BOUNDING THE CONSISTENCY STRENGTH OF A FIVE ELEMENT LINEAR BASIS

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Abstract. In [13] it was demonstrated that the Proper Forcing Axiom implies that there is a five element basis for the class of uncountable linear orders. The assumptions needed in the proof have consistency strength of at least infinitely many Woodin cardinals. In this paper we reduce the upper bound on the consistency strength of such a basis to something less than a Mahlo cardinal, a hypothesis which can hold in the constructible universe L.

A crucial notion in the proof is the *saturation of an Aronszajn tree*, a statement which may be of broader interest. We show that if all Aronszajn trees are saturated and $\mathrm{PFA}(\omega_1)$ holds, then there is a five element basis for the uncountable linear orders. We show that $\mathrm{PFA}(\omega_2)$ implies that all Aronszajn trees are saturated and that it is consistent to have $\mathrm{PFA}(\omega_1)$ plus every Aronszajn tree is saturated relative to the consistency of a reflecting Mahlo cardinal. Finally we show that a hypothesis weaker than the existence of a Mahlo cardinal is sufficient to force the existence of a five element basis for the uncountable linear orders.

1. Introduction

In [13] it was shown that the Proper Forcing Axiom implies that the class of uncountable linear orders has a five element basis, i.e., that there is a list of five uncountable linear orders such that every

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uncountable linear order contains an isomorphic copy of one of them. This basis consists of X, ω_1, ω_1^*, C, and C^* where X is any suborder of the reals of cardinality ω_1 and C is any Countryman line. In fact any five element basis for the uncountable linear orders must have this form.

Recall that a *Countryman line* is a linear order whose square is the union of countably many non-decreasing relations. These linear orders are necessarily Aronszajn. Their existence was proved by Shelah who conjectured that every Aronszajn line consistently contains a Countryman suborder [14]. In [1, p. 79] it is stated that, assuming an appropriate forcing axiom, every Aronszajn line contains a Countryman suborder if and only if the Coloring Axiom for Trees (CAT) holds:

> There is an Aronszajn tree T such that for every $K \subseteq T$ there is an uncountable antichain $X \subseteq T$ such that $\wedge(X)$ is either contained in or disjoint from K.

Here $\wedge(X)$ denotes the set of all pairwise meets of elements of X. Over time, this conjecture developed in the folklore and at some point it was known to be equivalent — modulo the forcing axiom for proper forcings of size ω_1 — to the assertion that the above list forms a basis for the class of uncountable linear orders.[1] The reader is referred to the final section of [15] for proofs of the above assertions. It is worth noting that the forcing axiom for proper partial orders of size ω_1 is relatively consistent with ZFC and hence no large cardinals are needed in this reduction.

In [13] it is shown that PFA implies CAT. In fact the conjunction of the Bounded Proper Forcing Axiom (PFA(ω_1)) [8] and the Mapping Reflection Principle (MRP) [12] suffices. Since the consistency of MRP requires considerable large cardinal assumptions[2] and since its use is isolated to Lemma 5.29 of [13], it is natural to ask what large cardinals, if any, are necessary for CAT. Here we reduce the consistency strength needed to something less than a Mahlo cardinal, a hypothesis which can hold in the constructible universe L.

The main object of this paper is to prove this Key Lemma from a greatly reduced hypothesis. A central notion in our analysis is that of the *saturation* of an Aronszajn tree T — whenever $\mathscr{A}$ is a collection of uncountable downward closed subsets of T which have pairwise countable intersection, then $\mathscr{A}$ has cardinality at most ω_1. When possible,

[1]The speculation of the consistent existence of a finite basis for the uncountable linear orders seems to have first been made in print in [3], although it still seems to have been unknown at that point that this was equivalent to Shelah's conjecture.

[2]The current upper bound is a supercompact cardinal [12].

this notion will be considered separately as it seems that this could be relevant in other contexts.

Central to the proof in [13] is the notion of *rejection*. This will be defined after recalling some preliminary definitions from [13]. For the moment, fix an Aronszajn tree $T \subseteq 2^{<\omega_1}$ which is coherent, special and closed under finite modifications,[3] and let K be a subset of T.

Definition 1.1. If P is a countable elementary submodel of $H(\omega_2)$ containing T, let $\mathscr{I}_P(T)$ be the collection of all $I \subseteq \omega_1$ such that for some uncountable $Z \subseteq T$ in P and some t in the downwards closure of Z having height $P \cap \omega_1$, I is disjoint from

$$\Delta(Z, t) = \{\Delta(s, t) : s \in Z\}.$$

It is always the case that $\mathscr{I}_P(T)$ is closed under finite unions and subsets. The former property uses the properties of T and is non-trivial; the argument is similar to the proof of [15, 4.1]. Similarly, $\mathscr{I}_P(T)$ remains the same if one takes t to be a fixed member of $T_{P\cap\omega_1}$ instead of letting t vary.

Definition 1.2. If X is a finite subset of T, $K(X)$ is the set of all γ which are less than the the heights of all elements of X and satisfy $s \restriction \gamma \in K$ for all s in X.

Definition 1.3. If X is a finite subset of T and P is a countable elementary submodel of $H(\omega_2)$, then P *rejects* X if $K(X \setminus P)$ is in $\mathscr{I}_P(T)$.

It is shown in [13] that the following lemma (Lemma 5.29 of [13]), taken in conjunction with PFA(ω_1), is sufficient to prove the existence of an uncountable antichain $X \subseteq T$ such that $\wedge(X)$ is contained in or disjoint from K.

Key Lemma 1.4. [13] *(*MRP*) If M is a countable elementary submodel of $H(2^{2^{\omega_1}+})$ which contains T and K and X is a finite subset of T, then there is a closed unbounded set E of countable elementary submodels of $H(\omega_2)$ such that E is in M and either every element of $E \cap M$ rejects X or no element of $E \cap M$ rejects X.*

We will begin by defining a combinatorial statement φ in Section 2 and showing that this statement implies the Key Lemma. The statement φ is a strengthening of *Aronszajn tree saturation* — the assertion that every Aronszajn tree is saturated. Moreover, φ is shown to be equivalent to Aronszajn tree saturation in the presence of PFA(ω_1). In Section 3 we will demonstrate that PFA(ω_2) implies φ. This reduces

[3]The tree $T(\varrho_3)$ of [18] is such an example.

the upper bound on the consistency strength of Shelah's conjecture to something less than the existence of $0^\sharp$ but greater than a weakly compact cardinal. Section 4 further refines the argument to show that an instance of φ can be forced by a proper forcing without a need for large cardinal assumptions. This is then implemented in Section 5 to further optimize the upper bound on the consistency strength of a five element basis for the uncountable linear orders to something less than the existence of a Mahlo cardinal.

The notation and terminology used in this paper is fairly standard. All ordinals are von Neumann ordinals; they are the set of their predecessors. The cardinal $\beth_\alpha$ is defined recursively so that $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, and $\beth_\delta = \sup_{\alpha<\delta} \beth_\alpha$ for limit δ. The reader is referred to [9] as a general reference for set theory. In this paper *Aronszajn tree* or *A-tree* will mean an uncountable tree in which all levels and chains are countable. A *subtree* of an A-tree T is an uncountable downward closed subset of T. The reader is referred to [10] or [17] for further reading on bounded fragments of PFA.

2. Aronszajn tree saturation

Recall the notion of the *saturation of* $\mathscr{P}(\omega_1)/\mathrm{NS}$:

> Any collection of stationary sets which have pairwise non-stationary intersection has cardinality at most ω_1.

Now consider the following statement $\psi_{\mathrm{NS}}(\mathscr{A})$ for a collection $\mathscr{A}$ of subsets of ω_1:

> There is a club $E \subseteq \omega_1$ and a sequence $\langle A_\xi : \xi < \omega_1 \rangle$ of elements of $\mathscr{A}$ such that for all δ in E, there is a $\xi < \delta$ with δ in A_ξ.

The assertion ψ_{NS} that $\psi_{\mathrm{NS}}(\mathscr{A})$ holds for every predense set $\mathscr{A} \subseteq \mathscr{P}(\omega_1)/\mathrm{NS}$ is in fact equivalent to the saturation of $\mathscr{P}(\omega_1)/\mathrm{NS}$. This was used to prove that Martin's Maximum implies that $\mathscr{P}(\omega_1)/\mathrm{NS}$ is saturated [7]. The significance of $\psi_{\mathrm{NS}}(\mathscr{A})$ from our point of view is that it implies that $\mathscr{A}$ is predense and is Σ_1 in complexity and therefore upwards absolute.

In this section we will be interested in analogous assertions about subtrees of an A-tree T.[4]

[4]Actually, all statements about the saturation of an A-tree make sense in the broader context of all ω_1-trees. We will not need this generality and in fact the saturation of all A-trees implies the more general case by Todorcevic's construction presented in Section 2 of [4].

Definition 2.1. An A-tree T is *saturated* if whenever $\mathscr{A}$ is a collection of subtrees T which have pairwise countable intersection, $\mathscr{A}$ has cardinality at most ω_1.

This statement follows from the stronger assertion shown by Baumgartner to hold after Levy collapsing an inaccessible cardinal to ω_2 [4].

> For every A-tree T, there is a collection $\mathscr{B}$ of subtrees of T such that $\mathscr{B}$ has cardinality ω_1 and every subtree of T contains an element of $\mathscr{B}$.

Unlike in the case of $\mathscr{P}(\omega_1)/\mathrm{NS}$, the maximality of an ω_1-sized antichain of subtrees can be shown not to be upwards absolute.[5] This leaves open the question, however, of how to obtain the consistency of A-tree saturation in the presence of a forcing axiom. It is not difficult to show that both Chang's Conjecture and the saturation of $\mathscr{P}(\omega_1)/\mathrm{NS}$ each imply that all A-trees are saturated. Therefore Martin's Maximum implies all A-trees are saturated. We will pursue a different proof, however, which is weaker in terms of consistency strength and somewhat different in character.

If $\mathscr{F}$ is a collection of subtrees of T, then $\mathscr{F}^\perp$ is the collection of all subtrees B of T such that for every A in $\mathscr{F}$, $A \cap B$ is countable. If $\mathscr{F}^\perp$ is empty, then $\mathscr{F}$ is said to be *predense*. For $\mathscr{F}$ a collection of subtrees of an A-tree T, we define the following assertions:

$\psi(\mathscr{F})$: There is a closed unbounded set $E \subseteq \omega_1$ and a continuous chain $\langle N_\nu : \nu \in E\rangle$ of countable subsets of $\mathscr{F}$ such that for every ν in E and t in T_ν there is a $\nu_t < \nu$ such that if $\xi \in (\nu_t, \nu) \cap E$, then there is $A \in \mathscr{F} \cap N_\xi$ such that $t \restriction \xi$ is in A.

$\varphi(\mathscr{F})$: There is a closed unbounded set $E \subseteq \omega_1$ and a continuous chain $\langle N_\nu : \nu \in E\rangle$ of countable subsets of $\mathscr{F} \cup \mathscr{F}^\perp$ such that for every ν in E and t in T_ν either
1. there is a $\nu_t < \nu$ such that if $\xi \in (\nu_t, \nu) \cap E$, then there is $A \in \mathscr{F} \cap N_\xi$ such that $t \restriction \xi$ is in A, or
2. there is a B in $\mathscr{F}^\perp \cap N_\nu$ such that t is in B.

It is not difficult to show that $\psi(\mathscr{F})$ implies that $\mathscr{F}$ is predense. It is also clear that $\psi(\mathscr{F})$ is a Σ_1-formula in the parameters $\mathscr{F}$ and T. While $\varphi(\mathscr{F})$ and $\psi(\mathscr{F})$ are equivalent if $\mathscr{F}$ is predense, $\varphi(\mathscr{F})$ is in general not a Σ_1-formula in $\mathscr{F}$ and T. Let φ be the assertion that whenever T is an A-tree and $\mathscr{F}$ is a family of subtrees T, $\varphi(\mathscr{F})$ holds and let ψ be the analogous assertion but with quantification only over $\mathscr{F}$ which are predense. As noted, φ implies ψ.

[5] This can be derived from the arguments in [16, §8] and the construction in [4, §2]. An argument is explicit in [11].

We will now show the relevance of φ to our main goal. Before proceeding, it will be useful to reformulate the notion of rejection presented in the introduction.

Definition 2.2. Let $T^{[n]}$ denote the collection of all elements τ of T^n such that every coordinate of τ has the same height and, when considered as a sequence of elements of T, the coordinates of τ are non-decreasing in the lexicographical order on T. $T^{[n]}$ will be considered as a tree with the coordinate-wise partial order induced by T.

Remark 2.3. Morally, elements of $T^{[n]}$ are n-element subsets of T. In order to ensure that $T^{[n]}$ is closed under taking restrictions, it is necessary to allow for n-element sets with repetition and the above definition is a formal means to accommodate this. We will abuse notation and identify elements of $T^{[n]}$ which have distinct coordinates with the set of their coordinates. In our arguments, only the range of these sequences will be relevant.

Definition 2.4. Let $n < \omega$ be fixed. For any uncountable set $Z \subseteq T$, let R_Z be downward closure of the set of elements Y of $T^{[n]}$ such that $\Delta(Z, t) \cap K(Y) = \emptyset$ for some element t of the downward closure of Z with $\operatorname{height}(t) = \operatorname{height}(Y)$. Let $\mathscr{R}_n$ denote the collection of all R_Z as Z ranges over the uncountable subsets of T.

Lemma 2.5. *Suppose that P is a countable elementary submodel of $H(\omega_2)$ which has T as a member. For any $Y \in T^{[n]}$ with* $\operatorname{height}(Y) \geq P \cap \omega_1$*, P rejects Y if and only if $Y \restriction (P \cap \omega_1)$ is in R_Z for some $Z \in P$.*

Proof. Suppose first that P rejects Y. Then there exist $Z \subseteq T$ in P and $t \in (Z \cap T_{P\cap\omega_1}) \setminus P$ such that $\Delta(Z, t) \cap K(Y) = \emptyset$. Since $K(Y) \cap (P \cap \omega_1) = K(Y \restriction (P \cap \omega_1))$, $Y \restriction (P \cap \omega_1)$ is in R_Z.

In the other direction, if $Y \restriction (P \cap \omega_1)$ is in R_Z for some $Z \in P$, then there exist $Y' \geq Y \restriction (P \cap \omega_1)$ in R_Z and $t \in Z \cap T$ such that $\operatorname{height}(t) \geq \operatorname{height}(Y')$ and $\Delta(Z, t) \cap K(Y') = \emptyset$. Then $K(Y') \cap (P \cap \omega_1) = K(Y \restriction (P \cap \omega_1))$, and $\Delta(Z, t) \cap (P \cap \omega_1) = \Delta(Z, t \restriction (P \cap \omega_1))$, so $\Delta(Z, t \restriction (P \cap \omega_1)) \cap K(Y) = \emptyset$, which means that P rejects Y. $\square$

Theorem 2.6. *Suppose that T is a coherent A-tree which is closed under finite changes. If $\varphi(\mathscr{R}_n)$ holds for every $n < \omega$, then the Key Lemma holds for T.*

Proof. Assume the hypothesis of the lemma and let M and X be given as in the statement of the Key Lemma. Without loss of generality, we may assume that X is in $T^{[n]}_{M\cap\omega_1}$ for some $n < \omega$. Note that $\mathscr{R}_n$

is Σ_1-definable using parameters for T and K and therefore there is an $\bar{N} = \langle N_\nu : \nu \in E\rangle$ in M which witnesses $\varphi(\mathscr{R}_n)$. Either there is a $\nu_X < M \cap \omega_1$ such that for each $\xi \in E \cap (\nu_X, M \cap \omega_1)$ there is an $R \in N_\xi \cap \mathscr{F}$ with $X \restriction \xi \in R$ or there is a B in ${\mathscr{R}_n}^\perp \cap N_{M\cap\omega_1}$ with $X \in B$.

In the first case, let E be the set of countable P elementary submodels of $H(\omega_2)$ which satisfy $\bar{N} \in P$ and $P \cap \omega_1 > \nu_X$. Then every member P of E contains $N_{P\cap\omega_1}$ and thus contains an $R \in \mathscr{R}_n$ with $X \restriction (P \cap \omega_1)$ in R. By Claim 2.5, P rejects X.

In the second case, let E be the set of all P countable elementary submodels of $H(\omega_2)$ with K, T, and B in P. If P in $E \cap M$ were to reject X, there would be an R in $\mathscr{R}_n \cap P$ with $X \restriction (P \cap \omega_1) \in R$. It would follow that $X \restriction (P \cap \omega_1)$ is in $B \cap R$, which by elementarity of P would imply that $B \cap R$ is uncountable which is contrary to B being in ${\mathscr{R}_n}^\perp$. Hence no element of $E \cap M$ rejects X. □

3. PFA(ω_2) implies φ

In this section we will show that PFA(ω_2) implies φ. If λ is a cardinal, then PFA(λ) is the fragment of PFA in which only antichains of size at most λ are considered [8]. We will use the following reformulation which is due to Miyamoto [10]:

> PFA(λ)**:** For every A in $H(\lambda^+)$ and Σ_0-formula ϕ, if some proper partial order forces $\exists X \phi(X, A)$, then there is a stationary set of N in $[H(\lambda^+)]^{\omega_1}$ such that A is in N and $H(\omega_2)$ satisfies $\exists X \phi(X, \pi_N(A))$ where π_N is the transitive collapse of N.

In [10] it is also shown that PFA(ω_2) is equiconsistent with the existence of a cardinal κ which is $H(\kappa^+)$-reflecting. Such cardinals are larger than weakly compact cardinals and still relativize to L and hence do not imply the existence of $0^\sharp$. In this section we show that PFA(ω_2) implies that every A-tree is saturated. First we recall some definitions from [12].

Definition 3.1. Let θ be a regular cardinal, let X be uncountable, and let M be a countable subset of $H(\theta)$ such that $[X]^\omega$ is in M. A subset Σ of $[X]^\omega$ is *M-stationary* iff for all E in M such that $E \subseteq [X]^\omega$ is club, $\Sigma \cap E \cap M$ is non-empty.

The *Ellentuck topology* on $[X]^\omega$ is obtained by declaring a set *open* iff it is the union of sets of the form

$$[x, N] = \{Y \in [X]^\omega : x \subseteq Y \subseteq N\}$$

where $N \in [X]^\omega$ and $x \subseteq N$ is finite. When we say 'open' in this paper we refer to this topology.

Definition 3.2. A set mapping Σ is *open stationary* iff there is an uncountable set $X = X_\Sigma$ and a regular cardinal $\theta = \theta_\Sigma$ such that $[X]^\omega \in H(\theta)$, $\mathrm{dom}(\Sigma)$ is a club in $[H(\theta)]^\omega$ and $\Sigma(M) \subseteq [X]^\omega$ is open and M-stationary, for every M in the domain of Σ.

Definition 3.3. Suppose Σ is an open stationary set mapping. We say that $\langle N_\xi : \xi < \omega_1 \rangle$ is a *reflecting sequence for* Σ if it a continuous $\in$-chain contained in the domain of Σ such that for all limit $\nu < \omega_1$, there is a $\nu_0 < \nu$ such that if $\nu_0 < \xi < \nu$, then $N_\xi \cap X$ is in $\Sigma(N_\nu)$.

The *Mapping Reflection Principle* (MRP) is the assertion that every open stationary set mapping which is defined on a club admits a reflecting sequence. In [12] it was shown that MRP follows from PFA by demonstrating the following theorem which will be useful to us here.

Theorem 3.4. [12] *If Σ is an open stationary set mapping defined on a club, then there is a proper forcing which adds a reflecting sequence for Σ.*

Lemma 3.5. *Suppose that T is a saturated A-tree. Then for every family $\mathscr{F}$ of subtrees of T there is a subfamily $\mathscr{F}'$ of $\mathscr{F}$ of cardinality at most ω_1 such $\mathscr{F}^\perp = (\mathscr{F}')^\perp$.*

Proof. Supposing that the lemma is false, for each $\alpha < \omega_2$, recursively choose subtrees F_α, R_α of T such that each $F_\alpha \in \mathscr{F}$, each $R_\alpha \cap F_\alpha$ is uncountable, and $R_\alpha \cap F_\beta$ is countable for each $\beta < \alpha$. Then the trees $F_\alpha \cap R_\alpha$ form an antichain of cardinality ω_2. □

Lemma 3.6. *Let κ be a cardinal greater than or equal to $(2^{\omega_1})^+$. Suppose that T is an A-tree, $\mathscr{F}$ is a collection of subtrees of T and M is a countable subset of $H(\kappa)$ which has T and $\mathscr{F}$ as members and satisfies all axioms of* ZFC *except the power set axiom. If t is an element of T of height $M \cap \omega_1$ and*

$$\{P \in [H(\omega_2)]^\omega : \exists A \in \mathscr{F} \cap P \, (t \restriction (P \cap \omega_1) \in A)\}$$

is not M-stationary, then there is an S in $M \cap \mathscr{F}^\perp$ which contains t.

Proof. Let $\delta = M \cap \omega_1$. Let $E \in M$ be a club of countable elementary submodels of $H(\omega_2)$ such that $E \cap M$ is disjoint from

$$\{P \in [H(\omega_2)]^\omega : \exists A \in \mathscr{F} \cap P \, (t \restriction (P \cap \omega_1) \in A)\}.$$

Let S be the set of all $s \in T$ such that there exist no P in E with $P \cap \omega_1 < \mathrm{height}(s)$ and A in $\mathscr{F} \cap P$ such that $s \restriction (P \cap \omega_1)$ is in A. We claim that $t \in S$. Otherwise, there would exist a P in E with $P \cap \omega_1 < \delta$ and an A in $\mathscr{F} \cap P$ such that $t \restriction (P \cap \omega_1)$ is in A. Letting $\gamma = P \cap \omega_1$, this is a statement about $t \restriction \gamma$, which is an element of

M, so by the elementarity of M there would exist a P in $E \cap M$ with $P \cap \omega_1 = \gamma$ and an A in $\mathscr{F} \cap P$ such that $t \restriction \gamma$ is in A, contradicting our choice of E. Therefore, t is in S.

Clearly S is downwards closed and it is uncountable since it is an element of M but not a subset of M. We are finished once we show that $S \cap A$ is countable for every element A of $\mathscr{F}$. Suppose not. Since S is in M, by elementarity there must be such an A in $M \cap \mathscr{F}$. Let P be an element of E which contains both A and S. Since $A \cap S$ is uncountable and downwards closed, there must be an s in $A \cap S$ of height $(P \cap \omega_1) + 1$. But this contradicts the definition of S. $\square$

Theorem 3.7. PFA(ω_2) *implies* φ.

Proof. In [12], it is shown that PFA(ω_1) implies that $2^{\omega_1} = \omega_2$ and hence PFA(ω_2) is equivalent to PFA(2^{ω_1}). Let $T = \{\tau(\alpha, i) : \alpha < \omega_1 \text{ and } i < \omega\}$ be such that $\tau(\alpha, i)$ is of of height α for every α and i. Define $\Sigma^i_{\mathscr{F}}$ as follows. The domain of $\Sigma^i_{\mathscr{F}}$ is the set of all countable subsets of $H(2^{\omega_1+})$ such that $M \cap H(\omega_2)$ is an elementary submodel of $H(\omega_2)$, $\mathscr{F}$ and $\mathscr{F}^\perp$ are in M, and M satisfies that $\mathscr{F} \cup \mathscr{F}^\perp$ is predense. If M is in the domain of $\Sigma^i_{\mathscr{F}}$ and there is no S in $M \cap \mathscr{F}^\perp$ with $\tau(M \cap \omega_1, i)$ in S, then let $\Sigma^i_{\mathscr{F}}(M)$ be the collection of all $P \in [H(\omega_2)]^\omega$ such that either $P \cap \omega_1$ is not an ordinal or else there is an A in $P \cap \mathscr{F}$ such that $\tau(M \cap \omega_1, i) \restriction (P \cap \omega_1)$ is in A. If there is an S in $M \cap \mathscr{F}^\perp$ with $\tau(M \cap \omega_1, i)$ in S, put $\Sigma^i_{\mathscr{F}}(M)$ to be all of $[H(\omega_2)]^\omega$. Lemma 3.6 implies that $\Sigma^i_{\mathscr{F}}$ is open and M-stationary.

We will now argue that PFA(2^{ω_1}) implies that each $\Sigma^i_{\mathscr{F}}$ admits a reflecting sequence. By the proof of Theorem 3.1 of [12], there is a proper forcing which introduces a reflecting sequence for each $\Sigma^i_{\mathscr{F}}$. Let $\phi_i(\vec{N}, \mathscr{F}, \mathscr{F}^\perp, \tau)$ assert that $\vec{N}$ is a reflecting sequence for $\Sigma^i_{\mathscr{F}}$. While $\Sigma^i_{\mathscr{F}}$ is not an element of $H(2^{\omega_1+})$, it is a definable class within this structure and hence $\phi_i(\vec{N}, \mathscr{F}, \mathscr{F}^\perp, \tau)$ satisfies the hypothesis of PFA(2^{ω_1}). Applying Miyamoto's reformulation of PFA(2^{ω_1}), there is an elementary submodel M of $H(2^{\omega_1+})$ of size ω_1 which contains ω_1 as a subset, $\mathscr{F}$, $\mathscr{F}^\perp$, τ as elements and is such that $H(\omega_2)$ satisfies there is a $\vec{N}$ such that $\phi_i(\vec{N}, \pi(\mathscr{F}), \pi(\mathscr{F}^\perp), \pi(\tau))$ where π is the transitive collapse of M. Notice that since ω_1 is a subset of M, the collapsing map fixes elements of $H(\omega_2)$ and in particular τ and elements of $\mathscr{F}$ and $\mathscr{F}^\perp$. It follows that the postulated $\vec{N}$ really is a reflecting sequence for $\Sigma^i_{\mathscr{F}}$.

Now fix, for each i, a reflecting sequence $\langle M^i_\nu : \nu < \omega_1 \rangle$ for $\Sigma^i_{\mathscr{F}}$. Let E be the collection of all ν such that $M^i_\nu \cap \omega_1 = \nu$, for each i. Letting

$$N_\nu = (\mathscr{F} \cup \mathscr{F}^\perp) \cap \bigcup_{i<\omega} M^i_\nu,$$

it is easily checked that $\langle N_\nu : \nu \in E\rangle$ is a witness to $\varphi(\mathscr{F})$. □

Corollary 3.8. *For a given family $\mathscr{F}$ of subtrees of an Aronszajn tree T, there is a proper forcing extension which satisfies $\varphi(\mathscr{F})$.*

Proof. Construct a countable support iteration of length ω such that at the i^{th} stage of the iteration, a reflecting sequence is added to $\dot{\Sigma}^i_{\mathscr{F}}$ by a proper forcing. It is easily checked that the iteration generated a generic extension which satisfies $\varphi(\mathscr{F})$. □

Remark 3.9. The reader is cautioned that it does not immediately follow that if $\mathscr{F}$ is moreover predense then there is a proper forcing extension in which $\psi(\mathscr{F})$ holds since *a priori* $\mathscr{F}$ may fail to be predense in the generic extension. This is addressed in the next section.

Corollary 3.10. *If* $\mathrm{PFA}(\omega_1)$ *holds and T is a saturated A-tree, then $\varphi(\mathscr{F})$ is true for all families $\mathscr{F}$ of subtrees of T.*

Proof. Let T be an A-tree and let $\mathscr{F}$ be a family of subtrees of $\mathscr{F}$. Applying Lemma 3.5, fix a subfamily $\mathscr{F}'$ of $\mathscr{F}$ of cardinality at most ω_1 such that $\mathscr{F}^\perp = (\mathscr{F}')^\perp$. Then $\varphi(\mathscr{F}')$ implies $\varphi(\mathscr{F})$, and $\varphi(\mathscr{F}')$ is a Σ_1-statement in a parameter listing T and the members of $\mathscr{F}'$. Theorem 3.7 shows that there is a proper forcing making this Σ_1-statement hold. □

4. Forcing instances of ψ

As already noted, Corollary 3.8 comes short of showing that, for a given predense $\mathscr{F}$, there is a proper forcing extension in which $\psi(\mathscr{F})$ holds. Upon forcing a reflecting sequence for $\Sigma^0_{\mathscr{F}}$, $\mathscr{F}$ may fail to be predense.

A similar problem arises in the context of $\mathscr{P}(\omega_1)/\mathrm{NS}$. For a given antichain $\mathscr{A}$ in $\mathscr{P}(\omega_1)/\mathrm{NS}$, there is a stationary subset S of $[\omega_1 \cup \mathscr{A}]^\omega$ such that $\psi_{\mathrm{NS}}(\mathscr{A})$ is equivalent to S strongly reflecting in the sense of [5, p. 57]. Furthermore, there is a semi-proper forcing $\mathscr{Q}$ such that if generic absoluteness holds for $\mathscr{Q}$ in the sense of the previous section, then S strongly reflects. This does not ensure, however, that $\psi_{\mathrm{NS}}(\mathscr{A})$ holds after forcing with $\mathscr{Q}$. In fact, while semi-proper forcing can always be iterated with revised countable support while preserving ω_1, there are models such as L in which there is no set forcing which makes $\mathscr{P}(\omega_1)/\mathrm{NS}$ saturated. Hence, in the case of $\mathscr{P}(\omega_1)/\mathrm{NS}$, the discrepancy between forcability and the consequences of generic absoluteness can represent an insurmountable difficulty.

In this section we will see that the saturation of A-trees is fundamentally different in this regard. We will show that there is a single set mapping associated with a given predense $\mathscr{F}$ such that if the set mapping reflects, $\psi(\mathscr{F})$ is true.

Lemma 4.1. *Suppose that $n \in \omega$, T is an A-tree, $\mathscr{F}$ is a predense collection of subtrees of T and M is a countable elementary submodel of $H((\beth_{n+2})^+)$ which contains T and $\mathscr{F}$ as elements. Let $\delta = M \cap \omega_1$. Suppose X is an n-element subset of the δ-th level. Then*

$$\{P \in H(\omega_2) : \forall t \in X \exists A \in \mathscr{F} \cap P (t \restriction (P \cap \omega_1) \in A)\}$$

is M-stationary.

Proof. The proof is by induction on n. In the base case $n = 0$, there is nothing to show. Now suppose that the lemma is true for n and let M be a countable elementary submodel of $H((\beth_{n+3})^+)$ and X be an $n+1$-element subset of the δ-th level of T. Let E be a given closed unbounded subset of $[H(\omega_2)]^\omega$. Let t be any element of X and let $X_0 = X \setminus \{t\}$. The set of elements of $[H(\omega_2)]^\omega$ of the form $N \cap H(\omega_2)$ for some countable elementary submodel N of $H((\beth_{n+2})^+)$ is a club set in M, so, applying Lemma 3.6, there is a countable elementary submodel N of $H((\beth_{n+2})^+)$ such that $\mathscr{F}$, E are in N and there is an A in $\mathscr{F} \cap N$ such that $t \restriction (N \cap \omega_1)$ is in A. Let E^* be the set of all P in E such that A is in P. Clearly, E^* is a club and belongs to N. Applying the inductive hypothesis to N and X_0, there is a P in $E^* \cap N$ such that for every s in X_0, there is a A' in $\mathscr{F} \cap P$ such that $s \restriction (P \cap \omega_1)$ is in A'. But A is also in P and $t \restriction (P \cap \omega_1)$ is in A as well since it is downward closed. $\square$

Lemma 4.2. *If $\mathscr{F}$ is a predense family of subtrees of T, then there is an open stationary set mapping $\Sigma_{\mathscr{F}}$ such that if $\Sigma_{\mathscr{F}}$ admits a reflecting sequence, then $\psi(\mathscr{F})$ is true.*

Proof. Fix, for each limit $\alpha < \omega_1$, a cofinal $C_\alpha \subseteq \alpha$ of order type ω. Let $T = \{t(\alpha, i) : \alpha < \omega_1 \text{ and } i < \omega\}$ be such that for every α and i, the height of $t(\alpha, i)$ is α. If M is a countable elementary submodel of $H((\beth_{\omega+1})^+)$, define $\Sigma_{\mathscr{F}}(M)$ as follows. Let $\delta = M \cap \omega_1$ and let $\Sigma_{\mathscr{F}}(M)$ be the set of all $P \in [H(\omega_2)]^\omega$ such that either P is not an elementary submodel of $H(\omega_2)$ or else for every $i < |C_{M \cap \omega_1} \cap P|$ there is an A in $P \cap \mathscr{F}$ which contains $t(\delta, i) \restriction (P \cap \omega_1)$.

It is easily checked that $\Sigma_{\mathscr{F}}(M)$ is open for every M. It should be clear that a reflecting sequence of $\Sigma_{\mathscr{F}}$ can easily be modified to produce a witness $\langle N_\nu : \nu \in E \rangle$ to $\psi(\mathscr{F})$. Therefore it remains to show that $\Sigma_{\mathscr{F}}(M)$ is M-stationary for all M in the domain of $\Sigma_{\mathscr{F}}$. To see this, let $E \subseteq H(\omega_2)$ be a club. Find a countable elementary submodel N of $H((\beth_\omega)^+)$ which is an element of M and contains E as a member.

Put $n = |C_{M \cap \omega_1} \cap N|$ and apply Lemma 4.1 to N and n to find a P in $E \cap \Sigma_{\mathscr{F}}(M) \cap M$. $\square$

Corollary 4.3. *If $\mathscr{F}$ is a predense family of subtrees of an A-tree, then there is a proper forcing extension in which $\psi(\mathscr{F})$ holds.*

Proof. By the proof of Theorem 3.1 of [12], there is a proper forcing which adds a reflecting sequence to the $\Sigma_{\mathscr{F}}$ of Lemma 4.2. $\square$

5. Relative consistency results

In this section we will present a number of iterated forcing constructions aimed at proving upper bounds on the consistency of φ and the existence of a five element basis for the uncountable linear orders. Throughout this section we will utilize the following standard facts about L.

Theorem 5.1. *(see [6, 2.2]) Suppose $V = L$. If κ is an uncountable regular cardinal and E is a stationary subset of κ, then $\diamondsuit_E(\kappa)$ holds: there is a sequence $\langle A_\xi : \xi \in E\rangle$ such that for all $X \subseteq \kappa$,*

$$\{\xi \in E : X \cap \xi = A_\xi\}$$

is stationary.

Remark 5.2. If $\diamondsuit_E(\kappa)$ holds and every element of E is an inaccessible cardinal, then $\diamondsuit_E(\kappa)$ is equivalent to the following stronger statement: There is a sequence $\langle A_\delta : \delta \in E\rangle$ of elements of $H(\kappa)$ such that if X_i $(i < n)$ is a finite sequence of subsets of $H(\kappa)$, then there is a stationary set of δ in E such that

$$A_\xi = \langle X_i \cap H(\delta) : i < n\rangle.$$

It is easily checked that if κ is a regular cardinal in V, then it is also a regular cardinal in L. Hence if κ is inaccessible (Mahlo), then L satisfies that κ is inaccessible (Mahlo). Reflecting cardinals also relativize to L [8].

Theorem 5.3. *Suppose that there is a Mahlo cardinal. Then there is a forcing extension of L which satisfies φ.*

Proof. Let κ be Mahlo and note that κ is also Mahlo in L; from now on, work in L. Let E be the stationary set of inaccessible cardinals less than κ and, applying Theorem 5.1, let $\langle A_\delta : \delta \in E\rangle$ be a $\diamondsuit_E(\kappa)$-sequence in the revised sense stated in Remark 5.2. Construct a countable support iteration $\langle \mathscr{P}_\alpha; \mathscr{Q}_\alpha : \alpha < \kappa\rangle$ of proper forcing notions of size $< \kappa$. If $\alpha \in E$ and $A_\alpha = (\dot{T}, \dot{\mathscr{F}})$ where $\dot{T}$ is a $\mathscr{P}_\alpha$-name for an A-tree and $\dot{\mathscr{F}}$ is a $\mathscr{P}_\alpha$-name for a family of subtrees of $\dot{T}$, then we let $\dot{\mathscr{Q}}_\alpha$ be a proper forcing in $H(\kappa)$ which first forces $\psi(\dot{\mathscr{F}} \cup \dot{\mathscr{F}}^{\perp})$ and then forces $\varphi(\dot{\mathscr{F}})$. In other cases we can let $\dot{\mathscr{Q}}_\alpha$ be any proper forcing in $H(\kappa)$. Let $\mathscr{P}_\kappa$ be the limit of the iteration. By standard arguments the forcing $\mathscr{P}_\kappa$ is proper and κ-c.c. [9].

Suppose now $\dot{T}$ is a $\mathscr{P}_\kappa$-name for an A-tree and $\dot{\mathscr{F}}$ is a $\mathscr{P}_\kappa$-name for a family of subtrees of $\dot{T}$. Let $\dot{\mathscr{F}}_\delta$ be the set of all $\mathscr{P}_\delta$-names $\dot{S}$ which are forced by every condition to be in $\dot{\mathscr{F}}$. Since κ is Mahlo and each of the iterands of $\mathscr{P}_\kappa$ has cardinality less than κ, there is a relative closed and unbounded set D of δ in E such that $\mathscr{P}_\delta$ has the δ-c.c., $\dot{T}$ is a $\mathscr{P}_\delta$-name, and if $\dot{S}$ is a $\mathscr{P}_\delta$-name for a subtree of $\dot{T}$ which has countable intersection with every element of $\dot{\mathscr{F}}_\delta$, then $\dot{S}$ is forced to be in $\dot{\mathscr{F}}^{\perp}$. Since $\langle A_\alpha : \alpha \in E\rangle$ is a $\diamondsuit_E(\kappa)$-sequence, there is a δ in D such that $A_\delta = (\dot{T}, \dot{\mathscr{F}}_\delta)$. At stage δ the partial order $\dot{\mathscr{Q}}_\delta$ forces both $\psi(\dot{\mathscr{F}}_\delta \cup \dot{\mathscr{B}})$ and $\varphi(\dot{\mathscr{F}}_\delta)$ where $\dot{\mathscr{B}}$ is $\dot{\mathscr{F}}_\delta^{\perp}$ computed after forcing with $\mathscr{P}_\delta$. By choice of δ, $\dot{\mathscr{B}}$ is forced to be a subset of $\dot{\mathscr{F}}^{\perp}$ and therefore $\mathscr{P}_\kappa$ forces that $\dot{\mathscr{F}}_\delta^{\perp} = \dot{\mathscr{F}}^{\perp}$. Consequently, $\varphi(\dot{\mathscr{F}}_\delta)$ implies $\varphi(\dot{\mathscr{F}})$. $\square$

Theorem 5.4. *If there is a cardinal which is both reflecting and Mahlo, then there is a proper forcing extension of L which satisfies the conjunction of* PFA(ω_1) *and φ. In particular the forcing extension satisfies that the uncountable linear orders have a five element basis.*

Proof. This is very similar to the proof of Theorem 5.3, except that at stages $\alpha < \kappa$ which are not in E, we force with partial orders in $H(\kappa)$ given by an appropriate book keeping device. Following [8], it is possible to arrange that PFA(ω_1) holds in the generic extension as well. $\square$

Theorem 5.5. *Suppose that there is an inaccessible cardinal κ such that for every $\kappa_0 < \kappa$, there is an inaccessible cardinal $\delta < \kappa$ such that κ_0 is in $H(\delta)$ and $H(\delta)$ satisfies there are two reflecting cardinals which are greater than κ_0. Then there is a proper forcing extension in which κ is ω_2 and the uncountable linear orders have a five element basis.*

First observe that by taking a direct sum of trees, Theorems 4.2 implies that if $\vec{T}$ is an ω-sequence of A-trees and $\vec{\mathscr{F}}$ is an ω-sequence such that $\mathscr{F}_n$ is a family of subtrees of T_n, then there is a single set mapping $\Sigma_{\vec{\mathscr{F}}}$ such that if $\Sigma_{\vec{\mathscr{F}}}$ admits a reflecting sequence, then $\psi(\mathscr{F}_n \cup {\mathscr{F}_n}^{\perp})$ holds for all $n < \omega$. Theorem 5.5 can be proved by iterating the forcings provided by the following theorem with appropriate book keeping to yield a model of CAT together with the forcing axiom for proper partial orders of size ω_1. By mixing in appropriate σ-closed collapsing forcings as needed, we may ensure that the iteration has the κ-c.c. but collapses every uncountable cardinal less than κ to ω_1.

Theorem 5.6. *Suppose that T is an A-tree, K is a subset of T, and there is a δ such that $H(\delta)$ satisfies that there are two reflecting cardinals. Then there is a proper forcing in $H(\delta)$ which forces the instance of* CAT *for K and T.*

Proof. If λ is a reflecting cardinal in $H(\delta)$, let $\mathscr{P}_\lambda$ denote the proper forcing which satisfies the λ-c.c. and forces that $H(\delta)^{V^{\mathscr{P}_\lambda}}$ satisfies PFA(ω_1). If $\vec{\mathscr{F}}$ is an ω-sequence of families of subtrees of A-trees, let $\mathscr{Q}_{\vec{\mathscr{F}}}$ be the proper forcing which forces the conjunction of $\psi(\mathscr{F}_n \cup {\mathscr{F}_n}^\perp)$ and $\varphi(\mathscr{F}_n)$ for all n. Let $\lambda_0 < \lambda_1$ be the two reflecting cardinals in $H(\delta)$. We claim that

$$(\mathscr{P}_{\lambda_0} * \dot{\mathscr{Q}}_{\vec{\mathscr{R}}}) * \dot{\mathscr{P}}_{\lambda_1}$$

is the desired proper forcing where $\mathscr{R}_n$ is the family of subtrees of $T^{[n]}$ defined in Section 2. Clearly this forcing is proper and an element of $H(\delta)$. It suffices to show that it forces the instance of CAT for T and K. The key observation is that, after forcing with $\mathscr{P}_{\lambda_0}$, if S is an element of ${\mathscr{R}_n}^\perp$ for some n, then S remains in ${\mathscr{R}_n}^\perp$ after any proper forcing which is in $H(\delta)^{V^{\mathscr{P}_{\lambda_0}}}$. This is because asserting that S is not in $\mathscr{R}_n^\perp$ is a Σ_1-statement with parameters T, K, and S. By arguments given in the proof of Theorem 5.3, $\mathscr{Q}_{\vec{\mathscr{R}}}$ forces $\varphi(\dot{\mathscr{R}}_n)$ is true for all n and that moreover this statement remains true after further forcing with $\mathscr{P}_{\lambda_1}$. Applying Theorem 2.6 in the extension by

$$(\mathscr{P}_{\lambda_0} * \dot{\mathscr{Q}}_{\vec{\mathscr{R}}}) * \dot{\mathscr{P}}_{\lambda_1},$$

both PFA(ω_1) and the Key Lemma for T and K hold. Therefore, by theorems from [13], the instance of CAT for T and K is true. $\square$

6. Concluding remarks and questions

Observe that the property of κ in the statement of Theorem 5.5 is expressible by a Σ_0-formula with no parameters. Hence the least such cardinal is not reflecting and it is therefore possible — if such cardinals exist at all — to produce a forcing extension of L in which Shelah's conjecture is true and ω_2 is not reflecting in L.

Question 6.1. *Suppose that the uncountable linear orders have a five element basis. Is there a $\delta < \omega_2$ such that L_δ satisfies "there is a reflecting cardinal?"*

Another intriguing question is the following.

Question 6.2. *Does* PFA(ω_1) *imply Aronszajn tree saturation?*

The only known direct construction of a failure of A-tree saturation is given in [4, §2] and is based upon the existence of a Kurepa tree. Baumgartner has shown that PFA(ω_1) implies that there are no Kurepa trees [2, 7.11].[6] This is likely closely related to the consistency strength of φ.

Question 6.3. *If φ is true, must ω_2 be Mahlo in L?*

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