SPECIAL TREES ON REALS

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Abstract. In [5], Steel shows that in the presence of large cardinals, trees on reals in $L(\mathbb{R})$ without uncountable branches in V have an absolute impediment preventing a branch from being added by forcing. We generalize this result to trees coded by arbitrary universally Baire sets, using Woodin's analysis of HOD in place of inner model theory.

In [5], Steel proves the following, where given a tree T, we let T^+ denote the set of sequences whose proper initial segments are all in T. We think of the trees on reals in this paper as set of reals.

Theorem 0.1. *Assume that there exist infinitely many Woodin cardinals below a measurable cardinal. Let $T \subset \mathbb{R}^{<\omega_1}$ be a tree in $L(\mathbb{R})$. Then exactly one of the following holds.*

- *There is an uncountable branch of T in V.*
- *There is a function $f : T^+ \to \omega^\omega$ in $L(\mathbb{R})$ such that for each $p \in T^+$, $f(p)$ codes a wellordering of ω in ordertype $dom(p)$.*

In this note, we generalize this fact (in a suitable context) to trees coded by arbitrary universally Baire sets. We can prove one of the two directions in a slightly more general context than the other.

We will be using the following well-known facts, though some of them are unpublished.

Theorem 0.2 (Woodin)**.** *If AD^+ holds and S is a set of ordinals, there is a $Q \subset \theta$ such that $L[Q] = HOD_S$ and $L[Q, x] = HOD_{S,x}$ for every countable $x \subset \omega_1$.*

Theorem 0.3 (Woodin)**.** *If AD^+ holds and Q is a set of ordinals, then for a Turing cone of reals x, $\omega_2^{L[Q,x]}$ is Woodin in $HOD_Q^{L[Q,x]}$.*

Note that if Q is a set of ordinals and x is a real, then every set of ordinals in $L[Q]$ is in $HOD_Q^{L[Q,x]}$.

Theorem 0.4 (Martin; see [2])**.** *If AD holds, then every set of Turing degrees contains or is disjoint from a cone, and the intersection of countably many cones is nonempty.*

Our proof uses a simple version of Woodin's stacking argument.

Theorem 0.5. *Let $T \subset \mathbb{R}^{<\omega_1}$ be a tree, let S be a set of ordinals coding trees on the ordinals projecting to T and its complement, and suppose that $L(S, \mathbb{R}) \models AD^+$. Then at least one of the following two statements is true.*

(1) There is an uncountable branch of T in V.

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(2) *There is a function* $f\colon T^+ \to \omega^\omega$ *in* $L(S,\mathbb{R})$ *such that for each* $p \in T^+$, $f(p)$ *codes a wellordering of* ω *in ordertype* $dom(p)$.

Furthermore, if there exists a Woodin cadinal δ *and every set of reals in* $L(S,\mathbb{R})$ *is* δ^+ *weakly homogeneously Suslin in* V, *at least one of (1) and (2) is false.*

Proof. We work in $L(S,\mathbb{R})$. First suppose that (2) fails. We show that (1) holds. Since there are wellorderings of $\mathcal{P}(\omega)^{HOD_{S,p}}$ uniformly definable from p, there must be a $p \in T^+$ which is uncountable in $HOD_{S,p}$. Fix also $Q \subset \theta$ as in Theorem 0.2, with respect to S. Then p is uncountable in $L[Q,p]$. We have then by Theorem 0.3 that for a Turing cone of reals x, $\omega_2^{L[Q,p,x]}$ is a Woodin cardinal in $HOD_{Q,p}^{L[Q,p,x]}$. We would like to see that for a cone of x,

$$\omega_1^{HOD_{Q,p}^{L[Q,p,x]}} \leq \omega_1^{L[Q,p]}.$$

This follows from the fact that each model of the form $HOD_{Q,p}^{L[Q,p,x]}$ has a definable (uniformly from p) wellordering $<_x$ of its reals, and so the following set of triples is definable from Q and p: the set of (α,i,j) such that for a cone of x,

$$\omega_1^{HOD_{Q,p}^{L[Q,p,x]}} > \alpha$$

and the pair (i,j) is in the $<_x$-least element of ω^ω in $HOD_{Q,p}^{L[Q,p,x]}$ coding a wellordering of ω of ordertype α. By the countable completeness of the cone measure (Theorem 0.4), this gives a sequence of surjections in $L[Q,p]$ from ω onto each ordinal α which is less than

$$\omega_1^{HOD_{Q,p}^{L[Q,p,x]}}$$

for a cone of x.

Now, fix an x such that

$$\omega_1^{HOD_{Q,p}^{L[Q,p,x]}} \leq \omega_1^{L[Q,p]}$$

and $\omega_2^{L[Q,p,x]}$ is a Woodin cardinal in $HOD_{Q,p}^{L[Q,p,x]}$. Let

$$M = HOD_{Q,p}^{L[Q,p,x]}.$$

Then by replacing p with an initial segment if necessary, we may assume that p has height ω_1^M. Let $\delta = \omega_2^{L[Q,p,x]}$. Since δ is countable in V, we can choose an M-generic filter g for $Coll(\omega_1,{<}\delta)^M$. Then the nonstationary ideal is presaturated in $M[g]$. Furthermore, there are trees in $M[g]$ projecting in V to T and its complement. This means that $M[g]$ is T-iterable. Stepping outside of $L(S,\mathbb{R})$ to a model of Choice and taking any iteration j of $M[g]$ of length ω_1, then, $j(p)$ is an uncountable member of T^+.

To see the last part of the Theorem, suppose that T and f are coded by δ^+-weakly homogeneously Suslin sets of reals, and suppose that p is an uncountable path through T. Then there is a countable elementary submodel X of some large enough initial segment of the universe containing δ, T, f and p whose transitive collapse M has the property that (letting $\bar{\delta}$ be the image of δ under the collapse), if $M[g]$ is a forcing extension of M by $Coll(\omega_1,\bar{\delta})^M$, then $M[g]$ is (T,f)-iterable ([7, 4, 1]). Letting $\bar{p}$ be the image of p under the collapse, then, every forcing extension of $M[g]$ by $(\mathcal{P}(\omega_1)/NS_{\omega_1})^{M[g]}$ has $f(\bar{p})$ as an element, which means that $M[g]$ has $f(\bar{p})$ as an element, giving a contradiction. □

The following theorems can be used to show that if there exists a proper class of Woodin cardinals and the tree T is a weakly homogeneously Suslin set of reals, then there is a model of the form $L(S, \mathbb{R})$ satisfying AD^+, where S is a set of ordinals coding trees projecting to T and its complement. In this context, then, exactly one of (1) and (2) above hold.

Theorem 0.6 (Steel [6, 3])**.** *Suppose that there exist proper class many Woodin cardinals. Then universally Baire sets of reals have universally Baire scales.*

Theorem 0.7 (Woodin)**.** *Suppose that there exist proper class many Woodin cardinals, and let A be a universally Baire set of reals. Then $L(A, \mathbb{R}) \models AD^+$.*

Theorem 0.8 (Woodin)**.** *Suppose that AD^+ holds and that M is an inner model containing the reals. Then $M \models AD^+$.*

References

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