SQUARE PRINCIPLES IN $\mathbb{P}_{\max}$ EXTENSIONS

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ABSTRACT. By forcing with $\mathbb{P}_{\max}$ over strong models of determinacy, we obtain models where different square principles at ω_2 and ω_3 fail. In particular, we obtain a model of $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \neg\square(\omega_2) + \neg\square(\omega_3)$.

CONTENTS

1. INTRODUCTION

The forcing notion $\mathbb{P}_{\max}$ was introduced by W. Hugh Woodin in the early 1990s, see Woodin [Woo10]. When applied to models of the Axiom of Determinacy, it achieves a number of effects not known to be obtainable by traditional forcing methods.

Recall that $\mathsf{AD}_{\mathbb{R}}$ asserts the determinacy of all length ω perfect information two player games where the players alternate playing real numbers, and Θ denotes the least ordinal that is not a surjective image of the reals. As usual, MM denotes the maximal forcing axiom, Martin's Maximum. By $\mathsf{MM}(\mathfrak{c})$ we denote its restriction to partial orders of size at most continuum.

For the strengthening $\mathsf{MM}^{++}(\mathfrak{c})$ of this latter principle, see Woodin [Woo10, Definition 2.47].

Woodin [Woo10, Theorem 9.39] shows that when $\mathbb{P}_{\max}$ is applied to a model of $\mathsf{AD}_{\mathbb{R}} +$ "Θ is regular", the resulting extension satisfies $\mathsf{MM}^{++}(\mathfrak{c})$. A natural question is to what extent one can extend this result to partial orders of size $\mathfrak{c}^+$. For many partial orders of this size, obtaining the corresponding forcing axiom from a determinacy hypothesis should greatly reduce the known upper bound for its large cardinal consistency strength. Moreover, using the Core Model Induction, a method pioneered by Woodin (see Schindler-Steel [SSa] and Sargsyan [Sara]), one can find lower bounds for the consistency strength of $\mathsf{MM}^{++}(\mathfrak{c}^+)$ and its consequences, which leads to the possibility of proving equiconsistencies.

In this paper we apply $\mathbb{P}_{\max}$ to theories stronger than $\mathsf{AD}_{\mathbb{R}} +$ "Θ is regular" and obtain some consequences of $\mathsf{MM}^{++}(\mathfrak{c}^+)$ on the extent of square principles, as introduced by Ronald B. Jensen [Jen72]. We recall the definitions:

Definition 1.1. Given a cardinal κ, the principle $\square_\kappa$ says that there exists a sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ such that for each $\alpha < \kappa^+$,

  (1) Each C_α is club in α;
  (2) For each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$; and
  (3) The order type of each C_α is at most κ.

Definition 1.2. Given an ordinal γ, the principle $\square(\gamma)$ says that there exists a sequence $\langle C_\alpha \mid \alpha < \gamma \rangle$ such that

  (1) For each $\alpha < \gamma$,
      • Each C_α is club in α;
      • For each limit point β of C_α, $C_\beta = C_\alpha \cap \beta$; and
  (2) There is no *thread* through the sequence, i.e., there is no club $E \subseteq \gamma$ such that $C_\alpha = E \cap \alpha$ for each limit point α of E.

We refer to sequences witnessing these principles as $\square_\kappa$-sequences or $\square(\gamma)$-sequences, respectively. Note that $\square_\kappa$ implies $\square(\kappa^+)$, and that $\square_\omega$ is true.

Remark 1.3. Suppose that κ is uncountable. A key distinction between these principles is that $\square_\kappa$ persists to outer models that agree about κ^+, while $\square(\kappa^+)$ need not. This seems to be folklore; since we could not locate an argument in the literature, we sketch one below.

For example, consider the poset $\mathbb{P}$ that attempts to add a $\square(\kappa^+)$-sequence with initial segments. Note that $\mathbb{P}$ is $(\kappa + 1)$-strategically closed.

Let G be V-generic for $\mathbb{P}$, and assume for the moment that the generic sequence added by $\mathbb{P}$ is indeed a $\square(\kappa^+)$-sequence, say $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$. Then one can thread it by further forcing over $V[G]$ with the poset $\mathbb{Q}$ whose conditions are closed bounded subsets c of κ^+ such that $\max(c)$ is a limit point of c and, for every $\alpha \in \mathrm{lim}(c)$, we have $c \cap \alpha = C_\alpha$.

This threading does not collapse κ^+, because $\mathbb{Q}$ is κ^+-distributive in $V[G]$, by a standard argument. In fact, the forcing $\mathbb{P} * \dot{\mathbb{Q}}$ has a κ^+-closed dense

set consisting of conditions of the form $(p, \dot{q})$ where p decides the value $\dot{q}$ to be $p(\alpha)$, for α the largest ordinal in $\mathrm{dom}(p)$. This can also be verified by a standard density argument.

Assume now that PFA holds in V, so $\square(\kappa^+)$ fails (as does any $\square(\gamma)$ for $\mathrm{cf}(\gamma) > \omega_1$, by Todorcevic [Tod84]). Since PFA is preserved by ω_2-closed forcing, by König-Yoshinobu [KY04], it holds in the extension by $\mathbb{P} * \dot{\mathbb{Q}}$. (One could argue similarly starting from a universe where κ is indestructibly supercompact.)

It remains to argue that the sequence $\vec{C}$ added by $\mathbb{P}$ is a $\square(\kappa^+)$-sequence. The (standard) argument verifying this was suggested by James Cummings, and simplifies our original approach, where a more elaborate poset than $\mathbb{P}$ was being used. Assume instead that the generic sequence is threadable, and let $\dot{c}$ be a name for a thread. Now inductively construct a descending sequence of conditions p_n, and an increasing sequence of ordinals γ_n, for $n \in \omega$, such that, letting α_n be the length of p_n, we have:

(1) $\alpha_n < \gamma_n < \alpha_{n+1}$,
(2) $p_{n+1} \Vdash \gamma_n \in \dot{c}$, and
(3) p_{n+1} determines the value of $\dot{c} \cap \alpha_n$.

Let $\gamma = \sup_n \gamma_n = \sup_n \alpha_n$, and let p' be the union of all p_n. Then p' is not a condition, but can be made it into one, call it p^*, by adding at γ as the value $p^*(\gamma)$, some cofinal subset of γ of order type ω that is distinct from $\dot{c} \cap \gamma$.

Then p^* forces that the γ-th member of $\vec{C}$ is different from $\dot{c} \cap \gamma$. But γ is forced by p^* to be a limit point of both $\dot{c}$ and the γ-th member of $\vec{C}$, and therefore p^* forces that $\dot{c}$ is not a thread through $\vec{C}$.

Viewing this as a density argument, we see that densely many conditions force that $\dot{c}$ is not a thread thorough $\vec{C}$. Thus, $\vec{C}$ is a $\square(\kappa^+)$-sequence in $V[G]$, as we wanted.

This shows that neither $\square(\kappa^+)$, nor its negation, is upward absolute to models that agree on κ^+. See also the discussion on *terminal square* in Schimmerling [Sch07, §6].

Question 1.4. Assuming that $\square(\kappa^+)$ fails, can it be made to hold by κ^+-closed forcing? This seems unlikely, though we do not see a proof at the moment. If the answer is no, then the argument above can be simplified, as there is no need to assume PFA or any such hypothesis on the background universe.

Via work of Stevo Todorcevic [Tod84, Tod02], it is known that $\mathsf{MM}^{++}(\mathfrak{c})$ implies $2^{\aleph_1} = \aleph_2 + \neg\square(\omega_2)$, and $\mathsf{MM}^{++}(\mathfrak{c}^+)$ implies $\neg\square(\omega_3)$. Through work of Ernest Schimmerling [Sch07] and Steel (via the Core Model Induction, see Schindler-Steel [SSa]) it is known that the following statement implies that the Axiom of Determinacy holds in the inner model $L(\mathbb{R})$:

$$(1) \qquad \neg\square(\omega_2) + \neg\square_{\omega_2} + 2^{\aleph_1} = \aleph_2.$$

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The following theorem gives a lower bound for the hypotheses needed for the results in this paper. As usual, HOD denotes the inner model of all hereditarily ordinal definable sets, see Jech [Jec03].

Theorem 1.5. *Assume* $\mathsf{AD}_{\mathbb{R}} +$ *"Θ is regular" and that there is no* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *such that* $L(\Gamma, \mathbb{R}) \models$ *"Θ is Mahlo in HOD". Then* $\square_{\omega_2}$ *holds in the* $\mathbb{P}_{\max}$ *extension.*

To prove Theorem 1.5, one shows that an appropriate version of $\square(\Theta)$ holds in the determinacy model, and therefore that $\square_{\omega_2}$ holds in the corresponding $\mathbb{P}_{\max}$ extension (the cardinal Θ of the inner model of determinacy becomes ω_3 in the $\mathbb{P}_{\max}$ extension). The argument is in essence a standard adaptation of the usual proof of square principles in fine structural models. The HOD analysis of Sargsyan [Sara] makes this adaptation possible, see Section 3.

The moral is that a stronger determinacy hypothesis is needed to force the failure of $\square_{\omega_2}$ via $\mathbb{P}_{\max}$. On the other hand, in Section 4 we show that weak square at ω_3, the statement $\square_{\omega_2, \omega_2}$ (see Definition 4.1), always holds in $\mathbb{P}_{\max}$ extensions of models of $\mathsf{AD}^+ +$ "Θ is regular".

The hypothesis that we use to force the negation of $\square_{\omega_2}$ is in the end just slightly stronger than the assumption of Theorem 1.5, see Theorem 6.1:

> $\mathsf{AD}_{\mathbb{R}} +$ "Θ is regular" $+$ "$\{\kappa \mid \kappa$ is regular in HOD, is a member of the Solovay sequence, and has cofinality $\omega_1\}$ is stationary in Θ".

We show from this hypothesis that $\square_{\omega_2}$ fails in the extension given by $\mathbb{P}_{\max}$ followed by a natural forcing well-ordering the power set of the reals. That $2^{\aleph_1} = \aleph_2$ and $\neg\square(\omega_2)$ also hold follows from Woodin's work. Similarly, a stronger hypothesis allows us to conclude that even $\square(\omega_3)$ fails in the final extension, see Theorem 7.5.

If we do not require choice to hold in the final model, then already $\mathsf{AD}_{\mathbb{R}} +$ "Θ is Mahlo in HOD" suffices to make $\square_{\omega_2}$ fail in the $\mathbb{P}_{\max}$ extension (and therefore Theorem 1.5 is optimal), see Theorem 5.1. As for the failure of $\square(\Theta)$ itself in the determinacy model, see Theorem 5.3.

With the possible exception of the assumptions in Section 7, the determinacy hypotheses we use are all weaker in consistency strength than a Woodin cardinal that is limit of Woodin cardinals. This puts them within the region suitable to be reached from current techniques by a Core Model Induction. Moreover, these hypotheses are much weaker than the previously known upper bounds on the strength of (1) and similar theories.

Prior to our work, two methods were known to show the consistency of (1): It is a consequence of $\mathsf{PFA}(\mathfrak{c}^+)$, the restriction of the proper forcing axiom to partial orders of size $\mathfrak{c}^+$ (see Todorcevic [Tod84]), and it can be forced directly from the existence of a *quasicompact* cardinal. Quasicompactness was introduced by Jensen, see Cummings [Cum05] and Jensen [Jena]; Sean Cox (unpublished), and possibly others, observed that the classical argument from James E. Baumgartner [Bau76] obtaining the consistency of "every

stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$ reflects" from weak compactness adapts straightforwardly to this setting.

We expect that the HOD analysis (see Sargsyan [Sara]) should allow us to extend the Core Model Induction to establish the precise consistency strength of (1). The question of whether it is possible to obtain $\mathsf{MM}^{++}(\mathfrak{c}^+)$ or even $\mathsf{PFA}(\mathfrak{c}^+)$ in a $\mathbb{P}_{\max}$ extension of some determinacy model remains open.

Since forcing axioms are connected with *failures* of square principles, we want to suggest some notation to refer to these negations in a positive way, highlighting their *compactness* character, and solving the slight notational inconvenience that refers to the square principle at a cardinal successor κ^+ as $\square_\kappa$, as if it were a property of its predecessor.

Definition 1.6. Let γ be an ordinal. We say that γ **is threadable** if and only if $\square(\gamma)$ fails.

If $\gamma = \lambda^+$ is a successor cardinal, and $\square_\lambda$ fails, we say that γ **is square inaccessible**.

For general background on descriptive set theory, we refer to Kechris [Kec95] and Moschovakis [Mos09]; the latter is also a good reference for basic determinacy results. In addition, for determinacy and Woodin's AD^+ theory, we also refer to Woodin [Woo10], Jackson [Jac10], Caicedo-Ketchersid [CK11], Ketchersid [Ket11], and references therein. Basic knowledge of determinacy will be assumed in what follows.

We also assume some ease with $\mathbb{P}_{\max}$ arguments, although the properties of $\mathbb{P}_{\max}$ that we require could be isolated and treated as black boxes. We refer to Woodin [Woo10] and Larson [Lar10] for background.

2. From HOD to $\mathrm{HOD}_{\mathcal{P}(\mathbb{R})}$

Some of our results use hypotheses on inner models of the form $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$, where $\mathcal{P}_\kappa(\mathbb{R})$ denotes the collection of sets of reals of Wadge rank less than κ. We refer the reader to Kechris [Kec95] for background on the Wadge hierarchy. If A is a set of reals, we let $|A|_W$ denote its Wadge rank.

We show in this section how some of these hypotheses follow from statements about HOD. With the possible exception of Theorem 2.4, none of the results in this section is new. The key technical tool is given by Lemma 2.2.

Recall that Θ is the least ordinal that is not a surjective image of the reals, and that the *Solovay sequence*, introduced by Robert M. Solovay [Sol78], is the unique increasing sequence of ordinals $\langle \theta_\alpha \mid \alpha \le \gamma \rangle$ such that

- θ_0 is the least ordinal that is not the surjective image of the reals by an ordinal definable function;
- for each $\alpha < \gamma$, $\theta_{\alpha+1}$ is the least ordinal that is not the surjective image of the reals by a function definable from an ordinal and a set of reals of Wadge rank θ_α;
- for each limit ordinal $\beta \le \gamma$, $\theta_\beta = \sup\{\theta_\alpha \mid \alpha < \beta\}$; and
- $\theta_\gamma = \Theta$.

Remark 2.1. The proof of Solovay [Sol78, Lemma 0.2] shows that, under AD, whenever γ is an ordinal, and $\phi\colon \mathbb{R} \to \gamma$ is a surjection, there exists a set of reals of Wadge rank γ definable from ϕ. From this it follows that if $|F|_W = \theta_\alpha$, then every set of reals of Wadge rank less than $\theta_{\alpha+1}$ is definable from F, a real and an ordinal. In turn, it follows from this that there is no surjection from $\mathbb{R}$ to $\theta_{\alpha+1}$ ordinal definable from any set of reals of Wadge rank less than $\theta_{\alpha+1}$, and moreover that each θ_α is the Θ of $\mathrm{HOD}_{\mathcal{P}_{\theta_\alpha}(\mathbb{R})}$.

The following is due to Woodin, building on work of Petr Vopěnka (see Jech [Jec03, Theorem 15.46]):

Lemma 2.2 (ZF). *For each ordinal ξ there exists a complete Boolean algebra $\mathbb{B}$ in HOD such that for each $E \subseteq \xi$ there is a HOD-generic filter $H \subseteq \mathbb{B}$ with*

$$\mathrm{HOD}_E = \mathrm{HOD}[H].$$

Furthermore, if AD holds, $\xi < \Theta$ and θ is the least member of the Solovay sequence greater than ξ, then $\mathbb{B}$ can be taken to have cardinality at most θ in V.

Proof. Fix an ordinal γ such that every ordinal definable subset of $\mathcal{P}(\xi)$ is ordinal definable in V_γ. Let $\mathbb{B}_0$ be the following version of the *Vopěnka algebra*: $\mathbb{B}_0$ is the Boolean algebra consisting of all sets of the form

$$A_{\phi,s} = \{x \subseteq \xi \mid \mathrm{V}_\gamma \models \phi(x,s)\},$$

where ϕ is a formula and s is a finite subset of γ, ordered by inclusion.

The relation $A_{\phi,s} = A_{\phi',s'}$ is ordinal definable, and there is an ordinal definable well-ordering of the corresponding equivalence classes. Let η be the length of this well-ordering, let $h\colon \eta \to \mathbb{B}_0$ be the corresponding inverse rank function, and let $\mathbb{B}_1$ be the Boolean algebra with domain η induced by h. Since the relation $A_{\phi,s} \subseteq A_{\phi',s'}$ is ordinal definable, $\mathbb{B}_1$ is in HOD.

Given a filter $G \subseteq \mathbb{B}_0$, let $E(G)$ be the set of $\alpha \in \xi$ such that

$$\{E \subseteq \xi \mid \alpha \in E\} \in G.$$

Then for any $E \subseteq \xi$, $E(\{A \in \mathbb{B}_0 \mid E \in A\}) = E$. According to Vopěnka's Theorem (see Caicedo-Ketchersid [CK11]),

(1) For every $E \subseteq \xi$, $H_E = h^{-1}[\{A \in \mathbb{B}_0 \mid E \in A\}]$ is HOD-generic for $\mathbb{B}_1$;

(2) There exists a $\mathbb{B}_1$-name $\dot{E} \in \text{HOD}$ such that if $H \subseteq \mathbb{B}_1$ is HOD-generic and $G = h[H]$, then $E(G) = \dot{E}_H$.

Now suppose that $H \subseteq \mathbb{B}_1$ is HOD-generic, and let $E = \dot{E}_H$. Let us see first that $\text{HOD}_E \subseteq \text{HOD}[H]$. Suppose that A is a set of ordinals in HOD_E, and fix an ordinal δ such that A and E are both subsets of δ. Then there is an ordinal definable relation $T \subseteq \delta \times \mathcal{P}(\delta)$ such that $A = \{\zeta < \delta \mid T(\zeta, E)\}$. Define the relation T^* on $\delta \times \mathbb{B}_1$ by setting $T^*(\zeta, p)$ if and only if $T(\zeta, D)$ holds for all $D \in h(p)$. Then $T^* \in \text{HOD}$, and A is in $\text{HOD}[H]$, since A is the set of $\zeta < \sup(A)$ such that there exists a p in H for which $T^*(\zeta, p)$ holds.

Now, for any set $E \subseteq \xi$, H_E is in $\text{HOD}_E \subseteq \text{HOD}[H]$, and some condition in $\mathbb{B}_1$ decides whether or not the generic filter H will be equal to $H_{\dot{E}_H}$. However, for any condition $p \in \mathbb{B}_1$, if F is any element of $h(p)$ then $\dot{E}_{H_F} = F$, which means that p cannot force the generic filter H to be different frorm $H_{\dot{E}_H}$. It follows then that H is in HOD_E so $\text{HOD}[H] = \text{HOD}_E$, whenever $H \subseteq \mathbb{B}_1$ is HOD-generic and $E = \dot{E}_H$.

Finally, assume that AD holds, and let θ be as in the statement of the lemma. Then θ is either θ_0 or $\theta_{\alpha+1}$ for some α. Let F be empty if the first case holds, and a set of reals of Wadge rank θ_α otherwise.

Let us see that the cardinality of η is at most θ in V. By Remark 2.1 and the Moschovakis Coding Lemma (see Koellner-Woodin [KW10, Theorem 3.2]) there is a surjection $\pi \colon \mathbb{R} \to \mathcal{P}(\xi)$ definable from ξ and F. If A is an ordinal definable subset of $\mathcal{P}(\xi)$, then $\pi^{-1}[A]$ is ordinal definable from F, which means that $|\pi^{-1}[A]|_W < \theta$, which again by Remark 2.1 implies that A is definable from F, a real and a finite subset of θ. For each fixed finite $a \subset \theta$, the definability order on the sets $A \subseteq \mathcal{P}(\xi)$ definable from F, a, and a real, induces a pre-well- ordering of the reals definable from F and a, which must then have order type less than θ. It follows from this that η also has cardinality at most θ. □

Under AD^+, every set of reals is ordinal definable from a set of ordinals, and in fact this set can be taken to be a bounded subset of Θ (see Woodin [Woo10, Theorem 9.5]). Combining this fact with Lemma 2.2 gives the following folklore result.

Theorem 2.3. *Assuming* AD^+, *if* θ *is a member of the Solovay sequence and* θ *is regular in* HOD, *then* θ *is regular in* $\text{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.

Proof. If $\theta = \theta_0$ or θ is a successor member of the Solovay sequence, there is a set of reals A from which Wadge-cofinally many sets of reals are ordinal definable. It follows then that θ is regular in $\text{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ without any assumption on HOD, since for each function $f \colon \mathbb{R} \to \Theta$ there is a surjection from $\mathbb{R}$ to $\sup(f[\mathbb{R}])$ definable from A and f.

Now suppose that θ is a limit in the Solovay sequence, and that $f\colon \alpha \to \theta$ is a cofinal function in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$, for some $\alpha < \theta$. Then f is ordinal definable from some set of reals in $\mathcal{P}_\theta(\mathbb{R})$ that itself is ordinal definable from a bounded subset A of θ. Pick $\theta_\xi < \theta$ such that A is bounded in θ_ξ. By Lemma 2.2, there is a set H, generic over HOD via a partial order of cardinality less than $\theta_{\xi+1}$ in HOD, and such that $\mathrm{HOD}_A \subseteq \mathrm{HOD}[H]$. Then $f \in \mathrm{HOD}[H]$. By cardinality considerations, the regularity of θ in HOD is preserved in $\mathrm{HOD}[H]$, giving a contradiction. $\square$

With a little more work, one gets Theorem 2.4 below. Note that if AD holds and Θ is a limit ordinal in the Solovay sequence then $\mathsf{AD}_\mathbb{R}$ holds – for instance, this is Woodin [Woo10, Theorem 9.24], modulo the fact that $\mathsf{AD}_\mathbb{R}$ reflects to $L(\mathcal{P}(\mathbb{R}))$.

Given a cardinal θ, a filter on θ is θ-*complete* if it is closed under intersections of cardinality less than θ, and $\mathbb{R}$-*complete* if it is closed under intersections indexed by $\mathbb{R}$.

Theorem 2.4. *Assume that* AD *holds, and that θ is a limit on the Solovay sequence. Let F be a θ-complete filter on θ in* HOD, *and let F' be the set of elements of $\mathcal{P}(\theta) \cap \mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ containing some member of F. Then the following hold:*

  (1) The filter F' is $\mathbb{R}$-complete in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.
  (2) If F is an ultrafilter in HOD, *then F' is an ultrafilter in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.*
Moreoever, every θ-complete filter on θ in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ is $\mathbb{R}$-complete.

Proof. Every element of $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ is ordinal definable from a set of reals of Wadge rank less than θ, and thus (since every set of reals is Suslin in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$) from a bounded subset of θ. By Lemma 2.2, every bounded subset of θ exists in a generic extension of HOD by a partial order of cardinality less than θ in HOD. The second conclusion of the lemma follows, as well as the fact that F' is θ-complete in $\mathrm{HOD}_{\mathcal{P}(\mathbb{R})}$. It suffices then to prove the last part of the theorem.

Let F' be a θ-complete filter on θ, and fix $G\colon \mathbb{R} \to F'$ in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$. For each $\alpha < \theta$, let $B_\alpha = \{x \in \mathbb{R} \mid \alpha \in G(x)\}$. Then $\bar{B} = \langle B_\alpha \mid \alpha < \theta \rangle$ is ordinal definable from a set of reals of Wadge rank less than θ.

Since $\mathsf{AD}_\mathbb{R}$ holds in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$, and θ is the Θ of this model (see Remark 2.1), there is no θ-sequence in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ consisting of sets of reals of Wadge rank unbounded in θ, since this would mean that there is one set of reals A such that every set of reals is ordinal definable from A and real, contradicting Uniformization.

On the other hand, if unboundedly many of the B_α were to be distinct sets with Wadge rank below some fixed set of reals, then one could define from this situation a pre-well-ordering of length θ, which is impossible. So $\bar{B}$ must contain fewer than θ many distinct sets.

Suppose that $B \subseteq \mathbb{R}$ is such that $\{\alpha < \theta \mid B_\alpha = B\}$ is F'- positive. For each $x \in \mathbb{R}$, $G(x) \in F'$, so there is an $\alpha < \theta$ for which $x \in B_\alpha$ and $B_\alpha = B$.

It follows that $B = \mathbb{R}$. Since F' is θ-complete,

$$\{\alpha \mid B_\alpha = \mathbb{R}\} \in F'.$$

Since $\bigcap_{x \in \mathbb{R}} G(x) = \{\alpha \mid B_\alpha = \mathbb{R}\}$, we are done. $\qquad\square$

3. SQUARE IN $\mathbb{P}_{\max}$ EXTENSIONS OF WEAK MODELS OF DETERMINACY

We show that square principles at ω_3 do hold sometimes in $\mathbb{P}_{\max}$ extensions. In this section, we outline a proof of Theorem 1.5, showing that, in order to obtain negative results, we must begin with strong assumptions. We restate the theorem, for the reader's convenience.

Theorem. *Assume* $\mathsf{AD}_{\mathbb{R}}$ + *"Θ is regular" and that there is no* $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ *such that* $L(\Gamma, \mathbb{R}) \models$ *"Θ is Mahlo in* HOD*". Then* $\square_{\omega_2}$ *holds in the* $\mathbb{P}_{\max}$ *extension.*

The heart of the construction takes place in HOD and is a straightforward combination of standard constructions of square sequences as developed in Jensen [Jen72], Schimmerling-Zeman [SZ04], and Zeman [Zem10], adapted to the context of strategic extender models as developed in Sargsyan [Sara]. In order to stay close to the constructions in Schimmerling-Zeman [SZ04] and Zeman [Zem10], we use fine structure notation and terminology as in Zeman [Zem02]; the rest of the notation and terminology is consistent with that in Mitchell-Steel [MS94], Steel [Ste96], and Sargsyan [Sara].

Proof. The key technical tool is a condensation lemma for initial segments of HOD which can be proved using the standard argument modified to the strategic extender models from Sargsyan [Sara]. Unlike the square constructions in Schimmerling-Zeman [SZ04] and Zeman [Zem10], our situation is specific in the sense that the initial segments of HOD used for the definition of the elements of our square sequence are never pluripotent (see Schimmerling-Zeman [SZ04] and Zeman [Zem10]), that is, they do not give rise to protomice. This makes it possible to run the construction without analysis of extender fragments, so the construction does not differ too much from that in L. The reason why this is the case is a consequence of the following corollary of our smallness assumption, see Sargsyan [Sara]:

(2)
If $\theta_\alpha < \Theta$, then θ_α is not overlapped by an extender on the HOD-sequence.

We now formulate the condensation lemma. Recall that if $\theta_\alpha < \Theta$, then Σ_α is the iteration strategy for $\text{HOD} \,|\, \theta_\alpha$ in HOD.

Lemma 3.1. *Assume* $\mathcal{N}$ *is an initial segment of* HOD. *Let* $\theta_\alpha < \Theta$, *and let* $\mathcal{M}$ *be a sound* Σ_α-*premouse such that* $\rho_{\mathcal{M}}^{n+1} = \theta_\alpha$. *Let finally* $\sigma : \mathcal{M} \to \mathcal{N}$ *be a* $\Sigma_0^{(n)}$-*preserving map with critical point* θ_α, $\sigma(\theta_\alpha) = \theta_\beta$, *and* $\sigma \in$ HOD. *Then* $\mathcal{M}$ *is an initial segment of* HOD.

Proof. (Sketch.) Since $\sigma \restriction \theta_\alpha = \mathsf{id}$, the model $\mathcal{M}$ agrees with HOD below θ_α.

By (2), all critical points of the iteration tree on the $\mathcal{M}$-side of the comparison of $\mathcal{M}$ against HOD are strictly larger than θ_α. The preservation degree of σ guarantees that $\mathcal{M}$ is iterable when using extenders with critical points larger than θ_α, so $\mathcal{M}$ can be compared with HOD. By the theory developed in Sargsyan [Sara], HOD wins the comparison against $\mathcal{M}$, the assumption $\sigma \in$ HOD is used here. (We sketch the argument below, using freely notation and results form Sargsyan [Sara].)

But then $\mathcal{M}$ is not moved in the coiteration, as it projects to θ_α, is sound, and all critical points on the $\mathcal{M}$-side are larger than θ_α. This gives the result. This is an instance of a more general result from Sargsyan [Sara], namely, that HOD thinks that it is *full*: Letting λ^{HOD} denote the order type of the set of Woodin cardinals in HOD and their limits, if $\alpha < \lambda^{\mathrm{HOD}}$, and $\eta \in [\theta_\alpha, \theta_{\alpha+1})$ is a cutpoint, then any sound Σ_α-mouse $\mathcal{M}$ over HOD$\,|\,\eta$ with $\rho(\mathcal{M}) = \eta$, *is* an initial segment of HOD.

To see that HOD wins the comparison, let Σ be the strategy of $\mathcal{N}$, so Σ respects Σ_β, the iteration strategy for HOD$\,|\,\theta_\beta$ in HOD. By the arguments of Sargsyan [Sara], all we need to check is that Σ^σ respects Σ_α. This follows from hull condensation.

$$(3) \qquad \begin{array}{ccc} \mathcal{N} & \xrightarrow{\ \mathcal{T}^\sigma\ } & \mathcal{N}' \\ {\scriptstyle\sigma}\big\uparrow & & \big\uparrow{\scriptstyle\sigma'} \\ \mathcal{M} & \xrightarrow[\ \mathcal{T}\ ]{} & \mathcal{M}' \end{array}$$

To see this, suppose that $\mathcal{T}$ is the tree arising on the $\mathcal{M}$-side of the comparison, as in diagram (3), so $\mathcal{T}$ has critical point above θ_α. Let $\mathcal{U}$ be on the sequence of $\mathcal{M}'$. We must argue that $\mathcal{U}$ is according to Σ_α. We use σ' to produce $\sigma'(\mathcal{U})$, and note that the pointwise image $\sigma'[\mathcal{U}]$ is a hull of $\sigma'(\mathcal{U})$. But then $\mathcal{U}$ is also a hull of $\sigma'(\mathcal{U})$, and since $\sigma \in$ HOD, so is $\sigma' \in$ HOD. This allows us to invoke hull condensation, as claimed. □

Note that, by assumption, $\Theta = \theta_\Omega$ for some Ω that is inaccessible but not Mahlo in HOD.

We modify the construction in Zeman [Zem10] to the current context to obtain a global square sequence below Θ in HOD of the form $\langle C_\tau \mid \tau \in C^* \rangle$ where $C^* \in$ HOD is a closed subset of Θ fixed in advance such that

$$\theta_\tau = \tau \text{ and } \tau \text{ is singular in HOD whenever } \tau \in C^*.$$

The sequence $\langle C_\tau \mid \tau \in C^* \rangle$ will have the following properties:
   (a) Each C_τ is a club subset of τ,
   (b) $C_{\bar\tau} = C_\tau \cap \bar\tau$ whenever $\bar\tau$ is a limit point of τ,
   (c) $\mathsf{otp}(C_\tau) < \tau$, and
   (d) $C_\tau \subseteq C^*$ unless $\mathsf{otp}(C_\tau) = \omega$.

By construction, $\mathsf{otp}(C_\tau)$ may be larger than ω_2. In the $\mathbb{P}_{\max}$ extension, we have $\Theta = \omega_3$, so it is possible to extend the sequence $\langle C_\tau \mid \tau \in C^* \rangle$ to all limit ordinals in the interval (ω_2, ω_3) such that (a)–(c) still hold. For auxiliary purposes, set $C_\tau = \tau$ whenever $\tau \leq \omega_2$ is a limit ordinal. It is then easy to verify that the following recursive construction from Jensen [Jen72] turns the sequence $\langle C_\tau \mid \tau \in \lim \cap \omega_3 \rangle$ into a $\square_{\omega_2}$-sequence $\langle c_\tau \mid \tau \in \lim \cap \omega_3 \rangle$. Let $\pi_\tau : \mathsf{otp}(C_\tau) \to C_\tau$ be the unique order isomorphism. Then set

$$c_\tau = \pi_\tau[c_{\mathsf{otp}(C_\tau)}].$$

In the rest of this section, we describe the modifications to the construction in Zeman [Zem10] that yield the sequence $\langle C_\tau \mid \tau \in C^* \rangle$. The point of our description is to separate aspects of the construction that can be accomplished by abstract fine structural considerations from those that are specific to strategic extender models. By "abstract fine structural considerations", we mean here methods within the framework of Zeman [Zem02, Chapter 1]. Our notation is consistent with that in Zeman [Zem10] with two exceptions: First, in our case Θ plays the same role the class of ordinals plays in Zeman [Zem10], and second, the sets C_τ from Zeman [Zem10] do not correspond to the sets denoted by C_τ here.

The construction in Zeman [Zem10] is carried out separately on two disjoint sets $\mathcal{S}^0, \mathcal{S}^1$. The class $\mathcal{S}^1$ consists of all those ordinals for which the singularizing structure is a protomouse. In our case, protomice do not arise in the construction due to (2), so we have $\mathcal{S}^1 = \varnothing$. This greatly simplifies the situation, as the verification that $C_\tau \subseteq \mathcal{S}^i$ whenever $\tau \in \mathcal{S}^i$ $(i = 0, 1)$ and $\mathsf{otp}(C_\tau) > \omega$, involved a substantial amount of work in Zeman [Zem10]. Thus we will refer only to the portion of the construction in Zeman [Zem10] that concerns the set $\mathcal{S}^0$.

To each $\tau \in C^*$, we assign the singularizing level of HOD for τ, which we denote by $\mathcal{N}_\tau$. We then define the auxiliary objects for $\mathcal{N}_\tau$ exactly as in Zeman [Zem10]:

- By $\tilde{h}_\tau^k$, we denote the uniform $\Sigma_1^{(k-1)}$-Skolem function for $\mathcal{N}_\tau$; this is a partial function from $\omega \times \mathcal{N}_\tau$ into $\mathcal{N}_\tau$.
- We write $\tilde{h}_\tau^k(\gamma \cup \{p\})$ to denote the set of all values $\tilde{h}_\tau^k(i, \langle \xi, p \rangle)$, where $i \in \omega$ and $\xi < \gamma$.
- We let p_τ be the standard parameter of $\mathcal{N}_\tau$.
- We let n_τ be the complexity degree of a singularizing function for τ over $\mathcal{N}_\tau$, or equivalently the least n such that $\tilde{h}_\tau^{n+1}(\gamma \cup \{p_\tau\})$ is cofinal in τ for some $\gamma < \tau$.
- $\tilde{h}_\tau = \tilde{h}_\tau^{n_\tau+1}$, and
- We let α_τ be the largest $\alpha < \tau$ such that $\tilde{h}_\tau(\alpha \cup \{p_\tau\}) \cap \tau = \alpha$.

We then define the sets B_τ, which are the first approximations to C_τ, analogously as in Zeman [Zem10]. Recall that B_τ may be bounded in τ if τ is countably cofinal and even empty, but on the other hand B_τ will be "almost" coherent also at successor points.

To be precise: Recall that an ordinal ζ is in p_τ if and only if some generalized solidity witness for ζ with respect to $\mathcal{N}_\tau$ and p_τ is an element of $\mathcal{N}_\tau$, and the standard solidity witness for ζ can be reconstructed from any generalized solidity witness for ζ inside $\mathcal{N}_\tau$. Here, by the *standard* solidity witness, we mean the transitive collapse of the hull $\tilde{h}_\tau^{k+1}(\zeta \cup \{p_\tau - (\zeta+1)\})$, where $\rho_{\mathcal{N}_\tau}^{k+1} \leq \zeta < \rho_{\mathcal{N}_\tau}^k$.

Now let $\bar{\tau} \in B_\tau$ if and only if the following hold:

(1) $\mathcal{N}_{\bar{\tau}}$ is a hod-premouse of the same type as $\mathcal{N}_\tau$,
(2) $n_{\bar{\tau}} = n_\tau$, and
(3) There is a $\Sigma_1^{(n)}$-preserving embedding $\sigma : \mathcal{N}_{\bar{\tau}} \to \mathcal{N}_\tau$ such that:
   - $\sigma \upharpoonright \bar{\tau} = \mathsf{id}$,
   - $\sigma(\bar{\tau}) = \tau$ if $\bar{\tau} \in \mathcal{N}_{\bar{\tau}}$,
   - $\sigma(p_{\bar{\tau}}) = p_\tau$ and,
   - For each $\xi \in p_\tau$, there is some generalized solidity witness Q for ξ with respect to $\mathcal{N}_\tau$ and p_τ, such that $Q \in \mathrm{rng}(\sigma)$.
(4) $\alpha_{\bar{\tau}} = \alpha_\tau$.

The following facts are proved by means of abstract fine structure theory, and the proofs look exactly as the corresponding proofs in Zeman [Zem10]:

(A) σ is unique, and we denote it by $\sigma_{\bar{\tau},\tau}$.
(B) $\sigma_{\bar{\tau},\tau}[\omega\rho_{\mathcal{N}_{\bar{\tau}}}^{n_{\bar{\tau}}}]$ is bounded in $\omega\rho_{\mathcal{N}_\tau}^{n_\tau}$.
(C) If $\tau^* < \bar{\tau}$ are in B_τ then there is a unique map $\sigma_{\tau^*,\bar{\tau}} : \mathcal{N}_{\tau^*} \to \mathcal{N}_{\bar{\tau}}$ with the list of properties stipulated above, and with $(\tau^*, \bar{\tau})$ in place of $(\bar{\tau}, \tau)$.
(D) If $\tau^* < \bar{\tau} < \tau'$ are in $B_\tau \cup \{\tau\}$, then $\sigma_{\bar{\tau},\tau'} \circ \sigma_{\tau^*,\bar{\tau}} = \sigma_{\tau^*,\tau'}$.
(E) If $\bar{\tau} \in B_\tau$, then $B_\tau \cap \bar{\tau} = B_{\bar{\tau}} - \min(B_\tau)$.

The proof of unboundedness of B_τ, for τ of uncountable cofinality, is similar as that in Zeman [Zem10], but uses the Condensation Lemma 3.1 where the argument from Zeman [Zem10] used condensation for $L[E]$-models:

For λ regular and sufficiently large, given $\tau' < \tau$, and working in HOD, we construct a countable elementary substructure $X \prec H_\lambda$, such that

$$\mathcal{N}_\tau, \tau', C^* \in X.$$

Letting $\tilde{\tau} = \sup(X \cap \tau)$, notice that $\tau' < \tilde{\tau} < \tau$ and $\tilde{\tau} \in C^*$, as C^* is closed. Let $\bar{\mathcal{N}}$ be the transitive collapse of $\mathcal{N}_\tau$, and let $\sigma : \bar{\mathcal{N}} \to \mathcal{N}_\tau$ be the inverse of the collapsing isomorphism. Set $\tilde{\mathcal{N}} = \mathsf{Ult}^{n_\tau}(\bar{\mathcal{N}}, \sigma \upharpoonright (\bar{\mathcal{N}} \,|\, \bar{\tau}))$, where $\sigma(\bar{\tau}) = \tau$, and let $\sigma' : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$ be the factor map, that is, $\sigma' \circ \tilde{\sigma} = \sigma$, and $\sigma' \upharpoonright \tilde{\tau} = \mathsf{id}$.

Exactly as in Zeman [Zem10], one can show the following facts by means of abstract fine structural considerations:

(F) $\tilde{\sigma}$ is $\Sigma_0^{(n)}$-preserving, and maps $\omega\rho_{\bar{\mathcal{N}}}^{n_\tau}$ cofinally into $\omega\rho_{\tilde{\mathcal{N}}}^{n_\tau}$. Similarly, σ' is $\Sigma_0^{(n)}$-preserving, but maps $\omega\rho_{\tilde{\mathcal{N}}}^{n_\tau}$ boundedly into $\omega\rho_{\tilde{\mathcal{N}}}^{n_\tau}$.
(G) $\tilde{\mathcal{N}}$ is a sound and solid hod-premouse.
(H) $\tilde{\mathcal{N}}$ is a singularizing structure for $\tilde{\tau}$ with singularization degree n_τ.

(I) $\sigma'(\tilde{\tau}) = \tau$, $\sigma'(p_{\tilde{\mathcal{N}}}) = p_\tau$, and α_τ is the largest $\alpha < \tilde{\tau}$ satisfying

$$\tilde{h}_{\tilde{\mathcal{N}}}^{n_\tau + 1}(\alpha \cup \{p_{\tilde{\mathcal{N}}}\}) \cap \tilde{\tau} = \alpha.$$

Since the entire construction took place inside HOD, Lemma 3.1 can be applied to the map $\sigma' : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$. The rest follows again by abstract fine structural considerations, literally as in Zeman [Zem10]. In particular, these considerations can be used to show that $\tilde{\mathcal{N}} = \mathcal{N}_{\tilde{\tau}}$ and $\sigma'(p_{\tilde{\mathcal{N}}}) = p_\tau$, and hence $\sigma' = \sigma_{\tilde{\tau},\tau}$. Additionally, $\alpha_{\tilde{\tau}} = \alpha_\tau$. This shows that $\tilde{\tau} \in B_\tau$.

In Zeman [Zem10] it is only proved that B_τ is closed on a tail-end; this was again caused by the fact that one has to consider protomice. Here, we prove that B_τ is itself closed:

Given a limit point $\tilde{\tau}$ of B_τ, notice first that $\tilde{\tau} \in C^*$, let $\tilde{\mathcal{N}}$ be the direct limit of $\langle \mathcal{N}_{\bar{\tau}}, \sigma_{\tau^*, \bar{\tau}} \mid \tau^* < \bar{\tau} < \tilde{\tau} \rangle$, and let $\sigma' : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$ be the direct limit map. Again, by abstract fine structural considerations that are essentially identical to those in Zeman [Zem10], we establish the conclusions analogous to (F)–(I) for the current version of $\tilde{\mathcal{N}}$ and σ', and then as above apply Lemma 3.1 to $\sigma' : \tilde{\mathcal{N}} \to \mathcal{N}_\tau$, and conclude that $\tilde{\mathcal{N}}$ is an initial segment of HOD. As above, we then conclude that $\tilde{\mathcal{N}} = \mathcal{N}_{\tilde{\tau}}$ and $\sigma' = \sigma_{\tilde{\tau},\tau}$.

Having established closedness and unboundedness of B_τ, we follow the construction from Zeman [Zem10], and obtain fully coherent sets B_τ^* by "stacking" the sets B_τ; so

$$B_\tau^* = B_{\tau_0} \cup B_{\tau_1} \cup \cdots \cup B_{\tau_{\ell_\tau}},$$

where $\tau_{i+1} = \min(B_{\tau_i})$, and ℓ_τ is the least ℓ such that $B_\ell = \varnothing$. We then define C_τ^* as the set of all ordinals τ_ι, defined inductively as follows:

$$
\begin{aligned}
\tau_0 &= \min(B_\tau), \\
\xi_\iota^\tau &= \text{the least } \xi < \tau \text{ such that } \tilde{h}_\tau(\{\xi\} \cup \{p_\tau\}) \not\subseteq \mathrm{rng}(\sigma_{\tau,\tau}), \\
\tau_{\iota+1} &= \text{the least } \bar{\tau} < \tau \text{ such that } \tilde{h}_\tau(\{\xi_\iota^\tau\} \cup \{p_\tau\}) \subseteq \mathrm{rng}(\sigma_{\bar{\tau},\tau}), \\
\tau_\iota &= \sup\{\tau_{\bar{\iota}} \mid \bar{\iota} < \iota\} \text{ for limit } \iota.
\end{aligned}
$$

The proof that the C_τ^* are fully coherent, unbounded in τ whenever τ has uncountable cofinality in HOD, and $\mathsf{otp}(C_\tau^*) < \tau$, can be carried out using abstract fine structural considerations, and is essentially the same as the argument in Zeman [Zem10].

Finally, we let $C_\tau = \lim(C_\tau^*)$ whenever $\lim(C_\tau^*)$ is unbounded in τ, and otherwise let C_τ be some randomly chosen cofinal ω-sequence in τ. (Recall again that here our notation diverges from that in Zeman [Zem10], where C_τ denoted sets that are fully coherent but not necessarily cofinal at countably cofinal τ.) $\qquad\square$

Remark 3.2. Above, we worked in the theory $\mathsf{AD}_\mathbb{R} + \Theta$ is regular, as it directly relates to our negative results but, as long as we are in the situation where there is no $\Gamma \subseteq \mathcal{P}(\mathbb{R})$ such that $L(\Gamma, \mathbb{R}) \models$ "Θ is Mahlo in HOD", Theorem 1.5 holds assuming only AD^+.

We briefly sketch the additional arguments. We need to consider two cases: Suppose first that $\Theta = \theta_\Omega$ for some singular limit ordinal Ω. In this case, $\Theta < \omega_3$ in the $\mathbb{P}_{\max}$-extension, and the construction of the $\square_{\omega_2}$-sequence takes place essentially above Θ and is almost literally the same as that in L.

Finally, if $\Theta = \theta_{\Omega+1}$ for some ordinal Ω, one can use that
    (a) Models that admit HOD-analysis are of the form $\mathbf{K}^\Sigma(\mathbb{R})$, and
    (b) Under our minimality assumption, the HOD-analysis applies to our V.

That (a) and (b) hold follows from unpublished arguments by Sargsyan and Steel on "capturing by $\mathbb{R}$-mice", and we omit the details. Assuming that this is the case, let G be $\mathbb{P}_{\max}$-generic over V. By the S-construction from Sargsyan [Sara], the model $\mathbf{K}^\Sigma(\mathbb{R})[G]$ can be rearranged into the form $\mathbf{K}^\Sigma(\mathbb{R}, G)$. Since G well-orders $\mathbb{R}$ in order type ω_2, there is $A \subseteq \omega_2$ such that $\mathbf{K}^\Sigma(\mathbb{R}, G) = \mathsf{Lp}(A)$. The standard construction of the canonical $\square_{\omega_2}$-sequence in $\mathsf{Lp}(A)$ thus yields a $\square_{\omega_2}$-sequence in the $\mathbb{P}_{\max}$-extension.

4. Weak squares in $\mathbb{P}_{\max}$ extensions

In this section we prove that there is a limit to how far $\mathbb{P}_{\max}$ arguments can reach on the extent of squares. We show that the weak square principle $\square_{\omega_2, \omega_2}$ always holds in $\mathbb{P}_{\max}$ extensions of models of $\mathsf{AD}^+ +$ "Θ is regular". As in Section 3, we emphasize the case where $\mathsf{AD}_\mathbb{R}$ holds. Recall:

Definition 4.1. Given cardinals κ and λ, the principle $\square_{\kappa, \lambda}$ says that there exists a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \kappa^+ \rangle$ of nonempty sets such that for each $\alpha < \kappa^+$,
    (1) $|\mathcal{C}_\alpha| \leq \lambda$;
    (2) Each element of $\mathcal{C}_\alpha$ is club in α, and has order type at most κ; and
    (3) For each member C of $\mathcal{C}_\alpha$, and each limit point β of C, $C \cap \beta \in \mathcal{C}_\beta$.

We call a sequence witnessing the above principle a $\square_{\kappa, \lambda}$-sequence. As shown in Cummings-Magidor [CM11], it is consistent with Martin's Maximum that $\square_{\omega_2, \omega_2}$ holds.

Before stating our result, we recall a result of Woodin that will prove very useful in what follows. The notion of A-*iterability* for A a set of reals, crucial in the theory of $\mathbb{P}_{\max}$, is introduced in Woodin [Woo10, Definition 3.30]. For $X \prec H(\omega_2)$, Woodin denotes by M_X its transitive collapse. Woodin [Woo10, §3.1] presents a series of covering theorems for $\mathbb{P}_{\max}$ extensions. In particular, we have:

Theorem 4.2 (Woodin [Woo10, Theorem 3.45]). *Suppose that M is a proper class inner model that contains all the reals and satisfies* $\mathsf{AD} + \mathsf{DC}$. *Suppose that for any $A \in \mathcal{P}(\mathbb{R}) \cap M$, the set*

$$\{X \prec H(\omega_2) \mid X \text{ is countable, and } M_X \text{ is } A\text{-iterable}\}$$

is stationary. Let X in V be a bounded subset of Θ^M of size ω_1. Then there is a set $Y \in M$, of size ω_1 in M, and such that $X \subseteq Y$.

Typically, we apply this result as follows: We start with $M = L(\mathcal{P}(\mathbb{R}))$, a model of $\mathsf{AD}^+ + \mathsf{DC}$, and force with $\mathbb{P}_{\max}$ to produce an extension $M[G]$. The technical stationarity assumption is then true in $M[G]$ by virtue of Woodin [Woo10, Theorem 9.32], and we can then apply Theorem 4.2 in this setting.

Theorem 4.3. *Assume that* $V \models \mathsf{AD}_{\mathbb{R}} + $ *"Θ is regular". Let* $G * H$ *be* $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)$*-generic over* V*. Then* $\square_{\omega_2, \omega_2}$ *holds in* $V[G][H]$*.*

Proof. It follows from the assumption that $V \neq L(A, \mathbb{R})$ for any $A \subseteq \mathbb{R}$. Work in $V[G][H]$. For $\alpha < \Theta^V = \omega_3^{V[G][H]}$, let $A \in V$ be a set of reals of Wadge rank α, and define $\mathcal{C}_\alpha$ as the collection of all club subsets of α of order type at most ω_2 that belong to $L(A, \mathbb{R})[G]$. Note that this definition is independent of the representative A that we choose. Then each $\mathcal{C}_\alpha$ is nonempty, and

$$|\mathcal{P}(\alpha) \cap L(A, \mathbb{R})[G]| \leq \omega_2$$

in $V[G][H]$.

We claim that $\langle \mathcal{C}_\alpha \mid \alpha \in \omega_3 \rangle$ is a $\square_{\omega_2, \omega_2}$-sequence, for which only coherence needs to be verified. To check this, suppose that $\alpha < \beta < \omega_3$, that $A, B \in \mathcal{P}(\mathbb{R}) \cap V$, and that $|A|_W = \alpha$ and $|B|_W = \beta$. Then $A \in L(B, \mathbb{R})$, so $L(A, \mathbb{R})[G] \subseteq L(B, \mathbb{R})[G]$. Therefore, if $C \in C_\beta$ and $\alpha \in \lim(C)$, then $\mathrm{cf}(\alpha) \leq \omega_1$ (in $L(B, \mathbb{R})[G]$).

Using now the Covering Theorem 4.2, we see that $|C \cap \alpha| \leq \aleph_1$, so there is a $D \in V$ with $C \cap \alpha \subseteq D$ and $|D| \leq \aleph_1$ in V. By the Coding Lemma, $D \in L(A, \mathbb{R})$ and $|D| \leq \aleph_1$ in $L(A, \mathbb{R})$. Therefore, since $\mathcal{P}(\omega_1) \subseteq L(\mathbb{R})[G]$, we see that $C \cap \alpha \in L(A, \mathbb{R})[G]$, so $C \cap \alpha \in \mathcal{C}_\alpha$. $\qquad\square$

Remark 4.4. Note that if AD^+ holds and Θ is regular, then either we are in the situation above, or else Θ is a successor in the Solovay sequence. In the latter case, we can modify the above construction by considering at each α a pre-well-ordering of length α, and letting $\mathcal{C}_\alpha$ consist of the realizations in $V[G][H]$ of all those names in V that (in the codes) are inductive in A, in the sense of Moschovakis [Mos09, §7.C]. The proof above then gives us that $\square_{\omega_2, \omega_2}$ also holds in $V[G][H]$ in this case.

5. Choiceless extensions where square fails

In this section we present two results showing that ω_3 is square inaccessible, or even threadable, in the $\mathbb{P}_{\max}$ extension of suitable models of determinacy (see Definition 1.6). Missing from these results is an argument that the subsequent forcing $\mathrm{Add}(\omega_3, 1)$ that adds a Cohen subset of ω_3, and in the process well-orders $\mathcal{P}(\mathbb{R})$, does not ("accidentally") add a square sequence. This issue is addressed in Section 6.

The existence of a model M_1 as in the following theorem follows from $\mathsf{AD}_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R})) + $ "Θ is Mahlo in HOD".

Theorem 5.1. *Assume that* AD^+ *holds and that* θ *is a limit on the Solovay sequence such that that there are cofinally many* $\kappa < \theta$ *that are limits of the*

Solovay sequence and are regular in HOD. *Then ω_3 is square inaccessible in the $\mathbb{P}_{\max}$ extension of* $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.

Proof. Fix σ a $\mathbb{P}_{\max}$ name in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ for a $\square_{\omega_2}$-sequence. Fix $\kappa < \theta$, a limit of the Solovay sequence and regular in HOD, such that σ is ordinal definable from some $A \in \mathcal{P}_\kappa(\mathbb{R})$.

By Theorem 2.3, κ is regular in $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$. The forcing relation for the κ-th member of the $\square_{\omega_2}$-sequence with name σ,

$$\{(p, \alpha) \mid p \Vdash \alpha \in \dot{C}_\kappa\},$$

is definable from σ and κ, and therefore belongs to $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$.

Then the interpretation C_κ of this κ-th club belongs to the $\mathbb{P}_{\max}$ extension of $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$. But this is a contradiction, since, on the one hand, C_κ is club in κ and has order type at most ω_2, which is less than κ, and, on the other hand, the regularity of κ is preserved in the $\mathbb{P}_{\max}$ extension of $\mathrm{HOD}_{\mathcal{P}_\kappa(\mathbb{R})}$, as $|\mathbb{P}_{\max}| = \mathfrak{c}$. $\square$

Question 5.2. Does forcing with $\mathrm{Add}(\Theta, 1)$ over the model $V[G]$ in Theorem 5.1 force $\square_{\omega_2}$, or possibly $\square(\omega_3)$?

Recall that a cardinal θ is *weakly compact* if and only if any $f \colon [\theta]^2 \to 2$ is constant on a set of the form $[A]^2$ for some $A \in [\theta]^\theta$. Equivalently, θ is weakly compact if and only if for every $S \subset \mathcal{P}(\theta)$ of cardinality θ there is a θ-complete filter F on θ such that $\{A, \theta \setminus A\} \cap F \neq \emptyset$ for each $A \in S$. It follows easily from this formulation that θ is threadable if it is weakly compact.

Theorem 5.3. *Assume that $\mathsf{AD}_{\mathbb{R}}$ holds, and that θ is a limit on the Solovay sequence, and weakly compact in* HOD. *Then θ remains weakly compact in the $\mathbb{P}_{\max}$ extension of* $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.

Proof. Let τ be a $\mathbb{P}_{\max}$ name in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ for a collection of θ many subsets of θ. Then τ is definable in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ from a set of reals that is itself definable from a bounded subset S of θ. For each $\mathbb{P}_{\max}$ condition p and each ordinal $\alpha < \theta$, let $A_{p,\alpha}$ be the set of ordinals forced by p to be in the α-th set represented by τ.

Fix an ordinal $\delta < \theta$ such that $S \subseteq \delta$, and let

$$C \colon \mathcal{P}(\omega) \times \mathcal{P}(\delta) \to \mathcal{P}(\omega + \delta)$$

be defined by letting $C(a, b) \cap \omega = a$, and, for all $\gamma < \delta$, $\omega + \gamma \in C(a, b)$ if and only if $\gamma \in b$. Fix also a coding of elements of $H(\aleph_1)$ by subsets of ω.

By Lemma 2.2, there is a partial order Q of cardinality less than θ in HOD such that for each $a \subseteq \omega$, $C(a, S)$ is HOD-generic for Q. The proof of Lemma 2.2 shows that there is a sequence $\langle \sigma_\alpha \mid \alpha < \theta \rangle$ in HOD, consisting of Q-names, such that for each $\alpha < \theta$ and each $a \subseteq \omega$ coding some $p \in \mathbb{P}_{\max}$, the realization of σ_α by $C(a, S)$ is the set $A_{p,\alpha}$.

To see this, following the argument from Lemma 2.2, let T be an ordinal definable relation on $\mathcal{P}(\omega + \delta) \times \theta \times \theta$ such that for all $a \subseteq \omega$ coding $p \in \mathbb{P}_{\max}$

and all $(\alpha, \beta) \in \theta \times \theta$, we have that $\beta \in A_{p,\alpha}$ if and only if $T(C(a, S), \alpha, \beta)$. Define T^* on $\theta \times \theta \times Q$ by letting $T^*(\alpha, \beta, q)$ hold if and only if $T(D, \alpha, \beta)$ holds for all $D \in h(q)$, where h is the function from the proof of Lemma 2.2. Then T^* is (essentially) the desired sequence $\langle \sigma_\alpha : \alpha < \theta \rangle$.

Applying the weak compactness of θ, we can find in HOD a θ-complete filter F on θ, such that, for each $\alpha < \theta$ and each $q \in Q$, F contains either the set $\{\beta < \theta \mid q \Vdash \check{\beta} \in \sigma_\alpha\}$, or its complement. Since $|Q| < \theta$ in HOD, for each $p \in \mathbb{P}_{\max}$ and each $\alpha < \theta$, either $A_{p,\alpha}$ or its complement is in F.By Theorem 2.4, the filter generated by F is $\mathbb{R}$-complete in $\text{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$. It follows then that F measures all the sets in the realization of τ in the $\mathbb{P}_{\max}$ extension. $\quad\square$

6. Forcing the square inaccessibility of ω_3

A *partial $\square_\kappa$-sequence* is a sequence $\langle C_\alpha \mid \alpha \in A \rangle$ (for A a subset of κ^+) satisfying the three conditions in Definition 1.1 for each $\alpha \in A$. Note that condition (2) implies that $\beta \in A$ whenever $\alpha \in A$ and β is a limit point of C_α.

Theorem 6.1 below is the main result of this paper. The partial order $\text{Add}(\omega_3, 1)$ adds a subset of ω_3 by initial segments. When $\mathfrak{c} = \aleph_2$, as in a $\mathbb{P}_{\max}$ extension, $\text{Add}(\omega_3, 1)$ well-orders $\mathcal{P}(\mathbb{R})$ in order type ω_3. Note that the hypotheses of the theorem imply that Θ is regular, since any singularizing function would exist in HOD_A for some set of reals A, and, by Theorem 2.3, this would give a club of singular cardinals in HOD below Θ.

Theorem 6.1. *Assume that* $\text{AD}_\mathbb{R}$ *holds, that* $V = L(\mathcal{P}(\mathbb{R}))$, *and that stationarily many elements* θ *of cofinality* ω_1 *in the Solovay sequence are regular in* HOD. *Then in the* $\mathbb{P}_{\max} * \text{Add}(\omega_3, 1)$-*extension there is no partial* $\square_{\omega_2}$-*sequence defined on all points of cofinality at most* ω_1.

Proof. Suppose that τ is a $\mathbb{P}_{\max} * \text{Add}(\omega_3, 1)$-name in $L(\mathcal{P}(\mathbb{R}))$ whose realization is forced by some condition p_0 to be such a partial $\square_{\omega_2}$-sequence. We may assume that τ is coded by a subset of $\mathcal{P}(\mathbb{R})$, and (by using the least ordinal parameter defining a counterexample to the theorem) that (τ, p_0) is definable from some $A \subseteq \mathbb{R}$. Using our hypothesis (and Theorem 2.3 for item (4)), we get $\theta < \Theta$ with A in $\mathcal{P}_\theta(\mathbb{R})$, and ordinals ξ_0 and ξ_1 such that

(1) $\theta < \xi_0 < \Theta < \xi_1$;
(2) $L_{\xi_1}(\mathcal{P}(\mathbb{R}))$ satisfies a sufficiently large (finite) fragment T of ZF;
(3) θ is a limit element of the Solovay sequence of $L(\mathcal{P}(\mathbb{R}))$ of cofinality ω_1;
(4) θ is regular in $\text{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$;
(5) $p_0 \in L_{\xi_0}(\mathcal{P}_{\xi_0}(\mathbb{R}))$;
(6) every element of $L_{\xi_0}(\mathcal{P}_{\xi_0}(\mathbb{R}))$ is definable in $L_{\xi_0}(\mathcal{P}_{\xi_0}(\mathbb{R}))$ from a set of reals in $\mathcal{P}_{\xi_0}(\mathbb{R})$;
(7) in $L_{\xi_1}(\mathcal{P}(\mathbb{R}))$, τ is a $\mathbb{P}_{\max} * \text{Add}(\omega_3, 1)$-name whose realization is forced by p_0 to be a partial $\square_{\omega_2}$-sequence defined on the ordinals of cofinality at most ω_1; and

(8) there exist $\sigma \in L_{\xi_0}(\mathcal{P}_\theta(\mathbb{R}))$ and an elementary embedding

$$j \colon L_{\xi_0}(\mathcal{P}_\theta(\mathbb{R})) \to L_{\xi_1}(\mathcal{P}(\mathbb{R}))$$

with critical point θ such that $j(\sigma) = \tau$.

To see this, let ξ_1 be the least ordinal $\xi > \Theta$ such that $L_\xi(\mathcal{P}(\mathbb{R})) \models T$, and τ is definable from A in $L_\xi(\mathcal{P}(\mathbb{R}))$. Then every element of $L_{\xi_1}(\mathcal{P}(\mathbb{R}))$ is definable in $L_{\xi_1}(\mathcal{P}(\mathbb{R}))$ from a set of reals. For each $\alpha < \Theta$, let X_α be the set of elements of $L_{\xi_1}(\mathcal{P}(\mathbb{R}))$ definable from a set of reals of Wadge rank less than α. Then, by the definition of the Solovay sequence, and Remark 2.1, the order type of $X_\alpha \cap \Theta$ is always at most the least element of the Solovay sequence above α. Since Θ is regular, $X_\alpha \cap \Theta$ is bounded below Θ. Let $f(\alpha) = \sup(X_\alpha \cap \Theta)$. Then f is continuous, and we can find θ satisfying items (3) and (4) above, and such that $f(\theta) = \theta$. Let ξ_0 be the order type of $X_\theta \cap \xi_1$, so that $L_{\xi_0}(\mathcal{P}_\theta(\mathbb{R}))$ is the transitive collapse of X_θ.

Let $M_0 = L_{\xi_0}(\mathcal{P}_\theta(\mathbb{R}))$ and $M_1 = L_{\xi_1}(\mathcal{P}(\mathbb{R}))$. Let G be $\mathbb{P}_{\max}$-generic over M_1, containing the first coordinate of p_0. Then j lifts to

$$j \colon M_0[G] \to M_1[G],$$

and $j(\mathrm{Add}(\theta, 1)^{M_0[G]}) = \mathrm{Add}(\Theta, 1)^{M_1[G]}$. Since $\mathbb{P}_{\max}$ is countably closed, θ has cofinality ω_1 in $M_1[G]$. Since $M_0[G]$ and $M_1[G]$ have the same reals, $\mathrm{Add}(\theta, 1)^{M_0[G]}$ is ω-closed in $M_1[G]$.

Claim 6.2. *If H is $M_1[G]$-generic over $\mathrm{Add}(\theta, 1)^{M_0[G]}$, then σ_{G*H} has a thread in $M_1[G][H]$.*

Since θ has uncountable cofinality in $M_1[G][H]$, there can be at most one thread through σ_{G*H} in $M_1[G][H]$. The thread, being unique, would be in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}[G][H]$ (note that $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ has the same sets of reals as M_0, and that every $\mathbb{P}_{\max}$-name for a subset of $\mathrm{Add}(\theta, 1)^{M_0[G]}$ in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$ is coded by a subset of $\mathcal{P}_\alpha(\mathbb{R})$, and thus by a set of reals in V). This leads to a contradiction, as θ would be collapsed in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}[G][H]$, which is impossible since θ is regular in $\mathrm{HOD}_{\mathcal{P}_\theta(\mathbb{R})}$.

It suffices then to prove the claim.

Proof. Towards a contradiction, suppose that the claim were false. By the homogeneity of $\mathrm{Add}(\theta, 1)^{M_0[G]}$, it suffices to consider the case where H contains the second coordinate of p_0. Let $C \in M_1$ be club in θ, o.t.$(C) = \omega_1$. By the Coding Lemma, C is in $L(B, \mathbb{R})$ for some set of reals B with $|B|_W = \theta$. If H is $M_1[G]$-generic as above, then σ_{G*H} is a coherent sequence of length θ with no thread in $M_1[G]$. We can fix $(\mathbb{P}_{\max} * \mathrm{Add}(\theta, 1)^{M_0[G]})$-names ρ and ψ in $L(B, \mathbb{R})$ such that

- ρ_{G*H} is the tree of attempts to build a thread through σ_{G*H} along C (i.e., the relation consisting of those pairs (α, β) from C for which the β-th member of σ_{G*H} extends the α-th member), and

- ψ_{G*H} is the poset that specializes ρ_{G*H} (i.e., which consists of finite partial functions mapping C to ω in such a way that ρ_{G*H}-compatible elements of C are mapped to distinct elements of ω, ordered by inclusion).

Since σ_{G*H} has no thread in $M_1[G][H]$, $\mathrm{Add}(\theta,1)^{M_0[G]} * \psi$ is ω-closed$*$c.c.c., and thus proper, in $M_1[G]$.

Subclaim 6.3. *In $M_1[G]$, there are H, f such that*
- *H is $\mathrm{Add}(\theta,1)^{M_0[G]}$-generic over $M_0[G]$, and*
- *f specializes ρ_{G*H}.*

Proof. In $M_0[G]$, $\mathrm{Add}(\theta,1)$ is $< \theta$-closed.

For each $\alpha \in C$ and each ternary formula ϕ, let $E_{\alpha,\phi}$ be the collection of sets of the form $\{x \mid M_0[G] \models \phi(A, G, x)\}$ that are dense subsets of $\mathrm{Add}(\theta,1)^{M_0[G]}$, where A is an element of $\mathcal{P}_\alpha(\mathbb{R})$. Then $E_{\alpha,\phi}$ has cardinality less than θ in $M_0[G]$, so there is a dense subset $D_{\alpha,\phi}$ of $\mathrm{Add}(\theta,1)^{M_0[G]}$ refining all the members of $E_{\alpha,\phi}$.

Since $\mathsf{PFA}(\mathfrak{c})$ holds in $M_1[G]$ (by Woodin [Woo10, Theorem 9.39]), there is a filter $H * K$ on $\mathrm{Add}(\theta,1)^{M_0[G]} * \psi$ in $M_1[G]$ such that H meets $D_{\alpha,\phi}$, and $H * K$ meets the dense sets guaranteeing that K determines a specializing function f for ρ_{G*H}. That gives the subclaim. □

Let H and f be as in the subclaim. Then in $M_1[G]$, H is a condition in $\mathrm{Add}(\Theta,1)$. We can therefore find a generic H_1 over $M_1[G]$ such that j lifts to

$$j \colon M_0[G][H] \to M_1[G][H_1].$$

But then σ_{G*H} has the thread $(\tau_{G*H_1})_\theta$. But f is in $M_1[G]$, so ω_1 was collapsed by going to $M_1[G][H_1]$, giving a contradiction. □

This completes the proof of Theorem 6.1. □

Remark 6.4. In consistency strength, the assumption of Theorem 6.1 is below a Woodin limit of Woodin cardinals; this follows from the proof of Sargsyan [Sara, Theorem, 3.7.3]. In turn, this assumption is equiconsistent with a determinacy statement that is easier to state. In fact, something stronger than mere equiconsistency holds, as we proceed to sketch:

Assume $\mathsf{V} = L(\mathcal{P}(\mathbb{R}))$ and that $\mathsf{AD}_\mathbb{R}$ holds. We claim that, at least in the presence of the HOD analysis, see Sargsyan [Sara], the key requirement that there are stationarily many θ in the Solovay sequence that have cofinality ω_1 and are regular in HOD is actually equivalent to (perhaps restricting to an inner model) requiring that Θ is Mahlo to measurables of HOD, that is, that there is a stationary subset of Θ consisting of cardinals that are measurable in HOD.

To see this, assume we can use the HOD analysis. If Θ is Mahlo to measurables of HOD, then we may assume the cardinals in the relevant stationary set are all in the Solovay sequence. Moreover, in V they must

have cofinality at least ω_1. The HOD analysis is used here to lift Steel [Ste10, Lemma 8.25] to our context.

It then follows that, for some $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, in $L(\Gamma, \mathbb{R})$ we have $\mathsf{AD}_{\mathbb{R}}$, and a stationary set of θ in the Solovay sequence that have cofinality *precisely* ω_1 in V and are regular in HOD. In effect, we can take $\Gamma = \mathcal{P}(\mathbb{R})$, unless there is a θ_α regular in HOD and of cofinality strictly larger than ω_1 in V, in which case we let α_0 be the least such α, and take $\Gamma = \mathcal{P}_{\theta_{\alpha_0}}(\mathbb{R})$. By the coding lemma and the results of Section 2, it follows that the assumptions of Theorem 6.1 hold in $M = L(\Gamma, \mathbb{R})$. That the set of $\theta < \theta_{\alpha_0} = \Theta^M$ that are regular in HOD^M and of cofinality ω_1 in M is stationary is also a consequence of the HOD analysis; the point is that non-overlapped measurable cardinals of Mitchell order zero have cofinality ω_1, so the minimality of α_0 gives the result.

Conversely, from the assumptions of Theorem 6.1, and the HOD analysis, it follows (again, just as in Steel [Ste10, Lemma 8.25]) that any cardinal in the relevant stationary set is actually measurable in HOD.

7. STRONGER HYPOTHESES, AND THE THREADABILITY OF ω_3

In this section we present some applications of hypotheses stronger than the one used in Theorem 6.1. These assumptions are natural strengthenings of determinacy, though at the moment we do not know how their consistency strength compares with traditional large cardinal assumptions. In particular, we do not know whether the assumptions we consider here can be proved consistent from a Woodin limit of Woodin cardinals.

The following lemma will be used in the proofs of Theorems 7.3 and 7.5:

Lemma 7.1. *Suppose that $M_0 \subseteq M_1$ are models of AD^+ such that*

- $\mathcal{P}(\mathbb{R})^{M_0}$ *is a proper subset of $\mathcal{P}(\mathbb{R})^{M_1}$;*
- $M_1 \models$ "Θ is regular";
- Θ^{M_0} *has cofinality at least ω_2 in M_1.*

Let $G \subset \mathbb{P}_{\max}$ be an M_1-generic filter. Then $\mathrm{Add}(\omega_3, 1)^{M_0[G]}$ is closed under ω_1-sequences in $M_1[G]$.

Proof. Since $M_0 \models \mathsf{AD}^+$ and G is an M_0-generic filter for $\mathbb{P}_{\max}$, $\omega_3^{M_0[G]} = \Theta^{M_0}$. Note first that $\mathrm{cf}(\Theta^{M_0}) = \omega_2$ in $M_1[G]$. Since we have assumed that $\mathrm{cf}(\Theta^{M_0}) \geq \omega_2$ in M_1, equality follows from the Covering Theorem 4.2: Given any bounded subset X of Θ^{M_1} of cardinality $\aleph_1$ in $M_1[G]$, let $A \in M_1$ be a set of reals of Wadge rank at least $\sup(X)$, and apply Theorem 4.2 with $M = L(A, \mathbb{R})$.

Let $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence in $M_1[G]$ consisting of conditions in $\mathrm{Add}(\omega_3, 1)^{M_0[G]}$. Since $\mathrm{cf}(\Theta^{M_0}) = \omega_2$ in $M_1[G]$, we may fix a $\gamma < \Theta^{M_0}$ such that each p_α is a subset of γ. Since ω_2-DC holds in $M_1[G]$ by Woodin [Woo10, Theorem 3.9], we may find in $M_1[G]$ a sequence $\langle \tau_\alpha \mid \alpha < \omega_1 \rangle$ consisting of $\mathbb{P}_{\max}$-names in M_0 such that $\tau_{\alpha,G} = p_\alpha$ for all $\alpha < \omega_1$.

Via a pre-well-ordering R of length γ in M_0, we may assume that each τ_α is coded by R and a set of reals S_α in M_0, in such a way that $\langle S_\alpha \mid \alpha < \omega_1 \rangle \in M_1[G]$. Letting $\eta < \Theta^{M_0}$ be a bound on the Wadge ranks of the sets S_α, we have that the sequence $\langle S_\alpha \mid \alpha < \omega_1 \rangle$ is coded by a single set of reals E in M_0 and an ω_1-sequence of reals in $M_1[G]$. Finally, since $\mathbb{R}^{\omega_1} \cap M_0[G] = \mathbb{R}^{\omega_1} \cap M_1[G]$ (again by Theorem 4.2) we have that $\langle p_\alpha \mid \alpha < \omega_1 \rangle \in M_0[G]$. □

Definition 7.2. Given an ordinal γ and a cardinal λ, the principle $\square(\gamma, \lambda)$ says that there exists a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \gamma \rangle$ of nonempty sets such that

(1) For each $\alpha < \gamma$,
   - $|\mathcal{C}_\alpha| \leq \lambda$;
   - Each element of $\mathcal{C}_\alpha$ is club in α;
   - For each member C of $\mathcal{C}_\alpha$, and each limit point β of C, $C \cap \beta \in \mathcal{C}_\beta$; and
(2) There is no thread through the sequence, i.e., there is no club $E \subseteq \gamma$ such that $E \cap \alpha \in \mathcal{C}_\alpha$ for every limit point α of E.

Again, we refer to sequences witnessing the above principle as $\square(\gamma, \lambda)$-sequences. Notice that $\square_{\kappa, \lambda}$ implies $\square(\kappa^+, \lambda)$.

The arguments of Todorcevic [Tod84, Tod02] show that $\mathsf{MM}^{++}(\mathfrak{c})$ implies the failure of $\square(\gamma, \omega_1)$ for any ordinal of cardinality and cofinality ω_2.

Note that the hypothesis of Theorem 7.3 below is stronger than that of Theorem 6.1. By Woodin [Woo10, Theorem 9.10], the hypotheses of the theorem imply that AD^+ holds in M_0.

Theorem 7.3. *Suppose that $M_0 \subseteq M_1$ are models of ZF containing the reals such that, letting $\Gamma_0 = \mathcal{P}(\mathbb{R}) \cap M_0$, the following hold:*

   - $M_0 \subseteq \mathrm{HOD}^{M_1}_{\Gamma_0}$;
   - $M_1 \models \mathsf{AD}_{\mathbb{R}} + $ "Θ is regular";
   - Θ^{M_0} is on the Solovay sequence of M_1 and has cofinality at least ω_2 in M_1.

Let $G \subset \mathbb{P}_{\max}$ be M_1-generic and let $H \subset \mathrm{Add}(\omega_3, 1)^{M_0[G]}$ be $M_1[G]$-generic. Then, in $M_0[G][H]$, ω_3 is square inaccessible and, in fact, $\square_{\omega_2, \omega}$ fails.

Proof. Suppose that τ is a $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3, 1)$-name in M_0 for a $\square_{\omega_2, \omega}$-sequence. We may assume that the realization of τ comes with an indexing of each member of the sequence in order type at most ω. In M_0, τ is ordinal definable from some set of reals S. Since Θ^{M_0} is on the Solovay sequence of M_1, $\mathrm{HOD}^{M_1}_{\Gamma_0}$ has the same sets of reals as M_0.

By Woodin [Woo10, Theorem 9.39], $\mathsf{MM}^{++}(\mathfrak{c})$ holds in $M_1[G]$. Since $\mathcal{P}(\omega_1)^{M_1[G]}$ is contained in $L(\mathbb{R})[G]$ (Woodin [Woo10, Theorem 9.23]), Θ^{M_0} has cofinality at least ω_2 in $M_1[G]$. Forcing with $< \omega_2$-directed closed partial orders of size at most $\mathfrak{c}$ preserves $\mathsf{MM}^{++}(\mathfrak{c})$ (see Larson [Lar00]). It follows then from Lemma 7.1 that $\mathsf{MM}^{++}(\mathfrak{c})$ holds in the $\mathrm{Add}(\omega_3, 1)^{M_0[G]}$-extension

of $M_1[G]$, and thus that in this extension every candidate for a $\square(\Theta^{M_0}, \omega)$-sequence is actually threaded.

Since Θ^{M_0} has cofinality at least ω_2 in the $\mathrm{Add}(\omega_3, 1)^{M_0[G]}$-extension of $M_1[G]$ (which satisfies Choice), each such sequence has at most ω many threads, since otherwise one could find a $\mathcal{C}_\alpha$ in the sequence with uncountably many members. Therefore, some member of some $\mathcal{C}_\alpha$ in the realization of τ will be extended by a unique thread through the sequence, and since the realization of τ indexes each $\mathcal{C}_\alpha$ in order type at most ω, there is in M_1 a name, ordinal definable from S, for a thread through the realization of τ. This name is then a member of $\mathrm{HOD}^{M_1}_{\Gamma_0}$. This gives a contradiction, since $\mathrm{HOD}^{M_1}_{\Gamma_0}$ having the same sets of reals as M_0 implies that $\mathrm{Add}(\omega_3, 1)$ is the same partial order in $M_0[G]$ and $\mathrm{HOD}^{M_1}_{\Gamma_0}$, so $\mathrm{HOD}^{M_1}_{\Gamma_0}[G]$ thinks that $\mathrm{Add}(\omega_3, 1)^{M_0[G]}$ is ω_2-closed and thus preserves ω_3. $\square$

The following principle is introduced in Woodin [Woo10, §9.5]. If one assumes that I is the nonstationary ideal, then one gets Todorcevic's reflection principle $\mathsf{SRP}(\kappa)$. The principle $\mathsf{SRP}(\omega_2)$ follows easily from $\mathsf{MM}^{++}(\mathfrak{c})$.

Definition 7.4. Given a cardinal $\kappa \geq \omega_2$, $\mathsf{SRP}^*(\kappa)$ is the statement that there is a proper, normal, fine ideal $I \subseteq \mathcal{P}([\kappa]^{\aleph_0})$ such that for all stationary $T \subseteq \omega_1$,

$$\{X \in [\kappa]^{\aleph_0} \mid X \cap \omega_1 \in T\} \notin I,$$

and such that for all $S \subseteq [\kappa]^{\aleph_0}$, if S is such that for all stationary $T \subseteq \omega_1$,

$$\{X \in S \mid X \cap \omega_1 \in T\} \notin I,$$

then there a set $Y \subseteq \kappa$ such that

- $\omega_1 \subseteq Y$;
- $|Y| = \aleph_1$;
- $\mathrm{cf}(\sup(Y)) = \omega_1$; and
- $S \cap [Y]^{\aleph_0}$ contains a club in $[Y]^{\aleph_0}$.

Before we state Theorem 7.5, we remark that the only difference between its hypothesis and that of Theorem 7.3, is that now we assume $M_0 = \mathrm{HOD}^{M_1}_{\Gamma_0}$ while, in Theorem 7.3, we assumed only $M_0 \subseteq \mathrm{HOD}^{M_1}_{\Gamma_0}$.

Theorem 7.5. *Suppose that $M_0 \subseteq M_1$ are models of ZF with the same reals such that, letting $\Gamma_0 = \mathcal{P}(\mathbb{R}) \cap M_0$, the following hold:*

- *$M_0 = \mathrm{HOD}^{M_1}_{\Gamma_0}$;*
- *$M_1 \models \mathsf{AD}_\mathbb{R} + $ "Θ is regular";*
- *$\Theta^{M_0} < \Theta^{M_1}$; and*
- *Θ^{M_0} has cofinality at least ω_2 in M_1.*

Let $G \subset \mathbb{P}_{\max}$ be M_1-generic, and let $H \subset \mathrm{Add}(\omega_3, 1)^{M_0[G]}$ be $M_1[G]$-generic. Then the following hold in $M_0[G][H]$:

- *ω_3 is threadable; in fact, we have $\neg\square(\omega_3, \omega)$;*
- *$\mathsf{SRP}^*(\omega_3)$;*

Proof. The beginning of the proof, through the proof of $\neg\square(\omega_3,\omega)$, is almost exactly like the proof of Theorem 7.3. We sketch it for the reader's convenience. Again, $\mathsf{MM}^{++}(\mathfrak{c})$ holds in $M_1[G]$, and AD^+ holds in M_0.

Again, $\mathsf{MM}^{++}(\mathfrak{c})$ holds in the $\mathrm{Add}(\omega_3,1)^{M_0[G]}$-extension of $M_1[G]$, so in this extension every candidate for a $\square(\Theta^{M_0},\omega)$-sequence is threaded. Since Θ^{M_0} has cofinality at least ω_2 in the $\mathrm{Add}(\omega_3,1)^{M_0[G]}$-extension of $M_1[G]$, each such sequence has a unique thread, which means that there is a name for such a thread definable from any name for such a sequence.

Now suppose that τ is a $\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1)$-name in M_0 for a $\square(\omega_3,\omega)$-sequence. Then, in M_1, τ is ordinal definable from an element of Γ_0.

Since in M_1 there is a $(\mathbb{P}_{\max} * \mathrm{Add}(\omega_3,1))^{M_0}$-name for a thread through the realization of τ, and since each thread is the unique thread extending some $\mathcal{C}_\alpha$ on the sequence, $\mathrm{HOD}_{\Gamma_0}^{M_1}$ also sees a name for a thread through the realization of τ.

That $\mathsf{SRP}^*(\omega_3)$ holds in $M_0[G]$ follows from the fact that the nonstationary ideal on $[\Theta^{M_0}]^{\aleph_0}$ as defined in $M_1[G][H]$ is an element of $M_0[G][H]$, and the facts that $|\Theta^{M_0}|^{M_1[G][H]} = \aleph_2$, and $M_1[G][H]$ satisfies $\mathsf{SRP}(\omega_2)$. $\qquad\square$

Question 7.6. Can one improve the theorems in this section to obtain the failures of $\square_{\omega_2,\omega_1}$ and $\square(\omega_3,\omega_1)$?

References

[Bau76] James E. Baumgartner, *A new class of order types*, Ann. Math. Logic **9 (3)** (1976), 187–222. MR 0416925 (54 #4988)

[CK11] Andrés E. Caicedo, and Richard Ketchersid, *A trichotomy theorem in natural models of* AD^+, in **Set theory and its applications**, Babinkostova, Caicedo, Geschke, Scheepers, eds., Contemporary Mathematics **533**, American Mathematical Society, Providence, RI, 2011, pp. 227–258. MR 2777751

[Cum05] James Cummings, *Compactness and incompactness phenomena in set theory*, in **Logic Colloquium '01**, Baaz, Friedman, Krajíček, eds., Lect. Notes Log. **20**, Assoc. Symbol. Logic, Urbana, IL, 2005, pp. 139–150. MR 2143895 (2006j:03061)

[CM11] James Cummings, and Menachem Magidor, *Martin's maximum and weak square*, Proc. Amer. Math. Soc. **139 (9)** (2011), 3339–3348. MR 2811288 (2012e:03114)

[Jac10] Steve Jackson, *Structural consequences of* AD, in **Handbook of set theory. Vols. 1, 2, 3**, Foreman, Kanamori, eds., Springer, Dordrecht, 2010, pp. 1753–1876. MR 2768700

[Jec03] Thomas Jech, **Set theory**, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded. MR 1940513 (2004g:03071)

[Jen72] Ronald B. Jensen, *The fine structure of the constructible hierarchy. With a section by Jack Silver*, Ann. Math. Logic **4** (1972), 22–308; erratum, ibid. **4** (1972), 443. MR 0309729 (46 #8834)

[Jena] ———, *Corrections and remarks*, unpublished manuscript (1998). Available at http://www.mathematik.hu-berlin.de/~raesch/org/jensen.html

[Ket11] Richard Ketchersid, *More structural consequences of* AD, in **Set theory and its applications**, Babinkostova, Caicedo, Geschke, Scheepers, eds., Contemporary Mathematics **533**, American Mathematical Society, Providence, RI, 2011, pp. 71–105. MR 2777745

[Kec95]   Alexander S. Kechris, **Classical descriptive set theory**, Graduate Texts in
          Mathematics **156**, Springer-Verlag, New York, 1995. MR 1321597 (96e:03057)
[KW10]   Peter Koellner, and W. Hugh Woodin, *Large cardinals from determinacy*, in
          Handbook of set theory. Vols. 1, 2, 3, Foreman, Kanamori, eds., Springer,
          Dordrecht, 2010, pp. 1951–2119. MR 2768702
[KY04]   Bernhard König, and Yasuo Yoshinobu, *Fragments of Martin's maximum in
          generic extensions*, MLQ Math. Log. Q. **50 (3)** (2004), 297–302. MR 2050172
          (2005b:03117)
[Lar00]   Paul Larson, *Separating stationary reflection principles*, J. Symb. Logic **65 (1)**
          (2000), 247–258. MR 1782117 (2001k:03094)
[Lar10]   ———, *Forcing over models of determinacy*, in **Handbook of set theory.
          Vols. 1, 2, 3**, Foreman, Kanamori, eds., Springer, Dordrecht, 2010, pp. 2121–
          2177. MR 2768703
[MS94]   William J. Mitchell, and John R. Steel, **Fine structure and iteration trees**,
          Lecture Notes in Logic **3**, Springer, 1994. MR 1300637 (95m:03099)
[Mos09]   Yiannis N. Moschovakis, **Descriptive set theory**, 2nd. edn., Mathematical
          Surveys and Monographs **155**, American Mathematical Society, Providence, RI,
          2009. MR 2526093 (2010f:03001)
[Sara]   Grigor Sargsyan, *A tale of hybrid mice*, preprint (2011). Available at
          http://grigorsargis.weebly.com/papers-and-cv.html
[Sch07]   Ernest Schimmerling, *Coherent sequences and threads*, Adv. Math. **216 (1)**
          (2007), 89–117. MR 2353251 (2009b:03141)
[SZ04]   Ernest Schimmerling, and Martin Zeman, *Characterization of $\square_\kappa$ in core models*,
          J. Math. Log. **4 (1)** (2004), 1–72. MR 2081183 (2005f:03067)
[SSa]   Ralf Schindler, and John R. Steel, **The core model induction**, unpublished
          book (2010). Available at
          http://wwwmath.uni-muenster.de/u/rds/
[Sol78]   Robert M. Solovay, *The independence of* DC *from* AD, Cabal Seminar 76–
          77 (Proc. Caltech-UCLA Logic Sem., 1976–77), Lecture Notes in Math. **689**,
          Springer, Berlin, 1978, pp. 171–183. MR 526918 (80e:03065)
[Ste96]   John R. Steel, **The Core Model Iterability Problem**, Lecture Notes in Logic
          8, Springer, 1996. MR 1480175 (99k:03043)
[Ste10]   ———, *An outline of inner model theory*, in **Handbook of set theory. Vols.
          1, 2, 3**, Foreman, Kanamori, eds., Springer, Dordrecht, 2010, pp. 1595–1684.
          MR MR2768698
[Tod84]   Stevo Todorcevic, *A note on the proper forcing axiom*, in **Axiomatic set theory
          (Boulder, Colorado, 1983)**, Baumgartner, Martin, Shelah, eds., Contempo-
          rary Mathematics **31**, American Mathematical Society, Providence, RI, 1984,
          pp. 209–218. MR 0763902 (86f:03089)
[Tod02]   ———, *Localized reflection and fragments of* PFA, in **Set theory. Proceed-
          ings of the International Conference held in honor of András Hajnal
          at Rutgers University, Piscataway, NJ, October 15–17, 1999**, Thomas,
          ed., DIMACS Series in Discrete Mathematics and Theoretical Computer Sci-
          ence **58**, American Mathematical Society, Providence, RI, 2002, pp. 135–148.
          MR 1903856 (2003g:03081)
[Woo10]   W. Hugh Woodin, **The axiom of determinacy, forcing axioms, and the
          nonstationary ideal**, 2nd. edn., De Gruyter Series in Logic and its Applications
          1, De Gruyter, 2010. MR 2723878 (2011g:03002)
[Zem02]   Martin Zeman, **Inner Models and Large Cardinals**, De Gruyter Series in
          Logic and its Applications **5**, De Gruyter, 2002. MR 1876087 (2003a:03004)
[Zem10]   ———, *Global square in extender models*, Ann. Pure Appl. Logic **161 (7)** (2010),
          956–985. MR 2601024 (2011h:03098)

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