example is the question of whether it is \aleph_2 -c.c., in which case it is said to be *saturated*. Steel and Van Wesep gave the first consistency proof of the saturation of NS_{ω_1} with ZFC by deriving the conclusion of the following theorem from the consistency of a strong form of determinacy. Woodin later improved the assumption to the Axiom of Determinacy (AD).

Theorem 1.1. (*[18]; [23]*) *If $ZF + AD$ is consistent then ZFC is consistent with the statement that the Boolean algebra $\mathcal{P}(\omega_1)/NS_{\omega_1}$ has a dense set order-isomorphic to $Coll(\omega, <\omega_2)$.*

The partial order $Coll(\omega, <\omega_2)$ consists of all finite partial functions p from $\omega_2 \times \omega$ to ω_2 , with the stipulation that $p(\alpha, i) \in \alpha$ for all (α, i) in the domain of p (ordered by inclusion). A straightforward Δ -system argument shows that this partial order is \aleph_2 -c.c.

The following result, giving an alternate form of the Boolean algebra with a stronger saturation property, was later proved using a variation of the partial order \mathbb{P}_{max} , which we will define in Section 2. The partial order $Coll(\omega, \omega_1)$ consists of all finite partial functions from ω to ω_1 , ordered by inclusion.

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Theorem 1.2. ([24]) *If the Axiom of Determinacy is consistent with ZF, then it is consistent with ZFC that the Boolean algebra $\mathcal{P}(\omega_1)/NS_{\omega_1}$ has a dense set order-isomorphic to $Coll(\omega, \omega_1)$.*

The following results show that the forms of the nonstationary ideal in the previous theorems are incompatible with MA_{\aleph_1} , as $Coll(\omega, <\omega_2)$ and $Coll(\omega, \omega_1)$ are homogeneous, and $Coll(\omega, \omega_1)$ is trivially \aleph_1 -dense.

Theorem 1.3. ([19]) *If MA_{\aleph_1} holds, then NS_{ω_1} is not \aleph_1 -dense.*

Theorem 1.4. ([4]; see also [10]) *If MA_{\aleph_1} holds, then for any two distinct stationary sets $A, B \subset \omega_1$, the Boolean algebras $\mathcal{P}(A)/NS_{\omega_1}$ and $\mathcal{P}(B)/NS_{\omega_1}$ do not have isomorphic dense sets.*

The partial order \mathbb{P}_{max} was invented by W. Hugh Woodin in the early 1990's. When applied to a model of determinacy, it produces a model which is in some ways maximal for the powerset of ω_1 . Furthermore, the theory of this model is computed in the inner model $L(\mathbb{R})$. It is an empirical fact that AD answers all natural combinatorial questions about $L(\mathbb{R})$, and one might expect then that every natural combinatorial question about the \mathbb{P}_{max} extension should similarly be answerable. The paper is an attempt to test that expectation on some questions about NS_{ω_1} .

Our investigation was partially inspired by the following axioms of Peter Nyikos [16].

1.5 Definition. Axiom F is the statement that for any function

$$F: \omega_2 \times \omega_1 \rightarrow \omega$$

there is an infinite $a \subset \omega_2$ and an $n \in \omega$ such that

$$\bigcap_{\alpha \in a} \{\beta < \omega_1 \mid F(\alpha, \beta) < n\}$$

is stationary.

1.6 Definition. Axiom F^+ is the statement that for any function

$$F: \omega_2 \times \omega_1 \rightarrow \omega$$

there is an infinite $a \subset \omega_2$ and an $n \in \omega$ such that

$$\bigcap_{\alpha \in a} \{\beta < \omega_1 \mid F(\alpha, \beta) = n\}$$

is stationary.

These axioms follow immediately from the \aleph_1 -density of $\mathcal{P}(\omega_1)/NS_{\omega_1}$, and are easily seen to fail if $\mathcal{P}(\omega_1)/NS_{\omega_1}$ has a dense set order-isomorphic to the partial order $Coll(\omega, <\omega_2)$. It is not known whether Martin's Maximum [4] implies either of Axiom F or Axiom F^+ or their negations (though it is known that

they are equivalent under MA_{\aleph_1} [16]). Part of this paper is an as yet unsuccessful attempt to determine whether these axioms hold in the \mathbb{P}_{max} extension. In Section 4 we show that when NS_{ω_1} is saturated Axiom F^+ is equivalent to the assertion that the restriction of NS_{ω_1} to some stationary set has caliber (\aleph_2, \aleph_0) (an ideal I has *caliber* (κ, γ) if every κ -sized collection of I -positive sets has a subcollection of cardinality γ with an I -positive lower bound). Section 5 shows that NS_{ω_1} satisfies a weak form of caliber (\aleph_2, \aleph_0) in the \mathbb{P}_{max} extension.

Another motivating question was the following, asked of us by Stuart Zoble.

1.7 Question. Is the square of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ \aleph_2 -c.c. in the \mathbb{P}_{max} extension?

A positive answer to Question 1.7 would follow from a positive answer to the following.

1.8 Question. In the \mathbb{P}_{max} extension, does every subset of $\mathcal{P}(\omega_1) \setminus NS_{\omega_1}$ of cardinality \aleph_2 contain a pairwise compatible set of cardinality \aleph_2 ?

Again, we do not know the answer to either of these questions, though in Section 7 we rule out a strong positive answer to Question 1.8 by showing that $\mathcal{P}(\omega_1) \setminus NS_{\omega_1}$ is not the union of \aleph_1 -many pairwise compatible sets in the \mathbb{P}_{max} extension.

Questions about the saturation properties of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ tend to rest on questions in the partition calculus on ω_1 where one asks for stationary sets with certain properties. The classical result of this type is Fodor's Lemma, which says that every regressive function on a stationary set is constant on a stationary set. In proofs of the saturation of NS_{ω_1} , one shows that every predense subset of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ contains a subset whose diagonal intersection contains a club. By Fodor's Lemma, then, there can be no stationary set which has nonstationary intersection with every member of this subset. An \aleph_1 -sized collection of stationary subsets of ω_1 whose diagonal union contains a club is therefore a type of obstruction to finding a counterexample to saturation. One recurring theme in the investigations in this paper is: what other sorts of obstructions can exist? Section 6 rules out one possible type of obstruction, and Questions 6.11 and 7.3 ask about two others.

Finally, we were motivated by a remark by Foreman, Magidor and Shelah in [4] that they had not found a characterization for $\mathcal{P}(\omega_1)/NS_{\omega_1}$ under Martin's Maximum. Section 3 puts a lower bound on the complexity of any such characterization in the \mathbb{P}_{max} extension, by showing that there is no dense subset of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ order-isomorphic to a set in the ground model. This is in contrast to Theorems 1.1 and 1.2, in which $\mathcal{P}(\omega_1)/NS_{\omega_1}$ has dense sets isomorphic to sets in L . We note again that the forms given in Theorems 1.1 and 1.2 make the question of whether Axiom F^+ holds or whether $(\mathcal{P}(\omega_1)/NS_{\omega_1})^2$ is \aleph_2 -c.c. almost trivial. One hope of the investigations in this paper, also unrealized, is that some one fact about $\mathcal{P}(\omega_1)/NS_{\omega_1}$ in the \mathbb{P}_{max} extension could be found from which the solutions to all natural questions about it could be easily derived. On the other hand, this may be too much to hope for.

As always with Martin's Maximum and \mathbb{P}_{max} , it would be interesting to know which of the results presented here carry over to the context of Martin's Maximum.

Notational Remarks: Given $A \subset \omega_1$, we let $[A]$ denote the corresponding member of $\mathcal{P}(\omega_1)/NS_{\omega_1}$, i.e., $\{B \subset \omega_1 \mid A \triangle B \in NS_{\omega_1}\}$. For a real x and an ordinal γ , we let $li(x, \gamma)$ denote the least Silver indiscernible of x above γ .

2 \mathbb{P}_{max}

The partial order \mathbb{P}_{max} was introduced in [24]. As much as possible, we will rely on that book and [13] for standard \mathbb{P}_{max} details (in particular for the definitions of *iterable* and *iteration* below). The theory ZFC* below is a weak fragment of ZFC. We will work with models of ZFC in this paper.

2.1 Definition. The partial order \mathbb{P}_{max} consists of all pairs $\langle (M, I), a \rangle$ such that

1. M is a countable transitive model of $ZFC^* + MA_{\aleph_1}$,
2. $I \in M$ and in M , I is a normal uniform ideal on ω_1 ,
3. (M, I) is iterable,
4. $a \in \mathcal{P}(\omega_1)^M$,
5. there exists an $x \in \mathcal{P}(\omega)^M$ such that $\omega_1^M = \omega_1^{L[a, x]}$.

The order on \mathbb{P}_{max} is as follows: $\langle (M, I), a \rangle < \langle (N, J), b \rangle$ if $N \in H(\omega_1)^M$ and there exists an iteration $j: (N, J) \rightarrow (N^*, J^*)$ such that

- $j(b) = a$,
- $j, N^* \in M$,
- $I \cap N^* = J^*$.

By a \mathbb{P}_{max} *pre-condition* we mean a pair (M, I) such that $\langle (M, I), a \rangle \in \mathbb{P}_{max}$ for some a .

The requirement that M satisfies MA_{\aleph_1} gives that whenever $\langle (M, I), a \rangle$ is a \mathbb{P}_{max} condition and b is a set of ordinals, there is at most one iteration of (M, I) sending a to b . It follows that if $G \subset \mathbb{P}_{max}$ is a filter, there is for each $\langle (M, I), a \rangle$ in G a unique iteration of (M, I) sending a to

$$A_G = \bigcup \{b \mid \langle (N, J), b \rangle \in G\}.$$

This induces one bit of standard but slightly ambiguous \mathbb{P}_{max} terminology: if $\langle (M, I), a \rangle$ is a \mathbb{P}_{max} condition in a filter G and $d \in \mathcal{P}(\omega_1)^M$, then we let d^* denote the image of d under this unique iteration of (M, I) .

Our base theory for working with \mathbb{P}_{max} is $ZF + DC_{\mathbb{R}} + AD^+$. The axiom AD^+ is an ostensibly strong form of determinacy and $DC_{\mathbb{R}}$ is “dependent choice for reals”; we refer the reader to [24, 13] for the definitions. In any generic extension of such a model by \mathbb{P}_{max} there exists a wellordering of $H(\omega_2)$, but in general Choice does not need to hold. Since the questions we are considering are all at the level of the Π_2 theory of $\mathcal{P}(\omega_2)$ or below, the (possible) lack of full Choice does not come up in this paper.

We note here some of the basic results about \mathbb{P}_{max} in the context of AD^+ (all due to Woodin, and appearing in [24, 13]). If A is a set of reals, we say that a precondition (M, I) is *A-iterable* if $A \cap M \in M$ and $j(A \cap M) = A \cap M^*$ for every iteration $j: (M, I) \rightarrow (M^*, I^*)$.

Theorem 2.2. *(ZF + DC $_{\mathbb{R}}$ + AD $^+$) If A is a set of reals, then for every real x and every $n \in \omega$ there exists a countable transitive model M of ZFC + “there exist n Woodin cardinals” such that $x \in M$ and*

$$\langle H(\omega_1)^M, A \cap M, \in \rangle \prec \langle H(\omega_1), A, \in \rangle,$$

and, whenever M' is a forcing extension of M satisfying MA_{\aleph_1} and I' is an ideal on $\omega_1^{M'}$ which is precipitous and normal in M' then (M', I') is an A -iterable \mathbb{P}_{max} precondition.

Theorem 2.3. *Suppose that W is a model of $ZF + DC_{\mathbb{R}} + AD^+$, and let $G \subset \mathbb{P}_{max}$ be an W -generic filter. Then $W[G]$ satisfies $ZF +$ “there exists a wellordering of $H(\omega_2)$ ” + “ NS_{ω_1} is saturated.” Furthermore, if every set in W is a surjective image in W of $\mathbb{R} \times \gamma$ for some ordinal γ , then $W[G]$ satisfies ZFC.*

Also, for every $D \in \mathcal{P}(\omega_1) \cap W[G]$ there exist a condition $\langle (M, I), a \rangle$ in G and a $d \in \mathcal{P}(\omega_1)^M$ such that $d^ = D$. Finally, for every $\langle (M, I), a \rangle$ in G and $d \in \mathcal{P}(\omega_1)^M$, $d^* \in NS_{\omega_1}$ if and only if $d \in I$.*

The following theorem states that \mathbb{P}_{max} is homogeneous, which implies that the theory of its extension can be computed in the ground model. See [9] for a discussion of daggers.

Theorem 2.4. *Suppose that the dagger of each real exists, and let $\langle (M_0, I_0), a_0 \rangle$ and $\langle (M_1, I_1), a_1 \rangle$ be \mathbb{P}_{max} conditions. Then there exist N, J, b_0 and b_1 such that $\langle (N, J), b_0 \rangle$ is a \mathbb{P}_{max} condition below $\langle (M_0, I_0), a_0 \rangle$ and $\langle (N, J), b_1 \rangle$ is a \mathbb{P}_{max} condition below $\langle (M_1, I_1), a_1 \rangle$. Furthermore, whenever N, J, b_0 and b_1 are such that $\langle (N, J), b_0 \rangle$ and $\langle (N, J), b_1 \rangle$ are \mathbb{P}_{max} conditions, the suborders of \mathbb{P}_{max} below them are isomorphic.*

The \mathbb{P}_{max} extension has many more striking properties, including its maximality, which we will not try to summarize here. We note the following useful fact about iterations which is key to the \mathbb{P}_{max} analysis.

Theorem 2.5. *If (M, I) is an iterable pair and x is a real coding (M, I) , then for any iteration of $j: (M, I) \rightarrow (M^*, I^*)$ of length α , $Ord^M < li(x, \alpha)$.*

It follows that the indiscernibles of any such x are on the critical sequence of any sufficiently long iteration of (M, I) .

3 Nondefinability of the quotient $\mathcal{P}(\omega_1)/NS_{\omega_1}$

Here we show that in the \mathbb{P}_{max} extension the Boolean algebra $\mathcal{P}(\omega_1)/NS_{\omega_1}$ does not have a dense set order-isomorphic to a set in the ground model. We let \leq_N be the partial order on $\mathcal{P}(\omega_1)/NS_{\omega_1}$ defined by setting $[S] \leq_N [T]$ if $S \setminus T$ is nonstationary.

Theorem 3.1. *Let W be a model of $ZF + DC_{\mathbb{R}} + AD^+$, and let $G \subset \mathbb{P}_{max}$ be W -generic. Then in $W[G]$, there is no isomorphism between a partial order in W and a dense subset of the partial order $(\mathcal{P}(\omega_1)/NS_{\omega_1}, \leq_N)$.*

Proof. Let (X, \leq_X) be a partial order in W , and suppose towards a contradiction that τ is a \mathbb{P}_{max} -name in W for an isomorphism between a dense subset of $(\mathcal{P}(\omega_1)/NS_{\omega_1}, \leq_N)$ and (X, \leq_X) . Fix p, d, x such that $p = \langle (M, I), a \rangle$ is a condition in \mathbb{P}_{max} , $\{d, \omega_1^M \setminus d\} \subset I^+$, $x \in X$ and $p \Vdash ([d^*], \check{x}) \in \tau$.

Let $q = \langle (N, J), b \rangle$ be any condition (strictly) below p , and let j be the iteration of (M, I) witnessing that $q \leq p$. Since d and $\omega_1^M \setminus d$ are both I -positive, there exists in N an iteration j' of (M, I) such that (letting $b' = j'(a)$) $q' = \langle (N, J), b' \rangle$ is also a condition below p , as witnessed by j' , and such that $j(d) \triangle j'(d) \notin J$ (in fact, one can just as easily have $j(d) \cap j'(d) \in J$; this follows from the freedom allowed by the basic iteration lemma for \mathbb{P}_{max} , see for instance, the game-theoretic formulation of the basic iteration lemma for preconditions in [13], using a strategy for $\mathcal{G}(\omega_1^N \setminus j(d))$). Then $q \Vdash ([j(d)^*], \check{x}) \in \tau$ and $q' \Vdash ([j'(d)^*], \check{x}) \in \tau$.

Now suppose that $G \subset \mathbb{P}_{max}$ is W -generic, with $q \in G$. Then there is a \mathbb{P}_{max} -generic filter $G' \in W[G]$ such that $W[G] = W[G']$ and $q' \in G'$ (by Theorem 2.4, for instance). Then τ_G and $\tau_{G'}$ are isomorphisms between dense sets D and D' of the Boolean algebra $(\mathcal{P}(\omega_1)/NS_{\omega_1}, \leq_N)^{W[G]}$ and the partial order (X, \leq_X) . Therefore, $\tau_{G'}^{-1} \circ \tau_G$ is an isomorphism between dense subsets of $(\mathcal{P}(\omega_1)/NS_{\omega_1}, \leq_N)$, which, since NS_{ω_1} is saturated in the \mathbb{P}_{max} extension, and so $\mathcal{P}(\omega_1)/NS_{\omega_1}$ is a complete Boolean algebra there, induces an automorphism. Furthermore, $j(d)^* \in D$, $j'(d)^* \in D'$, $\tau_G(j(d)^*) = x = \tau_{G'}(j'(d)^*)$ and $j(d)^* \triangle j'(d)^*$ is stationary, which means that this automorphism is nontrivial. But this contradicts the fact that MA_{\aleph_1} holds in $W[G]$, by Theorem 1.4. \square

The study of \mathbb{P}_{max} (more generally, of large cardinals and determinacy) tends to divide the low levels of the set theoretic hierarchy into the simply definable sets (which tend to be regular) and the sets given by the Axiom of Choice (which tend to be chaotic). Theorem 3.1 suggests that, considered as a partial order, the Boolean algebra $\mathcal{P}(\omega_1)/NS_{\omega_1}$ in the \mathbb{P}_{max} extension does not belong to the simply definable part. But belonging to the other half typically means serving as a parameter defining stationary-costationary subsets of ω_1 , ω_1 -sequences of reals and other AC-type objects. One could ask then whether the chaotic nature of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ (in the \mathbb{P}_{max} extension, considered as a partial order) enables some form of coding. The following is a test question in this direction.

3.2 Question. Suppose that P is a partial order in a \mathbb{P}_{max} extension which is isomorphic to the restriction of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ to some dense set. Does it follow that $L(P)$ contains a stationary, co-stationary subset of ω_1 ?

Theorem 3.3 suggests that one might be able to prove Theorem 3.1 locally, by finding a suborder of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ of smaller cardinality which by itself shows that the entire order does not have a dense subset in the ground model. We note the following consequence of the results of [10] which points in the opposite direction (and which may very well have been known long before): under MA_{\aleph_1} every partial order on \mathbb{R} of cardinality \aleph_1 is the restriction to its domain of a Σ_1^1 partial order (very briefly, the main theorem of [10] applies to (possibly illfounded) iterations satisfying the (first order) property that every real is represented by an injective function; since each real appears in a unique shortest iteration (if any), the union of the images of the partial order under all such iterations of the transitive collapse of any suitable countable elementary submodel of a rank initial segment of the universe is the desired Σ_1^1 set).

Theorem 3.3. *Suppose that W is a model of $ZF + DC_{\mathbb{R}} + AD^+$ and $G \subset \mathbb{P}_{max}^W$ is a W -generic filter. Then for every $A \in \mathcal{P}(\omega_2)^{W[G]}$, either $A \in W$ or there exists a $\gamma < \omega_2^{W[G]}$ such that $A \cap \gamma \notin W$.*

Our proof of Theorem 3.3 goes through the following combinatorial fact. Recall that the ground model of a \mathbb{P}_{max} extension contains the reals of the extension and satisfies AD^+ , so in particular it does not contain any stationary costationary subsets of ω_1 .

Lemma 3.4. *Suppose that κ is a measurable cardinal and let $\langle x_\alpha : \alpha < \omega_1 \rangle$ be a sequence of reals such that for all $\alpha < \beta < \omega_1$, $li(x_\alpha, \omega_1) < li(x_\beta, \omega_1)$. Let $\beta = \sup\{li(x_\alpha, \omega_1) : \alpha < \omega_1\}$. Define a function h on $\mathbb{R} \times \omega_1$ by letting $h(x, \alpha)$ be $i(\alpha)$ where i is the canonical bijection between ω_1 and $li(x, \omega_1)$ given by $x^\#$.*

Let f be a function with domain $2^{<\omega_1}$ such that

1. *each $f(p)$ is a bounded subset of β ,*
2. *$p \subset q$ implies $f(q)$ end-extends $f(p)$;*
3. *if neither of p and q extends the other, then neither of $f(p)$ and $f(q)$ extends the other.*

Let B be the subset of β induced by f and a V -generic $b \subset \omega_1$ added by initial segments. Then there exist $\gamma_0, \gamma_1 < \kappa$ such that the following sets are both stationary in $V[b]$:

- $\{z \in [\max\{\gamma_0, \gamma_1\}]^{\aleph_0} \mid h(x_{o.t.(z \cap \gamma_0)}, o.t.(z \cap \gamma_1)) \in B\},$
- $\{z \in [\max\{\gamma_0, \gamma_1\}]^{\aleph_0} \mid h(x_{o.t.(z \cap \gamma_0)}, o.t.(z \cap \gamma_1)) \notin B\}.$

Proof. By condition (3) on f , B is not in the ground model. Now suppose that some $p \in 2^{<\omega_1}$ forces the conclusion of the lemma to be false. Then for

each pair $\gamma_0, \gamma_1 < \kappa$ there are $2^{<\omega_1}$ -names for clubs $C_{\gamma_0, \gamma_1} \subset [\max\{\gamma_0, \gamma_1\}]^{\aleph_0}$ such that p forces that for each γ_0, γ_1 either $h(x_{o.t.(z \cap \gamma_0)}, o.t.(z \cap \gamma_1)) \in B$ for all $z \in C_{\gamma_0, \gamma_1}$ or $h(x_{o.t.(z \cap \gamma_0)}, o.t.(z \cap \gamma_1)) \notin B$ for all $z \in C_{\gamma_0, \gamma_1}$. Let X be a countable elementary submodel of $H((2^\kappa)^+)$ containing as elements $\kappa, p, \langle x_\alpha : \alpha < \omega_1 \rangle$ and the function taking each pair γ_0, γ_1 to the corresponding name for C_{γ_0, γ_1} , and let $b \subset X \cap \omega_1$ be an X -generic condition for the partial order $2^{<\omega_1}$. Now, using the fact that κ is measurable (see, for instance, [11]), let Y be an elementary submodel of $H((2^\kappa)^+)$ containing X such that $Y \cap \omega_1 = X \cap \omega_1$, $Y \cap 2^{2^\omega} = X \cap 2^{2^\omega}$ and $o.t.(Y \cap \kappa) = \omega_1$. Then b is Y -generic as well. Then for each $\gamma_0, \gamma_1 \in Y \cap \kappa$, some initial segment of b decides whether

$$h(x_{o.t.(z \cap \gamma_0)}, o.t.(z \cap \gamma_1)) \in B$$

for all $z \in C_{\gamma_0, \gamma_1}$ or $h(x_{o.t.(z \cap \gamma_0)}, o.t.(z \cap \gamma_1)) \notin B$ for all $z \in C_{\gamma_0, \gamma_1}$. Since for each $\gamma_0, \gamma_1 \in Y \cap \kappa$, b forces that $X \cap \max\{\gamma_0, \gamma_1\} \in C_{\gamma_0, \gamma_1}$, this means that b decides for each γ_0, γ_1 in $Y \cap \kappa$ whether $h(x_{o.t.(X \cap \gamma_0)}, o.t.(X \cap \gamma_1)) \in B$. But since

$$\beta = \{h(x_\alpha, \eta) : \alpha, \eta < \omega_1\},$$

this means that b decides all of B , giving a contradiction. \square

The form of the proof of Theorem 3.3 is standard. Recall that u_2 , the second uniform indiscernible, is equal to $\sup\{li(x, \omega_1) : x \in \mathbb{R}\}$ when the sharp of each real exists, as it does in our context.

Proof of Theorem 3.3. Let τ be a \mathbb{P}_{max} -name in the ground model for a subset of ω_2 not in the ground model. Let X be a set of reals coding the pairs (p, b) , where $p = \langle (M, I), a \rangle$ is a \mathbb{P}_{max} condition, $b \in \mathcal{P}(\gamma)$ for some $\gamma < \omega_2^M$ and p forces that $j(b)$ is an initial segment of the realization of τ , for j the unique iteration of (M, I) sending a to the generic set A_G . Applying Theorem 2.2, let p be a \mathbb{P}_{max} condition and let N be a countable transitive model of ZFC + CH such that

1. $p \in H(\omega_1)^N$;
2. $\langle H(\omega_1)^N, \in, X \cap N \rangle \prec \langle H(\omega_1), \in, X \rangle$;
3. there is a Woodin cardinal δ in N ;
4. if N' is any set generic extension of N by a partial order of cardinality less than δ in N , and NS_{ω_1} is precipitous in N' , then $(N', NS_{\omega_1}^{N'})$ is X -iterable.

Since CH holds in N we may find a sequence $\langle \beta_\alpha : \alpha < \omega_1^N \rangle$ in N with supremum u_2^N . We can also build in N a perfect $2^{<\omega_1^N}$ tree of \mathbb{P}_{max} conditions $p_s = \langle (M_s, I_s), a_s \rangle$ ($s \in 2^{<\omega_1^N}$) such that

- if s extends t , then $p_s \leq p_t$;

- if s has length α , then there exists a set b_s such that (p_s, b_s) is coded by a real in $X \cap N$ and there is a real $y_s \in M_s$ such that $li(y_s, \omega_1^N) > \beta_{dom(s)}$ and $\sup(b_s) > li(y_s, \omega_1^{M_s})$ (given p_t for all proper initial segments t of s , there exists such a b_s by condition (2) above, and the fact that $\beta_{dom(s)} < u_2^N$; more specifically, it follows from (2) that there is a real $y \in N$ such that $li(y, \omega_1^N) > \beta_{dom(s)}$ and a pair (p, b) coded by a real in $X \cap N$ such that p is below p_t for all proper initial segments t of s , $y \in M_p$ where $p = \langle (M_p, I_p), a_p \rangle$ and $\sup(b) > li(y, \omega_1^{M_p})$);
- incompatible s, s' have incompatible $b_s, b_{s'}$;
- each I_s -positive set is tagged to a $NS_{\omega_1^N}$ -positive set in such a way that for every element e of $(2^{\omega_1})^N$ extending s , every element of $j_s^e(I_s^+)$ is stationary in N , where j_s^e is the unique iteration of (M_s, I_s) sending a_s to $a^e = \bigcup_{\alpha < \omega_1^N} a_e \upharpoonright \alpha$ (and further this persists to forcing extensions of N preserving stationary subsets of ω_1^N) (this part of the construction is standard [24, 13]; one can for instance choose a family $\langle B_{s,i} : s \in 2^{<\omega_1^N}, i < \omega \rangle$ of pairwise disjoint stationary subsets of ω_1 in N , let $\langle E_{s,i} : i < \omega \rangle$ enumerate the I_s -positive sets in each M_s and construct the iterations $j_{s,s'}$ (witnessing $p_{s'} < p_s$ for each such pair) so that for each γ on the critical sequence of such an iteration, if $\gamma \in B_{s,i}$ for some $i < \omega$, then $\gamma \in j_s^e(E_{s,i})$).

Let b be an N -generic branch through 2^{ω_1} . Let $B = \bigcup \{j_b^b(b_b \upharpoonright \alpha) : \alpha < \omega_1^N\}$. Let κ_0 be the least measurable cardinal in N . Then by Lemma 3.4 there exist $\gamma_0, \gamma_1 < \kappa_0$ such that the following sets are both stationary in $N[b]$.

- $\{z \in [\max\{\gamma_0, \gamma_1\}]^{\kappa_0} \mid h(x_{o.t.(z \cap \gamma_0)}, o.t.(z \cap \gamma_1)) \in B\}$
- $\{z \in [\max\{\gamma_0, \gamma_1\}]^{\kappa_0} \mid h(x_{o.t.(z \cap \gamma_0)}, o.t.(z \cap \gamma_1)) \notin B\}$

Letting $N[b][c]$ be a $Coll(\omega_1, \max\{\gamma_0, \gamma_1\})^N$ -generic forcing extension of N , these sets are still both stationary in $N[b][c]$. Letting $N[b][c][d]$ be a $Coll(\omega_1, <\delta)^{N[b][c]}$ -generic forcing extension of $N[b][c]$ (since δ is Woodin in $N[b][c]$, NS_{ω_1} is pre-saturated in $N[b][c][d]$; see [5]), we have that

$$\langle (N[b][c][d], NS_{\omega_1}^{N[b][c][d]}), a^b \rangle$$

is a \mathbb{P}_{max} condition forcing that the image of B under the iteration of

$$(N[b][c][d], NS_{\omega_1}^{N[b][c][d]})$$

sending a^b to A_G is an initial segment of τ_G . Now let $\langle (N', J'), a' \rangle$ be a condition below $\langle (N[b][c][d], NS_{\omega_1}^{N[b][c][d]}), a^b \rangle$, as witnessed by an iteration j' . Then there is a real y in N' coding

$$\langle (N[b][c][d], NS_{\omega_1}^{N[b][c][d]}), a^b \rangle,$$

and in N' , $L[y^\#, j'(B)]$ contains a stationary, costationary subset of ω_1 , since $y^\#$ codes a bijection between $\omega_1^{N'}$ and $j'(\max\{\gamma_0, \gamma_1\})$ by Theorem 2.5. Since the ground model contains all the reals and contains no stationary, costationary subsets of ω_1 , then, it follows that $\langle(N', J'), a'\rangle$ forces that $j(j'(B))$ is not in the ground model, where j is the iteration of (N', J') sending a' to A_G . \square

We note the following combinatorial fact, proved by a simplified form of the argument for Theorem 3.3.

Theorem 3.5. *In the \mathbb{P}_{max} extension, every perfect countably closed subtree of $2^{<\omega_1}$ contains a branch such that if A is the corresponding subset of ω_1 then there is some real x such that $L[A, x]$ contains a stationary, costationary subset of ω_1 .*

Forcing with T and then $Coll(\omega_1, \gamma)$ as in the proof of Lemma 3.7 gives the following partial answer to a question of Woodin in [6], where $Coll^{+2}$ is the statement that for any γ , any collection D of \aleph_1 many dense subsets of $Coll(\omega_1, \gamma)$ and any two $Coll(\omega_1, \gamma)$ -names τ_0 and τ_1 for stationary subsets of ω_1 there is a filter $g \subset Coll(\omega_1, \gamma)$ such that g intersects each member of D and τ_{0g} and τ_{1g} are both stationary. Arguments similar to the ones in the proofs of Theorem 3.3 and Lemma 3.4 appear in [11, 14]. Similar issues are taken up in Section 8.

Theorem 3.6. *If $Coll^{+2}$ holds, there exists a measurable cardinal, $\delta_2^1 = \omega_2$ and the club filter on ω_1 is an ultrafilter in $L(\mathbb{R})$, then every countably closed perfect subtree of $2^{<\omega_1}$ has a branch not in $L(\mathbb{R})$.*

This can be proved by the following ostensibly weaker version of Lemma 3.4 (in which $\beta = \omega_1$ and h is the second-coordinate projection map).

Lemma 3.7. *Suppose that κ is a measurable cardinal and T is a countably-closed perfect subtree of $2^{<\omega_1}$. Let A be the subset of ω_1 induced by a V -generic branch b through T . Then there is a $\gamma < \kappa$ such that the following sets are both stationary in $V[b]$:*

- $\{x \in [\gamma]^{\aleph_0} \mid o.t.(x) \in A\};$
- $\{x \in [\gamma]^{\aleph_0} \mid o.t.(x) \notin A\}.$

As a sort of converse to the results presented here, we note the following, which uses the construction from Example 2.4.7 of [12].

Theorem 3.8. *Suppose that κ is a measurable cardinal, and A is a subset of ω_1 such that for each $\gamma < \kappa$, the set of $x \in [\gamma]^{\aleph_0}$ with $o.t.(x) \in A$ is either club or nonstationary. Then there is a countable transitive set M with $A \in L[M]$.*

Proof. Let X be a countable elementary submodel of $H((2^\kappa)^+)$ with $A \in X$, let μ be a normal fine measure on κ in X and let $\langle X_\alpha : \alpha \leq \omega_1 \rangle$ be a continuous increasing chain of elementary submodels of $H((2^\kappa)^+)$ such that $X_0 = X$ and each

$$X_{\alpha+1} = \{f(\eta_\alpha) : f : \kappa \rightarrow H((2^\kappa)^+) \wedge f \in X_\alpha\},$$

where $\eta_\alpha = \min \bigcap (X_\alpha \cap \mu)$. For each $\alpha \leq \omega_1$, let M_α be the transitive collapse of X_α . Let μ_X be the image of μ under the transitive collapse of X . Then $\langle M_\alpha : \alpha \leq \omega_1 \rangle$ is the iteration of M_0 by μ_0 of length $\omega_1 + 1$, so M_{ω_1} is in $L[M_0]$. For each $\beta < \omega_1$, let γ_β be the β th element of $X_{\omega_1} \cap \kappa$. Let $a = A \cap X$. Then, for each $\beta < \omega_1$,

$$M_{\omega_1} \models \text{“}\{x \in [\beta]^{\aleph_0} \mid o.t.(x) \in a\} \text{ is stationary”}$$

if and only if $\{x \in [\gamma_\beta]^{\aleph_0} \mid o.t.(x) \in A\}$ is stationary, which holds if and only if $o.t.(X_{\omega_1} \cap \gamma_\beta) \in A$. But $o.t.(X_{\omega_1} \cap \gamma_\beta) = \beta$, so this shows that A is in $L[M_0]$. \square

4 Axiom F^+ and caliber (\aleph_2, \aleph_0)

In our context, Nyikos’s axioms are equivalent to a simpler property of NS_{ω_1} . For a collection \mathcal{A} of \aleph_2 many stationary subsets of ω_1 , we let $\overline{\lim} \mathcal{A}$ be the least element of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ which is \leq_N -greater than all but \aleph_1 many members of \mathcal{A} , if such an element exists. Note that $\overline{\lim} \mathcal{A}$ always exists when $\mathcal{P}(\omega_1)/NS_{\omega_1}$ is a complete Boolean algebra, which it is when NS_{ω_1} is saturated. Recall that we are working in a context where $2^{\aleph_1} = \aleph_2$; when $2^{\aleph_1} > \aleph_2$ other definitions may be more useful.

Theorem 4.1. *Suppose that NS_{ω_1} is saturated. Then Axiom F^+ holds if and only if the restriction of NS_{ω_1} to some stationary set has caliber (\aleph_2, \aleph_0) .*

Proof. If the restriction of NS_{ω_1} to some stationary set has caliber (\aleph_2, \aleph_0) , then Axiom F^+ clearly holds.

Now, if the restriction of NS_{ω_1} to no stationary set has caliber (\aleph_2, \aleph_0) , then there is a collection \mathcal{A} of \aleph_2 -many stationary subsets of ω_1 with the following properties:

- no countable subset of \mathcal{A} has stationary intersection,
- $\overline{\lim} \mathcal{A} = [\omega_1]$.

To see this, let \mathcal{A}_0 be the set of A for which there exists a $\mathcal{B}_A \subset \mathcal{P}(A)$ with no infinite stationary intersections and $\overline{\lim} \mathcal{B}_A = A$. Then \mathcal{A}_0 is predense. Let the sequence $\langle A_\alpha : \alpha < \omega_1 \rangle$ enumerate a predense subset of \mathcal{A}_0 of cardinality \aleph_1 such that each $A_\alpha \setminus \bigcup_{\beta < \alpha} A_\beta$ is stationary. Now let \mathcal{A} be the set of all stationary sets of the form $B \setminus \bigcup_{\beta < \alpha} A_\beta$, for some $\alpha < \omega_1$ and $B \in \mathcal{B}_{A_\alpha}$.

Shrinking the members of \mathcal{A} (modulo NS_{ω_1}) if necessary and using the fact that NS_{ω_1} is saturated, we may write \mathcal{A} as $\{A_\beta^\alpha : \alpha < \omega_2, \beta < \omega_1\}$ such that for each $\alpha < \omega_2$, $\{A_\beta^\alpha : \beta < \omega_1\}$ is a disjoint collection with diagonal union containing a club C_α .

For each $\gamma < \omega_1$, let $\pi_\gamma : \gamma \rightarrow \omega$ be a bijection, and for each $\beta < \omega_1$ and $i < \omega$, let

$$B_i^\beta = \{\gamma \in \omega_1 \setminus (\beta + 1) \mid \pi_\gamma(\beta) = i\}.$$

Note that $B_i^\beta \cap B_i^{\beta'} = \emptyset$ whenever $\beta < \beta' < \omega_1$ and $i < \omega$. For each $\alpha < \omega_2$ and $i < \omega$, let

$$E_i^\alpha = \nabla \{A_\beta^\alpha \cap B_i^\beta : \beta < \omega_1\}.$$

Then for each $\alpha < \omega_2$ and each $\gamma \in C_\alpha$ there exist a $\beta < \gamma$ such that $\gamma \in A_\beta^\alpha$ and an $i \in \omega$ (i.e., $\pi_\gamma(\beta)$) such that $\gamma \in B_i^\beta$ (so $\gamma \in E_i^\alpha$). Note that Axiom F^+ is equivalent to the version where for each $\alpha < \omega_2$ there are only club many $\beta < \omega_1$ such that $F(\alpha, \beta)$ is defined. We now see that $\{E_i^\alpha : (\alpha, i) \in \omega_2 \times \omega\}$ gives a counterexample to this version via the function F defined on those $(\alpha, \beta) \in \omega_2 \times \omega_1$ such that $\beta \in C_\alpha$ by the formula

$$F(\alpha, \beta) = i \Leftrightarrow \beta \in E_i^\alpha.$$

Suppose towards a contradiction that $i < \omega$ and $a \in [\omega_2]^\omega$ are such that $\bigcap_{\alpha \in a} E_i^\alpha$ is stationary. If γ is in E_i^α , then there exists a $\beta < \gamma$ such that $\gamma \in A_\beta^\alpha \cap B_i^\beta$, and since the B_i^β 's are pairwise disjoint (for fixed i) this β is unique. Furthermore, if γ is in $E_i^{\alpha_0}$ and $E_i^{\alpha_1}$, then if $\beta_0, \beta_1 < \gamma$ are such that

$$\gamma \in A_{\beta_0}^{\alpha_0} \cap B_i^{\beta_0} \cap A_{\beta_1}^{\alpha_1} \cap B_i^{\beta_1},$$

then $\beta_0 = \beta_1$. It follows then that for every $\gamma \in \bigcap_{\alpha \in a} E_i^\alpha$ there is a fixed $\beta_\gamma < \gamma$ such that $\gamma \in A_{\beta_\gamma}^\alpha$ for all $\alpha \in a$, and so by pressing down we have a $\beta < \omega_1$ such that $\bigcap_{\alpha \in a} A_\beta^\alpha$ is stationary, giving a contradiction. \square

We take this opportunity to record a well-known fact about the \mathbb{P}_{max} extension which is related to the questions we consider here. The conclusion of the following theorem can also be shown to follow from Martin's Maximum (let $\{B_\alpha : \alpha < \omega_2\}$ enumerate $[NS_{\omega_1}]^+$ and let $\{A_\alpha : \alpha < \omega_2\}$ be elements of \mathcal{A} such that for all $\alpha < \beta < \omega_2$, $B_\alpha \setminus A_\beta \in NS_{\omega_1}^+$; consider Namba forcing followed by shooting a club the complement of $\bigcap_{\beta \in b} A_\beta$, where b is the cofinal ω -sequence given by Namba forcing).

Theorem 4.2. *Let W be a model of $ZF + DC_{\mathbb{R}} + AD^+$ and let $G \subset \mathbb{P}_{max}^W$ be a W -generic filter. Then the following holds in $W[G]$: suppose that \mathcal{A} is a collection of \aleph_2 -many stationary subsets of ω_1 with the property that every countable subset of \mathcal{A} has stationary intersection; then there exists $\mathcal{B} \in [NS_{\omega_1}^+]^{\aleph_1}$ such that for every $A \in \mathcal{A}$ there is a $B \in \mathcal{B}$ such that $B \setminus A \in NS_{\omega_1}$.*

Proof. We work in W . Let τ be a \mathbb{P}_{max} -name for an \aleph_2 -sized set of stationary subsets of ω_1 for which no such \mathcal{B} exists as in the conclusion of the theorem. Let X be the set of reals coding pairs (p, b) where $p = \langle \langle M, I \rangle, a \rangle$ is in \mathbb{P}_{max} , $b \in \mathcal{P}(\omega_1)^M$ and $p \Vdash b^* \in \tau$. Let p be a condition in \mathbb{P}_{max} and let (N, J) be an X -iterable \mathbb{P}_{max} pre-condition with $p \in N$ such that

$$\langle H(\omega_1)^N, X \cap N, \in \rangle \prec \langle H(\omega_1), X, \in \rangle.$$

Working in N , construct a descending sequence of \mathbb{P}_{max} conditions $p_i = \langle \langle M_i, I_i \rangle, a_i \rangle$ ($i \in \omega$) with $p_0 = p$ and a collection of sets b_i , ($i \in \omega \setminus \{0\}$) such that for each

nonzero i , $(p_i, b_i) \in X$, and such that p_i forces that b_i^* does not contain modulo NS_{ω_1} any e^* , $e \in \mathcal{P}(\omega_1)^{M_{i-1}}$. Let $\langle (M'_i, I'_i) : i < \omega \rangle, a' \rangle$ be the limit sequence derived from the p_i 's, and for each nonzero i let b'_i be the corresponding image of b_i . Still working in N , construct an iteration j of $\langle M'_i : i < \omega \rangle$ such that $\langle (N, J), j(a') \rangle$ is below each p_i , in the standard way ([24]; in particular we collaborate with a winning strategy for player I the game $\mathcal{G}(\omega_1^N)$ for limit sequences in [13]). This iteration consists of sequences $\langle M'_{\alpha, i} : i < \omega \rangle$ and filters G_α for each $\alpha < \omega_1^N$. For each $\alpha < \omega_1^N$, we let the strategy for player I pick first some $e \in (NS_{\omega_1}^+)^{M'_{i, \alpha}}$, for some $i < \omega$, to be in G_α . Then, letting $b'_{i+1, \alpha}$ be the image of b_{i+1} in $M'_{i, \alpha}$, we have that $e \setminus b'_{i+1, \alpha} \in (NS_{\omega_1}^+)^{M'_{i+1, \alpha}}$, so we can (and do) put $e \setminus b'_{i+1, \alpha} \in (NS_{\omega_1}^+)^{M'_{i+1, \alpha}}$ in G_α . The rest of G_α can be chosen in any fashion.

Having completed the construction, the critical sequence of j is a club in N disjoint from $\bigcap_{i \in \omega} j(b'_i)$, and $\langle (N, J), j(a') \rangle$ forces that $\bigcap_{i \in \omega} b_i^* \in NS_{\omega_1}$. \square

5 A saturation property

In this section we derive a saturation property of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ in the \mathbb{P}_{max} extension. One form of this property is: given an \aleph_2 -sized family \mathcal{A} consisting of stationary subsets of ω_1 , either some subcollection of \mathcal{A} of cardinality \aleph_1 has stationary diagonal intersection, or there is a \aleph_1 -sized collection \mathcal{B} of stationary sets such that every member of \mathcal{A} has nonstationary intersection with some member of \mathcal{B} . By way of comparison, we note that Nyikos [16] has shown that MA_{\aleph_1} , which holds in the \mathbb{P}_{max} extension, implies that for any infinite collection of stationary sets, either some infinite subcollection has infinite intersection, or infinitely many all have nonstationary intersection with some fixed stationary set.

5.1 Definition. We say that a set \mathcal{A} of stationary subsets of ω_1 satisfies $(**)$ if there exists some stationary $B \subset \omega_1$ such that for every $\mathcal{E} \subset \mathcal{P}(B) \setminus NS_{\omega_1}$ of cardinality \aleph_1 the set of $A \in \mathcal{A}$ such that $A \cap E$ is stationary for every $E \in \mathcal{E}$ has cardinality \aleph_2 .

We will show that in the \mathbb{P}_{max} extension every collection of stationary sets satisfying $(**)$ has a subcollection of size \aleph_1 with stationary diagonal intersection. Finding a collection in the \mathbb{P}_{max} extension which fails to satisfy $(**)$ appears to require a little work, which we do in the next section.

The use of property $(**)$ is given in the following lemma.

Lemma 5.2. *Suppose that N_i, J_i, f_i ($i < \omega$) and b are such that*

- *each N_i is a countable transitive model of ZFC;*
- *each J_i is a normal ideal on $\omega_1^{N_i}$ in N_i ;*
- *each $\omega_1^{N_i} = \omega_1^{N_0}$;*
- *each $J_i = J_{i+1} \cap N_i$;*

- $b \in \mathcal{P}(\omega_1)^{N_0} \setminus J_0$;
- $f_0 \cap b \in \mathcal{P}(\omega_1)^{N_0} \setminus J_0$;
- each $f_i \in \mathcal{P}(\omega_1)^{N_i} \setminus J_i$;
- for each $i < \omega$, each $z \in \mathcal{P}(b)^{N_i} \setminus J_i$, and each $i' > i$, $f_{i'} \cap z \in J_{i'}^+$.

Then there exists a filter $G \subset \bigcup_{i < \omega} \mathcal{P}(\omega_1)^{N_i} \setminus \bigcup_{i < \omega} J_i$ such that $b \in G$, each f_i is in G , and every function in some N_i which is regressive on a member of G is constant on a member of G .

Proof. To construct G , let h_i ($i < \omega$) enumerate the regressive functions on $\omega_1^{N_0}$ in $\bigcup_{i < \omega} N_i$, in such a way that each $h_i \in N_i$. We will choose sets x_i to generate G , with each $x_i \in \mathcal{P}(f_i)^{N_i} \setminus J_i$ and h_i constant on x_i . For the first stage of the construction, let x_0 be any subset of $f_0 \cap b$ in J_0^+ on which h_0 is constant. Having chosen $x_i \in \mathcal{P}(b)^{N_i} \setminus J_i$, $f_{i+1} \cap x_i$ is in J_{i+1}^+ , so we may let x_{i+1} be any J_{i+1} -positive subset of $f_{i+1} \cap x_i$ in N_{i+1} on which h_{i+1} is constant. \square

Many of the ingredients of the following proof are standard.

Theorem 5.3. *Suppose that W is a model of $ZF + DC_{\mathbb{R}} + AD^+$, and let $G \subset \mathbb{P}_{max}^W$ be W -generic. Then, in $W[G]$, if \mathcal{A} is a family of stationary subsets of ω_1 satisfying (**), then there is a $\mathcal{D} \in [\mathcal{A}]^{\aleph_1}$ such that $\triangle \mathcal{D}$ is stationary.*

Proof. Let τ be a \mathbb{P}_{max} name in W for a collection of subsets of ω_1 of cardinality \aleph_2 . Fix a \mathbb{P}_{max} condition $p = \langle (M, I), a \rangle$ and a set $b \in \mathcal{P}(\omega_1)^M \setminus I$ such that p forces that b^* will have the property stated for B in the statement of (**) with respect to the set given by τ .

Let X be a set of reals in W coding the set of pairs (q, d) such that q is in \mathbb{P}_{max} , $q \leq p$, $q \Vdash d^* \in \tau$, where $q = \langle (M_q, I_q), a_q \rangle$ and $d \in \mathcal{P}(\omega_1)^{M_q}$. Let (N, J) be an X -iterable \mathbb{P}_{max} pre-condition with $p \in H(\omega_1)^N$ and

$$\langle H(\omega_1), X \cap N, \in \rangle \prec \langle H(\omega_1), X, \in \rangle.$$

Working in N , we will construct a decreasing sequence $\langle p_\alpha : \alpha < \omega_1 \rangle$ of \mathbb{P}_{max} conditions below p . We will let $j_{\alpha\beta}$ be the embeddings witnessing the order on this sequence (allowing $\beta = \omega_1$ for embeddings generated by composition), and we will let $\langle (M_\alpha, I_\alpha), a_\alpha \rangle$ denote the α th member of this sequence, with $p = p_0$.

As we do this, we will choose sets e_α ($\alpha \in S$, letting S denote the set of successor ordinals below ω_1^N) such that

- $e_\alpha \in \mathcal{P}(\omega_1)^{M_\alpha}$;
- $e_\alpha \cap j_{0\alpha}(b) \in I_\alpha^+$;
- $p_\alpha \Vdash e_\alpha^* \in \tau$.

We will also at each stage $\alpha < \omega_1^N$ choose a set $f_\alpha \in \mathcal{P}(\omega_1)^{M_\alpha} \setminus I_\alpha$ which will be contained modulo I_α in $j_{0\alpha}(b)$, each $j_{\beta\alpha}(e_\beta)$, ($\beta \in (\alpha+1) \cap S$) and each $j_{\beta\alpha}(f_\beta)$, ($\beta < \alpha$). Furthermore, each f_α will have intersection in I_α^+ with each member of $j_{\beta\alpha}(\mathcal{P}(j_{0\beta}(b))^{M_\beta} \setminus I_\beta)$ for each $\beta < \alpha$. Lastly, each $f_{\alpha+1}$ will be equal to $e_{\alpha+1} \cap j_{\alpha(\alpha+1)}(f_\alpha)$.

We will let \bar{a} denote $\bigcup_{\alpha < \omega_1^N} j_{0\alpha}(a)$. Let $f_0 = b$.

To ensure that $\langle (N, J), \bar{a} \rangle$ is below each p_α , we fix in N a partition of ω_1^N into J -positive sets A_α ($\alpha < \omega_1^N$), and associate to each member of each $\mathcal{P}(\omega_1)^{M_\alpha} \setminus I_\alpha$ some $A_{\alpha'}$, for $\alpha' \neq 0$, which its image will contain on a club. (Again, this is standard, see the discussion of the game \mathcal{G}_{ω_1} in [13]; the construction here can be seen as a run of $\mathcal{G}_{\omega_1}((\omega_1 \setminus A_0) \cup S)$; more details are given below).

Finally, we will construct this sequence in such a way that the diagonal intersection of the sets $j_{\alpha\omega_1^N}(f_\alpha)$ ($\alpha < \omega_1^N$) will contain A_0 modulo a club in N .

For the successor stages of the construction, suppose that we have p_β and f_β for all $\beta \leq \alpha$ and e_β for all successor $\beta \leq \alpha$, satisfying the conditions above. We have that p forces τ to satisfy (**). In the \mathbb{P}_{max} extension, the final model of the iteration of (M_α, I_α) sending a_α to A_G will have cardinality \aleph_1 , so there will be in the extension an $A \in \tau_G$ having stationary intersection with every stationary subset of b^* in this final model. Then, applying the fact that $\langle H(\omega_1), X \cap N, \in \rangle \prec \langle H(\omega_1), X, \in \rangle$, choose $p_{\alpha+1} = \langle (M_{\alpha+1}, I_{\alpha+1}), a_{\alpha+1} \rangle$ and $e_{\alpha+1} \in \mathcal{P}(\omega_1)^{M_{\alpha+1}}$ so that $p_{\alpha+1} \leq p_\alpha$ and $p_{\alpha+1}$ forces that $e_{\alpha+1}^*$ will have these properties of A . Then $(p_{\alpha+1}, e_{\alpha+1})$ is in X , and $f_{\alpha+1} = e_{\alpha+1} \cap j_{\alpha(\alpha+1)}(f_\alpha)$ will have intersection in $I_{\alpha+1}^+$ with every member of $j_{\alpha(\alpha+1)}(\mathcal{P}(f_\alpha) \setminus I_\alpha)$, which since f_α has intersection in I_α^+ with each member of $j_{\beta\alpha}(\mathcal{P}(j_{0\beta}(b))^{M_\beta} \setminus I_\beta)$ for each $\beta < \alpha$, means that $f_{\alpha+1}$ will have intersection in $I_{\alpha+1}^+$ with each member of $j_{\beta\alpha+1}(\mathcal{P}(j_{0\beta}(b))^{M_\beta} \setminus I_\beta)$ for each $\beta < \alpha+1$. That $f_{\alpha+1}$ is contained modulo $I_{\alpha+1}$ in every $j_{\beta(\alpha+1)}(e_\beta)$ ($\beta \in (\alpha+1) \cap S$) follows similarly from the fact that f_α satisfies this condition with respect to α .

At a limit stage α choose a choose an increasing cofinal sequence η_i ($i < \omega$) below α . Let $\langle (N_i, J_i) : i < \omega \rangle, \bar{b} \rangle$ be the limit sequence (this term was introduced in [13]) corresponding to the system $\{p_{\eta_i}, j_{\eta_i\eta_j} : i < j < \omega\}$ and let j_i^* be the corresponding embedding of M_{η_i} into N_i . The key point is that each $j_i^*(f_{\eta_i})$ has intersection in J_i^+ with each member of $\mathcal{P}(j_0^*(j_{0\eta_0}(b)))^{N_{i'}} \setminus J_{i'}$ for each $i' < i$, and this property is carried forward by iterations of $\langle (N_i, J_i) : i < \omega \rangle$. So then, still working in N , we can choose a pre-condition (M_α, I_α) with $\langle (N_i, J_i) : i < \omega \rangle$ and $\langle j_i^*(f_{\eta_i}) : i < \omega \rangle$ in M_α . Now, working in M_α , construct an iteration j^+ of $\langle (N_i, J_i) : i < \omega \rangle$ of length $\omega_1^{M_\alpha}$ such that

1. for each $i < \omega$, $I_\alpha \cap j^+(\mathcal{P}(\omega_1)^{N_i}) = j^+(J_i)$;
2. if there is some z in some $\mathcal{P}(\omega_1)^{M_\beta} \setminus I_\beta$, for some $\beta < \alpha$ such that z is associated to some A_ξ and $\omega_1^{N_0} \in A_\xi$, then, letting i' be least such that $\beta \leq \eta_{i'}$, $j_{i'}^*(j_{\beta\eta_{i'}}(z))$ is in the first filter of this iteration;
3. if $\omega_1^{N_0} \in A_0$, then each $j_i^*(f_{\eta_i})$ is in the first filter of this iteration;
4. $\bigcap_{i < \omega} j^+(j_i^*(f_{\eta_i})) \in I_\alpha^+$.

The first two parts of this are standard. That we can meet the second two follow from Lemma 5.2, and the properties given above for the sets $j_i^*(f_{\alpha_i})$. Condition (1) here ensures that, letting $a_\alpha = j^+(b)$, $p_\alpha = \langle (M_\alpha, I_\alpha), a_\alpha \rangle$ is below each p_β ($\beta < \alpha$).

Let ξ_α denote the ordinal $\omega_1^{N_0}$ at stage α .

Having constructed the sequence $\langle p_\alpha : \alpha < \omega_1^N \rangle$, the set

$$C = \{\xi_\alpha : \alpha < \omega_1 \text{ limit}\}$$

is a club. Let j be the limit of the embeddings $j_{0\alpha}$. Condition (2) above ensures that $\langle (N, J), j(a) \rangle$ is a condition below each p_α . Condition (3) ensures that the diagonal intersection of the images of the f_α 's, and thus the diagonal intersection of the images of the e_α 's contains $A_0 \cap C$, and thus is in J^+ .

Now suppose that $\langle (N, J), j(a) \rangle$ is in a W -generic filter $G \subset \mathbb{P}_{max}$. Then there is a unique iteration $k: (N, J) \rightarrow (N^*, J^*)$ sending $j(a)$ to A_G . It is a standard fact here (also in [13]), and not hard to check, that each member of $k(\langle p_\alpha : \alpha < \omega_1^N \rangle)$ is in G . Furthermore, since (N, J) is X -iterable, each member of the sequence $k(\langle e_\alpha : \alpha \in S \rangle)$ is in τ_G . Since $\langle e_\alpha : \alpha \in S \rangle$ has stationary diagonal intersection in N , $k(\langle e_\alpha : \alpha \in S \rangle)$ has stationary diagonal intersection in N^* , and thus in the extension.

Thus $\langle (N, J), j(a) \rangle$ forces that the conclusion of the theorem will hold with respect to the realization of τ in the extension. \square

The following is a weak version of Theorem 5.3.

Corollary 5.4. *If W is a model of $ZF + DC_{\mathbb{R}} + AD^+$ and $G \subset \mathbb{P}_{max}^W$ is a W -generic filter, then, in $W[G]$, for every collection \mathcal{A} of stationary subsets of ω_1 of cardinality \aleph_2 , either \mathcal{A} has a subcollection of cardinality \aleph_1 whose diagonal intersection is stationary, or there exist stationary subsets of ω_1 , E_α ($\alpha < \omega_1$), such that each costationary member of \mathcal{A} has nonstationary intersection with some E_α .*

6 Cofinitely new functions on ω_1

Theorem 5.3 raises the question of whether any ω_2 -sequence of stationary sets in the \mathbb{P}_{max} extension fails to satisfy (**). To show that there are such sequences failing (**), we show that, in the \mathbb{P}_{max} extension, given any collection of \aleph_1 -many partitions \mathcal{E} of ω_1 into \aleph_1 -many pieces there is a stationary set $A \subset \omega_1$ having nonstationary intersection with at least one member of each \mathcal{E}_α . From this one can easily build such a sequence by letting $\langle \mathcal{E}_\alpha : \alpha < \omega_2 \rangle$ enumerate the partitions of ω_1 into \aleph_1 -many pieces, letting A_β (for each $\beta < \omega_2$) be as above with respect to $\{\mathcal{E}_\alpha : \alpha < \beta\}$ and letting \mathcal{A} be the set $\{A_\beta : \beta < \omega_2\}$.

It turns out that the existence of such an A follows from the existence of a partial order preserving stationary subsets of ω_1 which adds a function from ω_1 to ω_1 which intersects each such ground model function in only finitely many places. It has been known for some time that there are proper forcings that do

this. The argument here is essentially a subroutine of arguments in [20]. Similar arguments appear in [25].

Let P be the partial order consisting of triples $(a, \mathcal{F}, \mathcal{X})$, where

- a is a finite partial function from ω_1 to ω_1 ,
- \mathcal{F} is a finite set of functions from ω_1 to ω_1 ,
- \mathcal{X} is a finite \in -chain of countable elementary submodels of $H(\omega_2)$,

with the properties that

- for each $X \in \mathcal{X}$ and each $\alpha \in \text{dom}(a) \cap X$, $a(\alpha) \in X$,
- for each $X \in \mathcal{X}$ and each $\alpha \in \text{dom}(a) \setminus X$, the pair $(\alpha, a(\alpha))$ is not in any member of $X \cap \omega_1^{\omega_1}$.

The order on the conditions is: $(a, \mathcal{F}, \mathcal{X}) \leq (b, \mathcal{G}, \mathcal{Y})$ if $b \subset a$, $\mathcal{G} \subset \mathcal{F}$, $\mathcal{Y} \subset \mathcal{X}$ and, for all $\alpha \in \text{dom}(a) \setminus \text{dom}(b)$ and for all $g \in \mathcal{G}$, $a(\alpha) \neq g(\alpha)$.

Forcing with P adds a function from ω_1 to ω_1 which disagrees cofinitely with each such function in the ground model. We claim that P is proper (see [17, 1] for background on proper forcing). The key point is the following.

Lemma 6.1. *Let $(a, \mathcal{F}, \mathcal{X})$ be a condition in P , and let D be a dense subset of P below $(a, \mathcal{F}, \mathcal{X})$. Then there exists an $\alpha < \omega_1$ such that for every finite set of functions \mathcal{F}' from ω_1 to ω_1 there is a $(b, \mathcal{G}, \mathcal{Y})$ in D below $(a, \mathcal{F}, \mathcal{X})$ with $\sup(\text{dom}(b)) < \alpha$ such that for every $\beta \in \text{dom}(b) \setminus \text{dom}(a)$ and $f \in \mathcal{F}'$, $b(\beta) \neq f(\beta)$.*

Proof. Fix $(a, \mathcal{F}, \mathcal{X})$ and D as in the statement of the lemma. Towards a contradiction, let $\langle \mathcal{F}_\alpha : \alpha < \omega_1 \rangle$ be a sequence of finite sets of functions from ω_1 to ω_1 with the property that for every $\alpha < \omega_1$ and every $(b, \mathcal{G}, \mathcal{Y})$ in D below $(a, \mathcal{F}, \mathcal{X})$ with $\sup(\text{dom}(b)) < \alpha$ there exist a $\beta \in \text{dom}(b) \setminus \text{dom}(a)$ and an $f \in \mathcal{F}_\alpha$ such that $b(\beta) = f(\beta)$. We may assume that each \mathcal{F}_α has the same size n and is enumerated $\langle f_\alpha^i : i < n \rangle$. Let U be a uniform ultrafilter on ω_1 . For each $p = (c, \mathcal{H}, \mathcal{Z}) \in D$ there exist a $\beta_p \in \text{dom}(c) \setminus \text{dom}(a)$ and an $i_p < n$ such that for U -many $\alpha < \omega_1$, $c(\beta_p) = f_\alpha^{i_p}(\beta_p)$. Note that for any pair $p_0 = (c_0, \mathcal{H}_0, \mathcal{Z}_0), p_1 = (c_1, \mathcal{H}_1, \mathcal{Z}_1)$ of conditions in D , if $\beta_{p_0} = \beta_{p_1}$ and $i_{p_0} = i_{p_1}$ then $c_0(\beta_{p_0}) = c_1(\beta_{p_1})$. Therefore, we may define functions $h_i : \omega_1 \rightarrow \omega_1$ ($i < n$) by letting $h_i(\beta) = \gamma$ if there exists a $p = (c, \mathcal{H}, \mathcal{Z}) \in D$ such that $\beta_p = \beta$, $i_p = i$ and $c(\beta) = \gamma$, and 0 no such γ exists. Now let $p = (c, \mathcal{F}, \mathcal{X})$ be a condition in D below $(a, \mathcal{F} \cup \{h_i : i < n\}, \mathcal{X})$. Then $h_{i_p}(\beta_p) = c(\beta_p)$, giving a contradiction. \square

Theorem 6.2. *The partial order P is proper.*

Proof. Let $p = (a, \mathcal{F}, \mathcal{X})$ be a condition. Let X be a countable elementary submodel of $H((2^{|P|})^+)$ with $p \in X$, and let

$$p' = (a, \mathcal{F}, \mathcal{X} \cup \{X \cap H(\omega_2)\}).$$

Let $q = (b, \mathcal{G}, \mathcal{Y})$ be any condition below p' , and let D be a dense subset of P in X . By Lemma 6.1, there exists an $\alpha < \omega_1$ such that for any finite set of functions \mathcal{F}' from ω_1 to ω_1 there is a member $(c, \mathcal{F}^*, \mathcal{Y}^*)$ of D below $p'' = (b \cap X, \mathcal{G} \cap X, \mathcal{Y} \cap X)$ with $\text{dom}(c) \subset \alpha$ such that for each $f \in \mathcal{F}'$ and each $\beta \in \text{dom}(c) \setminus \text{dom}(b \cap X)$, $c(\beta) \neq f(\beta)$. This is a statement about D and p'' , so there exists such an α in X . Applying this property of α with $\mathcal{F}' = \mathcal{G}$, let $(c, \mathcal{F}^*, \mathcal{Y}^*)$ be a member of D below p'' with $\text{dom}(c) \subset \alpha$ such that for each $f \in \mathcal{G}$ and each $\beta \in \text{dom}(c) \setminus \text{dom}(b \cap X)$, $c(\beta) \neq f(\beta)$. Then the fact that there exist $\mathcal{F}^+, \mathcal{Y}^+$ with $(c, \mathcal{F}^+, \mathcal{Y}^+)$ in D and below p'' is a true statement about c , D and p'' , and so we may fix such $\mathcal{F}^+, \mathcal{Y}^+$ in X . Then $(c \cup b, \mathcal{F}^+ \cup \mathcal{G}, \mathcal{Y}^+ \cup \mathcal{Y})$ is a condition below both q and $(c, \mathcal{F}^+, \mathcal{Y}^+)$. \square

Theorem 6.3. *Suppose that f_α ($\alpha < \omega_1$) are functions from ω_1 to ω_1 , and define functions g_β ($\beta < \omega_1$) by letting $g_\beta(\alpha) = f_\alpha(\beta)$. Suppose that $h: \omega_1 \rightarrow \omega_1$ is a function such that for each $\beta < \omega_1$, $\{\alpha < \omega_1 \mid g_\beta(\alpha) = h(\alpha)\}$ is finite. Then there exist stationary sets S_e ($e \in E$), for E a collection of finite subsets of ω_1 such that $\bigcup_{e \in E} S_e$ contains a club and such that for each $e \in E$ and each $\alpha < \omega_1$ there is a $\gamma < \omega_1$ such that $f_\alpha^{-1}(\gamma) \cap S_e$ is contained in $\alpha + 1$.*

Proof. For each finite $e \subset \omega_1$, let S_e be the set of $\beta < \omega_1$ for which

$$\{\alpha < \beta \mid g_\beta(\alpha) = h(\alpha)\} = e.$$

Letting E be the set of such e for which S_e is stationary, by pressing down we see that the diagonal union of the stationary S_e 's contains a club (letting the sequence $\langle e_\gamma : \gamma < \omega_1 \rangle$ enumerate the finite subsets of ω_1 , for club many $\beta < \omega_1$, $\langle e_\gamma : \gamma < \beta \rangle$ enumerates the finite subsets of β , and for each such β there is a $\gamma < \beta$ such that $\beta \in S_{e_\gamma}$). Now fix one $e \in E$, and fix $\alpha < \omega_1$. If $\alpha \in e$, then $f_\alpha(\beta) = g_\beta(\alpha) = h(\alpha)$ for all $\beta > \alpha$ in S_e , so $f_\alpha^{-1}(\gamma) \cap S_e \subset (\alpha + 1)$ for any countable $\gamma \neq h(\alpha)$. If $\alpha \notin e$, then $f_\alpha(\beta) = g_\beta(\alpha) \neq h(\alpha)$ for all $\beta > \alpha$ in S_e , so $f_\alpha^{-1}(h(\alpha)) \cap S_e \subset (\alpha + 1)$. \square

Corollary 6.4. *In the \mathbb{P}_{\max} extension there is an ω_2 -sequence of stationary subsets of ω_1 not satisfying (**).*

Corollary 6.5. *The Proper Forcing Axiom implies that there is an ω_2 -sequence of stationary subsets of ω_1 not satisfying (**).*

The following corollary shows that in the \mathbb{P}_{\max} extension, and under PFA, there are functions $F: \omega_2 \times \omega_1 \rightarrow \omega$ such that for each $n < \omega$ the collection of sets of the form $\{\alpha < \omega_1 : F(\beta, \alpha) < n\}$ ($\beta < \omega_2$) fails to satisfy (**), ruling out a proof of Axiom F in the \mathbb{P}_{\max} extension along the lines of the proof of Theorem 5.3.

Corollary 6.6. *Suppose that $M \subset N$ are models of ZFC such that $\omega_1^M = \omega_1$ and $(2^{\omega_1})^M$ has cardinality \aleph_1 in N . Suppose that $h: \omega_1 \rightarrow \omega_1$ exists in a forcing extension of N , and h intersects each member of $\omega_1^{\omega_1}$ in N finitely. Then in $N[h]$ there is a partition of ω_1 into sets E_i ($i \in \omega$) such that for every partition of ω_1 $\langle A_\alpha : \alpha < \omega_1 \rangle$ in M and every $n \in \omega$ there is an $\alpha < \omega_1$ such that $E_i \cap A_\alpha$ is countable for each $i \leq n$.*

Proof. Let $e: \omega_1 \times \omega \rightarrow \omega_1$ be a bijection in N , and for each $i < \omega_1$, define $h_i: \omega_1 \rightarrow \omega_1$ by letting $h_i(\alpha) = h(e(\alpha, i))$. For each $f: \omega_1 \rightarrow \omega_1$ in M , define $f_e: \omega_1 \rightarrow \omega_1$ by letting $f_e(e(\alpha, i)) = f(\alpha)$. Then since $h \cap f_e$ is finite, $h_i \cap f$ is nonempty for only finitely many i .

Let $\langle A_\alpha^\beta : \alpha < \omega_1 \rangle : \beta < \omega_1 \rangle$ be an enumeration in N of all the \aleph_1 -sized partition of ω_1 in M with $A_\alpha^\beta \cap \alpha = \emptyset$ for all α, β . For each $\alpha < \omega_1$, define $a_\alpha: \alpha \rightarrow \alpha + 1$ by letting $a_\alpha(\beta) = \gamma$ if $\alpha \in A_\gamma^\beta$. Then $a_\alpha \in N$. Working in $N[h]$, put $\alpha \in E_i$ if i is least such that for all $j \geq i$, $h_j \cap a_\alpha$ is empty.

Now fix $n \in \omega$ and $\beta < \omega_1$. Then if $\alpha \in E_i$ for some $i \leq n$ and $\alpha > \beta$, then $h_n(\beta) \neq a_\alpha(\beta)$, so α is not in $A_{h_n(\beta)}^\beta$. Thus $E_i \cap A_{h_n(\beta)}^\beta$ is countable for all $i \leq n$. \square

We note a couple additional consequences of Theorem 6.3 with outside interest.

Corollary 6.7. *For any $A \subset \omega_1$ there is an ω_1 -preserving forcing in whose extension $L[A]$ does not contain a partition of ω_1 into ω_1 many pieces all stationary in V .*

Proof. First force to wellorder the functions from ω_1 to ω_1 in $L[A]$ in ordertype ω_1 , apply Theorem 6.3, and finally shoot a club through one of the corresponding stationary sets S_e as above. \square

Corollary 6.8. *There is a proper forcing in whose extension, for every function $c: [\omega_1]^2 \rightarrow \omega_1$ there is a stationary set B such that for all $\alpha < \omega_1$,*

$$\{c(\alpha, \beta) \mid \beta \in B \setminus (\alpha + 1)\} \neq \omega_1.$$

Proof. Iterate to take care of each such c , at each stage letting f_α (for each $\alpha < \omega_1$) be a function taking value $c(\alpha, \beta)$ at each $\beta > \alpha$, and applying Theorems 6.2 and 6.3. \square

By contrast, results of Todorćević [21] give the following (see also [22]). We note that the second of these has recently been improved by Moore [15].

Theorem 6.9. ([21]) *Every inner model which is correct about ω_1 contains a partition of ω_1 into ω many pieces all stationary in V .*

Theorem 6.10. ([21]) *There exists a $c: [\omega_1]^2 \rightarrow \omega$ such that for every stationary set B there exists an $\alpha < \omega_1$ such that $\{c(\alpha, \beta) \mid \beta \in B \setminus (\alpha + 1)\} = \omega$.*

As we mentioned in the introduction, we do not know whether NS_{ω_1} has caliber (\aleph_2, \aleph_0) in the \mathbb{P}_{max} extension. The following question appears to be linked to this problem.

6.11 Question. Suppose that A_β^α ($\alpha, \beta < \omega_1$) are stationary subsets of ω_1 such that for each α , $\{A_\beta^\alpha : \beta < \omega_1\}$ is a maximal antichain in $\mathcal{P}(\omega_1)/NS_{\omega_1}$. Let f_γ ($\gamma < \omega_1$) be functions from ω_1 to ω_1 . Does there exist a forcing construction preserving stationary subsets of ω_1 and adding a stationary $B \subset \omega_1$ such that for all $\gamma < \omega_1$, $\{\alpha \mid A_{f_\gamma(\alpha)}^\alpha \cap B \text{ is stationary}\}$ is nonstationary?

7 Unions of \aleph_1 -many linked sets

As a modest first step towards answering the question of whether the square of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ is \aleph_2 -c.c. in the \mathbb{P}_{max} extension, we show that it is not the union of \aleph_1 -many linked sets. One key point is the following fact. An ideal is *nowhere* \aleph_1 -dense if its restriction to no positive set is \aleph_1 -dense.

Lemma 7.1. *A σ -ideal I on ω_1 is nowhere \aleph_1 -dense if and only if for each family $\mathcal{A} \in [I^+]^{\aleph_1}$ there exist disjoint sets B_0, B_1 such that for each $A \in \mathcal{A}$, $B_0 \cap A$ and $B_1 \cap A$ are both in I^+ .*

Proof. The reverse direction (more specifically, its contrapositive), is immediate. For the other direction, by a result of [3], since I is nowhere \aleph_1 -dense the members of \mathcal{A} can be refined to a disjoint collection \mathcal{A}' . Then B_0 and B_1 are easily constructed by splitting each member of \mathcal{A}' into two disjoint I -positive sets. \square

The consequence of Lemma 7.1 that we need is the following: if I is a nowhere \aleph_1 -dense ideal, \mathcal{A} is in $[I^+]^{\aleph_1}$ and \mathcal{D} is a pairwise compatible subset of I^+ , then there exist disjoint sets B_0 and B_1 having I^+ -positive intersection with every element of \mathcal{A} , which means that they cannot both contain an element of \mathcal{D} .

Theorem 7.2. *If W is a model of $ZF + DC_{\mathbb{R}} + AD^+$ and $G \subset \mathbb{P}_{max}^W$ is W -generic, then, in $W[G]$, $\mathcal{P}(\omega_1)/NS_{\omega_1}$ is not the union of \aleph_1 -many pairwise compatible sets.*

Proof. Fix a name τ for a function from $\mathcal{P}(\omega_1)/NS_{\omega_1}$ to ω_1 , and suppose towards a contradiction that some condition p forces the the preimage of each countable ordinal under τ will be linked. Let X be set of reals coding the set of triples $\langle \langle (M, I), a \rangle, b, \alpha \rangle$ where $\langle (M, I), a \rangle \in \mathbb{P}_{max}$, $b \in \mathcal{P}(\omega_1)^M \setminus I$, $\alpha \in \omega_1$ and $\langle (M, I), a \rangle \Vdash ([b^*], \alpha) \in \tau$. Let (N, J) be an X -iterable \mathbb{P}_{max} precondition with $p \in H(\omega_1)^N$. Working in N , build a decreasing ω_1 -sequence $p_\alpha = \langle (M_\alpha, I_\alpha), a_\alpha \rangle$ ($\alpha < \omega_1$) of \mathbb{P}_{max} conditions below p , along with sets $b_\alpha \in \mathcal{P}(\omega_1)^{M_\alpha} \setminus I_\alpha$ such that, letting $j_{\alpha\beta}$ ($\alpha \leq \beta < \omega_1^N$) be the embeddings witnessing the order on the p_α 's,

1. each b_α is forced by p_α to have no subsets in $\tau^{-1}(\check{\alpha})$,
2. each b_α has intersection in I_α^+ with every member of $j_{\beta\alpha}(\mathcal{P}(\omega_1)^{M_\beta} \setminus I_\beta)$, for each $\beta < \alpha$,
3. for all $\beta < \alpha < \omega_1$, $b_\alpha \setminus j_{\beta\alpha}(b_\beta) \in I_\alpha$.

These conditions can be met at successor stages, by the Lemma 7.1. At limit stages α , we need only ensure that $\bigcap_{\beta < \alpha} j_{\beta\alpha}(b_\beta)$ is I_α -positive. Condition (2) above makes this possible, as in the proof of Theorem 5.3. For each $\alpha < \omega_1^N$, let j_α^N be iteration of (M_α, I_α) sending a_α to $a = \bigcup_{\alpha < \omega_1^N} a_\alpha$. Again as in the proof of Theorem 5.3, we can also ensure that the diagonal intersection b of the $j_\alpha^N(b_\alpha)$'s is J -positive. Then $\langle (N, J), a \rangle$ forces that b^* will not be a member of any $\tau^{-1}(\check{\alpha})$, giving a contradiction. \square

A negative answer to Question 1.8 should follow from a positive one to the following. The question asks for a version of the results in Section 6 where the sets $\{A_\beta^\alpha : \beta < \omega_1\}$ are not presumed to be antichains.

7.3 Question. Suppose that $\{A_\beta^\alpha : \alpha, \beta < \omega_1\}$ are stationary subsets of ω_1 , and for each $\alpha < \omega_1$ and each countable partition \mathcal{B} of ω_1 there exist a $\beta < \omega_1$ and a $B \in \mathcal{B}$ such that $B \setminus A_\beta^\alpha$ is stationary. Does there exist a forcing construction preserving stationary subsets of ω_1 and adding a stationary $E \subset \omega_1$ such that for each $\alpha < \omega_1$ there is a $\beta < \omega_1$ such that $A_\beta^\alpha \cap E$ is nonstationary?

8 Uniqueness of iterations

Finally, we show that on a dense suborder of \mathbb{P}_{max} , compatibility of compatible conditions is witnessed by a unique pair of iterations. We came across this fact while searching for a proof of Theorem 3.1 - the hope was that for a given pair stationary, costationary sets in the models of two compatible conditions, it could be undecided whether their images would have stationary intersection. Theorem 8.2 below shows that this is not the case, at least on a dense set.

A function $g: \omega_1 \rightarrow \omega_1$ is said to be a *canonical function* for an ordinal $\gamma \in [\omega_1, \omega_2)$ if for any bijection $\pi: \omega_1 \rightarrow \gamma$, the set of α such that $g(\alpha) = o.t.(\pi[\alpha])$ contains a club. Such a g represents γ in any generic ultrapower derived by forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$. The statement ψ_{AC} [24] says that for any pair A, B of stationary, costationary subsets of ω_1 , there is a $\beta \in [\omega_1, \omega_2)$ such that for every bijection $f: \omega_1 \rightarrow \omega_2$, the set $\{\alpha < \omega_1 \mid \alpha \in A \Leftrightarrow o.t.(f[\alpha]) \in B\}$ contains a club. Equivalently, ψ_{AC} says that for any pair A, B of stationary, costationary subsets of ω_1 , there is a canonical function g such that the set of $\alpha < \omega_1$ such that $\alpha \in A$ if and only if $g(\alpha) \in B$ contains a club. Yet another way to say the same thing is that for each pair A, B of stationary, costationary subsets of ω_1 there is a $\gamma \in [\omega_1, \omega_2)$ such that A is the Boolean value that $\gamma \in j(B)$, for j the generic ultrapower induced by forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$.

It is shown in [24, 13] that ψ_{AC} holds in the \mathbb{P}_{max} extension, and since it can be forced by a semi-proper forcing from a strongly inaccessible limit of measurable cardinals [24, 13, 2] it holds in densely many \mathbb{P}_{max} conditions. We will use the following strengthening of ψ_{AC} which we will call $(++)$:

Suppose that A is a subset of ω_1 such that $j(A)$ is not the same for every generic embedding j derived by forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$. Then for every stationary, costationary $B \subset \omega_1$ there is a canonical function g such that the set of α such that $\alpha \in B$ if and only if $g(\alpha) \in A$ contains a club.

Lemma 8.1. *If ψ_{AC} and $u_2 = \omega_2$ hold, then so does $(++)$.*

Proof. Since ψ_{AC} holds, canonical function bounding holds [2] and $j(\omega_1) = \omega_2$ for any generic embedding j derived from forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$. Fix $A \subset \omega_1$ and $\eta \in [\omega_1, \omega_2)$ such that the Boolean value that $\eta \in j(A)$ is neither 0 nor 1. Let $B \subset \omega_1$ be stationary, costationary. Let x be a real such that $li(x, \omega_1) > \eta$.

Fix an integer $n \in \omega \setminus \{0\}$, a function $f: Ord^n \rightarrow Ord$ and an $(n-1)$ -tuple b of countable x -indiscernibles such that

- f is definable in $L[x^\#]$ from $x^\#$ and no other parameters;
- for all $a \in \omega_1^n$, $f(a) \in \omega_1$;
- $f(b, \omega_1) = \eta$.

Since f is definable in $L[x^\#]$, for any n -tuple of functions g_1, \dots, g_n from ω_1 to the ordinals, $f([g_1]_G, \dots, [g_n]_G) = [h]_G$, where $G \subset \mathcal{P}(\omega_1)/NS_{\omega_1}$ is a generic filter, $[e]_G$ denotes the element of the induced ultrapower represented by a function e , and $h: \omega_1 \rightarrow Ord$ is the function defined by letting $h(\alpha) = f(g_1(\alpha), \dots, g_n(\alpha))$.

Since $f(b, \omega_1) = \eta$, the set of α such that $f(b, \alpha)$ is in A is stationary, costationary. Call it E . Fix a stationary, costationary $B \subset \omega_1$. Since ψ_{AC} holds, there exist a canonical function g for an ordinal ξ and a club $D \subset \omega_1$ such that for all $\alpha \in D$, $\alpha \in B$ if and only if $g(\alpha) \in E$. Let $h: \omega_1 \rightarrow \omega_1$ be defined by letting $h(\alpha) = f(b, g(\alpha))$. Then h is a canonical function for $f(b, \xi)$, and for club many $\alpha < \omega_1$, $\alpha \in B$ if and only if $g(\alpha) \in E$ if and only if $h(\alpha) \in A$. \square

Suppose that M is a model of ZFC, $x \in \mathcal{P}(\omega)^M$ and $a \in \mathcal{P}(\omega_1)^M$ are such that $\omega_1^{L[x, a]} = \omega_1^M$. Then in M there are stationary, costationary subsets of ω_1 definable in $L[x, a]$ from x and a . Since the image of these stationary costationary sets is not the same under every M -generic elementary embedding derived from forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$, the image of a is not the same under every M -generic elementary embedding derived from forcing with $\mathcal{P}(\omega_1)/NS_{\omega_1}$.

Now suppose that M is a model of ZFC and a is an element of $\mathcal{P}(\omega_1)^M$ such that in M , for every stationary, costationary $b \subset \omega_1$ there exist a $\gamma < \omega_1$ and a bijection $\pi_\gamma: \omega_1 \rightarrow \gamma$ and a club $c \subset \omega_1$ such that for all $\alpha \in c$, $\alpha \in b$ if and only if $o.t.(\pi_\gamma[\alpha]) \in a$. Then the following $\mathcal{P}(\omega_1)/NS_{\omega_1}$ -antichain $\mathcal{S}(a) = \langle x_i : i < \omega \rangle$ is definable in M from a : the unique sequence $\langle x_i : i < \omega \rangle$ of elements of $\mathcal{P}(\omega_1)/NS_{\omega_1}$ such that for each $i < \omega$, if γ_i is the least element of $[\omega_1, \omega_2)$ such that for some bijection $\pi_{\gamma_i}: \omega_1 \rightarrow \gamma_i$ the set

$$\{x_j : j < i\} \cup \{[\alpha < \omega_1 \mid o.t.(\pi_{\gamma_i}[\alpha]) \in a]\}$$

forms a nonmaximal antichain in $\mathcal{P}(\omega_1)/NS_{\omega_1}$ then

$$x_i = [\alpha < \omega_1 \mid o.t.(\pi_{\gamma_i}[\alpha]) \in a].^1$$

It follows that if M, N are two such models with respect to the same set a , and M and N are both correct about ω_1 and stationary subsets of ω_1 , then $\mathcal{S}(a)^M = \mathcal{S}(a)^N$. From this it follows by standard arguments that $\omega_2^M \leq \omega_2^N \Rightarrow \mathcal{P}(\omega)^M \subset N$: for each $z \subset \omega$ in M , there is a $\gamma \in [\omega_1, \omega_2)^M$ and a bijection $\pi_\gamma: \omega_1^M \rightarrow \gamma$ such that, in M , letting $\mathcal{S}(a)^M = \langle x_i : i < \omega \rangle$, $\bigvee \{x_{i+1} : i \in z\}$ is

¹A version of this construction arose in conversations between the author and David Asperó.

the Boolean value in $\mathcal{P}(\omega)/NS_{\omega_1}$ that $\gamma \in j(A)$ in the corresponding embedding, and z is the set of i such that x_{i+1} is compatible with $\llbracket \gamma \in j(A) \rrbracket$ in N .

The argument below uses the main result of [10]. An interesting aspect of the theorem is that it uses all three of the principles MA_{\aleph_1} , ψ_{AC} and $u_2 = \omega_2$.

Theorem 8.2. *Suppose that $p_0 = \langle (M_0, I_0), a_0 \rangle$ and $p_1 = \langle (M_1, I_1), a_1 \rangle$ are compatible \mathbb{P}_{max} conditions such that M_0 and M_1 both satisfy $ZFC + \psi_{AC}$ and $u_2 = \omega_2$. Then there exist iterations*

$$i_0: (M_0, I_0) \rightarrow (M_0^*, I_0^*), \quad i_1: (M_1, I_1) \rightarrow (M_1^*, I_1^*)$$

such that one of $\mathcal{P}(\omega_1)^{M_0^}$ and $\mathcal{P}(\omega_1)^{M_1^*}$ contains the other, and such that for every pair of iterations j_0, j_1 witnessing that p_0 and p_1 are compatible, i_0 is an initial segment of j_0 and i_1 is an initial segment of j_1 .*

Proof. Without loss of generality, we may assume that

$$\sup\{li(x, \omega_1) : x \in \mathbb{R}^{M_0}\} \leq \sup\{li(x, \omega_1) : x \in \mathbb{R}^{M_1}\}.$$

Let j_0 and j_1 be a pair of iterations witnessing that both p_0 and p_1 are above some condition $\langle \langle N, J \rangle, b \rangle$. Since

$$\sup\{li(x, \omega_1) : x \in \mathbb{R}^{M_0}\} \leq \sup\{li(x, \omega_1) : x \in \mathbb{R}^{M_1}\},$$

$j_0(\omega_2^{M_0}) \leq j_1(\omega_2^{M_1})$, and since M_0 and M_1 satisfy ψ_{AC} , $u_2 = \omega_2$ and MA_{\aleph_1} ,

$$j_0(\mathcal{P}(\omega_1)^{M_0}) \subset j_1(\mathcal{P}(\omega_1)^{M_1}).$$

This relationship must hold then for any pair of iterations witnessing the compatibility of p_0 and p_1 .

Now, there is a unique longest iteration i_0^0 of (M_0, I_0) such that $i_0^0(a_0)$ is an initial segment of a_1 , and by [10] there is a unique shortest iteration i_1^0 of (M_1, I_1) such that $i_0^0(\mathbb{R}^{M_0}) \subset i_1^0(\mathbb{R}^{M_1})$. Furthermore, these two iterations must be initial segments of any pair of iterations witnessing the compatibility of p_0 and p_1 .

We can continue in this way, letting i_{k+1}^0 be the longest iteration of (M_0, I_0) such that $i_{k+1}^0(a_0)$ is an initial segment of $i_k^1(a_1)$, and letting i_{k+1}^1 be the unique shortest iteration of (M_1, I_1) such that $i_{k+1}^0(\mathbb{R}^{M_0}) \subset i_{k+1}^1(\mathbb{R}^{M_1})$. These iterations will extend one another, and will still be initial segments of any pair of iterations witnessing the compatibility of p_0 and p_1 .

For any iterable pair (M, I) such that $M \models u_2 = \omega_2$, the countable ordinals which are indiscernibles of every real of M are on the critical sequence of every sufficiently long iteration of (M, I) [24]. It follows that since each $i_k^0(\mathbb{R}^{M_0}) \subset i_k^1(\mathbb{R}^{M_1})$, $i_{k+1}^0(\omega_1^{M_0}) = i_k^1(\omega_1^{M_1})$, since i_{k+1}^0 is an initial segment of an iteration i (j_0 , for instance) such that $i_k^1(a_1)$ is an initial segment of $i(a_0)$, and so $i_{k+1}^0(\omega_1^{M_0}) < i_k^1(\omega_1^{M_1})$ would imply that the first step of i extending i_{k+1}^0 sends $i_{k+1}^0(\omega_1^{M_0})$ above $i_k^1(\omega_1^{M_1})$, which is impossible since $i_k^1(\omega_1^{M_1})$ is an indiscernible for every real in M_0 .

Now let i^0 be the union of the i_k^0 's and let i^1 be the union of the i_k^1 's. Then $i^0(\omega_1^{M_0}) = i^1(\omega_1^{M_1})$ and $i^0(\mathbb{R}^{M_0}) = i^1(\mathbb{R}^{M_1})$. Then $i^0(a_0) = i^1(a_1)$, and so

$$i^0(\mathcal{P}(\omega_1)^{M_0}) \subset i^1(\mathcal{P}(\omega_1)^{M_1}).$$

□

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