ON THE HEREDITARY PARACOMPACTNESS OF LOCALLY COMPACT, HEREDITARILY NORMAL SPACES

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ABSTRACT. Assuming the consistency of a supercompact cardinal, it is shown consistent that if X is a locally compact, hereditarily normal space which includes no perfect pre-image of ω_1, then X is hereditarily paracompact.

This is the fifth in a series of papers ([LTo], [To], [LT], [T] being the previous ones) that establish powerful topological consequences in models of set theory obtained by starting with a particular kind of Souslin tree S, iterating partial orders that don't destroy S, and then forcing with S. The particular case of the theorem stated in the abstract when X is perfectly normal (and hence has no perfect pre-image of ω_1) was proved in [LT], using essentially that locally compact perfectly normal spaces are locally hereditarily Lindelöf and first countable. Here we avoid these two last properties by combining the methods of [B_2] and [T]. To apply [B_2], we establish the new set-theoretic result that Fleissner's "Axiom R" [F] holds in a model of the form "PFA^{++}(S)[S]". This notation is explained below; the model is a prima facie strengthening of those used in the previous four papers.

It is easy to find locally compact, hereditarily normal spaces which are not paracompact – ω_1 is one such. Non-trivial perfect pre-images of ω_1 may also be hereditarily normal, but are not paracompact. Our result says that consistently, any example must in fact include such a canonical example.

Theorem 1. *If it is consistent that there is a supercompact cardinal, it's consistent that every locally compact, hereditarily normal space that does not include a perfect pre-image of ω_1 is (hereditarily) paracompact.*

This is not a ZFC result, since there are many consistent examples of locally compact, perfectly normal spaces which are not paracompact.

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Let us state some axioms we will be using.

PFA^{++}: *Suppose P is a proper partial order, $\{D_\alpha\}_{\alpha<\omega_1}$ is a collection of dense subsets of P, and $\{\dot{S}_\alpha : \alpha < \omega_1\}$ is a sequence of terms such that $(\forall\alpha < \omega_1) \Vdash \dot{S}_\alpha$ is stationary in ω_1. Then there is a filter $G \subseteq P$ such that*

(i) $(\forall\alpha < \omega_1)\ G \cap D_\alpha \neq 0$,

and (ii) $(\forall\alpha < \omega_1)\ S_\alpha(G) = \{\xi < \omega_1 : (\exists p \in G) p \Vdash \xi \in \dot{S}_\alpha\}$ *is stationary in* ω_1.

Baumgartner [Ba] introduced this axiom and called it "PFA$^+$". Since then, others have called this "PFA^{++}", using "PFA$^+$" for the weaker one-term version. As Baumgartner observed, the usual consistency proof for PFA, which uses a supercompact cardinal, yields a model for what we are calling PFA^{++}.

Definition. $\Gamma \subseteq [X]^{<\kappa}$ *is* **tight** *if whenever $\{C_\alpha : \alpha < \delta\}$ is an increasing sequence from Γ, and $\omega < cf\delta < \kappa$, then $\bigcup\{C_\alpha : \alpha < \delta\} \in \Gamma$.* **Axiom R:** *if $\Sigma \subseteq [X]^\omega$ is stationary and $\Gamma \subseteq [X]^{<\omega_2}$ is tight and unbounded, then there is a $Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma$ is stationary in $[Y]^\omega$.* **Axiom R^{++}:** *if $\Sigma_\alpha(\alpha < \omega_1)$ are stationary subsets of $[X]^\omega$ and $\Gamma \subseteq [X]^{<\omega_2}$ is tight and unbounded, then there is a $Y \in \Gamma$ such that $\mathcal{P}(Y) \cap \Sigma_\alpha$ is stationary in $[Y]^\omega$ for each $\alpha < \omega_1$.*

Fleissner introduced Axiom R in [F] and showed it held in the usual model for PFA.

Σ^+: *Suppose X is a countably tight compact space, $\mathcal{L} = \{L_\alpha\}_{\alpha<\omega_1}$ a collection of disjoint compact sets such that each L_α has a neighborhood that meets only countably many L_β's, and $\mathcal{V}$ is a family of $\leq \aleph_1$ open subsets of X such that:*

a) $\bigcup\mathcal{L} \subseteq \bigcup\mathcal{V}$
b) For every $V \in \mathcal{V}$ there is an open U_V such that $\overline{V} \subseteq U_V$ and U_V meets only countably many members of $\mathcal{L}$.

Then $\mathcal{L} = \bigcup\limits_{n<\omega} \mathcal{L}_n$, *where each $\mathcal{L}_n$ is a discrete collection in $\bigcup\mathcal{V}$.*

Balogh [B_1] proved that MA_{ω_1} implies the restricted version of Σ^+ in which we take the L_α's to be points. We will call that "Σ'".

Definition. *A space is (strongly) κ-collectionwise Hausdorff if for each closed discrete subspace $\{x_d\}_{d\in D}$, $|D| \leq \kappa$, there is a disjoint (discrete) family of open sets $\{U_d\}_{d\in D}$ with $d \in U_d$. A space is (strongly) collectionwise Hausdorff if it is (strongly) κ-collectionwise Hausdorff for all κ.*

It is easy to see that normal ($\kappa-$)collectionwise Hausdorff spaces are strongly ($\kappa-$)collectionwise Hausdorff.

Balogh [B_2] proved:

Lemma 2. *MA_{ω_1} + Axiom R implies locally compact hereditarily strongly $\aleph_1$-collectionwise Hausdorff spaces which do not include a perfect pre-image of ω_1 are paracompact.*

The consequences of MA_{ω_1} he used are Σ' and Szentmiklóssy's result [S] that *compact spaces with no uncountable discrete subspaces are hereditarily Lindelöf.* Our plan is to find a model in which these two consequences and Axiom R hold, as well as normality implying (strongly) $\aleph_1$-collectionwise Hausdorffness for the spaces under consideration. The model we will consider is of the same genre as those in [LTo], [To], [LT], and [T]. One starts off with a particular kind of Souslin tree S, a *coherent* one, which is obtainable from $\diamondsuit$ or by adding a Cohen real. One then iterates in standard fashion as in establishing MA_{ω_1} or PFA, but omitting partial orders that adjoin uncountable antichains to S. In the PFA case for example, this will establish *PFA(S)*, which is like PFA except restricted to partial orders that don't kill S. In fact it will also establish $\mathrm{PFA}^{++}(\mathrm{S})$, which is the corresponding modification of PFA^{++}. We then force with S. We write "$\mathrm{PFA}^{++}(\mathrm{S})[\mathrm{S}]$ implies ϕ" (and similar notation) to mean that ϕ holds in any model formed by forcing with a coherent Souslin tree over a model of $\mathrm{PFA}^{++}(\mathrm{S})$.

In [T] it is established that:

Lemma 3. *PFA(S)[S] implies that locally compact normal spaces are $\aleph_1$-collectionwise Hausdorff.*

By doing some preliminary forcing (see [LT]), one can actually get full collectionwise Hausdorffness, but we won't need that here.

We will assume all spaces are Hausdorff, and use "X^*" to refer to the one-point compactification of a locally compact space X.

There is a bit of a gap in Balogh's proof of Lemma 2. Balogh asserted that:

Lemma 4. *If X is locally compact and does not include a perfect pre-image of ω_1, then X^* is countably tight.*

and referred to [B_1] for the proof. However in [B_1], he only proved this for the case in which X is countably tight. It is not obvious that that hypothesis can be omitted, but in fact it can. We need a definition and lemma.

Definition. *A space Y is* **ω-bounded** *if each separable subspace of Y has compact closure.*

Lemma 5. [G], [Bu]. *If Y is ω-bounded and does not include a perfect pre-image of ω_1, then Y is compact.*

We then can establish Lemma 4 as follows.

Proof. By Lemma 5, every ω-bounded subspace of X is compact. By [B_1], it suffices to show X is countably tight. Suppose, on the contrary, that there is a $Y \subseteq X$ which is not closed, but is such that for all countable $Z \subseteq Y$, $\overline{Z} \subseteq Y$. Since X is a k-space, there is a compact K such that $K \cap Y$ is not closed. Then $K \cap Y$ is not ω-bounded, so there is a countable $Z \subseteq K \cap Y$ such that $\overline{Z} \cap K \cap Y$ is not compact. But $\overline{Z} \subseteq Y$, so $\overline{Z} \cap K \cap Y = \overline{Z} \cap K$, which is compact, contradiction.

Lemma 3 takes care of the hereditary strong $\aleph_1$-collectionwise Hausdorffness we need, since if open subspaces are $\aleph_1$-collectionwise Hausdorff, all subspaces are, and open subspaces of locally compact spaces are locally compact. The proposition

$\mathbf{\Sigma}$: *in a compact countably tight space, locally countable subspaces of size $\aleph_1$ are σ-discrete.*

was established from PFA(S)[S] in [To].

From Σ' it is standard to get the result of Szentmiklóssy quoted earlier: since the compact space has no uncountable discrete subspace, it has countable tightness. If it were not hereditarily Lindelöf, it would have a right-separated subspace of size $\aleph_1$. But Σ' implies it has an uncountable discrete subspace, contradiction.

Σ' is established by essentially the same forcing as for Σ. Σ^+, however, is not so clear, and has not yet been proved. Thus, instead of using it to get $\aleph_1$-collectionwise Hausdorffness in locally compact normal spaces with no perfect pre-image of ω_1, as we did in an earlier version of this paper, we are instead quoting Lemma 3, which is a new result of the second author, based on methods of [To].

Thus all we have to do is prove that PFA^{++}(S)[S] implies Axiom R. In order to prove that PFA^{++}(S)[S] implies Axiom R, we first note that a straightforward argument using the forcing *Coll* (ω_1, X) (whose conditions are countable partial functions from ω_1 to X, ordered by inclusion) shows that PFA^{++}(S) implies Axiom R^{++}.

It then suffices to prove:

Lemma 6. *If Axiom R^{++} holds and S is a Souslin tree, then Axiom R^{++} still holds after forcing with S.*

Proof. First note that if X is a set, P is a c.c.c. forcing and τ is a P-name for a tight unbounded subset of $[X]^{<\omega_2}$, then the set of $a \in [X]^{<\omega_2}$ such that every condition in P forces that a is in the realization of τ is itself tight and unbounded. The tightness of this set is immediate. To see that it is unbounded, let b_0 be any set in $[X]^{<\omega_2}$. Define sets b_α $(\alpha \leq \omega_1)$ and σ_α $(\alpha < \omega_1)$ recursively by letting σ_α be a P-name for a member of the realization of τ containing b_α and letting $b_{\alpha+1}$ be the set of members of X which are forced by some condition in P to be in σ_α. For limit ordinals $\alpha \leq \omega_1$, let b_α be the union of the b_β $(\beta < \alpha)$. Then b_{ω_1} is forced by every condition in P to be in τ.

Since we are assuming that the Axiom of Choice holds, Axiom R^{++} does not change if we require X to be an ordinal. Fix an ordinal γ and let $\rho_\alpha(\alpha < \omega_1)$

be S-names for stationary subsets of $[\gamma]^\omega$. Let T be a tight unbounded subset of $[\gamma]^{<\omega_2}$. For each countable ordinal α and each node $s \in S$, let $\tau_{s,\alpha}$ be the set of countable subsets a of γ such that some condition in S extending s forces that α is in the realization of ρ_α. Applying Axiom R^{++}, we have a set $Y \in [\tau]^{<\omega_2}$ such that each $\mathcal{P}(Y) \cap \tau_{s,\alpha}$ is stationary in $[Y]^\omega$.

Since S is c.c.c., every club subset of $[Y]^\omega$ that exists after forcing with S includes a club subset of $[\gamma]^\omega$ existing in the ground model. Letting $\rho_{\alpha G}$ (for each $\alpha < \omega_1$) be the realization of ρ_α, we have by genericity then that after forcing with S, each $\mathcal{P}(Y) \cap \rho_{\alpha G}$ will be stationary in $[Y]^\omega$.

This completes the proof of Theorem 1.

We do not know the answer to the following question; a positive answer would likely enable us to dispense with Axiom R, and possibly with the supercompact cardinal.

Problem. Does MA_{ω_1} imply every locally compact, hereditarily strongly collectionwise Hausdorff space which does not include a perfect pre-image of ω_1 is paracompact?

We also do not know whether in our main result, we can replace "perfect pre-image of ω_1" by "copy of ω_1".

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