AN INTRODUCTION TO AD^+

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This survey is based on notes taken by Luke Serafin, on lectures given by the author on June 4th, 6th, 11th and 13th 2019, at the Workshop on Set Theory, Recursion Theory and their Interaction, held at the Institute for Mathematical Sciences at the University of Singapore. It covers an introductory portion of the classical theory of Woodin's AD^+. There is significant overlap between the material presented here and in the paper [8]. The lectures were based on parts of the author's book in preparation [12].

1. Determinacy axioms and forms of Choice

Woodin's axiom AD^+ is a strengthening of the classical Axiom of Determinacy (AD), which is inconsistent with the Axiom of Choice (AC). We define AD below, but refer the reader to [3, 5, 11] for a more thorough introduction, including an account of the history of this axiom. Before giving the definition of AD^+ we will quickly review AD and its strengthening $\mathsf{AD}_{\mathbb{R}}$, and forms of the Axiom of Choice consistent with AD, such as DC and $\mathsf{Uniformization}$. We will also discuss structural consequences of determinacy, such as the Wadge hierarchy and Moschovakis Coding Lemma, and ways of defining subsets of the real line using sets of ordinals, such as Suslin representations and ∞-Borel codes. With these definitions in hand we define AD^+ in Section 3.3. In the same section we list some of the major open questions surrounding AD^+, and discuss some of the original motivation for its study. For now we note that part of the motivation for studying AD^+ is that it allows the lifting of some of the consequences of AD in $L(\mathbb{R})$ to larger models of determinacy.

Collectively these notes present preliminary material toward some of Woodin's major theorems of the 1990's (since then significant progress in the study of AD^+ has been made by applying the stationary tower and inner model theory; we do not discuss these results here). Some of these theorems are listed in Section 3.3. For the most part we sketch this material at a high level, giving short proofs when appropriate. Complete proofs of (most of) the theorems discussed here appear in the author's book.

Our base theory throughout the paper is ZF. Additional axioms will be stated as needed.

1.1. AD and $\mathsf{AD}_{\mathbb{R}}$.

Let X be an arbitrary set and $A \subseteq X^\omega$. In the game $\mathcal{G}_X(A)$, two players, I and II, alternate playing elements of X. Player I plays a first element $x(0)$, player II a second element $x(1)$, and so on, with a play at each natural number "time". The entire run of the game produces an $x \in X^\omega$. Player I wins if and only if $x \in A$ (so player II wins if $x \in X^\omega \setminus A$).

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$$\begin{array}{c|ccccccc} \mathrm{I} & x(0) & & x(2) & & x(4) & & \ldots \\ \hline \mathrm{II} & & x(1) & & x(3) & & \ldots & \end{array}$$

The game $\mathcal{G}_X(A)$; I wins if and only if x is in A.

A *strategy* is a function $\sigma : X^{<\omega} \to X$, which we think of as determining a move for a player given the history of the game up to the point where the strategy is applied. Given a strategy σ and an $x \in X^\omega$, $\sigma * x$ is the result a run of $\mathcal{G}_X(A)$ where player I plays according to σ and x lists player II's moves. That is, $(\sigma*x)(0) = \sigma(\langle\rangle)$ and, for all $k \in \omega$, $(\sigma * x)(2k+1) = x(k)$ and $(\sigma * x)(2k+2) = \sigma((\sigma * x) \restriction 2k+1$. The result of a run where x lists player I's moves and player II plays according to σ is defined analogously, and denoted $x * \sigma$. A strategy σ is *winning* for player I if and only if $\sigma * x \in A$ for all $x \in X^\omega$, and winning for player II if $x * \sigma \notin A$, for all such x. The game $\mathcal{G}_X(A)$ is said to be *determined* if and only if one of the players has a winning strategy.

We list below three forms of determinacy defined in terms of the framework just given. The first two are consistent with ZF, relative to the consistency of certain large cardinal axioms (see Theorems 1.1 and 1.2). The second implies the first, since an integer game can be coded by a game on ω^ω. The third is refuted by ZF, as we discuss at the beginning of Section 3.2.

- AD: $\mathcal{G}_\omega(A)$ is determined for all $A \subseteq \omega^\omega$
- $\mathsf{AD}_\mathbb{R}$: $\mathcal{G}_{\omega^\omega}(A)$ is determined for all $A \subseteq (\omega^\omega)^\omega$
- AD_{ω_1} : $\mathcal{G}_{\omega_1}(A)$ is determined for all $A \subseteq (\omega_1)^\omega$

We refer the reader to [3, 5] for discussion of the following standard consequences of AD. The first two of the these will be used in these notes, at least implicitly, but not the third.

- All sets of reals have the perfect set property.
- All sets of reals have the property of Baire.
- All sets of reals are Lebesgue measurable.

The assertion that all sets of reals have the perfect set property implies that there are no injections from ω_1 into ω^ω. We will write this latter statement as $\aleph_1 \not\leq 2^{\aleph_0}$. We will also make use of (generalizations of) Solovay's theorem that, under AD, ω_1 is a measurable cardinal (see [3, 5]).

Large cardinals imply the existence of inner models of AD and $\mathsf{AD}_\mathbb{R}$, as shown by the two following theorems (due to Martin, Steel and Woodin; see [14, 19, 10]).

Theorem 1.1. *If there are infinitely many Woodin cardinals with a measurable cardinal above them all, then $L(\mathbb{R}) \models \mathsf{AD}$.*

Theorem 1.2. *If there exists a cardinal κ which is a limit of Woodin cardinals and of strong-to-κ cardinals and there is a measurable cardinal $\lambda > \kappa$, then there is a $\Gamma \subseteq \mathcal{P}(\omega^\omega)$ such that $L(\Gamma, \mathbb{R}) \models \mathsf{AD}_\mathbb{R}$.*

Since strategies in integer games can be coded by real numbers, AD implies that AD holds in every inner model containing the reals, for instance, in the inner model $L(\mathbb{R})$. Similarly, since strategies in real games can be coded by sets of real numbers, $\mathsf{AD}_\mathbb{R}$ implies that $\mathsf{AD}_\mathbb{R}$ holds in every inner model containing $\mathcal{P}(\mathbb{R})$. However, a model of the form $L(A, \mathbb{R})$, where A is a subset of $L(\mathbb{R})$, can never satisfy $\mathsf{AD}_\mathbb{R}$, since such a model can never satisfy both $\mathsf{Uniformization}$ and $\aleph_1 \not\leq 2^{\aleph_0}$, which are consequences of $\mathsf{AD}_{\mathbb{R}}$.

1.2. **Uniformization.** Given a set $A \subseteq \omega^\omega \times \omega^\omega$, and an $x \in \omega^\omega$, we let A_x denote the *section* of A at x, i.e., $\{y \in \omega^\omega : (x, y) \in A\}$. $\mathsf{Uniformization}$ is the assertion that every subset of the plane contains a function with the same domain:

$$\forall A \subseteq \omega^\omega \times \omega^\omega \, \exists B \subseteq A \, \forall x \in \omega^\omega \, (A_x \neq \emptyset \Rightarrow |B_x| = 1).$$

$\mathsf{Uniformization}$ follows from the Axiom of Choice (AC). It also follows from $\mathsf{AD}_{\mathbb{R}}$ via a game with one round, where player I plays $x \in \omega^\omega$, player II plays $y \in \omega^\omega$, and II wins if and only if $A_x = \emptyset$ or $y \in A_x$. Player I cannot have a winning strategy in this game, and a winning strategy for player II gives a uniformizing function for A.

A diagonal argument (given in the proof of Theorem 1.3) shows that the statement $\mathsf{Uniformization}$ fails in models of the form $L(A, \mathbb{R})$ for any set $A \subseteq L(\mathbb{R})$, assuming $\aleph_1 \not\leq 2^{\aleph_0}$. It follows that models of this form cannot satisfy $\mathsf{AD}_{\mathbb{R}}$. For any set or class X, we let OD_X denote the class of sets which are definable from an ordinal and a finite sequence from X (i.e., which are *ordinal definable* from a finite sequence from X). We write OD for OD_X when $X = \emptyset$. If OD_X contains a wellordering of X then it contains a wellordering of each element of $\mathrm{OD}X$.

Theorem 1.3 (Solovay). *If $\aleph_1 \not\leq 2^{\aleph_0}$ and A is a set contained in $L(\mathbb{R})$, then $L(A, \mathbb{R}) \models \neg\mathsf{Uniformization}$.*

Proof. For each set a in $L(A, \mathbb{R})$ there is an $x \in \omega^\omega$ such that a is in $\mathrm{OD}_{\{A,x\}}$. Since $\aleph_1 \not\leq 2^{\aleph_0}$, each set of the form $\omega^\omega \cap \mathrm{OD}_{\{A,x\}}$ is countable. Let B be the set of $(x, y) \in \omega^\omega \times \omega^\omega$ such that y is not in $\mathrm{OD}_{\{A,x\}}$. Any uniformizing function f for B would be in $\mathrm{OD}_{\{A,x\}}$ for some $x \in \omega^\omega$, which would mean that $f(x)$ would also be in $\mathrm{OD}_{\{A,x\}}$. $\square$

$\mathsf{Uniformization}$ is then one difference between AD and $\mathsf{AD}_{\mathbb{R}}$, as it is implied by the latter and not the former. There is a sense in which it may be the only difference, as results of Becker, Martin and Woodin show that $\mathsf{AD} + \mathsf{DC} + \mathsf{Uniformization}$ implies $\mathsf{AD}_{\mathbb{R}}$ (see Theorem 3.12). We now briefly review DC, which is a weak form of the Axiom of Choice.

1.3. **DC and $\mathsf{DC}_{\mathbb{R}}$.** For each set X, DC_X is the statement that each nonempty tree $T \subseteq X^{<\omega}$ without terminal nodes has an infinite branch. The Axiom of Dependent Choice,DC, is the statement that DC_X holds for every X.

$\mathsf{Uniformization}$ implies $\mathsf{DC}_{\mathbb{R}}$, since for any such tree $T \subseteq (\omega^\omega)^{<\omega}$ it implies the existence of a function choosing a successor (when there is one) for each node. It follows that $\mathsf{AD}_{\mathbb{R}}$ implies $\mathsf{DC}_{\mathbb{R}}$. Whether or not AD implies $\mathsf{DC}_{\mathbb{R}}$ is an open question. At the end of Section 11 we will review Solovay's theorem showing that $\mathsf{AD}_{\mathbb{R}}$ does not imply that DC holds in $L(\mathcal{P}(\mathbb{R}))$.

Since an infinite path through a tree on ω^ω can be coded by a single real, if $\mathsf{DC}_{\mathbb{R}}$ holds then it holds in any transitive model M containing $\mathbb{R}$. In particular, it holds in $L(\mathbb{R})$ if the Axiom of Choice holds in the full universe V.

While our current goal is the formulation of AD^+, we digress briefly to review some important structural properties of $\mathcal{P}(\omega^\omega)$ under AD.

2. The Wadge hierarchy

Given $A, B \subseteq \omega^\omega$, we write $A \leq_{\mathrm{W}} B$ (and say that A is *Wadge below* B) to mean that there is a continuous function $f : \omega^\omega \to \omega^\omega$ such that $A = f^{-1}[B]$. We write $A \equiv_{\mathrm{W}} B$ for the conjunction $A \leq_{\mathrm{W}} B \wedge B \leq_{\mathrm{W}} A$. The equivalence class $[A]_{\mathrm{W}}$ of A with respect to $\equiv_{\mathrm{W}}$ is the *Wadge degree* of A. The Wadge order gives an almost linear hierarchy on the set of Wadge degrees, in the following sense

Theorem 2.1 (Wadge). AD *implies that, for all* $A, B \subseteq \omega^\omega$, $A \leq_W B$ *or* $B \leq_W \omega^\omega \setminus A$.

Proof. Consider the game where players I and II play nonempty finite sequences of integers in each turn, with I building $x \in \omega^\omega$, and II building y. Say that I wins if and only if the statement $x \in B \Leftrightarrow y \in A$ holds. A winning strategy for I induces a witness to the statement $A \leq_{\mathrm{W}} B$. A winning strategy for II witnesses $B \leq_{\mathrm{W}} \omega^\omega \setminus A$. $\square$

Determinacy for the class of games in the proof just given is called Wadge Determinacy. It is an open question whether Wadge Determinacy implies AD.

Martin proved that the Wadge order is wellfounded, assuming Wadge Determinacy, $\mathsf{DC}_{\mathbb{R}}$ and the assumption that every set of reals has the Baire property. It is not known if $\mathsf{DC}_{\mathbb{R}}$ is needed for this result, or if AD alone implies that the Wadge hierarchy is wellfounded. The wellfoundedness of the Wadge order induces the notion of Wadge rank, where the Wadge rank of $A \subseteq \omega^\omega$ is the ordinal rank of $[A]_{\mathrm{W}}$ in the Wadge order.

A Wadge class $[A]_{\mathrm{W}}$ is said to be *selfdual* if $[A]_{\mathrm{W}} = [\omega^\omega \setminus A]_{\mathrm{W}}$. We mention a few important results from the highly-developed theory of Wadge classes (see, for instance [20]).

A standard diagonalization along the lines of Cantor's theorem shows that there is no largest Wadge degree. The proof below uses the following notation, which appears also in the next section. Using a fixed recursive bijection between ω and $\omega^{<\omega} \times \omega^{<\omega}$, we can think of an element c of ω^ω as listing a seqeunce of basic open sets in $\omega^\omega \times \omega^\omega$, and let fc be the complement of the union of these open sets. Letting CF be the (projective) set of c for which f_c is a continuous function from ω^ω to ω^ω, we have that $\{f_c : c \in CF\}$ is the set of all such functions. In particular there is a surjection from ω^ω to the set of continuous functions from ω^ω to ω^ω.

Theorem 2.2. *There is no largest Wadge degree.*

Proof. Given $A \subseteq \omega^\omega$, let B be the set of $c \in CF$ for which $f_c(c) \notin A$. Then B is not Wadge below A. $\square$

For each pair of complementary non-selfdual degrees there is a minimal selfdual degree above them.

Theorem 2.3. *For a non-selfdual class* $[A]_W$*, there is a minimal Wadge degree above* A *and* $\omega^\omega \setminus A$.

Proof. Let B be the set of $x \in \omega^\omega$ of the form $0^\frown x$ for $x \in A$ or $1^\frown x$ for $x \notin A$. Then $A \leq_{\mathrm{W}} B$ and $(\omega^\omega \setminus A) \leq_{\mathrm{W}} B$ and, for any $C \subseteq \omega^\omega$, if $A \leq_{\mathrm{W}} C$ and $(\omega^\omega \setminus A) \leq_{\mathrm{W}} C$ then $B \leq_{\mathrm{W}} C$. $\square$

The corresponding fact for selfdual degrees follows from the wellfoundedness of the Wadge hierarchy.

Theorem 2.4. *If* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ *holds then each selfdual Wadge degree has a non-selfdual successor in the Wadge order.*

Theorem 2.5 (whose proof includes contributions from Steel and Van Wesep) says when a Wadge degree of limit rank is seldual.

Theorem 2.5. *Assuming Wadge Determinacy and that all subsets of* ω^ω *have the property of Baire, limit classes under the Wadge order are selfdual precisely when the Wadge order below them has countable cofinality.*

2.1. **The ordinal Θ.** The ordinal Θ is defined to be the least ordinal which is not a surjective image of ω^ω. Clearly, Θ is a cardinal; the Moschovakis Coding Lemma (see Section 4) implies that it is a limit cardinal if AD holds. Assuming that $A \subseteq \omega^\omega$ has Wadge rank α (greater than 0), there is a surjection from ω^ω to $\alpha+1$, assigning each $c \in CF$ the Wadge rank of $f_c^{-1}[A]$. Thus, assuming $\mathrm{AD}+\mathrm{DC}_{\mathbb{R}}$, each $A \subseteq \omega^\omega$ has Wadge rank less than Θ. Similarly, one can use a surjection from ω^ω to an ordinal α to build a set of reals of Wadge rank at least α. In fact, ZF proves that for every ordinal $\alpha < \Theta$ there is a $<_{\mathrm{W}}$-increasing sequence of length α, where $<_{\mathrm{W}}$ is the Wadge order. This gives the following theorem, where we let $\mathrm{WR}(A)$ denote the Wadge rank of a set $A \subseteq \omega^\omega$.

Theorem 2.6 (Solovay)**.** *Assuming* $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, $\Theta = \{\mathrm{WR}(A) : A \subseteq \mathcal{P}(\mathbb{R})\}$.

We return now to our goal of defining AD^+.

3. The AD^+ Axioms

3.1. **Suslin Sets and ∞-Borel Sets.** Given a set X and a tree $T \subseteq X^{<\omega}$, $[T]$ denotes the set of infinite paths through T. Given sets X and Y and a tree $T \subseteq (X \times Y)^{<\omega}$, the *projection* $\mathrm{p}[T]$ is defined to be the set of $x \in X^\omega$ for which there exists an $f \in Y^\omega$ such that $(x, f) \in [T]$.

Definition 3.1. *Given an ordinal* γ *and a set* X*, a set* $A \subseteq X^\omega$ *is* γ-Suslin *if* $A = \mathrm{p}[T]$ *for some tree* $T \subseteq (X \times \gamma)^{<\omega}$*. The set* A *is* Suslin *if it is* γ*-Suslin for some ordinal* γ*, and* co-Suslin *if its complement in* X^ω *is Suslin.*

Continuous preimages of γ-Suslin sets are γ-Suslin, so the Suslin sets form an initial segment of the Wadge hierarchy.

If $A \subseteq \omega^\omega \times \omega^\omega$ (identified with the corresponding subset of $(\omega \times \omega)^\omega$) is Suslin then it can be uniformized. To see this, suppose that $A = \mathrm{p}[T]$, where $T \subseteq (\omega \times \omega \times \gamma)^{<\omega}$. For each $x \in A$ one can recursively find the lexicographically least $y \in (\omega \times \gamma)^{<\omega}$ such that $(x, y) \in [T]$. Models of the form $L(A, \mathbb{R})$ for $A \subseteq L(\mathbb{R})$ will not satisfy the assertion that all sets of reals are Suslin as they don't satisfy $\mathsf{Uniformization}$.

The following theorems, whose proofs are beyond the scope of these notes, relate the Suslin property and $\mathsf{Uniformization}$ with $\mathsf{AD}_{\mathbb{R}}$. The first is a combination of results of Becker and Woodin. The second is due to Martin and Woodin independently.

Theorem 3.2. $\mathsf{AD} + \mathsf{DC} + \mathsf{Uniformization}$ *implies all subsets of* ω^ω *are Suslin.*

Theorem 3.3. *If* AD *holds and all subsets of* ω^ω *are Suslin, then* $\mathsf{AD}_{\mathbb{R}}$ *holds.*

We say that a tree T is wellfounded if it does not have an infinite path, i.e., if $[T]$ is the emptyset. Assuming DC (or the wellorderability of T), wellfoundedness of a tree T is witnessed by a ranking function ρ from T to the ordinals, where $\rho(s) > \rho(t)$ whenever t extends s. If T is wellfounded, then a ranking function witnessing this can be constructed recursively in the model $L[T]$. It follows that, given an ordinal γ, a tree T on $\omega \times \gamma$, and an $x \in \omega^\omega$, $x \in \mathrm{p}[T]$ if and only if $L[T, x] \models x \in p[T]$. This property is generalized in the following definition (in Section 6.1 we discuss an equivalent definition which is more closely related to the usual notion of Borel set).

Definition 3.4. *A set $A \subseteq \omega^\omega$ is* ∞-Borel *if and only if there exist a formula ϕ and a set $S \subseteq \mathrm{Ord}$ such that $A = \{x \in \omega^\omega : L[S, x] \models \phi(S, x)\}$. We call the pair (S, ϕ) an ∞-Borel* code *for A.*

Again, Suslin sets are ∞-Borel. As with the Suslin sets, the ∞-Borel sets form an initial segment of the Wadge hierarchy, which can be seen as follows. If (S, ϕ) is an ∞-Borel code for A, and c is in CF, then there is an ∞-Borel code for $f_c^{-1}[A]$ of the form (S', ϕ'), where S' codes the pair (S, c) and $\phi'(S', x)$ is the formula $\phi(S, f_c(x))$ (phrased in terms of the coding of f_c by c).

If $A \subseteq \omega^\omega$ is ∞-Borel, then this is witnessed by a pair (S, ϕ) with S a bounded subset of Θ. This can be shown by mapping ω^ω onto an elementary submodel (containing ω^ω) of a suitable model of the form $L_\alpha(T, \mathbb{R})$, where (T, ϕ) is a given ∞-Borel code for A. Since every element of $L_\alpha(T, \mathbb{R})$ is definable in $L_\alpha(T, \mathbb{R})$ from T, an ordinal and an element of ω^ω, such a submodel can be built assuming only ZF.

If A is ∞-Borel and $\aleph_1 \not\leq 2^{\aleph_0}$, then A has the property of Baire and is Lebesgue measurable. To see this for the property of Baire, let $\mathbb{P}$ be Cohen forcing and let $\dot{c}$ be the canonical $\mathbb{P}$-name for the generic Cohen real. Suppose that (S, ϕ) is an ∞-Borel code for A. Since $\aleph_1 \not\leq 2^{\aleph_0}$ and $L[S] \models \mathsf{AC}$, $\omega^\omega \cap L[S]$ is countable, so the set of reals which are Cohen-generic over $L[S]$ is comeager. For each such real x, let g_x be the filter for which $\dot{c}_{g_x} = x$. Let A' be the set of x which are Cohen-generic over $L[S]$ for which there is a condition $p \in g_x$ forcing that $L[S, \dot{c}] \models \phi(S, \dot{c})$. Then A' is Borel, and the symmetric difference $A \bigtriangleup A'$ is contained in the set of reals which are not Cohen-generic over $L[S]$. For Lebesgue measurability the argument is the same, using random forcing instead of Cohen forcing. One can use the same argument with Mathias forcing to show that A (under the same assumptions) satisfies the Ramsey property. Unlike the cases of the Baire property and Lebesgue measurability, it is still unknown whether AD alone implies that every subset of 2^ω has the Ramsey property.

One part of AD^+ is the statement that all $A \subseteq \omega^\omega$ are ∞-Borel. We discuss proofs of the following theorems (due to Woodin) in Sections 5 and 6.2.

Theorem 3.5. $\mathsf{AD} + V = L(\mathbb{R})$ *implies that all subsets of ω^ω are ∞-Borel.*

Theorem 3.6. *If $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$ holds and A is ∞-Borel then it is ∞-Borel in $L(A, \mathbb{R})$.*

We list three theorems of Woodin from the 1990's on ∞-Borel sets, whose proofs are too involved to include in these notes. Since $\mathsf{DC}_\mathbb{R}$ implies that DC holds in models of the form $L(S, \mathbb{R})$ whenever S is a set of ordinals, Theorem 3.9 implies Theorem 3.8.

Theorem 3.7. $\mathsf{AD} + \mathsf{Uniformization}$ *implies that all subsets of ω^ω are ∞-Borel.*

Theorem 3.8. *Assuming* $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$*, for* $A \subseteq \omega^\omega$*,* A *is* ∞*-Borel if and only if there is an* $S \subseteq \mathrm{Ord}$ *with* $A \in L(S, \mathbb{R})$*.*

Theorem 3.9. *Assuming* $\mathsf{AD}+\mathsf{DC}$*, for every ordinal* $\lambda < \Theta$*, all subsets of* ω^ω *in* $L(\mathcal{P}(\lambda))$ *are* ∞*-Borel.*

There is still one part of AD^+ left to be defined, concerning the determinacy of games on the ordinals.

3.2. **γ-Determinacy.** Let γ be an ordinal, and $f\colon \gamma^\omega \to \omega^\omega$ be a continuous function, where γ^ω is given the product topology induced by the discrete topology on γ. Given $A \subseteq \omega^\omega$, we let $\mathcal{G}(f, A)$ be the game where players I and II alternately choose the values of some $x \in \gamma^\omega$, with player I winning if and only if $f(x) \in A$. We let γ-Determinacy be the statement that each such game $\mathcal{G}(f, A)$ is determined. For all $\gamma \geq 2$, γ-Determinacy implies AD.

$$\begin{array}{c|cccccccc} \mathrm{I} & x(0) & & x(2) & & x(4) & & \ldots & \\ \hline \mathrm{II} & & x(1) & & x(3) & & \ldots & & \end{array}$$

The game $\mathcal{G}(f, A)$; I wins if and only if $f(x)$ is in A.

Although we call this statement γ-Determinacy, we do not consider arbitrary subsets of γ^ω as payoff sets. The reason for this is that ZF implies the existence of a nondetermined game on ω_1^ω (ie., that AD_{ω_1} is false, as mentioned above). To see this, consider the game where player I plays a countable ordinal α in round 1 and makes no subsequent plays, and player II in her turns plays natural numbers x_i ($i \in \omega$) collectively forming $x \in \omega^\omega$, with player I wins if x codes a wellordering of ω in ordertype α. Since player I cannot have a winning strategy, determinacy of this game implies the existence of an injection from ω_1 to ω^ω, which implies the existence of a set of reals without the perfect set property, and therefore contradicts AD, as noted in Section 1.1. One interesting aspect of this argument is that it does not produce a single game on ω_1 which is provably undetermined (in particular, player II has a winning strategy in the game just described if AC holds). Woodin has shown that in fact determinacy can hold for all definable games on ω_1.

A typical game of the type considered in the statement of γ-Determinacy is when A is a subset of ω^ω, τ is a $\mathrm{Col}(\omega, \gamma)$-name for an element of ω^ω, and players I and II play a descending sequence of (codes for) conditions in $\mathrm{Col}(\omega, \gamma)$ collectively realizing τ, with player I winning if and only if the given realization of τ is in the set A.

We let $<\Theta$-Determinacy be the statement that γ-Determinacy holds for every ordinal $\gamma < \Theta$. We have now defined all the parts of the axiom AD^+.

3.3. **AD^+.**

Definition 3.10. AD^+ *is the conjunction of the following three statements.*

- $\mathsf{DC}_{\mathbb{R}}$
- *Every subset of* ω^ω *is* ∞*-Borel.*
- $<\Theta$-Determinacy

Some authors (including Woodin) do not include $\mathsf{DC}_{\mathbb{R}}$ in the statement of AD^+, but consider AD^+ (defined instead as the conjunction of the other two parts) only in the context of $\mathsf{DC}_{\mathbb{R}}$. The axiom $\mathsf{DC}_{\mathbb{R}}$ is different from the other two parts of AD^+ in that it is witnessed by individual real numbers, as opposed to bounded subsets of Θ.

Here is a brief listing of implications, known non-implications, and open questions regarding the statements discussed so far.

- $\mathsf{AD}_{\mathbb{R}}$ implies $\mathsf{AD} + \mathsf{Uniformization}$ (Sections 1.1 and 1.2), which implies that all subsets of ω^ω are ∞-Borel (Theorem 3.7).
- $\mathsf{AD}_{\mathbb{R}} + \mathsf{DC}$ implies that all subsets of ω^ω are Suslin (Theorem 3.2).
- AD plus "all subsets of ω^ω are Suslin" implies AD^+ (Theorem 3.19 for $<\Theta$-$\mathsf{Determinacy}$).
- If all subsets of ω^ω are Suslin, then $\mathsf{Uniformization}$ holds (Section 1.2).
- $\mathsf{Uniformization}$ implies $\mathsf{DC}_{\mathbb{R}}$ (Section 1.2).
- $<\Theta$-$\mathsf{Determinacy}$ implies AD, so AD^+ does, too (Section 3.2).
- AD^+ does not imply $\mathsf{Uniformization}$ (Theorem 1.3 and Section 5).
- It is open whether AD implies any or all of the parts of AD^+.
- It is open whether $\mathsf{AD}_{\mathbb{R}}$ implies $<\Theta$-$\mathsf{Determinacy}$ (it does imply the other two parts of AD^+).
- It is open whether $<\Theta$-$\mathsf{Determinacy}$ implies the other two parts of AD^+ (the other two parts are consequences of AC, so they don't imply $<\Theta$-$\mathsf{Determinacy}$).

The study of AD^+ was inspired by the following question. Suppose that $M \subseteq N$ are models of $\mathsf{ZF}+\mathsf{AD}$ with the same reals, and that every set of reals in M is Suslin in N. What consequences does this entail for M? As a partial answer, we have that in this situation $M \models \mathsf{AD}^+$. To see this for the case of $<\Theta$-$\mathsf{Determinacy}$, we use the Moschovakis Coding Lemma and the fact that AD implies the existence of cofinally many strong partition cardinals below Θ (see Sections 4 and 3.4). Similar arguments give the following theorem. As seen above, the corresponding theorem fails for $\mathsf{AD}_{\mathbb{R}}$.

Theorem 3.11 (Woodin)**.** *If AD^+ holds, then it holds in every inner model containing ω^ω.*

More generally, for all three parts of AD^+, if $A \subseteq \omega^\omega$ and $L(A, \mathbb{R}) \subseteq M$ where A satisfies the given part of AD^+ in M, then it satisfies this part in $L(A, \mathbb{R})$ also. For $\mathsf{DC}_{\mathbb{R}}$ this follows from the fact, observed above, that $\mathsf{DC}_{\mathbb{R}}$ is witnessed by individual real numbers. For $<\Theta$-$\mathsf{Determinacy}$, this is discussed at the end of Section 4. The corresponding fact for ∞-Borel sets is covered in Section 6.2.

Combining these remarks with Theorems 3.2 and 3.3, we have the following.

Theorem 3.12. *Each of the following statements implies the ones below it, and the first two are equivalent. Assuming* DC *all four statements are equivalent.*

(1) AD + *"all subsets of $\mathcal{P}(\omega^\omega)$ are Suslin"*
(2) AD^+ + *"all subsets of $\mathcal{P}(\omega^\omega)$ are Suslin"*
(3) $\mathsf{AD}_{\mathbb{R}}$
(4) $\mathsf{AD} + \mathsf{Uniformization}$

While the question of whether AD implies AD^+ is open, it is known that the consistency strength of $\mathsf{AD} + \neg\mathsf{AD}^+$ is greater than that of AD. To illustrate this, let Γ_+ be the set of $A \subseteq \omega^\omega$ for which $L(A, \mathbb{R}) \models \mathsf{AD}^+$. By Theorem 3.11, Γ_+ is an initial segment of the Wadge hierarchy. Since the implication $\mathsf{AD} \Rightarrow \mathsf{AD}^+$ is known to hold in $L(\mathbb{R})$ (see Section 5), AD implies that Γ_+ is nonempty. The following theorem (whose proof we will not discuss in these notes) shows that a model of $\mathsf{AD} + \neg\mathsf{AD}^+$ would contain a model of $\mathsf{AD}_\mathbb{R}$.

Theorem 3.13 (Woodin)**.** *Assuming* $\mathsf{AD}+\mathsf{DC}_\mathbb{R}$, $L(\Gamma_+, \mathbb{R}) \models \mathsf{AD}^+$. *If* $\Gamma_+ \neq \mathcal{P}(\omega^\omega)$ *then* $L(\Gamma_+, \mathbb{R}) \models \mathsf{DC} + \mathsf{AD}_\mathbb{R}$.

Woodin's Derived Model Theorem (Theorem 3.14) is the standard method for producing models of AD^+. The model $L(\Gamma, \mathbb{R}^*)$ from the statement of the theorem is called the *derived model*. For any set or class X, the inner model HOD_X is the class of all sets a such that the transitive closure of $\{a\}$ is contained in $\mathrm{OD}_{\mathrm{TC}(X)}$, where $\mathrm{TC}(X)$ denotes the transitive closure of X. When X is ordinal definable from one of its finite subsets, HOD_X is a model of ZF. If a wellordering of X is ordinal definable from a finite subset of X, HOD_X is a model of ZFC. A proof of an early version of the theorem below can be found in [19].

Theorem 3.14 (Woodin)**.** *Let δ be a limit of Woodin cardinals and let G be a V-generic filter for* $\mathrm{Col}(\omega, <\delta)$. *Let*

$$\mathbb{R}^* = \bigcup\{\mathbb{R}^{V[G\restriction\lambda]} : \lambda < \delta\}$$

and

$$\Gamma = \{A \subseteq \mathbb{R}^* : A \in \mathrm{HOD}_{V,\mathbb{R}^*},\ L(A, \mathbb{R}) \models \mathsf{AD}^+\}.$$

Then $L(\Gamma, \mathbb{R}^*) \models \mathsf{AD}^+$.

This is the standard way to produce models of AD^+, and in a sense it is the only way, since the following theorem, a reversal of the Derived Model Theorem, says that every model of AD^+ arises as a derived model.

Theorem 3.15 (Woodin)**.** *Assuming* $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$, $L(\Gamma_+, \mathbb{R})$ *is the derived model of some inner model which exists in some forcing extension of V.*

Finally we mention Theorem 3.16 below, which is a generalization of the Solovay Basis Theorem (Theorem 5.3). Among other things, the forward direction of the theorem enables one to carry about the basic analysis of Woodin's $\mathbb{P}_{\max}$ forcing assuming AD^+ (see [21]).

Theorem 3.16 (Woodin)**.** *Assuming* AD, AD^+ *is equivalent to the statement that every true $\utilde{\Sigma}^2_1$ statement has a witness which is Suslin and co-Suslin.*

In the theorem above, a sentence is $\utilde{\Sigma}^2_1$ if it has the form $\exists A \subseteq \omega^\omega \phi(A, a)$, where a is an element of ω^ω and all quantifiers in ϕ range over ω^ω (i.e., ϕ is *projective*). More generally, given a set $B \subseteq \omega^\omega$, a set X of some Polish space Y is $\utilde{\Sigma}^2_1(B)$ if there exist a projective formula ϕ and an $a \in \omega^\omega$ such that

$$X = \{y \in Y : \exists A \subseteq \omega^\omega \phi(A, B, x, a)\}.$$

When $B = \emptyset$ we write $\utilde{\Sigma}^2_1$. The set X is $\utilde{\Delta}^2_1(B)$ if X and $Y \setminus X$ are both $\utilde{\Sigma}^2_1(B)$. The ordinal $\utilde{\delta}^1_2(B)$ is defined to be the supremum of the ordertypes of the $\utilde{\Delta}^2_1(B)$ prewellorderings, where a*prewellordering* a reflexive, transitive, and wellfounded binary relation.

3.4. **Strong Partition Cardinals.** Given ordinals α and β, we let $[\alpha]^\beta$ be the collection of subsets of α of ordertype β. The existence of an infinite κ satisfying the following definition contradicts the Axiom of Choice. It appears to be an open question whether one gets an equivalent definition by replacing γ with 2.

Definition 3.17. *A cardinal κ is a* strong partition cardinal *if for every $\gamma < \kappa$ and every $f : [\kappa]^\kappa \to \gamma$ there is an $A \in [\kappa]^\kappa$ such that $f \restriction [A]^\kappa$ is constant.*

The converse of the following theorem holds in $L(\mathbb{R})$, but not in general [2]. The proof of the theorem involves a deep analysis of pointclasses under AD.

Theorem 3.18 (Kechris-Kleinberg-Moschovakis-Woodin [7])**.** *The Axiom of Determinacy implies that the strong partition cardinals are cofinal in Θ, and in particular that, for each $A \subseteq \omega^\omega$, $\underset{\sim}{\delta}^1_2(A)^{L(A,\mathbb{R})}$ is a strong partition cardinal.*

The existence of strong partition cardinals, in conjunction with Suslin representations, can be used to prove $<\Theta$-$\mathsf{Determinacy}$. In particular, assuming $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, $<\Theta$-determinacy holds for games $\mathcal{G}(f, A)$ where A is Suslin and co-Suslin.

Theorem 3.19 (Moschovakis, Woodin)**.** *Suppose that $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ holds. Let $\gamma < \kappa < \lambda$ be ordinals, with λ a strong partition cardinal. Let $A \subseteq \omega^\omega$ be such that A and $\omega^\omega \setminus A$ are both κ-Suslin and let $f \colon \gamma^\omega \to \omega^\omega$ be continuous. Then $\mathcal{G}(f, A)$ is determined.*

The proof of Theorem 3.19 uses the Moschovakis Coding Lemma, which is discussed in Section 4. With the Coding Lemma in hand we will outline in Section 5 a proof that the implication $\mathsf{AD} \Rightarrow \mathsf{AD}^+$ holds in $L(\mathbb{R})$.

4. The Moschovakis Coding Lemma

A prewellordering of a set X is the order induced by a surjection from X to some ordinal. We call this ordinal the *length* of the prewellording. For each $x \in X$, the image of x under this surjection is called the *rank* of x in the prewellordering. We write $\mathrm{rank}_{\leq}(x)$ for the rank of x in $\leq$ when $\leq$ is a prewellordering of a set with x as a member. We shall be interested in prewellorderings of ω^ω. Note that Θ is the supremum of the lengths of prewellorderings of ω^ω.

The following simplified version of the Moschovakis Coding Lemma is sufficient for our purposes. A more precise version replaces "projective" with a Σ^1_1 condition in parameters derived from the prewellordering $\leq$.

Theorem 4.1 (Moschovakis)**.** *Assume that AD holds and let $\leq$ be a prewellordering of ω^ω of length γ. Then there is a projective formula ϕ with $\leq$ as its only parameter such that for every $A \subseteq \gamma$ there is $x \in \omega^\omega$ such that*

$$A = \{\mathrm{rank}_{\leq}(y) : y \in \omega^\omega \wedge \phi(x, y)\}.$$

One important corollary of the Moschovakis Coding Lemma is that, assuming AD, whenever $A \subseteq \omega^\omega$ and γ is an ordinal below $\Theta^{L(A,\mathbb{R})}$, every subset of γ is in $L(A, \mathbb{R})$. Moreover, $\mathcal{P}(\gamma)$ is a surjective image of ω^ω in $L(A, \mathbb{R})$, which implies that Θ is a limit cardinal (under AD).

Corollary 4.2. *Assuming AD, if $A \subseteq \omega^\omega$ and $\gamma < \Theta^{L(A,\mathbb{R})}$ then $\mathcal{P}(\gamma) \subseteq L(A, \mathbb{R})$ and there is a surjection from ω^ω to $\mathcal{P}(\gamma)$ in $L(A, \mathbb{R})$.*

Suppose now that AD holds, $\leq$ is a prewellordering of ω^ω of length γ and $f : \gamma^\omega \to \omega^\omega$ is continuous and $A \subseteq \omega^\omega$. Strategies in the game $\mathcal{G}(f, A)$ can be coded by subsets of γ. It follows from the Moschovakis Coding Lemma then that any winning strategy for the game $\mathcal{G}(f, A)$ exists in $L(\leq, \mathbb{R})$. This gives the $<\Theta$-$\mathsf{Determinacy}$ part of Theorem 3.11.

5. AD^+ in $L(\mathbb{R})$

The fact that, in $L(\mathbb{R})$, AD implies AD^+ follows from several classical results from the Cabal Seminar. First, we have Kechris's theorem, giving $\mathsf{DC}_\mathbb{R}$.

Theorem 5.1 (Kechris [6]). $L(\mathbb{R}) \models (\mathsf{AD} \Rightarrow \mathsf{DC})$.

The following theorem of Martin and Steel characterizes the Suslin sets of $L(\mathbb{R})$ under AD.

Theorem 5.2 (Martin-Steel [13]). *Assuming* $\mathsf{AD} + V = L(\mathbb{R})$*, the Suslin subsets of* ω^ω *are exactly the* $\utilde{\Sigma}^2_1$ *sets.*

Finally, we have the Solovay Basis Theorem, which reflects potential counterexamples to AD^+ into the Suslin, co-Suslin sets (see Section 2.4 of [9] for further discussion).

Theorem 5.3 (Solovay). *In* $L(\mathbb{R})$*,* $\utilde{\Sigma}^2_1$ *facts have* $\utilde{\Delta}^2_1$ *witnesses.*

By the Moschovakis Coding Lemma, the assertion that there exist a pair (f, A) giving a counterexample to $<\Theta$-$\mathsf{Determinacy}$ is equivalent to a Σ^2_1-statement. If $<\Theta$-$\mathsf{Determinacy}$ fails in $L(\mathbb{R})$, then, the Solovay Basis Theorem implies that there is a counterexample where A is Suslin and co-Suslin. Theorem 3.19 however implies that this is impossible. We have then that AD implies $<\Theta$-$\mathsf{Determinacy}$ in $L(\mathbb{R})$. A similar argument applies to the statement that every subset of ω^ω is ∞-Borel, using the material in Section 6.2.

6. ∞-Borel sets

6.1. **Infinitary ∞-Borel codes.** In this section we give an alternate (equivalent) definition of ∞-Borel set which corresponds to the classical notion of Borel set. This definition defines the ∞-Borel subsets of 2^ω using an infinitary propositional language.

Definition 6.1. *For* κ *an infinite cardinal,* $\mathcal{L}_{\kappa,0}$ *is the language with propositional variables* $\{p_n : n \in \omega\}$ *which is closed under the unary connective of negation and under wellordered disjunctions and conjunctions indexed by subsets of* κ*. We let* $\mathcal{L}_{\infty,0}$ *be the union over all cardinals* κ *of the languages* $\mathcal{L}_{\kappa,0}$*.*

An element $x \in 2^\omega$ may be viewed as an $\mathcal{L}_{\infty,0}$-structure, where $x \models p_n$ if and only if $x(n) = 1$.

Definition 6.2. *A set* $A \subseteq 2^\omega$ *is* κ-Borel *if there is a formula* $\phi \in \mathcal{L}_{\kappa,0}$ *such that* $A = \{x \in 2^\omega : x \models \phi\}$*.*

We let A_ϕ denote the set of $x \in 2^\omega$ such that $x \models \phi$. Then ϕ gives a description of a construction of A in terms of negations and wellordered unions and intersections, starting with the basic open sets in 2^ω. Using a definable paring function on the ordinals we can code elements of $\mathcal{L}_{\infty,0}$ by sets of ordinals, and for each suitable

$S \subseteq \mathrm{Ord}$ we let ϕ_S be the sentence coded by S. When κ is an infinite cardinal each ϕ in $\mathcal{L}_{\kappa,0}$ is ϕ_S for some $S \subseteq \kappa$. Then there is a first-order formula ψ such that (S, ψ) is an ∞-Borel code for A_ϕ in the sense defined previously. Our two notions of ∞-Borel set are in fact equivalent in ZF.

Theorem 6.3 (Woodin)**.** *For all $A \subseteq 2^\omega$ the following are equivalent:*

(1) *There exist $S \subseteq \mathrm{Ord}$ and a first-order formula ϕ such that*
$$A = \{x \in 2^\omega : L[S,x] \models \phi(S,x)\}.$$
(2) *There is $S \subseteq \mathrm{Ord}$ such that $A = \{x \in 2^\omega : x \models \phi_S\}$.*

We have just sketched the reverse direction of this theorem. The proof of the forward direction is more involved, and proceeds by induction on γ (observing that if $A = \{x \in 2^\omega : L[S,x] \models \phi(S,x)\}$ then there is an ordinal γ such that $A = \{x \in 2^\omega : L_\gamma[S,x] \models \phi(S,x)\}$) and a second induction on the complexity of ϕ.

The restriction to subsets of 2^ω here (instead of ω^ω) is a technical convenience in the case where $\kappa = \omega$. There are several ways to consider subsets of ω^ω as ∞-Borel in our new sense. For instance, we can assume instead that the propositional variables in $\mathcal{L}_{\infty,0}$ can take arbitrary values in ω. We skip the details here and treat both notions of ∞-Borel set as equivalent and meaningful for subsets of ω^ω.

The ∞-Borel sets are not always the same as the smallest class containing the open sets and closed under negations and wellordered unions. In particular, it is consistent with ZF that the collection of ∞-Borel sets is not closed under wellordered unions. In order to show that a given wellordered union of ∞-Borel sets is ∞-Borel one must be able to choose the codes, which is not possible in general.

In general ∞-Borel sets cannot be uniformized. In particular, models of the form $L(A, \mathbb{R})$ (where A is a subset of $L(\mathbb{R})$) can satisfy the statement that every subset of ω^ω is ∞-Borel, even though they cannot satisfy $\mathsf{Uniformization}$. Similarly (and again, unlike the case of Suslin representations), it can be that S is an ∞-Borel code for a nonempty set of reals which has no members in the model $L[S]$. For instance, the set of reals which are Cohen-generic over L has an ∞-Borel code in L.

In [8] and [12] the type of ∞-Borel code defined in Definition 3.4 is called an ∞-Borel* code; here we will not make a distinction, and hope that the intended meaning will be clear from context. In practice one needs both definitions for the ∞-Borel sets, both types of ∞-Borel code and the ability to translate between them. In Section 6.2 we see that if a set $A \subseteq \omega^\omega$ has an ∞-Borel code, then it has one coded by a set of reals projective in A.

6.2. **Local ∞-Borel codes.** We show in this section that (assuming $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$) each ∞-Borel set $A \subset \omega^\omega$ has an ∞-Borel code which is not much more complicated than A. In particular, the assertion that a given set $A \subseteq \omega^\omega$ is ∞-Borel is projective in a parameter for A. This fact will complete the argument from Section 5 that AD implies AD^+ in $L(\mathbb{R})$.

We define the following ordinals, all of which are at most Θ:

- χ_{B}, the least χ such that all ∞-Borel sets are χ-Borel;
- κ_{B}, the supremum of the lengths of wellordered sequences of ∞-Borel codes for nonempty disjoint sets;
- λ_{B}, the supremum of the lengths of ∞-Borel prewellorderings.
- ρ_{B}, the supremum of the Wadge ranks of ∞-Borel sets.

Assuming $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ (or just Wadge Determinacy plus the wellfoundedness of the Wadge hierarchy), $\rho_{\mathrm{B}}=\Theta$ is equivalent to the statement that every subset of ω^ω is ∞-Borel. We will see below that $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ implies that all four ordinals are the same. Our proof of this fact will use Theorem 3.9, whose proof is not discussed in these notes. It may be that there is an easier proof.

Our goal in this section is to prove Theorem 6.7 below assuming only $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$. We begin make a few preliminary observations on the four ordinals defined above. We first prove that $\chi_{\mathrm{B}}\leq\kappa_{\mathrm{B}}\leq\lambda_{\mathrm{B}}$, using a function $\phi\mapsto\phi^*$ on $\mathcal{L}_{\infty,0}$ sentences, defined recursively as follows:

- $p_n^*=p_n$;
- $(\neg\phi)^*=\neg\phi^*$;
- whenever Y is a set of ordinals,
$$\left(\bigvee_{\alpha\in Y}\phi_\alpha\right)^*=\bigvee_{\alpha\in X}\phi_\alpha^*,$$
where $X=\{\alpha\in Y : A_{\phi_\alpha}\not\subseteq\bigcup_{\beta\in Y\cap\alpha}A_{\phi_\beta}\}$, and
$$\left(\bigwedge_{\alpha\in Y}\phi_\alpha\right)^*=\bigwedge_{\alpha\in X}\phi_\alpha^*,$$
where $X=\{\alpha\in Y : A_{\phi_\alpha}\not\supseteq\bigcap_{\beta\in Y\cap\alpha}A_{\phi_\beta}\}$.

This function recursively removes from each conjunction and disjunction any terms which do not affect the final result. In particular, for any $\phi\in\mathcal{L}_{\kappa,0}$, $A_\phi=A_{\phi^*}$.

Parts of the proofs of Theorem 6.4 and 6.5 will be reused in the proof of Theorem 6.7.

Theorem 6.4 (Woodin)**.** *Assuming* $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$*,* $\chi_{\mathrm{B}}\leq\kappa_{\mathrm{B}}\leq\lambda_{\mathrm{B}}$*.*

Proof. To see that $\chi_{\mathrm{B}}\leq\kappa_{\mathrm{B}}$, let ϕ be any sentence in $\mathcal{L}_{\kappa,0}$. For any disjunction $\bigvee_{\alpha<\gamma}\psi_\alpha$ in ϕ^* the sentences $\psi_\alpha\wedge\neg(\bigvee_{\beta<\alpha}\psi_\beta)$ are ∞-Borel codes for nonempty disjoint sets. A similar remark applies to the conjunctions. It follows that ϕ^* is in $\mathcal{L}_{\kappa_{\mathrm{B}},0}$.

To see that $\kappa_{\mathrm{B}}\leq\lambda_{\mathrm{B}}$, note that one can convert a sequence of ∞-Borel codes for nonempty disjoint sets to an ∞-Borel code for a prewellordering in a uniform fashion by means of uniformly coding the product. Consider $\langle\phi_\alpha : \alpha<\gamma\rangle$ with the A_{ϕ_α}'s disjoint and nonempty. Then
$$\bigvee_{\alpha\leq\beta<\gamma}A_{\phi_\alpha}\times A_{\phi_\beta}$$
is a prewellordering of length γ. If ξ is such that each ϕ_α is in $\mathcal{L}_{\xi,0}$, then the resulting code is in $\mathcal{L}_{|\xi\cup\gamma|,0}$. $\square$

We observed above that $\rho_{\mathrm{B}}=\Theta$ if and only if all subsets of ω^ω are ∞-Borel. The same is true for λ_{B} (and, as well shall see, for all four ordinals). Given $A\subseteq\omega^\omega$, we let δ_A be the supremum of the lengths of the prewellorderings which are Wadge-below either A or $\omega^\omega\setminus A$

Theorem 6.5 (Woodin)**.** *Assuming* $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$*,* $\lambda_{\mathrm{B}}=\Theta$ *if and only if every subset of* ω^ω *is* ∞*-Borel.*

Proof. By definition, Θ is the supremum of the lengths of the prewellorderings of ω^ω, so if every subset of ω^ω is ∞-Borel then $\lambda_{\rm B} = \Theta$. For the other direction, suppose that $A \subseteq \omega^\omega$ is not ∞-Borel and that P is an ∞-Borel prewellorder. Then $A \not\leq_W P$ as the ∞-Borel sets form an initial segment of the Wadge hierarchy. Thus either $P \leq_W A$ or $P \leq_W \omega^\omega \setminus A$, and in either case the length of P is less than δ_A, which is less than Θ. $\square$

Putting together the results above, we have the following.

Theorem 6.6 (Woodin)**.** *Assuming* $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$*, the following statements are equivalent.*

- *At least one of* $\chi_{\rm B}$*,* $\kappa_{\rm B}$*,* $\lambda_{\rm B}$ *and* $\rho_{\rm B}$ *is* Θ*.*
- *All subsets of* ω^ω *are* ∞*-Borel*
- $\chi_{\rm B} = \kappa_{\rm B} = \lambda_{\rm B} = \rho_{\rm B} = \Theta$

Proof. All that remains to be shown is that if all subsets of ω^ω are ∞-Borel then $\chi_{\rm B} = \Theta$. This follows from the Moschovakis Coding Lemma, which implies that for each $\chi < \Theta$ there is a surjection from ω^ω to $\mathcal{P}(\chi)$, and therefore one from ω^ω to the χ-Borel sets. $\square$

Finally, we have the existence of local ∞-Borel codes.

Theorem 6.7 (Woodin)**.** *Assume that* $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ *holds. If* $A \subseteq \omega^\omega$ *is* ∞*-Borel then it is* δ_A*-Borel.*

Proof. If $\delta_A \geq \chi_B$ we are done. Supposing otherwise, there is a $\phi \in \mathcal{L}_{\infty,0}$ such that ϕ^* contains a disjunction or conjunction of length δ_A. Take a minimal example and cut it off at length δ_A. Thus we get $\langle \phi_\alpha : \alpha < \delta_A \rangle$ where each ϕ_α is in $\mathcal{L}_{\delta_A,0}$ and the sets A_{ϕ_α} are nonempty and disjoint. As in the second half of the proof of Theorem 6.4, we obtain a δ_A-Borel prewellordering P of length δ_A. Then P is not Wadge-below A or $\omega^\omega \setminus A$. Hence $A \leq_{\rm W} P$, and since P is δ_A-Borel A is too. $\square$

With the Coding Lemma and the fact that $\delta_A < \Theta^{L(A,\mathbb{R})}$ we have that $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ implies that if A is ∞-Borel then it is ∞-Borel in $L(A,\mathbb{R})$, and moreover that the assertion that $A \subseteq \omega^\omega$ is ∞-Borel is projective in A. In particular, the assertion that there is a subset of ω^ω which is not ∞-Borel is Σ^2_1. With Theorems 5.2 and 5.3, and the fact that Suslin sets are ∞-Borel, we get that AD implies that in $L(\mathbb{R})$ every subset of ω^ω is ∞-Borel.

We conclude this section by showing that the four ordinals defined above are all the same under $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$.

Theorem 6.8. *Assume that* $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$ *holds. Then*

$$\chi_{\rm B} = \kappa_{\rm B} = \lambda_{\rm B} = \rho_{\rm B} = \Theta^{L(\mathcal{P}(\chi_B))}.$$

Proof. By Theorem 6.6, the theorem holds in the case where all subsets of ω^ω are ∞-Borel. Suppose then that some $A \subseteq \omega^\omega$ is not ∞-Borel. Since $\mathsf{DC}_{\mathbb{R}}$ holds, $L(A,\mathbb{R})$ satisfies DC. Since the ∞-Borel sets form an initial segment of the Wadge hierarchy, they are all in $L(A,\mathbb{R})$. By Theorem 6.7, every ∞-Borel subset of ω^ω is ∞-Borel in $L(A,\mathbb{R})$. By the Moschovakis Coding Lemma, $L(A,\mathbb{R})$ contains all bounded subsets of its Θ. It follows that $\chi_{\rm B} = \chi_{\rm B}^{L(A,\mathbb{R})}$, $\lambda_{\rm B} = \lambda_{\rm B}^{L(A,\mathbb{R})}$ and $\rho_{\rm B} = \rho_{\rm B}^{L(A,\mathbb{R})}$.

By Theorem 3.9 (applied in $L(A,\mathbb{R})$), every subset of ω^ω in $L(\mathcal{P}(\chi_{\rm B}))$ is ∞-Borel. It follows then that the ∞-Borel sets are exactly the subsets of ω^ω in $L(\mathcal{P}(\chi_{\rm B}))$, all of which are ∞-Borel in $L(\mathcal{P}(\chi_{\mathrm{B}}))$. Moreover, $\chi_{\mathrm{B}} = \chi_{\mathrm{B}}^{L(\mathcal{P}(\chi_{\mathrm{B}}))}$, $\lambda_{\mathrm{B}} = \lambda_{\mathrm{B}}^{L(\mathcal{P}(\chi_{\mathrm{B}}))}$ and $\rho_{\mathrm{B}} = \rho_{\mathrm{B}}^{L(\mathcal{P}(\chi_{\mathrm{B}}))}$. The theorem then follows from Theorems 6.4 (applied in V) and 6.6 (applied in $L(\mathcal{P}(\chi_{\mathrm{B}}))$). □

7. Cone Measures

This section begins by presenting Martin's theorem that under AD the cone measure on the Turing degrees is an ultrafilter. This result lifts to larger degree notions, in particular those induced by contructibility relative to a fixed set of ordinals. This in turn induces the partial order $\leq_{\mathcal{D}}$ on degree notions, which turns out to be connected to $\mathsf{Uniformization}$ and the Suslin property.[1]

7.1. **Turing reducibility.** Let HF denote the set of hereditarily finite sets. Given subsets x and y of HF, we say that x is *Turing-reducible* to y if x is Δ^0_1 definable over HF with a predicate for y. For x and y in ω^ω we write $x \leq_{\mathrm{Tu}} y$ to indicate that x is Turing-reducible to y, and $x \equiv_{\mathrm{Tu}} y$ to mean that $x \leq_{\mathrm{Tu}} y$ and $y \leq_{\mathrm{Tu}} x$.

Given $x \in \omega^\omega$, we let $[x]_{\mathrm{Tu}} = \{y \in \omega^\omega : x \equiv_{\mathrm{Tu}} y\}$. Sets of the form $[x]_{\mathrm{Tu}}$ (for some $x \in \omega^\omega$) are called *Turing degrees*. We write $\mathcal{D}_{\mathrm{Tu}}$ for the set of Turing degrees. The sets $\{y \in \omega^\omega : y \geq_{\mathrm{Tu}} x\}$ or $\{[y]_{\mathrm{Tu}} : y \in \omega^\omega,\, y \geq_{\mathrm{Tu}} x\}$ are both referred to as the *Turing cone* above x (for any $x \in$ HF, which is called a *base* for the cone).

Given an equivalence relation E on a set X, we say that a set $A \subseteq X$ is E-*invariant* if it is a union of E-classes, that is, if, for all $x, y \in X$ such that xEy, $x \in A$ if and only if $y \in A$. A subset of ω^ω is said to be *Turing-invariant* if it is invariant for the restriction of $\equiv_{\mathrm{Tu}}$ to ω^ω. The following theorem of Martin shows that, under AD, every Turing-invariant subset of ω^ω either contains or is disjoint from a Turing cone.

Theorem 7.1 (Martin)**.** *Let $A \subseteq \omega^\omega$ be Turing-invariant, let σ be a strategy in $\mathcal{G}_\omega(A)$ and let $x \in \omega^\omega$ be such that $x \geq_{\mathrm{Tu}} \sigma$. If σ is a winning strategy for player I then A contains the Turing cone above x. If σ is a winning strategy for player II then A is disjoint from the Turing cone above x.*

We let TD be the statement that each Turing-invariant subset of ω^ω either contains or is disjoint from a cone. Countable Choice for Reals ($\mathsf{CC}_{\mathbb{R}}$) is the restriction of the Axiom of Choice to countable subsets of $\mathcal{P}(\mathbb{R})$. Mycielski [15] proved that AD implies $\mathsf{CC}_{\mathbb{R}}$. A theorem of Peng and Yu [16] shows that $\mathsf{CC}_{\mathbb{R}}$ follows from TD.[2] It is an open question whether TD implies AD. Woodin has shown that this implication does hold in $L(\mathbb{R})$.

We let μ_{Tu} be the set of $A \subseteq \mathcal{D}_{\mathrm{Tu}}$ which contain a Turing cone. Martin's theorem implies μ_{Tu} is an ultrafilter on the Turing degrees, and $\mathsf{CC}_{\mathbb{R}}$ implies this ultrafilter is countably complete. The following theorem shows that TD implies that $\aleph_1 \not\leq 2^{\aleph_0}$.

Theorem 7.2. *If TD holds then ω_1 is measurable.*

Proof. Let U be the set of $A \subseteq \omega_1$ for which the set

$$\{[x]_{\mathrm{Tu}} : x \in \omega^\omega,\, \omega_1^{L[x]} \in A\}$$

is in μ_{Tu}. Then U is a countably complete ultrafilter on ω_1. □

[1] Add the $\mathsf{Uniformization}$ part and give references.

[2] Talk about the extension to higher degree notions?

These results above apply to a larger class of equivalence relations, including those described by the following definition. We will use a nonstandard definition.

Definition 7.3. *An equivalence relation E on ω^ω is* good *if it contains $\equiv_{\mathrm{Tu}}$ and there exists a preorder $\leq_E$ on ω^ω such that*

- $E = \leq_E \cap \geq_E$;
- *for every $x \in \omega^\omega$ the set $\{y \in 2^\omega : y \leq_E x\}$ is countable;*
- *for all $x, y \in \omega^\omega$, if $x \leq_E y$ then there is $z \geq_{\mathrm{Tu}} x$ such that $z \equiv_E y$.*

In particular, for $S \subseteq \mathrm{Ord}$ the following equivalence relations are good.

- $x \equiv_S y \Leftrightarrow L[S,x] = L[S,y]$ (induced by setting $x \leq_S y$ to be $x \in L[S,y]$);
- $x \equiv_{\mathrm{HOD}_S} y \Leftrightarrow \mathrm{HOD}_{\{S,x\}} = \mathrm{HOD}_{\{S,y\}}$ (induced by setting $x \leq_S^{\mathrm{HOD}} y$ to be $x \in \mathrm{HOD}_{\{S,y\}}$).

Under AD, Martin's theorem implies that if E is a good equivalence relation, then every set of E-degrees either contains or is disjoint from a $\leq_E$-cone, and that the corresponding cone measures for these relations are (countably complete) ultrafilters. We let $\mathcal{D}_S$ be the set of $\equiv_S$-degrees, and μ_S be the corresponding cone measure. We call the corresponding cones S-cones. In the next section we will consider ultraproducts by these measures.

7.2. **Cone Measures and Ultrapowers.** We have seen that AD implies TD and that the Turing measure μ_{Tu} is countably complete, and that the same holds for any good equivalence relation, including the relations $\equiv_S$ and $\equiv_{\mathrm{HOD}_S}$. We will be looking at ultraproducts of the form

$$\prod_{d \in \mathcal{D}_{\mathrm{Tu}}} M_d/\mu_{\mathrm{Tu}}$$

or

$$\prod_{d \in \mathcal{D}_S} M_d/\mu_S,$$

for S a set of ordinals.

If the ultraproduct by the Turing measure is wellfounded, then so is the ultraproduct by any good equivalence relation, since any ultraproduct for a good equivalence relation embeds into the corresponding ultraproduct by the Turing measure. By countable completeness, these ultraproducts are wellfounded provided $\mathsf{TD} + \mathsf{DC}$ holds. Recall that the implication $\mathsf{DC}_{\mathbb{R}} \Rightarrow \mathsf{DC}$ holds in models of the form $L(A, \mathbb{R})$ when A is a subset of $L(\mathbb{R})$. The following theorem of Solovay shows that $\mathsf{DC}_{\mathbb{R}}$ follows from the wellfoundedness of the ultrapower of the ordinals by the cone measure for a good equivalence relation.[3]

Theorem 7.4 (Solovay)**.** *If $(E, \leq_E)$ is a good equivalence relation, μ_E is the corresponding cone measure and the ultrapower of the ordinals by μ_E is wellfounded, then $\mathsf{DC}_{\mathbb{R}}$ holds.*

Woodin has shown that if $V = L(\mathcal{P}(\mathbb{R}))$ and AD^+ holds then the wellfoundedness of the ultrapower of the ordinals by the cone measure on the Turing degrees. The proof uses the reversal of the Derived Model Theorem.

Theorem 7.5 (Woodin)**.** *If $V = L(\mathcal{P}(\mathbb{R}))$ and AD^+ holds then the ultrapower of the ordinals by μ_{Tu} is wellfounded*

[3]Check : The theorem is essentially proved in [**?**], and proof is also give in [**?**].

We will eventually (in Theorem 9.6) be interested in the ordinal represented by the function $x \mapsto \omega_2^{L[S,x]}$ in the μ_S-ultrapower, for a fixed set $S \subseteq \mathrm{Ord}$. First we look at functions of the form $x \mapsto \omega_1^{L[S,x]}$. Given a set S of ordinals, and function f on ω^ω, we say that f is *S-invariant* if $f(x) = f(y)$ whenever $x \equiv_S y$.

Let WO be the set of wellorderings of ω. The proof of Theorem 7.6 uses the following classical fact, known as $\utilde{\Sigma}^1_1$ boundedness: for every analytic $A \subseteq \mathrm{WO}$ there is $\beta < \omega_1$ such that for every $x \in A$, the ordertype of x is less than β. We write $\mathrm{otp}(x)$ for the ordertype of a wellordering x.

Theorem 7.6 (Woodin)**.** *Assume that* AD *holds. Let S be a set of ordinals, define the function $f\colon \omega^\omega \to \omega_1$ by setting $f(x)$ to be $\omega_1^{L[S,x]}$. Then f represents ω_1 in the ultrapower $\prod \mathrm{Ord}/\mu_S$.*

Proof. Let $[f]_S$ denote the element of $\prod \mathrm{Ord}/\mu_S$ represented by f. The countable completeness of μ_S implies that ω_1^V is (isomorphic to) an initial segment of the ultrapower $\prod \mathrm{Ord}/\mu_S$. For each $\alpha \in \omega_1$, $\{[x]_S : f(x) > \alpha\} \in \mu_S$, so $[f]_S \geq \alpha$.

To see that $[f]_S \leq \omega_1$ suppose that $g : \omega^\omega \to \omega_1$ is S-invariant and that for every $x \in \omega^\omega$ we have that $g(x) < \omega_1^{L[S,x]}$. We want to find an $\alpha < \omega_1$ such that $\{[x]_S : g(x) < \alpha\} \in \mu_S$.

Consider the game where

- player I plays $x(i)$ in round i, collectively defining $x \in \omega^\omega$;
- player II plays $(y(i), z(i)) \in \omega \times (\omega \times \omega)$ in round i, collectively defining $y \in \omega^\omega$ and $z \subseteq \omega \times \omega$;
- player II wins if and only if $y \geq_S x$, $z \in \mathrm{WO}$, and the ordertype of z is greater than $g(y)$.

If σ is a strategy for player I, player II can defeat it by playing a $y \in \omega^\omega$ such that $y \geq_{\mathrm{Tu}} \sigma$, and a $z \in \mathrm{WO} \cap L[S,y]$ with ordertype greater than $g(y)$ (such a z exists since $g(y) < \omega_1^{L[S,y]}$). Thus by AD we may fix a winning strategy σ for player II. For each $x \in \omega^\omega$, let $\sigma_y(x)$ denote the $y \in \omega^\omega$ that σ produces when player I plays x, and let $\sigma_z(x)$ denote the set $z \subseteq \omega \times \omega$ that σ produces when player I plays x. Then $\{\sigma_z(x) : x \in \omega^\omega\}$ is analytic and included in WO, so there is $\alpha < \omega_1$ such that for every $x \in \omega^\omega$, the ordertype of $\sigma_z(x)$ is less than α. The set of $x \in \omega^\omega$ such that $x \geq_{\mathrm{Tu}} \sigma$ is an S-cone. For any such x, $\sigma_y(x) \equiv_S x$, and

$$\alpha > \mathrm{otp}(\sigma_z(x)) > g(\sigma_y(x)) = g(x).$$

□

The following application of cone measures is used in the proof of Theorem 9.6, which is the key theorem for producing Suslin representations from ∞-Borel codes. Theorem 7.7 was first proved by Steel assuming AD [18] and later by Woodin from TD by the proof given below. Almost the same proof gives the corresponding result for $\Diamond$. Whether one can also get $\Box$ or forms of condensation seems to be open.

Theorem 7.7. *Suppose that $\aleph_1 \not\leq 2^{\aleph_0}$ and let S be a set of ordinals such that μ_S is an ultrafilter. Then*

$$\{[y]_S : L[S,y] \cap V_{\omega_1} \models \mathsf{GCH}\} \in \mu_S.$$

Proof. We first prove the theorem with CH in place of GCH. It suffices to show that for each $y \in \omega^\omega$ there is $x \in \omega^\omega$ such that $L[S,y][x] \models \mathsf{CH}$. Since $\aleph_1 \not\leq 2^{\aleph_0}$, for each $y \in \omega^\omega$ there exist $L[S,y]$-generic filters for each partial order in $L[S,y] \cap V_{\omega_1}$.

Fix $y \in \omega^\omega$ and force over $L[S,y]$ with the iteration $\mathrm{Col}^*(\omega_1, \omega^\omega) * \mathbb{Q}$ (as defined in $L[S,y]$), where Col^* adds a bijection f from ω_1 to ω^ω by countable approximations and $\mathbb{Q}$ is the almost disjoint coding forcing (see [4]) to make the relation

$$\{(a,b) \in (\omega^\omega)^2 : f^{-1}(x) \leq f^{-1}(y)\}$$

an F_σ subset of $(\omega^\omega)^2$. If $a \in \omega^\omega$ is a real in the corresponding generic extension coding this F_σ set, then $L[S,y][a] \models \mathsf{CH}$.

For the general case of GCH, let $x_0 \in \omega^\omega$ be such that for every $y \geq_S x_0$, $L[S,y] \models \mathsf{CH}$. Fix $y \geq_S x_0$ and an $L[S,y]$-cardinal $\gamma < \omega_1^V$. Let $g \subseteq \mathrm{Col}(\omega,\gamma)$ be an $L[S,y]$-generic filter. Then $L[S,y][g] \models \mathsf{CH}$ so $L[S,y] \models 2^\gamma = \gamma^+$. □

7.3. **A degree notion for sets of ordinals.** The proof of Theorem 9.6 uses the following order on sets of ordinals, induced by their corresponding degree notions. Given $S, T \subseteq \mathrm{Ord}$, we write $S \leq_{\mathcal{D}} T$ to mean that

$$\{[x]_{\mathrm{Tu}} : \omega^\omega \cap L[S,x] \subseteq L[T,x]\} \in \mu_{\mathrm{Tu}}.$$

We write $S <_{\mathcal{D}} T$ for $S \leq_{\mathcal{D}} T \wedge \neg(T \leq_{\mathcal{D}} S)$. A reflection argument shows that for any set S of ordinals there is a bounded $T \subseteq \Theta$ with $S \leq_{\mathcal{D}} T$. Ultimately we will be concerned only with the restriction of $\leq_{\mathcal{D}}$ to bounded subsets of Θ.

Theorem 7.8 shows that $\leq_{\mathcal{D}}$ is a total order when μ_{Tu} is a countably complete ultrafilter. As with Theorem 7.7, Theorem 7.8 was first proved by Steel in [18] assuming AD and later by Woodin from $\mathsf{TD} + \mathsf{CC}_{\mathbb{R}}$. Woodin's proof uses a variation of Mathias forcing.

Theorem 7.8. *Assume that* TD *holds. If* $\neg(T \leq_{\mathcal{D}} S)$ *then*

$$\{[x]_{\mathrm{Tu}} : \omega^\omega \cap L[S,x] \in H(\aleph_1)^{L[T,x]}\} \in \mu_{\mathrm{Tu}}.$$

The proof of Theorem 7.10 uses the following classical result of Solovay. In many of our applications we will have $M = M'$, but we will use the general version at the end of Section 7.4.

Theorem 7.9 (Solovay)**.** *If $M \subseteq M'$ are both transitive models of* ZF*, $\mathbb{P} \in M$ is a partial order, $\mathcal{P}(\mathbb{P}) \cap M'$ is countable, and $x \in M[G]$ for all M'-generic $G \subseteq \mathbb{P}$, then x is in M.*

Theorem 7.10 lists some consequences of Theorem 7.8 which will be used in Section 7.4. Parts (1a) and (2a) follow from Theorems 7.8 and 7.9 by a collapsing argument as in the second half of the proof of Theorem 7.7. Note that both parts apply to all ordinals γ below ω_1^V, not just $\omega_1^{L[T,x]}$. Part (1b) follows from part (1a) and Theorem 7.9, by another collapsing argument. Part (2b) follows from part (2a). Note also that neither part of the theorem assumes (or implies) that S is in $L[T,y]$. A natural application of part (1b) is when S is an ∞-Borel code.

Theorem 7.10. *Assume that* TD *holds, and let S and T be sets of ordinals.*

(1) *If x is a base of a cone witnessing that $S \leq_{\mathcal{D}} T$, then the following hold for all $y \geq_{\mathrm{Tu}} x$.*
   (a) *For every $\gamma < \omega_1^V$, $\mathcal{P}(\gamma) \cap L[S,y] \subseteq L[T,y]$.*
   (b) *For every formula ϕ,*

$$\{z \in \omega^\omega : L[S,z] \models \phi(S,z)\} \cap L[T,y] \in L[T,y].$$

(2) *If $x \in \omega^\omega$ is a base of a Turing cone witnessing that $S <_{\mathcal{D}} T$, then the following hold for all $y \geq_{\mathrm{Tu}} x$.*

(a) *For every* $\gamma < \omega_1^V$*,*

$$\mathcal{P}(\gamma) \cap L[S, y] \in H(|\gamma|^+)^{L[T,y]}.$$

(b) $\omega_1^{L[T,y]}$ *is a strongly inaccessible cardinal in* $L[S, y]$*.*

7.4. **Forcing with sets of reals.** Given a set S of ordinals (and assuming that μ_S induces a wellfounded ultrapower), we let δ_S^∞ be the ordinal represented by the function $x \mapsto \omega_2^{L[S,x]}$. Theorem 7.13 (in conjunction with part (2b) of Theorem 7.10) shows that if S is not $\leq_{\mathcal{D}}$-maximal, then $\delta_S^\infty < \Theta$, which is one of the hypotheses of Theorem 9.7. The proof of Theorem 7.13 uses a result of Martin on pointed perfect trees.

Given a set S of ordinals, a set $A \subseteq \omega^\omega$ is *S-positive* if it intersects every S-cone. A tree is *perfect* if each node has an incompatible pair of extensions.

Definition 7.11. *For* $S \subseteq \mathrm{Ord}$*, an* S-pointed *(perfect) tree is a perfect tree* $a \subseteq \omega^{<\omega}$ *such that for every* $x \in [a]$*,* $a \in L[S, x]$*.*

Theorem 7.12 (Martin)**.** AD *implies that for every* $B \subseteq \omega^\omega$ *and every* $S \subseteq \mathrm{Ord}$*,* B *is S-positive if and only if there is an S-pointed perfect tree* a *with* $[a] \subseteq B$*.*

Given a set T of ordinals, we define the relation $<_T^{\mathrm{c}}$ on $(\omega^\omega)^2$ by setting $(x, y) <_T^{\mathrm{c}} (z, w)$ if $x = z$, $y, w \in L[T, x]$ and y comes before w in the constructibility order on $L[T, x]$ using T and x as parameters. Theorem 7.13 says (in conjunction with part (2) of Theorem 7.10) that if $S \subseteq \mathrm{Ord}$ is not $\leq_{\mathcal{D}}$-maximal, and f is an S-invariant function on ω^ω sending each x to $H(\gamma)^{L[S,x]}$, for some γ below the least strongly inaccessible cardinal of $L[S, x]$, then the structure represented by f in the μ_S ultrapower is coded by a subset of ω^ω. Although it is not the case we are interested in, this fact holds even in the case where this structure is illfounded.

Theorem 7.13 (Woodin)**.** *Suppose that* AD *holds, and let* S *and* T *be sets of ordinals. Suppose that* f *is an S-invariant function on* ω^ω *such that for every* $x \in \omega^\omega$*,* $f(x)$ *is a transitive set in* $H(\aleph_1)^{L[T,x]}$*. Then* $\prod f(x)/\mu_S$ *is isomorphic to a relation projective in* $\leq_S \times <_T^{\mathrm{c}}$*.*

Let S^∞ denote the set represented by the constant function $x \mapsto S$ in the ultrapower given by μ_S. The countable completeness of μ_S give that for all $x, y \in \omega^\omega$, $x \in L[S, y]$ if and only if $x \in L[S^\infty, y]$. It follows that a tree is S-pointed if and only if it is S^∞-pointed.

Let $\mathbb{P}_S$ be the partial order of (not necessarily S-invariant) S-positive sets under containment. Theorem 7.12 implies that $\mathbb{P}_S$ is forcing-equivalent to the partial order of S^∞-pointed perfect trees, also under containment. Therefore any x_G produced by a V-generic filter $G \subseteq \mathbb{P}_S$ is also generic over $L(S^\infty, \mathbb{R})$ for this partial order (which is in $L(S^\infty, \mathbb{R})$ while $\mathbb{P}_S$ may not be).

Since μ_S is countably complete, whenever A is an S-positive set and n is in ω there exists an $s \in \omega^n$ such that $\{x \in A : x \restriction n = s\}$ is S-positive. It follows that if G is a V-generic filter for $\mathbb{P}_S$ then there is a unique $x \in \omega^\omega$ such that, for each $n \in \omega$ the set $\{y \in A : y \restriction n = x \restriction n\}$ is in G. We let x_G denote the unique such x. A standard genericity argument shows that x_G will be Turing-above every element of $\omega^\omega \cap V$. This shows that $\mathbb{P}_S$ collapses every cardinal below Θ^V.

A generic filter G for $\mathbb{P}_S$ induces an ultrapower $\prod_{x \in \omega^\omega} V/G$, whose elements are represented by functions with domain ω^ω. Although we will not prove it here, it turns out that, assuming AD plus the wellfoundednes of the μ_S-ultrapower, the models $\prod_{x\in\omega^\omega} L[S,x]/G$ and $L[S^\infty, x_G]$ are isomorphic.

Now suppose that $A \subseteq \omega^\omega$ is such that $A \cap L[S,x] \in L[S,x]$ for all $x \in \omega^\omega$. Then the function $A \mapsto A \cap L[S,x]$ represents in $\prod_{x\in\omega^\omega} L[S,x]/G$ a set $A^* \subseteq \omega^\omega$ in $V[G]$ such that $A^* \in L[S^\infty, x_G]$ and $A^* \cap (\omega^\omega)^V = A$. It follows that for any V-generic x_G we have $A \in L(S^\infty, \mathbb{R})[x_G]$, so $A \in L(S^\infty, \mathbb{R})$ by Theorem 7.9. Along with part (1b) of Theorem 7.10, this gives Theorem 7.14.

Theorem 7.14 (Woodin)**.** *Assume that* AD *holds, S is a set of ordinals, and the μ_S-ultrapower of the ordinals is wellfounded. If S is a $\leq_{\mathcal{D}}$-maximal set of ordinals, then every ∞-Borel set is in $L(S^\infty, \mathbb{R})$.*

Theorem **??** below shows that, assuming AD^+, the existence of a $\leq_{\mathcal{D}}$-maximal set of ordinals is equivalent to the failure of $\mathsf{Uniformization}$.

By Theorem 6.7, the ∞-Borel sets are all ∞-Borel in $L(S^\infty, \mathbb{R})$, assuming the hypotheses of Theorem 7.14. Wellfoundedness of the μ_S-ultrapower follows from $\mathsf{AD} + \mathsf{DC}$, and also from AD^+, by Theorem 7.5. If AD holds and $A \subseteq \omega^\omega$ is not ∞-Borel, then every ∞-Borel set is ∞-Borel in $L(A, \mathbb{R})$ by Theorem 6.7, and one can apply the theorem in $L(S, A, \mathbb{R})$ assuming only $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. That is, under these assumptions the model $L(S, A, \mathbb{R})$ satisfies the statement that the μ_S-ultrapower is wellfounded. In this case, however, the S^∞ in the statement of the theorem is as computed in $L(S, A, \mathbb{R})$.

8. The Vopěnka Algebra

Let X be a set with $X \in \mathrm{OD}_X$, let z be an element of HOD_X and let λ be the least ordinal such that $\mathcal{P}(\mathcal{P}(z)) \cap \mathrm{OD}_X \subseteq \mathrm{OD}_X^{V_\lambda}$. We let $P_{X,z}$ be the set of triples $(n, \bar{x}, \bar{\alpha})$ such that

- n is the Gödel number of a formula ϕ_n of arity $|\bar{x}| + |\bar{\alpha}| + 1$,
- $\bar{x} \in X^{<\omega}$,
- $\bar{\alpha} \in \lambda^{<\omega}$ and
- the set $b_{n,\bar{x},\bar{\alpha}} = \{C \subseteq z : V_\lambda \models \phi_n(C, \bar{x}, \bar{\alpha})\}$ is nonempty.

Order $P_{X,z}$ by setting $(n, \bar{x}, \bar{\alpha}) \leq (m, \bar{y}, \bar{\beta})$ to hold if $b_{n,\bar{x},\bar{\alpha}} \subseteq b_{m,\bar{y},\bar{\beta}}$. We write $[(n, \bar{x}, \bar{\alpha})]$ for the corresponding equivalence class of $(n, \bar{x}, \bar{\alpha})$ and let $\mathcal{V}_{X,z}$ denote the corresponding partial order on the equivalence classes. Then $\mathcal{V}_{X,z} \in \mathrm{HOD}_X$. Furthermore, for each $a \in z$, the set $T_{X,z}$ consisting of the pairs $(a, [(n, \bar{x}, \bar{\alpha})])$ for which $b_{n,\bar{x},\bar{\alpha}} = \{C \subseteq z : a \in C\}$ is in HOD_X.

For each $C \subseteq z$, we let $G_C^{X,z}$ be the set of $[(n, \bar{x}, \bar{\alpha})]$ for which V_λ satisfies $\phi_n(C, \bar{x}, \bar{\alpha})$. Vopěnka's Theorem says that under these assumptions on X and z, each set $G_C^{X,z}$ is a HOD_X-generic filter for $\mathcal{V}_{X,z}$. The key point in the proof is that union of the sets $b_{n,\bar{x},\bar{\alpha}}$ corresponding to any maximal antichain in $\mathcal{V}_{X,z}$ is all of $\mathcal{P}(z)$.

Theorem 8.1 (Vopěnka)**.** *Let X be a set with $X \in \mathrm{OD}_X$, and let z be an element of HOD_X. For each $C \subseteq z$ the set $G_C^{X,z}$ is a HOD_X-generic filter in $\mathcal{V}_{X,\gamma}$, and C is in $\mathrm{HOD}_X[G_C^{X,z}]$.*

For the rest of this section we restrict to the case where $X = \{S\}$, for some set S of ordinals, and $z = \omega$, and consider the Vopěnka algebra in models of the form $L[S,x]$, where $x \in \omega^\omega$. Using the definability order in $\mathrm{HOD}_{\{S\}}^{L[S,x]}$ we can definably copy $\mathcal{V}^{L[S,x]}_{\{S\},\omega}$ and $T^{L[S,x]}_{\{S\},\omega}$ to a partial order Q^S_x on Ord and a corresponding set $K^S_x \subseteq \omega \times \mathrm{Ord}$, both in $\mathrm{HOD}^{L[S,x]}_{\{S\}}$. Thus for every $c \in \mathcal{P}(\omega) \cap L[S,x]$ there exists a $L[S, Q^S_x, K^S_x]$-generic filter $G_c \subseteq Q^S_x$ with $c \in L[S, Q^S_x, K^S_x][G_c]$.

It follows immediately from either definition that the collection of ∞-Borel sets of ω^ω (which generalizes naturally to subsets of $(\omega^\omega)^n$ for any $n \in \omega$) is closed under complements. We will now outline a proof (assuming $\mathsf{TD} + \mathsf{DC}_{\mathbb{R}}$) that it is also projectively closed. This amounts to showing that if $A \subseteq (\omega^\omega)^2$ is an ∞-Borel set, and $B = \{x \in \omega^\omega : \exists y\, (x,y) \in A\}$, then B is also ∞-Borel. We start with the following lemma.

Lemma 8.2 (Woodin)**.** *Let S be a set of ordinals, and suppose that $\aleph_1 \not\leq 2^{\aleph_0}$. Let ϕ be a ternary formula. Then for all $x \in \omega^\omega$,*

$$\exists y \in \omega^\omega\, L[S,x,y] \models \phi(S,x,y)$$

if and only if, for a μ_S-cone of $z \in \omega^\omega$, the model $L[S, Q^S_z, K^S_z, x]$ satisfies the statement that some partial order of cardinality at most $2^{|Q^S_z|}$ forces the existence of a $y \in \omega^\omega$ such that $L[S,x,y] \models \phi(S,x,y)$.

Proof. The reverse direction follows from the fact that each partial order of cardinality at most $2^{|Q^S_z|}$ in $L[S, Q^S_z, K^S_z, x]$ is countable in V, so $L[S, Q^S_z, K^S_z, x]$-generic filters exist.

For the other direction, fix $x \in B$ and $y \in \omega^\omega$ such that $L[S,x,y] \models \varphi(S,x,y)$. Let $z \in \omega^\omega$ be $\leq_S$-above both x and y. Then z is in the model $L[S, Q^S_z, K^S_z][G^{\{S\},\omega}_z]$, which is a generic extension of $L[S, Q^S_z, K^S_z, x]$ by a quotient of the regular open algebra of Q^S_z. $\square$

The μ_S-ultrapower now gives an ∞-Borel code for the set coded by the pair (S, ϕ).

Theorem 8.3 (Woodin)**.** *Let S be a set of ordinals, and suppose that $\mathsf{DC}_{\mathbb{R}}$ holds and μ_S is an ultrafilter. Let ϕ be a ternary formula and let*

$$B = \{x \in \omega^\omega : \exists y \in \omega^\omega\, L[S,x,y] \models \phi(S,x,y)\}.$$

Then there exist a binary formula ψ and a set T of ordinals such that $T \in \mathrm{OD}_{\{S\}}$ and

$$B = \{x \in \omega^\omega : L[T,x] \models \psi(T,x)\}.$$

Proof. Work in $L(S, \mathbb{R})$, and let

$$S^\infty = \prod S/\mu_S,$$

$$Q^\infty_S = \prod Q^S_z/\mu_S$$

and

$$K^\infty_S = \prod K^S_z/\mu_S.$$

For each $x \in \omega^\omega$, the constant function $y \mapsto x$ represents x in the ultrapower by μ_S. By Lemma 8.2, then, we have that for every $x \in \omega^\omega$, $x \in B$ if and only if $L[S^\infty, Q^\infty_S, K^\infty_S, x]$ satisfies the statement that there exists a partial order of cardinality at most $2^{|Q^\infty_S|}$ forcing the existence of a $y \in \omega^\omega$ such that

$$L[S^\infty, x, y] \models \phi(S^\infty, x, y).$$

$\square$

Restating Theorem 8.3, we have the following.

Theorem 8.4. *If* $\mathsf{TD} + \mathsf{DC}_{\mathbb{R}}$ *holds, then the set of* ∞*-Borel subsets of* $(\omega^\omega)^{<\omega}$ *is projectively closed.*

9. Strong generic codes

Given a nice $\mathbb{P}$-name τ for an element of ω^ω, we let, for each $n \in \omega$, $D_{\tau,n}$ be the set

$$\{p \in \mathbb{P} : \exists m \in \omega\, (p, (n, m)) \in \tau\}.$$

When $\{D_{\tau,n} : n \in \omega\} \subseteq B$, we write $A_{\mathbb{P},B,\tau}$ for the set of values τ_g, where g ranges over the set of all B-generic filters contained in $\mathbb{P}$ (see Section **??** for the definitions of nice name and B-generic filter, and our corresponding notational conventions). When $\mathbb{P}$ is a partial order, we write $\mathrm{dom}(\mathbb{P})$ for the underlying set. Similarly, if B is a sequence, we write $\mathrm{dom}(B)$ for the corresponding index set, and $\mathrm{range}(B)$ for the set indexed by B.

Definition 9.1. *Given* $A \subseteq \omega^\omega$*, an ordinal* α*, a partial order* $\mathbb{P}$*, a sequence* B *of dense open subsets of* $\mathbb{P}$ *and a nice* $\mathbb{P}$*-name* τ *for an element of* ω^ω*, we say that* $(\mathbb{P}, B, \tau)$ *is a* generic code *for* A *if* $\{D_{\tau,n} : n \in \omega\} \subseteq \mathrm{range}(B)$ *and* $A_{\mathbb{P},B,\tau} = A$*. If we say that* C *is a generic code, then we mean that* C *is a triple of the form* $(\mathbb{P}_C, B_C, \tau_C)$*, and write* A_C *for* $A_{\mathbb{P}_C,B_C,\tau_C}$*. If in addition* α *is an ordinal containing* $\mathrm{dom}(\mathbb{P}) \cup \mathrm{dom}(B)$*, then we say that* $(\mathbb{P}, B, \tau)$ *is a* generic α-code *for* A*. We say that* $(\mathbb{P}, B, \tau)$ *is a* generic ∞-code *if it is a generic* α*-code for some ordinal* α*.*

Theorem 8.1 shows that if (S, ψ) is an ∞-Borel* code for a set $A \subseteq \omega^\omega$, then $(\mathcal{V}_{\{S\},\omega^2}\restriction p, B, \tau\restriction p)$ is a generic code for A, where, letting τ be the $\mathcal{V}_{\{S\},\omega^2}$-name for the associated generic real (given by the set K in the proof of Theorem 8.1),

- p is the $\mathcal{V}_{\{S\},\omega}$-condition corresponding to the set
$$\{x \in \omega^\omega : L[S, x] \models \psi(S, x)\},$$
- $\mathcal{V}_{\{S\},\omega^2}\restriction p$ and $\tau\restriction p$ are the corresponding restrictions of $\mathcal{V}_{\{S\},\omega^2}$ and τ below p and
- B is the set of dense open subsets of $\mathcal{V}_{\{S\},\omega^2}\restriction p$ in $\mathrm{HOD}_{\{S\}}$.

Using the definability order on $\mathrm{HOD}_{\{S\}}$, one can convert this generic code to a generic ∞-code. In the proof of Theorem 9.4 we use the fact that, by Theorem 7.7, for a Turing cone of $x \in \omega^\omega$ there exists in $L[S, x]$ a generic $\omega_2^{L[S,x]}$-code for $A \cap L[S, x]$.

Definition 9.2. *Suppose that* $(\mathbb{P}, B, \tau)$ *is a generic* ∞*-code. For any set* X *of ordinals, we let* $\mathbb{P}_X$ *be* $\mathbb{P}\restriction\mathrm{dom}(\mathbb{P}) \cap X$*,* B_X *be* $\langle b_\beta : \beta \in X \cap \mathrm{dom}(B)\rangle$ *and* τ_X *be* $((\mathrm{dom}(\mathbb{P}) \cap X) \times \omega^2) \cap \tau$*. Similarly, if* $C = (\mathbb{P}, B, \tau)$*, we write* C_X *for* $(\mathbb{P}_X, B_X, \tau_X)$*.*

Definition 9.3. *Suppose that* $C = (\mathbb{P}, B, \tau)$ *is a generic* α*-code for a set* A*, for some ordinal* α*. We associate to* $(\mathbb{P}, B, \tau)$ *a game on* $\mathrm{dom}(\mathbb{P}) \cup \mathrm{dom}(B)$*, called* $\mathcal{G}_{\mathbb{P},B,\tau}$ *or* $\mathcal{G}_C$*, where I and II collaborate to build a countable subset* $\sigma \subseteq \mathrm{dom}(\mathbb{P}) \cup \mathrm{dom}(B)$*, and I wins if* $C_\sigma = (\mathbb{P}_\sigma, B_\sigma, \tau_\sigma)$ *is a generic code for a subset of* A*. We say that* $(\mathbb{P}, B, \tau)$ *is a* strong generic code *(or* strong α-generic code*) for* A_ϕ *if II does not have a winning strategy in* $\mathcal{G}_{\mathbb{P},B,\tau}$*.*

Note that a strong generic α-code is also a strong generic β-code for any ordinal $\beta \geq \alpha$.

For any infinite cardinal κ and κ-generic code $(\mathbb{P}, B, \tau)$, κ-$\mathsf{Determinacy}$ implies the determinacy of the game $\mathcal{G}_{\mathbb{P},B,\tau}$, since a run of the game continuously builds a subset of ω coding a generic ω-code.

Unlike ∞-Borel codes, strong generic codes witness Suslinity, in the context of $<\Theta$-$\mathsf{Determinacy}$.

Theorem 9.4. *Let κ be an infinite cardinal and let C be a generic κ-code for a set $A \subseteq \omega^\omega$. If player I has a winning strategy in the game $\mathcal{G}_C$, then A is κ-Suslin. Moreover, if Σ is a winning strategy for player I in $\mathcal{G}_C$, then $A = \mathrm{p}[T]$, for some tree $T \subseteq (\omega \times \kappa \times 2)^{<\omega}$ which is definable from C and Σ.*

Proof. (Sketch) Let C be $(\mathbb{P}, B, \tau)$, and fix a winning strategy Σ for player I in $\mathcal{G}_C$. Then for all $x \in \omega^\omega$, $x \in A$ if and only if there exist a countable $\sigma \subseteq \mathrm{dom}(\mathbb{P}) \cup \mathrm{dom}(B)$ produced by a run of $\mathcal{G}_C$ where I plays according to Σ and a B_σ-generic filter $g \subseteq \mathbb{P}_\sigma$ such that $x = \tau_{\sigma,g}$.

The set of (x, y, z) such that

- $x \in \omega^\omega$,
- $y \in \kappa^\omega$ is a run of $\mathcal{G}_C$ where I plays according to Σ,
- $z \in 2^\omega$ and
- $y[z^{-1}[\{1\}]]$ is a filter on $y[\omega] \cap \mathbb{P}$ for which the realization of
$$\tau_{y[\omega],y[z^{-1}[\{1\}]]}$$
is x

is the set of paths through a tree on $\omega \times \kappa \times 2$ whose projection is A. $\square$

There is a natural join operation on wellordered sequences of generic ∞-codes, which we now define. Suppose that ζ is an ordinal and $\langle C_\alpha : \alpha < \zeta \rangle$ is a sequence of generic ∞-codes. Let C_α be $(\mathbb{P}_\alpha, B_\alpha, \tau_\alpha)$, for each $\alpha < \zeta$. For each $\alpha < \zeta$, let $\leq_\alpha$ be the order on $\mathrm{dom}(\mathbb{P}_\alpha)$ given by $\mathbb{P}_\alpha$. Let X be the set $\{(\alpha, \gamma) : \gamma \in \mathrm{dom}(\mathbb{P}_\alpha)\}$. Let $\leq_X$ the partial order on X defined by setting $(\alpha, \gamma) \leq_X (\alpha', \gamma')$ to hold if $\alpha = \alpha'$ and $\gamma \leq_\alpha \gamma'$. Let $\mathbb{P}_X$ denote the corresponding partial order on X.

For each $\alpha < \zeta$, let B_α be the sequence $\langle b_{\alpha,\delta} : \delta \in \mathrm{dom}(B_\alpha) \rangle$. Let Y be the set $\{(\alpha, \delta) : \delta \in \mathrm{dom}(B_\alpha)\}$. For each pair (α, δ) in Y, let $e_{\alpha,\delta}$ be the set of $(\beta, \gamma) \in X$ such that either $\beta \neq \alpha$ or $\gamma \in b_{\alpha,\delta}$.

Let Z be the set of pairs $((\alpha, \gamma), (n, m))$ such that $(\alpha, \gamma) \in X$ and the pair $(\gamma, (n, m))$ is in τ_α. Then Z is a nice $\mathbb{P}_X$-name for an element of ω^ω. Let B_* be
$$\{e_{\alpha,\delta} : (\alpha, \delta) \in Y\} \cup \{D_{Z,n} : n \in \omega\}.$$
Then $(\mathbb{P}_X, B_*, Z)$ is a generic code for $\bigcup_{\alpha<\zeta} A_{C_\alpha}$.

Next we map $(\mathbb{P}_X, B_*, Z)$ over to a generic ∞-code. Let $f : \eta \to X$ and $g : \xi \to Y$ (for some ordinals η, ξ) be the bijections induced by the Gödel ordering on pairs of ordinals. Let $\mathbb{P}$ be the partial order $f(\gamma) \leq_X f(\gamma')$ on η. Let $B = \langle b_\delta : \delta < \omega + \xi \rangle$ be defined by setting each b_n $(n \in \omega)$ to be $f^{-1}[D_{Z,n}]$ for each $n \in \omega$, and each $b_{\omega+\nu}$ to be $f^{-1}[b_{\alpha,\delta}]$, where $g(\nu) = (\alpha, \delta)$. Let τ be the set of pairs $(\gamma, (n, m))$ for which $(f(\gamma), (n, m)) \in Z$. Let C be $(\mathbb{P}, B, \tau)$. Then C is a generic $(\eta \cup \xi)$-code for $\bigcup_{\alpha<\kappa} A_{C_\alpha}$. We call C the *join* of $\langle C_\alpha : \alpha < \zeta \rangle$. Note that if ζ is an infinite cardinal then $(\eta \cup \xi) = \zeta$.

Theorem 9.5 below shows that the join operation just defined preserves strong generic codes, under the appropriate form of ordinal determinacy.

Theorem 9.5 ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$)**.** *Let κ be an infinite cardinal below Θ such that κ-*Determinacy *holds, and let $\bar{C} = \langle C_\alpha : \alpha < \kappa \rangle$ be a sequence of strong κ-codes. Then the join of $\bar{C}$ is a strong generic κ-code for $\bigcup_{\alpha<\kappa} A_{C_\alpha}$.*

Proof. Let $C = (\mathbb{P}, B, \tau)$ be the join of $\bar{C}$. To see that C is a strong generic code, fix a set $A \subseteq \omega^\omega$ of Wadge rank greater than κ. It suffices to show that player II does not have a winning strategy in $\mathcal{G}_C$ in $L(A, \mathbb{R})$, since $L(A, \mathbb{R})$ contains all subsets of κ, by the Coding Lemma. Since $\mathsf{DC}_{\mathbb{R}}$ holds, DC holds in $L(A, \mathbb{R})$. Fixing a strategy Σ for player II, we can find by DC a countable $\sigma \subseteq \eta \cup \xi$ and winning strategies ρ_α for player I in the games $\mathcal{G}_{C_\alpha}$ ($\alpha \in \sigma \cap \eta$) such that

- $\omega \subseteq g[\sigma]$,
- σ is the result of a run of $\mathcal{G}_C$ where II has played by Σ and
- for each $\alpha \in \sigma \cap \eta$, the set

  $$\sigma_\alpha = \{\gamma \in \mathrm{dom}(\mathbb{P}_\alpha) : (\alpha, \gamma) \in f[\sigma]\} \cup \{\delta \in \mathrm{dom}(B_\alpha) : (\alpha, \delta) \in g[\sigma]\}$$

  is the result of a run of $\mathcal{G}_{C_\alpha}$ where player I has played by ρ_α.

Then

$$A_{C_\sigma} = \bigcup_{\alpha \in \sigma} A_{C_{\alpha, \sigma_\alpha}} \subseteq \bigcup_{\alpha \in \sigma} A_{C_\alpha},$$

showing that I wins this run of the game. $\square$

Next we see how to convert ∞-Borel codes into strong generic codes.

In this section we use the terms μ_S and $\mathcal{D}_S$ from Definition **??**, for S a set of ordinals. Recall from Section **??** that δ_S^∞ denotes $\prod \omega_2^{L[S,x]}/\mu_S$. Recall also from Theorem 9.4 that subsets of ω^ω with strong generic codes are Suslin. The following theorem produces strong generic codes.

Theorem 9.6. *Let S be a set of ordinals. Suppose that the following hold.*

- *The ultrapower $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$ is wellfounded, and $\delta_S^\infty < \Theta$.*
- δ_S^∞-Determinacy.

Then if S is an ∞-Borel code for a set $A \subseteq \omega^\omega$, then A has a strong generic δ_S^∞-code which is contained definable from S.

Aside from the use of AD at the beginning of the proof, the only use of δ_S^∞-Determinacy in the proof of Theorem 9.6 was to show the determinacy of the game $\mathcal{G}_C$. Since each C_y in the proof is a generic code in $L[S, y]$ for a subset of A containing $A \cap L[S, y]$, C is a generic code for A. If A happens to be Suslin, then, $\mathcal{G}_C$ is determined, then, by Remark **??**. It follows that the assumption of δ_S^∞-Determinacy can be removed from the statement of Theorem 9.6, if one assumes instead that A is Suslin. The point then is that the theorem gives a way of picking Suslin representations using ∞-Borel representations, for Suslin sets.

Theorem 9.4 gives the following.

Theorem 9.7 ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + {<}\Theta$-$\mathsf{Determinacy}$)**.** *Every subset of ω^ω which has an ∞-Borel code which is not $\leq_{\mathcal{D}}$-maximal is Suslin.*

Proof. Suppose that $A \subseteq \omega^\omega$ has an ∞-Borel code which is not $\leq_{\mathcal{D}}$-maximal. A reflection argument shows that there exist bounded subsets S and T of Θ such that $S <_{\mathcal{D}} T$ and S is an ∞-Borel code for A. By Theorem **??**, and part (2b) of Theorem

7.10, $\delta^\infty_S < \Theta$. Letting $B \subseteq$ have Wadge rank greater than δ^∞_S, the hypothesis of Theorem 9.6 (with respect to S) are satisfied in $L(B, \mathbb{R})$. It follows from Theorem 9.4 that A is Suslin. □

In addition, one gets the following equivalences under AD^+.

Theorem 9.8 ($\mathsf{ZF} + \mathsf{AD}^+$). *The following are equivalent.*

(1) *There is no $\leq_{\mathcal{D}}$-maximal set of ordinals.*
(2) *The Suslin cardinals are cofinal in Θ.*
(3) *Every subset of ω^ω is Suslin.*
(4) Uniformization

Proof. The equivalence of (2) and (3) is Corollary **??**, which requires only AD. By Theorem 9.7, (1) implies (3). That (3) implies (4) follows from Theorem **??**. The equivalence of (4) and (1) follows from Corollary **??**. □

Recall from Theorem **??** that if every subset of ω^ω has a Wadge rank, and *LipschitzDeterminacy* and Uniformization hold, then the length of the Solovay sequence is a limit ordinal. A version of the converse holds assuming AD^+.

Theorem 9.9. *If AD^+ holds and the Solovay sequence has limit length, then every subset of ω^ω is Suslin.*

Proof. By Theorem 9.7 it suffices to show every subset of ω^ω has an ∞-Borel code which is not $\leq_{\mathcal{D}}$-maximal. Suppose that $A \subseteq \omega^\omega$ is a counterexample. Applying Remark **??**, fix an ∞-Borel code S for A which is a bounded subset of Θ. The set S_∞ from Section **??** is definable from S, and it follows from Theorem 7.14 that every subset of ω^ω is definable from S_∞ and a real. If $B \subseteq \omega^\omega$ has Wadge rank on the Solovay sequence, and greater than the supremum of S, it follows by the Moschovakis Coding Lemma that S, and thus every set of reals, is definable from B and a real. This implies that Θ is the least member of the Solovay sequence above the Wadge rank of B, contradicting our hypothesis that the Solovay sequence has limit length. □

In the remaining two sections we will briefly sketch some structural consequences of AD, AD^+ and $\mathsf{AD}_{\mathbb{R}}$.

10. Suslin Cardinals

A *Suslin cardinal* is an ordinal γ for which there exists an $A \subseteq \omega^\omega$ which is γ-Suslin but not η-Suslin for any $\eta < \gamma$. Any such γ must be a cardinal.

A reflection argument shows that if $A \subseteq \omega^\omega$ is Suslin, then A is γ-Suslin for some $\gamma < \Theta$. It follows that all Suslin cardinals are below Θ. The Moschovakis Coding Lemma implies that, for each $\gamma < \Theta$, there is a surjection from ω^ω onto the set of γ-Suslin sets. It follows that if every subset of ω^ω is Suslin, then the Suslin cardinals are cofinal in Θ. The converse is also true, although its proof is more involved.

Theorem 10.1. *Assuming AD, every subset of ω^ω is Suslin if and only if the Suslin cardinals are cofinal in Θ.*

A join argument shows that the supremum of a countable sequence of Suslin cardinals is Suslin. A theorem of Steel and Woodin shows that if AD holds then the set of Suslin cardinals is closed below its supremum, which by Theorem 10.1 is below Θ if and only if some subset of ω^ω is not Suslin. The following theorem of Woodin shows that under AD^+ the supremum of the Suslin cardinals is also Suslin if it is less than Θ.

Theorem 10.2 (Woodin)**.** *Assuming AD^+, the set of Suslin cardinals is closed below Θ.*

Woodin has shown the converse as well. The proof is well beyond the scope of these notes.

Theorem 10.3 (Woodin)**.** *If $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ holds, then AD^+ holds if and only if the supremum of the Suslin cardinals is either Θ or a Suslin cardinal.*

We will sketch the proof of Theorem 10.2, using Theorems 10.4 and 10.5 below. The proof of Theorem 10.4, which we will not give, uses a classical pointclass analysis for the first conclusion, and an analysis of ultrapowers by cone measures (using the first conclusion) for the second. We say that a subset of ω^ω is $<\kappa$-Borel if it is γ-Borel for some $\gamma < \kappa$, and $<\kappa$-Suslin if it is γ-Suslin for some $\gamma < \kappa$. The ordinal δ^∞_S was defined to be $\prod \omega_2^{L[S,x]}/\mu_{\mathrm{Tu}}$ at the beginning of Section 7.4.

Theorem 10.4. *If AD^+ holds and κ is a limit of Suslin cardinals and the cofinality of κ is uncountable, then every $<\kappa$-Borel set is $<\kappa$-Suslin, and for all bounded $S \subseteq \kappa$, $\delta^\infty_S < \kappa$.*

The following (ZF) theorem of Kunen and Martin generalizes Σ^1_1-boundedness, and follows from applying Σ^1_1-boundedness in a forcing extension by $\mathrm{Col}(\omega, \gamma)$.

Theorem 10.5 (Kunen-Martin)**.** *For any cardinal γ, every γ-Suslin prewellordering of ω^ω has length less than γ^+.*

Proof of Theorem 10.2. It suffices to show that if $\kappa < \Theta$ is a limit of Suslin cardinals of uncountable cofinality, then κ is a Suslin cardinal. Fixing such a κ, there is by the Moschovakis Coding Lemma a set $A \subseteq \omega^\omega$ which is not κ-Borel. Since A is ∞-Borel, the $*$-operation from Section 6.2 applied to an ∞-Borel code (in $\mathcal{L}_{\infty,0}$) for A gives a κ-sequence of ∞-Borel codes for disjoint nonempty subsets of ω^ω. The proof of the second half of Theorem 6.4 then produces a κ-length disjunction of $\mathcal{L}_{\kappa,0}$-sentences defining a prewellordering of length κ. A uniform version of Theorem 9.6 applied to the members of this disjunction, along with Theorem 9.4, gives a strong ∞-Borel code in $\mathcal{L}_{\kappa,0}$ for this prewellordering. Theorem 9.4 then implies that this prewellordering is κ-Suslin. Theorem 10.5 implies that the prewellordering cannot be γ-Suslin for any $\gamma < \kappa$. $\square$

11. The Solovay Sequence

The *Solovay sequence* [17] is the unique continuous sequence $\langle \theta_\alpha : \alpha \leq \beta \rangle$ such that

- θ_0 is the least ordinal which is not the surjective image of ω^ω under an OD function;
- for every ordinal α such that $\alpha + 1 \leq \beta$, $\theta_{\alpha+1}$ is the least ordinal which is not the surjective image of ω^ω under an $\mathrm{OD}_{\{A\}}$ function for any $A \subseteq \omega^\omega$ of Wadge rank θ_α;
- $\theta_\beta = \Theta$.

We call β the *length* of the Solovay sequence. In $L(\mathbb{R})$, $\theta_0 = \Theta$, so $\beta = 0$. For each $\alpha \leq \beta$, if we let Γ_α be the set of subsets of ω^ω of Wadge rank less than θ_α, then $\mathrm{HOD}_{\Gamma_\alpha}$ is a model of ZF containing ω^ω whose subsets of ω^ω are exactly the members of Γ_α. Every successor member of the Solovay sequence below Θ has cofinality ω ([12] contains a proof).

The proof of Theorem 1.3 shows that if $\mathsf{AD}+\mathsf{Uniformization}$ holds then β is a limit ordinal. On the other hand, Theorems 7.14 and 9.7 (noting that every bounded subset of Θ is in $L(A, \mathbb{R})$ for some $A \subseteq \omega^\omega$, and that the μ_S ultrapower of a set S of ordinals ordinal definable from S) give the following theorem, the conclusion of which implies $\mathsf{AD}_\mathbb{R}$ by Theorem 3.3.

Theorem 11.1 (Woodin)**.** *If* AD^+ *holds and the Solovay sequence has limit length, then every subset of* ω^ω *is Suslin.*

If the length of the Solovay sequence is at least $\alpha + 1$, then there is a largest Suslin cardinal below $\theta_{\alpha+1}$. The largest Suslin cardinal below $\theta_{\alpha+1}$ is either θ_α or $\delta^2_1(A)$ for any $A \subseteq \omega^\omega$ of Wadge rank α. These cases are mutually exclusive, and the former can happen only when α is a limit ordinal.

As a final note, we observe that the length of the Solovay sequence in a model of $\mathsf{AD} + \mathsf{DC}_\mathbb{R} + V = L(\mathcal{P}(\mathbb{R}))$ determines whether or not DC holds in the model. If Θ has countable cofinality, then DC fails. If the Solovay sequence has length 0, or successor length, then DC holds in $L(\mathcal{P}(\mathbb{R}))$ if $\mathsf{DC}_\mathbb{R}$ does. Solovay [17] showed that, assuming $\mathsf{AD}_\mathbb{R}$, DC holds in $L(\mathcal{P}(\mathbb{R}))$ if and only the cofinality of Θ is uncountable. In conjunction with Theorem 11.1 we get that under AD^+, DC fails in $L(\mathcal{P}(\mathbb{R}))$ if and only if, in $L(\mathcal{P}(\mathbb{R}))$, the length of the Solovay sequence is a limit ordinal of countable cofinality.

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