Polar forcings and measured extensions*

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Abstract

An ideal I on a Polish space X is said to be polar if I is the intersection of the null ideals for some family of Borel probability measures on X. We study polar ideals where the corresponding family of measures is analytic and the induced forcing of Borel sets modulo I is proper. We show that for a broad class of examples this property is closed under iterations, and that the universally measurable sets of the ground model reinterpret as universally measurable sets in the corresponding extensions.

1 Introduction

In this note, we isolate a rather natural class of proper partial orders, namely the orders for which one can naturally assign a Borel probability measure to each condition. We show that partial orders of this type can be iterated with countable support, preserving the property in question. As a result, the iterated extensions possess interesting and uncommon properties.

In Section 3 we isolate the class of partial orders in question. These are quotient partial orders of the form P_I, the Borel I-positive sets ordered by inclusion, for a σ-ideal I on a Polish space X which is the intersection of null ideals associated with an analytic set of Borel probability measures on X; such quotient posets we call $\mathbf{\Sigma}^1_1$-*polar* (Definition 3.1). There are many natural examples as well as sophisticated ones.

In Section 4 we show that countable support iterations of countable length of $\mathbf{\Sigma}^1_1$-polar posets are again $\mathbf{\Sigma}^1_1$-polar (Theorem 4.1). This makes it possible to prove preservation theorems for iterations of arbitrary length.

In Section 5, we take an axiomatic approach to preservation theorems for iterations of $\mathbf{\Sigma}^1_1$-polar forcings. We isolate the notion of a measured extension (Definition 5.1). Measured extensions may not be generated by polar forcings

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or may not even be forcing extensions at all. However, iterations of polar forcings do generate measured extensions (Theorem 5.12). We abstractly derive several properties of measured extensions. Not surprisingly, such extensions are bounding and preserve outer Lebesgue measure.

It turns out that in measured extensions the interpretations of the ground model universally measurable sets are universally measurable (Theorem 5.6). This is helpful in the analysis of universally measurable sets in such extensions, but it is also in some sense a negative result. One major open question in the theory of universally measurable sets is whether consistently all universally measurable sets can have the Baire property [12]. Under the Continuum Hypothesis (and certain weakenings) there exist medial limits [13, 2], which induce universally measurable ideals without the Baire property. In a forcing extension in which universally measurable sets reinterpret as universally measurable sets, medial limits reinterpret as medial limits. This shows that any forcing extension of a model of CH in which all universally measurable sets have the property of Baire must not be measured.

The authors are very happy to be able to contribute to this volume in honor of Kenneth Kunen, whose [8] has introduced generations of set theorists to the field. In addition, the analysis of the measure algebra in Kunen's [9] was instrumental in [10], which lead in part to the current paper.

2 Interpretations

The notation used in this note follows the set theoretic standard of [4]. In addition, given transitive models $N_0 \subseteq N_1$ of set theory, and in N_0 a Polish space X and a Borel set $B \subseteq X$, we denote by $i(X)$ and $i(B)$ their interpretations in the model N_1 (see [5, 15]). Interpretations of Borel sets commute with complements, continuous preimages, countable products, and countable unions and intersections, among other things. If the models $N_0 \subseteq N_1$ in addition have the same ordinals, and in N_0, $A \subseteq X$ is an analytic set, we write $i(A)$ for its interpretation in the model N_1. By a Shoenfield absoluteness argument, interpretation of analytic sets commutes with continuous images: if $f\colon X_0 \to X_1$ is a continuous function and $A_0 \subset X_0$, $A_1 \subset X_1$ are analytic sets in the model N_0 such that $A_1 = f''A_0$, then $i(A_1) = i(f)''i(A_0)$ holds in the model N_1.

A σ-ideal I of analytic sets on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every Polish space Y and every analytic set $A \subseteq Y \times X$ the set $\{y \in Y \colon A_y \in I\}$ is coanalytic. Every $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal I has a natural reinterpretation in every generic extension $V[G]$. To define the reinterpretation, just choose a universal analytic set $A \subset 2^\omega \times X$ and in $V[G]$, reinterpret the analytic set

$$C = \{y \in 2^\omega \colon A_y \notin I\},$$

and define the reinterpretation of I as the set of all subsets of vertical sections A_y, for $y \notin i(C)$. The nontrivial point here is that this is indeed a σ-ideal of analytic sets which does not depend on the choice of the universal analytic set A [5, Theorem 2.4.12].

By a measure on a Polish space X we always mean a Borel probability measure. The Polish space of Borel probability measures on X is denoted by $P(X)$. If X, Y are Polish spaces, μ is a measure on X, and $f\colon X \to Y$ is a Borel function, then the *pushforward measure* $f_*(\mu)$ on Y is defined by $f_*(\mu)(B) = \mu(f^{-1}B)$ for all Borel sets $B \subseteq Y$. We repeatedly use a basic complexity calculation:

Fact 2.1. (Kondô–Tugué, [6, Theorem 29.26]) *If X, Y are Polish spaces, $A \subseteq Y \times X$ is an analytic set, and ε is a nonnegative real number, then the set $\{\langle y, \mu\rangle \in Y \times P(X)\colon \mu(A_y) > \varepsilon\} \subseteq Y \times P(X)$ is analytic.*

3 Polar forcings

We start with a central definition.

Definition 3.1. Let I be a σ-ideal on a Polish space X which is generated by analytic sets. We say that I is *polar* if there is a set $M \subseteq P(X)$ such that the following are equivalent for every analytic set $A \subseteq X$:

1. $A \notin I$;
2. there is a measure $\mu \in M$ such that $\mu(A) > 0$;
3. there is a measure $\mu \in M$ such that $\mu(A) = 1$;

We say that an ideal I is a $\mathbf{\Sigma}^1_1$*-polar* if there exists an analytic $M \subseteq P(X)$ witnessing that I is polar. If the quotient forcing P_I is proper in all forcing extensions, then we say that I is an *iterable* $\mathbf{\Sigma}^1_1$*-polar ideal.* The quotient poset P_I is then a $\mathbf{\Sigma}^1_1$*-polar forcing.*

Some remarks are in order. If I is a polar ideal as witnessed by a set $M \subseteq P(X)$, then clearly I can be recovered from M as the σ-ideal of all analytic sets $A \subseteq X$ such that $\mu(A) = 0$ for all $\mu \in M$. Item (3) is a normalization demand which is included just for convenience: if we find a set M of measures satisfying the equivalence of (1) and (2), it is possible to extend it to a collection M' of measures satisfying the equivalence of (1, 2, 3) by letting $\mu \in M'$ if there exist a measure $\nu \in M$ and a countable collection of pairwise disjoint ν-positive compact sets C_n for $n \in \omega$ such that for every basic open set $O \subseteq X$,

$$\mu(O) = \mu(\bigcup_n C_n \cap O)/\mu(\bigcup_n C_n).$$

Clearly, if M is analytic, then so is M'. If $\nu(A) > 0$ then $\{C_n : n \in \omega\}$ can be chosen so that $\mu(A) = 1$.

If the ideal I is $\mathbf{\Sigma}^1_1$-polar as witnessed by an analytic set of measures $M \subseteq P(X)$, then I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ by Fact 2.1. The analytic set M cannot be easily recovered from I: the natural candidate $M' = \{\mu \in P(X)\colon$ for every analytic set $A \in I$, $\mu(A) = 0\}$ is coanalytic by Fact 2.1 again, but it apparently does not have to be analytic. We do get an additional piece of information from the set M'. Since $M \subseteq M'$ holds, by Suslin's theorem there is a Borel set $M'' \subseteq P(X)$ such that $M \subseteq M'' \subseteq M'$ and then M' is in fact a Borel set of measures witnessing that I is $\mathbf{\Sigma}^1_1$-polar.

The consideration of all forcing extensions is awkward, but apparently necessary, and it is trivial in all specific cases. Since the ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, it has a natural reinterpretation in all forcing extensions, as outlined in Section 2. The reinterpretation of the witnessing analytic set of measures will witness the polarity of the ideal in any extension. Absoluteness of properness of the quotient forcing is a wide open question even though for all known examples it is easily seen to hold.

We illustrate the definition with several salient examples.

Example 3.2. Let I be the ideal of countable sets on an uncountable Polish space X. Then I is an iterable $\mathbf{\Sigma}^1_1$-polar ideal, as witnessed by the set M of all nonatomic measures μ on X. The quotient forcing is Sacks forcing.

Example 3.3. Let ν be a Borel probability measure on a Polish space X, and let I be the ideal of ν-null sets. Then I is an iterable $\mathbf{\Sigma}^1_1$-polar idea by its definition. The quotient forcing is the corresponding notion of random forcing.

Example 3.4. Let I be the ideal on $X = (2^\omega)^\omega$ generated by those analytic sets $A \subseteq X$ which do not contain a subset of the form $\prod_n C_n$ where each $C_n \subset 2^\omega$ is a perfect set. It is well-known that I is a σ-ideal and the quotient poset P_I is just the full support product of countably many copies of the Sacks forcing [14, Theorem 5.2.6]. To see that it is proper, first let μ be the product of ω many copies of the usual Haar measure on the Cantor group. A standard argument shows that every analytic μ-positive subset contains a product $\prod_n C_n$ of countably many perfect sets, and so is I-positive. Now, for each I-positive set $A \subseteq X$ there are continuous injective functions $f_n \colon 2^\omega \times 2^\omega$ such that the range of the function $\prod_n f_n$ is a subset of A. Thus, I is $\mathbf{\Sigma}^1_1$-polar as witnessed by the set M of all pushforwards of the measure μ along all continuous functions of this type. It is iterable as properness of the countable support product of Sacks forcing is a theorem of ZFC.

Example 3.5. Let $X = 2^\omega$, let $\mathbb{E}_0$ be the modulo finite equality of elements of X, and let I be the σ-ideal generated by analytic partial $\mathbb{E}_0$-selectors. Then I is a $\mathbf{\Sigma}^1_1$-polar ideal. To see this, first argue that the usual Haar probability measure μ on the Cantor group gives zero mass to all analytic $\mathbb{E}_0$-selectors. In addition, if $A \notin I$ is an analytic set then there is a function $f \colon \omega \times 2 \to 2^{<\omega}$ such that for each $n \in \omega$, $f(n, 0), f(n, 1)$ are distinct binary strings of the same length and the continuous function $\hat{f} \colon 2^\omega \to 2^\omega$ given by letting $f(x)$ be the concatenation of $f(n, x(n))$ for $n \in \omega$, has range included in A. Now one can let M be the set of all pushforwards of the measure μ along all functions of this type. In addition, the poset P_I is proper [14, Section 4.7.1] so I is an iterable $\mathbf{\Sigma}^1_1$-polar ideal.

Example 3.6. Let ϕ be a strongly subadditive outer regular Choquet capacity on a Polish space X. Then ϕ is an envelope of measures [1]: $\phi(B) =$

$\sup\{\mu(B)\colon \mu \in P(X) \text{ and } \mu \leq \phi\}$ holds for all analytic sets $B \subseteq X$. Thus, the ideal $I = \{A \subseteq X\colon \phi(A) = 0\}$ is $\mathbf{\Sigma}^1_1$-polar as witnessed by the set $M = \{\mu \in P(X)\colon$ for every basic open set $O \subseteq X$, $\mu(O) \leq \phi(O)\}\}$. For many outer regular strongly subadditive capacities, such as the Newtonian capacity on $X = \mathbb{R}^3$, the quotient forcing P_I is proper [14, Section 4.3.2, 4.3.3] so the ideal I is in fact iterable $\mathbf{\Sigma}^1_1$.

Example 3.7. Let X be the unit circle in the complex plane, and let M be the (analytic) collection of Rajchman measures on it. Let I be the σ-ideal of all sets which are μ-null for all measures $\mu \in M$. The ideal I is σ-generated by closed sets by a theorem of Debs and Saint Raymond [7, VIII.3, Theorem 1]; therefore, the quotient forcing is proper by [14, Theorem 4.1.2]. The ideal I is the ideal of sets of extended uniqueness.

Naturally, we want to see also some examples of σ-ideals which are not polar. Some such examples are relatively unsophisticated.

Example 3.8. Let I be the σ-ideal on the Baire space ω^ω generated by compact sets. Then I is not polar: for every measure μ on ω^ω, by its regularity there are compact sets $K_n \subset \omega^\omega$ such that $\mu(\bigcup_n K_n) = 1$. At the same time, $\bigcup_n K_n \in I$.

Other examples are significantly more difficult to verify.

Example 3.9. The σ-ideal I of σ-porous subsets of $\mathbb{R}$ is not polar. Preiss and Humke [3] found a closed I-positive set $C \subset \mathbb{R}$ such that for every measure μ on C there is a Borel set $B \subset C$ in the σ-ideal I such that $\mu(B) = 1$.

4 Polar iterations

The main theorem of this section shows that countable support iterations of iterable $\mathbf{\Sigma}^1_1$-polar ideals result in iterable $\mathbf{\Sigma}^1_1$-polar ideals.

Theorem 4.1. *Let X be a Polish space and I be an iterable $\mathbf{\Sigma}^1_1$-polar ideal. Let $\alpha \in \omega_1$ be an ordinal. Then I^α is an iterable $\mathbf{\Sigma}^1_1$-polar ideal on X^α.*

Proof. We need an iteration operation on coanalytic sets of measures. Let X_0, X_1 be Polish spaces and M_0, M_1 be analytic subsets of $P(X_0), P(X_1)$ respectively. We define $M_0 * M_1$ to be the set of all measures μ on $X_0 \times X_1$ such that there exist a measure $\mu_0 \in M_0$ and a Borel function $f\colon X_0 \to P(X_1)$ with range contained in M_1 such that for every Borel set $B \subseteq X_0 \times X_1$, $\mu(B) = \int f(x)(B_x)\, d\mu_0(x)$.

Claim 4.2. *The set $M_0 * M_1 \subseteq P(X_0 \times X_1)$ is analytic.*

Proof. Recall the measure disintegration theorem [6, Exercise 17.35]: for every measure μ on $X_0 \times X_1$, letting μ_0 be the pushforward measure on X_0 induced by the projection map, there exists a Borel function $f\colon X \to P(X_1)$ such that for every Borel set $B \subseteq X_0 \times X_1$, $\mu(B) = \int f(x)(B_x)\, d\mu_0(x)$. Moreover, the evaluation of the pushforward measure is a continuous function from $P(X_0 \times X_1)$

to $P(X_0)$ and the Borel function f is unique modulo μ_0. Let $C \subseteq P(X_1) \times \omega^\omega$ be a closed set projecting to M_1 and let $\pi \colon C \to P(X_1)$ be the projection map. It is not difficult to see that for a measure μ on $X_0 \times X_1$, $\mu \in M_0 * M_1$ holds if and only if both of the following hold:

- the pushforward μ_0 belongs to M_0;
- there exist a collection of pairwise disjoint compact sets $K_n \subseteq X_0$ for $n \in \omega$ and continuous functions $g_n \colon K_n \to C$ such that for all basic open sets $O_0 \subseteq X_0$ and $O_1 \subseteq X_1$,

$$\mu(O_0 \times O_1) = \sum_n \int_{x \in K_n \cap O_0} (\pi \circ g_n(x))(O_1)\ d\mu_0(x).$$

This proves the claim since both items are clearly analytic statements. $\square$

The $*$ operation is easily seen to be noncommutative, even accounting for the induced reversal of coordinates. However, it is easily seen to be associative. The operation also extends to include infinite iterations. Let X_n for $n \in \omega$ be Polish spaces and $M_n \subseteq P(X_n)$ be analytic sets. Let $*_n M_n \subseteq P(\prod_n X_n)$ be the set of measures μ on $\prod_n X_n$ such that for each $n \in \omega$, the pushforward of μ to $\prod_{m \in n} X_m$ belongs to $M_0 * M_1 * \cdots * M_{n-1}$. It is clear that the set $*_n M_n \subseteq P(\prod_n X_n)$ is analytic.

Now we apply the operation of iteration of sets of measures to the iteration of polar ideals.

Claim 4.3. *Let I_0 and I_1 be iterable $\mathbf{\Sigma}^1_1$-polar ideals on respective Polish spaces X_0 and X_1. Then $I_0 * I_1$ is an iterable $\mathbf{\Sigma}^1_1$-polar ideal.*

Proof. The whole argument below should be applied in an arbitrary forcing extension. Since I_0 and I_1 are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, so is the iteration $I_0 * I_1$ as proved in [14, Section 5.1.3]. The quotient forcing is proper as it is the iteration of the quotient forcings P_{I_0} and P_{I_1}. We need only verify then that if M_0 and M_1 are analytic sets of measures witnessing the respective polarity of ideals I_0 and I_1 then $M_0 * M_1$ witnesses the polarity of the ideal $I_0 * I_1$.

Let $B \subseteq X_0 \times X_1$ be an analytic $I_0 * I_1$-positive set. Shrinking the set if necessary, we may assume that B is in fact Borel, the projection $B_0 \subseteq X$ of B to X_0 is Borel and I_0-positive, and for all $x \in B_0$ the vertical section B_x is I_1-positive. Let $\mu_0 \in M_0$ be a measure assigning the set B_0 mass one. The set $D = \{\langle x, \nu \rangle \in X_0 \times P(X_1) \colon x \in B_0, \nu \in M_1, \nu(B_x) = 1\}$ is analytic by Fact 2.1, with all sections above B_0 nonempty. By the Jankov–von Neumann uniformization theorem [6, Theorem 18.1] there is a Borel set $C \subseteq B_0$ of full μ_0-mass and a Borel function $f \colon C \to P(X_1)$ such that for all $x \in C$, $\langle x, f(x) \rangle \in D$. Let μ be the measure on $X_0 \times X_1$ defined by $\mu(E) = \int f(x)(E_x)\ d\mu_0(x)$, observe that $\mu \in M_0 * M_1$, and $\mu(B) = 1$ as desired.

For the other direction, let $B \subseteq X_0 \times X_1$ be an analytic set in the ideal $I_0 * I_1$ and $\mu \in M_0 * M_1$ be a measure. The set $B_0 = \{x \in X_0 \colon B_x \notin I_1\}$ is analytic as the ideal I_1 is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. It is also in the ideal I_0 as $B \in I_0 * I_1$ holds. Thus, the pushforward μ_0 of μ to X_0 assigns it mass zero, as $\mu_0 \in M_0$. Let $f\colon X_0 \to M_1$ be the disintegration function for μ as posited in the definition of $M_0 * M_1$. For all $x \in X_0 \setminus B_0$ it is the case that $f(x)(B_x) = 0$ as $B_x \in I_1$ and $f(x) \in M_1$. We conclude that $\mu(B) = \int f(x)(B_x)\, d\mu_0(x) = 0$ as desired. $\square$

Claim 4.4. *Let X_n for $n \in \omega$ be Polish spaces and for each $n \in \omega$ let I_n be an iterable $\mathbf{\Sigma}^1_1$-polar ideal on X_n. Then $*_n I_n$ is an iterable $\mathbf{\Sigma}^1_1$-polar ideal.*

Proof. The whole argument below should be applied in an arbitrary forcing extension. Let $\pi_m\colon \prod_{n\in\omega} X_n \to \prod_{n\in m} X_n$ denote the projection maps for each $m \in \omega$. Since I_n are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, so is the iteration $*_n I_n$ as proved in [14, Section 5.1.3]. The quotient forcing is proper as it is the iteration of the quotient forcings P_{I_n} for $n \in \omega$. We show that $*_n M_n$ witnesses the polarity of the ideal $*_n I_n$.

Let $B \subset \prod_n X_n$ be an analytic $*_n I_n$-positive set. Shrinking the set B if necessary, we may assume [14, Theorem 5.1.9] that

- the set B is Borel,
- the projections $\pi_n'' B$ are Borel for each $n \in \omega$,
- for each $n \in \omega$ and each $x \in \pi_n'' B$ the set $\{y \in X_n\colon x^\frown y \in \pi_{n+1}'' B\}$ is I_n-positive, and
- for each point $x \in \prod_n X_n$, if $x \restriction n \in \pi_n'' B$ for all $n \in \omega$ then $x \in B$.

Now, by induction on $n \in \omega$ let μ_n be a measure on $\prod_{m\in n} X_m$ such that $\mu_n \in *_{m\in n} M_m$ and $\mu_n(\pi_n'' B) = 1$ and the pushforward of μ_{n+1} to $\prod_{m\in n} X_m$ is μ_n; such a sequence is obtained by an application of the proof of Claim 4.3. Let μ be the limit of the measures μ_n: that is, μ_n is a measure on $\prod_n X_n$ such that for every basic open set $O = \{x \in \prod_n X_n\colon x(m) \in P\}$ for some open set $P \subseteq X_m$, $\mu(O) = \mu_{m+1}(P)$. Clearly, $\mu \in *_n M_n$; we need to conclude that $\mu(B) = 1$. However, this is immediate from the fact that $B = \bigcap_n C_n$ where $C_n = \pi_n'' B \times \prod_{m\geq n} X_m$, and each of the sets C_n has full μ-mass.

Now let $B \subset \prod_n X_n$ be an analytic set in the ideal $*_n I_n$. By the proof of [14, Theorem 5.1.9], there are coanalytic sets $C_n \subseteq \prod_{m\in n+1} X_m$ for $n \in \omega$ such that for each $n \in \omega$ and each $x \in \prod_{m\in n}$ the vertical section $(C_n)_x$ is Borel and belongs to the ideal I_n, and for each $x \in B$ there exists $n \in \omega$ such that $x \restriction n+1 \in C_n$. Let $\mu \in *_n M_n$; we have to show that $\mu(B) = 0$. Let μ_n be the pushforward of μ to $\prod_{m\in n} X_m$, so $\mu_n \in *_{m\in n} M_m$. The definition of the set $*_{m\in n+1} M_m$ as well as the fact that the measures in M_n assign mass zero to all Borel sets in the ideal I_n shows that $\mu_{n+1}(C_n) = 0$. Thus, for each $n \in \omega$ $\mu(\{x \in X\colon x \restriction n+1 \in C_n\}) = 0$ and $\mu(B) = 0$ as desired. $\square$

The theorem now follows by an easy transfinite induction on α. $\square$

5 Measured extensions

The axiomatic treatment for preservation theorems for polar forcings starts with the following abstract notion.

Definition 5.1. Let $N_0 \subseteq N_1$ be transitive models of set theory with the same ordinals. We say that N_1 is a *measured extension of* N_0 if the following hold.

1. Every set $a_1 \in N_1$ such that $a_1 \subseteq N_0$ and $N_1 \models a_1$ is countable is a subset of a set $a_0 \in N_0$ such that $N_0 \models a_0$ is countable.
2. Suppose that $\varepsilon > 0$ is a rational number, $X \in N_0$ is a Polish space and $N_0 \models F$ is a collection of Borel subsets of X. If there is a Borel probability measure μ_1 on $i(X)$ in the model N_1 such that for every $B \in F$, $\mu_1(i(B)) \geq \varepsilon$, then there is a Borel probability measure μ_0 in the model N_0 such that for every $B \in F$, $\mu_0(B) \geq \varepsilon$.

Propositions 5.3-5.5 below outline some of the basic properties of measured extensions. We start with a simple technical proposition which will be used many times over in our arguments.

Proposition 5.2. *Suppose that in N_0, for each Borel probability measure μ on X, $B_\mu \subseteq X$ is a Borel set of μ-mass smaller than one. Let $x \in N_1$ be a point in $i(X)$. Then there is μ such that $x \notin i(B_\mu)$ holds.*

Proof. In the model N_0, let $F = \{B_\mu \colon \mu \in P(X)\}$. There is no Borel probability measure on X which gives each set in F mass one, because F contains a null set for each Borel probability measure. Suppose towards a contradiction that $x \in \bigcap_\mu i(B_\mu)$ holds. Then in N_1, the Borel probability measure giving the singleton $\{x\}$ mass one would give each set $i(B)$ for $B \in F$ mass one, contradicting the assumption that N_1 is a measured extension of N_0. □

Next we present three classical and not particularly surprising properties of measured extensions.

Proposition 5.3. *Measured extensions are bounding.*

Proof. This is to say that if N_0 is a transitive model of ZFC and N_1 is a measured extension of N_0 then every function in $\omega^\omega \cap N_1$ is bounded by a function in $\omega^\omega \cap N_0$. Indeed, work in the model N_0, let $X = \omega^\omega$, and for each Borel probability measure μ on X find a K_σ-subset $B_\mu \subseteq X$ of μ-mass one. By Proposition 5.2, for every point $x \in \omega^\omega \cap N_1$ there must be a measure $\mu \in N_0$ such that $x \in i(B_\mu)$. This means that $x \in i(K)$ for some compact set $K \subset \omega^\omega$ in the model N_0. Each compact subset of X is dominated by a single function, so there is $y \in \omega^\omega \cap N_0$ such that x is dominated by y as desired. □

Proposition 5.4. *Measured extensions preserve outer measure.*

Proof. This is to say that if

- N_0 is a transitive model of ZFC and N_1 is a measured extension of N_0,
- X is a Polish space in the model N_0 and μ is a Borel probability measure on X,
- $\varepsilon < 1$ is a rational number and
- $A \subseteq X$ is a set of outer μ-mass greater than ε,

then in N_1, A still has outer $i(\mu)$-mass greater than ε. To see this, let Y be the set of all open subsets of X of μ-mass $\leq \varepsilon$. Equip Y with a natural Polish topology. For each Borel probability measure ν on Y, the Fubini theorem implies that the set $C_\nu = \{x \in X : \{y \in Y : x \in y\}$ has ν-mass smaller than $1\}$ has μ-mass at least $1 - \varepsilon$. Pick a point $x_\nu \in A \cap C_\nu$ and let $B_\nu = \{y \in Y : x_\nu \in y\}$. By Proposition 5.2, in the model N_1 for every point $y \in i(Y)$ there is a measure $\nu \in N_0$ such that $y \notin i(B_\nu)$ so $x_\nu \notin y$. It follows that outer μ-mass of A in the model N_1 is greater than ε. $\square$

Proposition 5.5. *The relation of being a measured extension is transitive.*

Proof. Let N_1 be a measured extension of N_0 and let N_2 be a measured extension of N_1; we must show that N_2 is a measured extension of N_0. To verify (1) of Definition 5.1, suppose that $a_2 \subseteq N_0$ is a set which belongs to N_2 and is countable there. Since N_2 is a measured extension of N_1, a_2 is covered by a set $a_1 \in N_1$ which is countable there. Since N_1 is a measured extension of N_0, the set $a_1 \cap N_0$ is covered by a set $a_0 \in N_0$ which is countable there. Clearly, a_0 witnesses (1).

To verify (2) of Definition 5.1, suppose that $X \in N_0$ is a Polish space and $F_0 \in M_0$ is a σ-complete filter of Borel subsets of X and $\varepsilon > 0$ is a rational number. Suppose that $\mu_2 \in N_2$ is a Borel probability measure on $i(X)$ such that $\mu_2(i(B)) \geq \varepsilon$ for all $B \in F_0$. Let $F_1 \in M_1$ be the collection $\{i(B) : B \in F\}$. Clearly, $\mu_2(i(C)) \geq \varepsilon$ holds for all $C \in F_1$. Since N_2 is a measured extension of N_1, there is a Borel probability measure $\mu_1 \in N_1$ such that $\mu(C) \geq \varepsilon$ for all $C \in F_1$. Since N_1 is a measured extension of N_0, there is a Borel probability measure $\mu_0 \in N_0$ such that $\mu_0(B) \geq \varepsilon$ for all $B \in F_0$. The proof of (2) is complete. $\square$

As a less classical feature of measured extensions, we will show that in measured extensions the σ-algebra of universally measurable sets allows a unique interpretation commuting with many natural operations on universally measurable sets. Let X be a Polish space. A set $A \subseteq X$ is *universally measurable* if for every Borel probability measure μ on X there is a Borel set $B \subseteq X$ such that $\mu(A \Delta B) = 0$. The universally measurable subsets of X form a σ-algebra, denoted by $UM(X)$.

Theorem 5.6. *Let $N_0 \subseteq N_1$ be a transitive model of ZFC and its measured extension. Let $X \in N_0$ be a Polish space. Then there is a unique map from $(UM(X))^{N_0}$ to $(UM(i(X)))^{N_1}$ which commutes with the σ-algebra operations and extends the interpretation map on open sets. Moreover, the map commutes with products, continuous preimages, and the Suslin operation.*

Proof. For a universally measurable set $A \subseteq X$ in the model N_0 let $\pi(A) = \bigcup\{i(B) \colon B \subseteq X$ is a Borel set in N_0 which is a subset of $A\}$ (this is called the *Borel reinterpretation* of A in [10]). We claim that the map π works and it is unique. It appears that we should first prove that the set $\pi(A)$ is universally measurable in N_1. In fact, this is nearly the last thing we can prove after a series of informative claims. At first, it is only clear that π preserves inclusion and coincides with the usual interpretation map on Borel sets.

Claim 5.7. *The map π commutes with complements.*

Proof. Let $A \subseteq X$ be a universally measurable set in N_0. We need to show that the sets $\pi(A), \pi(X \setminus A)$ form a partition of $i(X)$ in N_1. It is clear that the two sets are disjoint. To see this, suppose towards contradiction that $x \in i(X)$ is a point in $\pi(A) \cap \pi(X \setminus A)$. Then there must be Borel sets $B, C \in N_0$ such that $B \subseteq A$, $C \subseteq X \setminus A$, and $x \in i(B)$ and $x \in i(C)$. But then, in the model N_0, $B \cap C = 0$ and by Mostowski absoluteness $i(B) \cap i(C) = 0$, contradicting the assumption that x belongs to both $i(B)$ and $i(C)$.

To show that the two sets are complementary, work in the model N_0 and for each Borel probability measure μ on X pick a Borel μ-null set $B_\mu \subseteq X$ such that $A \setminus B_\mu \subseteq X$ is a Borel set. Suppose that $x \in i(X)$ is a point. By Proposition 5.2, there must be a measure μ in N_0 such that $x \notin i(B_\mu)$ holds. Now, $i(X) = i(B_\mu) \cup i(A \setminus B_\mu) \cup i(X \setminus (A \cup B_\mu))$ so either $x \in i(A \setminus B_\mu)$ (and then $x \in \pi(A)$) or $x \in i(X \setminus (A \cup B_\mu))$ (and then $x \in \pi(X \setminus A)$). ☐

It follows that the map π, if it works, must be unique satisfying the requirements of the theorem. To see this, recall [15] that the interpretation of Borel sets is unique if it is to commute with the σ-algebra operations and extend interpretation of open sets. It follows that any other map $\hat{\pi}$ satisfying the demands of the theorem must satisfy $\pi(A) \subseteq \hat{\pi}(A)$ for every universally measurable set $A \subseteq X$ in N_0 in order to preserve the inclusion between A and its Borel subsets. At the same time, $\hat{\pi}(A)$ must have empty intersection with $\hat{\pi}(X \setminus A)$ and therefore with $\pi(X \setminus A)$, leaving $\hat{\pi}(A) = \pi(A)$ as the only possibility in view of the claim.

Claim 5.8. *The map π commutes with countable intersections.*

Proof. Suppose that in the model N_0, $A_n \subseteq X$ are universally measurable sets for all $n \in \omega$. We need to show that $\pi(\bigcap_n A_n) = \bigcap_n \pi(A_n)$. The left-to-right inclusion is clear as the map π preserves inclusion. For the right-to-left inclusion, suppose that $x \in \bigcap_n \pi(A_n)$. Then there are Borel sets $B_n \subseteq A_n$ in N_0 such that $x \in i(B_n)$, for all $n \in \omega$. Since N_1 is a measured extension of N_0, there is in N_0 a countable set a such that $\{B_n \colon n \in \omega\} \subseteq a$. Working in the model N_0, let $C_n = \bigcup\{B \colon B \in a$ is a Borel subset of X and $B \subseteq A_n\}$. The set $C_n \subseteq A_n$ is a Borel set for each $n \in \omega$, $\bigcup_n C_n \subset \bigcup_n A_n$ is a Borel set, and $x \in i(\bigcap_n C_n) \subset \pi(\bigcap_n A_n)$ as desired. ☐

It follows that the map π commutes with countable unions as unions can be reconstructed from intersections and complements.

Claim 5.9. *The map π commutes with continuous preimages.*

Proof. Suppose that $f\colon X \to Y$ is a continuous map between Polish spaces and $A \subseteq Y$ is a universally measurable set, all in the model N_0. Suppose that $x \in X$ is a point in $i(f)^{-1}\pi(A)$. Then $i(f)(x) \in \pi(A)$, so by the definition of the map π there must be a Borel set $B \subseteq A$ such that $i(f)(x) \in i(B)$. Since interpretation of Borel sets commutes with continuous preimages [15] we have that $x \in i(f^{-1}B)$. However, $f^{-1}B \subseteq f^{-1}A$ is a Borel set, so $i(f^{-1}B) \subset \pi(f^{-1}A)$ and $x \in \pi(f^{-1}A)$.

Suppose on the other hand that $x \in \pi(f^{-1}A)$ holds. Then there must be a Borel set $B \subseteq f^{-1}A$ such that $x \in i(B)$. Move to the model N_0. For every Borel probability measure μ on X choose a μ-null Borel set $C_\mu \subseteq X$ such that $f''(B \setminus C_\mu) \subseteq Y$ is a Borel set. By Proposition 5.2, there must be a measure μ in N_0 such that $x \notin i(C_\mu)$. By a Shoenfield absoluteness argument $i(f''(B \setminus C_\mu)) = i(f)''i(B \setminus C_\mu)$ holds. The former set is a subset of $\pi(A)$ by the definition of the map π. The point $i(f)(x) \in Y$ belongs to the latter set so $x \in i(f)^{-1}\pi(A)$. $\square$

It follows that the map π commutes with countable products: If $A_n \subseteq X_n$ are universally measurable subsets of their respective Polish spaces for $n \in \omega$, then $\prod_n A_n$ is the intersection of the preimages of the sets A_n under the continuous projection functions from $\prod_n X_n$ to the separate coordinates.

Claim 5.10. *The map π commutes with the Suslin operation.*

Proof. Recall that for a collection $\{A_t\colon t \in \omega^{<\omega}\}$ of subsets of a Polish space X, the Suslin operation returns the set $B = \{x \in X\colon \exists y \in \omega^\omega\ \forall n\ x \in A_{y\restriction n}\}$, which we also denote by $S(A_t\colon t \in \omega^{<\omega})$.

Suppose that the collection $\{A_t\colon t \in \omega^{<\omega}\}$ is in N_0 and consists of universally measurable sets there. Suppose that $x \in i(X)$ is a point in the model N_1 and $x \in \pi(B)$ holds. This means that in N_0 there is a Borel set $C \subseteq B$ such that $x \in i(C)$. Working in the model N_0, for each Borel probability measure μ on X let $D_\mu \subseteq X$ be a Borel μ-null set such that for each $t \in \omega^{<\omega}$, $A_t \setminus D_\mu$ is a Borel set. By Proposition 5.2, there is $\mu \in N_0$ such that $x \notin D_\mu$. Observe that $C \setminus D_\mu \subseteq S(A_t \setminus D_\mu)$. By a Shoenfield absoluteness argument, $i(C \setminus D_\mu) \subseteq S(i(A_t \setminus D_\mu)\colon t \in \omega^{<\omega})$, the latter set is by definition a subset of $S(\pi(A_t)\colon t \in \omega^{<\omega})$, so $x \in S(\pi(A_t)\colon t \in \omega^{<\omega})$ as desired.

Suppose on the other hand that $x \in S(\pi(A_t)\colon t \in \omega^{<\omega})$ holds. Working in N_1, pick $y \in \omega^{<\omega}$ such that for all $n \in \omega$, $x \in \pi(A_{y\restriction n})$. Pick for each $n \in \omega$ a Borel set $B_n \in N_0$ such that $B_n \subseteq A_{y\restriction n}$ and $x \in i(B_n)$. By item (1) of Definition 5.1, there is a countable set $a \in N_0$ such that $B_n \in a$ holds for all $n \in \omega$. Move to the model N_0. For each $t \in \omega^{<\omega}$ let $C_t = \bigcup\{D \in a_0\colon D \subseteq A_t, D$ Borel$\}$ and let $E = S(C_t\colon t \in \omega^{<\omega})$. Note that the sets C_t are Borel and the set E is analytic. For each Borel probability measure μ on X let $E_\mu \subseteq X$ be a Borel μ-null set such that $E \setminus E_\mu$ is Borel. By Proposition 5.2, there is a Borel probability measure μ such that $x \notin i(E_\mu)$. Note that $E \setminus E_\mu = S(C_t \setminus E_\mu\colon t \in \omega^{<\omega}) \subseteq B$ and by Shoenfield absoluteness, $i(E \setminus E_\mu) = S(i(C_t \setminus E_\mu)\colon t \in \omega^{<\omega})$.

Note that x belongs to the set on the right side of this equation, so it belongs to the set on the left side as well. It follows that $x \in \pi(B)$ by the definition of π as desired. $\square$

Finally, we are ready for the cinch.

Claim 5.11. *The map π sends universally measurable sets to universally measurable sets.*

Proof. Suppose that A is a universally measurable subset of a Polish space X in the model N_0. To show that $\pi(A)$ is universally measurable in N_1, work for N_0 for a while. Consider the Polish space Y of all Borel probability measures on X, and the universally measurable set $B = A \times Y \subseteq X \times Y$. For each probability measure μ on Y let $\hat{\mu}$ be the Borel probability measure on $X \times Y$ defined by $\hat{\mu}(C) = \int \nu(C_\nu)\ d\mu(\nu)$. Since the set B is universally measurable, for each measure μ on Y there exist a $\hat{\mu}$-null Borel set $C_\mu \subseteq X \times Y$ such that $B \setminus C_\mu$ is Borel, and a μ-null Borel set $D_\mu \subseteq Y$ such that for all $\nu \in Y \setminus D_\mu$, the horizontal section $(C_\mu)_\nu$ is ν-null.

Move to the model N_1 and let ν be a Borel probability measure on X. By Proposition 5.2, there must be a Borel probability measure μ in the model N_0 such that $\nu \notin i(D_\mu)$. Working in N_0, consider the Borel subsets $B \setminus C_\mu$, $(X \times Y) \setminus (B \cup C_\mu)$ and C_μ of the space $X \times Y$. They cover the space $X \times Y$, the first is a subset of $B = A \times Y$ and the second is a subset of $(X \times Y) \setminus B = (X \setminus A) \times Y$. Now look at the model N_1. By the definition of the map π, $i(B \setminus C_\mu) \subseteq \pi(B) = \pi(A) \times i(Y)$ and $i(X \times Y) \setminus (B \cup C_\mu)) \subseteq \pi(X \setminus A) \times i(Y)$. It follows that the Borel horizontal sections $i(B \setminus C_\mu)_\nu$ and $i((X \times Y) \setminus (B \cup C_\mu))_\nu$ are subsets of $\pi(A)$ and $\pi(X \setminus A)$ respectively. So, $\pi(A) \setminus (i(C_\mu))_\nu = i(B \setminus C_\mu)_\nu$ is Borel. By a standard absoluteness argument, the ν-mass of the horizontal section $(i(C_\mu))_\nu$ must be zero, so $(i(C_\mu))_\nu$ witnesses that $\pi(A)$ is ν-measurable. $\square$

This concludes the proof of the theorem. $\square$

Finally, we record the most important theorem which ties the previous sections together.

Theorem 5.12. *Let I be an iterable $\mathbf{\Sigma}^1_1$-polar ideal on a Polish space X. Let α be any ordinal. Let P be the countable support iteration of the quotient forcing P_I of length α. The P-extension is a measured extension of the ground model.*

Proof. Recall that the iteration can be presented in the following way. A condition in P is a set p such that, for some countable set $\mathrm{supp}(p) \subset \alpha$, p is a Borel $I^{\mathrm{supp}(p)}$-positive subset of $X^{\mathrm{supp}(p)}$. The ordering is defined by $q \leq p$ if $\mathrm{supp}(p) \subset \mathrm{supp}(q)$ and for all $x \in q$, $x \restriction \mathrm{supp}(p) \in p$. The iteration adds a generic sequence $\dot{x}_{gen} \in X^\alpha$ such that the generic filter is exactly the set of all conditions $p \in P$ such that $\dot{x}_{gen} \restriction \mathrm{supp}(p) \in p$. For every Polish space Y and every condition $p \in P$ and every name τ such that $p \Vdash \tau \in Y$ there is a condition $q \leq p$ and a Borel function $f \colon q \to Y$ such that $q \Vdash \tau = f(\dot{x}_{gen} \restriction \mathrm{supp}(q))$.

Now, suppose that

- Y is a Polish space,
- F is a set of Borel subsets of Y,
- ε is a rational number,
- $p \in P$ is a condition and
- τ is a P-name for a measure on Y

such that, for all Borel sets $B \in F$, $p \Vdash \tau(B) \geq \varepsilon$. We must find a measure μ on Y in the ground model such that for every Borel set $B \in F$, $\mu(B) \geq \varepsilon$ holds. To this end, find a condition $q \leq p$ and a Borel function $f\colon q \to P(Y)$ such that $q \Vdash \tau = f(\dot{x}_{gen} \restriction \mathrm{supp}(p))$. By Theorem 4.1, there is a probability measure ν on $X^{\mathrm{supp}(p)}$ such that $\nu(q) = 1$ and all analytic sets in the ideal $I^{\mathrm{supp}(p)}$ have ν-mass zero.

Let μ be the Borel probability measure on Y defined by the formula

$$\mu(B) = \int f(x)(B)\ d\nu(x).$$

It will be enough to argue that for all sets $B \in F$, $\mu(B) \geq \varepsilon$. Suppose towards a contradiction that this fails for some $B \in F$. Then the set $r = \{x \in q\colon f(x)(B) < \varepsilon\}$ is Borel and must have ν-positive mass. Thus, $r \leq q$ is a condition in P and by a standard absoluteness argument it forces $f(\dot{x}_{gen} \restriction \mathrm{supp}(q))(B) = \tau(B) < \varepsilon$. This contradicts the initial assumptions on p and τ. $\square$

The paper [11] presents a class of forcing iterations which give rise to measured extensions but which are not evidently polar.

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