FAILURE OF SQUARE PRINCIPLES IN $\mathbb{P}_{\max}$ EXTENSIONS

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Abstract. We obtain a model where $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \neg\square(\omega_2) + \neg\square(\omega_3)$ by forcing with $\mathbb{P}_{\max}$ over a strong model of determinacy.

1. Introduction

The partial order $\mathbb{P}_{\max}$ was invented by W. Hugh Woodin in the early 1990s, see [Woo10]. When applied to models of the Axiom of Determinacy, it achieves a number of effects not known to be obtainable by traditional forcing methods. The axiom $\mathsf{AD}_{\mathbb{R}}$ asserts the determinacy of perfect information length ω games between two players that alternate playing real numbers. By convention, Θ denotes the least ordinal not the surjective image of the reals.

Woodin showed that when $\mathbb{P}_{\max}$ is applied to a model of the theory $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular", the resulting extension satisfies $\mathsf{MM}^{++}(\mathfrak{c})$, the restriction of the maximal forcing axiom to partial orders of size the continuum. There is one extra step in this context, as one must use another forcing after $\mathbb{P}_{\max}$ to recover the full Axiom of Choice.

It would be desirable to determine whether Woodin's result can be extended to partial orders of size $\mathfrak{c}^+$. Doing this should greatly reduce the large cardinal consistency strength known to be sufficient to achieve such results. Moreover, using the Core Model Induction, a method pioneered by Woodin, one can find lower bounds for the consistency strength of $\mathsf{MM}^{++}(\mathfrak{c}^+)$ and its consequences, which leads to the possibility of proving equiconsistencies.

In this paper we obtain by $\mathbb{P}_{\max}$ techniques some consequences of $\mathsf{MM}^{++}(\mathfrak{c}^+)$ on the extent of square principles. To state our results, we need to review two related principles due to Ronald Jensen [Jen72]:

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Definition 1.1. Given a cardinal κ, the principle $\square_\kappa$ says that there exists a sequence $\langle C_\alpha : \alpha < \kappa^+\rangle$ such that for each $\alpha < \kappa^+$,

- Each C_α is a closed cofinal subset of α;
- For each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$;
- The order type of each C_α is at most κ.

Definition 1.2. The principle $\square(\kappa)$ says that there exists a sequence $\langle C_\alpha : \alpha < \kappa\rangle$ such that

- For each $\alpha < \kappa$,
  - Each C_α is a closed cofinal subset of α;
  - For each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$;
- There is no *thread* through the sequence, i.e., there is no closed unbounded $E \subseteq \kappa$ such that $C_\alpha = E \cap \alpha$ for every limit point α of E.

Note that $\square_\kappa$ implies $\square(\kappa^+)$. A key distinction between them is that $\square_\kappa$ persists to outer models which agree about κ^+, while $\square(\kappa^+)$ need not. Via work of Stevo Todorčević [Tod84, Tod02], it is known that $\mathsf{MM}^{++}(\mathfrak{c})$ implies $2^{\aleph_1} = \aleph_2 + \neg\square(\omega_2)$ and $\mathsf{MM}^{++}(\mathfrak{c}^+)$ implies $\neg\square(\omega_3)$. Through work of Ernest Schimmerling [Sch07] and Steel (via the Core Model Induction, see [SS]) it is known that the statement implies that the Axiom of Determinacy holds in the inner model $L(\mathbb{R})$:

$$\neg\square(\omega_2) + \neg\square(\omega_3) + 2^{\aleph_1} = \aleph_2. \tag{1}$$

We prove that $\square(\Theta)$ holds in the minimal model of $\mathsf{AD}_\mathbb{R}$ + "Θ is regular" and therefore $\square_{\omega_2}$ holds in the corresponding $\mathbb{P}_{\max}$ extension (the cardinal Θ of the inner model of determinacy becomes ω_3 in the $\mathbb{P}_{\max}$ extension). The moral is that a stronger determinacy hypothesis is needed to force the failure of this principle.

The hypothesis that we use in the end is just slightly stronger, namely:

> $\mathsf{AD}_\mathbb{R}$ + "Θ is regular" + "$\{\kappa \mid \kappa$ is regular in HOD, is a member of the Solovay sequence, and has cofinality $\omega_1\}$ is stationary in Θ."

We show from this hypothesis that $\square_{\omega_2}$ fails in the extension consisting of $\mathbb{P}_{\max}$ followed by a forcing well-ordering the power set of the reals.

As far as we know, this is the first application of $\mathbb{P}_{\max}$ which uses a theory stronger than $\mathsf{AD}_\mathbb{R}$ + "Θ is regular." In fact, if we do not require choice to hold in the final model, then a weaker theory than the one isolated above suffices, namely, $\mathsf{AD}_\mathbb{R}$+"Θ is Mahlo in HOD." Similarly, a stronger hypothesis allows us to conclude that even $\square(\omega_3)$ fails in the final extension.

The key technical tools in our arguments are the interaction between an inner model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular" and another, larger, such model, usually coupled with some amount of correctness relating both models, and the fact that the extension of the larger model satisfies $\mathsf{MM}^{++}(\mathfrak{c})$.

The determinacy hypotheses we have mentioned are all weaker in consistency strength than a Woodin cardinal that is limit of Woodin cardinals. This puts them within the region suitable to be reached from current techniques by a Core Model Induction. Moreover, these hypotheses are much weaker than the previously known upper bounds on the strength of (1). Prior to our work, two methods were known to show the consistency of (1): It is a consequence of $\mathsf{PFA}(\mathfrak{c}^+)$, the restriction of the proper forcing axiom to partial orders of size $\mathfrak{c}^+$, and can be forced directly from the existence of a *quasicompact* cardinal.

It is not clear what the consistency strength of $\mathsf{PFA}(\mathfrak{c}^+)$ is. We expect that the HOD analysis should allow us to extend the Core Model Induction to establish the precise consistency strength of (1). The project of obtaining $\mathsf{MM}^{++}(\mathfrak{c}^+)$ or even $\mathsf{PFA}(\mathfrak{c}^+)$ in a $\mathbb{P}_{\max}$ extension of some determinacy model is a very interesting one; what seems to be one of the key requirements is an appropriate notion of $\mathcal{B}$-*iterability* for $\mathcal{B}$ a subset of $\mathcal{P}(\mathbb{R})$.

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2. A note on hypotheses

Some of our results here use hypotheses on inner models of the form $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$, where $\mathcal{P}_\kappa(\mathbb{R})$ denotes the collection of sets of reals of Wadge rank less than κ. We note here that some of these hypotheses follow from statements about HOD. The main point is that, for each θ on the Solovay sequence of a model of AD, every bounded subset of θ is generic over HOD via a partial order (the Vopenka algebra) of cardinality at most θ in HOD.

To see this, suppose that AD holds, and fix a member θ of the Solovay sequence, an ordinal $\xi < \theta$, and a set $A \subseteq \xi$. Let $\mathbb{B}_0$ be the Boolean algebra consisting of those subsets of $\mathcal{P}(\xi)$ which are ordinal definable, ordered by inclusion. Given a filter $G \subseteq \mathbb{B}_0$, let $E(G)$ be the set of $\alpha \in \xi$ such that $\{E \subseteq \xi \mid \alpha \in E\} \in G$. Then, according to

Vopěnka's Theorem, there exist a Boolean algebra $\mathbb{B}_1$ in HOD, a $\mathbb{B}_1$-name $\dot{E} \in$ HOD and an isomorphism $h\colon \mathbb{B}_0 \to \mathbb{B}_1$ such that

(1) for every $E \subseteq \xi$, $H_E = h[\{A \in \mathbb{B}_0 \mid E \in A\}]$ is HOD-generic for $\mathbb{B}_1$;
(2) if $H \subseteq \mathbb{B}_1$ is HOD-generic and $G = h^{-1}[H]$, then $E(G) = \dot{E}_H$.

Furthermore, $\mathbb{B}_1$ has cardinality at most θ in HOD. To see this, note first of all that if we fix a set of reals F of Wadge rank ξ, then by the Moschovakis Coding Lemma there is a surjection from the sets of reals ordinal definable from F onto the ordinal definable subsets of $\mathcal{P}(\xi)$. By the definition of the Solovay sequence, each set of reals which is ordinal definable from F has Wadge rank less than θ, since otherwise there would be a prewellordering of length at least θ ordinal definable from F. Similarly, for all $\beta < \theta$, the collection of sets of reals of Wadge rank at most β which are ordinal definable from F is wellorderable, and any wellorder must have ordertype less than θ. It follows that $\mathbb{B}_1$ is a union of θ many sets of cardinality less than θ in HOD.

Now suppose that $H \subseteq \mathbb{B}_1$ is HOD-generic, and let $E = \dot{E}_H$. Then HOD$[E]$ = HOD$[H]$, by items (1) and (2) above.[1] Furthermore, HOD$[H]$ = HOD$_E$. The forward direction of this equality is immediate. For the other, suppose that A is a set of ordinals in HOD_E, and fix an ordinal γ such that A and E are both subsets of γ. Then there an OD relation $T \subseteq \gamma \times \mathcal{P}(\gamma)$ such that $A = \{\xi < \gamma | T(\xi, E)\}$. Define the relation T^* on $\gamma \times \mathbb{B}_1$ by setting $T^*(\xi, p)$ if and only if $T(\xi, C)$ holds for all $C \in h^{-1}(p) \cap \mathcal{P}(\gamma)$. Then $T^* \in$ HOD, and A is in HOD$[H]$, since A is the set of $\xi < \sup(A)$ such that there exists a p in H for which $T^*(\xi, p)$ holds.

Now suppose that $\mathsf{AD}_{\mathbb{R}}$ holds. Then every set of reals has a scale, so is $<\Theta$-Suslin and thus ordinal definable from a bounded subset of Θ. From this and the argument above we get the following facts.

Theorem 2.1. *Assuming* $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$*, if* θ *is a member of the Solovay sequence and* θ *is regular in* HOD*, then* θ *is regular.*

Proof. If $\theta = \theta_0$ or θ is a successor member of the Solovay sequence, then θ is regular without any assumption on HOD, since in these cases there is one set of reals from which Wadge-cofinally many sets of reals are ordinal definable. Suppose that Θ is a limit in the Solovay sequence.

[1]Let's do the Vopenka algebra in terms of sets $A_{\phi,s} = \{x \mid \phi(x, s)\}$. Also, the forward containment of the fact just given is easy, and all that we need, but the reverse direction needs more spelling out if we want to do it.

Then $\mathsf{AD}_{\mathbb{R}}$ holds.[2] Suppose that $f\colon \alpha \to \Theta$ is cofinal, for some $\alpha < \Theta$. Then f is ordinal definable from some set of reals, which itself is ordinal definable from a bounded subset A of Θ. Pick $\theta_\xi < \Theta$ such that A is bounded in θ_ξ. Let H be the generic filter for the Vopenka algebra corresponding to A. As above, this partial order has cardinality at most θ_ξ in HOD. Moreover, since $\mathrm{HOD}_A \subseteq \mathrm{HOD}[H]$, $f \in \mathrm{HOD}[H]$. On the other hand, the regularity of Θ in HOD is preserved in $\mathrm{HOD}[H]$, giving a contradiction. $\square$

Essentially the same argument shows the following.

Theorem 2.2. *Assume* $\mathsf{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$, *and suppose that* θ *is a limit member of the Solovay sequence such that, in* HOD, *the set of regular cardinals below* θ *is stationary. Then the set of regular cardinals below* θ *is stationary.*

With a little more work, one gets the following.

Theorem 2.3. *Assuming* $\mathsf{AD}_{\mathbb{R}}$, *if* Θ *is measurable in* HOD *then in* $\mathrm{HOD}_{\mathcal{P}(\mathbb{R})}$ *there is an* $\mathbb{R}$-*complete measure on* Θ.

Proof. Let μ be a measure on Θ in HOD. For each bounded subset s of Θ, we can let μ_s be the lift of μ to $\mathrm{HOD}[H_s]$, by the Vopenka algebra argument above. Let ν be the union of all such μ_s's. Then ν is equal to

$$\{A \subseteq \Theta : \exists B \in \mu(B \subseteq A)\}. \tag{2}$$

Every partition of Θ in $\mathrm{HOD}_{\mathcal{P}(\mathbb{R})}$ is ordinal definable from some s as above, and is therefore measured by some μ_s. Hence, ν is Θ-complete.

We claim then that this gives that ν is $\mathbb{R}$-complete. For if $\langle A_x : x \in \mathbb{R}\rangle$ is a collection of ν-positive sets, let $B_\alpha = \{x \in \mathbb{R} \mid \alpha \in A_x\}$, for each $\alpha < \Theta$. Then $\bar{B} = \langle B_\alpha : \alpha < \Theta\rangle$ is ordinal definable from a set of reals. Since $\mathsf{AD}_{\mathbb{R}}$ holds, there is no Θ-sequence in $\mathrm{HOD}_{\mathcal{P}(\mathbb{R})}$ consisting of sets of reals of unbounded Wadge rank. On the other hand, if unboundedly many B_α's were to be distinct sets with Wadge rank below some fixed set of reals, then one could define from this situation a prewellordering of length Θ, which is impossible. So $\bar{B}$ must contain fewer than Θ many distinct sets. Then B_α must be the same, and indeed, must be $\mathbb{R}$, for ν-many B_α's. $\square$

[2]To see this, note (using $V = L(\mathcal{P}(\mathbb{R}))$ that if there is a largest Suslin cadinal then $V = L(T, \mathbb{R})$ for a set of ordinals T which is coded by some set of reals, in which case there cannot be a $\theta_\alpha < \Theta$ above the Wadge rank of this set.

3. $\square_{\omega_2}$ holds in the $\mathbb{P}_{\max}$ extension of the minimal model of $\mathsf{AD}_{\mathbb{R}}$ + "Θ is regular"

From Grigor's email 30 Jul 2011:

Martin and I discussed square (not the failure) portion of the paper and it seems like there isn't much to do. It seemed it is just a straightforward generalization of the square proof in mice. We should probably write it later either in a separate paper or as a small section of this paper.

4. Squareless Choiceless extensions

Definition 4.1. Θ is called weakly compact if whenever M is a transitive model of ZF^- and $|M| = \mathcal{P}(\mathbb{R}) \times \Theta$, there is an $\mathbb{R} \times \Theta$-complete M-ultrafilter.

Theorem 4.2. *Assume* $\mathsf{AD}_{\mathbb{R}} + \Theta$ *is weakly compact. Then in the* $\mathbb{P}_{\max}$ *extension,* $\square(\omega_3)$ *fails.*

Proof. It suffices to argue that ω_3 has the tree property in the extension, because then any coherent sequence of length ω_3 is threadable.

It is enough to show that for any collection of ω_3 many subsets of ω_3 in the $\mathbb{P}_{\max}$ extension there is a measure that decides all of them. Then, given a tree T on ω_3, we can form an ultrapower and obtain extensions, thus showing that T must have a branch.

Let $(A_\alpha : \alpha < \Theta)$ be a collection of subsets of Θ. In V consider the sets $B_{p,\alpha} = \{\beta : p \Vdash \beta \in A_\alpha\}$ of the associated forcing relations. There is an $M = L_\eta(\mathcal{P}(\mathbb{R})$ containing all these sets with $\eta < \Theta^+$. By our assumption on Θ, it follows that we can find an M-measure μ that is $\mathbb{R} \times \Theta$-complete.

By a standard Levy-Solovay argument, the lifting $\mu_G = \{B \mid \exists A \in G\, (A \subseteq B)\}$ is an $M[G]$-measure that is Θ-complete, where G is the $\mathbb{P}_{\max}$-generic. $\square$

The existence of a model M_1 as in the following theorem follows from $\mathrm{AD}_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R})) +$ "Θ is Mahlo in HOD".

Theorem 4.3. *Assume* $V \models \mathrm{AD}_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))+$ *"There are cofinally many* $\kappa < \Theta$ *that are limits of the Solovay sequence and are regular in* HOD*." Let* G *be* $\mathbb{P}_{\max}$ *generic. Then* $\square_{\omega_2}$ *fails in* $V[G]$*.*

Proof. let $M_1 = V$. Fix σ a $\mathbb{P}_{\max}$ name for a $\square_{\omega_2}$-sequence. Fix κ a limit of the Solovay sequence, regular in HOD, and sufficiently large to ensure that $\sigma \in \mathrm{OD}(A)$ for some $A \in \mathcal{P}_\kappa(\mathbb{R})$. Let $\Gamma_0 = \mathcal{P}_\kappa(\mathbb{R})$ and $M_0 = \mathrm{HOD}^{M_1}_{\Gamma_0}$. Note that $\Gamma_0 = \mathcal{P}(\mathbb{R})^{M_0}$.

Since κ is regular in HOD, exactly the same argument with Vopenka forcing as in the previous section shows that it is regular in M_0. The forcing relation for the κ-th member of the $\square_{\omega_2}$-sequence with name σ,

$$\{(p, \alpha) \mid p \Vdash \alpha \in \dot{C}_\kappa\},$$

is definable from σ, $\mathbb{P}_{\max}$, and κ, and therefore belongs to M_0. Note that $M_0[G] \models \sigma_G$ is a $\square_\kappa$-sequence.

But then the interpretation C_κ of this κ-th club belongs to $M_0[G]$. But this is a contradiction: On the one hand, C_κ is club in $\kappa = \omega_3^{M_0[G]}$ and has order type at most ω_2. On the other hand, the regularity of κ is preserved from M_0 to $M_0[G]$, as $|\mathbb{P}_{\max}| = \mathfrak{c}$. $\square$

The following question should have a negative answer.

Question 4.4. Does forcing with $\mathrm{Add}(1, \Theta)$ over the model $V[G]$ in Theorem 4.3 force $\square(\omega_3)$?

5. Choice without square

Theorem 5.1. *Assume* $\mathsf{AD}_\mathbb{R} + V = L(\Gamma, \mathbb{R}) +$ "*S is stationary*", *where*

(3) $\quad S = \{\kappa : \kappa$ *is regular in* HOD *and has cofinality* ω_1 *in* $V\}$.

Let G be a $\mathbb{P}_{\max} * \mathrm{Add}(1, \Theta)$ *generic. Then* $V[G]$ *thinks* $\mathsf{MM}^{++}(\mathfrak{c})$ + "$\square_{\omega_2}$ *fails.*"

Theorem 5.2. *Assume* $\mathsf{AD}_\mathbb{R} + V = L(\Gamma, \mathbb{R}, \mu)$, *where μ is a Θ-complete ultrafilter concentrating on the set*

(4) $\quad S = \{\kappa : \kappa$ *is regular in* HOD *and has cofinality* ω_1 *in* $V\}$

is stationary. Let $A = \mathcal{P}(\Theta)$ *and let* $W = L(\Gamma, A, \mathbb{R})$. *Let G be a* $\mathbb{P}_{\max} * \mathrm{Add}(1, \Theta)$*-generic for W. Then* $W[G]$ *thinks* $\mathsf{MM}^{++}(\mathfrak{c}) + \square(\omega_3)$ *fails.*

Finally, $\square_{\omega,\omega_2}$ fails in Theorems 4.3 and 5.1. (5.1 and 5.2 ?)

6. Paul's remembrances

Lemma 6.1. *Suppose that $M_0 \subseteq M_1$ are models of AD^+ with the same reals such that*

- $M_1 \models$ "*Θ is regular*";
- Θ^{M_0} *has cofinality at least ω_2 in M_1.*

Let $G \subset \mathbb{P}_{\max}$ be M_1-generic. Then $\mathrm{Add}(1, \omega_3)^{M_0[G]}$ *is closed under ω_1-sequences in $M_1[G]$.*

Proof. Recall that $\omega_3^{M_0[G]} = \Theta^{M_0}$. We need that $cf(\Theta^{M_0}) = \omega_2$ in $M_1[G]$. For this we need that $cf(\Theta^{M_0}) \geq \omega_2$ in M_1, and Woodin's covering theorems for $\mathbb{P}_{\max}$ (for instance, Theorem 3.45 of [Woo10]). Let $\langle p_\alpha : \alpha < \omega_1\rangle$ be a sequence in $M_1[G]$ consisting of conditions in $\mathrm{Add}(1,\omega_3)^{M_0[G]}$. Since $cf(\Theta^{M_0}) = \omega_2$ in $M_1[G]$ we may fix a $\gamma < \Theta^{M_0}$ such that each p_α is a subset of γ. Since ω_2-DC holds in $M_1[G]$ by Theorem 3.9 of [Woo10], we may find in $M_1[G]$ a sequence $\langle \tau_\alpha : \alpha < \omega_1\rangle$ consisting of $\mathbb{P}_{\max}$-names in M_0 such that $\tau_{\alpha,G} = p_\alpha$ for all $\alpha < \omega_1$. Via a prewellordering R of the reals of length γ in M_0 we may assume that each τ_α is coded by R and a set of reals S_α in M_0, in such a way that $\langle S_\alpha : \alpha < \omega_1\rangle \in M_1[G]$. Letting $\eta < \Theta^{M_0}$ be a bound on the Wadge ranks of the S_α's, we have that the sequence $\langle S_\alpha : \alpha < \omega_1\rangle$ is coded by a single set of reals E in M_0 and an ω_1-sequence of reals in $M_1[G]$. Finally, since $\mathbb{R}^{\omega_1} \cap M_0[G] = \mathbb{R}^{\omega_1} \cap M_1[G]$ (see [Woo10, Theorem 9.32]) we have that $\langle p_\alpha : \alpha < \omega_1\rangle \in M_0[G]$. $\square$

Definition 6.2. Given cardinals κ and λ, the principle $\square_{\kappa,\lambda}$ says that there exists a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa^+\rangle$ such that for each $\alpha < \kappa^+$,

- $|\mathcal{C}_\alpha| \leq \lambda$;
- each element of $\mathcal{C}_\alpha$ is a closed cofinal subset of α of ordertype at most κ;
- for each limit point β of each member C of $\mathcal{C}_\alpha$, $C \cap \beta \in \mathcal{C}_\beta$;

Definition 6.3. Given cardinals κ and λ, the principle $\square(\kappa,\lambda)$ says that there exists a sequence $\langle \mathcal{C}_\alpha : \alpha < \kappa\rangle$ such that

- for each $\alpha < \kappa$,
  - $|\mathcal{C}_\alpha| \leq \lambda$;
  - each element of $\mathcal{C}_\alpha$ is a closed cofinal subset of α;
  - for each limit point β of each member C of $\mathcal{C}_\alpha$, $C \cap \beta \in \mathcal{C}_\beta$;
- there is no thread through the sequence, i.e., there is no closed unbounded $E \subseteq \kappa$ such that $E \cap \alpha \in \mathcal{C}_\alpha$ for every limit point α of E.

Theorem 6.4. *Suppose that $M_0 \subseteq M_1$ are models of* ZF *containing the reals such that, letting $\Gamma_0 = \mathcal{P}(\mathbb{R}) \cap M_0$, the following hold.*

- $M_0 \subseteq \mathrm{HOD}^{M_1}_{\Gamma_0}$;
- $M_1 \models \mathrm{AD}_{\mathbb{R}} +$ "Θ *is regular*";
- Θ^{M_0} *is on the Solovay sequence of M_1 and has cofinality at least ω_2 in M_1.*

Let $G \subset \mathbb{P}_{\max}$ be M_1-generic and let $H \subset \mathrm{Add}(1,\omega_3)^{M_0[G]}$ be $M_1[G]$-generic. Then $\square_{\omega_2}$ fails in $M_0[G][H]$.

Proof. First note that $\mathsf{MM}^{++}(\mathfrak{c})$ holds in $M_1[G]$ by [Woo10, Theorem 9.39]. By [Woo10, Theorem 9.10], AD^+ holds in M_0.

It follows from Lemma 6.1 and the fact that $< \omega_2$-directed closed forcing of size at most $\mathfrak{c}$ preserves $\mathsf{MM}^{++}(\mathfrak{c})$ (see [Lar00]) that $\mathsf{MM}^{++}(\mathfrak{c})$ holds in the $\mathrm{Add}(1,\omega_3)^{M_0[G]}$-extension of $M_1[G]$, and thus that in this extension every coherent sequence of clubs indexed by Θ^{M_0} is threaded. Since Θ^{M_0} has cofinality at least ω_2 in the $\mathrm{Add}(1,\omega_3)^{M_0[G]}$-extension of $M_1[G]$, each such sequence has a unique thread, which means that there is a name for such a thread definable from any name for such a sequence.

Now suppose that τ is a $\mathbb{P}_{\max} * \mathrm{Add}(1,\omega_3)$-name in M_0 for a $\square_{\omega_2}$-sequence. Then, in M_0, τ is $OD(S)$, for some set of reals S. Since Θ^{M_0} is on the Solovay sequence of M_1, $\mathrm{HOD}^{M_1}_{\mathcal{P}(\mathbb{R})\cap M_0}$ has the same sets of reals as M_0. Since in M_1 there is a $(\mathbb{P}_{\max} * \mathrm{Add}(1,\omega_3))^{M_0}$-name for a thread through the realization of τ, and since this thread is unique, $\mathrm{HOD}^{M_1}_{\Gamma_0}$ also sees a name for a thread through the realization of τ. This gives a contradiction, since $\mathrm{HOD}^{M_1}_{\Gamma_0}$ having the same sets of reals as M_0 implies that $\mathrm{HOD}^{M_1}_{\Gamma_0}[G]$ thinks that $\mathrm{Add}(1,\omega_3)^{M_0[G]}$ is ω_2-closed and thus preserves ω_3. □

Definition 6.5. $\mathrm{SRP}^*(\omega_3)$ is the statement that there is a proper, normal, fine ideal $I \subseteq \mathcal{P}([\omega_3]^{\aleph_0})$ such that for all stationary $T \subseteq \omega_1$,

$$\{X \in [\omega_3]^{\aleph_0} \mid X \cap \omega_1 \in T\} \notin I,$$

and such that for all $S \subseteq [\omega_3]^{\aleph_0}$ is such that for all stationary $T \subseteq \omega_1$,

$$\{X \in S \mid X \cap \omega_1 \in T\} \notin I,$$

then there a set $Y \subseteq \omega_3$ such that

- $\omega_1 \subseteq Y$;
- $|Y| = \aleph_1$;
- $\mathrm{cf}(\sup(Y)) = \omega_1$;
- $S \cap [Y]^{\aleph_0}$ contains a club in $[Y]^{\aleph_0}$.

See also [Lar00].

Theorem 6.6. *Suppose that $M_0 \subseteq M_1$ are models of ZF with the same reals such that, letting $\Gamma_0 = \mathcal{P}(\mathbb{R}) \cap M_0$, the following hold.*

- $M_0 = \mathrm{HOD}^{M_1}_{\Gamma_0}$*;*
- $M_1 \models \mathrm{AD}_{\mathbb{R}} +$ "Θ *is regular*"*;*
- $\Theta^{M_0} < \Theta^{M_1}$*;*
- Θ^{M_0} *has cofinality at least ω_2 in M_1.*

Let $G \subset \mathbb{P}_{\max}$ be M_1-generic and let $H \subset \mathrm{Add}(1,\omega_3)^{M_0[G]}$ be $M_1[G]$-generic. Then the following hold in $M_0[G][H]$.

- $\neg\Box(\omega_3)$;
- *SRP*$^*(\omega_3)$;

Proof. The beginning of the proof, through the proof of $\neg\Box(\omega_3)$, is almost exactly like the proof of Theorem 6.4. We repeat it for the reader's convenience. Again, $\mathrm{MM}^{++}(\mathfrak{c})$ holds in $M_1[G]$, and AD^+ holds in M_0.

Again, $\mathrm{MM}^{++}(\mathfrak{c})$ holds in the $\mathrm{Add}(1,\omega_3)^{M_0[G]}$-extension of $M_1[G]$, so in this extension every coherent sequence of clubs indexed by Θ^{M_0} is threaded. Since Θ^{M_0} has cofinality at least ω_2 in the $\mathrm{Add}(1,\omega_3)^{M_0[G]}$-extension of $M_1[G]$, each such sequence has a unique thread, which means that there is a name for such a thread definable from any name for such a sequence.

Now suppose that τ is a $\mathbb{P}_{\max} * \mathrm{Add}(1,\omega_3)$-name in M_0 for a $\Box_{\omega_2}$-sequence. Then, in M_1, τ is ordinal definable from a finite subset of Γ_0. Since in M_1 there is a $(\mathbb{P}_{\max} * \mathrm{Add}(1,\omega_3))^{M_0}$-name for a thread through the realization of τ, and since this thread is unique, $\mathrm{HOD}^{M_1}_{\mathcal{P}(\mathbb{R})\cap M_0}$ also sees a name for a thread through the realization of τ.

That $\mathrm{SRP}^*(\omega_3)$ holds in $M_0[G]$ follows from the fact that the nonstationary ideal on $[\Theta^{M_0}]^{\aleph_0}$ as defined in $M_1[G][H]$ is an element of $M_0[G][H]$, and the facts that $|\Theta^{M_0})|^{M_1[G][H]} = \aleph_2$ and $M_1[G][H]$ satisfies $\mathrm{SRP}(\omega_2)$.

$\Box$

Getting the failure of $\Box_{\omega_1,\omega_2}$ seems to require something like the hypothesis of 5.1. We seems to need more than the nonstationary ideal of the big model being an element of the small model. We need that it is closed under intersections of size $\aleph_2$ in the small model. Supposing this, the argument should go as follows. Suppose that $\bar{C} = \{C_\alpha : \alpha < \omega_3\}$ is such a sequence. It suffices to show that some member of some C_α is extended by a unique thread in $V[G][H]$. To show this is suffices to show that $\bar{C}$ has at most $\aleph_1$ many threads in $V[G][H]$.

To see this, let F be the restriction of the club filter on Θ^{M_0} in $V[G][H]$ to $M_0[G][H]$. $M_0[G][H]$ can define C'_α to be the set of elements of C_α which are extended by members of F-many C_β's. Since F is ω_2-complete, and since each C_α has size at most $\aleph_1$, there is no ω_2-sized antichain among the C'_α's. But if we take an elementary submodel X in $M_0[G][H]$ whose intersection with Θ^{M_0} has cofinality ω_2, we get paths through $X \cap \bar{C}$, and the branches off these paths give antichains of size $\aleph_2$.

7. Emails

For background on the core model induction see [Sar09, SS]

7.1. **Grigor.** What we have is the following. Let $j\colon L(\Gamma_0,\mathbb{R})\to L(\Gamma_1,\mathbb{R})$ be our embedding, μ be the $L(\Gamma_0,\mathbb{R})$-measure derived from j, G be the $\mathbb{P}_{\max}$ generic and H be $L(\Gamma_0,\mathbb{R})[G]$-generic for $\mathrm{Col}(\omega_3,\mathcal{P}(\mathbb{R}))$. What we showed is that if C in $L(\Gamma_0,\mathbb{R})[G][H]$ is any coherent sequence of length ω_3 it has a thread in

$$L(\Gamma_0,\mu,\mathbb{R})[G][H].$$

If we try to close our final model under such μ's we should get the failure of $\Box(\omega_3)$.

I think the following hypothesis does what we want.

Theorem 7.1. *Suppose that $V = L(A,\mathbb{R})$ for A a subset of reals, AD^+ holds, $w(A)$ is on the Solovay sequence, $L(\mathcal{P}_{w(A)}(\mathbb{R}),\mathbb{R})$ satisfies $\mathsf{AD}_{\mathbb{R}}+\Theta$ is regular, and the following holds. For each β let O_β be the OD subsets of θ_β. Then there is then a $\beta<w(A)$ such that there is*

$$j\colon L(\mathcal{P}_\beta(\mathbb{R}),O_\beta,\mathbb{R})\to L(\mathcal{P}_{w(A)}(\mathbb{R}),O_{w(A)},\mathbb{R}) \tag{5}$$

with critical point θ_β. Then $\Box(\omega_3)$ fails ....

Another which I haven't fully verified is the following. Let G be the $\mathbb{P}_{\max}$ generic and H be $\mathrm{Col}(\omega_3,\mathcal{P}(\mathbb{R}))$-generic over $M_0[G]$. Then whenever B substeq of θ_β in $M_0[G][H]$ then there is N in O_β as above such that $B\in L(\mathcal{P}_\beta,N,\mathbb{R})$.

Given this last claim we finish as before. Suppose $\vec{C}=\langle C_\alpha:\alpha<\omega_3\rangle$ in $M_0[G][H]$ is a non-threadable sequence. There is N in O_β such that $\vec{C}\in L(\mathcal{P}_\beta(\mathbb{R}),N,\mathbb{R})$. Let mu be $L(\mathcal{P}_\beta(\mathbb{R}),N,\mathbb{R})$-measure derived from j. Then mu is OD in M_1 and hence mu is in M_0. By our proof, $L(\mathcal{P}_\beta(\mathbb{R}),\mu,N,\mathbb{R})[G][H]$ sees the thread of $\vec{C}$. But μ is in M_0. Hence, $M_0[G][H]$ sees our thread.

I think what suffices is exactly what we needed to get that square fails in the $\mathbb{P}_{\max}$ extension, namely $\mathsf{AD}_{\mathbb{R}}+\Theta=\theta_{\alpha+\omega}+\theta_\alpha$ has cof $\omega_2+\theta_\alpha$ is regular in HOD. Again the name of the thread is OD from the name of the sequence and so we can repeat the argument.

(I actually think $\mathsf{AD}_{\mathbb{R}}$ at the top is not needed and $\Theta=\theta_{\alpha+1}$ would suffice).

(John: Yes, you're right. One has to check that there is a thread in $V[G][H]$, which is why you originally wanted $\mathsf{AD}_{\mathbb{R}}$ in V, I guess. But I guess you don't need full $\mathsf{MM}(\mathfrak{c})$ in $V[G]$, because the $P*Q$-dot to which you apply $\mathsf{MM}(\mathfrak{c})$ in $V[G]$ (as part of the argument by contradiction) is in $L(\mathcal{P}_\kappa(\mathbb{R}))$. Thus $P*Q$-dot, and the relevant dense sets, are given by terms which are Suslin-co-Suslin in V. So $\mathsf{MM}(\mathfrak{c})$ holds for $P * Q$-dot in $V[G]$.)

Question 7.2. Does the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$ satisify $\mathsf{MM}(\mathfrak{c})$ with respect to posets and sequences of dense sets given by Suslin-co-Suslin terms?[3]

7.2. **John.**

Theorem 7.3. *Assume $\mathsf{AD}_{\mathbb{R}}$ plus θ regular plus there is λ s.t. for arbitrarily large $\kappa < \lambda$ in the Solovay sequence, there is an elementary embedding*

$$j\colon \mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})} \to \mathrm{HOD}_{\mathcal{P}_\lambda(\mathbb{R})}, \tag{6}$$

with κ regular. Let $M_1 = HOD_{\mathcal{P}_\lambda(\mathbb{R})}$.

We claim that $M_1[G][H]$ models "ω_3 is threadable", where G is for $\mathbb{P}_{\max}$ and H is for $\mathrm{Col}(\Theta, \mathcal{P}(\mathbb{R}))^{V[G]}$.

Actually, the assumption that there is such an embedding is an overkill, and an appropriate instance of Π^1_1-reflection suffices (this is revisited in the last theorem before subsection Grigor 4).

(This isn't quite what you wrote, as your O_β doesn't allow sets of reals of Wadge rank less than β as parameters.)

Proof. Suppose not, and let τ be such that τ_{G*H} is an unthreadable coherent sequence. We may assume $\tau \subseteq \mathcal{P}(\mathbb{R})^V$, and $\tau \in M_1$. Pick κ and j as in our hypothesis such that τ is $\mathrm{OD}(A)$ for some A in $\mathcal{P}_\kappa(\mathbb{R})$. Let

$$M_0 = \mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}, \text{ and } j(\sigma) = \tau. \tag{7}$$

Let H be $\mathrm{Col}(\kappa, \mathcal{P}(\mathbb{R}))^{M_0[G]}$ generic over $M_1[G]$. The key argument shows σ_{G*H} is threaded in $M_1[G][H]$. But the thread is unique! (Which is where the attempt to prove stationary reflection comes to grief.) So there is a term ρ for the thread such that ρ is $\mathrm{OD}(\sigma)$ in M_1. (ρ is the forcing relation for the unique thread of τ.) Since ρ is $\mathrm{OD}(\sigma)^{M_1}$, ρ in M_0.

Hence $M_0[G][H]$ models σ_{G*H} is threadable. This contradicts the existence of j. □

I think we can just work with $\lambda = \Theta$ and $M_1 = V$ in the above.

The hypothesis above is well below LST, right? It's just a little past $\mathsf{AD}_{\mathbb{R}} + \Theta$ reg., isn't it?

[3]Don't you need a normal fine measure on $\mathcal{P}_{\aleph_1}(\mathbb{R})$? There is something relevant in the $\mathbb{P}_{\max}$ book, in the weak and strong reflections principles section.

7.3. **John 2.** In going over the proof myself, it seemed we used $\mathsf{AD}_{\mathbb{R}}$ + Θ Mahlo + $\mathcal{P}(\mathbb{R})^{\#}$ exists. I didn't see how to make do with Θ regular only.

With Θ Mahlo, I think it works. You don't need to extend j.

Proof. Suppose $L(\mathcal{P}(\mathbb{R}))$ models "if you force with $\mathbb{P}_{\max}$, then $\mathrm{Col}(\Theta, \mathcal{P}(\mathbb{R}))$, then $\square_{\omega_2}$ holds". Take τ in $L(\mathcal{P}(\mathbb{R}))$ a name for a square sequence in the final model. τ is $\mathrm{OD}(A)$ for some $A \subseteq \mathbb{R}$. Now let κ regular be in the Solovay sequence and let

$$j\colon L(\mathcal{P}_\kappa(\mathbb{R})) \to L(\mathcal{P}(\mathbb{R})) \tag{8}$$

be an elementary embedding, with $j(\sigma) = \tau$. I think you really need κ regular in V, not just $cof(\kappa) \geq \omega_2$ in V, so you really need Θ Mahlo.

Because of j, $L(\mathcal{P}_\kappa(\mathbb{R}))$ models "if you force with $\mathbb{P}_{\max} * \mathrm{Col}(\Theta, \mathcal{P}_\kappa(\mathbb{R}))$, then τ_{G*H} is a $\square_{\omega_2}$ sequence"

The rest of the argument gets a contradiction from this: you get an $\mathrm{OD}(\sigma)$ term ρ for a thread, then consider

$$L(\mathcal{P}_\kappa(\mathbb{R}), \rho)[G][H].$$

κ is ω_3 here because it started out regular in $L(\mathcal{P}_\kappa(\mathbb{R}), \rho)$. [Here's where you use Θ Mahlo.] On the other hand, the thread collapses κ. Contradiction. $\square$

....

What works is $\mathsf{AD}_{\mathbb{R}}$ plus "there are stat many κ in the Solovay sequence such that $\mathrm{cf}(\kappa) = \omega_2$ and κ is regular in $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$". (You just had one such κ, which doesn't seem enough.)[4]

I think that $\mathsf{AD}_{\mathbb{R}}$ + there are stationary many κ in the Solovay sequence such that κ is regular in $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$ is equivalent to just $\mathsf{AD}_{\mathbb{R}}$ plus "Θ is Mahlo in HOD".

Proving square below this point looks trickier. You can get square in the extension if you have club many HOD-singulars, I'm sure.[5]

7.4. **Grigor 3 : The iteration.** Lets start with $\mathsf{AD}_{\mathbb{R}} + \mathcal{P}(\mathbb{R})^{\#} + \Theta$ is regular + S is stationary, where S is the set of κ which are regular in HOD and a member of the Solovay sequence with cofinality ω_1. We claim that $\mathbb{P}_{\max}$ followed by a forcing that wellorders $\mathcal{P}(\omega_2)$ forces the failure of $\square_{\omega_2}$.

Let A be the members of the Solovay sequence of cofinality ω_2. We consider $\mathbb{P}_{\max} * Q$ where Q is the direct limit iteration of $\mathrm{Add}(1, \lambda)$'s for λ in A. We take inverse limits everywhere except at cof ω_2 and at

[4]Why not?

[5]Our square expert should prove this!

the end. So Q is ω_1-closed. Q preserves $\mathsf{MM}(\mathfrak{c})$ as any subset of ω_2 is added by an initial fragment of Q which is ω_1-closed.

Suppose now that $\mathbb{P}_{\max} * Q$ forces that σ is a $\square_{\omega_2}$ sequence. There is $B \subseteq \mathbb{R}$ such that σ is $\mathrm{OD}(A)$. We can find κ in S such that $\sigma \restriction \kappa$ is a $\mathbb{P}_{\max} * (Q \restriction \kappa)$ name for a square sequence and $\kappa > w(A)$. Let $P_\kappa = Q \restriction \kappa$. We claim that P_κ forces that $\sigma \restriction \kappa$ is threaded (we cannot use $\mathsf{MM}(\mathfrak{c})$ here anymore). But P_κ forces that the rest of Q is ω_1-closed and hence if it adds a thread to $\sigma \restriction \kappa$ it has to be in the ground model (this has to follow from the ω_1 covering). Notice that we again have a unique thread for $\sigma \restriction \kappa$. Let then τ be a name for the thread. Both τ and $\sigma \restriction \kappa$ are in $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$ and hence, $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}^{\mathbb{P}_{\max} * P_\kappa}$ collapses κ while it has to be ω_3 in there.

(John : I don't see why $\mathsf{MM}(\mathfrak{c})$ is preserved by Q. It adds new posets of size c. Why does $\mathsf{MM}(\mathfrak{c})$ hold for them?)

Maybe there is a problem for $\mathsf{MM}(\mathfrak{c})$ but as far as the failure of $\square(\omega_2)$ goes, we are ok, I think. Here is the proof.

Suppose $A \subseteq \omega_2$ codes a nonthreadable ω_1-coherent sequence after forcing with Q. Q preserves ω_3 (by a chain condition argument, I think). So if G is Q generic then A is decided by a set of condition of size ω_2 which cannot be cofinal in ω_3, i.e., if $B \subseteq G$ is this set of conditions deciding A then there is $\alpha < \omega_3$ such that for each p in B, support(p) is contained in α. So A is added by $Q \restriction \alpha$. But $Q \restriction \alpha$ is ω_1 directed closed and therefore it preserves $\mathsf{MM}(\mathfrak{c})$ and hence A has to be threaded after forcing $Q \restriction \alpha$.

For $\mathsf{MM}(\mathfrak{c})$, you are right, some posets might be stationary set preserving at the end but not initially. Can this actually happen?[6]

(John : The existence of a κ such that $\sigma \restriction \kappa$ is a $\mathbb{P}_{\max} * (Q \restriction \kappa)$-name follows from Q taking direct limits at cof $\geq \omega_2$ points, doesn't it?)

I now think that it is actually fine. My worry was the following.

It was an attempt to reduce the hypo, and we only had cof ω_1 or cof omega point of Solovay sequence regular in HOD. If we take inverse limit at cof ω_1 points then I thought this collapses cardinals when done locally. If κ is this reflection point which has cof ω_1 but is reg in HOD then, I thought, in $L(\mathcal{P}_\kappa(\mathbb{R}))^{Q \restriction \kappa}$, κ is most likely singular of cof ω_1 while we want it to be regular.

However, this cannot happen. $Q \restriction \kappa$ is ω_1 closed in $L(\mathcal{P}_\kappa(\mathbb{R}))^{Q \restriction \kappa}$. And so it can only change κ's cof to ω_2. But κ has cof ω_1 in the bigger model, so this cannot happen.

The part that is problematic now is that the bigger model has more ω_1 sequences than $L(\mathcal{P}_\kappa(\mathbb{R}))$, so $Q \restriction \kappa$ from the smaller models point of

[6]Not unless you destroy stationary sets.

view isn't the same as the one from the bigger models point of view. However, the $Q\restriction\kappa$ from the smaller models point of view is still ω_1 closed by Hugh's covering argument, no?

Let P be $Q\restriction\kappa$ of the smaller model. Then forcing with P over the bigger model adds a thread. Then we finish as before.

(John : But adding new sets of reals when you add back choice looks problematic to me.)

(John : I don't think we need to use your iteration Q to wellorder $\mathcal{P}(\mathbb{R})$, just the $\mathrm{Add}(1,\omega_3)$ poset we've been using. (Yours probably works too.))

7.5. **John 3.**

Theorem 7.4. *Assume that $\mathsf{AD}_{\mathbb{R}}$ holds, and that stationarily many elements κ of cofinality ω_1 in the Solovay sequence of $L(\mathcal{P}(\mathbb{R}))$ are regular in $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$. Then in the $\mathbb{P}_{\max} * \mathrm{Add}(1,\omega_3)$-extension of $L(\mathcal{P}(\mathbb{R}))$ there is no partial $\square_{\omega_2}$-sequence defined on all points of cofinality at most ω_1.*

Proof. Suppose that τ is a $\mathbb{P}_{\max} * \mathrm{Add}(1,\omega_3)$-name in $L(\mathcal{P}(\mathbb{R}))$ whose realization is forced by some condition p_0 to be a $\square_{\omega_2}$-sequence. We may assume (by using the least ordinal parameter defining a counterexample to the theorem) that τ is given by a subset of $\mathcal{P}(\mathbb{R})$, and that (τ, p_0) is $\mathrm{OD}(A)$ in $L(\mathcal{P}(\mathbb{R}))$ for some $A \subseteq \mathbb{R}$. Using our hypothesis, we get $\kappa < \Theta$ with A in $\mathcal{P}_\kappa(\mathbb{R})$, and ordinals ξ_0 and ξ_1 such that

(1) $\kappa < \xi_0 < \Theta < \xi_1$;
(2) κ is an element of the Solovay sequence of $L(\mathcal{P}(\mathbb{R}))$ of cofinality ω_1;
(3) κ regular in $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$;
(4) $p_0 \in L_{\xi_0}(\mathcal{P}_{\xi_0}(\mathbb{R}))$;
(5) every element of $L_{\xi_0}(\mathcal{P}_{\xi_0}(\mathbb{R}))$ is definable in $L_{\xi_0}(\mathcal{P}_{\xi_0}(\mathbb{R}))$ from a set of reals in $\mathcal{P}_{\xi_0}(\mathbb{R})$;
(6) in $L_{\xi_1}(\mathcal{P}(\mathbb{R}))$, τ is a $\mathbb{P}_{\max} * \mathrm{Add}(1,\omega_3)$-name whose realization is forced by p_0 to be a $\square_{\omega_2}$-sequence;
(7) there exist $\sigma \in L_{\xi_0}(\mathcal{P}_\kappa(\mathbb{R}))$ and elementary embedding
$$j \colon L_{\xi_0}(\mathcal{P}_\kappa(\mathbb{R})) \to L_{\xi_1}(\mathcal{P}(\mathbb{R}))$$
with critical point κ such that $j(\sigma) = \tau$.

Let $M_0 = L_{\xi_0}(\mathcal{P}_\kappa(\mathbb{R}))$ and $M_1 = L_{\xi_1}(\mathcal{P}(\mathbb{R}))$. Let G be $\mathbb{P}_{\max}$ generic over M_1, containing the first coordinate of p_0. Then j lifts to
$$j \colon M_0[G] \to M_1[G]$$
and $j(\mathrm{Add}(1,\kappa)^{M_0[G]}) = \mathrm{Add}(1,\Theta)^{M_1[G]}$. Note that $\mathrm{Add}(1,\kappa)^{M_0[G]}$ is ω-closed in $M_1[G]$.

Claim 7.5. *If H is $M_1[G]$-generic over* $\mathrm{Add}(1,\kappa)^{M_0[G]}$*, then σ_{G*H} has a thread in $M_1[G][H]$.*

The claim leads to a contradiction as the thread, being unique, would be in $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}[G][H]$, which would mean that κ is collapsed in this model, which is impossible as κ is regular in $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$. It suffices then to prove the claim.

Now suppose that the claim were false. By the homogeneity of $\mathrm{Add}(1,\kappa)^{M_0[G]}$, it suffices to consider the case where H contains the second coordinate of p_0. Let C be club in κ, $o.t.(C) = \omega_1$, and C in M_1. By the Coding Lemma, C in $L(B,\mathbb{R})$ where $w(B) = \kappa$. If H is $M_1[G]$ generic as above, then σ_{G*H} is a coherent sequence of length κ with no thread in $M_1[G]$. We can fix $(\mathbb{P}_{\max} * \mathrm{Add}(1,\kappa)^{M_0[G]})$-names ρ and ψ in $L(B,\mathbb{R})$ such that ρ_{G*H} be the tree of attempts to build a thread through σ_{G*H} along C, and ψ_{G*H} the poset which specializes ρ_{G*H}. Since σ_{G*H} has no thread in $M_1[G][H]$, $\mathrm{Add}(1,\kappa)^{M_0[G]} * \psi$ is ω-closed*c.c.c., and thus proper, in $M_1[G]$.

Subclaim 7.6. *In $M_1[G]$, there are H, f such that*

- *H is* $\mathrm{Add}(1,\kappa)^{M_0[G]}$*-generic over $M_0[G]$, and*
- *f specializes ρ_{G*H}.*

Proof of Subclaim. In $M_0[G]$, $\mathrm{Add}(1,\kappa)$ is $<\kappa$-closed. By item (5) above, the definition of the Solovay sequence, and the fact that κ has cofinality ω_1 in M_1, $M_0[G]$ can be decomposed in $M_1[G]$ into $\aleph_1$ many sets, each an element of $M_0[G]$, and each of cardinality less than κ in $M_0[G]$. Let $\{E_\eta : \eta < \omega_1\}$ be such a decomposition, and, for each $\eta < \omega_1$, let D_η be a dense subset of $\mathrm{Add}(1,\kappa)^{M_0[G]}$ refining all such sets in E_η.

Since $\mathsf{PFA}(\mathfrak{c})$ holds in $M_1[G]$, there is a filter $H*K$ on $\mathrm{Add}(1,\kappa)^{M_0[G]} * \psi$ in $M_1[G]$ such that H meets all the D_η's, and $H*K$ meets the dense sets which will guarantee that K determines a specializing function f for ρ_{G*H}. That gives the subclaim.

Let H and f be as in the subclaim. Then in $M_1[G]$, H is a condition in $\mathrm{Add}(1,\Theta)$. We can therefore find a generic H_1 over $M_1[G]$ so that j lifts to

$$j\colon M_0[G][H] \to M_1[G][H_1].$$

But then σ_{G*H} has the thread $(\tau_{G*H_1})_\kappa$. But f in $M_1[G]$, so ω_1 was collapsed by going to $M_1[G][H_1]$, contradiction. $\square$

(Regarding Theorem 5.2 above):

This isn't the theorem for Con($\mathsf{ZFC}+\mathsf{MM}(\mathfrak{c})+\omega_3$ is threadable) that I remember. In fact, I don't see why $W[G]$ models choice, if you aren't doing anything to wellorder $\mathcal{P}(\Theta)$.

I thought the result we had was: Con(AD^+ plus some $\kappa < \Theta$ in the Solovay sequence has cof ω_2 in V and is regular in HOD) implies Con($\mathsf{ZFC}+\mathsf{MM}(\mathfrak{c})+\omega_3$ is threadable). I'm not sure how the hypo here compares with that in Theorem 3 above. I'm not sure how you prove Theorem 3 above.

Aside 7.7. Isn't "ω_3 is threadable" equiconsistent with a weakly compact? The above is stronger than $\mathsf{AD}_\mathbb{R}$ plus "Θ is weakly compact in HOD". It holds if you cut off a bigger HOD at some element of the Solovay sequence which is measurable in HOD of Mitchell order one. (= two measures).

Definition 7.8. The Π^1_1-indescribability ideal is the ideal I on Θ defined by : $B \in I$ if and only if there exist a Π^1_1 sentence ϕ and $A \subseteq \Theta$ such that $(V_\Theta, A) \models \phi$ but for no $\beta < \Theta$ in B do we have $(V_\beta, A\cap\beta) \models \phi$.

Theorem 7.9. *Suppose that $\mathsf{AD}_\mathbb{R}$ holds, and that in* HOD *the set of $\alpha < \Theta$ which are measurable is I-positive, where I is the Π^1_1-indescribability ideal. Then in the $\mathbb{P}_{\max} * \ldots$ extension we get* ZFC + $\mathsf{MM}(\mathfrak{c})$ + *"ω_3 is threadable".*

The point is that you don't need to extend embeddings, you can just use the reflection given by the set of measurables being I-positive in HOD. (And of course, that every measurable of HOD has cof $\geq \omega_1$ in V.)

The corresponding hypo for ZFC + $\mathsf{MM}(\mathfrak{c})$ + "ω_3 is square-inaccessible" (i.e. $\neg\square_{\omega_2}$) would be $\mathsf{AD}_\mathbb{R}$ + Θ regular + HOD $\models$ "$\{\alpha < \theta | \alpha$ is measurable$\}$ is stationary."

7.6. **Grigor 4: The hierarchy.** Yes, in fact we seem to get a hierarchy of results. One in which there is no partial square sequence at cof ω_1's and another (from a stronger hypo) no partial square sequence at cof ω_2's.

The first is Theorem 7.4 above.

The second is

Theorem 7.10. *Assume $\mathsf{AD}_\mathbb{R}$ + Θ is regular + "S is stationary", where S is the set of members of the Solovay sequence of cofinality ω_2*

which are regular in HOD*. Then there is no partial square sequence at cof ω_2's (in the usual extension).*

I am not sure about the hypo for cof ω_2 points. It should probably be for partial $\square(\omega_3)$-sequences on cof ω_2's

(Two comments on Theorem 4.3.)

The hypo for the ZF result can be weakened further to the following.

Theorem 7.11. *Assume V is the minimal model of $\mathsf{AD}_{\mathbb{R}} + \Theta$ is Mahlo in HOD. Then in the $\mathbb{P}_{\max}$ extension $\square_{\omega_2}$ fails.*

Below this hypo we have $\square_{\omega_2}$ (or should have).

7.7. **Grigor 5: Let's go on.** Lets work with the following hypo which I think should hold a bit past LST+ there is a Woodin in HOD which is between the largest Suslin and Θ.

So the fantasy below is a bit different from John's fantasy.

First some notation. Assume AD^+ and λ be a member of the Solovay sequence. Let A be a set. We let

$$K_{A,\lambda} = HOD_{A^{\omega_1}}|(\lambda^+)^{HOD_{A^{<\omega_1}}}$$

We define a stack on HOD$|\lambda$. This is a sequence $\langle N_\xi : \xi \leq \omega_2\rangle$ defined as follows.

(1) $N_0 = K_{\lambda,\lambda}$
(2) $N_{\alpha+1} = K_{N_\alpha,\lambda}$
(3) $N_\nu = \cup_{\alpha<\nu} N_\alpha$ for nu limit.

We let $M_\lambda = N_{\omega_2}$.

Here is the proposed theorem.

Theorem 7.12. *Assume $\mathsf{AD}_{\mathbb{R}}$ plus there is κ on the Solovay sequence such that HOD_{N_κ} thinks κ is regular and it has new sets of reals. Let G be $\mathbb{P}_{\max}$ generic, H be $\mathrm{Add}(1,\kappa)$ generic and let K be $\mathrm{Add}(1,\kappa^+)$ generic. Then $\square_{\omega_3}$ fails in $\mathrm{HOD}_{N_\kappa}[G][H][K]$*

Proof. Let $W = \mathrm{HOD}_{N_\kappa}$.

Claim 7.13. $\mathrm{cf}((\kappa^+)^W) \geq \omega_2$ *in V.*

Proof of claim. First assume that the sequence didn't stabilize below ω_2, i.e., all N_ξ's are different. Then if $\mathrm{cf}((\kappa^+)^W) < \omega_2$ then we must have that there is $A \subseteq \kappa$ such that A is in W but not in N_κ. But such an A is OD from N_κ and some $b \in N_\kappa$ and hence A is in some N_ξ. So it must be that the sequence stabilized. Let then N_ξ be such that $N_{\xi+1} = N_\xi$. But then $(\kappa^+)^W$ is equal to the ordinal height of N_ξ and therefore we couldn't have $\mathrm{cf}(\kappa^+)^{L(N_\kappa)} < \omega_2$.

Suppose now that square ω_3 holds in $L(N_\kappa)[G][H][K]$. Notice that ω_4 of the extension is $(\kappa^+)^W$. Let λ be the ω_4 of the extension. Let ρ in W be a name for $\square_{\omega_3}$ in $W^{\mathbb{P}_{\max}*\mathrm{Add}(1,\kappa)*\mathrm{Add}(1,\lambda)}$.

We claim that in $V[G]$, $\mathrm{Add}(1,\kappa)*\mathrm{Add}(1,\lambda)$ forces that there is a thread for ρ. This follows from the fact that $\mathrm{Add}(1,\kappa)*\mathrm{Add}(1,\lambda)$ is ω_1 closed in $V[G]$ and because $\mathsf{MM}(\mathfrak{c})$ holds in $V[G]$, so forcing by $\mathrm{Add}(1,\kappa)*\mathrm{Add}(1,\lambda)$ preserves $\mathsf{MM}(\mathfrak{c})$.

Now let τ be a $\mathbb{P}_{\max}*\mathrm{Add}(1,\kappa)*\mathrm{Add}(1,\lambda)$ name for the thread. Then τ is $\mathrm{OD}(\sigma)$ and hence τ is in W. But then ω_4 is collapsed in $W[G][H][K]$. □

I just thought of a way of doing this when closing under omega sequences using John's idea. Of course we can keep proving $\square_{\omega_4}$ fails and etc. All we need now is to prove that this hypos are consistent.

I don't see how to do it by just closing under ω-sequences. The write-up below goes up to the place where I got stuck, if it is of any interest.

Also, for the other argument, I just realized that one needs to prove that $\mathrm{Add}(1,\omega_4)$ is ω_1 closed in the bigger model, but this should come from closing under ω_1 sequences.

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First some notation. Assume AD^+ and λ be a member of the Solovay sequence. Let A be a set. We let

$$K_{A,\lambda} = HOD_{A^\omega}|(\lambda^+)^{HOD_{A^{<\omega}}}$$

We define a stack on $\mathrm{HOD}|\lambda$. This is a sequence $\langle N_\xi : \xi \leq \omega\rangle$ defined as follows.

(1) $N_0 = K_{\lambda,\lambda}$
(2) $N_{\alpha+1} = K_{N_\alpha,\lambda}$
(3) $N_\nu = \cup_{\alpha<\nu} N_\alpha$ for nu limit.

We let $M_\lambda = N_{\omega_1}$.

Here is the situation we need for this proof. Suppose we have κ and λ such that there

$$j\colon L(N_\lambda) \to L(N_\kappa) \tag{9}$$

j is V. We would get such things under the appropriate hypo. So we force over $L(N_\lambda)$ via $\mathbb{P}_{\max}*P*\mathrm{Add}(1,\omega_4)$. And we claim that $\square_{\omega_3}$ fails. P here is the iterated forcing for adding for well-ordering $\mathcal{P}(\mathbb{R})$. Suppose not.

Let ρ be a $\mathbb{P}_{\max}*\mathrm{Add}(1,\omega_3)*\mathrm{Add}(1,\omega_4)$-name for a square sequence in $L(N_\lambda)$. Let $\nu = \omega_4$ of $L(N_\lambda)$. Let G be $\mathbb{P}_{\max}$ generic and let H be P generic over $L(N_\kappa)[G]$.

The claim is that in $L(N_\kappa)[G][H]$, $\mathrm{Add}(1,\lambda)$ forces that there is a thread. If not, using the argument from John's email and using the fact that λ now has cof ω_1, we have K, f such that K in V is $L(N_\kappa)[G][H]$-generic and f specializes the tree of attempts for building a thread for ρ_{G*H*K}.

We can then lift j to

$$j^+ \colon L(N_\lambda)[G][H][K] \to L(N_\kappa)[G][j^+(H)][S] \tag{10}$$

because $j"K$ is a directed set of size ω_2 and the top forcing on N_κ side is ω_3-directed closed (Why?)— seems like supercompactness kind of embedding is needed.

Yes, I think the ω_1-closure of $\mathrm{Add}(1,\lambda)$ of that email follows from closed under ω_1-sequence.

So our situation is that we have $L(N_\kappa)$ and $\lambda = \omega_3$ of Pmax. We want to see that in V[G][H], $\mathrm{Add}(1,\lambda)$ of $L(N_\kappa)[G][H]$ is ω_1-closed.

Let then $f\colon \omega_1 \to \mathrm{Add}(1,\lambda)$ of $L(N_\kappa)[G][H]$ be a map.

for each $\alpha < \omega_1$ let A_α be the set of triples (r, A, β) such that

- r is a $\mathbb{P}_{\max}$ condition,
- A is a subset of reals coding a name for a condition in $Add(1,\kappa)$, and
- (r, A) forces that β is in $f(a)$.

Notice that A_α is in $L(N_\kappa)$ for each α. Hence, $\langle A_\alpha : \alpha < \omega_1\rangle$ is in there too. So f is in $L(N_\kappa)[G][H]$.

Why do we need $\mathrm{cf}(\omega_2)$? I think there was an argument on Monday after the square showing this. Here is the proof again which uses John's simplification.

Theorem 7.14. *Assume AD^+ + there is κ in Solovay sequence such that κ has Mitchell order 2. Let $W = \mathrm{HOD}(\mathcal{P}_\kappa(\mathbb{R}))$. Let $G * H$ be $\mathbb{P}_{\max} * \mathrm{Add}(1,\kappa)$ generic. Then $W[G][H]$ thinks that $\square(\omega_3)$ fails.*

Proof. First of all $\mathrm{cf}(\kappa) = \omega_1$. We also have that in W there are stationary many λ of cof ω_1 which are regular in HOD and there is

$$j\colon HOD(\mathcal{P}_\lambda(\mathbb{R}))|\lambda^+ \to HOD(\mathcal{P}_\kappa(\mathbb{R}))|\kappa^+ \tag{11}$$

If we reflect the bad σ to one such λ then since the thread is unique we get that it must be in $\mathrm{HOD}(\mathcal{P}_\lambda(\mathbb{R}))$, contradiction. $\square$

Assuming the bigger model, however, isn't necessary. Just assuming Θ is measurable via a measure concentrating on the set of κ of cofinality ω_1 which are regular in HOD is enough and the hypo of Thm 3. $\mathcal{P}(\Theta)$ is well-ordered because it is essentially a mouse over $\mathcal{P}(\mathbb{R})$, so well ordering it well-orders $\mathcal{P}(\Theta)$. Then we again can reflect the bad σ and by frequent extensions lemma lift the mice that give as the name for the thread.

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