Scott processes

Paul B. Larson*

June 23, 2014

Abstract

The Scott process of a relational structure M is the sequence of sets of formulas given by the Scott analysis of M. We present axioms for the class of Scott processes of structures in a relational vocabulary τ, and use them to give a proof of an unpublished theorem of Leo Harrington from the 1970's, showing that a counterexample to Vaught's Conjecture has models of cofinally many Scott ranks below ω_2. Our approach also gives a theorem of Harnik and Makkai, showing that if there exists a counterexample to Vaught's Conjecture, then there is a counterexample whose uncountable models have the same $\mathcal{L}_{\omega_1,\omega}(\tau)$-theory, and which has a model of Scott rank ω_1. Moreover, we show that if ϕ is a sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$ giving rise to a counterexample to Vaught's Conjecture, then for every limit ordinal α greater than the quantifier depth of ϕ and below ω_2, ϕ has a model of Scott rank α.

1 Introduction

We fix for this paper a relational vocabulary τ, and distinct variable symbols $\{x_n : n < \omega\}$. For notational convenience, we assume that τ contains a 0-ary relation symbol, as well as the binary symbol $=$, which is always interpreted as equality. We refer the reader to [7, 6, 9] for the definition of the language $\mathcal{L}_{\infty,\omega}(\tau)$ and the languages $\mathcal{L}_{\kappa,\omega}(\tau)$, for κ an infinite cardinal. In this paper, all formulas will have only finitely many free variables. Formally, we consider conjunctions and disjunctions of formulas as unordered, even when we write them as indexed by an ordered set (in this way, for instance, a formula in $\mathcal{L}_{\omega_2,\omega}(\tau)$ becomes a member of $\mathcal{L}_{\omega_1,\omega}(\tau)$ in a forcing extension in which the ω_1 of the ground model is countable). We begin by recalling the standard definition of the Scott process corresponding to a τ-structure M (see [6, 9]), slightly modified to require the sequences $\bar{a}$ to consist of distinct elements.

1.1 Definition. Given a τ-structure M over a relational vocabulary τ, we define for each finite ordered tuple $\bar{a} = \langle a_0, \ldots, a_{|\bar{a}|-1}\rangle$ of distinct elements of M and each ordinal α the $|\bar{a}|$-ary formula $\phi^M_{\bar{a},\alpha} \in \mathcal{L}_{|M|^+,\omega}(\tau)$, as follows.

*Research supported in part by NSF Grants DMS-0801009 and DMS-1201494. The author thanks John Baldwin for his many comments on earlier drafts, and Leo Harrington for discussing the material in this paper with him.

1. Each formula $\phi^M_{\bar{a},0}$ is the conjunction of all expressions of the two following forms:

   - $R(x_{f(0)}, \ldots, x_{f(k-1)})$, for R a k-ary relation symbol from τ and f a function from k to $|a|$, such that $M \models R(a_{f(0)}, \ldots, a_{f(k-1)})$,
   - $\neg R(x_{f(0)}, \ldots, x_{f(k-1)})$, for R a k-ary relation symbol from τ and f a function from k to $|a|$, such that $M \models \neg R(a_{f(0)}, \ldots, a_{f(k-1)})$.

2. Each formula $\phi^M_{\bar{a},\alpha+1}$ is the conjunction of the following three formulas:

   - $\phi^M_{\bar{a},\alpha}$,
   - $\bigwedge_{c \in M \setminus \{a_0, \ldots, a_{|\bar{a}|-1}\}} \exists x_{|a|} \phi^M_{\bar{a}^\frown \langle c \rangle, \alpha}$,
   - $\forall x_{|\bar{a}|} \notin \{x_0, \ldots, x_{|a|-1}\} \bigvee_{c \in M \setminus \{a_0, \ldots, a_{|\bar{a}|-1}\}} \phi^M_{\bar{a}^\frown \langle c \rangle, \alpha}$.

3. For limit ordinals β, $\phi^M_{\bar{a},\beta} = \bigwedge_{\alpha < \beta} \phi^M_{\bar{a},\alpha}$.

We call $\phi^M_{\bar{a},\alpha}$ the *Scott formula* of $\bar{a}$ *in* M *at level* α.

The following well-known fact can be proved by induction on α (see Theorem 3.5.2 of [6]). Again, we refer the reader to [7, 6, 9] for the definition of the *quantifier depth* of a formula, and note that each formula $\phi^M_{\bar{a},\alpha}$ as defined above has quantifier depth exactly α.

Theorem 1.2. *Given τ-structures M and N, $n \in \omega$, an ordinal α and n-tuples $\bar{a}$ from M and $\bar{b}$ from N, each consisting of distinct elements, $\phi^M_{\bar{a},\alpha} = \phi^N_{\bar{b},\alpha}$ if and only if, for each n-ary $\mathcal{L}_{\infty,\omega}(\tau)$ formula ψ of quantifier depth at most α, $\bar{a}$ satisfies ψ in M if and only if $\bar{b}$ satisfies ψ in N.*

1.3 Definition. Given a τ-structure M and an ordinal β, we let $\Phi_\beta(M)$ denote the set of all formulas of the form $\phi^M_{\bar{a},\beta}$, for $\bar{a}$ a finite tuple of distinct elements of M. We call the class-length sequence $\langle \Phi_\alpha(M) : \alpha \in \mathrm{Ord} \rangle$ the *Scott process of M*.

This paper studies the class of Scott processes of all τ-structures (or, more formally, the class of set-length initial segments of Scott processes of τ-structures). Section 2 introduces an array of sets of formulas (properly) containing all the formulas appearing in the Scott process of any τ-structure, and vertical and horizontal projection functions acting on this array. Section 3 introduces our general notion of a Scott process (i.e., without regard to a fixed τ-structure). Section 4 develops some of the basic consequences of this definition, and Section 5 defines the rank of a Scott process. The material in these two sections checks that Scott process in general, as defined here, satisfy various basic properties of Scott processes of τ-structures. Section 6 shows that a Scott process of countable length whose last level is countable is an initial segment of the Scott process of some τ-structure. Section 7 shows how to build models of cardinality $\aleph_1$, for certain Scott processes (roughly, those corresponding to Scott sentences). Section 8 develops more basic material on Scott processes, studying the way they reflect finite blocks of existential quantifiers. Section 9 looks at extending Scott processes of limit length. Section 10 is largely disjoint from the rest of the paper, and presents an argument showing that in some cases (for instance, counterexamples to Vaught's Conjecture in $\mathcal{L}_{\omega_1,\omega}(\tau)$) a Scott process which exists in a forcing extension can be shown to exist in the ground model. Put together, the material in Sections 7, 9 and 10 gives Harrington's theorem that a counterexample to Vaught's Conjecture has models of cofinally many Scott ranks below ω_2. Our proof gives slightly more than Harrington's theorem as commonly stated, although we do not know if our version of his result is new.

The main results of the paper are summarized in Theorem 10.8. A proof of the Harnik-Makkai theorem on minimal counterexamples to Vaught's Conjecture is outlined in Remark 10.9.

The material in this paper was inspired by the slides of a talk given by David Marker on Harrington's theorem [10]. Our proof is different in some respects from the proof outlined there. Marker's talk outlines a recursion-theoretic argument, assuming the existence of a counterexample ϕ to Vaught's Conjecture, for finding a sentence in $\mathcal{L}_{\omega_2,\omega}$ which will be the Scott sentence of a model of ϕ (of suitably high Scott rank) in a forcing extension collapsing ω_1. This part of the proof is replaced here by a forcing-absoluteness argument in Section 10. The remainder of Harrington's proof builds a model of this Scott sentence. This we do in Section 7, guided by the argument in Marker's slides.

Another, different, proof of Harrington's theorem is due to appear in [1].

2 Formulas and projections

For each $n \in \omega$, let X_n denote the set $\{x_m : m < n\}$, and let i_n denote the identity function on X_n. For all $m \leq n \in \omega$, we let $\mathcal{I}_{m,n}$ denote the set of injections from X_m into X_n.

We start by defining a class of formulas which contains every formula appearing in the Scott process of any τ-structure (see Remark 2.5). The sets Ψ_α defined below also contain formulas that do not appear in the Scott process of any τ-structure. Definition 5.13 takes advantage of this extra generality, and in any case strengthening the definition to rule out such formulas would raise issues that we would rather defer. For the moment, the important point is that the sets Ψ_β ($\beta \in \mathrm{Ord}$) are small enough to carry the projection functions $V_{\alpha,\beta}$ and H^n_α defined below.

2.1 Definition. We define, for each ordinal α and each $n \in \omega$, the sets Ψ_α and Ψ^n_α, by recursion on α, as follows.

1. For each $n \in \omega$, Ψ^n_0 is the set of all conjunctions consisting of, for each atomic τ-formula using variables from X_n, either the formula or its negation, including an instance of the formula $x_i \neq x_j$ for each pair of distinct x_i, x_j from X_n.

2. For each ordinal α and each $n \in \omega$, $\Psi^n_{\alpha+1}$ is the set of formulas ϕ for which there exist a formula $\phi' \in \Psi^n_\alpha$ and a subset E of Ψ^{n+1}_α such that ϕ is the conjunction of ϕ' with the following two formulas.

   (a) $\bigwedge_{\psi \in E} \exists x_n \psi$;

   (b) $\forall x_n(x_n \notin \{x_0, \ldots, x_{n-1}\} \rightarrow \bigvee_{\psi \in E} \psi)$.

3. For each limit ordinal α and each $n \in \omega$, Ψ^n_α is the set of conjunctions which consist of exactly one formula ψ_β from each Ψ^n_β, for $\beta < \alpha$, satisfying the following conditions.

   (a) For each $\beta < \alpha$, ψ_β is the formula ϕ' with respect to $\psi_{\beta+1}$, as in condition (2).

   (b) For each limit ordinal $\beta < \alpha$, $\psi_\beta = \bigwedge\{\psi_\gamma : \gamma < \beta\}$.

4. For each ordinal α, $\Psi_\alpha = \bigcup_{n \in \omega} \Psi^n_\alpha$.

We can think of the sets Ψ^n_α as forming an array, with the rows indexed by α and the columns indexed by n. In the rest of this section we define the functions $V_{\alpha,\beta}$, which map between rows while preserving column rank, and the functions H^n_α which map between columns while preserving row rank.

2.2 Remark. Each Ψ_α is a set of $\mathcal{L}_{\infty,\omega}(\tau)$ formulas of quantifier depth α, so the sets Ψ_α are disjoint for distinct α. Similarly, for each $n \in \omega$ and each ordinal α, X_n is the set of free variables for each formula in each Ψ^n_α.

2.3 Remark. As we require our vocabulary to contain a 0-ary relation as well as the binary relation $=$, Ψ^n_α is nonempty for each ordinal α and each $n \in \omega$.

2.4 Definition. For each ordinal α, and each formula ϕ in $\Psi_{\alpha+1}$, we let $E(\phi)$ denote the set E from condition (2) of Definition 2.1.

2.5 Remark. If M is a τ-structure, α is an ordinal and $\bar{a}$ is a finite tuple of distinct elements of M, then the Scott formula of $\bar{a}$ in M at level α defined in Definition 1.1 (i.e., $\phi^M_{\bar{a},\alpha}$) is an element of $\Psi^{|\bar{a}|}_\alpha$. It follows that $\Phi_\alpha(M) \subseteq \Psi_\alpha$.

The functions $V_{\alpha,\beta}$, as defined below, are the *vertical projection functions.*

2.6 Definition. The functions $V_{\alpha,\beta} \colon \Psi_\beta \rightarrow \Psi_\alpha$, for all pairs of ordinals $\alpha \leq \beta$ are defined as follows.

1. Each function $V_{\alpha,\alpha}$ is the identity function on Ψ_α.

2. For each ordinal α, and each $\phi \in \Psi_{\alpha+1}$, $V_{\alpha,\alpha+1}(\phi)$ is the first conjunct of ϕ, i.e., the formula ϕ' in condition (2) of Definition 2.1.

3. For each limit ordinal β, each formula $\phi \in \Psi_\beta$, and each $\alpha < \beta$, $V_{\alpha,\beta}(\phi)$ is the unique conjunct of ϕ in Ψ_α.

4. For all ordinals $\alpha < \beta$, $V_{\alpha,\beta+1} = V_{\alpha,\beta} \circ V_{\beta,\beta+1}$.

2.7 Remark. Conditions (2) and (3) of Definition 2.1 imply the following stronger version of condition (4) of Definition 2.6 : for all ordinals $\alpha \leq \beta \leq \gamma$, $V_{\alpha,\gamma} = V_{\alpha,\beta} \circ V_{\beta,\gamma}$.

2.8 Remark. For all ordinals $\alpha \leq \beta$, each $n \in \omega$, and each $\phi \in \Psi^n_\beta$, $V_{\alpha,\beta}(\phi)$ is in Ψ^n_α, so ϕ and $V_{\alpha,\beta}(\phi)$ have the same free variables.

2.9 Remark. Since the domains of the functions $V_{\alpha,\beta}$ are disjoint for distinct β, one could drop β and simply write V_α (which would then be a definable class-sized function from $\bigcup_{\beta \in (\mathrm{Ord} \setminus \alpha)} \Psi_\beta$ to Ψ_α). We retain both subscripts for clarity.

We define the *horizontal projection functions* as follows.

2.10 Definition. The functions H^n_α, for each ordinal α and each $n \in \omega$, are defined recursively on α, as follows.

1. The domain of each H^n_α consists of all pairs (ϕ, j), where $\phi \in \Psi^n_\alpha$ and, for some $m \leq n$, $j \in \mathcal{I}_{m,n}$.
2. For all $m \leq n$ in ω, all formulas $\phi \in \Psi^n_0$, and all $j \in \mathcal{I}_{m,n}$, $H^n_0(\phi, j)$ is the conjunction of all conjuncts from ϕ whose variables are contained in the range of j, with these variables replaced by their j-preimages.
3. For each ordinal α, each $m \leq n$ in ω, each $\phi \in \Psi^n_{\alpha+1}$, and each $j \in \mathcal{I}_{m,n}$, $H^n_{\alpha+1}(\phi, j)$ is the formula $\psi \in \Psi^m_{\alpha+1}$ such that
$$V_{\alpha,\alpha+1}(\psi) = H^n_\alpha(V_{\alpha,\alpha+1}(\phi), j)$$
and $E(\psi) = H^{n+1}_\alpha[E(\phi) \times \{j \cup \{(x_m, y)\} \mid y \in (X_{n+1} \setminus \mathrm{range}(j))\}]$.
4. For each limit ordinal α, each $m \leq n$ in ω, each $j \in \mathcal{I}_{m,n}$ and each $\phi \in \Psi^n_\alpha$,
$$H^n_\alpha(\phi, j) = \bigwedge\{H^n_\beta(V_{\beta,\alpha}(\phi), j) : \beta < \alpha\}.$$

2.11 Remark. Since the domains of the functions H^n_α are disjoint for distinct pairs (α, n), one could drop α and n and simply write H. We retain them for clarity.

2.12 Remark. For all ordinals α, all $m \leq n$ in ω, all $j \in \mathcal{I}_{m,n}$ and all $\phi \in \Psi^n_\alpha$, $H^n_\alpha(\phi, j)$ is an element of Ψ^m_α, and $H^n_\alpha(\phi, i_n) = \phi$.

We leave it to the reader to verify (by induction on α) that if

- M is a τ-structure,
- α is an ordinal,
- $m \leq n$ are elements of ω,
- $\bar{b} = \langle b_0, \ldots, b_{n-1} \rangle$ is a sequence of distinct elements of M,

- $j^* \colon m \to n$ is an injection,
- $\bar{a}$ is the sequence $\langle b_{j^*(0)}, \ldots, b_{j^*(m-1)} \rangle$ and
- $j \in \mathcal{I}_{m,n}$ is such that $j(x_p) = x_{j^*(p)}$ for each $p < m$,

then $H^n_\alpha(\phi^M_{\bar{b},\alpha}, j) = \phi^M_{\bar{a},\alpha}$.

2.13 Remark. The following facts can be easily verified by induction on α.

1. For each ordinal α, each $n \in \omega$, each $\phi \in \Psi^n_\alpha$ and each $j \in \mathcal{I}_{n,n}$, $H^n_\alpha(\phi, j)$ is the result of replacing each free variable in ϕ (i.e., each member of X_n) with its j-preimage.
2. For each ordinal α, all $m \leq n \leq p$ in ω, all $\phi \in \Psi^p_\alpha$, all $j \in \mathcal{I}_{n,p}$ and all $k \in \mathcal{I}_{m,n}$, $H^n_\alpha(H^p_\alpha(\phi, j), k) = H^p_\alpha(\phi, j \circ k)$.

The following proposition shows that the vertical and horizontal projection functions commute appropriately.

Proposition 2.14. *For all ordinal $\alpha \leq \beta$, all $m \leq n \in \omega$, all $j \in \mathcal{I}_{m,n}$, and all $\phi \in \Psi^n_\beta$,*

$$V_{\alpha,\beta}(H^n_\beta(\phi, j)) = H^n_\alpha(V_{\alpha,\beta}(\phi), j).$$

Proof. When $\alpha = \beta$, both sides are equal to $H^n_\alpha(\phi, j)$. When $\beta = \alpha + 1$, the proposition is part of condition (3) of Definition 2.10. When β is a limit ordinal, it follows from condition (3) of Definition 2.6 and condition (4) of Definition 2.10. The remaining cases can be proved by induction on β, fixing α, using the induction hypotheses for the pairs α, β and $\beta, \beta + 1$ at successor stages of the form $\beta + 1$.

□

2.15 Example. Suppose that τ contains a single binary relation symbol R, along with $=$ and the 0-ary relation symbol S. The set Ψ^0_0 then consists of the sentences S and $\neg S$. The set Ψ^1_0 contains four formulas, $S \wedge R(x_0, x_0)$, $S \wedge \neg R(x_0, x_0)$, $\neg S \wedge R(x_0, x_0)$ and $\neg S \wedge \neg R(x_0, x_0)$. Call the first two of these formulas ψ^1_0 and ϕ^1_0, respectively. The set Ψ^2_0 then contains 32 formulas, for instance,

$$S \wedge \neg R(x_0, x_1) \wedge \neg R(x_1, x_0) \wedge R(x_0, x_0) \wedge R(x_1, x_1) \wedge x_0 \neq x_1$$

and

$$S \wedge \neg R(x_0, x_1) \wedge \neg R(x_1, x_0) \wedge R(x_0, x_0) \wedge \neg R(x_1, x_1) \wedge x_0 \neq x_1.$$

Call these formulas ψ^2_0 and ϕ^2_0, respectively. Then

$$H^2_0(\psi^2_0, i_1) = \psi^1_0$$

and

$$H^2_0(\phi^2_0, \{(x_0, x_1)\}) = \phi^1_0,$$

as defined in Definition 2.10. The set Ψ_0^3 then contains 2^{10} formulas, including the conjunction of S with every instance of $R(y,z)$ for $y,z \in X_3$. In general, Ψ_0^n contains $2^{(n^2+1)}$ formulas.

The set Ψ_1^0 contains the sentences

$$S \wedge (\exists x_0 S \wedge R(x_0,x_0)) \wedge (\forall x_0 S \wedge R(x_0,x_0))$$

(omitting one instance each of $\bigwedge$ and $\bigvee$, corresponding to a conjunction and a disjunction of of size 1) and

$$S \wedge ((\exists x_0 S \wedge R(x_0,x_0)) \wedge (\exists x_0 S \wedge \neg R(x_0,x_0))) \wedge (\forall x_0 (S \wedge R(x_0,x_0)) \vee (S \wedge \neg R(x_0,x_0))).$$

Call these sentences ψ_1^0 and ϕ_1^0, respectively. Then $E(\psi_1^0) = \{\psi_0^1\}$ and $E(\phi_1^0) = \{\psi_0^1, \phi_0^1\}$, as defined in Definition 2.4. The set Ψ_1^1 contains the formulas

$$\psi_0^1 \wedge (\exists x_1 \psi_0^2) \wedge (\forall x_1\, x_1 \neq x_0 \rightarrow \psi_0^2)$$

and

$$\psi_0^1 \wedge (\exists x_1 \phi_0^2) \wedge (\forall x_1\, x_1 \neq x_0 \rightarrow \phi_0^2),$$

again omitting an instance of each of $\bigwedge$ and $\bigvee$ in each formula. Call these formulas ψ_1^1 and ϕ_1^1. Then $E(\psi_1^1) = \{\psi_0^2\}$, $E(\phi_1^1) = \{\phi_0^2\}$,

$$V_{0,1}(\psi_1^1) = V_{0,1}(\phi_1^1) = \psi_0^1,$$

$H_1^1(\psi_1^1, i_0) = \psi_1^0$ and $H_1^1(\phi_1^1, i_0) = \phi_1^0$. Note that the function H_1^1 changes the bound variables (as well as the free variables, when the second coordinate of the input is the empty function).

3 Scott processes

This section introduces the central topic of the paper, the class of Scott processes (for a relational vocabulary τ).

3.1 Definition. A *Scott process* is a sequence $\langle \Phi_\alpha : \alpha < \delta \rangle$, for some ordinal δ (the *length* of the process), satisfying the following conditions, where for each ordinal α and each $n \in \omega$, Φ_α^n denotes the set $\Phi_\alpha \cap \Psi_\alpha^n$.

1. **The Formula Conditions**
   (a) Each Φ_α is a subset of the corresponding set Ψ_α.
   (b) For each ordinal of the form $\alpha + 1 < \delta$, and each $\phi \in \Phi_{\alpha+1}$, $E(\phi)$ is a subset of Φ_α.
   (c) For all $\alpha < \beta < \delta$, $\Phi_\alpha = V_{\alpha,\beta}[\Phi_\beta]$.
   (d) For all $\alpha < \delta$, all $n \in \omega$, all $j \in \mathcal{I}_{n,n}$ and all $\phi \in \Phi_\alpha^n$, $H_\alpha^n(\phi, j) \in \Phi_\alpha^n$.
   (e) For all $\alpha < \delta$, and all $m < n$ in ω, $\Phi_\alpha^m = H_\alpha^n[\Phi_\alpha^n \times \{i_m\}]$.
2. **The Coherence Conditions**

(a) For each ordinal of the form $\alpha+1$ below δ, each $n \in \omega$ and each $\phi \in \Phi^n_{\alpha+1}$,

$$E(\phi) = V_{\alpha,\alpha+1}[\{\psi \in \Phi^{n+1}_{\alpha+1} \mid H^{n+1}_{\alpha+1}(\psi, i_n) = \phi\}].$$

(b) For all $\alpha < \beta < \delta$, all $n \in \omega$ and all $\phi \in \Phi^n_\beta$,

$$E(V_{\alpha+1,\beta}(\phi)) \subseteq V_{\alpha,\beta}[\{\psi \in \Phi^{n+1}_\beta \mid H^{n+1}_\beta(\psi, i_n) = \phi\}].$$

(c) For all $\alpha < \delta$, n, m in ω, $\phi \in \Phi^n_\alpha$ and $\psi \in \Phi^m_\alpha$, there exist $\theta \in \Phi^{n+m}_\alpha$ and $j \in \mathcal{I}_{m,n+m}$ such that $\phi = H^{n+m}_\alpha(\theta, i_n)$ and $\psi = H^{n+m}_\alpha(\theta, j)$.

The sets Φ_α are called the *levels* of the Scott process.

3.2 Remark. Condition (2b) of Definition 3.1 includes the left to right inclusion in condition (2a). We prefer the given formulation of condition (2a), as it gives a better sense of the meaning of $E(\phi)$.

3.3 Remark. Proposition 4.4 shows that equality holds in condition (2b) of Definition 3.1, for any Scott process, so that conditions (2a) and (2b) could equivalently be replaced by condition (2b) alone with $=$ in place of $\subseteq$.

3.4 Remark. Conditions (1d) and (1e) of Definition 3.1 combine to give the following: for all $\alpha < \delta$, all $m \leq n$ in ω and all $j \in \mathcal{I}_{m,n}$, $\Phi^m_\alpha = H^n_\alpha[\Phi^n_\alpha \times \{j\}]$.

Proposition 3.5 follows from condition (2c) of Definition 3.1 and part (1) of Remark 2.13, which implies that $H^0_\alpha(\phi, i_0) = \phi$ for all ordinals α and all $\phi \in \Psi^0_\alpha$.

Proposition 3.5. *Whenever $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process, Φ^0_α has a unique element, for each $\alpha < \delta$.*

4 Consequences of coherence

In this section we prove some basic facts about Scott processes, primarily about sets of the form $E(\phi)$. The main result of the section is Proposition 4.4, which was referred to in Remark 3.3. We fix for this section a Scott process $\langle \Phi_\alpha : \alpha < \delta \rangle$.

Proposition 4.1 follows from Proposition 2.14 (i.e., the commutativity of the horizontal and vertical projections). The failure of the reverse inclusion is witnessed whenever a set of the form $V^{-1}_{\alpha,\beta}[\{\rho\}]$ has more than one member (consider $\theta \in E(\phi_1) \setminus E(\phi_2)$, for $\phi_1, \phi_2 \in V^{-1}_{\alpha,\beta}[\{\rho\}]$).

Proposition 4.1. *For all $\alpha \leq \beta < \delta$, all $m \leq n \in \omega$, all $j \in \mathcal{I}_{m,n}$, and all $\phi \in \Phi^m_\beta$,*

$$V_{\alpha,\beta}[\{\psi \in \Phi^n_\beta \mid H^n_\beta(\psi, j) = \phi\}] \subseteq \{\theta \in \Phi^n_\alpha \mid H^n_\alpha(\theta, j) = V_{\alpha,\beta}(\phi)\}.$$

The right-to-left inclusion in Proposition 4.2 says that every one-point extension of a formula ϕ at level α is a member of $E(\psi)$, for some $\psi \in V^{-1}_{\alpha,\alpha+1}[\phi]$. This proposition is used in Remark 5.14.

Proposition 4.2. *For each ordinal of the form $\alpha + 1$ below δ, each $n \in \omega$ and each $\phi \in \Phi^n_\alpha$,*

$$\bigcup\{E(\psi) \mid \psi \in V^{-1}_{\alpha,\alpha+1}[\{\phi\}]\} = \{\theta \in \Phi^{n+1}_\alpha \mid H^{n+1}_\alpha(\theta, i_n) = \phi\}.$$

Proof. The left-to-right inclusion follows from Proposition 4.1 and condition (2a) of Definition 3.1. The reverse inclusion follows from conditions (1c) and (2a) of Definition 3.1, and Proposition 2.14. □

Proposition 4.3 is the successor case of Proposition 4.4.

Proposition 4.3. *For all $\alpha \leq \beta$ such that $\beta + 1 < \delta$, and all $\phi \in \Phi_{\beta+1}$,*

$$E(V_{\alpha+1,\beta+1}(\phi)) = V_{\alpha,\beta}[E(\phi)].$$

Proof. Fix $n \in \omega$ such that $\phi \in \Phi^n_{\beta+1}$. For the forward direction, condition (2b) of Definition 3.1 gives that

$$E(V_{\alpha+1,\beta+1}(\phi)) \subseteq V_{\alpha,\beta+1}[\{\psi \in \Phi^{n+1}_{\beta+1} \mid H^{n+1}_{\beta+1}(\psi, i_n) = \phi\}],$$

which by condition (4) of Definition 2.6 is equal to

$$V_{\alpha,\beta}[V_{\beta,\beta+1}[\{\psi \in \Phi^{n+1}_{\beta+1} \mid H^{n+1}_{\beta+1}(\psi, i_n) = \phi\}]],$$

which by condition (2a) of Definition 3.1 is equal to $V_{\alpha,\beta}[E(\phi)]$.

For the reverse direction we have from condition (2a) of Definition 3.1 that $V_{\alpha,\beta}[E(\phi)]$ is equal to

$$V_{\alpha,\beta}[V_{\beta,\beta+1}[\{\psi \in \Phi^{n+1}_{\beta+1} \mid H^{n+1}_{\beta+1}(\psi, i_n) = \phi\}]],$$

which by Remark 2.7 is equal to

$$V_{\alpha,\alpha+1}[V_{\alpha+1,\beta+1}[\{\psi \in \Phi^{n+1}_{\beta+1} \mid H^{n+1}_{\beta+1}(\psi, i_n) = \phi\}]],$$

which by Proposition 4.1 is contained in

$$V_{\alpha,\alpha+1}[\{\theta \in \Phi^{n+1}_{\alpha+1} \mid H^{n+1}_{\alpha+1}(\theta, i_n) = V_{\alpha+1,\beta+1}(\phi)\}]],$$

which by condition (2a) of Definition 3.1 is equal to $E(V_{\alpha+1,\beta+1}(\phi))$. □

We now show that the reverse inclusion of condition (2b) of Definition 3.1 holds for any Scott process.

Proposition 4.4. *For all $\alpha < \beta < \delta$, for all $n \in \omega$ and all $\phi \in \Phi^n_\beta$,*

$$E(V_{\alpha+1,\beta}(\phi)) = V_{\alpha,\beta}[\{\psi \in \Phi^{n+1}_\beta \mid H^{n+1}_\beta(\psi, i_n) = \phi\}].$$

Proof. When β is a successor ordinal, this is Proposition 4.3, using condition (2a) of Definition 3.1. For any β, the left-to-right inclusion is condition (2b) of Definition 3.1. For the reverse inclusion,

$$V_{\alpha,\beta}[\{\psi \in \Phi_\beta^{n+1} \mid H_\beta^{n+1}(\psi, i_n) = \phi\}]$$

is equal to

$$V_{\alpha,\alpha+1}[V_{\alpha+1,\beta}[\{\psi \in \Phi_\beta^{n+1} \mid H_\beta^{n+1}(\psi, i_n) = \phi\}]]$$

by Remark 2.7, and this is contained in

$$V_{\alpha,\alpha+1}[\{\psi \in \Phi_{\alpha+1}^{n+1} \mid H_{\alpha+1}^{n+1}(\psi, i_n) = V_{\alpha+1,\beta}(\phi)\}],$$

by Proposition 4.1. Finally, this last term is equal to $E(V_{\alpha+1,\beta}(\phi))$ by condition (2a) of Definition 3.1. $\square$

5 Ranks and Scott sentences

The *Scott rank* of a τ-structure M is the least ordinal α such that $V_{\alpha,\alpha+1}$ is injective on $\Phi_{\alpha+1}(M)$ (see [6, 9], which use different terminology, of course, to define the same thing). If α is the Scott rank of M, then $V_{\beta,\beta+1}$ injective on $\Phi_{\beta+1}(M)$ for all $\beta \geq \alpha$ as well. Proposition 5.4 below verifies that Scott processes have the same property. We isolate the successor step of the proof as a separate proposition (the second part of the proposition is used in Remark 9.10).

Proposition 5.1. *Let β be an ordinal, and let $\langle \Phi_\alpha : \alpha \leq \beta + 2\rangle$ be a Scott process. If ϕ is an element of $\Phi_{\beta+1}$, then each of the following conditions implies that $V^{-1}_{\beta+1,\beta+2}[\{\phi\}] \cap \Phi_{\beta+2}$ is a singleton.*

1. *$V^{-1}_{\beta,\beta+1}[\{\psi\}] \cap \Phi_{\beta+1}$ is a singleton for each $\psi \in E(\phi)$.*
2. *There exists a $\psi \in E(\phi)$ such that $V^{-1}_{\beta,\beta+2}[\{\psi\}] \cap \Phi_{\beta+2}$ is a singleton.*

Proof. Suppose that $\phi' \in \Phi_{\beta+2}$ is such that $V_{\beta+1,\beta+2}(\phi') = \phi$. Assuming the first condition, by Proposition 4.3, $V_{\beta,\beta+1}[E(\phi')] = E(\phi)$. Since $V^{-1}_{\beta,\beta+1}[\{\psi\}] \cap \Phi_{\beta+1}$ is a singleton for each $\psi \in E(\phi)$, this implies that $E(\phi') = V^{-1}_{\beta,\beta+1}[E(\phi)] \cap \Phi_{\beta+1}$, which uniquely determines ϕ'.

For the second part, let ψ' be the unique member of $V^{-1}_{\beta,\beta+2}[\{\psi\}] \cap \Phi_{\beta+2}$. Since $\psi \in E(\phi)$, $V_{\beta+1,\beta+2}(\psi')$ is a member of $E(\phi')$, by Proposition 4.3. Let $n \in \omega$ be such that $\phi \in \Phi^n_{\beta+1}$. Then $\phi' = H^{n+1}(\psi', i_n)$, by part (2a) of Definition 3.1. $\square$

The following is a consequence of part (1) of Proposition 5.1.

Corollary 5.2. *Let β be an ordinal, and let $\langle \Phi_\alpha : \alpha \leq \beta+2\rangle$ be a Scott process. Suppose that $n \in \omega$ is such that $V_{\beta,\beta+1}$ is injective on $\Phi^{n+1}_{\beta+1}$. Then $V_{\beta+1,\beta+2}$ is injective on $\Phi^n_{\beta+2}$.*

5.3 Remark. It is natural to ask whether part (1) of Proposition 5.1 has a converse, in the sense that if $\langle \Phi_\alpha : \alpha \leq \beta + 1 \rangle$ is a Scott process and $\phi \in \Phi_{\beta+1}$ and $\psi \in E(\phi)$ are such that $V^{-1}_{\beta,\beta+1}[\{\psi\}]$ has at least two members then there must exist a set $\Phi_{\beta+2}$ such that $\langle \Phi_\alpha : \alpha \leq \beta + 2 \rangle$ is a Scott process and $V^{-1}_{\beta+1,\beta+2}[\{\phi\}]$ is not a singleton. This is not the case in general, however, as by Proposition 3.5, each function of the form $V_{\alpha,\alpha+1} \restriction \Phi^0_{\alpha+1}$ is always injective.

Proposition 5.4. *If $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process, $\beta < \gamma$ are ordinals with $\gamma + 1 < \delta$, and $V_{\beta,\beta+1} \restriction \Phi_{\beta+1}$ is injective, then $V_{\gamma,\gamma+1} \restriction \Phi_{\gamma+1}$ is injective.*

Proof. Letting η be such that $\gamma = \beta + \eta$, we prove the proposition by induction on η, for all β and δ simultaneously. Applying the induction hypotheses, the limit case follows from Remark 2.7, and the successor case follows from part (1) of Proposition 5.1 (and also from Corollary 5.2). $\square$

5.5 Definition. The *rank* of a Scott process $\langle \Phi_\alpha : \alpha < \delta \rangle$ is the least β such that $V_{\beta,\beta+1} \restriction \Phi_{\beta+1}$ is injective, if such a β exists, and undefined otherwise.

The rank of (any suitably long set-sized initial segment of) the Scott process of a τ-structure M is the same then as the Scott rank of M.

5.6 Remark. Suppose that β and γ are ordinals, and $n \in \omega$ is such that $\gamma > \beta + n$. Suppose that $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is a Scott process, and that $V_{\beta,\beta+1}$ is injective on $\Phi^m_{\beta+1}$, for all $m > n$ in ω. By Corollary 5.2, the rank of $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is at most $\beta + n$ (since each Φ^0_α is a singleton, $V_{\alpha,\alpha+1} \restriction \Phi^0_{\alpha+1}$ is injective for all α).

In the following definition, j can equivalently be replaced with i_n, by condition (1d) of Definition 3.1.

5.7 Definition. Let β and γ be ordinals such that $\gamma > \beta + 1$, and let

$$\langle \Phi_\alpha : \alpha < \gamma \rangle$$

be a Scott process. Let n be an element of ω, and let ϕ be an element of Φ^n_β. We say that the Scott process $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is *injective beyond* ϕ if for all $m \in \omega \setminus n$, all $j \in \mathcal{I}_{n,m}$ and all $\psi \in \Phi^m_\beta$ such that $\phi = H^m_\beta(\psi, j)$, $V^{-1}_{\beta,\beta+1}[\{\psi\}] \cap \Phi_{\beta+1}$ is a singleton.

5.8 Remark. Let $\beta < \delta < \gamma$ be ordinals, and let $\langle \Phi_\alpha : \alpha \leq \gamma \rangle$ be a Scott process. Let n be an element of ω, and let $\phi \in \Phi^n_\beta$ be such that $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is injective beyond ϕ. Then for all $m \in \omega \setminus n$, all $j \in \mathcal{I}_{n,m}$ and all $\psi \in \Phi^m_\beta$ such that $\phi = H^m_\beta(\psi, j)$, $V^{-1}_{\beta,\gamma}[\{\psi\}] \cap \Phi_\gamma$ is a singleton. This follows from part (1) of Proposition 5.1. It follows that $\langle \Phi_\alpha : \alpha \leq \gamma \rangle$ is injective beyond the unique member of $V^{-1}_{\beta,\delta}[\{\phi\}]$.

5.9 Remark. Let β be an ordinal, and n an element of ω. Suppose that

$$\langle \Phi_\alpha : \alpha \leq \beta + 1 \rangle$$

is a Scott process, and that $\phi \in \Phi^n_\beta$ is such that $\langle \Phi_\alpha : \alpha \leq \beta+1\rangle$ is injective beyond ϕ. The proof of Scott's Isomorphism Theorem (Theorem 2.4.15 of [9]; using $\bar{a}$ in place of $\emptyset$ at stage 0) shows that for any two countable τ-structures M and N whose Scott processes agree with $\langle \Phi_\alpha : \alpha \leq \beta+1\rangle$ though level $\beta+1$, if $\bar{a}$ is an n-tuple from M and $\bar{b}$ is an n-tuple from N, each satisfying ϕ in their respective models, then there is an isomorphism of M and N sending $\bar{a}$ to $\bar{b}$. Alternately, one can show that for each ordinal $\gamma > \beta+1$, there is a unique Scott process of length γ extending β, using either Remark 5.8 or Proposition 8.10.

5.10 Remark. In the situation of Definition 5.7, $\langle \Phi_\alpha : \alpha \leq \beta+1\rangle$ need not have rank β. To see this, consider the Scott process of a countably infinite undirected graph G consisting of an infinite set of nodes which are not connected to anything, and another infinite set of nodes which are all connected to each other, but not to themselves. The formula in $\Phi^2_0(G)$ corresponding to a connected pair has the property of ϕ in Remark 5.9, but the Scott rank of G is 1, not 0, since the unique member of $\Phi^1_0(G)$ has two successors in $\Phi^1_1(G)$.

The following definition is inspired by Remarks 5.9 and 5.10.

5.11 Definition. The *pre-rank* of a Scott process $\langle \Phi_\alpha : \alpha < \beta\rangle$ is the least $\gamma \leq \beta$ such that for all ordinals $\eta > \gamma$, there exists a unique Scott process of length η extending $\langle \Phi_\alpha : \alpha < \gamma\rangle$ (if such a γ exists). The Scott *pre-rank* of a τ-structure is the pre-rank of the sufficiently long initial segments of its Scott process.

The pre-rank of a Scott process is at most its rank, and Remark 5.10 shows that it can be smaller. By Proposition 9.17, if a Scott process has countable length, and all of its levels are countable, then its rank is at most ω more than its pre-rank. Proposition 5.12 gives a tighter bound in the situation of Definition 5.7.

Proposition 5.12. *Let β be an ordinal, and n an element of ω. Suppose that $\langle \Phi_\alpha : \alpha \leq \beta+\omega\rangle$ is a Scott process, and that $\phi \in \Phi^n_\beta$ is such that $\langle \Phi_\alpha : \alpha \leq \beta+\omega\rangle$ is injective beyond ϕ. Then $\langle \Phi_\alpha : \alpha \leq \beta+\omega\rangle$ has rank at most $\beta+n$.*

Proof. By Remark 5.8, for each $m \in \omega \setminus n$, $V^{-1}_{\beta,\beta+\omega}[\{\psi\}] \cap \Phi_{\beta+\omega}$ is a singleton for each $\psi \in \Phi^m_\beta$ such that $\phi = H^m_\beta(\psi, j)$ for some $j \in \mathcal{I}_{n,m}$. Let Υ be the set of $\psi \in \Phi_\beta$ for which $V^{-1}_{\beta,\beta+n+1}[\{\psi\}] \cap \Phi_{\beta+\omega}$ is a singleton. We show by induction on $p \leq n$ that if $q \in \omega$ and $\theta \in \Phi^q_{\beta+p}$ is such that $\theta = H^{q+p}_{\beta+p}(\rho, i_q)$ for some $\rho \in \Phi^{q+p}_{\beta+p}$ such that $V_{\beta,\beta+p}(\rho) \in \Upsilon$, then $V^{-1}_{\beta+p,\beta+n+1}[\{\theta\}]$ is a singleton. For $p = 0$ this follows from the definition of Υ. The induction step from p to $p+1$ (for some $\theta \in \Phi^q_{\beta+p+1}$, for some $q \in \omega$) follows from part (1) of Proposition 5.1, applied to

$$V_{\beta+p,\beta+p+1}(H^{q+p+1}_{\beta+p+1}(\rho, i_{q+1})),$$

where $\rho \in \Phi^{q+p+1}_{\beta+p+1}$ is such that $\theta = H^{q+p+1}_{\beta+p+1}(\rho, i_q)$ and $V_{\beta,\beta+p+1}(\rho) \in \Upsilon$. Finally, this statement for $n = p$ implies the proposition, by condition (2c) of Definition 3.1. $\square$

5.13 Definition. Given an ordinal δ and a set $\Phi \subseteq \Psi_\delta$, the *maximal completion* of Φ is the set of $\phi \in \Psi_{\delta+1}$ such that for some $n \in \omega$ and some $\phi' \in \Phi \cap \Psi^n_\delta$, $V_{\delta,\delta+1}(\phi) = \phi'$, and

$$E(\phi) = \{\psi \in \Phi \cap \Psi^{n+1}_\delta \mid H^{n+1}_\delta(\psi, i_n) = \phi'\}.$$

The extension of a Scott process of successor length by the maximal completion of its last level may not be a Scott process (see Proposition 5.18 below).

5.14 Remark. By Proposition 4.2, if $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process, and β is an ordinal such that $\beta + 1 < \delta$, then $V_{\beta,\beta+1} \restriction \Phi_{\beta+1}$ is injective if and only if $\Phi_{\beta+1}$ is the maximal completion of Φ_β.

The following definition describes the situation in which no formula ϕ has incompatible horizontal extensions.

5.15 Definition. Given an ordinal δ, a set $\Phi \subseteq \Psi_\delta$ satisfies the *amalgamation property* (or *amalgamates*) if for all $m < n \in \omega$, $\phi \in \Phi \cap \Psi^{m+1}_\delta$, and $\psi \in \Phi \cap \Psi^n_\delta$ such that $H^{m+1}(\phi, i_m) = H^n_\delta(\psi, i_m)$, there exist $\theta \in \Phi \cap \Phi^{n+1}_\delta$ and $y \in X_{n+1} \setminus X_m$ such that $H^{m+1}(\theta, i_m \cup \{(x_m, y)\}) = \phi$ and $H^{m+1}(\theta, i_n) = \psi$.

5.16 Remark. Given an ordinal δ and a set $\Phi \subseteq \Phi_\delta$ satisfying condition (1d) of Definition 3.1 (i.e., closure under the functions H^n_δ $(n \in \omega)$), the amalgamation property for a set $\Phi \subseteq \Psi_\delta$ is equivalent to the statement that for all $m \leq n \in \omega$, $\phi \in \Phi \cap \Psi^m_\delta$, $j \in \mathcal{F}_{m,n}$ and $\psi \in \Phi \cap \Psi^n_\delta$ such that $\phi = H^n_\delta(\psi, j)$,

$$\{\theta \in \Phi \cap \Psi^{m+1}_\delta \mid H^{m+1}_\delta(\theta, i_m) = \phi\}$$

is the same as

$$H^{n+1}_\delta[\{\rho \in \Phi \cap \Psi^{n+1}_\delta \mid H^{n+1}_\delta(\rho, i_n) = \psi\} \times \{j \cup \{(x_m, y)\} \mid y \in (X_{n+1} \setminus \mathrm{range}(j))\}].$$

This follows immediately from the definitions (using part (3) of Definition 2.10).

5.17 Remark. The set in the second displayed formula in Definition 5.15 is always contained in the set in the first, by part (2) of Remark 2.13.

Proposition 5.18. *The extension of a nonempty Scott process of nonlimit length by the maximal completion of its last level induces a Scott process if and only if its last level amalgamates.*

Proof. Let $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ be a Scott process. Conditions (1a)-(1c) of Definition 3.1 are always satisfied by the extension by the maximal completion. The other conditions depend on whether the functions $H^n_{\delta+1}$ $(n \in \omega)$ lift the actions of the functions H^n_δ $(n \in \omega)$, i.e., whether whenever $n \in \omega$, $\psi \in \Phi^n_\delta$ and ψ' is the unique member of $V^{-1}_{\delta,\delta+1}[\{\psi\}]$ in the maximal completion of Φ_δ,

$$V_{\delta,\delta+1}[H^n_{\delta+1}(\psi', j)] = H^n_\delta(\psi, j).$$

Comparing the condition (3) of Definition 2.10 with Definition 5.13 shows that is exactly the statement that Φ_δ amalgamates as expressed in Remark 5.16. $\square$

We conclude this section by giving a restatement of the amalgamation property which will be useful in Section 7. A failure of amalgamation gives a counterexample to Proposition 5.19 with $n = m + 1$.

Proposition 5.19. *Suppose that $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is Scott process whose last level amalgamates, and that $m, n, p \in \omega$ are such that $m \leq \min\{n, p\}$. Suppose now that $j \in \mathcal{I}_{m,n}$, $k \in \mathcal{I}_{m,p}$, $\psi \in \Phi^n_\delta$ and $\theta \in \Phi^p_\delta$ are such that*

$$H^n_\delta(\psi, j) = H^p_\delta(\theta, k).$$

Then there exist $q \in \omega \setminus \max\{n, p\}$, a formula $\rho \in \Phi^q_\delta$ and functions $j' \in \mathcal{I}_{n,q}$ and $k' \in \mathcal{I}_{p,q}$ such that $X_q = \mathrm{range}(j') \cup \mathrm{range}(k')$, $j' \circ j = k' \circ k$, $\psi = H^q_\delta(\rho, j')$ and $\theta = H^q_\delta(\rho, k')$.

Proof. Fixing m and p, we prove the proposition by induction on n. If $n = m$, then we can let $q = p$, $\rho = \theta$, $k' = i_p$ and $j' = k \circ j^{-1}$. Suppose that the proposition holds for n. Let $j \in \mathcal{I}_{m,n+1}$, $k \in \mathcal{I}_{m,p}$, $\psi \in \Phi^{n+1}_\delta$ and $\theta \in \Phi^p_\delta$ be such that $H^{n+1}_\delta(\psi, j) = H^p_\delta(\theta, k)$. Let $f \in \mathcal{I}_{n+1,n+1}$ be the identity function if $x_n \notin \mathrm{range}(j)$; otherwise, fix n' such that $x_{n'} \notin \mathrm{range}(j)$ and let f map x_n and $x_{n'}$ to each other and fix the rest of X_{n+1}. Let $\psi_0 = H^{n+1}_\delta(\psi, f)$. Then $x_n \notin \mathrm{range}(f \circ j)$. Let $\psi_1 = H^{n+1}_\delta(\psi_0, i_n)$. By the second part of Remark 2.13,

$$\begin{array}{rcl} H^n_\delta(\psi_1, f \circ j) & = & H^n_\delta(H^{n+1}_\delta(\psi_0, i_n), f \circ j) \\ & = & H^{n+1}_\delta(\psi_0, i_n \circ (f \circ j)) \\ & = & H^{n+1}_\delta(\psi_0, f \circ j) \\ & = & H^{n+1}_\delta(H^{n+1}_\delta(\psi_0, f), j) \\ & = & H^{n+1}_\delta(\psi, j) \\ & = & H^p_\delta(\theta, k). \end{array}$$

Applying the induction hypothesis to $f \circ j$, k, ψ_1 and θ, we get $q_0 \in \omega \setminus \max\{n, p\}$, a formula $\rho_0 \in \Phi^{q_0}_\delta$ and functions $j_0 \in \mathcal{I}_{n,q_0}$ and $k' \in \mathcal{I}_{p,q_0}$ such that

$$X_{q_0} = \mathrm{range}(j_0) \cup \mathrm{range}(k'),$$

$j_0 \circ (f \circ j) = k' \circ k$, $\psi_1 = H^{q_0}_\delta(\rho_0, j_0)$ and $\theta = H^{q_0}_\delta(\rho_0, k')$.

Suppose first that there exists a $y \in X_{q_0} \setminus \mathrm{range}(j_0)$ such that

$$\psi_0 = H^{q_0}_\delta(\rho_0, j_0 \cup \{(x_n, y)\}).$$

Then q_0, ρ_0 and k' are as desired. If $f = i_{n+1}$, then we can let $j' = j_0 \cup \{(x_n, y)\}$ and we are done. Otherwise, let j' send $x_{n'}$ to y, x_n to $j_0(x_{n'})$ and every other member of X_n to the same place that j_0 does (i.e., let $j' = (j_0 \cup \{(x_n, y)\}) \circ f$). Then $j' \circ j = k' \circ k$, and

$$\begin{array}{rcl} \psi & = & H^{n+1}_\delta(\psi_0, f) \\ & = & H^{n+1}_\delta(H^q_\delta(\rho, j_0 \cup \{(x_n, y)\}), f) \\ & = & H^q_\delta(\rho, (j_0 \cup \{(x_n, y)\}) \circ f) \\ & = & H^q_\delta(\rho, j'), \end{array}$$

as desired.

Finally, suppose that there is no such $y \in X_{q_0} \setminus \mathrm{range}(j_0)$. Putting together the amalgamation property of Φ and the equation $\psi_1 = H^{n+1}_{\delta+1}(\psi_0, i_n)$, we get that there exist a formula $\rho \in \Phi^{q_0+1}_\delta$ such that $H^{q_0+1}_\delta(\rho, i_{q_0}) = \rho_0$ and a $y \in X_{q_0+1} \setminus \mathrm{range}(j_0)$ such that $H^{q_0+1}_\delta(\rho, j_0 \cup \{(x_n, y)\}) = \psi_0$. Then k', ρ, and $q = q_0 + 1$ are as desired. If $f = i_{n+1}$, then we can let $j' = j_0 \cup \{(x_n, y)\}$, and we are done. Otherwise, as above, let $j' = (j_0 \cup \{(x_n, y)\}) \circ f$. Then again $j' \circ j = k' \circ k$ and $\psi = H^q_\delta(\rho, j')$, as desired. □

6 Building countable models

In this section we show that any Scott process of successor length has a countable model if its last level is countable. This in turn implies that such a sequence can be extended to any given ordinal length (although the rank of the Scott process of length ω_1 corresponding to a countable model is countable).

6.1 Definition. Given an ordinal β, and a countable set $\Phi \subseteq \Psi_\beta$, a *thread* through Φ is a set of formulas $\{\phi_n : n \in \omega\} \subseteq \Phi$ such that

1. for all $n \in \omega$, $\phi_n \in \Psi^n_\beta$;
2. for all $m < n$ in ω, $\phi_m = H^n_\beta(\phi_n, i_m)$;
3. for all $m \in \omega$, all $\alpha < \beta$, and all $\psi \in E(V_{\alpha+1,\beta}(\phi_m))$, there exist an $n \in \omega \setminus (m+1)$ and a $y \in X_n \setminus X_m$ such that $\psi = V_{\alpha,\beta}(H^n_\beta(\phi_n, i_m \cup \{(x_m, y)\}))$.

6.2 Remark. If β is a successor ordinal, condition (3) of Definition 6.1 is equivalent to the restriction of the condition to the case where $\alpha = \beta - 1$. This follows from Proposition 4.3. Similarly, condition (3) of Definition 6.1 is equivalent to the restriction of the condition to the set of α in any cofinal subset of β.

6.3 Remark. Suppose that $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process, and $\beta < \delta$ is such that $V_{\beta,\delta} \restriction \Phi_\delta$ is injective. Then the $V_{\beta+1,\delta}$-preimage of a thread through $\Phi_{\beta+1}$ is a thread through Φ_δ. This follows from Remark 2.8, Proposition 2.14 and Proposition 4.3.

Proposition 6.4. *If $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process with Φ_δ countable, then there exists a thread through Φ_δ.*

Proof. By Remark 6.3, if suffices to consider the cases where δ is either a successor ordinal or an ordinal of cofinality ω. Let A be $\{\delta - 1\}$ in the case where δ is a successor ordinal, and a countable cofinal subset of δ otherwise. We choose the formulas ϕ_n recursively, meeting instances of condition (3) of Definition 6.1 for $\alpha \in A$ while satisfying condition (2). Note that ϕ_0 is the unique element of Φ^0_δ. To satisfy an instance of condition (3), we need to see that if $m \leq n$ are in ω, $\alpha \in A$, ϕ_n has been chosen, and $\psi \in E(V_{\alpha+1,\beta}(\phi_m))$ is not equal to $V_{\alpha,\beta}(H^n_\beta(\phi_n, i_m \cup \{(x_m, y)\}))$ for any $y \in X_n \setminus X_m$, then ϕ_{n+1} can be chosen so that

$$\psi = V_{\alpha,\beta}(H^{n+1}_\beta(\phi_{n+1}, i_m \cup \{(x_m, x_n)\}))$$

(since Φ_δ is countable, the set of such formulas ψ is also countable). The existence of such a ϕ_{n+1} follows from condition (3) of Definition 2.10 applied to $V_{\alpha+1,\beta}(\phi_n)$ and i_m, giving a $\theta \in E(V_{\alpha+1,\beta}(\phi_n))$ such that

$$H^{n+1}_\alpha(\theta, i_m \cup \{(x_m, x_n)\}) = \psi,$$

followed by condition (2b) of Definition 3.1 applied to ϕ_n, giving ϕ_{n+1} as desired. □

Theorem 6.5. *Given a Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ with Φ_δ countable, a thread $\langle \phi_n : n \in \omega \rangle$ through Φ_δ and a set $C = \{c_n : n \in \omega\}$, there is a τ-structure with domain C in which each tuple $\langle c_m : m < n \rangle$ satisfies ϕ_n.*

Proof. Let each tuple $\langle c_m : m < n \rangle$ satisfy all the atomic formulas indicated by $V_{0,\delta}(\phi_n)$. We show by induction on α that each tuple $\langle c_m : m < n \rangle$ satisfies the formula $V_{\alpha,\delta}(\phi_n)$. This follows immediately for limit stages. For the induction step from α to $\alpha + 1$, $\langle c_m : m < n \rangle$ satisfies $V_{\alpha+1,\delta}(\phi_n)$ if and only if

$$E(V_{\alpha+1,\delta}(\phi_n)) = V_{\alpha,\delta}[\{H^p_\delta(\phi_p, i_n \cup \{(x_n, y)\}) : p \in \omega \setminus (n+1), y \in X_p \setminus X_n\}].$$

That is, checking that $\langle c_m : m < n \rangle$ satisfies $V_{\alpha+1,\delta}(\phi_n)$ means showing that the left side of the equality is the set of formulas from Φ^{n+1}_α satisfied by extensions of $\langle c_m : m < n \rangle$ by one point, which by the induction hypothesis is what the right side is. The left-to-right containment follows from condition (3) of Definition 6.1. For the other direction, note first that by Proposition 4.4,

$$E(V_{\alpha+1,\delta}(\phi_n)) = V_{\alpha,\delta}[\{\theta \in \Phi^{n+1}_\delta \mid H^{n+1}_\delta(\theta, i_n) = \phi_n\}].$$

That

$$\{H^p_\delta(\phi_p, i_n \cup \{(x_n, y)\}) : p \in \omega \setminus (n+1), y \in X_p \setminus X_n\}$$

is contained in $\{\theta \in \Phi^{n+1}_\delta \mid H^{n+1}_\delta(\theta, i_n) = \phi_n\}$ follows from the assumption that $\phi_n = H^p_\delta(\phi_p, i_n)$. □

6.6 Definition. Given an ordinal β, and a countable set $\Phi \subseteq \Psi_\beta$, a thread $\{\phi_n : n \in \omega\}$ through Φ is *complete* if for all $m \in \omega$ and all $\psi \in \Phi \cap \Psi^m_\beta$, there exist $n \in \omega$ and $j \in \mathcal{I}_{m,n}$ such that $\psi = H^n_\beta(\phi_n, j)$.

6.7 Remark. The thread through Φ_δ given by Proposition 6.4 induces a model (via Theorem 6.5) whose Scott process contains $\langle \Phi_\alpha : \alpha < \delta \rangle$, and for which the δ-th level of the corresponding Scott process is contained in the given Φ_δ. The δ-th level is equal to Φ_δ if and only if the thread is complete. Condition (2c) of Definition 3.1 implies that one can add stages to the construction in Proposition 6.4 to produce a complete thread.

6.8 Definition. A τ-structure M is a *model* of a Scott process $\langle \Phi_\alpha : \alpha < \delta \rangle$ if $\Phi_\alpha = \Phi_\alpha(M)$ for all $\alpha < \delta$.

Proposition 6.4, Theorem 6.5 and Remark 6.7 give the following.

Theorem 6.9. *Every Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ with Φ_δ countable has a countable model.*

6.10 Remark. Theorem 9.8 gives a stronger version of Theorem 6.9, showing that every Scott process with all levels countable (and possibly of limit length) has a model.

6.11 Remark. If $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process, $\gamma < \delta$ and $\{\phi_n : n \in \omega\}$ is a thread through Φ_δ, then $\{V_{\gamma,\delta}(\phi_n) : n \in \omega\}$ is a thread through Φ_γ (this follows from Proposition 2.14). This thread induces (as in the proof of Theorem 6.5) the same class-length Scott process as $\{\phi_n : n \in \omega\}$.

We insert here an argument for constructing a pair of models. The issue of extending this theorem to uncountable models is discussed in Remark 7.13.

Theorem 6.12. *Let γ be a countable ordinal, and suppose that $\langle \Phi_\beta : \beta \leq \gamma \rangle$ is a Scott process with Φ_γ countable. Let Φ^* be a subset of Φ_γ such that the extension of $\langle \Phi_\beta : \beta \leq \gamma \rangle$ by Φ^* is also a Scott process. Then there exists τ-structures M and N such that M is a substructure of N, N is a model of $\langle \Phi_\beta : \beta \leq \gamma \rangle$ and M is a model of the extension of $\langle \Phi_\beta :< \gamma \rangle$ by Φ^*.*

Proof. By Theorem 6.5, it suffices to find a complete thread $\langle \phi_n : n \in \omega \rangle$ through Φ_γ and a infinite set $Y \subseteq \omega$ such that, letting (for each $n \in \omega$)

- j_n be the order preserving map from X_n to the first n elements of the set $\{x_m : m \in Y\}$,
- k_n be the least element of ω such that $|Y \cap k_n| = n$,

$\langle H^n_\gamma(\phi_{k_n}, j_n) : n \in \omega \rangle$ is a complete thread through Φ^*.

A construction of such a pair $\langle \phi_n : n \in \omega \rangle$, Y can be carried out in essentially the same manner as the proof of Theorem 6.4 (as above, completeness can be achieved using Condition (2c) of Definition 3.1). The construction of $\langle \phi_n : n \in \omega \rangle$ is exactly the same, except with stages inserted to ensure that the formulas $H^n_\gamma(\phi_{k_n}, j_n)$ are as desired. That is, if $k_m \leq n$ (as determined so far), $\alpha \in A$ (as in the proof of Theorem 6.4), ϕ_n has been chosen, and $\psi \in E(V_{\alpha+1,\beta}(H^n_\gamma(\phi_{k_m}, j_m))$ is not equal to $V_{\alpha,\beta}(H^n_\beta(\phi_n, j_m \cup \{(x_m, y)\}))$ for any $x_p \in X_n \setminus X_m$ with p already chosen to be in Y, then ϕ_{n+1} can be chosen so that

$$\psi = V_{\alpha,\beta}(H^{n+1}_\beta(\phi_{n+1}, j_m \cup \{(x_m, x_n)\})).$$

As in the proof of Theorem 6.4, the existence of such a ϕ_{n+1} follows from condition (3) of Definition 2.10 applied to $V_{\alpha+1,\beta}(\phi_n)$ and j_m, giving a $\theta \in E(V_{\alpha+1,\beta}(\phi_n))$ such that

$$H^{n+1}_\alpha(\theta, j_m \cup \{(x_m, x_n)\}) = \psi,$$

followed by condition (2b) of Definition 3.1 applied to ϕ_n, giving ϕ_{n+1} as desired. We then put n into Y and continue the construction. $\square$

7 Models of cardinality $\aleph_1$

In this section we show how to build models for Scott processes of length a successor ordinal, under the assumption that the last level of the process amalgamates and has cardinality at most $\aleph_1$.

Given two finite sets of ordinals $a \subseteq b$ with $a = \{\alpha_0, \ldots, \alpha_{n-1}\}$ (listed in increasing order), let $j_{a,b}$ be the function j in $\mathcal{I}_{n,|b|}$ such that $j(x_m) = x_{|b \cap \alpha_m|}$ for all $m < n$.

In the case $\kappa = \omega$, the following definition is essentially the same as Definition 6.1, as the formulas $\{\phi_n : n \in \omega\}$ of the weaving then satisfy Definition 6.1.

7.1 Definition. Suppose that δ is an ordinal, κ is an infinite cardinal and Φ is a subset of Ψ_δ of cardinality κ. A *weaving* through Φ is a set of formulas $\{\phi_a : a \in [\kappa]^{<\omega}\} \subseteq \Phi$ such that the following hold.

1. each $\phi_a \in \Psi_\delta^{|a|}$.
2. For all $a \subseteq b \in [\kappa]^{<\omega}$, $\phi_a = H_\delta^{|b|}(\phi_b, j_{a,b})$.
3. for all $a \in [\kappa]^{<\omega}$, all $\alpha < \delta$, and all $\psi \in E(V_{\alpha+1,\delta}(\phi_a))$, there exist a $b \in [\kappa]^{|a|+1}$ containing a and a $y \in X_{|b|} \setminus \mathrm{range}(j_{a,b})$ such that
$$\psi = V_{\alpha,\delta}(H_\delta^{|a|+1}(\phi_b, j_{a,b} \cup \{(x_{|a|}, y)\})).$$

The proof of Theorem 7.2 is an adaptation of the proof of Theorem 6.5.

Theorem 7.2. *Given a Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ with δ and Φ_δ of cardinality κ, a weaving $\langle \phi_a : a \in [\kappa]^{<\omega} \rangle$ through Φ_δ and a set $C = \{c_\alpha : \alpha < \kappa\}$, there is a τ-structure with domain C in which, for each $a \in [\kappa]^{<\omega}$, the tuple $\langle c_\alpha : \alpha \in a \rangle$ satisfies ϕ_a.*

Proof. For each $a \in [\kappa]^{<\omega}$, let the tuple $\langle c_\alpha : \alpha \in a \rangle$ satisfy all the atomic formulas indicated by $V_{0,\delta}(\phi_a)$. We show by induction on $\beta < \delta$ that each tuple $\langle c_\alpha : \alpha \in a \rangle$ satisfies the formula $V_{\beta,\delta}(\phi_a)$. This follows immediately for limit stages. For the induction step from β to $\beta + 1$, $\langle c_\alpha : \alpha \in a \rangle$ satisfies $V_{\beta+1,\delta}(\phi_a)$ if and only if $E(V_{\beta+1,\delta}(\phi_a))$ is equal to
$$V_{\beta,\delta}[\{H_\delta^{|b|}(\phi_b, j_{a,b} \cup \{(x_{|a|}, y)\}) : a \subseteq b \in [\kappa]^{<\omega}, y \in X_{|b|} \setminus \mathrm{range}(j_{a,b})\}].$$
The left-to-right containment follows from condition (3) of Definition 7.1. For the other direction, note first that by Proposition 4.4,
$$E(V_{\beta+1,\delta}(\phi_a)) = V_{\beta,\delta}[\{\theta \in \Phi_\delta^{|a|+1} \mid H_\delta^{|a|+1}(\theta, i_{|a|}) = \phi_a\}].$$
That
$$\{H_\delta^{|b|}(\phi_b, j_{a,b} \cup \{(x_{|a|}, y)\}) : a \subseteq b \in [\kappa]^{<\omega}, y \in X_{|b|} \setminus \mathrm{range}(j_{a,b})\}$$
is contained in $\{\theta \in \Phi_\delta^{|a|+1} \mid H_\delta^{|a|+1}(\theta, i_{|a|}) = \phi_a\}$ follows from the assumption that $\phi_a = H_\delta^{|b|}(\phi_b, j_{a,b})$. $\square$

7.3 Definition. Suppose that δ is an ordinal, κ is an infinite cardinal and Φ is a subset of Ψ_δ of cardinality κ. A weaving $\{\phi_a : a \in [\kappa]^{<\omega}\} \subseteq \Phi$ through Φ is *complete* if for all $n \in \omega$ and all $\psi \in \Phi \cap \Psi^n_\delta$, there exist $a \in [\kappa]^n$ and $j \in \mathcal{I}_{n,n}$ such that $\psi = H^n_\delta(\phi_a, j)$;

7.4 Remark. As in Remark 6.7, given a Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ and a weaving through Φ_δ, the proof of Theorem 7.2 gives a model whose Scott process has $\langle \Phi_\alpha : \alpha < \delta \rangle$ as an initial segment, and for which the δ-th level of its Scott process is contained in the given Φ_δ. The δ-th level is equal to Φ_δ if and only if the weaving is complete.

It remains to find a complete weaving through a Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$, assuming that Φ_δ amalgamates and has cardinality $\aleph_1$.

7.5 Definition. Suppose that δ is an ordinal, κ is an infinite cardinal and Φ is a subset of Ψ_δ of cardinality κ. A *strong weaving* through Φ is a set

$$\{\phi_a : a \in [\kappa]^{<\omega}\} \subseteq \Phi$$

satisfying conditions (1) and (2) of Definition 7.1 plus the following condition: for all $a \in [\kappa]^{<\omega}$, and all $\psi \in \Phi \cap \Psi^{|a|+1}_\delta$ such that $H^{|a|+1}_\delta(\psi, i_{|a|}) = \phi_a$, there exist a $b \in [\kappa]^{|a|+1}$ containing a and a $y \in X_{|b|} \setminus \mathrm{range}(j_{a,b})$ such that

$$\psi = H^{|a|+1}_\delta(\phi_b, j_{a,b} \cup \{(x_{|a|}, y)\})).$$

7.6 Remark. In condition (3) of Definition 7.1 and in Definition 7.5, the variable y is in fact the unique member of $X_{|b|} \setminus \mathrm{range}(j_{a,b})$.

Proposition 7.7. *Suppose that δ is an ordinal, κ is an infinite cardinal and Φ is a subset of Ψ_δ of cardinality κ. A strong weaving through Φ is both a weaving and complete.*

Proof. That a strong weaving satisfies condition (3) of Definition 7.1 follows from condition (2b) of Definition 3.1. Completeness for formulas in $\Phi \cap \Psi^n_\delta$ follows by induction on n. □

A subset S of a collection C of sets is $\subseteq$-*cofinal* in C if every member of C is contained in a member of S.

Proposition 7.8. *Suppose that $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process such that Φ_δ amalgamates. Let $\mathcal{W} = \{\phi_a : a \in [\kappa]^{<\omega}\}$ be a subset of Φ satisfying conditions (1) and (2) of Definition 7.1, such that the set of $a \in [\kappa]^{<\omega}$ for which the condition in Definition 7.5 is satisfied is $\subseteq$-cofinal in $[\kappa]^{<\omega}$. Then $\mathcal{W}$ is a strong weaving.*

Proof. Suppose that we have $a \subseteq b \in [\kappa]^{<\omega}$, and that the condition in Definition 7.5 holds for b. Suppose that $\psi \in \Phi^{|a|+1}_\delta$ is such that $H^{|a|+1}_\delta(\psi, i_{|a|}) = \phi_a$. By Proposition 5.19, there is formula $\theta \in \Phi^{|b|+1}$ such that $H^{|b|+1}_\delta(\theta, i_{|b|}) = \phi_b$

and $H_\delta^{|b|+1}(\theta, j_{a,b} \cup \{(x_{|a|}, x_{|b|})\}) = \psi$. Then there exist a $\beta \in \kappa \setminus b$ a $y \in X_{|b|+1} \setminus \mathrm{range}(j_{b,b\cup\{\beta\}})$ such that

$$\theta = H_\delta^{|b|+1}(\phi_{b\cup\{\beta\}}, j_{b,b\cup\{\beta\}} \cup \{(x_{|b|}, y)\}),$$

which implies that

$$\psi = H_\delta^{|a|+1}(\phi_{a\cup\{\beta\}}, j_{a,a\cup\{\beta\}} \cup \{(x_{|a|}, y)\}),$$

for y the unique element of $X_{|a|+1} \setminus \mathrm{range}(j_{a,a\cup\{\beta\}})$. □

Proposition 7.9. *If $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process such that Φ_δ amalgamates and has cardinality $\aleph_1$, then there is a strong weaving through Φ_δ.*

Proof. We recursively pick suitable formulas ϕ_a, for $a \in [\omega_1]^{<\omega}$. To begin with, let ϕ_n ($n \in \omega$) be any elements of Φ_δ with the property that $H_\delta^n(\phi_n, i_m) = \phi_m$, for all $m \leq n < \omega$. Suppose now that we have $\alpha < \omega_1$ and that ϕ_a has been chosen for each finite subset of α (note that a choice of ϕ_a determines a choice of ϕ_b for each subset of b, where a is a finite subset of ω_1). Following some bookkeeping, we fix the least pair a, ψ as in Definition 7.5 for which the corresponding condition has not been met, and let $\phi_{a\cup\{\alpha\}}$ be this ψ. Fixing a bijection $\pi\colon \omega \to (\alpha \setminus a)$, we now successively choose the formulas $\phi_{a\cup\{\alpha\}\cup\pi[n]}$. For each positive n, the choice of $\phi_{a\cup\{\alpha\}\cup\pi[n]}$ requires amalgamating $\phi_{a\cup\{\alpha\}\cup\pi[n-1]}$ with $\phi_{a\cup\pi[n]}$, which have already been chosen. The fact that Φ_δ amalgamates (via Proposition 5.19) implies that there exists a suitable choice of $\phi_{a\cup\{\alpha\}\cup\pi[n]}$. Since $\phi_{a\cup\{\pi(n-1)\}}$ did not satisfy third condition of Definition 7.5 with respect to a and ψ, this choice of $\phi_{a\cup\{\alpha\}\cup\pi[n]}$ does not require identifying $\pi(n-1)$ and α. Proceeding in this fashion completes the construction of the desired strong weaving. □

Putting together Theorem 7.2 with Propositions 7.7 and 7.9, we have the following.

Theorem 7.10. *If $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process, Φ_δ amalgamates and $|\Phi_\delta| \leq \aleph_1$, then $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ has a model.*

7.11 Remark. One difference between the construction in this section and the construction in Section 6 is that in Proposition 6.4 there were (in some cases) many options for the thread produced, as every τ-structure whose Scott process extends $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is given by a thread through Φ_δ. In this section, with uncountably many tasks to complete, we need to rely on the fact that Φ_δ amalgamates, which means that we construct models of Scott rank δ only. The assumption that Φ_δ amalgamates also enables the simplification given by Proposition 7.7.

7.12 Remark. One might naturally try to adapt the proof of Theorem 7.2 to build a model of size $\aleph_2$ by assigning a formula from Φ_δ to each finite tuple from ω_2. Doing this in the manner of the proof of Theorem 7.2, one finds oneself with an uncountable $\alpha < \omega_2$ such that formulas have been assigned for all finite subsets of α, but not for $\{\alpha\}$. Choosing formulas for all finite subsets of $\alpha + 1$, one comes to a point where, for some countably infinite $B \subseteq \alpha$, formulas have been chosen for all sets of the form $\{\alpha\} \cup b$, for b a finite subset of B. Then, for some $\beta \in \alpha \setminus B$, one would like to chose a formula for some finite superset c of $\{\alpha, \beta\}$ intersecting B. Finally, consider $\gamma \in B \setminus c$. We have at this point that formulas have been chosen for $\{\alpha, \gamma\}$, $\{\beta, \gamma\}$ and c, but not for $\{\alpha, \beta, \gamma\}$, and our assumptions do not give us suitable choice for $\{\alpha, \beta, \gamma\}$ that extends the choices already made. One can naturally define a notion of 3-amalgamation such that this construction could succeed under the assumption that this property holds.

7.13 Remark. The natural attempt to combine the proofs of Theorem 6.12 and Proposition 7.9 to produce a version of Theorem 6.12 for models of size $\aleph_1$ runs into a problem similar to the one in Remark 7.12. In this case, we have a Scott process $\langle \Phi_\alpha : \alpha \leq \beta \rangle$, for some $\beta \in [\omega_1, \omega_2)$ such that, letting Φ^* be the set of isolated threads in Φ_β,

- Φ^* is a proper subset of Φ_β,
- the extension of $\langle \Phi_\alpha : \alpha < \beta \rangle$ by Φ^* gives a Scott process.

We could then try to build a strong weaving $\{\phi_a : a \in [\omega_1]^{<\omega}\}$ through Φ_β, and an an uncountable set $Y \subseteq \omega_1$ such that $\{\phi_a : a \in [Y]^{<\omega}\}$ is a strong weaving through Φ^* (or, more precisely, induces one via some bijection between Y and ω_1). Carrying out this construction, we come to a point where, for some infinite $\gamma < \omega_1$, ϕ_a has been chosen for every finite subset of γ, and for $\{\gamma\} \cup a$, for some finite $a \subseteq \gamma$ intersecting Y as so far constructed, but not contained in it. At some stages it will also be that this γ has been put into Y. Now suppose that δ is in $Y \cap \gamma$, as constructed so far, but that no formula for $\{\delta, \gamma\}$ has been chosen. Then we need to choose a formula for $a \cup \{\delta, \gamma\}$ such that the induced formula for $(a \cap Y) \cup \{\delta, \gamma\}$ is in Φ^*. Since Φ_δ amalgamates, we can choose a formula for $a \cup \{\delta, \gamma\}$, but we can't guarantee that the induced formula for $(a \cap Y) \cup \{\delta, \gamma\}$ will be in Φ^*. Similarly, since Φ^* amalgamates we can choose a formula for $(a \cap Y) \cup \{\delta, \gamma\}$ in Φ^*. Then we have the same 3-amalgamation issue as in Remark 7.12, as we would then need to amalgamate the chosen formulas for $(a \cap Y) \cup \{\delta, \gamma\}$, $a \cup \{\delta\}$ and $a \cup \{\gamma\}$ in Φ_β.

8 Finite existential blocks

The function E defined in Definition 2.4 corresponds to a single existential quantifier. In this section we extend E to the function F which corresponds to finite blocks of existential quantifiers. The analysis of F in this section is used in the following section. Most of this section consists of consequences of Proposition 8.4.

8.1 Definition. For each ordinal β, each $m \in \omega$ and each $\phi \in \Psi^m_\beta$, $F(\phi)$ is the set of ψ such that for some $n \in \omega$ and some ordinal α with $\alpha + n \leq \beta$, $\psi \in \Psi^{m+n}_\alpha$ and there exist $\psi_0, \ldots, \psi_n$ such that

- $\psi_0 = \psi$;
- for all $p \in \{0, \ldots, n-1\}$, $\psi_p \in E(\psi_{p+1})$;
- $\psi_n = V_{\alpha+n,\beta}(\phi)$.

8.2 Remark. Suppose that $\beta < \delta$, $\phi \in \Phi_\beta$ and $\psi_0, \ldots, \psi_n$ are as in Definition 8.1. Then by condition (1b) of Definition 3.1, each ψ_i is in $\Phi^{m+n-i}_{\alpha+i}$.

8.3 Remark. Given α, β, ϕ and ψ as in Definition 8.1, the issue of whether or not ψ is in $F(\phi)$ depends only on $V_{\alpha+n,\beta}(\phi)$ (as opposed to ϕ). It follows that $\psi \in F(\theta)$ for any formula $\theta \in \Psi^m_\gamma$ (for some ordinal $\gamma \geq \alpha + n$) such that $V_{\alpha+n,\gamma}(\theta) = V_{\alpha+n,\beta}(\phi)$.

Fix for rest of this section a Scott process $\langle \Phi_\alpha : \alpha < \delta \rangle$.

Proposition 8.4. *Suppose that $m, n \in \omega$ and $\alpha, \beta < \delta$ are such that $\alpha + n \leq \beta$. Let ϕ and ψ be elements of Φ^m_β and Φ^{m+n}_α, respectively. Then $\psi \in F(\phi)$ if and only if there is a formula $\theta \in \Phi^{m+n}_\beta$ such that $H^{m+n}_\beta(\theta, i_m) = \phi$ and $V_{\alpha,\beta}(\theta) = \psi$.*

Proof. By induction on n. In the case $n = 1$, $\psi \in F(\phi)$ if and only if $\psi \in E(V_{\alpha+1,\beta}(\phi))$. In this case, the proposition is Proposition 4.4. The induction step from $n = p$ to $n = p + 1$ follows from the induction hypothesis in the cases $n = p$ and $n = 1$. $\square$

8.5 Remark. Applying Proposition 8.4 and condition (1e) of Definition 3.1, we get that if $m, n, p \in \omega$ and $\alpha, \beta < \delta$ are such that $\alpha + n + p \leq \beta$, and if $\phi \in \Phi^m_\beta$, then for each $\psi \in \Phi^{m+n}_\alpha \cap F(\phi)$ there exists a $\rho \in \Phi^{m+n+p}_\alpha \cap F(\phi)$ such that $H^{m+n+p}_\alpha(\rho, i_{m+n}) = \psi$.

8.6 Remark. Fix $m, n \in \omega$ and suppose $\alpha, \beta < \delta$ are such that $\alpha + n \leq \beta$. Let ϕ be an element of Φ^{m+n}_β, let f be an element of $\mathcal{I}_{m,m+n}$ and let g be any element of $\mathcal{I}_{m+n,m+n}$ extending f. Then, by Proposition 8.4 and part (2) of Remark 2.13, $V_{\alpha,\beta}(H^{m+n}_\beta(\phi, g))$ is in $F(H^{m+n}_\beta(\phi, f))$.

Proposition 8.7 follows from Proposition 8.4 and Remark 2.7.

Proposition 8.7. *Fix $\phi \in \Phi^m_\beta$, for some $\beta < \delta$ and $m \in \omega$. Let $\psi \in \Phi^{m+n}_\alpha$ be an element of $F(\phi)$, for some $n \in \omega$ and some ordinal α with $\alpha + n \leq \beta$. Then for all $\gamma < \alpha$, $V_{\gamma,\alpha}(\psi) \in F(\phi)$.*

Propositions 2.14 and 8.4 imply that members of $F(\phi)$ project horizontally to vertical projections of ϕ.

Proposition 8.8. *Suppose that $\alpha < \beta < \delta$, $m \leq n \in \omega$, $\phi \in \Phi^m_\beta$ and $\psi \in \Phi^n_\alpha \cap F(\phi)$. Then $H^n_\alpha(\psi, i_m) = V_{\alpha,\beta}(\phi)$.*

Proposition 8.9 is used in the proof of Theorem 9.8.

Proposition 8.9. *Suppose that $m, n \in \omega$, $\alpha < \beta$ are such that $\beta + n < \delta$, $\phi \in \Phi^m_{\beta+n}$ and $\psi \in \Phi^{m+n}_\alpha \cap F(\phi)$. Then there exists a $\psi' \in V^{-1}_{\alpha,\beta}[\{\psi\}] \cap F(\phi)$.*

Proof. By Proposition 8.4, there is a $\theta \in \Phi^{m+n}_{\beta+n}$ such that $V_{\alpha,\beta+n}(\theta) = \psi$ and $H^{m+n}_{\beta+n}(\theta, i_m) = \phi$. By Proposition 8.4 again, $V_{\beta,\beta+n}(\theta) \in F(\phi)$. $\square$

A consequence of the following proposition is that every formula in a limit level of a Scott process determines the entire process below that level (note that $F(\phi)$ depends only on ϕ). This fact is used in Remark 9.7.

Proposition 8.10. *Suppose that $\langle \Phi_\alpha : \alpha < \beta \rangle$ is a Scott process. Fix $\alpha_0 < \beta$, $n \in \omega$ and $\phi \in \Phi^n_{\alpha_0}$. Then for each $\alpha < \beta$ and $m \in \omega$ such that $\alpha + m \leq \alpha_0$, the set Φ^m_α is equal to $\{H^n_\alpha(\psi, f) : \psi \in F(\phi) \cap \Phi^{n+m}_\alpha, f \in \mathcal{I}_{m,n+m}\}$.*

Proof. Let ψ be a member of Φ^m_α. By condition (1c) of Definition 3.1, there is a $\psi' \in \Phi^m_{\alpha_0}$ such that $V_{\alpha,\alpha_0}(\psi') = \psi$. By condition (2c) of Definition 3.1, there exist $f \in \mathcal{I}_{m,n+m}$ and $\theta \in \Phi^{n+m}_{\alpha_0}$ such that $H^{n+m}_{\alpha_0}(\theta, i_n) = \phi$ and $H^{n+m}_{\alpha_0}(\theta, f) = \psi'$. Then $V_{\alpha,\alpha_0}(\theta)$ is in $F(\phi)$ by Proposition 8.4, and is as desired by Proposition 2.14. $\square$

The following proposition shows that members of $F(\phi)$ can be combined, in suitable situations.

Proposition 8.11. *For all $m, n, p \in \omega$, all $\alpha, \beta < \delta$ such that $\beta \geq \alpha+n+p$, and all $\phi \in \Phi^m_\beta$, $\psi \in \Phi^{m+n}_\alpha \cap F(\phi)$ and $\theta \in \Phi^{m+p}_\alpha \cap F(\phi)$, there exist $j \in \mathcal{I}_{m+p,m+n+p}$ and $\rho \in \Phi^{m+n+p}_\alpha \cap F(\phi)$ such that*

- *$j \circ i_m = i_m$;*
- *$H_\alpha(\rho, i_{m+n}) = \psi$;*
- *$H_\alpha(\rho, j) = \theta$.*

Proof. This can be proved by induction on p, for all m and n simultaneously. In the case where $p = 0$ there is nothing to show, so suppose that p is positive. Since $\theta \in F(\phi)$, there is a $\theta' \in \Phi^{m+p-1}_{\alpha+1}$ such that $\theta \in E(\theta')$. By Proposition 8.11, there is a $\psi' \in \Phi^{m+n}_{\alpha+1}$ such that $V_{\alpha,\alpha+1}(\psi') = \psi$ and $H^{m+n}_{\alpha+1}(\psi', i_m) = V_{\alpha+1,\beta}(\phi)$. Let $\rho' \in \Phi^{m+p+p-1}_{\alpha+1}$ be the result of applying the induction hypothesis to ψ' and θ'. Since $\theta \in E(\theta')$, the desired ρ can be found in $E(\rho')$ by applying condition (3) of Definition 2.10. $\square$

The following proposition is not used in this paper. It does, however, illustrate a ways in which the function F acts as expected. The proposition follows immediately from Propositions 2.14 and 8.4.

Proposition 8.12. *Suppose that $\alpha < \beta < \delta$, $m, n \in \omega$, $\phi \in \Phi^m_\beta$ and $\psi \in \Phi^{m+n}_\alpha$ are such that $\alpha + n \leq \beta$ and $\psi \in F(\phi)$. Fix $p \in [m, m+n]$ and let $j \in \mathcal{I}_{p,m+n}$ be such that $j \restriction X_m = i_m$. Then $H^{m+n}_\alpha(\psi, j) \in F(\phi)$.*

9 Extending a process of limit length

9.1 Definition. Given a limit ordinal β and a sequence $\langle \Phi_\alpha : \alpha < \beta \rangle$ such that each Φ_α is a subset of Ψ_α, a *path through* $\langle \Phi_\alpha : \alpha < \beta \rangle$ is a formula ϕ in Ψ_β such that $V_{\alpha,\beta}(\phi) \in \Phi_\alpha$ for each $\alpha < \beta$.

9.2 Remark. For each limit ordinal α, Ψ_α is the set of paths through the sequence $\langle \Psi_\beta : \beta < \alpha \rangle$.

9.3 Definition. Let β be a limit ordinal β and let $\langle \Phi_\alpha : \alpha < \beta \rangle$ be such that each Φ_α is a subset of Ψ_α. Let ϕ be a path through $\langle \Phi_\alpha : \alpha < \beta \rangle$, and let $n \in \omega$ be such that $\phi \in \Psi^n_\beta$. The *minimal set* of ϕ according to $\langle \Phi_\alpha : \alpha < \beta \rangle$ is the set of paths υ through $\langle \Phi_\alpha : \alpha < \beta \rangle$ for which there exist

- $m \in \omega \setminus n$;
- $p \in m + 1$;
- $\alpha_0 < \beta$;
- $\psi_0 \in \Phi^m_{\alpha_0} \cap F(\phi)$;
- $f \in \mathcal{I}_{p,m}$;

such that for all $\alpha \in [\alpha_0, \beta)$ and all $\psi \in \Phi^m_\alpha \cap F(\phi)$ such that $V_{\alpha_0,\alpha}(\psi) = \psi_0$, $H^m_\alpha(\psi, f) = V_{\alpha,\beta}(\upsilon)$.

9.4 Remark. The conclusion of the Definition 9.3 can equivalently be replaced by "such that for all $\psi \in \Psi_\beta$ such that $V_{\alpha_0,\beta}(\psi) = \psi_0$ and $H^m_\beta(\psi, i_n) = \phi$, $H^m_\beta(\psi, f) = \upsilon$."

9.5 Remark. Let $p \leq n$ be elements of ω, let f be an element of $\mathcal{I}_{p,n}$, let β be a limit ordinal, and let ϕ and ϕ be paths through a Scott process $\langle \Phi_\alpha : \alpha < \beta \rangle$, with ψ in the minimal set of ϕ (with respect to this Scott process) and $\psi \in \Psi^n_\beta$. Then $H^n_\beta(\psi, f)$ is an element of the minimal set of ϕ.

9.6 Remark. Let the *weakly minimal set* of a formula ϕ (in the context of Definition 9.3) be the set of formulas $\upsilon \in \Psi^p_\beta$ for which membership in the minimal set of ϕ is witnessed with $f = i_p$. One obtains an equivalent definition of the minimal set of ϕ by taking the closure of the weakly minimal set under permutations of free variables (i.e., including all formulas of the form $H^p_\beta(\upsilon, f)$, where $\upsilon \in \Psi^p_\beta$ is in the weakly minimal set of ϕ and f is in $\mathcal{I}_{p,p}$. This follows from the second part of Remark 2.2, and condition (1d) of Definition 3.1.

9.7 Remark. Suppose that β is a limit ordinal, $\langle \Phi_\alpha : \alpha \leq \beta \rangle$ is a Scott process and ρ is an element of Φ_β. Then every member of the minimal set of ρ according to $\langle \Phi_\alpha : \alpha < \beta \rangle$ is a member of Φ_β. This follows from Proposition 8.4. Furthermore, by Proposition 8.10, ρ determines $\langle \Phi_\alpha : \alpha < \beta \rangle$, so the expression "according to $\langle \Phi_\alpha : \alpha < \beta \rangle$" is (in this case, where ϕ is part of a Scott process) unnecessary.

We write $\mathrm{ms}(\phi)$ for the minimal set of ϕ.

Theorem 9.8. *Suppose that δ is a limit ordinal of countable cofinality and $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process such that each Φ_α is countable. Let ρ be a path through $\langle \Phi_\alpha : \alpha < \delta \rangle$. Then there exists a countable $\Phi_\delta \subseteq \Psi_\delta$ such that $\rho \in \Phi_\delta$ and $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process.*

Furthermore, if Υ is a countable subset of Ψ_δ disjoint from $\mathrm{ms}(\rho)$, Φ_δ can be chosen to be disjoint from Υ.

Proof. In order to make $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ a Scott process, we need to pick Φ_δ so that conditions (1c), (1d), (1e), (2b) and (2c) of Definition 3.1 are satisfied. Let $\langle \gamma^0_p : p < \omega \rangle$ be an increasing sequence cofinal in δ. We will recursively pick formulas θ_p $(p < \omega)$, a nondecreasing sequence of ordinals γ_p $(p < \omega)$ below δ and a nondecreasing unbounded sequence of integers n_p $(p < \omega)$ such that $\rho \in \Psi^{n_0}_\delta$ and such that, for each $p \in \omega$,

- $\gamma_p \geq \gamma^0_p$;
- $\theta_p \in \Phi^{n_p}_{\gamma_p} \cap F(\rho)$;
- $H^{n_{p+1}}_{\gamma_p}(V_{\gamma_p,\gamma_{p+1}}(\theta_{p+1}), i_{n_p}) = \theta_p$.

The first of these conditions implies that $\theta_0 = V_{\gamma_0,\delta}(\rho)$.

Having chosen the θ_p's, for each $n \in \omega$ we let ϕ_n be the path through $\langle \Phi_\alpha : \alpha < \delta \rangle$ determined by $\{H^{n_p}_{\gamma_p}(\theta_p, i_n) \mid p \in \omega,\, n_p \geq n\}$. Then for all $m \leq n \in \omega$ we will have that $\phi_m = H_\delta(\phi_n, i_m)$, and we will let

$$\Phi_\delta = \bigcup_{n<\omega} \{H^n_\delta(\phi_n, j) : m \leq n,\, j \in \mathcal{I}_{m,n}\}.$$

This is enough to ensure that conditions (1d), (1e) and (2c) from Definition 3.1 are met. For condition (1d) this is immediate. For condition (1e), the right-to-left containment follow from condition (1d). For the other direction, fix $m \leq n$ in ω. An arbitrary formula $\psi \in \Phi^m_\delta$ has the form $H^q_\delta(\phi_q, j)$, for some $q \in \omega \setminus m$ and some $j \in \mathcal{I}_{m,q}$. Since $\phi_n = H^p_\delta(\phi_p, i_n)$ for all $p \geq n$ in ω, we may assume that $q \geq n$. Letting $j' \in \mathcal{I}_{n,q}$ be such that $j \restriction X_m = j' \restriction X_m$, we have that $H^q_\delta(\phi_q, j') \in \Phi^n_\delta$, and that $\psi = H^n_\delta(H^q_\delta(\phi_q, j'), i_m)$, by part (2) of Remark 2.13.

To see that condition (2c) holds, fix $n, m \in \omega$, $\phi \in \Phi^n_\delta$ and $\psi \in \Phi^m_\delta$. Then there exist $p, q \in \omega$, $j \in \mathcal{I}_{n,p}$ and $k \in \mathcal{I}_{m,q}$ such that $\phi = H^p_\delta(\phi_p, j)$ and $\psi = H^q_\delta(\phi_q, k)$. Since

$$\phi_p = H^{\max\{p,q\}}_\delta(\phi_{\max\{p,q\}}, i_p)$$

and

$$\phi_q = H^{\max\{p,q\}}_\delta(\phi_{\max\{p,q\}}, i_q),$$

we may assume by part 2 of Remark 2.13 that $p = q$. Similarly, we may assume that $p \geq m+n$. Let A be a subset of X_p of size $m+n$ which contains the ranges of both j and k. Let $j' \colon X_{m+n} \to A$ be a bijection such that $j = j' \circ i_n$. Then

$$\phi = H^p_\delta(\phi_p, j' \circ i_n) = H^{m+n}_\delta(H^p_\delta(\phi_p, j'), i_n),$$

by part 2 of Remark 2.13, and $H^p_\delta(\phi_p, j') \in \Phi_\delta$. Finally, let $k' \in \mathcal{I}_{m,m+n}$ be such that $k = j' \circ k'$. Then $\psi = H^{m+n}_\delta(H^p_\delta(\phi_p, j'), k')$, as desired.

To complete the proof, we show how to choose the formulas θ_p so that conditions (1c) and (2b) of Definition 3.1 are satisfied, and also so that no member of Υ is in Φ_δ. We let $\theta_0 = V_{\gamma_0,\delta}(\rho)$, as above. Suppose that $p \in \omega$ is such that θ_p has been chosen, but θ_{p+1} has not.

To satisfy condition (1c), let γ_{p+1} be the least member of $\{\gamma^0_q : q \in \omega\}$ which is at least as big as both γ_p and γ^0_{p+1}, and suppose that ψ is an element of Φ^m_α, for some $\alpha \leq \gamma_{p+1}$ and some $m \in \omega$. By Proposition 8.4, we can find a formula $\theta'_p \in \Phi^{n_p}_{\gamma_{p+1}+m+n_p}$ such that $V_{\gamma_p,\gamma_{p+1}+m+n_p}(\theta'_p) = \theta_p$ and

$$H^{n_p}_{\gamma_{p+1}+m+n_p}(\theta'_p, i_{n_0}) = V_{\gamma_{p+1}+m+n_p,\delta}(\rho).$$

By condition (1c), there is a $\psi' \in \Phi^m_{\gamma_{p+1}+m+n_p}$ such that $V_{\alpha,\gamma_{p+1}+m+n_p}(\psi') = \psi$. Applying condition (2c) of Definition 3.1, we can choose $\theta''_p \in \Phi^{m+n_p}_{\gamma_{p+1}+m+n_p}$ and $j \in \mathcal{I}_{m,n_p+m}$ such that

$$H^{m+n_p}_{\gamma_{p+1}+m+n_p}(\theta''_p, i_{n_p}) = \theta'_p$$

and

$$H^{m+n_p}_{\gamma_{p+1}+m+n_p}(\theta''_p, j) = \psi'.$$

Then $\theta_{p+1} = V_{\gamma_{p+1},\gamma_{p+1}+m+n_p}(\theta''_p)$ is as desired, by Propositions 2.14 and 8.4.

To satisfy condition (2b), suppose that we have $m \leq n_p$ and $\alpha < \gamma_p$. We can represent an arbitrary $\phi \in \Phi^m_\delta$ as $H^{n_p}_\delta(\phi_p, j)$ for some $j \in \mathcal{I}_{m,n_p}$, in which case $V_{\alpha+1,\delta}(\phi)$ will be $V_{\alpha+1,\gamma_p}(H^{n_p}_{\gamma_p}(\theta_p, j))$. So it suffices to fix such a j and a formula ψ in $E(V_{\alpha+1,\gamma_p}(H^{n_p}_{\gamma_p}(\theta_p, j)))$. By Proposition 8.9, it suffices to find a $\theta'_p \in \Phi^{n_p+1}_{\gamma_p} \cap F(\rho)$ such that $H^{n_p+1}_{\gamma_p}(\theta'_p, i_{n_p}) = \theta_p$, and such that

$$H^{n_p+1}_\alpha(V_{\alpha,\gamma_p}(\theta'_p), j \cup \{(x_m, y)\}) = \psi$$

for some $y \in X_{n_p+1} \setminus \mathrm{range}(j)$. By Proposition 2.14,

$$V_{\alpha+1,\gamma_p}(H^{n_p}_{\gamma_p}(\theta_p, j)) = H^{n_p}_{\alpha+1}(V_{\alpha+1,\gamma_p}(\theta_p), j).$$

By condition (3) of Definition 2.10, there is a $\psi' \in E(V_{\alpha+1,\gamma_p}(\theta_p))$ such that

$$\psi = H^{n_p+1}_\alpha(\psi', j \cup \{(x_m, y)\})$$

for some $y \in X_{n_p+1} \setminus \mathrm{range}(j)$. By Proposition 8.9, there is a $\theta^*_p \in \Phi^{n_p}_{\gamma_p+1} \cap F(\rho)$ such that $\theta_p = V_{\gamma_p,\gamma_p+1}(\theta^*_p)$. By Proposition 4.3, there is a $\theta'_p \in E(\theta^*_p)$ such that $V_{\alpha,\gamma_p}(\theta'_p) = \psi'$. Then θ'_p is as desired.

Finally let us see how to avoid the members of Υ. Fix $m \leq n_p$, $f \in \mathcal{I}_{m,n_p}$ and $\upsilon \in \Upsilon \cap \Psi^m_\delta$. It suffices to show that we can find γ_{p+1} in the interval $(\max\{\gamma_p, \gamma^0_{p+1}\}, \delta)$ and a $\theta_{p+1} \in \Phi^{n_p}_{\gamma_{p+1}} \cap F(\rho)$ such that $H^{n_p}_\delta(\theta_{p+1}, f) \neq \upsilon$. Since Υ is disjoint from $\mathrm{ms}(\rho)$, there exists such a θ_{p+1} as desired. $\square$

9.9 Definition. Given a limit ordinal β and sets Φ_β ($\alpha < \beta$) such that each Φ_α is a subset of Ψ_α, a path $\bigwedge\{\psi_\alpha : \alpha < \beta\}$ through $\langle\Phi_\alpha : \alpha < \beta\rangle$ is *isolated* (with respect to $\langle\Phi_\alpha : \alpha < \beta\rangle$) if for some $\alpha_0 < \beta$, $|V^{-1}_{\alpha_0,\alpha}[\{\phi_\alpha\}]| = 1$ for all $\alpha \in (\alpha_0, \beta)$.

As in Remark 9.7, Proposition 8.10 shows that the term "with respect to $\langle\Phi_\alpha : \alpha < \beta\rangle$" is unnecessary in Definition 9.9, if $\langle\Phi_\alpha : \alpha < \beta\rangle$ is a Scott process.

9.10 Remark. Suppose that β is a limit ordinal, and $\mathcal{P} = \langle\Phi_\alpha : \alpha < \beta\rangle$ is a Scott process. Suppose that $m \leq n$ are elements of ω, $j \in \mathcal{I}_{m,n}$ and $\phi \in \Psi^n_\beta$ is an isolated path through $\mathcal{P}$. Then $H^n_\beta(\phi, j)$ is isolated. To see this, note first of all that the case $m = n$ follows from part (1) of Remark 2.13. This fact allow us to reduce to the case where $j = i_m$. Then a proof by induction reduces to the case where $n = m + 1$. This case follows part (2) of Proposition 5.1.

9.11 Remark. Given a limit ordinal β and sets Φ_β ($\alpha < \beta$) such that each Φ_α is a subset of Ψ_α, the isolated paths through $\langle\Phi_\alpha : \alpha < \beta\rangle$ are exactly the minimal set of the sentence formed by taking the conjunction of the unique members of each set Φ^0_α. This follows from Remark 9.10 and Proposition 8.4.

9.12 Definition. A Scott process $\langle\Phi_\alpha : \alpha < \beta\rangle$ is *scattered* if it there do not exist $n \in \omega$ and α_σ, ϕ_σ ($\sigma \in 2^{<\omega}$) such that

- each α_σ is an element of β;
- each ϕ_σ is an element of $\Phi^n_{\alpha_\sigma}$;
- whenever σ, τ in $2^{<\omega}$ are such that τ properly extends σ, $\alpha_\sigma < \alpha_\tau$ and $V_{\alpha_\sigma,\alpha_\tau}(\phi_\tau) = \phi_\sigma$;
- whenever σ, τ in $2^{<\omega}$ are such that neither of σ and τ extends the other,
  - if $\alpha_\sigma \leq \alpha_\tau$ then $V_{\alpha_\sigma,\alpha_\tau}(\phi_\tau) \neq \phi_\sigma$,
  - if $\alpha_\tau \leq \alpha_\sigma$ then $V_{\alpha_\tau,\alpha_\sigma}(\phi_\sigma) \neq \phi_\tau$;

9.13 Remark. Whether or not a Scott process $\langle\Phi_\alpha : \alpha < \beta\rangle$ is scattered is absolute between forcing extensions.

9.14 Remark. If $\langle\Phi_\alpha : \alpha < \delta\rangle$ is a Scott process of limit length having only countably many paths, it is scattered, and if it is scattered then every element of each Φ_α is part of an isolated path through $\langle\Phi_\alpha : \alpha < \delta\rangle$. Similarly, suppose that $\langle\Phi_\alpha : \alpha < \delta\rangle$ is a Scott process of limit length where δ is possibly uncountable, and that there exist a $\beta < \omega_1$ such that for $\gamma \in (\beta, \omega_1)$ there are only countably many Scott processes of length γ extending $\langle\Phi_\alpha : \alpha < \beta\rangle$. Then again every element of each Φ_α is part of an isolated path through $\langle\Phi_\alpha : \alpha < \delta\rangle$. Otherwise, one could find a $2^{<\omega}$-splitting family in $\bigcup\{\Phi_\alpha : \beta < \alpha < \delta\}$ such whenever X is a countable elementary submodel of a suitably large $H(\theta)$ with X containing this family, the image of this splitting family under the transitive collapse of X would give (using Theorem 9.8) a perfect set of Scott processes of the same countable length.

In Proposition 9.15, we do not require δ to have countable cofinality (whereas we did for Theorem 9.8).

Proposition 9.15. *Suppose that δ is a limit ordinal, and that $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process such that each element of $\bigcup\{\Phi_\alpha : \alpha < \delta\}$ is extended by an isolated path through $\langle \Phi_\alpha : \alpha < \delta \rangle$. Letting Φ_δ be the set of isolated paths through $\langle \Phi_\alpha : \alpha < \delta \rangle$, $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ is a Scott process. Furthermore, Φ_δ then satisfies amalgamation, and every Scott process properly extending $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ has rank at most δ.*

Proof. Checking that Φ_δ induces a Scott process involves checking conditions (1e), (2b) and (2c) of Definition 3.1. Remark 9.10 gives one direction of (1e). The other conditions can be shown by applying the corresponding fact at levels above the ordinal α_0 witnessing that the formulas in question are isolated.

That Φ_δ amalgamates also follows from the definition of the functions $H_{\alpha+1}$ ($n \in \omega$) for any ordinal α above the ordinal α_0 witnessing that the formulas in question are isolated. By Proposition 5.18, it also follows from the fact that some Scott properly extending $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ has rank δ, which follows from the next paragraph.

To see that every Scott process $\langle \Phi_\alpha : \alpha \leq \delta + 1 \rangle$ extending $\langle \Phi_\alpha : \alpha \leq \delta \rangle$ has rank δ, suppose that we have $n \in \omega$, $\phi \in \Phi^n_{\delta+1}$ and $\psi \in \Phi^{n+1}_\delta$ such that $H^{n+1}_\delta(\psi, i_n) = V_{\delta,\delta+1}(\phi)$. Let $\beta < \delta$ be such that $V_{\delta,\delta+1}(\phi)$ and ψ are the unique members of $V^{-1}_{\beta,\delta}[\{V_{\beta,\delta+1}(\phi)\}]$ and $V^{-1}_{\beta,\delta}[\{V_{\beta,\delta}(\psi)\}]$ respectively. Then

$$H^{n+1}_{\beta+1}(V_{\beta+1,\delta}(\psi), i_n) = V_{\beta+1,\delta+1}(\phi)$$

by Proposition 2.14, so $V_{\beta,\delta}(\psi) \in E(V_{\beta+1,\delta+1}(\phi))$ by condition (2a) of Definition 3.1. Then conditions (2a) and (3) of Definition 3.1 imply that $\psi \in E(\phi)$. $\square$

9.16 Remark. Theorem 9.8 shows that if δ is a limit ordinal and $\langle \Phi_\alpha : \alpha < \delta \rangle$ is a Scott process with just countably many paths, then for each such path ρ, letting Φ_δ be $\mathrm{ms}(\rho)$ we get a Scott process $\langle \Phi_\alpha : \alpha \leq \delta \rangle$. Since $\mathrm{ms}(\phi)$ and being scattered are absolute to forcing extensions, we get the same conclusion from the assumption that $\langle \Phi_\alpha : \alpha < \delta \rangle$ is scattered. In this context, then, since $\mathrm{ms}(\rho)$ is the smallest set one can add to $\langle \Phi_\alpha : \alpha < \delta \rangle$ to get a Scott processes with ρ in its last level, it follows (again, in the case where $\langle \Phi_\alpha : \alpha < \delta \rangle$ is scattered) that if ϕ and ψ are paths through $\langle \Phi_\alpha : \alpha < \delta \rangle$ with $\phi \in \mathrm{ms}(\psi)$, then $\mathrm{ms}(\phi)$ is a subset of $\mathrm{ms}(\psi)$.

In the following proposition, the countability assumption on the sets Φ_α can be replaced by the assumption that $\langle \Phi_\alpha : \alpha < \gamma \rangle$ is scattered, using Remark 9.16.

Proposition 9.17. *Let β be an ordinal, and let γ be the least limit ordinal greater than or equal to β. Suppose that $\langle \Phi_\alpha : \alpha \leq \gamma + 1 \rangle$ is a Scott process of pre-rank β, such that Φ_α is countable for all $\alpha < \gamma$. Then the rank of $\langle \Phi_\alpha : \alpha \leq \gamma + 1 \rangle$ is at most γ.*

Proof. Since Φ_α is countable for all $\alpha < \gamma$, β is countable. By the definition of pre-rank, $\langle \Phi_\alpha : \alpha \leq \gamma \rangle$ is the unique Scott process of length $\gamma + 1$ extending $\langle \Phi_\alpha : \alpha < \gamma \rangle$. By Theorem 9.8 that $\langle \Phi_\alpha : \alpha < \gamma \rangle$ has only countably many paths. By Proposition 9.15 that all of them are isolated, and $\langle \Phi_\alpha : \alpha \leq \gamma + 1 \rangle$ has rank at most γ. $\square$

Combining Remark 9.14 with Propositions 5.18, 6.4 and 9.15 and Theorems 1.2, 6.9 and 7.10, we get the following.

Theorem 9.18. *Let ϕ be a sentence of $\mathcal{L}_{\omega_1,\omega}(\tau)$ and let α be the quantifier depth of ϕ. Let $\beta \in (\alpha, \omega_2)$ be an ordinal such that ϕ has a model of Scott rank β, but only countably many models of Scott rank γ for each countable ordinal γ in the interval (α, β). Then for every limit ordinal $\delta \in (\alpha, \beta)$, ϕ has a model of Scott rank δ.*

10 A forcing-absoluteness argument

The set of τ-structures with domain ω is naturally seen as a Polish space X_τ, where a basic open set is given by the set of structures in which $R(i_0, \ldots, i_{n-1})$ holds, for R an n-ary relation symbol from τ and $i_0, \ldots, i_{n-1} \in \omega$ (see Section 11.3 of [3], for instance). Given a sentence $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$, the set of models of ϕ (with domain ω) is a Borel subset of X_τ. By a theorem of Lopez-Escobar [8], every Borel subset of X_τ which is closed under isomporphism is also the set of models of some $\mathcal{L}_{\omega_1,\omega}(\tau)$ sentence. Let us say that an *analytic counterexample to Vaught's Conjecture* is an analytic set of τ-structures on ω, closed under isomorphism, having uncountably many models up to isomorphism, but not a perfect set of nonisomorphic models. Steel [11] presents two examples of analytic counterexamples to Vaught's Conjecture (for certain relational vocabularies), one due to H. Friedman and the other to K. Kunen. In this section we use a forcing-absoluteness argument to prove the following.

Theorem 10.1. *Suppose that $\mathcal{A}$ is an analytic counterexample to Vaught's Conjecture, and let $x \subseteq \omega$ be such that $\mathcal{A}$ is Σ^1_1 in x. Fix $M \in \mathcal{A}$, and let β be an ordinal. Then $\langle \Phi_\alpha(M) : \alpha < \beta \rangle \in L[x]$.*

Applying this theorem in forcing extensions of V we get the following ostensibly stronger fact.

Corollary 10.2. *Suppose that $\mathcal{A}$ is an analytic counterexample to Vaught's Conjecture, and let $x \subseteq \omega$ be such that $\mathcal{A}$ is Σ^1_1 in x. Let M be a member of the reinterpreted version of $\mathcal{A}$ in a forcing extension of V, and let β be an ordinal. Then $\langle \Phi_\alpha(M) : \alpha < \beta \rangle \in L[x]$.*

Before beginning the proof of Theorem 10.1 (which is short), we make a couple remarks. In what follows we will talk of sufficient fragments of ZFC. The theory ZFC$^\circ$ from [2] is one such fragment.

10.3 Remark. Let $\mathcal{A}$ be an analytic family of τ-structures on ω, and fix $\beta < \omega_1$. The set of sequences $\langle \Phi_\alpha(M) : \alpha < \beta \rangle$ for $M \in \mathcal{A}$ is naturally coded by an analytic set of reals, which contains a perfect set if it is uncountable. If $\mathcal{A}$ is an analytic counterexample to Vaught's Conjecture, then, the set of such sequences is countable for each $\beta < \omega_1$.

For any analytic family of τ-structures, and any countable (possibly empty) set of Scott processes of length $\beta < \omega_1$, the assertion that there exists a member of the family whose Scott process up to length β is not in this countable set is Σ^1_1 in codes for β, the family and the countable set, and thus absolute to any model of (a sufficient fragment of) ZFC that contains them. Furthermore, if such a model thought that uncountably many such processes existed, it could build a perfect set of such processes.

It follows that if $\mathcal{A}$ is an analytic counterexample to Vaught's Conjecture then any inner model N of (a sufficient fragment of) ZFC containing a real parameter code for $\mathcal{A}$ contains all sequences of the form $\langle \Phi_\alpha(M) : \alpha < \beta \rangle$, for $M \in \mathcal{A}$ and $\beta < \omega_1^N$. This gives Theorem 10.1 for initial segments of Scott processes of length less than $\omega_1^{L[x]}$.

Recall that for any ordinal γ, $\mathrm{Col}(\omega, \gamma)$ is the partial order which adds a function (generically, a surjection) from ω to γ by finite pieces, ordered by inclusion.

Proof of Theorem 10.1. Let $\theta > \beta$ be a regular cardinal of $L[x]$ such that $L_\theta[x]$ satisfies a sufficient fragment of ZFC (for instance, let θ be a regular cardinal of V greater than $2^{2^{(|\beta|+\omega_1)}}$). Let X be a countable (in V) elementary submodel of $L_\theta[x]$ containing $\{x, \langle \Phi_\alpha : \alpha < \beta \rangle\} \cup \beta$. Let γ be such that the transitive collapse of X is $L_\gamma[x]$. By the last paragraph of Remark 10.3, whenever g is an $L_\gamma[x]$-generic filter for $\mathrm{Col}(\omega, \beta)$, $\langle \Phi_\alpha : \alpha < \beta \rangle$ is in $L_\gamma[x][g]$. This means that $\langle \Phi_\alpha : \alpha < \beta \rangle$ is in $L_\gamma[x]$ (this is a classical forcing fact; the point is that otherwise one could choose a generic filter while ensuring that each name in $L_\gamma[x]$ realizes to some value other than $\langle \Phi_\alpha : \alpha < \beta \rangle$). By elementarity, then, $\langle \Phi_\alpha : \alpha < \beta \rangle$ is in $L_\theta[x]$. $\square$

10.4 Remark. Let $\mathcal{A}$ be an analytic family of τ-structures on ω. The assertion that $\mathcal{A}$ is an analytic counterexample to Vaught's Conjecture is Π^1_2 in a real parameter x for $\mathcal{A}$, and therefore absolute to $L[x]$.[1] It follows that for every cardinal κ of $L[x]$, there are cofinally many ordinals below $(\kappa^+)^{L[x]}$ which are the Scott rank of a structure in $\mathcal{A}$, in any forcing extension of $L[x]$ via the partial order $\mathrm{Col}(\omega, \kappa)$. Applying Theorem 9.18, this gives (in the case where $\mathcal{A}$ is Borel) that this set of ordinals (in such a forcing extension) includes coboundedly many limit ordinals below $(\kappa^+)^{L[x]}$.

[1] There exist perfectly many nonisomorphic structures in $\mathcal{A}$ if and only if some wellfounded countable model of a sufficient fragment of ZFC thinks there exist perfectly many nonisomorphic structures in $\mathcal{A}$ (see the proof of Theorem 6.2 of [2], for instance), and this later statement is easily seen to be Σ^1_2. The statement that there are countable models in $\mathcal{A}$ of unboundedly many Scott ranks below ω_1 is easily seen to be Π^1_2.

Theorem 10.6 below gives an alternate proof of Theorem 10.1 (the idea behind this alternate proof is essentially the same, recast slightly). The proof of Theorem 10.6 in turns uses the following standard forcing fact, which is easily proved by induction on the rank of a given set in the intersection of $V[G]$ and $V[H]$.

Theorem 10.5. *Let Q_1 and Q_2 be partial orders, and suppose that (G, H) is a generic filter for $Q_1 \times Q_2$. Then the ground model V is the intersection of $V[G]$ and $V[H]$.*

Theorem 10.6. *Suppose that $\mathcal{A}$ is an analytic counterexample to Vaught's Conjecture, and that $x \subseteq \omega$ is such that $\mathcal{A}$ is Σ^1_1 in x. Let κ be an infinite cardinal of $L[x]$, fix $\gamma < (\kappa^+)^{L[x]}$, and suppose that σ is a $\mathrm{Col}(\omega, \kappa)$-name in $L[x]$ for a τ-structure in $\mathcal{A}$ of Scott rank γ. Then there is a Scott process in $L[x]$ which is forced by some condition in $\mathrm{Col}(\omega, \kappa)$ to be and initial segment of the Scott process of the realization of σ.*

Proof. Let P be the finite support product of $(\kappa^+)^{L[x]}$ many copies of $\mathrm{Col}(\omega, \kappa)$. Forcing with P over $L[x]$ makes κ countable, but leaves κ^+ uncountable. Moreover, κ^+ is the ω_1 of any P-extension. This follows from a standard Δ-system argument. Alternately, P is forcing-equivalent to the product of one copy of $\mathrm{Col}(\omega, \kappa)$ with the finite support product of $(\kappa^+)^{L[x]}$ many copies of Cohen forcing, the latter of which is c.c.c..

Let $\mu \in L[x]$ be a $\mathrm{Col}(\omega, \kappa)$-name for the Scott process of the realization of σ. The name μ induces $(\kappa^+)^{L[x]}$ many P-names, by copying μ respectively into each coordinate of P. Since P cannot force the existence of $(\kappa^+)^{L[x]}$ many distinct Scott processes of rank γ for elements of $\mathcal{A}$, there is a condition p in P forcing that the realizations of μ in two different coordinates will be the same. Let α and β be two such coordinates, and let $p(\alpha)$ and $p(\beta)$ be the values of p at these coordinates. Then $p(\alpha)$ and $p(\beta)$ are conditions in $\mathrm{Col}(\omega, \kappa)$. By Theorem 10.5, the condition $(p(\alpha), p(\beta))$ forces in $\mathrm{Col}(\omega, \kappa) \times \mathrm{Col}(\omega, \kappa)$ that the realization of μ in each coordinate will be members of the ground model. It follows that $p(\alpha)$ and $p(\beta)$ each decide all of μ, as desired. $\square$

Theorem 10.6 implies Theorem 10.1, since if Theorem 10.1 were false one could let σ be a name for a τ-structure whose Scott process (up to level α) is not in $L[x]$, and obtain a contradiction.

Theorems 7.10 and 9.18, along with Corollary 10.2 and Remark 10.4, give the following unpublished theorem of Leo Harrington from the 1970's.

Theorem 10.7 (Harrington)**.** *Suppose that τ is a countable relational vocabulary and that $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ gives a counterexample to Vaught's Conjecture. Then the the Scott ranks of the models of ϕ of cardinality $\aleph_1$ are cofinal in ω_2.*

The arguments we have given here give a slightly stronger version of Harrington's theorem, as follows. The theorem follows from Theorems 7.10 and 9.18, Corollary 10.2, Proposition 9.15 and Remark 10.4, which together show that for all limit ordinals β in the interval (α, ω_2), ϕ has a model of Scott rank

β whose formulas at level β are all isolated. We do not know if the result stated here is new.

Theorem 10.8. *Suppose that τ is a countable relational vocabulary and that $\phi \in \mathcal{L}_{\omega_1,\omega}(\tau)$ gives a counterexample to Vaught's Conjecture. Let α be the quantifier depth of ϕ. Then for every limit ordinal β in the interval (α, ω_2), ϕ has a model of Scott rank β.*

Standard arguments show that if there is a counterexample to Vaught's Conjecture then there is one of quantifier depth at most ω (in an expanded language).

10.9 Remark. The arguments here also give a proof of Theorem 1 of [4], showing that any counterexample to Vaught's Conjecture can be strengthened to a minimal counterexample. The point again is that if $\sigma \in \mathcal{L}_{\omega_1,\omega}(\tau)$ is a counterexample to Vaught's Conjecture, and α is the quantifier depth of σ, then there is a sentence $\sigma' \in \mathcal{L}_{\omega_1,\omega}(\tau)$ which is the unique member of $\Phi^0_\alpha(M)$ for uncountably many countable models M satisfying σ. Then all models of σ' are models of σ, by Theorem 1.2, and σ' is also a counterexample to Vaught's Conjecture. Let S be the set of all countable length Scott processes which have σ' as their unique sentence at level α and are initial segments of the Scott process of some model of uncountable Scott rank. Since σ' is a counterexample to Vaught's Conjecture, S is not empty. On the other hand, since σ' does not have perfectly many countable models, there will be a member of S without incompatible extensions in S. Since any extension of this member in S will have the same property, there is such a member of S with successor length. Let ϕ be the unique sentence in the last level of this process. Then ϕ is a counterexample to Vaught's Conjecture, and all uncountable models of ϕ satisfy the same $\mathcal{L}_{\omega_1,\omega}(\tau)$-theory.

Hjorth [5] showed that if there exists a counterexample to Vaught's Conjecture, then there is one with no model of cardinality $\aleph_2$. Recently, this has been extended by Baldwin, S. Friedman, Koerwien and Laskowski [1], who showed (among other things) that if there exists a counterexample to Vaught's Conjecture, then there is one with with the property that for some countable $\mathcal{L}_{\omega_1,\omega}$-fragment T, no model of cardinality $\aleph_1$ has a T-elementary extension.

References

[1] J.T. Baldin, S. Friedman, M. Koerwien, M.C. Laskowski, *Three red herrings around Vaught's Conjecture*, preprint

[2] J.T. Baldwin, P.B. Larson, *Iterated elementary embeddings and the model theory of infinitary logic*, in preparation

[3] S. Gao, **Invariant Descriptive Set Theory**, CRC Press, 2009

[4] V. Harnik, M. Makkai, *A tree argument in infinitary model theory*, Proceedins of the American Mathematical Society 67 (1977), 309-314

[5] G. Hjorth, *A note on counterexamples to the Vaught Conjecture*, Notre Dame J. Formal Logic Volume 48, Number 1 (2007), 49-51

[6] W. Hodges, **Model theory**, Encyclopedia of Mathematics and its Applications, 42. Cambridge University Press, Cambridge, 1993

[7] J. Keisler, **Model Theory of Infinitary Languages**, North-Holland 1971

[8] E. G. K. Lopez-Escobar, *An interpolation theorem for denumerably long formulas*, Fund. Math. 57 (1965), 253272

[9] D. Marker, **Model Theory : An Introduction**, Springer, 2002

[10] D. Marker. Scott ranks of counterexamples to Vaught's conjecture. Notes from 2011; `http://homepages.math.uic.edu/~marker/harrington-vaught.pdf`.

[11] J.R. Steel, *On Vaught's conjecture*, in : **Cabal Seminar 76-77**, Lecture Notes in Mathematics v. 689, 1978, 193-208