Extensions of the Axiom of Determinacy

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Contents

This draft includes notes to myself, in footnotes. Aside from these, the book should be close to its final form. Comments and corrections are more than welcome.

0.1 Introduction

The Axiom of Determinacy (AD) is the statement that all length-ω integer games of perfect information are determined. The beginning of Chapter 1 contains a more precise definition, but we expect the reader to be familiar with the classical theory of determinacy, as found in [8, 10], for instance. The axiom AD^+ is a generalization, due to W. Hugh Woodin, of the Axiom of Determinacy. We define it as the conjunction of three statements.

0.1.1 Definition. The axiom AD^+ is the conjunction of the following three statements.

1. $\mathsf{DC}_{\mathbb{R}}$
2. Every subset of ω^ω is ∞-Borel.
3. ($<\Theta$-$\mathsf{Determinacy}$) For all $\lambda < \Theta$, every $A \subseteq \omega^\omega$ and every continuous function $\pi\colon {}^\omega\lambda \to \omega^\omega$, $\pi^{-1}(A)$ is determined (as a subset of ${}^\omega\lambda$).

Our definition of AD^+ diverges from Woodin's own terminology, as he defines AD^+ to be the conjunction of items (2) and (3) above. As AD^+ is usually considered only in the context of $\mathsf{DC}_{\mathbb{R}}$ the choice of definition does not make a significant difference. One difference between $\mathsf{DC}_{\mathbb{R}}$ and the other two parts of AD^+ as we have defined it is that $\mathsf{DC}_{\mathbb{R}}$ is witnessed by real numbers, so, assuming the Axiom of Choice, $\mathsf{DC}_{\mathbb{R}}$ holds in any inner model containing the reals. The other two parts of AD^+ however are witnessed by bounded subsets of Θ.

The ∞-Borel sets are defined in Definition 9.1.1 for subsets of 2^ω, and in Remark 9.1.8 for subsets of ω^ω. We define $<\Theta$-$\mathsf{Determinacy}$ (which Woodin calls Ordinal Determinacy) in Chapter 7. The restriction of $<\Theta$-$\mathsf{Determinacy}$ to the case where π is the identity function on ω^ω is exactly AD.

It is an open question whether AD implies any or all of the parts of AD^+, and in fact whether AD plus any two parts of AD^+ imply the third. It also open whether $\mathsf{AD}_{\mathbb{R}}$ implies AD^+. The issue in this case is whether $\mathsf{AD}_{\mathbb{R}}$ implies $<\Theta$-$\mathsf{Determinacy}$, as $\mathsf{DC}_{\mathbb{R}}$ is easily seen to follow from $\mathsf{AD}_{\mathbb{R}}$, and the second part of AD^+ follows from $\mathsf{AD}_{\mathbb{R}}$ (moreover, from AD + $\mathsf{Uniformization}$) by Theorem 11.3.3.

If $M \subseteq N$ are models of AD with the same reals, and every set of reals in M is Suslin in N, then $M \models \mathsf{AD}^+$ (this is Theorem 9.2.9). In fact, this is the context which the axiom AD^+ was designed to describe; its original name was "within scales".

Here is a list of the major results of the book (all due to Woodin, except where noted):

- In $L(\mathbb{R})$, AD implies AD^+. This is Corollary 9.2.7.
- If AD holds and every true Σ^2_1 sentence is witnessed by a Suslin, co-Suslin set, then AD^+ holds. This is Theorem 9.2.8.
- If AD^+ holds then every inner model containing the reals satisfies AD^+. This is Theorem 9.2.9
- If AD holds then for all $A \subseteq \omega^\omega$, either $\mathcal{P}(\omega^\omega) \subseteq L(A, \mathbb{R})$ or $A^\#$ exists. This is Theorem 10.1.8.
- Assuming $\mathsf{AD}+\mathsf{DC}_\mathbb{R}$, a subset of ω^ω is ∞-Borel if and only if there is a set of ordinals S such that $A \in L(S, \mathbb{R})$. This follows from part (1) of Corollary 10.3.7.
- Assuming AD, if $A \subseteq \omega^\omega$ is ∞-Borel, then so is every set of reals in $L(A, \mathbb{R})$. See Remark 10.3.8.
- Assuming AD^+, $\mathsf{AD}_\mathbb{R}$ is equivalent to the assertion that the Solovay sequence has limit length. This is Theorem 11.2.5.
- If $<\Theta$-$\mathsf{Determinacy}$ and $\mathsf{Uniformization}$ hold, then every subset of ω^ω is Suslin. This follows from Theorem 11.3.3, Corollary 10.2.7 and Theorem 11.2.1.
- If AD^+ holds then the set of Suslin cardinals is closed below Θ. This follows from Theorem 11.4.1.
- If AD holds, then the set of Suslin cardinals is closed below its supremum. This is Theorem 11.4.5 (joint with John Steel).
- If AD^+ holds, then $\mathsf{AD}_\mathbb{R}$ fails if and only if there is a set of ordinals T such that $L(\mathcal{P}(\mathbb{R})) = L(T, \mathbb{R})$. This is Theorem 13.3.3.

In addition, we prove the follow theorem illustrating the relationships among AD, $\mathsf{AD}_\mathbb{R}$, the Suslin property and $\mathsf{Uniformization}$. The first sentence of the theorem is due to Woodin, aside from the implication from (1) to (3), which is due independently to Donald Martin. The implication from (4) to (1) under DC combines results of Woodin and Becker (see Chapter 12).

Theorem 0.1.2. *Each of the following statements implies the ones below it, and the first two statements are equivalent. If* DC *holds, then all four statements are equivalent.*

1. AD + *"every subset of ω^ω is Suslin"*
2. AD^+ + *"every subset of ω^ω is Suslin"*
3. $\mathsf{AD}_\mathbb{R}$
4. AD + $\mathsf{Uniformization}$

Proof. Since AD is the restriction of $<\Theta$-$\mathsf{Determinacy}$ to the case $\lambda = \omega$, (2) implies (1). Theorem 6.2.1 says that Suslin sets can be uniformized, and Remark 6.2.2 shows that $\mathsf{Uniformization}$ implies $\mathsf{DC}_{\mathbb{R}}$. The implication from (1) to (2) then follows from Theorems 5.1.4 and 7.0.3, and Remark 6.1.7. Theorem 13.0.1 says that (1) implies (3). The implication from (3) to (4) is covered in Remark 6.2.2. That (4) implies (1) under DC is Theorem 12.3.1. □

Part I of the book collects various facts (primarily from the Cabal seminar) which will be needed in Part II. It is not intended as a comprehensive introduction to the topics discussed there (for which, see [32, 15, 16, 17]). Part I also significantly overlaps [5]. A natural way to read the book might be to start at the beginning of Part II and refer back to Part I as needed.

The material in this book covers the development of the theory of AD^+ up to some point in the early 1990's. We plan to continue the story in future volumes, starting with a book on derived models and reversals. In particular, the following results of Woodin are not included in this book, although we hope to prove them in the next volume.

- AD^+ implies Σ_1 reflection into the Suslin, co-Suslin sets.
- AD^+ implies that the ultrapower of V by the Turing measure is wellfounded.
- Suppose that AD holds, and AD^+ fails, and let Γ be the pointclass of Suslin, co-Suslin sets. Then $L(\Gamma, \mathbb{R})$ satisfies $\mathsf{AD}_{\mathbb{R}}$ and contains every $A \subseteq \omega^\omega$ for which $L(A, \mathbb{R}) \models \mathsf{AD}^+$, and $\mathcal{P}(\omega^\omega) \cap L(\Gamma, \mathbb{R}) \subseteq \Gamma$.
- If AD holds and there is a largest Suslin cardinal, then AD^+ holds.
- If AD holds, and $A \subseteq \omega^\omega$ is Suslin, then $L(A, \mathbb{R}) \models \mathsf{AD}^+$.
- $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ implies that for all ordinals χ, every subset of ω^ω in $L(\mathcal{P}(\chi))$ is ∞-Borel.

The statement that the ultrapower of the V by the Turing measure is wellfounded was originally one of the axioms of AD^+, before Woodin proved that it follows from the other axioms. This statement is easily seen to follow from the assumption that $\mathsf{Turing\ Determinacy} + \mathsf{DC}$ holds, which in turn follows from the assumption that $\mathsf{Turing\ Determinacy} + \mathsf{DC}_{\mathbb{R}}$ holds, along with the assumption that there is some set X with the property that every set is definable from X, a real and an ordinal. This is the approach taken in this book, although in some instances we include the wellfoundedness of the Turing ultrapower (or some other related ultrapower) as an explicit assumption.

This book has a great deal of overlap with many sources, notably [6, 21, 22]. The writing of the book was supported by NSF grants DMS-1201494 and DMS-1764320.

0.2 Notation

We reserve the symbol $\mathbb{R}$ for the real line, which is never used directly. However, we use traditional notation such as $\mathsf{AD}_{\mathbb{R}}$, $\mathsf{DC}_{\mathbb{R}}$, $L(\mathbb{R})$ and so on, as these terms are equivalent to more relevant forms such as $\mathsf{AD}_{\omega^\omega}$, $\mathsf{DC}_{\omega^\omega}$ and $L(\omega^\omega)$. We informally use the word "real" to mean an element of the Baire space ω^ω.

We use the following notational conventions.

- We write Ord for the class of ordinals.
- We write X^Y to mean the set of functions from Y to X, and, when γ is an ordinal $X^{<\gamma}$ to mean $\bigcup_{\alpha\in\gamma} X^\alpha$.
- For an ordinal α and a set of ordinals X, $[X]^\alpha$ denotes the collection of subsets of X of ordertype α, and $[X]^{<\alpha}$ denotes the collection of subsets of X of ordertype less than α.
- Given a set X, we write $\exists^X$ and $\forall^X$ for existential and universal quantification over X, respectively.
- A *preorder* on a set is a binary relation which is reflexive and transitive. A preorder is *wellfounded* if every nonempty subset of its domain has a minimal element.
- If $\leq$ is a wellfounded preorder on a set X, the *canonical rank function* $\mathrm{rank}_{\leq}$ associated to $\leq$ assigns to each $x \in X$ the least ordinal α such that $\alpha > \mathrm{rank}_{\leq}(y)$ for all $y \in X$ such that $y \leq x$ and $x \not\leq y$.
- In any preorder, especially the natural order on the ordinals, the *strict supremum* of a set (if it exists) is the least element in the domain of the order which is strictly greater than every element of the set.
- Formally, a partial order $\mathbb{P}$ is a pair $(\mathrm{dom}(\mathbb{P}), \leq_{\mathbb{P}})$, where $\mathrm{dom}(\mathbb{P})$ is the domain or underlying set of $\mathbb{P}$, and $\leq_{\mathbb{P}}$ is a partial order on $\mathrm{dom}(\mathbb{P})$.
- Given a partial order $\mathbb{P}$, a *nice* $\mathbb{P}$-name for an element of ω^ω is a set τ such that
  - each element of τ is a pair $(p,(n,m))$, where $p \in \mathbb{P}$ and $n, m \in \omega$ (formally we should write $(p, \check{x})$, where x is the pair (n,m));
  - for each $p \in \mathbb{P}$ and each $n \in \omega$, there exist $q \leq p$ and $m \in \omega$ such that $(q,(n,m)) \in \tau$;
  - for all $p, p' \in \mathbb{P}$ and all $n, m, m' \in \omega$, if $(p,(n,m))$ and $(p',(n,m'))$ are both in τ, and p and p' are compatible, then $m = m'$.
- Given a partial order $\mathbb{P}$ and a set (or sequence) B, we say that a set $g \subseteq \mathbb{P}$ is *B-generic* if g intersects every dense open subset of $\mathbb{P}$ in B.
- We let $\leq_{\text{Gö}}$ be the Gödel order, i.e., the order on pairs of ordinals defined as follows : $(\alpha,\beta) \leq_{\text{Gö}} (\delta,\gamma)$ if any of the following holds:

  - $\max\{\alpha,\beta\} < \max\{\delta,\gamma\}$;
  - $\max\{\alpha,\beta\} = \max\{\delta,\gamma\}$ and $\alpha < \delta$;
  - $\max\{\alpha,\beta\} = \max\{\delta,\gamma\}$, $\alpha = \delta$ and $\beta \leq \gamma$.

- Given ordinals α and β, we write $\prec\alpha,\beta\succ$ for the ordinal ordertype of the set of predecessors of (α,β) in the order $\leq_{\text{Gö}}$.

- For a set X, we let $\text{TC}(X)$ denote the transitive closure of X.

- Given a cardinal κ, $H(\kappa)$ denotes the collection of sets of hereditary cardinality less than κ, i.e., the set of x for which $|\,\text{TC}(x)| < \kappa$.

We sometimes write HF for $H(\aleph_0)$. The members of HF are said to be *hereditarily finite*. A set $x \subseteq \text{HF}$ is *semi-recursive* if x is Σ_1-definable over HF, and *recursive* if x and $\text{HF}\setminus x$ are both semi-recursive. Given $A, B \subseteq \text{HF}$, we say that A is *Turing reducible* to B, and write $A \leq_{\text{Tu}} B$ if there is a semi-recursive function $f\colon \text{HF} \to \text{HF}$ such that $A = f^{-1}[B]$. We say that A is *Turing equivalent* to B if A and B are Turing-reducible to each other. We write $\leq_{\text{Tu}}$ and $\equiv_{\text{Tu}}$ respectively for the restrictions of Turing reducibility and Turing equivalence to ω^ω.

We also sometimes write HC for $H(\aleph_1)$. The members of HC are said to be *hereditarily countable*. We say that a set $x \in \omega^\omega$ HC-*codes* a set $y \in \text{HC}$ if $(\omega, \{(n,m) \in \omega \times \omega : x(2^n 3^m) = 0\})$ is isomorphic to $(\text{TC}(\{y\}), \in)$.

0.3 Prerequisites

We expect the reader to be familiar with Zermelo-Fraenkel set theory, Gödel numbering of formulas, relative constructibility, ordinal definability, inner models of the form HOD_X, ultrapowers, sharps and forcing, along with some elementary notions from descriptive set theory (e.g., Borel sets and projective sets). We do not expect the reader to be familiar with everything in the books [8, 24, 10, 19, 32, 35], but they make good references.

0.4 Forms of Choice

Our base theory in this book is Zermelo-Fraenkel set theory (ZF). Additional axioms will be stated as used. Although we will sometimes consider models of the Axiom of Choice (AC), our main interest is in models of the Axiom of Determinacy (AD), which contradicts AC. Weak forms of AC can (and do) hold in models of AD, however.

Given a set X, the principle of $\mathsf{Dependent\ Choice}$ for X (DC_X) is the statement that whenever $T \subseteq {}^{<\omega}X$ is a tree (i.e., a subset of ${}^{<\omega}X$ closed under initial segments) with the property that every element of T has a proper extension in T, there exists an infinite path through T. The principle of $\mathsf{Countable\ Choice}$ for X (CC_X) is the statement that for all countable sets Y, if $\langle A_y : y \in Y\rangle$ is a sequence of nonempty subsets of X, then there exists a function $f\colon Y \to X$ such that $f(y)$ is in A_y, for each $y \in Y$. The principle CC_X follows immediately from DC_X. A classical argument (due to Mycielski; see Remark 1.1.2) shows that $\mathsf{CC}_{\mathbb{R}}$ follows from AD. Whether or not $\mathsf{DC}_{\mathbb{R}}$ follows from AD is an open question. Note however that if $\mathsf{DC}_{\mathbb{R}}$ holds, then any inner model containing $\mathcal{P}(\omega)$ satisfies $\mathsf{DC}_{\mathbb{R}}$.

The following theorem is part of the result that, in $L(\mathbb{R})$, AD implies AD^+.

Theorem 0.4.1 (Kechris [12]). *Assuming $V{=}L(\mathbb{R})$, AD implies $\mathsf{DC}_{\mathbb{R}}$.*

The axiom of Dependent Choice (DC) asserts that DC_X holds for every set X. Similarly, the axiom of Countable Choice (CC) asserts that CC_X holds for every set X.

0.4.2 Remark. We make frequent use of the standard fact that if every set is definable from an ordinal and a member of X, then DC_X implies DC.

0.5 Partial orders

We list here some classical partial orders which are used as forcing notions throughout the book.

- $\mathrm{Col}(\kappa, X)$, where κ is an infinite cardinal and X is a set. Conditions are functions $f\colon \alpha \to X$, where $\alpha < \kappa$. The order is extension.

- $\mathrm{Col}^*(\kappa, X)$, where κ is an infinite cardinal and X is a set. Conditions are injective functions $f\colon \alpha \to X$, where $\alpha < \kappa$. The order is extension.

- Given a set X consisting of infinite subsets of ω, the classical *almost-disjoint coding* forcing for X (due to Jensen and Solovay [9]) consists of pairs (a, B), where a is a finite subset of ω and B is a finite subset of X, with the order $(a, B) \leq (c, D)$ if

  - c is either the emptyset or $a \cap (\max(c) + 1)$,
  - $D \subseteq B$ and
  - $(a \setminus c) \cap r = \emptyset$ for all $r \in D$.

  This partial order is c.c.c. and adds a subset of ω having finite intersection with each member of X and infinite intersection with each element of $\mathcal{P}(\omega)$ not contained mod-finite in the union of a finite subset of X.

Part I

Preliminaries

Chapter 1

Determinacy

In Chapter 1 we review the Axiom of Determinacy (AD) and the axiom of Turing Determinacy (TD). We expect that most readers will be familiar with the material in this chapter, and include it mostly for easy reference. Remark 1.1.2 below lists four classical consequences of AD which we will be using throughout the book. We make some original definitions in Section 1.2 to formalize the process of lifting the ultrafilter property from one cone measure to another.

1.1 The Axiom of Determinacy

Given a set X, a set $A \subseteq X^\omega$ is *determined* (as a subset of X^ω) if there is a function $\pi\colon X^{<\omega} \to X$ such that one of the two following statements holds.

1. For every $x \in X^\omega$, if $x(2n) = \pi(x\restriction 2n)$ holds for all $n \in \omega$, then $x \in A$.
2. For every $x \in X^\omega$, if $x(2n+1) = \pi(x\restriction(2n+1))$ hold for all $n \in \omega$, then $x \notin A$.

We let AD_X denote the statement that every subset of X^ω is determined (as a subset of X^ω). The Axiom of Determinacy (AD) is the statement AD_ω.

1.1.1 Remark. The following easily verifiable assertions (for arbitrary sets X and Y) show that AD implies that AD_X holds for each countable set X.

- If there is an injection from X to Y, then AD_Y implies AD_X.
- If X is wellorderable and there is a surjection from X to Y, then AD_X implies AD_Y.
- If AD holds, then so does AD_X for each countable set X.

It is convenient and conventional to rephrase determinacy in terms of games. A set $A \subseteq X^\omega$ corresponds to a game $\mathcal{G}_A$ between players I and II, who alternate picking members of X, with I winning if and only if the induced member of X^ω is in A. The set A is called the *payoff set* for $\mathcal{G}_A$.

I	$x(0)$		$x(2)$		$x(4)$		$\ldots$
II		$x(1)$		$x(3)$		$\ldots$	

A run of the game $\mathcal{G}_A$; I wins if and only if x is in A.

A function $\pi\colon X^{<\omega} \to X$ is then said to be a *strategy* in the game $\mathcal{G}_A$. A function π as in case (1) above is a *winning strategy* for player I; in case (2) it is a winning strategy for player II. If σ is a strategy and x is in X^ω, then we write $\sigma * x$ for combined output of the two players when I plays according to σ and II plays x, that is, the unique $y \in X^\omega$ such that

- $y(2n+1) = x(n)$ for all $n \in \omega$;
- $y(n) = \sigma(y{\restriction}n)$ for all even $n \in \omega$.

Similarly, we write $x * \sigma$ for the unique $y \in X^\omega$ such that

- $y(2n) = x(n)$ for all $n \in \omega$;
- $y(n) = \sigma(y{\restriction}n)$ for all odd $n \in \omega$.

The statement that a set $A \subseteq X^\omega$ is determined can then be rephrased as asserting the existence of a strategy σ such that one of the two following statements holds.

- For every $x \in X^\omega$, $\sigma * x$ is in A.
- For every $x \in X^\omega$, $x * \sigma$ is not in A.

1.1.2 Remark. We list four classical consequences of AD, the details of which are presented in Chapter 33 of [8] and Sections 27 and 28 of [10]. An abbreviated history of determinacy axioms can be found in [26].

1. Jan Mycielski observed that AD implies $\mathsf{CC}_{\mathbb{R}}$ (as defined in Section 0.4). To see this, let A_i ($i \in \omega$) be nonempty subsets of ω^ω, and consider the game where I plays $i \in \omega$ and then II must list the values of a member of A_i.

2. Morton Davis proved that AD implies that every uncountable subset of ω^ω contains a perfect set. To see this, consider a set $A \subseteq \omega^\omega$, and the game where players I and II collaborate to build an $x \in \omega^\omega$, with player II choosing individual digits as usual, but player I allowed to play finite sequences from ω, with I winning if the concatenation of these moves is in A. Player II has a winning strategy if and only if A is countable, and a winning strategy for I induces a perfect subset of A. It follows from ZF that if every uncountable subset of ω^ω contains a perfect set then there is no injection from ω_1 into ω^ω (such an injection would give a wellordering of ω^ω in ordertype ω_1). The nonexistence of an injection from ω_1 into ω^ω

(which we will denote by writing $\aleph_1 \not\leq 2^{\aleph_0}$) is equivalent to the assertion that for any inner model M satisfying Choice (equivalently, for all models of the form $L[S]$, for S a subset of ω_1), and any countable ordinal α, $\mathcal{P}(\alpha) \cap M$ is countable. Since ZF implies the existence of partition of $\mathcal{P}(\omega)$ into $\aleph_1$ many sets, this shows that the statement AD_{ω_1} is inconsistent with ZF.

3. Stefan Banach proved that AD implies that every subset of ω^ω (similarly, every subset of 2^ω) has the property of Baire, i.e., that for each $A \subseteq \omega^\omega$ there exists an open $U \subseteq \omega^\omega$ such that the symmetric difference $A \triangle U$ is meager. To see this, fix a bijection $\pi\colon \omega \to \omega^{<\omega}$ and, given $A \subseteq \omega^\omega$, let B be the set of $x \in \omega^\omega$ for which the concatenation of the values $\pi(x(i))$ $(i \in \omega)$ is in A. If player I was a winning strategy in $\mathcal{G}_B$, then A is relatively comeager in some open set; if player II does, then A is meager. Wacław Sierpiński that nonpricipal ultrafilters on ω (considered as subsets of 2^ω) do not have the property of Baire. It follows that nonpricipal ultrafilters on ω don't exist under AD, and from this that, under AD, every nonprincipal ultrafilter on any set is countably complete.

4. Robert Solovay proved AD implies that the club filter on ω_1 is an ultrafilter (which is countably complete by either part (1) or (3) above). This gives another proof that (under AD) there is no injection from ω_1 into $\mathcal{P}(\omega)$. We present a proof that AD implies the measurabilty of ω_1 (not Solovay's original proof) in Remark 1.2.7.

Given $\Gamma \subseteq \mathcal{P}(\omega^\omega)$, we will write $\mathsf{Baire}(\Gamma)$ for the assertion that every element of Γ has the property of Baire.

1.2 Turing determinacy

In this section we prove Martin's theorem that under AD the cone measure on the Turing degrees is an ultrafilter. We specify a class of relations for which the notion of cone measure applies, and call these *ordered equivalence relations*. We define a relative coarseness relation (being *as thick as*) on ordered equivalence relations for which the property of the associated cone measure being an ultrafilter is preserved upwards. We prove Martin's theorem for the smallest relation for which his original proof applies, which we call $\leq_{\mathrm{Ma}}$. In practice we will often use his theorem with Turing equivalence and coarser relations, in the sense introduced in Definition 1.2.3.

Fixing an enumeration $\langle \sigma_n : n < \omega \rangle$ of $\omega^{<\omega}$, we can associate to each $x \in \omega^\omega$ a function (strategy) $\pi_x\colon \omega^{<\omega} \to \omega$ defined by the formula $\pi_x(\sigma_n) = x(n)$.

Let even: $\omega^\omega \to \omega^\omega$ be the function defined by letting $\mathrm{even}(y)(n) = y(2n)$ for each $n \in \omega$ and let odd: $\omega^\omega \to \omega^\omega$ be the function defined by letting $\mathrm{odd}(y)(n) = y(2n+1)$ for each $n \in \omega$. Given $x \in \omega^\omega$, let

- $I^*_x\colon \omega^\omega \to \omega^\omega$ be the function defined by letting $I^*_x(y)$ be $\pi_y * x$, i.e., the result of playing x for player II against the strategy π_y for player I;

- $II^*_x\colon \omega^\omega \to \omega^\omega$ be the function defined by letting $II^*_x(y)$ be $x * \pi_y$, i.e., the result of playing x for player I against the strategy π_y for player II;

- $\mathcal{F}_x$ be the smallest set of functions on ω^ω which is closed under composition and contains the identity function, even, odd and the functions $I^*_{f(x)}$ and $II^*_{f(x)}$ for each $f \in \mathcal{F}_x$.

Let $\leq_{\mathrm{Ma}}$ be the reflexive binary relation on ω^ω defined by setting $y \leq_{\mathrm{Ma}} x$ if $y = f(x)$ for some $f \in \mathcal{F}_x$. Observe that if $y \leq_{\mathrm{Ma}} x$ then $\mathcal{F}_y \subseteq \mathcal{F}_x$; it follows from this that $\leq_{\mathrm{Ma}}$ is transitive. The functions even and odd can be used to show that $\leq_{\mathrm{Ma}}$ is countably directed.

We let $\equiv_{\mathrm{Ma}}$ denote the equivalence relation $\leq_{\mathrm{Ma}} \cap \geq_{\mathrm{Ma}}$. The ordered pairs $(\equiv_{\mathrm{Tu}}, \leq_{\mathrm{Tu}})$ and $(\equiv_{\mathrm{Ma}}, \leq_{\mathrm{Ma}})$ are instances of the following definition, with X as ω^ω.

1.2.1 Definition. An *ordered equivalence relation* is a pair $(E, \leq_E)$ where E is an equivalence relation on a set X and $\leq_E$ is a preorder order on X such that $E = \leq_E \cap \geq_E$. We say that $(E, \leq_E)$ is an ordered equivalence relation *on* X. We write $[x]_E$ for the E-equivalence class of x, and $\mathcal{C}_E$ for the set of E-equivalence classes (or E-*classes*).

Given an ordered equivalence relation $(E, \leq_E)$ on ω^ω, and $x \in \omega^\omega$, we define the *upward* $\leq_E$-*cone* of x to be $\mathcal{U}_E(x) = \{[y]_E : y \geq_E x\}$ and the *downward* $\leq_E$-cone of x to be $\mathcal{D}_E(x) = \{[y]_E : y \leq_E x\}$. In each case we say that x is a *base* for the corresponding cone. Since we are typically interested in upward cones, we often write "cone" for "upward cone". The $\leq_E$-*cone measure* is

$$\{A \subseteq \{[x]_E : x \in X\} : \exists x \in X\, \mathcal{U}_E(x) \subseteq A\}.$$

We will write $[x]_{\mathrm{Ma}}$, $\mathcal{C}_{\mathrm{Ma}}$, $[x]_{\mathrm{Tu}}$ and $\mathcal{C}_{\mathrm{Tu}}$ instead of the more cumbersome forms using $\equiv_{\mathrm{Ma}}$ and $\equiv_{\mathrm{Tu}}$. We also write μ_{Ma} and μ_{Tu} for the corresponding cone measures, which are ultrafilters under AD. We call μ_{Tu} the *Turing measure*. A *Turing cone* is an upward cone for $\leq_{\mathrm{Tu}}$. We also use the expression *Turing cone* for the union of the members of a Turing cone as just defined (i.e., for the set of reals corresponding to the set of equivalence classes).

Theorem 1.2.2 (Martin)**.** *Suppose that* AD *holds. Then* μ_{Ma} *is a countably complete ultrafilter.*

Proof. Since AD implies $\mathsf{CC}_{\mathbb{R}}$, μ_{Ma} is countably complete. It suffices then to show that each subset of $\mathcal{C}_{\mathrm{Ma}}$ either contains or is disjoint from a cone. Fix $A \subseteq \mathcal{C}_{\mathrm{Ma}}$ and consider the game $\mathcal{G}_{A^*}$, where A^* is the set of $x \in \omega^\omega$ such that $[x]_{\mathrm{Ma}}$ is in A. Let π be a winning strategy (for either player), and let $x \in \omega^\omega$ be such that $\pi = \pi_x$. Let $y \in \omega^\omega$ be such that $y \geq_{\mathrm{Ma}} x$. If π is a winning strategy for player I, then $[\pi_x * y]_{\mathrm{Ma}}$ is in A, and $[y]_{\mathrm{Ma}} = [\pi_x * y]_{\mathrm{Ma}}$ (one direction of the equivalence uses the function $I^*_y \circ g$, where $g \in \mathcal{F}_y$ is such that $x = g(y)$; the other uses the function odd). If π is a winning strategy for player II, then $[y * \pi_x]_{\mathrm{Ma}}$ is not in A, and $[y]_{\mathrm{Ma}} = [y * \pi_x]_{\mathrm{Ma}}$. $\square$

We will typically be considering equivalence relations satisfying the following definition, with $\leq_{\mathrm{Ma}}$ or $\leq_{\mathrm{Tu}}$ in the role of F, especially those of the form of the second type listed in Example 1.2.4, with S a set of ordinals.

1.2.3 Definition. Given two ordered equivalence relations $(E, \leq_E)$ and $(F, \leq_F)$ on the same underlying set X, say that $(E, \leq_E)$ is *as thick as* $(F, \leq_F)$ if, for all x, y in X,

1. if $x \leq_F y$ then $x \leq_E y$;
2. if $x \leq_E y$, then for some $z \in [y]_E$, $x \leq_F z$.

Every ordered equivalence relation is trivially as thick as itself. We leave it as an exercise to check that the "as thick as" relation is transitive. Given the first condition in Definition 1.2.3, the second condition is equivalent to saying that for all $x \in X$,

$$\mathcal{U}_E(x) = \{[y]_E : x \leq_F y\}.$$

If $\mathsf{CC}_{\mathbb{R}}$ holds and $(E, \leq_E)$ and $(F, \leq_F)$ are ordered equivalence relations, with F a subset of E and F countably directed, then E is also countably directed.

1.2.4 Example. The following are examples of ordered equivalence relations on ω^ω which are as thick as $(\equiv_{\mathrm{Ma}}, \leq_{\mathrm{Ma}})$.

1. $(\equiv_{\mathrm{Tu}}, \leq_{\mathrm{Tu}})$
2. For any set S, the equivalence relation $L(S, x) = L(S, y)$, under the order $x \in L(S, y)$. When $S = \emptyset$, the corresponding equivalence classes are called the *constructibility degrees*.
3. For any set S, the equivalence relation $\mathrm{HOD}_{S \cup \{x\}} = \mathrm{HOD}_{S \cup \{y\}}$, under the order $x \in \mathrm{HOD}_{S \cup \{y\}}$.

1.2.5 Remark. Suppose that $(E, \leq_E)$ and $(F, \leq_F)$ are ordered equivalence relations on a set X, $(E, \leq_E)$ is as thick as $(F, \leq_F)$, and μ_F is an ultrafilter on the set of F-classes. Then μ_E is an ultrafilter on the set of E-classes. To see this, let A be a set of E-classes, and let B be the set of F-classes contained in a member of A. Since μ_F is an ultrafilter, we can fix an $x \in \omega^\omega$ such that $\mathcal{U}_F(x)$ is either contained in or disjoint from B. Since $\mathcal{U}_E(x) = \{[y]_E : y \geq_F x\}$, $\mathcal{U}_E(x)$ is either contained in or disjoint from A.

We let $\mathsf{Turing\ Determinacy}$ (TD) denote the statement that the $\leq_{\mathrm{Tu}}$-cone measure is an ultrafilter on the $\equiv_{\mathrm{Tu}}$-classes. A recent result of Peng and Yu [33] shows that TD implies $\mathsf{CC}_{\mathbb{R}}$. Their proof gives the analogous result for the relation $x \in L[S, y]$, for S a set of ordinals and $x, y \in \omega^\omega$, if in addition the sharp of each such pair (S, y) exists. It is open whether either of AD and TD implies $\mathsf{DC}_{\mathbb{R}}$, or whether either of TD and $\mathsf{TD} + \mathsf{DC}_{\mathbb{R}}$ implies AD. Woodin has shown that $\mathsf{TD} + \mathsf{DC}_{\mathbb{R}} + V{=}L(\mathbb{R})$ implies AD [2]. Remark 1.2.5 and Theorem 1.2.2 (or, its proof, in conjunction with the Peng-Yu theorem) give the following.

Corollary 1.2.6 ($\mathsf{ZF} + \mathsf{TD}$). *If $(E, \leq_E)$ is an ordered equivalence relation on ω^ω which is as thick as $\equiv_{\mathrm{M}}$, then μ_E is a countably complete ultrafilter on $\mathcal{C}_E$.*

The following remark gives (from TD) Solovay's theorem that, assuming AD, ω_1 is measurable. It also shows that TD implies that $\aleph_1 \not\leq 2^{\aleph_0}$.

1.2.7 Remark. Let $(E, \leq_E)$ be an ordered equivalence relation on ω^ω such that each downward cone is countable, and suppose that μ_E is a countably complete ultrafilter on $\mathcal{C}_E$. For each $\alpha < \omega_1$, let X_α be the set of $x \in \omega^\omega$ which HC-code α, in the sense of Section 0.2. For each $\sigma \in \mathcal{C}_E$, let α_σ be $\sup\{\alpha < \omega_1 : \exists x \in X_\alpha\, \exists y \in \sigma\, x \leq_E y\}$. Let U be the set of $A \subseteq \omega_1$ for which $\{\sigma \in \mathcal{C}_E : \alpha_\sigma \in A\} \in \mu_E$. Then U is a countably complete ultrafilter on ω_1.

The following example gives consequences of the Turing measure being an ultrafilter. Similar arguments will appear in Chapters 8, 10 and 11.

1.2.8 Example. Let S be a set of ordinals, let $\leq_S$ be the binary relation on ω^ω given by the binary relation $x \in L[S, y]$, let $\equiv_S$ be the corresponding equivalence relation and let μ_S be the $\equiv_S$-cone measure. Assume that μ_S is an ultrafilter. It follows then by Remark 1.2.7 that $\aleph_1 \not\leq 2^{\aleph_0}$. For each $x \in \omega^\omega$, there exist y and z in ω^ω such that y is generic for Sacks forcing over $L[S, x]$ and z codes a wellordering of ω in ordertype $\omega_1^{L[S,x]}$. By standard facts about Sacks forcing [34], $\bigcup_{n \in \omega}\{(2n, x(n)), (2n + 1, y(n))\}$ is an immediate $\leq_S$-successor of x. It follows then that the set of $\leq_S$-successors contains a set in μ_S. Let $f \colon \omega^\omega \to \omega_1$ be the function defined by setting $f(x)$ to be $\omega_1^{L[S,x]}$. Since Sacks forcing preserves ω_1, we have that for every $x \in \omega^\omega$, there exist y, z in ω^ω both $\leq_S$-above x, such that $f(x) = f(y)$ and $f(x) < f(z)$.

Chapter 2

The Wadge hierarchy

In this chapter we review some basic concepts from classical descriptive set theory. In Section 2.1 we introduce Lipschitz determinacy and prove Martin's theorem that the corresponding hierarchy is wellfounded under $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. In Section 2.2 we review the relationship between the Lipschitz degrees and the Wadge degrees, and show that a Wadge class whose rank is a limit ordinal is selfdual if and only if its rank has countable cofinality. In Section 2.3 we introduce boldface and lightface pointclasses. Section 2.4 is about universal sets for pointclasses and the s-m-n property. Finally, Section 2.5 introduces Θ, the least (nonzero) ordinal which is not a surjective image of ω^ω.

2.1 Lipschitz determinacy

Given sets $A, B \subseteq \omega^\omega$, we say that A is *Wadge reducible* to B (and write $A \leq_{\mathrm{Wa}} B$) if there is a continuous function $f\colon \omega^\omega \to \omega^\omega$ such that $A = f^{-1}[B]$. A function $f\colon \omega^\omega \to \omega^\omega$ is *Lipschitz* if, for all $x, y \in \omega^\omega$ and $n \in \omega$, if $x\restriction n = y\restriction n$, then $f(x)\restriction n = f(y)\restriction n$. We say that A is *Lipschitz reducible* to B (and write $A \leq_{\mathrm{L}} B$) if there is a Lipschitz function $f\colon \omega^\omega \to \omega^\omega$ such that $A = f^{-1}[B]$. Since Lipschitz functions are continuous, $A \leq_{\mathrm{L}} B$ implies $A \leq_{\mathrm{W}} B$.

2.1.1 Remark. Since $A = f^{-1}[B]$ implies that $(\omega^\omega \setminus A) = f^{-1}[\omega^\omega \setminus B]$, $A \leq_{\mathrm{W}} B$ implies $(\omega^\omega \setminus A) \leq_{\mathrm{W}} (\omega^\omega \setminus B)$ and $A \leq_{\mathrm{L}} B$ implies $(\omega^\omega \setminus A) \leq_{\mathrm{L}} (\omega^\omega \setminus B)$.

The relations $\leq_{\mathrm{W}}$ and $\leq_{\mathrm{L}}$ are easily seen to be preorders, that is, reflexive and transitive. We write $=_{\mathrm{Wa}}$ and $=_{\mathrm{Li}}$ respectively for the equivalence relations $\leq_{\mathrm{W}} \cap \geq_{\mathrm{Wa}}$ and $\leq_{\mathrm{L}} \cap \geq_{\mathrm{Li}}$, whose equivalence classes are respectively called *Wadge classes* and *Lipschitz classes*. Given $A \subseteq \omega^\omega$, we write $[A]_{\mathrm{Wa}}$ for the Wadge class of A, and $[A]_{\mathrm{Li}}$ for the Lipschitz class of A. We say that a Wadge class or Lipschitz class is *selfdual* if it contains a pair of complements (in which case it is closed under complements, by Remark 2.1.1), and *nonselfdual* otherwise. We write $x <_{\mathrm{Wa}} y$ for $(x \leq_{\mathrm{W}} y) \wedge \neg(y \leq_{\mathrm{W}} x)$ and $x <_{\mathrm{Li}} y$ for $(x \leq_{\mathrm{L}} y) \wedge \neg(y \leq_{\mathrm{L}} x)$.

The following definition converts a pair $A, B \subseteq \omega^\omega$ into a set $A \ominus B \subseteq \omega^\omega$ such that a winning strategy for player II in $\mathcal{G}_{A\ominus B}$ gives a Lipschitz reduction of A to B.

2.1.2 Definition. Let A, B be subsets of ω^ω. We write $A \ominus B$ for the set of $\langle x_i : i \in \omega\rangle$ for which $\langle x_{2i} : i \in \omega\rangle \in A$ if and only if $\langle x_{2i+1} : i \in \omega\rangle \notin B$.

The game $\mathcal{G}_{A\ominus B}$ can then be seen as a game where I plays $x \in \omega^\omega$, II plays $y \in \omega^\omega$, and II wins if and only if $(x \in A \Leftrightarrow y \in B)$. We call this the *Lipschitz game* for the pair (A, B).

2.1.3 Remark. The relation $A \leq_{\mathrm{L}} B$ is equivalent to the assertion that player II has a winning strategy in the game $\mathcal{G}_{A\ominus B}$; it also follows from player I having a winning strategy in $\mathcal{G}_{B\ominus(\omega^\omega\setminus A)}$.

This observation leads to the following fundamental fact.

Theorem 2.1.4 (Wadge)**.** *Let A and B be subsets of ω^ω. If player I has a winning strategy in $\mathcal{G}_{A\ominus B}$, then $(\omega^\omega \setminus B) \leq_{\mathrm{L}} A$. If player II has a winning strategy in $\mathcal{G}_{A\ominus B}$, then $A \leq_{\mathrm{L}} B$.*

Wadge's Theorem (along with Remarks 2.1.1 and 2.1.3) has the following consequences. Proposition 2.1.5 (which can be proved directly from Wadge's Theorem) shows that the ordering on the Lipschitz degrees induced by $\leq_{\mathrm{Li}}$ is almost linear, the exceptions being pairs of the form $[A]_{\mathrm{Li}}$, $[\omega^\omega \setminus A]_{\mathrm{Li}}$, for nonselfdual classes $[A]_{\mathrm{Li}}$. The conclusion of the proposition is sometimes called the Semi-Linear Ordering Principle for Lipschitz maps.

Proposition 2.1.5. *Let A and B be subsets of ω^ω such that $A \ominus B$ and $B \ominus A$ are both determined. If $A \not\leq_{\mathrm{L}} B$ and $B \not\leq_{\mathrm{L}} A$, then $A =_{\mathrm{Li}} (\omega^\omega \setminus B)$.*

Proposition 2.1.6. *Let A and B be subsets of ω^ω such that the sets $B \ominus A$ and $B \ominus (\omega^\omega \setminus A)$ are both determined. If $A <_{\mathrm{Li}} B$, then player I wins both $\mathcal{G}_{B\ominus A}$ and $\mathcal{G}_{B\ominus(\omega^\omega\setminus A)}$.*

Proof. In the case of $B \ominus (\omega^\omega \setminus A)$, one gets otherwise that $A <_{\mathrm{Li}} B \leq_{\mathrm{L}} (\omega^\omega \setminus A)$, and therefore by Remark 2.1.1 that $A =_{\mathrm{Li}} (\omega^\omega \setminus A)$ and $B \leq_{\mathrm{L}} A$, giving a contradiction. □

Proposition 2.1.7. *Let A and B be subsets of ω^ω such that $A \ominus B$ and $B \ominus A$ are both determined. If $A <_{\mathrm{Wa}} B$ then $A <_{\mathrm{Li}} B$.*

Proof. Since $B \not\leq_{\mathrm{W}} A$, $B \not\leq_{\mathrm{L}} A$, so by Proposition 2.1.5, either $A =_{\mathrm{Li}} (\omega^\omega \setminus B)$ or $A \leq_{\mathrm{L}} B$. The former is impossible, as then $A \leq_{\mathrm{W}} B$ would imply that $(\omega^\omega \setminus B) \leq_{\mathrm{W}} (\omega^\omega \setminus A)$ and thus $B \leq_{\mathrm{W}} A$, giving a contradiction. □

We let Lipschitz Determinacy be the statement that $A \ominus B$ is determined for all subsets A, B of ω^ω.

2.1.8 Remark. It is not hard to see that $\mathsf{Lipschitz\ Determinacy}$ implies $\mathsf{CC}_{\mathbb{R}}$ (see [25], for instance; the proof there combines the proof of Theorem 2.5.4 with a standard strategy selection argument, which appears also at the end of the proof of Theorem 13.2.3). Similarly, $\mathsf{Lipschitz\ Determinacy}$ implies that every uncountable subset of ω^ω contains a perfect set. To see this, consider the game $\mathcal{G}_{A\ominus B}$, where A is a Borel set which is not a countable union of closed sets, and B is non-Borel. A winning strategy for player II produces an uncountable analytic subset of B. It follows then that $\mathsf{Lipschitz\ Determinacy}$ implies $\aleph_1 \not\leq 2^{\aleph_0}$.

The following gives Martin's theorem showing that $<_{\mathrm{Li}}$ is wellfounded (assuming $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, for instance). By Proposition 2.1.7, theorem implies the corresponding version for $<_{\mathrm{Wa}}$.

Theorem 2.1.9 (Martin)**.** *If* $\mathsf{Lipschitz\ Determinacy} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$ *holds, then there does not exist a sequence* $\langle A_i : i \in \omega\rangle$ *consisting of subsets of* ω^ω*, such that* $A_{i+1} <_{\mathrm{Li}} A_i$ *for each* $i \in \omega$*.*

Proof. Suppose towards a contradiction that such a sequence $\langle A_i : i < \omega\rangle$ does exist. Applying Proposition 2.1.6 and $\mathsf{CC}_{\mathbb{R}}$ we can fix winning strategies f_i^0, f_i^1 $(i \in \omega)$ for player I in the games $\mathcal{G}_{A_i \ominus A_{i+1}}$ and $\mathcal{G}_{A_i \ominus (\omega^\omega \setminus A_{i+1})}$, respectively.

For each $x \in {}^\omega 2$, define $y_i(x) \in \omega^\omega$ $(i \in \omega)$ by letting $y_i(x)(k)$ be

$$f_i^{x(i)}(\langle y_i(x)(0), y_{i+1}(x)(0), \ldots, y_i(x)(k-1), y_{i+1}(x)(k-1)\rangle).$$

Note that the values $y_i(x)(k)$ are defined recursively in k, for all x and i simultaneously. For each fixed x, each $y_i(x)$ is the set of values given by the strategy $f_i^{x(i)}$ when playing against the moves for player II listed by $y_{i+1}(x)$. It follows that for each $i \in \omega$, if $x(i) = 0$ then $y_i(x) \in A_i \Leftrightarrow y_{i+1}(x) \notin A_{i+1}$, and if $x(i) = 1$ then $y_i(x) \in A_i \Leftrightarrow y_{i+1}(x) \in A_{i+1}$.

Let Z be the set of $x \in 2^\omega$ for which $y_0(x) \in A_0$. It suffices to show that whenever $x, x' \in 2^\omega$ differ at exactly one point, $x \in Z$ if and only if $x' \notin Z$, since no subset of 2^ω with the property of Baire can have this property.

First observe that if $x, x' \in {}^\omega 2$ and $i \in \omega$ are such that $x(j) = x'(j)$ for all $j \geq i$, then $y_i(x) = y_i(x')$. If i_0 is the unique $i \in \omega$ such that $x(i) \neq x'(i)$, then $y_{i_0+1}(x) = y_{i_0+1}(x')$. Since $x(i_0) \neq x'(i_0)$, it follows that $y_{i_0}(x) \in A_{i_0}$ if and only if $y_{i_0}(x') \notin A_{i_0}$. Since $x(i) = x'(i)$ for all $i < i_0$, it follows that $y_i(x) \in A_i$ if and only if $y_i(x') \notin A_i$ for all such i. $\square$

Recall that a preorder is *wellfounded* if every nonempty subset of its domain has a minimal element. The additional assumption of $\mathsf{DC}_{\mathbb{R}}$ derives the wellfoundedness of $<_{\mathrm{Li}}$ from Theorem 2.1.9. Since every Wadge degree is a union of Lipschitz degrees, Corollary 2.1.10 holds for $<_{\mathrm{Wa}}$ as well. Theorem 2.2.8 shows (assuming $\mathsf{Lipschitz\ Determinacy}$ and $\mathsf{Baire}(\mathcal{P}(\omega^\omega))$ but not $\mathsf{DC}_{\mathbb{R}}$) that $<_{\mathrm{Li}}$ is wellfounded if and only if $<_{\mathrm{Wa}}$ is.

Corollary 2.1.10 (Martin)**.** *If* $\mathsf{DC}_{\mathbb{R}} + \mathsf{Lipschitz\ Determinacy} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$ *holds, then* $<_{\mathrm{Li}}$ *and* $<_{\mathrm{Wa}}$ *are wellfounded.*

Proof. If $<_{\mathrm{Li}}$ is illfounded, then $\mathsf{DC}_{\mathbb{R}}$ gives a sequence $\langle A_i : i \in \omega\rangle$ of subsets of ω^ω such that $A_{i+1} <_{\mathrm{Li}} A_i$ for each $i \in \omega$. $\square$

As noted in Section 0.4, it is open whether AD implies $\mathsf{DC}_{\mathbb{R}}$. It is also open whether $\mathsf{DC}_{\mathbb{R}}$ is needed for Corollary 2.1.10.

The *Lipschitz rank* $\mathrm{LR}(A)$ (respectively, *Wadge rank* $\mathrm{WR}(A)$) of a set $A \subseteq \omega^\omega$ is recursively defined to the be the least ordinal greater than the Lipschitz (Wadge) rank of every $B \subseteq \omega^\omega$ with $B <_{\mathrm{Li}} A$ ($B <_{\mathrm{Wa}} A$), if this is defined (that is, the functions LR and WR are the canonical rank functions on the wellfounded initial segments of $\leq_{\mathrm{Li}}$ and $\leq_{\mathrm{Wa}}$). By Proposition 2.1.5, for each ordinal α, the subsets of ω^ω of Lipschitz rank (Wadge rank) α (if there are any) consist either of a single Lipschitz (Wadge) class or a pair of Lipschitz (Wadge) classes corresponding to complements. Theorem 2.2.8 says more about the relationship between the two sets of classes.

2.2 Lipschitz degrees and Wadge degrees

Following [40], we give some more details on the structure of the Lipschitz and Wadge hierarchies, using $\mathsf{Lipschitz\ Determinacy}$ and the assumption that all sets of reals have the Baire property, but not $\mathsf{DC}_{\mathbb{R}}$. We begin by noting that $\emptyset$ and ω^ω are the only subsets of ω^ω of Lipschitz or Wadge rank 0, and that they are Wadge (and thus Lipschitz) inequivalent.

We write $s^\frown x$ for the concatenation of s and x, where s is a finite sequence and x is either finite or infinite. Given $A \subseteq \omega^\omega$ and $i \in \omega$, we write $A^{(i)}$ for $\{\langle i\rangle^\frown x : x \in A\}$ and $A_{(i)}$ for $\{x : \langle i\rangle^\frown x \in A\}$. Note that $(A^{(i)})_{(i)} = A$, while $(A_{(i)})^{(i)} = \{x \in A : x(0) = i\}$.

Proposition 2.2.1 shows (among other things) that $[A^{(i)}]_{\mathrm{Li}}$ is the least Lipschitz class above $[A]_{\mathrm{Li}}$ when A is selfdual.

Proposition 2.2.1 ($\mathsf{ZF}+\mathsf{Lipschitz\ Determinacy}$)**.** *Let $A \subseteq \omega^\omega$ be such that $[A]_{\mathrm{Li}}$ is selfdual, and let i be an element of ω. Then*

1. *$[A^{(i)}]_{\mathrm{Li}}$ is selfdual, and $[A^{(i)}]_{\mathrm{Li}}$ is the $\leq_{\mathrm{L}}$-least Lipschitz class above $[A]_{\mathrm{Li}}$;*
2. *$A_{(i)} <_{\mathrm{Li}} A$.*

Proof. The function $f(x) = \langle i\rangle^\frown x$ shows that $A \leq_{\mathrm{L}} A^{(i)}$ and $A_{(i)} \leq_{\mathrm{L}} A$. To see that $A <_{\mathrm{Li}} A^{(i)}$ and $A_{(i)} <_{\mathrm{Li}} A$, note that whenever $f\colon \omega^\omega \to \omega^\omega$ has the property that for each $n \in \omega$, $f(x)\restriction n$ depends only on $x\restriction n$, there is an $x \in \omega^\omega$ such that $x(0) = i$ and

$$f(x) = \langle x(1), x(2), x(3), \ldots\rangle.$$

It follows that no such f witnesses that $A^{(i)} \leq_{\mathrm{L}} (\omega^\omega \setminus A)$ or $(\omega^\omega \setminus A) \leq_{\mathrm{L}} A_{(i)}$, which is enough since $[A]_{\mathrm{Li}}$ is selfdual.

Using the assumption that A is selfdual, it is easy to that $[A^{(i)}]_{\mathrm{Li}}$ is selfdual. Finally, if $C \subseteq \omega^\omega$ is such that $A <_{\mathrm{Li}} C$, then player I wins $\mathcal{G}_{C\ominus(\omega^\omega\setminus A)}$, by

Proposition 2.1.6. Any strategy witnessing this can be used to show that $A^{(i)} \leq_{\mathrm{L}} C$ (using the fact that $C \neq \omega^\omega$ to deal with inputs whose first coordinates are not i). $\square$

2.2.2 Remark. For any $A \subseteq \omega^\omega$ and $i \in \omega$, $[A]_{\mathrm{Wa}} = [A^{(i)}]_{\mathrm{Wa}}$. For the direction left open by Proposition 2.2.1, consider the function on ω^ω which removes the first coordinate of its input.

2.2.3 Remark. An argument similar to the proof of Proposition 2.2.1 shows that $[\bigcup_{i\in\omega} A^{(i)}]_{\mathrm{Li}}$ is also the $\leq_{\mathrm{L}}$-least Lipschitz class above $[A]_{\mathrm{Li}}$ (again assuming $\mathsf{Lipschitz\ Determinacy}$). It is also not hard to show directly that $A^{(j)} =_{\mathrm{Li}} \bigcup_{i\in\omega} A^{(i)}$ for each $j \in \omega$.

2.2.4 Remark. Let $\pi \colon \omega \times \omega \to \omega$ be a bijection, and suppose that A_i $(i \in \omega)$ are subsets of ω^ω. Let $C \subseteq \omega^\omega$ be the set of functions of the form

$$\langle \pi(i, x(0)), x(1), x(2), \ldots \rangle$$

for $i \in \omega$ and $x \in A_i$. Then $A_i \leq_{\mathrm{L}} C$ for all $i \in \omega$. If $\mathsf{CC}_{\mathbb{R}}$ holds, then for all $D \subseteq \omega^\omega$, if $A_i \leq_{\mathrm{L}} D$ for all $i \in \omega$ then $C \leq_{\mathrm{L}} D$. If $\{A_i : i \in \omega\}$ does not have a $\leq_{\mathrm{L}}$-maximal element, then $A_i <_{\mathrm{Li}} C$ for all $i \in \omega$.

If $\mathsf{Lipschitz\ Determinacy}$ holds, then $[C]_{\mathrm{Li}}$ is nonselfdual exactly in the case where $\{A_i : i \in \omega\}$ has a $\leq_{\mathrm{L}}$-maximal element whose Lipschitz class is nonselfdual. To see this, note first of all that $[C]_{\mathrm{Li}}$ is clearly selfdual in the case where $\{A_i : i \in \omega\}$ has a $\leq_{\mathrm{L}}$-maximal element whose Lipschitz class is selfdual, as $[C]_{\mathrm{Li}}$ is equal to this class (similarly, if $\{A_i : i \in \omega\}$ has a $\leq_{\mathrm{W}}$-maximal element, then $[C]_{\mathrm{Wa}}$ is equal to this class). In the remaining case, $\{[A_i]_{\mathrm{Li}} : i \in \omega\}$ has the same supremum as

$$\{[A_i]_{\mathrm{Li}} : i \in \omega\} \cup \{[\omega^\omega \setminus A_i]_{\mathrm{Li}} : i \in \omega\}.$$

Running the construction of C above with the set $\{A_i : i \in \omega\} \cup \{\omega^\omega \setminus A_i : i \in \omega\}$ clearly gives a selfdual Lipschitz class (equal to $[C]_{\mathrm{Li}}$). This argument gives the following facts, for any $A \subseteq \omega^\omega$.

- If $[A]_{\mathrm{Li}}$ is nonselfdual, then the pair $[A]_{\mathrm{Li}}, [\omega^\omega \setminus A]_{\mathrm{Li}}$ has a $\leq_{\mathrm{L}}$-least upper bound, and this upper bound is selfdual.
- If $[A]_{\mathrm{Li}}$ is the $\leq_{\mathrm{L}}$-supremum of a countable set of Lipschitz classes strictly below it, and $\mathsf{Lipschitz\ Determinacy}$ holds, then $[A]_{\mathrm{Li}}$ is selfdual (Proposition 2.2.5 below gives the converse, for non-successor classes).

From Proposition 2.2.1 and Remarks 2.2.2 and 2.2.4 it follows that for all $A \subseteq \omega^\omega$ such that $[A]_{\mathrm{Li}}$ is selfdual, the first ω_1 many Lipschitz classes above $[A]_{\mathrm{Li}}$ are all selfdual and contained in $[A]_{\mathrm{Wa}}$. Proposition 2.2.5 shows (assuming $\mathsf{Lipschitz\ Determinacy}$ and that the Lipschitz rank of each subset of ω^ω exists) that the nonselfdual Lipschitz classes are exactly those whose Lipschitz rank is either 0 or an ordinal of uncountable cofinality.

Proposition 2.2.5 (ZF + Lipschitz Determinacy). *Let $A \subseteq \omega^\omega$ be such that $[A]_{\mathrm{Li}}$ is selfdual, and $[A]_{\mathrm{Li}}$ is not the $\leq_{\mathrm{L}}$-least Lipschitz class above any other class. Then $[A]_{\mathrm{Li}}$ is the $\leq_{\mathrm{L}}$-supremum of a countable set of Lipschitz classes strictly below it.*

Proof. Let $\pi\colon \omega \times \omega \to \omega$ be a bijection. For each $i \in \omega$, let B_i be the set of sequences of the form

$$\langle \pi(n, x(0)), x(1), x(2), \ldots \rangle$$

such that either n is even and $\langle i, x(0), x(1), x(2), \ldots \rangle$ is in A, or n is odd and $\langle i, x(0), x(1), x(2), \ldots \rangle$ is not in A. Then by Remark 2.2.4 $[B_i]_{\mathrm{Li}}$ is the least upper bound of the pair $\{[A_{(i)}]_{\mathrm{Li}}, [(\omega^\omega \setminus A)_{(i)}]_{\mathrm{Li}}\}$ and B_i is selfdual. By Proposition 2.2.1, and the assumption that $[A]_{\mathrm{Li}}$ is not the $\leq_{\mathrm{L}}$-least Lipschitz class above any other class, we have that $B_i <_{\mathrm{Li}} A$.

For each $i \in \omega$, let C_i be the set of $x \in \omega^\omega$ such that $x(0) = i$ and $\langle x(1), x(2), \ldots \rangle \in B_i$. By Proposition 2.2.1, each $[C_i]_{\mathrm{Li}}$ is the $<_{\mathrm{Li}}$-successor of the corresponding $[B_i]_{\mathrm{Li}}$. Since $[A]_{\mathrm{Li}}$ is not a successor class, it follows that $B_i <_{\mathrm{Li}} C_i <_{\mathrm{Li}} A$ holds, for each $i \in \omega$. Let D be the set constructed from $\{C_i : i \in \omega\}$ as in Remark 2.2.4. Then $[D]_{\mathrm{Li}}$ is the supremum of $\{[C_i]_{\mathrm{Li}} : i < \omega\}$, and it suffices to see that $A \leq_{\mathrm{L}} D$.

Let $g\colon {}^{<\omega}\omega \to \omega$ be such that for all $n \in \omega \setminus 2$,

$$g(\langle i_0, \ldots, i_n \rangle) = \langle \pi(i_0, i_0), \pi(0, i_1), i_2, \ldots, i_n \rangle.$$

Let $g^*\colon \omega^\omega \to \omega^\omega$ be the Lipschitz function induced by g. Let us see that g^* witnesses that $A \leq_{\mathrm{L}} D$. Fix $x \in \omega^\omega$, and let $g^*(x)$ have the form

$$\langle \pi(i_0, y(0)), y(1), y(2), \ldots \rangle$$

for some $i_0 \in \omega$ and some $y \in \omega^\omega$. Then $i_0 = x(0) = y(0)$, $y(1) = \pi(0, x(1))$ and $y(i) = x(i)$ for all $i \in \omega \setminus 2$. We want to see that $x \in A$ if and only if $g^*(x) \in D$. Now, $g^*(x)$ is in D if and only if $y \in C_{i_0}$, which in turn happens if and only if $\langle \pi(0, x(1)), x(2), x(3), \ldots \rangle$ is in B_{i_0}, which happens if and only if x is in A. $\square$

The following proposition completes the analysis of which Wadge classes are selfdual classes, as well as the relationship between the Lipschitz classes and the Wadge classes.

Theorem 2.2.6 (Steel, Van Wesep). *Suppose that*

$$\text{Lipschitz Determinacy} + \text{Baire}(\mathcal{P}(\omega^\omega))$$

holds. For all $A \subseteq \omega^\omega$, if $[A]_{\mathrm{Wa}}$ is selfdual, then so is $[A]_{\mathrm{Li}}$.

Proof. Suppose towards a contradiction that $[A]_{\mathrm{Wa}}$ is selfdual but $[A]_{\mathrm{Li}}$ is not. By Lipschitz Determinacy, if $[A]_{\mathrm{Li}}$ is not selfdual, then player I wins $\mathcal{G}_{A \ominus (\omega^\omega \setminus A)}$. Let σ be a strategy witnessing this.

Consider the following game between players I and II (which is called the *Wadge game*). For each $i \in \omega$, player I plays a value $x(i)$, and player II either passes or plays a value $y(j)$, for j the least $k \in i+1$ for which a value for $y(k)$ has not yet been chosen. If at the end of the game there is a $j \in \omega$ for which $y(j)$ has not been chosen, then II loses. Otherwise II wins if and only if $x \in A \Leftrightarrow y \notin A$. The statement that $A \leq_{\mathrm{W}} (\omega^\omega \setminus A)$ is equivalent to the existence of a winning strategy for II. Let τ be such a strategy.

For each positive $n \in \omega$ and each sequence $\bar{c} = \langle c_m : m < n\rangle \in 3^n$, there is a unique (possibly partial) function $s_{\bar{c}}$ on $n \times \omega$ satisfying the following conditions.

- For each $m < n$, the set of i for which $(m, i) \in \mathrm{dom}(s_{\bar{c}})$ is an ordinal $\alpha_m \in \omega + 1$. We let t_m be the function with domain α_m such that $t_m(i) = s_{\bar{c}}(m, i)$ for all $i < \alpha_m$.

- The largest $m < n$ for which $\alpha_m > 0$ is the largest $m < n$ for which $c_m = 2$, and for this m, $\alpha_m = 1$ and $s_{\bar{c}}(m, 0) = \sigma(\langle\rangle)$.

- For each $m < n - 1$ such that $c_m = 2$, $\alpha_m = \alpha_{m+1} + 1$, and for each $i < \alpha_m$, $s_{\bar{c}}(m, i)$ is the response given by σ when player II plays $t_m \restriction i$.

- For each $m < n - 1$ such that $c_m = 0$, $\alpha_m = \alpha_{m+1}$, and $t_m = t_{m+1}$.

- For each $m < n - 1$ such that $c_m = 1$, t_m is the longest sequence of nonpassing moves made by τ in response to t_{m+1}.

Choose integers i_k $(k \in \omega)$ so that $i_0 = 0$, $i_{k+1} > i_k + 1$ for all $k \in \omega$ and, for each sequence $\bar{c} = \langle c_m : m < i_{k+1}\rangle$ in 3^n, if $c_m = 2$ for all

$$m \in i_{k+1} \setminus \{i_p : p \leq k\},$$

then the corresponding value α_j is at least k for each $j \leq i_k$.

For each $x \in {}^\omega 2$ and each $m \in \omega$, let c^x_m be $x(k)$ if $m = i_k$ for some $k \in \omega$, and let c^x_m be 2 otherwise. For each such x there is a unique sequence $\langle y^x_m : m \in \omega\rangle \in (\omega^\omega)^\omega$ such that for each $m \in \omega$,

- if $c^x_m = 2$, then y^x_m is the sequence produced by σ when player II plays y^x_{m+1};

- if $c^x_m = 0$, then $y^x_m = y^x_{m+1}$;

- if $c^x_m = 1$, then y^x_m is the sequence produced by τ when player I plays y^x_{m+1}.

Then as in the proof of Theorem 2.1.9, if $x, x' \in 2^\omega$ and $k \in \omega$ are such that $x(p) = x'(p)$ for all $p > k$, then $y^x_m = y^{x'}_m$ for all $m > i_k$. So again, if we let Z be the set of x for which $y^x_0 \in A$, Z cannot have the property of Baire, since whenever x and x' disagree at exactly one point, exactly one of them will be in Z. $\square$

Theorem 2.2.6 has the following corollary, which shows that the nonselfdual Lipschitz classes and Wadge classes coincide.

Corollary 2.2.7. *Suppose that* $\mathsf{Lipschitz\ Determinacy} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$ *holds. Let* A *be a subset of* ω^ω. *If* $[A]_{\mathrm{Li}}$ *is nonselfdual, then* $[A]_{\mathrm{Li}} = [A]_{\mathrm{Wa}}$.

Proof. Supposing otherwise, fix $B \in [A]_{\mathrm{Wa}} \setminus [A]_{\mathrm{Li}}$. By Wadge's Theorem (Theorem 2.1.4), either $(\omega^\omega \setminus A) \leq_{\mathrm{L}} B$ or $(\omega^\omega \setminus B) \leq_{\mathrm{L}} A$. Each of these implies that $[A]_{\mathrm{Wa}}$ is selfdual, giving a condradiction by Theorem 2.2.6. □

Summarizing, we have the following.

Theorem 2.2.8. *Suppose that* $\mathsf{Lipschitz\ Determinacy} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$ *holds.*

1. *The minimal Wadge classes consist of the singletons* $\{\emptyset\}$ *and* $\{\omega^\omega\}$.
2. *Each selfdual Wadge class contains* $\aleph_1$ *many selfdual Lipschitz classes, and these are ordered in ordertype* ω_1 *by* $\leq_{\mathrm{L}}$.
3. *Each nonselfdual Wage class is equivalent to the corresponding Lipschitz class, and has a selfdual class as an immediate successor.*
4. *A Wadge class which is neither minimal nor a successor is selfdual if and only if it is the supremum of a countable set of classes strictly below it.*

As we show in Proposition 2.5.4, a straightforward diagonal argument (from [36]) shows that there is no largest Wadge degree, if $\mathsf{Lipschitz\ Determinacy}$ holds.

2.2.9 Remark. The material in this section does not give a definition for the least Wadge class above a given selfdual class, or show (without assuming $\mathsf{DC}_{\mathbb{R}}$) that such a class exists. Adding $\mathsf{DC}_{\mathbb{R}}$ to the hypotheses of Theorem 2.2.8 $[A]_{\mathrm{Wa}}$ has a pair of nonselfdual classes as immediate successors, the Wadge classes corresponding to the $<_{\mathrm{Li}}$-least Lipschitz classes above all the Lipschitz classes contained in $[A]_{\mathrm{Wa}}$.

2.3 Pointclasses

The notions of Wadge reducibility and Wadge rank naturally generalize to other topological spaces. In general, we say that for any pair of topological spaces X and Y, and any sets $A \subseteq X$ and $B \subseteq Y$, that $A \leq_{\mathrm{Wa}} B$ if there is a continuous function $f \colon X \to Y$ such that $A = f^{-1}[B]$. We will be concerned only with topological spaces of the form $X_1 \times \cdots \times X_n$ for some positive $n \in \omega$, where at least one X_i is ω^ω, and each X_i is either ω^ω or ω. Throughout this book, we let $\mathcal{X}$ be the collection of such spaces; these spaces are all homeomorphic with ω^ω.

A *pointset* is a subset of a member of $\mathcal{X}$. A *pointclass* is a set of pointsets. A *boldface pointclass* is a pointclass closed under continuous preimages, i.e., the union of an initial segment of the Wadge hierarchy on pointsets.

The *cofinality* of a pointclass Γ is the least ordinal cardinality of a cofinal subset of $\{[A]_{\mathrm{Wa}} : A \in \Gamma \cap \mathcal{P}(\omega^\omega)\}$ under the order induced by $<_{\mathrm{Wa}}$, if any such cardinality exists. We say that Γ has uncountably cofinality if its cofinality is not countable.

Given a finite sequence $s \in {}^{<\omega}\omega$, we let $[s] = \{x \in \omega^\omega \mid x \restriction |s| = s\}$. A *basic open interval* of a space $X_1 \times \cdots X_n$ in $\mathcal{X}$ is a product of the form $a_1 \times \cdots \times a_n$, where, for each $i \in \{1, \ldots, n\}$,

- a_i is of the form $[s]$, for some $s \in {}^{<\omega}\omega$, if $X_i = \omega^\omega$;
- a_i is either $\emptyset$, ω or $\{m\}$ for some $m \in \omega$ if $X_i = \omega$.

We say that continuous functions $f \colon X_1 \times \cdots \times X_n \to Y_1 \times \cdots \times Y_m$ between spaces in $\mathcal{X}$ is *recursive* if the set of pairs of basic open intervals $U \subseteq X$, $V \subseteq Y$ for which $f[U] \subseteq V$ is recursive (i.e., Δ_1-definable over HF). A *lightface pointclass* is a pointclass closed under preimages of continuous functions which are recursive. Under these definitions boldface pointclasses are also lightface. Our definitions may be nonstandard; in any case, lightface pointclasses are rarely mentioned in this book.

Given $X \in \mathcal{X}$ and $A \subseteq X$, we write $\check{A}$ for $X \setminus A$. Given a pointclass Γ, we write $\check{\Gamma}$ for $\{\check{A} : A \in \Gamma\}$. We say that a pointclass Γ is *selfdual* if $\Gamma = \check{\Gamma}$; otherwise it is *nonselfdual*. Observe that Γ is a boldface pointclass if and only if $\check{\Gamma}$ is.

2.3.1 Example. The collection of analytic subsets of spaces in $\mathcal{X}$ (i.e., $\underset{\sim}{\Sigma}^1_1$) is a nonselfdual boldface pointclass, as are the projective pointclasses $\underset{\sim}{\Sigma}^1_n$ and $\underset{\sim}{\Pi}^1_n$ for all $n \in \omega$. The pointclasses and $\underset{\sim}{\Delta}^1_n = \underset{\sim}{\Sigma}^1_n \cap \underset{\sim}{\Pi}^1_n$ are boldface and selfdual. Assuming $\mathsf{CC}_{\mathbb{R}}$, all of the classes mentioned in this example are closed under countable unions and intersections.

The following is an immediate, but useful, corollary of Theorem 2.1.4 (and Remark 2.1.1).

Corollary 2.3.2 ($\mathsf{ZF} + \mathsf{Lipschitz\ Determinacy}$)**.** *Let Γ be a nonselfdual boldface pointclass, and suppose that Λ is a boldface pointclass properly containing Γ. Then $\check{\Gamma} \subseteq \Lambda$.*

A member A of a pointclass Γ is *complete* for Γ (or Γ-*complete*) if every member of Γ is a continuous preimage of (e.g., Wadge below) A.

Proposition 2.3.3. *Suppose that* $\mathsf{Lipschitz\ Determinacy} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$ *holds. If Γ is a boldface pointclass, then every member of $\Gamma \setminus \check{\Gamma}$ is Γ-complete.*

Proof. Fix a set
$$A \in \mathcal{P}(\omega^\omega) \cap (\Gamma \setminus \check{\Gamma}).$$
Then $[A]_{\mathrm{Li}}$ is nonselfdual, so $[A]_{\mathrm{Li}} = [A]_{\mathrm{Wa}}$, by Corollary 2.2.7. Suppose towards a contradiction that there is a $B \in \Gamma$ such that $B \not\leq_{\mathrm{W}} A$. Then, by Theorem 2.1.4, $A \leq_{\mathrm{W}} (\omega^\omega \setminus B)$, contradicting our assumption that $A \notin \check{\Gamma}$. $\square$

We write $\omega^\omega \times \mathcal{X}$ for the set of $X \in \mathcal{X}$ of the form
$$\omega^\omega \times X_1 \times \cdots \times X_n,$$

for some $X_1, \ldots, X_n$ (all equal to either ω or ω^ω). Given $X \in \omega^\omega \times \mathcal{X}$ and $A \subseteq X$, we write $\exists^{\omega^\omega} A$ for the set

$$\{(x_1, \ldots, x_n) : \exists x_0 \in \omega^\omega \, (x_0, \ldots, x_n) \in A\},$$

and $\forall^{\omega^\omega} A$ for the set

$$\{(x_1, \ldots, x_n) : \forall x_0 \in \omega^\omega \, (x_0, \ldots, x_n) \in A\}.$$

Given a pointclass Γ, we write $\exists^{\omega^\omega} \Gamma$ for

$$\{\exists^{\omega^\omega} A : A \in \Gamma \wedge \exists X \in \omega^\omega \times \mathcal{X} \, A \subseteq X\}$$

and $\forall^{\omega^\omega} \Gamma$ for

$$\{\forall^{\omega^\omega} A : A \in \Gamma \wedge \exists X \in \omega^\omega \times \mathcal{X} \, A \subseteq X\}.$$

Similarly, for any ordinal δ, we write $\bigcup_\delta \Gamma$ for the collection of sets of the form $\bigcup_{\alpha<\delta} A_\alpha$, where each A_α is in Γ, and the A_α's are all subsets of the same element of $\mathcal{X}$.

A pointclass Γ is $\exists^{\omega^\omega}$-*closed* if $\exists^{\omega^\omega} \Gamma \subseteq \Gamma$, and $\forall^{\omega^\omega}$-*closed* if $\forall^{\omega^\omega} \Gamma \subseteq \Gamma$.

Proposition 2.3.4 ($\mathsf{ZF} + \mathsf{CC}_{\mathbb{R}}$)**.** *If Γ is an $\exists^{\omega^\omega}$-closed boldface pointclass with a complete set, then Γ is closed under countable unions. If in addition* Lipschitz Determinacy *holds, and Δ is a selfdual $\exists^{\omega^\omega}$-closed boldface pointclass with uncountable cofinality, then Δ is closed under countable unions and countable intersections.*

Proof. We prove the first part first. Let $A \subseteq \omega^\omega$ be Γ-complete, and let $B_i \subseteq \omega^\omega$ ($i \in \omega$) be elements of Γ. Applying $\mathsf{CC}_{\mathbb{R}}$, for each i, let f_i be a continuous function such that $B_i = f_i^{-1}[A]$. Let C be $\{(x, y) \in \omega^\omega \times \omega^\omega : y \in B_{x(0)}\}$ and define $g \colon \omega^\omega \times \omega^\omega \to \omega^\omega$ by setting $g(x, y)$ to be $f_{x(0)}(y)$. Then $C = g^{-1}[A]$, so $C \in \Gamma$. Since $\bigcup_{i \in \omega} B_i = \{y : \exists x \in \omega^\omega (x, y) \in C\}$, we are done.

For the second part, the assumptions imply that every countable subset of Δ is contained in a boldface pointclass Γ_0 having a complete set. Then $\exists^{\omega^\omega} \Gamma_0$ is contained in Δ and satisfies the assumptions of the first part. $\square$

2.3.5 Remark. The proof of Proposition 2.3.4 shows the following, without the assumption of $\mathsf{CC}_{\mathbb{R}}$

- If Γ is an $\exists^{\omega^\omega}$-closed boldface pointclass with a complete set, then Γ is closed under unions.
- If Lipschitz Determinacy holds and Δ is a selfdual $\exists^{\omega^\omega}$-closed boldface pointclass, then either Δ is closed under unions and intersections or there exists an $A \subseteq \omega^\omega$ such that Δ is the collection of pointsets which are Wadge reducible to either A or $\omega^\omega \setminus A$.

2.3.6 Remark. Let Γ be a boldface pointclass. If Γ is closed under unions, then so are $\exists^{\omega^\omega} \Gamma$ and $\forall^{\omega^\omega} \Gamma$ (the latter case can be verified by coding a pair of reals with a single real). Similarly, if Γ is closed under countable unions (and $\mathsf{CC}_{\mathbb{R}}$ holds, for the $\forall^{\omega^\omega}$ case), then so are $\exists^{\omega^\omega} \Gamma$ and $\forall^{\omega^\omega} \Gamma$.

2.4 Universal sets

Given an integer $n \in \omega \setminus 2$, a set $A \subseteq (\omega^\omega)^n$ is *universal* for a pointclass Γ (or Γ-*universal*) if $A \in \Gamma$, and for every $B \subseteq (\omega^\omega)^{n-1}$ in Γ there is an $x \in \omega^\omega$ such that

$$B = \{(y_1, \ldots, y_{n-1}) \mid (x, y_1, \ldots, y_{n-1}) \in A\}$$

(we call this set A_x). If Γ is closed under Wadge-equivalence, and n is in $\omega \setminus 2$, then there exists a universal subset of $(\omega^\omega)^n$ in Γ if and only if there is a universal subset of $(\omega^\omega)^2$ in Γ.

2.4.1 Example. Fixing an enumeration $\langle \sigma_n : n \in \omega \rangle$ of $\omega^{<\omega}$, the set of (x, y) in $(\omega^\omega)^2$ such that $y \in [\sigma_n]$ for some $n \in x^{-1}[\{0\}]$ is a universal open set. It follows that there exist a universal closed set $C \subseteq (\omega^\omega)^3$, and universal analytic and coanalytic subsets of $(\omega^\omega)^2$.

The usual diagonal argument shows that selfdual lightface pointclasses do not have universal sets.

Proposition 2.4.2. *If Δ is a selfdual lightface pointclass, then Δ does not have a universal set.*

Proof. Given a set $A \subseteq (\omega^\omega)^2$ in Δ, let B be the set of $x \in \omega^\omega$ for which $(x, x) \notin A$. Then for any $x \in \omega^\omega$, $x \in B$ if and only if $x \notin A_x$. $\square$

Nonselfdual boldface pointclasses have universal sets, under suitable determinacy assumptions.

Theorem 2.4.3. *Suppose that* Lipschitz Determinacy + Baire($\mathcal{P}(\omega^\omega)$) *holds. If Γ is a boldface pointclass, then Γ has a universal set if and only if it is nonselfdual.*

Proof. The selfdual case follows from Proposition 2.4.2.

For the other case, suppose that Γ is nonselfdual, and fix a set

$$A \in \mathcal{P}(\omega^\omega) \cap (\Gamma \setminus \check{\Gamma}).$$

Then $[A]_{\mathrm{Li}}$ is nonselfdual, so $[A]_{\mathrm{Li}} = [A]_{\mathrm{Wa}}$, by Corollary 2.2.7, and A is Γ-complete by Proposition 2.3.3. Fix a recursive bijection $\rho\colon \omega^{<\omega} \to \omega$. For each $x \in \omega^\omega$, define $f_x\colon \omega^\omega \to \omega^\omega$ by setting $f_x(y)(n)$ to be $x(\rho(y{\restriction}(n+1)))$.

Then each f_x is Lipschitz, and each Lipschitz function from ω^ω to ω^ω is equal to f_x for some $x \in \omega^\omega$. Now let U be the set of $(x, y) \in (\omega^\omega)^2$ such that $f_x(y) \in A$. Then $U =_{\mathrm{Wa}} A$, and U is Γ-universal. $\square$

2.4.4 Remark. Suppose that Γ is a boldface pointclass and $U \subseteq (\omega^\omega)^n$ is Γ-universal, for some $n \in \omega \setminus 3$. Then

$$\{(z_1, \ldots, z_{n-2}) \in (\omega^\omega)^{n-2} : \exists y \in \omega^\omega\ (x, y, z_1, \ldots, z_{n-2}) \in U\}$$

is universal for $\exists^{\omega^\omega}\Gamma$ and

$$\{(z_1, \ldots, z_{n-2}) \in (\omega^\omega)^{n-2} : \forall y \in \omega^\omega\ (x, y, z_1, \ldots, z_{n-2}) \in U\}$$

is universal for $\forall^{\omega^\omega}\Gamma$. It follows from Theorem 2.4.3 that if Γ is nonselfdual, then so are $\exists^{\omega^\omega}\Gamma$ and $\forall^{\omega^\omega}\Gamma$ (although either or both of these pointclasses could be equal to Γ).

In order to prove the Moschovakis Coding Lemma (in Chapter 3) we will need sequences of universal sets in different dimensions which are suitably coherent.

2.4.5 Definition. Let $\bar{U} = \langle U_n : n \in \omega \setminus \{0\}\rangle$ be such that each U_n is a subset of $(\omega^\omega)^{n+1}$.

- The sequence $\bar{U}$ has the *s-m-n property* if for each pair of positive integers $n < m$, there exists a continuous $s_{m,n} \colon (\omega^\omega)^{n+1} \to \omega^\omega$ such that, for all $x, y_1, \ldots, y_m \in \omega^\omega$,
$$(x, y_1, \ldots, y_m) \in U_m$$
if and only if $(s_{m,n}(x, y_1, \ldots, y_n), y_{n+1}, \ldots, y_m) \in U_{m-n}$.

- The sequence $\bar{U}$ has the *recursion property* (with respect to a pointclass Γ) if for each $n \in \omega \setminus \{0\}$ and each
$$A \in \mathcal{P}((\omega^\omega)^{n+1}) \cap \Gamma,$$
there exists an $x \in \omega^\omega$ such that for all $y_1, \ldots, y_n \in \omega^\omega$,
$$(x, y_1, \ldots, y_n) \in U_n$$
if and only if $(x, y_1, \ldots, y_n) \in A$ (i.e., such that $U_{n,x} = A_x$).

We will omit the phrase "with respect to Γ" when talking about sequences $\bar{U}$ with the recursion property, since Γ is recoverable from $\bar{U}$. Similarly, when we say that $\langle U_n : n \in \omega \setminus \{0\}\rangle$ is a sequence of sets with the s-m-n property, we will mean that each U_n is a subset of $(\omega^\omega)^{n+1}$.

2.4.6 Remark. The recursion property defined above of universal set U is a fixed point property. By the universality of U there is, for each set A in the definition, and each $x \in \omega^\omega$, a $z \in \omega^\omega$ such that $A_x = U_{n,z}$. The point of the definition is that there is some x such that $A_x = U_{n,x}$.

Theorems 2.4.7 and 2.4.8 below show that if Γ is a boldface pointclass with a universal set, then there exists a sequence of Γ-universal set with the s-m-n and recursion properties. The statement of Theorem 2.4.7, and its proof, are taken from [6].

Theorem 2.4.7. *If Γ is a boldface pointclass with a universal set then there exists a sequence of Γ-universal sets with the s-m-n property.*

Proof. Fix homeomorphisms $\pi_n \colon \omega^\omega \to (\omega^\omega)^n$ for each $n \in \omega \setminus 2$, and let $\pi_{m,n} \colon \omega^\omega \to \omega^\omega$ $(n < m \in \omega \setminus 2)$ be such that
$$\pi_m(x) = (\pi_{m,0}(x), \ldots, \pi_{m,m-1}(x))$$

for all $x \in \omega^\omega$, so that $\pi_{m,n}(\pi_m^{-1}(x_0, \ldots, x_{m-1})) = x_n$ for all $x_0, \ldots, x_{m-1} \in \omega^\omega$. Let $U \subseteq (\omega^\omega)^2$ be a universal set for Γ. For each $n \in \omega \setminus \{0\}$, let U_n be the set of $(x, y_1, \ldots, y_n) \in (\omega^\omega)^{n+1}$ such that

$$(\pi_{2,0}(x), \pi_{n+1}^{-1}(\pi_{2,1}(x), y_1, \ldots, y_n)) \in U.$$

We check first that each U_n is Γ-universal. Fix $n \in \omega \setminus \{0\}$ and $A \subseteq (\omega^\omega)^n$ in Γ. We want to find an $x \in \omega^\omega$ such that $U_{n,x} = A$. Fixing any $z \in \omega^\omega$, we have that $\pi_{n+1}^{-1}[\{(z, y_1, \ldots, y_n) : (y_1, \ldots, y_n) \in A\}]$ is in Γ. Since U is universal, there is a $w \in \omega^\omega$ such that $U_w = \pi_{n+1}^{-1}[\{(z, y_1, \ldots, y_n) : (y_1, \ldots, y_n) \in A\}]$. Let $x = \pi_2^{-1}(w, z)$. Then for all $y_1, \ldots, y_n \in \omega^\omega$, $(x, y_1, \ldots, y_n)$ is in U_n if and only if $(w, \pi_{n+1}^{-1}(z, y_1, \ldots, y_n))$ is in U, which holds if and only if $(y_1, \ldots, y_n)$ is in A.

To check that the s-m-n property holds, fix $m > n$ in ω. Let W be the set of $w \in \omega^\omega$ for which

$$(u(w), \pi_{m+1}^{-1}(v(w), r_1(w), \ldots, r_n(w), t_1(w), \ldots, t_{m-n}(w))) \in U,$$

where

- $u(w) = \pi_{2,0}(\pi_{n+1,0}(\pi_{m-n+1,0}(w)))$;
- $v(w) = \pi_{2,1}(\pi_{n+1,0}(\pi_{m-n+1,0}(w)))$;
- $r_i(w) = \pi_{n+1,i}(\pi_{m-n+1,0}(w))$ for $i \in \{1, \ldots, n\}$;
- $t_j(w) = \pi_{m-n+1,j}(w)$ for $j \in \{1, \ldots, m-n\}$.

Then $W \leq_{\mathrm{Wa}} U$, and, as U is universal for Γ, there exists a $z \in \omega^\omega$ such that $U_z = W$.

Define $s_{m,n} \colon (\omega^\omega)^{n+1} \to \omega^\omega$ by setting

$$s_{m,n}(x, y_1, \ldots, y_n) = \pi_2^{-1}(z, \pi_{n+1}^{-1}(x, y_1, \ldots, y_n))$$

for all $x, y_1, \ldots, y_n \in \omega^\omega$. Now fix $x, y_1, \ldots, y_m \in \omega^\omega$. Then $(x, y_1, \ldots, y_m) \in U_m$ if and only if

$$(\pi_{2,0}(x), \pi_{m+1}^{-1}(\pi_{2,1}(x), y_1, \ldots, y_m)) \in U,$$

and $(s_{m,n}(x, y_1, \ldots, y_n), y_{n+1}, \ldots, y_m) \in U_{m-n}$ if and only if

$$(\pi_{2,0}(s_{m,n}(x, y_1, \ldots, y_n)), \pi_{m-n+1}^{-1}(\pi_{2,1}(s_{m,n}(x, y_1, \ldots, y_n)), y_{n+1}, \ldots, y_m)) \in U.$$

Now, $\pi_{2,0}(s_{m,n}(x, y_1, \ldots, y_n)) = z$ and

$$\pi_{2,1}(s_{m,n}(x, y_1, \ldots, y_n)) = \pi_{n+1}^{-1}(x, y_1, \ldots, y_n).$$

Since $U_z = W$, we have that

$$(z, \pi_{m-n+1}^{-1}(\pi_{n+1}^{-1}(x, y_1, \ldots, y_n), y_{n+1}, \ldots, y_m)) \in U$$

if and only if $\pi^{-1}_{m-n+1}(\pi^{-1}_{n+1}(x, y_1, \ldots, y_n), y_{n+1}, \ldots, y_m) \in W$, which, letting w be $\pi^{-1}_{m-n+1}(\pi^{-1}_{n+1}(x, y_1, \ldots, y_n), y_{n+1}, \ldots, y_m)$, holds if and only if

$$(u(w), \pi^{-1}_{m+1}(v(w), r_1(w), \ldots, r_n(w), t_1(w), \ldots, t_{m-n}(w))) \in U,$$

which holds if and only if

$$(\pi_{2,0}(x), \pi^{-1}_{m+1}(\pi_{2,1}(x), y_1, \ldots, y_m)) \in U,$$

since

- $u(w) = \pi_{2,0}(x)$,
- $v(w) = \pi_{2,1}(x)$,
- $r_i(w) = y_i$ for $i \in \{1 \ldots, n\}$ and
- $t_j(w) = y_{n+i}$ for $j \in \{1, \ldots, m-n\}$.

□

The following is a version of Kleene's Recursion Theorem, saying that a sequence of Γ-universal sets with the s-m-n property has the $\bar{U}$ has the recursion property.

Theorem 2.4.8 (Kleene). *If Γ is a pointclass and $\bar{U} = \langle U_n : n \in \omega \setminus \{0\}\rangle$ is a sequence of Γ-universal sets with the s-m-n property, then $\bar{U}$ has the recursion property.*

Proof. Fix $n \in \omega \setminus \{0\}$ and $A \in \mathcal{P}((\omega^\omega)^{n+1}) \cap \Gamma$. Applying the assumption that U_{n+1} is Γ-universal, let $y \in \omega^\omega$ be such that for all $w, z_1, \ldots, z_n$ in ω^ω, $(y, w, z_1, \ldots, z_n) \in U_{n+1}$ if and only if $(s(w, w), z_1, \ldots, z_n) \in A$, where $s\colon (\omega^\omega)^2 \to \omega^\omega$ witnesses the s-m-n property for $\bar{U}$ in the role of $s_{n+1,1}$. Then for all $w, z_1, \ldots, z_n \in \omega^\omega$,

$$(s(y, w), z_1, \ldots, z_n) \in U_n$$

if and only if

$$(y, w, z_1, \ldots, z_n) \in U_{n+1}$$

if and only if

$$(s(w, w), z_1, \ldots, z_n) \in A.$$

Then $x = s(y, y)$ is as desired. □

Putting together Theorems 2.4.3, 2.4.7 and 2.4.8 we get the following.

Theorem 2.4.9. *Suppose that* Lipschitz Determinacy + Baire($\mathcal{P}(\omega^\omega)$) *holds. If Γ is a nonselfdual boldface pointclass, then there exists a sequence of universal Γ-sets with the s-m-n and recursion properties.*

Theorem 2.4.10 below is another version of the Recursion Theorem. A function $f\colon \omega^\omega \to \omega^\omega$ is in $\utilde{\Sigma}^1_1$ if it is $\utilde{\Sigma}^1_1$ as a subset of $\omega^\omega \times \omega^\omega$ (equivalently, if the f-preimage of each open set is in $\utilde{\Sigma}^1_1$.

Theorem 2.4.10 (Kleene)**.** *Suppose that Γ is a $\exists^{\omega^\omega}$-closed pointclass and that $\bar{U} = \langle U_n : n \in \omega \setminus \{0\}\rangle$ is a sequence of Γ-universal sets with the recursion property. For any $m \in \omega \setminus \{0\}$ and any $\utilde{\Sigma}^1_1$ function $f\colon \omega^\omega \to \omega^\omega$, there is an $x \in \omega^\omega$ such that $U_{m,x} = U_{m,f(x)}$.*

Proof. Since Γ is $\exists^{\omega^\omega}$-closed, the set

$$A = \{(x, y_1, \ldots, y_m) \in (\omega^\omega)^3 : (f(x), y_1, \ldots, y_m) \in U_m\}$$

is in Γ. Then for each $x \in \omega^\omega$, $A_x = U_{m,f(x)}$. Applying the recursion property for U_m, we get an $x \in \omega^\omega$ such that $U_{m,x} = A_x$. $\square$

Using a recursive bijection $\pi\colon \omega \to \omega \times \omega$, we can associate to each $y \in \omega^\omega$ an ω-sequence $\langle (y)_n : n \in \omega\rangle$ of elements of ω^ω by setting $(y)_n(m)$ to be $y(\pi^{-1}(n,m))$. Loosely following 7D.7 (page 430) of [32], we say that a set $B \subseteq X$ (for some X in $\mathcal{X}$) is *pos*-$\utilde{\Sigma}^1_1(A)$ (for some $A \subseteq \omega^\omega$) if

$$B = \{x \in X : \exists y \in \omega^\omega((\forall n \in \omega\, (y)_n \in A) \wedge (x,y) \in S)\},$$

for some $\utilde{\Sigma}^1_1$ set $S \subseteq X \times \omega^\omega$. These are essentially the sets which are $\utilde{\Sigma}^1_1$ using a predicate for A positively. If A is a subset of $(\omega^\omega)^n$ for some positive $n \in \omega$, we say that a set $B \subseteq X$ is in pos-$\utilde{\Sigma}^1_1(A)$ if it is in pos-$\utilde{\Sigma}^1_1(b[A])$ for some recursive bijection $b\colon (\omega^\omega)^n \to \omega^\omega$. If $A_1, \ldots, A_n$ are subsets of $(\omega^\omega)^n$, we write pos-$\utilde{\Sigma}^1_1(A_1, \ldots, A_n)$ for pos-$\utilde{\Sigma}^1_1(A)$, where A is the disjoint union of $A_1, \ldots, A_n$, i.e., the set of reals of the form $\langle \pi(i, x(0)), x(1), x(2), \ldots\rangle$ for $x \in A_i$ and $i \in \{1, \ldots, n\}$ (we call this set $A_1 \oplus \cdots \oplus A_n$).

2.4.11 Remark. For any $X \in \mathcal{X}$ and $A \subseteq \omega^\omega$, A is in pos-$\utilde{\Sigma}^1_1(A)$, and pos-$\utilde{\Sigma}^1_1(A)$ is a boldface pointclass closed under $\exists^{\omega^\omega}$, countable unions and countable intersections (assuming $\mathsf{CC}_{\mathbb{R}}$). The existence of universal sets for $\utilde{\Sigma}^1_1$ (as shown in Example 2.4.1) implies that each pointclass of the form pos-$\utilde{\Sigma}^1_1(A)$ has a universal set.

Given $A \subseteq \omega^\omega$, $\utilde{\Sigma}^1_1(A)$ the collection of subsets of spaces in $\mathcal{X}$ which can be defined by a $\utilde{\Sigma}^1_1$ formula in a predicate for A. This is the same as the pointclass pos-$\utilde{\Sigma}^1_1(A, \omega^\omega \setminus A)$. Similarly, we write $\utilde{\Sigma}^1_1(A_1, \ldots, A_n)$ for

$$\text{pos-}\utilde{\Sigma}^1_1(A_1, \ldots, A_n, \omega^\omega \setminus A_1, \ldots, \omega^\omega \setminus A_n).$$

Given a positive $n \in \omega$, $\utilde{\Pi}^1_n(A)$ is the set of complements of sets in $\utilde{\Sigma}^1_1(A)$, $\utilde{\Sigma}^1_{n+1}(A)$ is the set of continuous images of sets in $\utilde{\Pi}^1_n(A)$ and $uTDelta^1_n(A)$ is the intersection of $\utilde{\Sigma}^1_n(A)$ and $\utilde{\Pi}^1_n(A)$. We say that a set B is *projective in* A if it is in $\bigcup_{n\in\omega} \utilde{\Sigma}^1_{n+1}(A)$ and write $\Delta_\omega(A)$ for the collection of sets projective in A. We say that a pointclass Δ is *projectively closed* if for each $A \in \Delta \cap \mathcal{P}(\omega^\omega)$, every set projective in A is in Δ.

2.4.12 Remark. Let $\bar{U} = \langle U_n : n < \omega\rangle$ be a sequence of universal sets for $\underset{\sim}{\Sigma}^1_1$ with the s-m-n property. For any $A \subseteq \omega^\omega$, let $U_n(A)$ be the set

$$\{x \in (\omega^\omega)^{n+1} : \exists y \in \omega^\omega((\forall n \in \omega\, (y)_n \in A) \wedge (x, y) \in U_{n+1})\}.$$

Then each $U_n(A)$ is universal for pos-$\underset{\sim}{\Sigma}^1_1(A)$. Furthermore, if the functions $s_{m,n}$ ($n < m < \omega$) witness the s-m-n property for $\bar{U}$, then for all $n < m < \omega$, function $s_{m+1,n}$ witnesses the s-m-n property for $\bar{U}(A) = \langle U_n(A) : n < \omega\rangle$ for m and n. Similarly, given a function $f : \omega^\omega \to \omega^\omega$ and $n \in \omega \setminus \{0\}$, if $U_{n,x} = U_{n,f(x)}$, then $U_{n,x}(A) = U_n(A)_x = U_n(A)_{f(x)} = U_{n,f(x)}(A)$ for all $A \subseteq \omega^\omega$.

For $\bar{U}$ as Remark 2.4.12, $n \in \omega$ and $A_1, \ldots, A_m \subseteq \omega^\omega$, we write $U_n(A_1, \ldots, A_m)$ for $U_n(A_1 \oplus \cdots \oplus A_m)$.

2.5 The cardinal Θ

This section presents results of Solovay on the height of the Wadge hierarchy, taken from [36].

2.5.1 Definition. The ordinal Θ is defined to be the least nonzero ordinal which is not a surjective image of $\mathcal{P}(\omega)$.

It follows immediately from this definition that Θ is a cardinal.

2.5.2 Remark. We will often making use of the fact that continuous functions on ω^ω can be coded by members of ω^ω. While there are many ways of doing this, we fix one for concreteness and convenience. Recall from Example 2.4.1 that there is a universal set $C \subseteq (\omega^\omega)^3$ for the pointclass of closed subsets of spaces in $\mathcal{X}$. For each continuous function $f : \omega^\omega \to \omega^\omega$, there is an $x \in \omega^\omega$ such that $C_x = f$. Let $\mathcal{F}^{\mathrm{c}}$ be the set of x for which C_x is a continuous function from ω^ω to ω^ω, and for each $x \in \mathcal{F}^{\mathrm{c}}$ let f^{c}_x denote the set C_x. Then $\mathcal{F}^{\mathrm{c}}$ is in $\underset{\sim}{\Pi}^1_1$. Using universal closed sets $C_k \subseteq (\omega^\omega)^{k+3}$ ($k \in \omega$), we can in a similar fashion fix for each pair $(n, m) \in \omega \times \omega$ a set $\mathcal{F}^{\mathrm{c},n,m}$ of codes x for all continuous functions $f^{\mathrm{c},n,m}_x : (\omega^\omega)^n \to (\omega^\omega)^m$.

2.5.3 Remark. The usual diagonal argument shows that it is not possible to have an association of each $x \in \omega^\omega$ to a continuous function $f^*_x : \omega^\omega \to \omega^\omega$ in such a way that the map $(x, y) \mapsto f^*_x(y)$ is continuous on $(\omega^\omega)^2$. To see this, consider the function $g : \omega^\omega \to \omega^\omega$ defined by setting $g(x)(n) = f^*_x(x)(n) + 1$ for each $x \in \omega^\omega$, for any such map $x \mapsto f^*_x$.

Another diagonalization argument gives a Wadge class above a given selfdual class.

Theorem 2.5.4 (Solovay)**.** *If* Lipschitz Determinacy *holds, then there a function $j : \mathcal{P}(\omega^\omega) \to \mathcal{P}(\omega^\omega)$ such that, for all $A \subseteq \omega^\omega$, $A <_{\mathrm{W}} j(A)$.*

Proof. Fix $A \subseteq \omega^\omega$. We define $j(A)$ in two cases, depending on whether $[A]_{\mathrm{Wa}}$ is selfdual or not. In the nonselfdual case, let $j(A)$ be the set of reals of the form $\langle \pi(i, x(0)), x(1), x(2), \ldots \rangle$ where x is in A if and only if i is even. Then $j(A) >_{\mathrm{Wa}} A$ as in Remark 2.2.4 (recall that $[A]_{\mathrm{Wa}} = [A]_{\mathrm{Li}}$ in this case, by Theorem 2.2.6.

In the selfdual case, let $j(A)$ be the set of $x \in \mathcal{F}^{\mathrm{c}}$ such that $x \notin (f^{\mathrm{c}}_x)^{-1}[A]$. $\square$

2.5.5 Remark. The proof of Theorem 2.5.4 shows that if $A \subseteq \omega^\omega$ and Δ is the smallest selfdual boldface pointclass containing A, then

- if $[A]_{\mathrm{Wa}}$ is nonselfdual, then $[j(A)]_{\mathrm{Wa}} = \Delta$;
- if $[A]_{\mathrm{Wa}}$ is selfdual, and Δ contains a universal closed set, then $j(A)$ is in $\forall^{\omega^\omega}\Delta$;
- if $[A]_{\mathrm{Wa}}$ is selfdual and $\undertilde{\Pi}^1_1 \subseteq \Delta$, then $j(A)$ is in $\exists^{\omega^\omega}\Delta$.

The proof of Theorem 2.5.4 does not show how high the Wadge rank of $j(A)$ is relative to A. In particular, we have the following question (one could ask a similar question about the limit stages of the construction in the proof of Theorem 2.5.9).

2.5.6 Question. Does AD imply that the Wadge rank of $j(A)$ is defined whenever A is a subset of ω^ω whose Wadge rank is defined?

The main result of this section is the following fact. One direction is proved in Proposition 2.5.8, the other in Theorem 2.5.9.

Theorem 2.5.7 (Solovay). *If* $\mathsf{Lipschitz\ Determinacy}$ *holds then* Θ *is the supremum of the set of ordinals* γ *for which there exists a* $<_{\mathrm{Wa}}$*-increasing sequence*

$$\langle A_\alpha : \alpha < \gamma \rangle.$$

If one assumes in addition that $<_{\mathrm{Wa}}$ is wellfounded, then one gets that Θ is the supremum of $\{\mathrm{WR}(A) : A \subseteq \omega^\omega\}$ (Corollary 2.5.11).

The following proposition shows that if $\langle A_\alpha : \alpha \leq \gamma \rangle$ is a $\leq_{\mathrm{Wa}}$-increasing sequence, then there is a surjection from ω^ω to $\gamma + 1$ defined by mapping $x \in \omega^\omega$ to α if

$$[A_\alpha]_{\mathrm{Wa}} = [(f^{\mathrm{c}}_x)^{-1}[A_\gamma]]_{\mathrm{Wa}}$$

(if there exists such an α, and 0 otherwise). It follows that $\gamma < \Theta$. It shows moreover that for each $A \subseteq \omega^\omega$ whose Wadge rank is defined, $\mathrm{WR}(A) < \Theta$.

Proposition 2.5.8. *Assume that* $\mathsf{Lipschitz\ Determinacy}$ *holds, let* A *be a subset of* ω^ω*, and let* Δ *be the smallest selfdual boldface pointclass containing* A *and* $\mathcal{F}^{\mathrm{c}}$*. Let* $\leq$ *be the order on* $\mathcal{F}^{\mathrm{c}}$ *defined by setting* $x \leq y$ *if and only if*

$$(f^{\mathrm{c}}_x)^{-1}[A] \leq_{\mathrm{W}} (f^{\mathrm{c}}_y)^{-1}[A].$$

Then $(\mathcal{F}^{\mathrm{c}}, \leq)$ *is isomorphic to* $(\{B \subseteq \omega^\omega : B \leq_{\mathrm{W}} A\}, \leq_{\mathrm{W}})$*, and* $\leq$ *is in* $\exists^{\omega^\omega}\forall^{\omega^\omega}\Delta$*.*

Proof. That $(\mathcal{F}^{\mathrm{c}}, \leq)$ is isomorphic to $(\{B \subseteq \omega^\omega : B \leq_{\mathrm{W}} A\}, \leq_{\mathrm{W}})$ is immediate from the definitions. That $\leq$ is in $\forall^{\omega^\omega}\Delta$ follows noting that $x \leq y$ if and only if $\{x, y\} \subseteq \mathcal{F}^{\mathrm{c}}$ and there exists a $u \in \omega^\omega$ such that for all z, w, v, t in ω^ω, if $w = f^{\mathrm{c}}_x(z)$, $v = f^{\mathrm{c}}_u(z)$ and $t = f^{\mathrm{c}}_y(v)$, then $w \in A$ if and only if $t \in A$ (that is, $z \in (f^{\mathrm{c}}_x)^{-1}[A]$ if and only if $f^{\mathrm{c}}_u(z) \in (f^{\mathrm{c}}_y)^{-1}[A]$). ☐

Finally, we use Theorem 2.5.4 to building $<_{\mathrm{Wa}}$-increasing sequences of any length less than Θ.

Theorem 2.5.9 (Solovay)**.** *If* Lipschitz Determinacy *holds, γ is an ordinal and $g\colon \omega^\omega \to \gamma$ is a surjection, then there is a $<_{\mathrm{Wa}}$-increasing sequence $\langle A_\alpha : \alpha < \gamma\rangle$ definable from g.*

Proof. Let $g\colon \omega^\omega \to \gamma$ be a surjection, for some ordinal γ, and let $\pi\colon \omega \times \omega \to \omega$ be a recursive bijection. Recursively define the sequence $\langle A_\alpha : \alpha < \gamma\rangle$ by setting each A_α to be

$$j(\pi[\{(x, y) \in \omega^\omega \times \omega^\omega \mid g(y) < \alpha \text{ and } x \in A_{g(y)}\}]),$$

where j is as in Theorem 2.5.4. ☐

2.5.10 Remark. Let Δ be a $\exists^{\omega^\omega}$-closed selfdual pointclass containing $\mathcal{F}^{\mathrm{c}}$, let γ be a ordinal such that $\bigcup_\gamma \Delta \subseteq \Delta$ and let $f\colon \omega^\omega \to \gamma$ be a surjection such that $f^{-1}[\{\alpha\}] \in \Delta$ for all $\alpha < \gamma$. The proof of Theorem 2.5.9, along with Remark 2.5.5, shows that Δ contains a set whose Wadge rank is at least γ (or undefined).

It follows that if the Wadge rank of each subset of ω^ω is defined, then Θ is the supremum of the Wadge ranks of the elements of $\mathcal{P}(\omega^\omega)$.

Corollary 2.5.11 (Solovay)**.** *Suppose that* Lipschitz Determinacy *holds, and that* $\mathrm{WR}(A)$ *is defined for every $A \subseteq \omega^\omega$. Then $\Theta = \{\mathrm{WR}(A) \mid A \subseteq \omega^\omega\}$.*

Chapter 3

Coding Lemmas

This chapter presents the Moschovakis Coding Lemma, which is one of the fundamental theorems of the theory of determinacy and which will be used throughout this book. Among other things the Coding Lemma can be used to map ω^ω onto the powerset of any ordinal below Θ (see Corollary 3.0.2). Theorem 3.0.1 is the basic form of the lemma. Theorem 3.0.3 is a uniform version, which will be needed in Chapter 5. We follow [22]. Section 7D of [32] also contains proofs of the theorems in this section.

Given a strategy $\sigma\colon \omega^{<\omega} \to \omega$, and $x \in \omega^\omega$, we let $\sigma \circ x$ (similarly, $x \circ \sigma$) be the sequence of moves played by player I (II) when he plays according to σ and player II (I) plays x.

Theorem 3.0.1 (The Coding Lemma; Moschovakis)**.** *Assume that* AD *holds. Let*

- *X be a subset of ω^ω;*
- *Z be a subset of $X \times \omega^\omega$;*
- *f be a function from X to the ordinals;*
- *$<_f$ be $\{(y,z) \in X^2 : f(y) < f(z)\}$.*

For each $y \in X$, let $[y]_f$ denote $\{z \in X : f(y) = f(z)\}$. Then there exists a pos-$\undertilde{\Sigma}^1_1(<_f)$ set $A \subseteq Z$ such that for all $y \in X$,

$$A \cap ([y]_f \times \omega^\omega) = \emptyset \textit{ if and only if } Z \cap ([y]_f \times \omega^\omega) = \emptyset.$$

Proof. It suffices to consider the case where the range of f is an ordinal γ, and, proving the theorem by induction, we may assume that it holds for all smaller ordinals. As pos-$\undertilde{\Sigma}^1_1(<_f)$ is closed under unions and contains all finite subsets of ω^ω, we may assume that γ is a limit ordinal. Applying Theorem 2.4.7 and Remark 2.4.12, let $\bar{U} = \langle U_n : n \in \omega \setminus \{0\}\rangle$ be a sequence of universal sets for pos-$\undertilde{\Sigma}^1_1(<_f)$ with the s-m-n property. We seek an $x \in \omega^\omega$ such that

1. $U_{2,x} \subseteq Z$;
2. for all $y \in X$, if $Z \cap ([y]_f \times \omega^\omega) \neq \emptyset$ then $U_{2,x} \cap ([y]_f \times \omega^\omega) \neq \emptyset$.

Let $Y = \{x \in \omega^\omega : U_{2,x} \subseteq Z\}$. For each $x \in \omega^\omega$, let α_x be the least value $f(y)$ for any $y \in X$ witnessing the failure of item (2) for x, if there exists such a y; otherwise, let $\alpha_x = \gamma$. Consider the game between players I and II where player I builds $x_1 \in \omega^\omega$, player II builds $x_2 \in \omega^\omega$, and I wins if and only if $x_1 \in Y$ and either $x_2 \notin Y$ or $\alpha_{x_1} \geq \alpha_{x_2}$. We will show that a winning strategy for either player gives the desired conclusion.

First, suppose that σ is a winning strategy for player I. Then for all $x \in \omega^\omega$, $U_{2,\sigma \circ x} \subseteq Z$. The set $\bigcup_{x \in \omega^\omega} U_{2,\sigma \circ x}$ is in pos-$\utilde{\Sigma}^1_1(<_f)$, which is $\exists^{\omega^\omega}$-closed. By the assumption that the theorem holds for all ordinals smaller than γ, there exists for each $\alpha < \gamma$ an $x \in Y$ such that $\alpha_x \geq \alpha$. It follows then that $\bigcup_{x \in \omega^\omega} U_{2,\sigma \circ x}$ is as desired.

Now suppose that τ is a winning strategy for player II. For each $y \in X$, let $[<y]_f$ denote the set of $z \in X$ with $f(z) < f(y)$. The set of $(x, y, z, w) \in (\omega^\omega)^4$ for which $(x, z, w) \in U_2$, $(y, z) \in X^2$ and $z \in [<y]_f$ is in pos-$\utilde{\Sigma}^1_1(<_f)$. Fix $a_0 \in \omega^\omega$ such that this set is U_{4,a_0}. Let $s_{4,2} \colon (\omega^\omega)^3 \to \omega^\omega$ be a continuous function witnessing the s-m-n property of $\bar{U}$ for $n = 2$ and $m = 4$, and let $h_0 \colon (\omega^\omega)^2 \to \omega^\omega$ be defined by setting $h_0(x, y) = s_{4,2}(a_0, x, y)$. Then for all $(x, y) \in \omega^\omega \times X$,

$$U_{2,h_0(x,y)} = U_{2,x} \cap ([<y]_f \times \omega^\omega). \tag{3.1}$$

Similarly, the set of $(x, y, z) \in (\omega^\omega)^3$ for which $(h_0(x, y) \circ \tau, y, z) \in U_2$ is in pos-$\utilde{\Sigma}^1_1(<_f)$. Fix $a_1 \in \omega^\omega$ such that this set is U_{3,a_1}. Let $s_{3,1} \colon (\omega^\omega)^2 \to \omega^\omega$ be a continuous function witnessing the s-m-n property of $\bar{U}$ for $n = 1$ and $m = 3$, and let $h_1 \colon \omega^\omega \to \omega^\omega$ be defined by setting $h_1(x) = s_{3,1}(a_1, x)$. Then for all $x \in \omega^\omega$,

$$U_{2,h_1(x)} = \bigcup_{y \in X} (U_{2,h_0(x,y) \circ \tau} \cap (\{y\} \times \omega^\omega)). \tag{3.2}$$

By Theorem 2.4.10, there exists an $x_1 \in \omega^\omega$ such that $U_{2,x_1} = U_{2,h_1(x_1)}$. We show now that x_1 satisfies conditions (1) and (2). For condition (1), suppose toward a contradiction that the condition fails, and consider $(y, z) \in U_{2,x_1} \setminus Z$ with $f(y)$ minimal. Since $U_{2,x_1} = U_{2,h_1(x_1)}$, $(y, z) \in U_{2,h_0(x_1,y) \circ \tau}$. Then by equation (3.1), for all $(a, b) \in U_{2,h_0(x_1,y)}$, (a, b) is in U_{2,x_1} and $f(a) < f(y)$. By the minimality of $f(y)$, it follows that $(a, b) \in Z$ and therefore that $h_0(x_1, y) \in Y$. Since τ is a winning strategy for II, $h_0(x_1, y) \circ \tau \in Y$, which means that $(y, z) \in Z$, giving a contradiction.

For condition (2), suppose toward a contradiction that $\alpha_{x_1} < \gamma$. Let $y \in X$ be such that $f(y) = \alpha_{x_1}$. By equation (3.1), the fact that x_1 is in Y and the definition of α_{x_1}, $h_0(x_1, y) \in Y$, and $\alpha_{h_0(x_1,y)} = \alpha_{x_1}$. Since τ is a winning strategy for II, $\alpha_{h_0(x_1,y)\circ\tau} > \alpha_{h_0(x_1,y)} = \alpha_{x_1}$, which is impossible, as

$$(U_{2,h_0(x_1,y)\circ\tau} \cap ([y]_f \times \omega^\omega)) \subseteq U_{2,x_1},$$

by equation (3.2) and the choice of x_1. This completes the proof. $\square$

Corollary 3.0.2 lists two immediate consequences of the Coding Lemma. The first part of the corollary to the Coding Lemma follows from the fact that there is a surjection from ω^ω to $\underset{\sim}{\Sigma}^1_1$.

Corollary 3.0.2. *If* AD *holds, then each of the following hold.*

1. *For each $\lambda < \Theta$ there is a surjection from $\mathcal{P}(\omega)$ to $\mathcal{P}(\lambda)$.*

2. *Θ is a limit cardinal.*

We now prove a uniform version of Theorem 3.0.1 which is used in the proof of Theorem 5.2.3, which in turn is used to prove that the strong partition cardinals are cofinal below Θ. Given a sequence $\langle U_n : n < \omega \rangle$ of universal sets for $\underset{\sim}{\Sigma}^1_1$, and an $A \subseteq \omega^\omega$, we let $\langle U_n(A) : n < \omega \rangle$ be as in Remark 2.4.12.

Theorem 3.0.3 (The Uniform Coding Lemma)**.** *Assume that* AD *holds. Let*

- *$\bar{U} = \langle U_n : n < \omega \rangle$ be a sequence of universal sets for $\underset{\sim}{\Sigma}^1_1$ with the s-m-n property;*

- *X be a subset of ω^ω;*

- *Z be a subset of $X \times \omega^\omega$;*

- *f be a function from X to the ordinals.*

For each $y \in X$, let $[y]_f$ denote $\{z \in X : f(y) = f(z)\}$, let $[<y]_f$ denote $\{z \in X : f(z) < f(y)\}$ and let

$$C_y = \omega^\omega \setminus ([y]_f \cup [<y]_f).$$

Then there exists an $x \in \omega^\omega$ such that for all $y \in X$,

1. *$U_{2,x}([y]_f, C_y) \subseteq Z \cap ([y]_f \times \omega^\omega)$,*

2. *$U_{2,x}([y]_f, C_y) \neq \emptyset$ if and only if $Z \cap ([y]_f \times \omega^\omega) \neq \emptyset$.*

Proof. It suffices to consider the case where the range of f is an ordinal γ, and, proving the theorem by induction, we may assume that it holds for all smaller ordinals. As $\underset{\sim}{\Sigma}^1_1$ is closed under unions and contains all finite subsets of ω^ω, we may assume that γ is a limit ordinal. Let Y be the set

$$\{x \in \omega^\omega : \forall y \in X \, U_{2,x}([y]_f, C_y) \subseteq Z \cap ([y]_f \times \omega^\omega)\}.$$

For each $x \in \omega^\omega$, let α_x be the least value $f(y)$ for a $y \in X$ witnessing the failure of conclusion (2) of the theorem, with respect to x, if there exists such a y; otherwise, let $\alpha_x = \gamma$. Consider the game between players I and II where player I builds $x_1 \in \omega^\omega$, player II builds $x_2 \in \omega^\omega$, and I wins if and only if $x_1 \in Y$ and either $x_2 \notin Y$ or $\alpha_{x_1} \geq \alpha_{x_2}$. We will show that a winning strategy for either player gives the desired conclusion.

First, suppose that σ is a winning strategy for player I. Let $z_\sigma \in \omega^\omega$ be such that $U_{2,z_\sigma} = \bigcup_{x\in\omega^\omega} U_{2,\sigma\circ x}$. Then for all $P_1, P_2 \subseteq \omega^\omega$ (in particular, in the case $P_1 = [y]_f$, $P_2 = C_y$, for some $y \in X$),

$$U_{2,z_\sigma}(P_1, P_2) = \bigcup_{x\in\omega^\omega} U_{2,\sigma\circ x}(P_1, P_2).$$

Since σ is a winning strategy for I, z_σ is in Y. By our assumption that the theorem holds for all ordinals smaller than γ, there exist for each $\alpha < \gamma$ an $x \in Y$ such that $\alpha_x \geq \alpha$. It follows then that z_σ is as desired.

Now suppose that τ is a winning strategy for player II. Fix a recursive bijection $\pi\colon \omega \times \omega \to \omega$. For each $i \in \{1, 2\}$, let Q_i be the set of pairs $(y, r) \in \omega^\omega \times \omega^\omega$ such that

$$(r)_0 = \langle \pi(i, y(0)), y(1), y(2), \ldots \rangle$$

(this corresponds to how we defined the class pos-$\underset{\sim}{\Sigma}^1_1(A_1, A_2)$ in terms of our definition of pos-$\underset{\sim}{\Sigma}^1_1(A)$ in Section 2.4). Then for any $P, P' \subseteq \omega^\omega$ and $z \in \omega^\omega$,

- $z \in P$ if and only if $(z, r) \in Q_1$ holds for some $r \in \omega^\omega$ with $(r)_0 \in P \oplus P'$.
- $z \in P'$ if and only if $(z, r) \in Q_2$ holds for some $r \in \omega^\omega$ with $(r)_0 \in P \oplus P'$.

For each $r \in \omega^\omega$, let $r^- \in \omega^\omega$ be such that $(r^-)_n = (r)_{n+1}$ for all $n \in \omega$.

Let $a_0 \in \omega^\omega$ be such that U_{5,a_0} is the set of $(x, y, z, w, r) \in (\omega^\omega)^5$ for which $(x, z, w, r^-) \in U_3$ and $(y, r) \in Q_2$. Let $s_{5,2}\colon (\omega^\omega)^3 \to \omega^\omega$ be a continuous function witnessing the s-m-n property of $\bar{U}$ for $n = 2$ and $m = 5$, and let $h_0\colon (\omega^\omega)^2 \to \omega^\omega$ be defined by setting $h_0(x, y) = s_{5,2}(a_0, x, y)$. Then for all $(x, y) \in \omega^\omega \times \omega^\omega$, all $P, P' \subseteq \omega^\omega$ and all $(z, w) \in \omega^\omega \times \omega^\omega$,

$$(z, w) \in U_{2,h_0(x,y)}(P, P')$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (h_0(x, y), z, w, r) \in U_3)$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (a_0, x, y, z, w, r) \in U_5)$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (x, z, w, r^-) \in U_3 \wedge (y, r) \in Q_2)$$

if and only if

$$(z, w) \in U_{2,x}(P, P') \wedge y \in P'.$$

That is, we have the following fact (**) : for all $(x, y) \in \omega^\omega \times \omega^\omega$, and all $P, P' \subseteq \omega^\omega$ (in particular, sets of the form C_y), $U_{2,h_0(x,y)}(P, P')$ is $U_{2,x}(P, P')$ if y is in P' and $\emptyset$ otherwise.

Now let $a_1 \in \omega^\omega$ be such that U_{4,a_1} is the set of $(x, z, w, r) \in (\omega^\omega)^4$ such that, for some $y \in \omega^\omega$, $(z, w, r^-) \in U_{3,h_0(x,y)\circ\tau}$ and $(y, r) \in Q_1$ holds. Let $s_{4,1} \colon (\omega^\omega)^2 \to \omega^\omega$ be a continuous function witnessing the s-m-n property of $\bar{U}$ for $n = 1$ and $m = 3$, and let $h_1 \colon \omega^\omega \to \omega^\omega$ be defined by setting $h_1(x) = s_{4,1}(a_1, x)$.

Then for all $x \in \omega^\omega$, all $P, P' \subseteq \omega^\omega$ and all $(z, w) \in \omega^\omega \times \omega^\omega$,

$$(z, w) \in U_{2,h_1(x)}(P, P')$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (h_1(x), z, w, r) \in U_3)$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge (a_1, x, z, w, r) \in U_4)$$

if and only if

$$\exists r \in \omega^\omega((\forall n \in \omega\, (r)_n \in P \oplus P') \wedge \exists y \in \omega^\omega((z, w, r^-) \in U_{3,h_0(x,y)\circ\tau} \wedge (y, r) \in Q_1))$$

if and only if

$$\exists y \in P\, (z, w) \in U_{2,h_0(x,y)\circ\tau}(P, P').$$

Then for all $x \in \omega^\omega$ and all $P, P' \subseteq \omega^\omega$,

$$U_{2,h_1(x)}(P, P') = \bigcup_{y \in P} U_{2,h_0(x,y)\circ\tau}(P, P'). \tag{3.3}$$

By Theorem 2.4.10 and Remark 2.4.12, there exists an $x_1 \in \omega^\omega$ such that $U_{2,x_1}(P, P') = U_{2,h_1(x_1)}(P, P')$ for all $P, P' \subseteq \omega^\omega$. We show now that x_1 satisfies both conditions in the conclusion of the theorem. For condition (1) (i.e., the assertion that $x_1 \in Y$), fix $y^* \in X$ with

$$U_{2,x_1}([y^*]_f, C_{y^*}) \not\subseteq Z \cap ([y^*]_f \times \omega^\omega)$$

and $f(y^*)$ minimal. Since

$$U_{2,x_1}([y^*]_f, C_{y^*}) = U_{2,h_1(x_1)}([y^*]_f, C_{y^*}),$$

there exists by equation (3.3) a $y' \in [y^*]_f$ such that

$$U_{2,h_0(x_1,y')\circ\tau}([y^*]_f, C_{y^*}) \not\subseteq Z \cap ([y^*]_f \times \omega^\omega).$$

We want to see that $h_0(x_1, y') \in Y$, since then, as τ is a winning strategy for II, we will have that $h_0(x_1, y') \circ \tau \in Y$, giving a contradiction. For each $y \in X$, we have by (**) that

$$U_{2,h_0(x_1,y')}([y]_f, C_y) = U_{2,x_1}([y]_f, C_y) \subseteq Z \cap ([y]_f \times \omega^\omega)$$

if $f(y) < f(y')$, by the minimality of $f(y^*)$, and $U_{2,h_0(x_1,y')}([y^*]_f, C_y) = \emptyset$ otherwise. In either case, $U_{2,h_0(x_1,y')}([y]_f, C_y) \subseteq Z \cap ([y]_f \times \omega^\omega)$, which shows that $h_0(x_1, y') \in Y$ as desired.

For condition (2), suppose toward a contradiction that $\alpha_{x_1} < \gamma$. Let $y^* \in X$ be such that $f(y^*) = \alpha_{x_1}$. By (**), the fact that x_1 is in Y and the definition of α_{x_1}, $h_0(x_1, y^*) \in Y$, and $\alpha_{h_0(x_1,y^*)} = \alpha_{x_1}$. Since τ is a winning strategy for II, $\alpha_{h_0(x_1,y^*)\circ\tau} > \alpha_{h_0(x_1,y^*)} = \alpha_{x_1}$, which is impossible, as

$$(U_{2,h_0(x_1,y^*)\circ\tau}([y^*]_f, C_{y^*}) \subseteq U_{2,x_1}([y^*]_f, C_{y^*}),$$

by equation (3.3) and the choice of x_1. This completes the proof.

$\square$

Chapter 4

Properties of pointclasses

In this chapter we develop various properties of boldface pointclasses under AD, including closure and separation properties, and the prewellordering property. Our immediate goal is preparing for the proof of the existence of strong partition cardinals in Chapter 5. Some of the material in this chapter will also be used in Chapter 6, and in Part II of the book (in Remark 6.1.19, for instance). The material introduced in this chapter is part of a very deep and general theory. However, we are mostly interested in taking the shortest path to proving the existence of strong partition cardinals. Readers interested in this general theory are directed to [15, 16, 17] for an introduction. Much of this chapter is taken from [6, 5] and conversations with their author.

4.1 Separation and reduction

Given a pointclass Γ, and disjoint subsets A and B of the same space in $\mathcal{X}$, we say that A and B are Γ-separable if there exists a set $C \in \Gamma \cap \check{\Gamma}$ such that $A \subseteq C$ and $B \cap C = \emptyset$ (and Γ-inseparable otherwise). We generally use this property only with selfdual pointclasses Δ. We say that a pointclass Γ satisfies the *separation property* if all pairs of disjoint subsets of ω^ω in Γ are $(\Gamma \cap \check{\Gamma})$-separable. We write $\mathsf{Sep}(\Gamma)$ to indicate that Γ has the separation property.

We follow [38]. Let us say that a function $f\colon \omega^\omega \to \omega^\omega$ is *strongly Lipschitz* if for all $x, y \in \omega^\omega$ and $n \in \omega$, if $x{\restriction}n = y{\restriction}n$, then $f(x)(n) = f(y)(n)$. Strategies for player I in Lipschitz games induce strongly Lipschitz functions.

The following lemma is used in the proof of Theorem 4.1.2.

Lemma 4.1.1 ($\mathsf{ZF} + \mathsf{AD}$; Steel)**.** *Let Γ be a boldface pointclass and let (A_0, A_1) be a Γ-inseparable pair of subsets of ω^ω. Let B_0 and B_1 be disjoint subsets of ω^ω, both in in Γ or both in $\check{\Gamma}$. Then there is a strongly Lipschitz map f so that $f[B_0] \subseteq A_0$ and $f[B_1] \subseteq A_1$.*

Proof. Let $\Delta = \Gamma \cap \check{\Gamma}$. Consider the game where I plays x, II plays y, and I wins if $y \in B_0$ implies $x \in A_0$, and $y \in B_1$ implies $x \in A_1$. It suffices to see that II cannot have a winning strategy. Supposing that σ were a winning strategy for II, let g be the function $x \mapsto x \circ \sigma$ (this notation was defined at the beginning of Chapter 3). Then g is continuous, and $g^{-1}[B_0]$ is in Δ, since $\omega^\omega \setminus g^{-1}[B_0] = g^{-1}[B_1]$. Furthermore, $g^{-1}[B_0]$ separates A_0 and A_1. □

Theorem 4.1.2 (ZF+AD; Steel). *If Γ is a nonselfdual boldface pointclass, then at least one of Γ and $\check{\Gamma}$ satisfies the separation property.*

Proof. Suppose toward a contradiction that the pairs (A_0, A_1) and (C_0, C_1) form counterexamples to $\mathsf{Sep}(\Gamma)$ and $\mathsf{Sep}(\check{\Gamma})$, respectively. By Lemma 4.1.1, there is a strongly Lipschitz function f mapping A_0 into C_0 and A_1 into C_1. Let $B_0 = f^{-1}[C_0]$ and let $B_1 = f^{-1}[C_1]$. Then B_0 and B_1 are disjoint and both in $\check{\Gamma}$, and $A_0 \subseteq B_0$ and $A_1 \subseteq B_1$. Applying Lemma 4.1.1 again, there exist strongly Lipschitz functions f_0, f_1 and f_2 such that

- $f_0[A_0] \subseteq A_1$;
- $f_0[A_1] \subseteq A_0$;
- $f_1[A_0] \subseteq A_0$;
- $f_1[\omega^\omega \setminus B_0] \subseteq A_1$;
- $f_2[A_1] \subseteq A_1$;
- $f_2[\omega^\omega \setminus B_1] \subseteq A_0$.

Since f_0, f_1 and f_2 are all strongly Lipschitz, for each $r \in 3^\omega$ there is a unique sequence $\langle x^r_n : n < \omega\rangle$ of elements of ω^ω such that $x^r_n = f_{r(n)}(x^r_{n+1})$ for all $n \in \omega$. Let 3^ω have its induced topology as a subspace of ω^ω. By $\mathsf{Baire}(\mathcal{P}(\omega^\omega))$, every subset of 3^ω has the property of Baire, in particular the set $C = \{r \in 3^\omega : x^r_0 \notin A_0 \cup A_1\}$.

Suppose first that C is nonmeager. By the Baire property, we may fix an $s \in 3^{<\omega}$ such that $\{r \in 3^\omega : x^{s^\frown r}_0 \notin A_0 \cup A_1\}$ is comeager. However, for each $s \in 3^{<\omega}$ and $r \in 3^\omega$, if $x^r_0 \in A_0 \cup A_1$ then $x^{s^\frown r}_0 \in A_0 \cup A_1$. It follows then that $x^r_0 \notin A_0 \cup A_1$, for comeagerly many $r \in 3^\omega$. Since $B_0 \cap B_1 = \emptyset$, we may fix an $i^* \in \{0, 1\}$ such that $x^r_0 \notin B_{i^*}$, for nonmeagerly many $r \in 3^\omega$. Inspecting f_0, f_1 and f_2, we see that for each $i \in \{0, 1\}$ and each $r \in 3^\omega$, if $x^r_0 \notin B_i$ then $x^{\langle i+1\rangle^\frown r}_0 \in A_{1-i}$. It follows then that $x^r_0 \in A_0 \cup A_1$ for nonmeagerly many $r \in 3^\omega$, which gives a contradiction.

Suppose on the other hand that C is comeager. Choose $i^* \in \{0, 1\}$ so that $\{r \in 3^\omega : x^r_0 \in A_{i^*}\}$ is nonmeager. Choose $s \in 3^{<\omega}$ so that $\{r \in 3^\omega : x^{s^\frown r}_0 \in A_{i^*}\}$ is comeager. For any $r \in 3^\omega$ such that $x^r_0 \in A_{i^*}$, $x^{s^\frown\langle 0\rangle^\frown r}_0 \in A_{1-i^*}$. It follows that $x^{s^\frown r}_0 \in A_{1-i^*}$ for nonmeagerly many $r \in 3^\omega$, giving another contradiction. □

4.1.3 Remark. Theorems 5.2 and 5.3 of [40] show that if AD holds then for any nonselfdual pointclass Γ, at most one of Γ and $\check{\Gamma}$ has the separation property.

A pointclass Γ is said to have the *reduction property* if for all $A, B \subseteq \omega^\omega$ in Γ, there exist disjoint $A', B' \in \Gamma$ such that $A' \subseteq A$, $B' \subseteq B$ and $A' \cup B' = A \cup B$. In this case we say that the sets A' and B' *reduce* the sets A and B. We write $\mathsf{Red}(\Gamma)$ to indicate that the pointclass Γ has the reduction property. It follows easily from the definitions that $\mathsf{Red}(\Gamma)$ implies $\mathsf{Sep}(\check{\Gamma})$. Along with Remark 4.1.3, this shows that, under AD, $\mathsf{Red}(\Gamma)$ and $\mathsf{Sep}(\Gamma)$ cannot both hold for a nonselfdual pointclass. The following (taken from 4B.12 of [32]) gives a direct proof.

Theorem 4.1.4. *Assume that* $\mathsf{Lipschitz\ Determinacy} + \mathsf{Baire}(\mathcal{P}(\omega^\omega))$ *holds. If* Γ *is a nonselfdual boldface pointclass and* $\mathsf{Red}(\Gamma)$ *holds, then* $\mathsf{Sep}(\Gamma)$ *does not hold.*

Proof. By Theorem 2.4.3 there exist a Γ-universal set $U \subseteq (\omega^\omega)^2$. Let $\pi\colon \omega^\omega \to (\omega^\omega)^2$ be a continuous bijection, and let π_0, π_1 be the functions on ω^ω such that $\pi(x) = (\pi_0(x), \pi_1(x))$ for all $x \in \omega^\omega$. Let A be the set of $x \in \omega^\omega$ such that $(\pi_0(x), x) \in U$ and let B be the set of $x \in \omega^\omega$ such that $(\pi_1(x), x) \in U$. Let A' and B' in Γ witness the reduction property for A and B.

Suppose now that $C \in \Gamma \cap \check{\Gamma}$ witnesses the separation property for A' and B' (i.e., C contains A' and is disjoint from B'). Let $x \in \omega^\omega$ be such that $U_{\pi_0(x)} = \omega^\omega \setminus C$ and $U_{\pi_1(x)} = C$. If $x \in C$, then $x \in B \setminus A$, so $x \in B'$, giving a contradiction. Similarly, if $x \notin C$, then $x \in A \setminus B$, so $x \in A'$, giving another contradiction. □

4.1.5 Remark. Theorems 4.1.2 and 4.1.4 together imply, assuming AD, that if Γ is a nonselfdual boldface pointclass, then at most one of Γ and $\check{\Gamma}$ has the reduction property. Theorem 4.1.6, which is Theorem 5.5 of [40], shows that if in addition Γ is closed under finite unions and intersections, then the separation property holds for exactly one of Γ and $\check{\Gamma}$, and the reduction property holds for the other.

Theorem 4.1.6 ($\mathsf{ZF} + \mathsf{AD}$; Van Wesep)**.** *If* Γ *is a boldface pointclass closed under finite intersections, and* $\mathsf{Sep}(\check{\Gamma})$ *holds, then* $\mathsf{Red}(\Gamma)$ *holds.*

The proof of Theorem 4.1.6 (which we give below) uses the following lemma.

Lemma 4.1.7. *Suppose that* AD *holds, and that* Γ *is a boldface pointclass closed under finite intersections. If* $\mathsf{Red}(\Gamma)$ *does not hold, then for all* A, B *in* $\mathcal{P}(\omega^\omega) \cap \check{\Gamma}$ *there exist* $C, D \in \Gamma$ *such that*

- $A \setminus B \subseteq C \setminus D$;
- $B \setminus A \subseteq D \setminus C$;
- $C \cup D = \omega^\omega$.

Proof. Let (E, F) be subsets of ω^ω witnessing $\neg\mathsf{Red}(\Gamma)$, and fix A, B in $\mathcal{P}(\omega^\omega) \cap \check{\Gamma}$. Consider the game $\mathcal{G}(E, F, A, B)$ in which player I produces $x \in \omega^\omega$, player II produces $y \in \omega^\omega$ and player II wins if and only if the following conditions are met:

- $y \in E \cup F$;
- if $x \in A \setminus B$ then $y \in E \setminus F$;
- if $x \in B \setminus A$ then $y \in F \setminus E$.

Suppose toward a contradiction that σ is a winning strategy for player I. Let $E'' = \{y \in \omega^\omega : \sigma \circ y \notin A\}$ and let $F'' = \{y \in \omega^\omega : \sigma \circ y \notin B\}$. Then we have the following:

- $E \cup F \subseteq E'' \cup F''$;
- $(E \cup F) \cap (E'' \cap F'') = \emptyset$;
- $E \setminus F \subseteq E''$;
- $F \setminus E \subseteq F''$;
- $E'', F'' \in \Gamma$.

Let $E' = E \cap E''$ and let $F' = F \cap F''$. Then E' and F' are disjoint sets in Γ, and $E \cup F = E' \cup F'$. This contradicts the assumption that E and F witness the failure of $\mathsf{Red}(\Gamma)$.

It follows then that there is a winning strategy τ for player II. Let $C = \{x \in \omega^\omega : x \circ \tau \in E\}$ and let $D = \{x \in \omega^\omega : x \circ \tau \in F\}$. Then C and D are as desired. $\square$

Proof of Theorem 4.1.6. Supposing that $\mathsf{Red}(\Gamma)$ fails, we show that $\mathsf{Red}(\check{\Gamma})$ holds, contradicting Theorem 4.1.4. Let A and B be subsets of ω^ω in Γ. Let C and D be as given by Lemma 4.1.7. Applying $\mathsf{Sep}(\check{\Gamma})$, let $E \in \Gamma \cap \check{\Gamma}$ be such that $\omega^\omega \setminus C \subseteq E$ and $E \cap (\omega^\omega \setminus D) = \emptyset$. Let $C' = C \setminus E$ and let $D' = D \cap E$. Then

$$D' \cap (A \setminus B) = C' \cap (B \setminus A) = C' \cap D' = \emptyset$$

and $A \cup B \subseteq C' \cup D'$, so $A \setminus D'$ and $B \setminus C'$ satisfy the conclusion of the reduction property with respect to A and B. $\square$

The following result (which we need for Theorem 6.1.11) is sometimes called the 0th Periodicity Theorem. Recall that $x \leq_{\mathrm{Tu}} y$ and $x \equiv_{\mathrm{Tu}} y$ refer to Turing reducibility and were defined in Section 0.2.

Theorem 4.1.8 ($\mathsf{ZF} + \mathsf{AD}$; Kechris)**.** *Suppose that Γ is a boldface pointclass closed under countable intersections and countable unions, and that $\mathsf{Red}(\Gamma)$ holds.*

- *If $\exists^{\omega^\omega}\Gamma \subseteq \Gamma$, then $\mathsf{Red}(\forall^{\omega^\omega}\Gamma)$ holds.*
- *If $\forall^{\omega^\omega}\Gamma \subseteq \Gamma$, then $\mathsf{Red}(\exists^{\omega^\omega}\Gamma)$ holds.*

Proof. Let A and B be subsets of $(\omega^\omega)^2$ in Γ. For the first part we will find a reduction for $A^* = \{x : \forall y \in \omega^\omega\, (x, y) \in A\}$ and $B^* = \{x : \forall y \in \omega^\omega\, (x, y) \in B\}$. Let

$$A' = \{(x, z) \in (\omega^\omega)^2 : \forall y \leq_{\mathrm{Tu}} z\, (x, y) \in A\}$$

and let

$$B' = \{(x, z) \in (\omega^\omega)^2 : \forall y \leq_{\mathrm{Tu}} z\, (x, y) \in B\}.$$

Since Γ is closed under countable intersections, A' and B' are in Γ; fix sets $C \subseteq A'$ and $D \subseteq B'$ reducing them. Let

$$C' = \{(x, z) \in (\omega^\omega)^2 : \exists z' \equiv_{\mathrm{Tu}} z\, (x, z') \in C\}$$

and let

$$D' = \{(x, z) \in (\omega^\omega)^2 : \forall z' \equiv_{\mathrm{Tu}} z\, (x, z') \in D\}.$$

Then C' and D' are in Γ, since Γ is closed under countable intersections and countable unions, and they reduce A' and B'.

Let $\mathcal{G}_A(x)$ (for $x \in \omega^\omega$) be the game where players I and II collaborate to build $z \in \omega^\omega$, where I wins if and only if $(x, z) \in C'$, and let E be the set of x for which I has a winning strategy in $\mathcal{G}_A(x)$. Let $\mathcal{G}_B(x)$ (for $x \in \omega^\omega$) be the game where players I and II collaborate to build $z \in \omega^\omega$, where I wins if and only if $(x, z) \in D'$, and let F be the set of x for which I has a winning strategy in $\mathcal{G}_B(x)$. Then E and F are in $\forall^{\omega^\omega}\Gamma$ (via the assertion that player II does not have a winning strategy, and the assumption that $\exists^{\omega^\omega}\Gamma \subseteq \Gamma$), and they are disjoint, since C' and D' are.

To see that E ad F reduce A^* and B^*, fix $x \in \omega^\omega$ and suppose that at least one of A_x and B_x is all of ω^ω. In the former case, $A'_x = \omega^\omega$, and in the latter case $B'_x = \omega^\omega$. Then for each $z \in \omega^\omega$, (x, z) is either in C' or D', and one of the sets $\{[z]_{\mathrm{Tu}} : (x, z) \in C'\}$ and $\{[z]_{\mathrm{Tu}} : (x, z) \in D'\}$ contains a Turing cone. Then player I has a winning strategy in one of the games $\mathcal{G}_A(x)$ and $\mathcal{G}_B(x)$.

For the second part (redirecting our labels) we find a reduction for $A^* = \{x : \exists y \in \omega^\omega\, (x, y) \in A\}$ and $B^* = \{x : \exists y \in \omega^\omega\, (x, y) \in B\}$. Let

$$A' = \{(x, z) \in (\omega^\omega)^2 : \exists y \leq_{\mathrm{Tu}} z\, (x, y) \in A\}$$

and let

$$B' = \{(x, z) \in (\omega^\omega)^2 : \exists y \leq_{\mathrm{Tu}} z\, (x, y) \in B\}.$$

Since Γ is closed under countable unions, A' and B' are in Γ, so there exist C and D in Γ reducing them. Let

$$C' = \{(x, z) \in (\omega^\omega)^2 : \exists z' \equiv_{\mathrm{Tu}} z\, (x, z') \in C\}$$

and let

$$D' = \{(x, z) \in (\omega^\omega)^2 : \forall z' \equiv_{\mathrm{Tu}} z\, (x, z') \in D\}.$$

Then C' and D' are in Γ, since Γ is closed under countable intersections and countable unions, and they reduce A' and B'.

Let $\mathcal{G}_A(x)$ (for $x \in \omega^\omega$) be the game where players I and II collaborate to build $z \in \omega^\omega$, where I wins if and only if $(x,z) \in C'$, and let E be the set of x for which I has a winning strategy. Let $\mathcal{G}_B(x)$ (for $x \in \omega^\omega$) be the game where players I and II collaborate to build $z \in \omega^\omega$, where I wins if and only if $(x,z) \in D'$, and let F be the set of x for which I has a winning strategy. Then E and F are in $\exists^{\omega^\omega}\Gamma$ (via the assertion that player I has a winning strategy, and the assumption that $\forall^{\omega^\omega}\Gamma \subseteq \Gamma$), and they are disjoint.

To see that E and F reduce A^* and B^*, fix $x \in \omega^\omega$ and suppose that at least one of A_x and B_x is nonempty. Then again one of the sets $\{[z]_{\mathrm{Tu}} : (x,z) \in C'\}$ and $\{[z]_{\mathrm{Tu}} : (x,z) \in D'\}$ contains a Turing cone. Then player I has a winning strategy in one of the games $\mathcal{G}_A(x)$ and $\mathcal{G}_B(x)$. $\square$

4.2 The prewellordering property

Given a subset P of a space in $\mathcal{X}$, a *norm* on P is a function from P to the ordinals. A norm is *regular* if its range is an ordinal. We write $o(\phi)$ for the ordertype of the range of a norm ϕ, and call this the *length* or the *rank* of ϕ.

A *prewellordering* on a set P is a binary relation $\leq$ on which is reflexive, transitive, total (so for all $x, y \in X$, at least one of $x \leq y$ and $y \leq x$ holds) and wellfounded. Equivalently (via the canonical rank function for a wellfounded preorder), a prewellordering on a set P is a set of the form $\{(x,y) \in P : f(x) \leq f(y)\}$, where f is a (regular) norm on P. Similarly, a *strict prewellordering* of P is a set of the form $\{(x,y) \in P \times P : f(x) < f(y)\}$, for some norm f on P. The *length* of a (strict) prewellordering then is the ordertype of the range of any such f. Given a pointclass Γ, we let $\delta(\Gamma)$ be the supremum of the lengths of the prewellorderings in $\Gamma \cap \check{\Gamma}$.

Given a pointclass Γ, a space X in $\mathcal{X}$ and a set $P \subseteq X$, a Γ-*norm* on P is norm ϕ on P for which each of the following sets are in Γ :

- $\leq^*_\phi$, the set of pairs $(x,y) \in X \times X$ such that $x \in P$ and either $y \notin P$ or $\phi(x) \leq \phi(y)$;
- $<^*_\phi$, the set of pairs $(x,y) \in X \times X$ such that $x \in P$ and either $y \notin P$ or $\phi(x) < \phi(y)$.

4.2.1 Remark. Suppose that Γ is a boldface pointclass, and ϕ is a Γ-norm on a subset P of a space X in $\mathcal{X}$. Let Y be a space in $\mathcal{X}$ and let $f\colon Y \to X$ be a continuous function. Then $\phi \circ f$ is a Γ-norm on $f^{-1}[P]$. In particular, if P is Γ-complete, and P is the domain of a Γ-norm, then every member of Γ is the domain of some Γ-norm.

4.2.2 Remark. Let Γ be a pointclass. If Γ is selfdual and closed under intersections, then, for each $A \in \Gamma$, any constant function from A to the ordinals is a Γ-norm. If Γ is a lightface pointclass, and for each $A \in \Gamma$ there is a constant Γ-norm on A, then Γ is selfdual.

4.2.3 Remark. Let Γ be a pointclass, let P be an element of Γ contained in a space X in $\mathcal{X}$. Let ϕ be a Γ-norm on P. For points x, y in X, the negation of $x \leq^*_\phi y$ is equivalent to the statement

$$x \notin P \vee (x, y \in P \wedge \phi(y) < \phi(x))$$

and the negation of $x <^*_\phi y$ is equivalent to the statement

$$x \notin P \vee (x, y \in P \wedge \phi(y) \leq \phi(x)).$$

4.2.4 Remark. It follows from Remark 4.2.3 that if Γ is a boldface pointclass closed under intersections and unions, ϕ is a Γ-norm on a set P and $z \in P$, then $\{(x, y) \in P^2 : \phi(x) \leq \phi(y) < \phi(z)\}$ is a prewellordering in $\Gamma \cap \check{\Gamma}$.

Lemma 4.2.5. *Suppose that Γ is a $\forall^{\omega^\omega}$-closed boldface pointclass closed under unions, and that ϕ is a Γ-norm on a set $P \in \Gamma \setminus \check{\Gamma}$. Let Q be a subset of P in $\check{\Gamma}$. Then there exists a $y \in P$ such that $\phi(x) < \phi(y)$ for all $x \in Q$.*

Proof. Supposing otherwise, $P \in \check{\Gamma}$, as $y \in P$ if and only if there exists an $x \in Q$ such that $x <^*_\phi y$ fails. □

A pointclass Γ has the *prewellordering property* if every member of Γ has a Γ-norm. We write $\mathsf{PWO}(\Gamma)$ to mean that Γ has the prewellordering property.

Theorem 4.2.6. *If Γ is a boldface pointclass closed under intersections and unions and satisfying the prewellordering property, then $\mathsf{Red}(\Gamma)$ holds.*

Proof. Let A and B be subsets of ω^ω in Γ. For each $i \in 2$, let $f_i : \omega^\omega \to \omega^\omega$ be such that, for each $x \in \omega^\omega$, $f_i(x)(0) = i$ and $f_i(x)(n+1) = x(n)$ for all $n \in \omega$. Let ϕ be a Γ-norm on $f_0[A] \cup f_1[B]$. Let $A' = \{x \in \omega^\omega : f_0(x) \leq^*_\phi f_1(x)\}$ and let $B' = \{x \in \omega^\omega : f_1(x) <^*_\phi f_0(x)\}$. Then A' and B' reduce A and B. □

4.2.7 Remark. The pointclass $\underset{\sim}{\Pi}^1_1$ has the prewellordering property (see 4B.2 of [32]). Adapting an argument of Novikov, Moschovakis showed that if Γ is a $\forall^{\omega^\omega}$-closed boldface pointclass and $\mathsf{PWO}(\Gamma)$ holds, then $\mathsf{PWO}(\exists^{\omega^\omega}\Gamma)$ also holds. The First Periodicity Theorem (due to Martin and Moschovakis) says that, assuming $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$, if Γ is an $\exists^{\omega^\omega}$-closed boldface pointclass closed under intersections and $\mathsf{PWO}(\Gamma)$ holds, then $\mathsf{PWO}(\forall^{\omega^\omega}\Gamma)$ also holds. See 4B.3 and 6B.1 of [32].

4.2.8 Remark. Suppose that Γ is a boldface pointclass, and P is an element of $\Gamma \setminus \check{\Gamma}$, and ϕ is a Γ-norm on P with range some limit ordinal γ. By Remark 4.2.4, ϕ gives rise to a γ-sequence of sets in $\Gamma \cap \check{\Gamma}$ whose union is P. In particular, $\check{\Gamma}$ is not closed under γ-unions.

Given a set $A \subseteq \omega^\omega$, a $\underset{\sim}{\Sigma}^2_1(A)$ set is a set of the form

$$\{(x_1, \ldots, x_n) \in (\omega^\omega)^n : \exists B \subseteq \omega^\omega\, \phi(A, B, x_1, \ldots, x_n, y)\},$$

where the quantifiers in ϕ range over the hereditarily countable sets, and y is an element of ω^ω. If $\phi(A, B, x_1, \ldots, x_n, y)$ holds for suitable A, B $x_1, \ldots, x_n$ and

y, we say that the formula $\exists B \subseteq \omega^\omega\, \phi(A, B, x_1, \ldots, x_n, y)$ is *witnessed* by B. The pointclass $\underset{\sim}{\Sigma}^2_1(A)$ is the smallest boldface pointclass containing each $\underset{\sim}{\Sigma}^2_1(A)$ subset of ω^ω. It straightforward to verify that $\underset{\sim}{\Sigma}^2_1(A)$ is closed under $\exists^{\omega^\omega}$ and intersections. We do not know whether it is closed under $\forall^{\omega^\omega}$, although it is in many natural models of AD. If the cardinal Θ is regular (i.e., there is no cofinal function from ω^ω to Θ, then the $\forall^{\omega^\omega}$-closure of $\underset{\sim}{\Sigma}^2_1(A)$ can be shown by taking a witness B of sufficiently high Wadge rank. The pointclasses $\underset{\sim}{\Sigma}^2_1(A)$ play an important role in Chapter 5. We introduce them here to give another example of a class of pointclasses with the prewellordering property (see Remark 4.2.10).

4.2.9 Remark. A set $C \subseteq \omega^\omega$ is $\underset{\sim}{\Sigma}^2_1(A)$ if, for some first order formula ϕ in the language of set theory, and some $y \in \omega^\omega$, C is the set of $x \in \omega^\omega$ for which there is an ω-structure (M, E) containing $\omega^\omega \cup \{A\}$ (where E is the $\in$-relation of M) such that $(M, E) \models \phi(A, y, x)$. Modifying this formulation one can define a universal $\underset{\sim}{\Sigma}^2_1(A)$ set from the set of $(n, x, y) \in \omega \times \omega^\omega \times \omega^\omega$ for which there exists an ω-structure (M, E) containing $\omega^\omega \cup \{A\}$ and a set T (naturally coded by a set of reals) such that T is the theory of M (in parameters from M, using the Gödel numbering for first order formulas) and n is the Gödel number of a formula ϕ such that $(M, E) \models \phi(A, x, y)$. It follows from Proposition 2.4.2 that $\underset{\sim}{\Sigma}^2_1(A)$ is nonselfdual.

We write $\Pi^2_1(A)$ for $\check{\underset{\sim}{\Sigma}}^2_1(A)$ (the class of complements of members of $\underset{\sim}{\Sigma}^2_1(A)$), and $\underset{\sim}{\Delta}^2_1$ for $\underset{\sim}{\Sigma}^2_1(A) \cap \underset{\sim}{\Pi}^2_1(A)$. We write $\underset{\sim}{\delta}^2_1(A)$ for $\delta(\underset{\sim}{\Sigma}^2_1(A))$. Proposition 2.5.8 shows that for each $A \subseteq \omega^\omega$ whose Wadge rank is defined, if $\underset{\sim}{\Sigma}^2_1(A)$ is $\forall^{\omega^\omega}$-closed then $\underset{\sim}{\delta}^2_1(A)$ is at least the Wadge rank of A.

4.2.10 Remark. Assuming $\mathsf{AD}+\mathsf{DC}_{\mathbb{R}}$, the pointclass $\underset{\sim}{\Sigma}^2_1(A)$ has the prewellordering property for each $A \subseteq \omega^\omega$. To see this, fix a $\underset{\sim}{\Sigma}^2_1(A)$ set X of the form

$$\{x \in \omega^\omega : \exists B \subseteq \omega^\omega\, \psi(A, B, x, z)\}.$$

Orders $\leq^*$ and $<^*$ witnessing the prewellordering property for X can be defined using the Wadge rank of a witness B. That is, define $x \leq^* y$ to hold if there exists a $B \subseteq \omega^\omega$ such that $\psi(A, B, x, z)$ holds, and for every continuous function $f \colon \omega^\omega \to \omega^\omega$ such that $f^{-1}[B] <_{\mathrm{W}} A$, $\psi(A, f^{-1}[B], y, z)$ fails. Similarly, define $x <^* y$ to hold if there exists a $B \subseteq \omega^\omega$ such that $\psi(A, B, x, z)$ holds, and for every continuous function $f \colon \omega^\omega \to \omega^\omega$, $\psi(A, f^{-1}[B], y, z)$ fails.

Other examples of pointclasses with the prewellordering property are given by the following theorem.

Theorem 4.2.11 (ZF + $\mathsf{Lipschitz\ Determinacy}$; Martin)**.** *Let Δ be a selfdual boldface pointclass closed under unions. Suppose that Δ is not closed under wellordered unions, and let ρ be the least ordinal γ such that $\bigcup_\gamma \Delta \not\subseteq \Delta$. Then $\bigcup_\rho \Delta$ has the prewellordering property. Moreover, each element of $\bigcup_\rho \Delta \setminus \Delta$, has a prewellordering of length ρ whose proper initial segments are all in Δ.*

Proof. Since Δ is closed under unions, ρ is an infinite regular cardinal. By Wadge's Theorem and Remark 4.2.1, it suffices to find a $\bigcup_\rho \Delta$-norm as desired for an arbitrary A in $\mathcal{P}(\omega^\omega) \cap \bigcup_\rho \Delta \setminus \Delta$. Fix such an A. By the minimality of ρ and the assumptions on Δ, there exists a function $f\colon \rho \to \Delta \setminus \{0\}$ such that

- for all $\alpha < \beta < \rho$, $f(\alpha) \cap f(\beta) = \emptyset$;
- $A = \bigcup_{\alpha<\rho} f(\alpha)$.

Define $\phi\colon A \to \rho$ by setting $\phi(x)$ to be the unique $\alpha < \rho$ such that $x \in f(\alpha)$. Then ϕ is a $\bigcup_\rho \Delta$-norm on A. To see this, note first that, for all x, y in ω^ω, $x <^*_\phi y$ if and only if there exists an $\alpha < \rho$ such that $x \in f(\alpha)$ and $y \notin \bigcup_{\beta\leq\alpha} f(\beta)$, and this set is in $\bigcup_\rho \Delta$. Similarly, $x \leq^*_\phi y$ if and only if there exists an $\alpha < \rho$ such that $x \in f(\alpha+1)$ and $y \notin \bigcup_{\beta\leq\alpha} f(\beta)$, and this set is also in $\bigcup_\rho \Delta$. Finally, for each $\gamma < \rho$, the set $\{(x, y) \in A : \phi(x) \leq \phi(y) < \gamma\}$ is a member of Δ, being a union of fewer than ρ many sets from Δ. $\square$

4.2.12 Remark. Assuming that Lipschitz Determinacy holds, Theorem 4.2.11 implies that if Δ is a selfdual boldface pointclass containing a countable Wadge-cofinal subset but no Wadge-maximal set, then $\bigcup_\omega \Delta$ has the prewellordering property. This holds in particular whenever Δ is the pointclass of sets projective in a fixed subset of ω^ω. This shows moreover that if Δ is a selfdual boldface pointclass closed under countable unions, and every element of Δ is a member of subset of Δ with a countable Wadge-cofinal subset, then Δ has the prewellordering property.

Along with Proposition 2.3.3, the following theorem shows, under the additional assumptions of Lipschitz Determinacy and Baire($\mathcal{P}(\omega^\omega)$), that at most one member of any complementary pair of nonselfdual boldface pointclasses can have the prewellordering property. The proof is an adaptation of the proof of Theorem 4.1.4.

Theorem 4.2.13. *If Γ is a boldface pointclass which is closed under intersections and has a universal set, then Γ and $\breve{\Gamma}$ do not both have the prewellordering property.*

Proof. Let $U \subseteq \omega^\omega \times \omega^\omega$ be a universal Γ-set, and let ϕ be a Γ-norm on U. Let $\pi\colon \omega^\omega \to \omega^\omega \times \omega^\omega$ be a continuous bijection, and let π_0 and π_1 be such that $\pi(x) = (\pi_0(x), \pi_1(x))$ for all $x \in \omega^\omega$. Let B be the set of (x, y) such that

$$(\pi_0(x), y) <^*_\phi (\pi_1(x), y),$$

and let C be the set of (x, y) such that

$$(\pi_1(x), y) <^*_\phi (\pi_0(x), y).$$

Then B and C are disjoint and in Γ.

Let $\rho\colon \omega \times 2 \to \omega$ be a continuous bijection. For each $i \in 2$, let ρ^* be the function on $\omega^\omega \times \omega^\omega$ defined by setting $\rho_i^*(x, y)$ to be $(\langle \rho(x(0), i), x_1, x_2, \ldots \rangle, y)$. Let P be the set

$$\rho_0^*[(\omega^\omega \times \omega^\omega) \setminus B] \cup \rho_1^*[(\omega^\omega \times \omega^\omega) \setminus C].$$

Then P is in $\check{\Gamma}$. Let ψ be a $\check{\Gamma}$-norm on P. Let E be the set of $(x, y) \in \omega^\omega \times \omega^\omega$ such that

$$\rho_0^*(x, y) <_\psi^* \rho_1^*(x, y).$$

Since $B \cap C = \emptyset$, E is also the set of (x, y) such that

$$\neg(\rho_1^*(x, y) \leq_\psi^* \rho_0^*(x, y)),$$

so E is in Δ.

We derive a contradiction to Theorem 2.4.2 by showing that E is universal for Δ. Fixing $D \in \Delta$, let $x \in \omega^\omega$ be such that $\omega^\omega \setminus D = U_{\pi_0(x)}$ and $D = U_{\pi_1(x)}$. Then $\omega^\omega \setminus D = B_x$ and $D = C_x$, since, for all $y \in \omega^\omega$,

$$y \in D \Leftrightarrow ((\pi_1(x), y) \in U \wedge (\pi_0(x), y) \notin U).$$

Finally, for each $y \in \omega^\omega$,

$$(x, y) \in C \Rightarrow (\rho_0^*(x, y) \in P \wedge \rho_1^*(x, y) \notin P) \Rightarrow (x, y) \in E$$

and

$$(x, y) \in B \Rightarrow (\rho_0^*(x, y) \notin P \wedge \rho_1^*(x, y) \in P) \Rightarrow (x, y) \notin E.$$

It follows that $E_x = D$. □

Finally, the following theorem shows that for certain pointclasses, every witness to the prewellordering property has the same length. Part of the proof of Theorem 4.2.14 is reused for Theorem 4.3.3.

Theorem 4.2.14 (Moschovakis). *Suppose that Γ is a $\forall^{\omega^\omega}$-closed boldface pointclass closed under unions, and that $\langle U_n : n \in \omega \setminus \{0\} \rangle$ is a sequence of Γ-universal sets with the recursion property. Then for every $n \in \omega \setminus \{0\}$ and every regular Γ-norm ϕ on U_n, $o(\phi) = \delta(\Gamma)$.*

Proof. Fix $n \in \omega \setminus \{0\}$, and let ϕ be a Γ-norm on U_n. By Remark 4.2.4, and the fact that Γ is closed under intersections and unions, we get that $o(\phi) \leq \delta(\Gamma)$.

For the reverse inequality, let $\prec \in \check{\Gamma}$ be a strict prewellordering on a set $Y \subseteq (\omega^\omega)^n$ (this is implied by the corresponding order $\preceq$ being in $\Gamma \cap \check{\Gamma}$). Since any two spaces in $\mathcal{X}$ are homeomorphic, it suffices to show that the rank of $\prec$ is at most $o(\phi)$. Let Q be the set of $(x, y) \in \omega^\omega \times (\omega^\omega)^n$ such that for all $z \in (\omega^\omega)^n$, $z \prec y$ implies $(x, z) <_\phi^* (x, y)$. Then $Q \in \Gamma$. Applying the recursion property for U_n, we get an $x^* \in \omega^\omega$ such that $U_{n,x^*} = Q_{x^*}$.

We want to see that for all $y \in Y$, $y \in U_{n,x^*}$ and, for all $z \in (\omega^\omega)^n$, $z \prec y$ implies $\phi(x^*, z) < \phi(x^*, y)$. This we do by induction on the $\prec$-rank of y. Fix $y \in Y$, and suppose that for all $z \in (\omega^\omega)^n$ with $z \prec y$, z is in U_{n,x^*} and, for all

$w \in (\omega^\omega)^n$, if $w \prec z$ then $\phi(x^*, w) < \phi(x^*, z)$. To see that y is in U_{n,x^*}, we need to see that for all $z \in (\omega^\omega)^n$, if $z \prec y$ then $(x^*, z) <^*_\phi (x^*, y)$. Fixing $z \in (\omega^\omega)^n$ such that $z \prec y$, have that the failure of $(x^*, z) <^*_\phi (x^*, y)$ implies that either $(x^*, z) \notin U_n$ or $(x^*, y) \in U_n$ and $\phi(x^*, y) \leq \phi(x^*, z)$. Since (x^*, z) is in U_n, we have that (x^*, y) is in U_n. Finally, fix once again a $z \in (\omega^\omega)^n$ such that $z \prec y$. Since $(x^*, y) \in U_n$, we have that $(x^*, z) <^*_\phi (x^*, y)$, and (again applying the fact that (x^*, y) is in U_n) this means that $\phi(x^*, z) < \phi(x^*, y)$, as desired. □

4.2.15 Remark. Suppose that Γ is a nonselfdual boldface pointclass closed under unions and intersections. Then $\delta(\Gamma)$ is a limit ordinal, and (by Theorem 2.4.3), Γ has a universal set. Suppose that in addition $\mathsf{Lipschitz\ Determinacy}$ and $\mathsf{Baire}(\mathcal{P}(\omega))$ hold, and that Γ is $\forall^{\omega^\omega}$-closed and has the prewellordering property. Theorems 2.4.9 and 4.2.14 and Remark 4.2.4 together then imply that $\check{\Gamma}$ is not closed under unions of length $\delta(\Gamma)$.

4.3 Prewellorderings and wellfounded relations

Theorems 4.3.1 and 4.3.2 connect the length of prewellorderings in a boldface pointclass Γ with the closure of Γ under wellfounded unions. Aside from Theorem 4.3.5, this section is based on material from [6]

Theorem 4.3.1 ($\mathsf{ZF} + \mathsf{AD}$; Martin)**.** *Let Γ be a nonselfdual boldface pointclass closed under $\exists^{\omega^\omega}$ and intersections. Let ρ be the length of some strict prewellordering in Γ. Then $\bigcup_\rho \Gamma \subseteq \Gamma$.*

Proof. Fix $f\colon \rho \to \Gamma \cap \mathcal{P}(\omega^\omega)$, and let $F = \bigcup_{\alpha<\rho} f(\alpha)$. We want to see that F is in Γ. Let $\prec$ be a strict prewellordering in Γ of length ρ. Let X be the domain of $\prec$, and for each $x \in X$ let α_x denote the $\prec$-rank of x. Applying Theorem 2.4.3, let $U \subseteq (\omega^\omega)^2$ be a universal Γ-set, and let Z be $\{(x, y) \in X \times \omega^\omega : U_y = f(\alpha_x)\}$.

By the Coding Lemma (Theorem 3.0.1), there is a set $A \subseteq Z$ in pos-$\underset{\sim}{\Sigma}^1_1(\prec)$ (which is contained in Γ, since Γ is $\exists^{\omega^\omega}$-closed and closed under intersections) such that for all $x \in X$, if $Z_x \neq \emptyset$ then there is an $x' \in X$ with $\alpha_x = \alpha_{x'}$ and $A_{x'} \neq \emptyset$. Then for all $z \in \omega^\omega$, $z \in F$ if and only if there exist $(x, y) \in A$ with $(y, z) \in U$. □

Theorem 4.3.2 ($\mathsf{ZF} + \mathsf{AD}$; Martin)**.** *Let Γ be a nonselfdual pointclass closed under $\forall^{\omega^\omega}$ and unions, and assume that Γ has the prewellordering property. Let $\Delta = \Gamma \cap \check{\Gamma}$. Then Δ is closed under wellordered unions and intersections of length less than $\delta(\Gamma)$.*

Proof. Since Δ is closed under complements, it suffices to see that it is closed under unions of length less than $\delta(\Gamma)$. Supposing that it is not, let $\rho < \delta(\Gamma)$ be minimal such that there is exists a function $f\colon \rho \to \Delta$ with $\bigcup_{\alpha<\rho} f(\alpha) \notin \Delta$. Since Δ is closed under unions, ρ is a limit ordinal. By Theorem 4.2.11, the pointclass $\bigcup_\rho \Delta$ has the prewellordering property. By Theorems 2.4.9 and 4.2.14, and Remark 4.2.4, there is a prewellordering in Δ of length ρ. Since Δ is closed under complements and intersections, the corresponding strict prewellordering is in Δ

and therefore in $\check{\Gamma}$. By Theorem 4.3.1, $\bigcup_\rho \Delta \subseteq \check{\Gamma}$. Since $\bigcup_\rho \Delta$ is not contained in Δ, it must be (by Lipschitz Determinacy) that $\bigcup_\rho \Delta = \check{\Gamma}$. We then have a contradiction to Theorem 4.2.13. □

The following theorem relates the lengths of prewellorderings in certain pointclasses to the ranks of the wellfounded relations in the same class.

Theorem 4.3.3 (Moschovakis)**.** *Suppose that* Lipschitz Determinacy *holds, and that every subset of ω^ω has the property of Baire. Let Γ be a nonselfdual boldface pointclass closed under $\forall^{\omega^\omega}$ and unions, and suppose that* $\mathsf{PWO}(\Gamma)$ *holds. Then every strict wellfounded relation in $\check{\Gamma}$ has length less than $\delta(\Gamma)$.*

Proof. Let $\prec$ be a strict wellfounded relation in $\check{\Gamma}$ on a set $X \in \check{\Gamma} \cap \mathcal{P}(\omega^\omega)$. We will show that the rank of $\prec$ is at most $\delta(\Gamma)$. This will suffice, since there are strict wellfounded relations in $\check{\Gamma}$ whose rank is more than that of $\prec$ (one more, for instance), and the argument will apply to these relations also.

Applying Theorem 2.4.9, let $\bar{U} = \langle U_n : n < \omega \setminus \{0\}\rangle$ be a sequence of universal sets for Γ with the recursion property. Let ϕ be a regular Γ-norm on U_2. By Theorem 4.2.14, $o(\phi) = \delta(\Gamma)$.

Let A be the set of $(x, y) \in \omega^\omega \times \omega^\omega$ such that, for all $z \in X$, if $z \prec y$ then $(x, z) <^*_\phi (x, y)$. By the recursion property of U_1, there is an $x \in \omega^\omega$ such that $A_x = U_{1,x}$.

We verify first by induction on the $\prec$-rank of $y \in \omega^\omega$ that $(x, y) \in U_1$ for all $y \in X$. First note that if $(x, y) \notin U_1$ and (x, z) is in U_1 for all $z \prec y$, then (x, y) is in A, which implies that in fact (x, y) is in U_1. It follows then that for all $y, z \in X$, if $z \prec y$ then $\phi(x, z) < \phi(x, y)$, as desired. □

In conjunction with Theorem 4.3.5, the following theorem shows that the supremum of the Wadge ranks of the sets in a pointclass of the form $\underset{\sim}{\Sigma}^2_1(A)$ (assuming that the Wadge hierarchy is wellfounded) is a regular cardinal.

Theorem 4.3.4 ($\mathsf{ZF} + \mathsf{AD}$)**.** *If Γ is a nonselfdual boldface pointclass satisfying the prewellordering property and closed under $\forall^{\omega^\omega}$ and unions then the supremum of the ranks of the wellfounded relations in $\check{\Gamma}$ is a regular cardinal.*

Proof. Let κ be the supremum of the ranks of the wellfounded relations in $\check{\Gamma}$. Since $\check{\Gamma}$ is closed under unions, κ is a limit ordinal, and all wellfounded relations in $\check{\Gamma}$ have rank less than κ. Let ρ be the cofinality of κ, suppose that $\rho < \kappa$, and let $f \colon \rho \to \kappa$ be cofinal. Applying Theorem 4.3.3, let $\prec$ be a strict prewellordering in $\Gamma \cap \check{\Gamma}$ of length ρ. Let X be the domain of $\prec$, and, for each $x \in X$, let α_x be the $\prec$-rank of x. Applying Theorem 2.4.3, let $U \subseteq (\omega^\omega)^3$ be a universal $\check{\Gamma}$ set, and let Z be the set of $(x, y) \in X \times \omega^\omega$ for which U_y is a wellfounded relation of length at least $f(\alpha_x)$. By the Coding Lemma (Theorem 3.0.1), there exists a pos-$\underset{\sim}{\Sigma}^1_1(\prec)$-set $A \subseteq Z$ such that, for all $x \in X$, if $Z_x \neq \emptyset$ then $A_x \neq \emptyset$. Then A is in $\check{\Gamma}$. Now define the relation $<$ on $(\omega^\omega)^3$ by setting $(x, y, z) < (a, b, c)$ if and only if $(x, y) \in A$, $(x, y) = (a, b)$ and $(y, z, c) \in U$. Then $<$ is a wellfounded relation in Γ of rank κ, giving a contradiction. □

A *projective algebra* is a selfdual boldface pointclass containing all closed subsets of spaces in $\mathcal{X}$ and closed under real quantification (some authors, e.g. [18], include closure under wellordered unions in the definition of projective algebra). Given a pointclass Γ, we write $o(\Gamma)$ for the set of Wadge ranks of elements of $\Gamma \cap \mathcal{P}(\omega^\omega)$. The following is a weak version of a theorem from [18]. The use of $\mathsf{DC}_{\mathbb{R}}$ in the theorem is only to ensure that each element of Δ has a Wadge rank.

Theorem 4.3.5 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *Let Γ be a nonselfdual boldface pointclass closed under $\forall^{\omega^\omega}$ and unions, and suppose that $\mathsf{PWO}(\Gamma)$ holds. Let $\Delta = \Gamma \cap \check{\Gamma}$, and suppose that Δ is a projective algebra. Then the following three ordinals are equivalent:*

- *$o(\Delta)$;*
- *$\delta(\Gamma)$;*
- *The supremum of the ranks of the wellfounded relations in Δ.*

Proof. Let κ be the supremum of the ranks of the wellfounded relations in Δ. That $\delta(\Gamma) \leq \kappa$ follows immediately; that $\delta(\Gamma) = \kappa$ follows from Theorem 4.3.3. That $o(\Delta) \leq \delta(\Gamma)$ follows from Proposition 2.5.8. For $\delta(\Gamma) \leq o(\Delta)$, note first by Theorem 4.3.2 that Δ is closed under unions of length less than $\delta(\Gamma)$. One can then construct sets in Δ of Wadge ranks cofinal in $\delta(\Gamma)$ by following Solovay's construction, as in Theorem 2.5.9 and Remark 2.5.10. □

4.4 Closure under wellordered unions

This section gives applications of the material in Section 4.2 and 4.3, showing that certain pointclasses are closed under wellordered unions. These results will be used in Chapters 6 and 9. Theorem 4.4.1 is Theorem 2.1 of [7].

Theorem 4.4.1 (ZF+AD)**.** *Let Γ be a boldface nonselfdual pointclass containing $\utilde{\Sigma}^1_1$ which is closed under $\exists^{\omega^\omega}$ and $\forall^{\omega^\omega}$. Then at least one of Γ and $\check{\Gamma}$ is closed under wellordered unions.*

Proof. Supposing otherwise, let κ be the least ordinal γ such that Γ is not closed under γ-unions, and let λ be the least ordinal γ such that $\check{\Gamma}$ is not closed under γ-unions. Without loss of generality, $\lambda \leq \kappa$. We will find a strict prewellordering in $\check{\Gamma}$ of length κ. By Theorem 4.3.1, this will contradict the choice of λ. Note that κ is regular.

By Corollary 2.3.2, $\check{\Gamma} \subseteq \bigcup_\kappa \Gamma$. Applying Theorem 2.4.3, let $A \subseteq \omega^\omega$ be in $\check{\Gamma} \setminus \Gamma$, and let A_α ($\alpha < \kappa$) be sets in Γ such that $A = \bigcup_{\alpha<\kappa} A_\alpha$. Let $U \subseteq (\omega^\omega)^2$ be a universal Γ set, and let $B = \{x \in \omega^\omega : U_x \subseteq A\}$. Since $\check{\Gamma}$ is $\forall^{\omega^\omega}$-closed, $B \in \check{\Gamma}$. There exist then B_α ($\alpha < \kappa$) in Γ such that $B = \bigcup_{\alpha<\kappa} B_\alpha$. For each $\alpha < \kappa$, let $C_\alpha = \bigcup_{x \in B_\alpha} U_x$. Then $A = \bigcup_{\alpha<\kappa} C_\alpha$, and every subset of A in Γ is contained in C_α for some $\alpha < \kappa$. For each x such that $U_x \subseteq A$, let α_x be the least $\alpha < \kappa$ with $U_x \subseteq C_\alpha$.

Let $\mathcal{G}$ be the game where I plays $x \in \omega^\omega$, II plays y and z, and II wins if and only if $U_x \not\subseteq A$ or

$$\exists\alpha > \alpha_x(U_y = \bigcup_{\beta\leq\alpha} C_\beta \wedge z \in C_\alpha \setminus \bigcup_{\alpha<\beta} C_\beta).$$

Since every subset of A in Γ is contained in some C_α, I cannot have a winning strategy in $\mathcal{G}$. Let τ be a winning strategy for II. Given $x \in \omega^\omega$, let $\tau_y(x)$ and $\tau_z(x)$ denote, respectively, the y and z parts of the response to x (played by I) given by τ. Define the binary relation $\prec$ on B by setting $x_0 \prec x_1$ if and only if $\tau_z(x_1) \notin U_{\tau_y(x_0)}$. It follows that $\prec$ is a strict prewellordering in $\check{\Gamma}$ of length κ. ☐

4.4.2 Remark. Remark 4.2.15 and Theorem 4.4.1 together imply that if AD holds and Γ is a boldface nonselfdual pointclass closed under $\exists^{\omega^\omega}$ and $\forall^{\omega^\omega}$, and $\mathsf{PWO}(\Gamma)$ holds, then Γ is closed under wellordered unions.

Lemma 4.4.3 and Theorem 4.4.5 below concern pointclasses which are closed under $\exists^{\omega^\omega}$ but not $\forall^{\omega^\omega}$. The proofs of Lemma 4.4.3 and Theorem 4.4.4 are extracted from the proof of Lemma 2.18 of [6]. The proof of the first part of Lemma 4.4.3 is very similar to the proof of Theorem 4.4.1.

Lemma 4.4.3 $(\mathsf{ZF} + \mathsf{AD})$**.** *Suppose that Γ is a nonselfdual boldface pointclass containing $\utilde{\Sigma}^1_1$ and closed under $\exists^{\omega^\omega}$ and finite intersections, but not $\forall^{\omega^\omega}$ or wellordered unions.*

1. *Let Γ_1 be $\exists^{\omega^\omega}\check{\Gamma}$ and let κ be the least ordinal γ such that $\bigcup_\gamma \Gamma \not\subseteq \Gamma$. Then $\bigcup_\kappa \Gamma_1 = \Gamma_1 = \bigcup_\kappa \Gamma$ and Γ_1 is nonselfdual.*
2. *Let $\Delta_1 = \Gamma_1 \cap \check{\Gamma}_1$. Then Δ_1 is not closed under wellorded unions. Letting ρ be the least ordinal γ such that $\bigcup_\gamma \Delta_1 \not\subseteq \Delta_1$, $\rho \leq \kappa$, $\Gamma_1 = \bigcup_\rho \Delta_1$ and Γ_1 has the prewellordering property.*

Proof. Note that κ is a regular cardinal. Since Γ is $\exists^{\omega^\omega}$-closed, but not $\forall^{\omega^\omega}$-closed, Γ is nonselfdual. It follows by Theorem 2.4.3 and Remark 2.4.4 that Γ_1 is also nonselfdual. By Corollary 2.3.2, $\check{\Gamma} \subseteq \bigcup_\kappa \Gamma$ and $\Gamma \subseteq \Gamma_1$. Since $\bigcup_\kappa \Gamma$ is closed under $\exists^{\omega^\omega}$, Γ_1 is also contained in $\bigcup_\kappa \Gamma$. We will show first that there is a strict prewellordering of length κ in Γ_1. From this and Theorem 4.3.1 it will follow that $\bigcup_\kappa \Gamma_1 \subseteq \Gamma_1$, from which the first part of the lemma follows.

Let $A \subseteq \omega^\omega$ be any set in $\check{\Gamma} \setminus \Gamma$. Let $S \subseteq (\omega^\omega)^2$ be a universal $\utilde{\Sigma}^1_1$ set, and let $B = \{x \in \omega^\omega : S_x \subseteq A\}$. Since $\check{\Gamma}$ is closed under $\forall^{\omega^\omega}$ and unions, $B \in \check{\Gamma}$, so B can be written as $\bigcup_{\alpha<\kappa} B_\alpha$, where each B_α is in Γ. For each $\alpha < \kappa$, let $A_\alpha = \bigcup_{x\in B_\alpha} S_x$ Then $A = \bigcup_{\alpha<\kappa} A_\alpha$, and every $\utilde{\Sigma}^1_1$ subset of A (being S_x for some $x \in \omega^\omega$) is contained in some A_α. Since each A_α is in Γ, A is not contained in $\bigcup_{\alpha<\gamma} A_\alpha$ for any $\gamma < \kappa$. For each $x \in A$, let α_x be the least α with $x \in A_\alpha$.

Let $U \subseteq (\omega^\omega)^2$ be a universal Γ set. Let $\mathcal{G}$ be the game where I plays $x \in \omega^\omega$, II plays y and z, and II wins if and only if $x \notin A$ or

$$\exists\alpha > \alpha_x(U_y = \bigcup_{\beta\leq\alpha} A_\beta \wedge z \in A_\alpha \setminus \bigcup_{\beta<\alpha} A_\beta).$$

Since every $\utilde{\Sigma}^1_1$ subset of A is contained in some A_α, I cannot have a winning strategy in $\mathcal{G}$. Let τ be a winning strategy for II. Given $x \in \omega^\omega$, let $\tau_y(x)$ and $\tau_z(x)$ denote, respectively, the y and z parts of the response to x (played by I) given by τ. Define the binary relation $\prec$ on ω^ω by setting $x_0 \prec x_1$ if and only if $\{x_0, x_1\} \subseteq A$ and $\tau_z(x_1) \notin U_{\tau_y(x_0)}$. It follows that $\prec$ is a strict prewellordering in $\check{\Gamma}$ of length κ.

For the second part of the lemma, since $\Gamma \subseteq \Delta_1$ and $\bigcup_\kappa \Gamma$ is nonselfdual, Δ_1 is not closed under wellordered unions, and $\rho \leq \kappa$. Since $\bigcup_\rho \Delta_1$ properly contains Δ_1 and is contained in $\bigcup_\kappa \Gamma_1 = \Gamma_1$, $\Gamma_1 = \bigcup_\rho \Delta_1$, by Proposition 2.3.3. It follows from Theorem 4.2.11 that Γ_1 has the prewellordering property. $\square$

The last paragraph of the proof of the following theorem is taken from the proof of Lemma 2.18 of [6].

Theorem 4.4.4 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *Suppose that Δ is a projective algebra such that $\bigcup_\omega \Delta \not\subseteq \Delta$. Then $\bigcup_\omega \Delta$ is nonselfdual, has the prewellordering property and is closed under wellordered unions.*

Proof. Since Δ is a projective algebra and $\bigcup_\omega \Delta \not\subseteq \Delta$, there exist $B_i \subseteq \omega^\omega$ ($i \in \omega$) which are collectively Wadge cofinal in Δ. Let $\pi \colon \omega^\omega \to (\omega^\omega)^\omega$ be a continuous bijection, and let π_i ($i \in \omega$) be the functions on ω^ω such that $\pi(x) = \langle \pi_i(x) : i \in \omega \rangle$ for all $x \in \omega^\omega$. Recalling our coding in Remark 2.5.2 of continuous functions f^{c}_x by $x \in \mathcal{F}^{\mathrm{c}} \subseteq \omega^\omega$, for each $i \in \omega$ let C_i be the set of $(x, y) \in (\omega^\omega)^2$ such that $\pi_i(x) \in \mathcal{F}^{\mathrm{c}}$ and $f^{\mathrm{c}}_{\pi_i(x)}(y) \in B_i$. Since Δ is a projective algebra, each C_i is in Δ. Let $C = \bigcup_{i \in \omega} C_i$. We claim that C is universal for the pointclass $\bigcup_\omega \Delta$. To verify this, consider a set $\bigcup_{i\in\omega} A_i$, where each A_i is in Δ. Applying $\mathsf{CC}_{\mathbb{R}}$, we may find E_i and x_i ($i \in \omega$) such that each x_i is in $\mathcal{F}^{\mathrm{c}}$, each E_i is $(f^{\mathrm{c}}_{x_i})^{-1}[B_i]$ and $\bigcup_{i\in\omega} A_i = \bigcup_{i\in\omega} E_i$. Then $\bigcup_{i\in\omega} E_i = C_x$. Since $\bigcup_\omega \Delta$ has a universal set, it is nonselfdual, by Proposition 2.4.2.

Let $\Gamma = \bigcup_\omega \Delta$. Then Γ is closed under $\exists^{\omega^\omega}$ and intersections. It follows by Theorem 4.2.11 that Γ has the prewellordering property. By the First Periodicity Theorem (see Remark 4.2.7), $\check{\Gamma}_1 = \forall^{\omega^\omega}\Gamma$ does as well. It follows by Theorem 4.2.13 that Γ_1 does not have the prewellordering property. It follows then by the second part of Lemma 4.4.3 that Γ must be $\forall^{\omega^\omega}$-closed or closed under wellordered unions. However, it follows by Remark 4.4.2 that Γ is closed under wellordered unions even if it is $\forall^{\omega^\omega}$-closed. $\square$

The following is Lemma 2.4 of [18]. It implies Corollary 4.4.6 below, which we will use in Chapter 6 to analyze pointclasses of Suslin sets.

Theorem 4.4.5 ($\mathsf{ZF}{+}\mathsf{AD}$)**.** *Let Γ be a $\exists^{\omega^\omega}$-closed boldface pointclass containing $\utilde{\Sigma}^1_1$ which is closed under countable unions and intersections but not $\forall^{\omega^\omega}$-closed. If $\mathsf{Red}(\Gamma)$ holds, then Γ is closed under wellordered unions.*

Proof. Since Γ is $\exists^{\omega^\omega}$-closed and not $\forall^{\omega^\omega}$-closed, Γ is nonselfdual. Working toward a contradiction, let κ be the least ordinal γ such that for some $f \colon \gamma \to \Gamma$, $\bigcup_{\alpha<\gamma} f(\alpha) \notin \Gamma$. Then κ is a regular uncountable cardinal. Fix such a function

f, and let $F = \bigcup_{\alpha<\kappa} f(\alpha)$. By Theorem 2.1.4, the boldface pointclass $\bigcup_\kappa \Gamma$ contains both Γ and $\check{\Gamma}$. Letting $\Gamma_1 = \exists^{\omega^\omega}\check{\Gamma}$ and $\Delta_1 = \Gamma_1 \cap \check{\Gamma}_1$, we have the following by Lemma 4.4.3:

- $\Gamma \subseteq \Gamma_1 = \bigcup_\kappa \Gamma_1 = \bigcup_\kappa \Gamma$;
- Γ_1 is nonselfdual;
- $\Gamma \subseteq \Delta_1$, which is not closed under wellordered unions;
- letting ρ be the least ordinal γ such that $\bigcup_\rho \Delta_1 \not\subseteq \Delta_1$, $\rho \leq \kappa$, $\Gamma_1 = \bigcup_\rho \Delta_1$;
- Γ_1 has the prewellordering property, and thus the reduction property, by Theorem 4.2.6.

By Theorem 4.1.8 and the assumption that Γ has the reduction property, $\check{\Gamma}_1$ has the reduction property. We now have a contradiction, by Remark 4.1.5. $\square$

Corollary 4.4.6 (ZF + AD)**.** *If Δ is a projective algebra then every member of Δ is an element of a nonselfdual pointclass contained in Δ and closed under wellordered unions.*

Proof. Fix $A \in \Delta \cap \mathcal{P}(\omega^\omega)$. The pointclasses $\utilde{\Sigma}^1_1(A)$ and $\utilde{\Sigma}^1_2(A)$ have universal sets, so they are nonselfdual and properly contained in Δ. Moreover, they are closed under countable unions and intersections but not $\forall^{\omega^\omega}$-closed. By Remark 4.1.5 and Theorems 4.1.8 and 4.4.5, one of these two pointclasses satisfies the reduction property and (by Theorem 4.4.5) is closed under wellordered unions. $\square$

Chapter 5

Strong Partition Cardinals

In this chapter we prove that under AD the strong partition cardinals are cofinal below Θ. In Section 5.1 we introduce strong partition cardinals and the pointclasses $\utilde{\Sigma}^2_1(A)$. Section 5.2 completes the proof. Again, the material presented here is a small part of a much larger story.

5.1 Strong partition cardinals

Given an ordinal α and a set X of ordinals, $[X]^\alpha$ denotes the collection of subsets of X of ordertype α (for notational convenience, we often identify each element of $[X]^\alpha$ with the corresponding order-preserving function on α which enumerates it.). Given ordinals α, β, δ and γ the formula $\alpha \to (\beta)^\delta_\gamma$ asserts that for every partition of $[\alpha]^\delta$ into γ many pieces, there exists an $X \in [\alpha]^\beta$ such that $[X]^\delta$ is contained in one piece. We write $\alpha \to (\beta)^\delta$ when $\gamma = 2$. A *strong partition cardinal* is a regular uncountable cardinal κ such that $\kappa \to (\kappa)^\kappa_\mu$ holds for all $\mu < \kappa$.

5.1.1 Remark. Some authors define a strong partition cardinal using the ostensibly weaker property $\kappa \to (\kappa)^\kappa$. We do not know whether ZF implies that these two definitions are equivalent. However, if κ is a regular uncountable cardinal such that $\kappa \to (\kappa)^\kappa$, then $\forall \mu < \kappa(\kappa \to (\kappa)^\mu_\mu)$ holds. This can be seen, for instance, by considering, given an $f \colon [\kappa]^\mu \to \mu$, the function $f' \colon [\kappa]^\kappa \to 2$ defined by setting $f'(X) = 0$ if and only if $f(a) = f(b)$, where a is the initial segment of X of ordertype μ, and b is the initial segment of $X \setminus a$ of ordertype μ. The property $\forall \mu < \kappa(\kappa \to (\kappa)^\mu_\mu)$ (in particular its consequence given in Theorem 5.1.3 below) is enough for our application of strong partition cardinals in Theorem 7.0.3.

5.1.2 Remark. A theorem of Kleinberg (Theorem 28.10 of [10]) shows that if $\kappa \to (\kappa)^{\omega_1+\omega_1}_2$, then κ is measurable.

The following notation will be used in Theorem 5.1.3, and again in the proof of Theorem 7.0.3. Given an ordinal-valued function g on $\omega \cdot \alpha$, for some ordinal

α, let g^* be the function on α defined by setting $g^*(\xi)$ to be the supremum of $\{g(\omega\cdot\xi+n) : n \in \omega\}$. For each set X of ordinals, let $X^*(\alpha)$ be $\{g^* : g \in [X]^{\omega\cdot\alpha}\}$. Given an ordinal δ, let $\mu_{\delta,\alpha}$ be the set of $A \subseteq [\delta]^\alpha$ for which there exists a club $C \subseteq \delta$ with $C^*(\alpha) \subseteq A$.

Theorem 5.1.3. *Let δ be a regular uncountable cardinal and let $\alpha < \delta$ be such that $\delta \to (\delta)^{\omega\cdot\alpha}_\rho$ holds for all $\rho < \delta$. Then $\mu_{\delta,\alpha}$ is a δ-complete ultrafilter on $[\delta]^\alpha$.*

Proof. Fix $\rho < \delta$ and suppose that $[\delta]^\alpha$ is the union of sets A_β ($\beta < \rho$). For each $\beta < \rho$, let B_β be $\{g \in [\delta]^{\omega\alpha} : g^* \in A_\beta\}$. Since $\delta \to (\delta)^{\omega\cdot\alpha}_\rho$ holds, there exist an $H \in [\delta]^\delta$ and a $\beta < \rho$ such that $[H]^{\omega\cdot\alpha} \subseteq B_\beta$. Let C be the set of limit points of H. Then C is a club subset of δ. Given $h \in [C]^{\omega\cdot\alpha}$, define $g \in [H]^{\omega\cdot\alpha}$ by setting $g(\gamma)$ to be $\min(H \setminus h(\gamma))$. Then $h^* = g^*$, which is in A_β. It follows that $C^*(\alpha) \subseteq \mathcal{A}_\beta$. $\square$

Our aim in this chapter is to prove the following theorem, which is due to Kechris, Kleinberg, Moschovakis and Woodin [14].

Theorem 5.1.4. *If* AD *holds then the strong partition cardinals are cofinal below Θ.*

The proof uses the pointclasses $\utilde{\Sigma}^2_1(A)$ and their associated ordinals $\utilde{\delta}^2_1(A)$ as defined in Section 4.2. We do not know whether AD implies that cardinals of the form $\utilde{\delta}^2_1(A)$ are strong partition cardinals. One problem is that we do not know how to show that the pointclass $\utilde{\Sigma}^2_1(A)$ has the prewellordering property without some additional assumption. In Remark 4.2.10 we show this using the additional assumption of $\mathsf{DC}_{\mathbb{R}}$, which implies that the Wadge hierarchy is wellfounded. A second problem is that we do know whether AD implies that $\utilde{\Sigma}^2_1(A)$ is $\forall^{\omega^\omega}$-closed, although it is in many natural models of AD. Adding these two properties as hypotheses, we get the following.

Theorem 5.1.5 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *For each $A \subseteq \omega^\omega$ for which $\utilde{\Sigma}^2_1(A)$ is $\forall^{\omega^\omega}$-closed, $\utilde{\delta}^2_1(A)$ is a strong partition cardinal.*

To get around these issues, we relativize $\utilde{\Sigma}^2_1(A)$ to the inner model $L(A,\mathbb{R})$. We write loc-$\utilde{\Sigma}^2_1(A)$ for

$$\utilde{\Sigma}^2_1(A)^{L(A,\mathbb{R})},$$

loc-$\utilde{\Delta}^2_1(A)$ for $\utilde{\Delta}^2_1(A)^{L(A,\mathbb{R})}$ and loc-$\utilde{\delta}^2_1(A)$ for $\utilde{\delta}^2_1(A)^{L(A,\mathbb{R})}$. As with $\utilde{\Sigma}^2_1(A)$, each pointclass loc-$\utilde{\Sigma}^2_1(A)$ is nonselfdual, has a universal set and closed under $\exists^{\omega^\omega}$ and intersections. Since Θ is regular in models of the form $L(A,\mathbb{R})$, pointclasses of the form loc-$\utilde{\Sigma}^2_1(A)$ are also $\forall^{\omega^\omega}$-closed.

The Solovay Basis Theorem (Theorem 5.1.6 below) says that for each $A \subseteq \omega^\omega$, loc-$\utilde{\Delta}^2_1(A)$ is Σ_1-elementary in $\mathcal{P}(\omega^\omega) \cap L(A,\mathbb{R})$. The proof of the theorem, which we leave to the reader, follows from the fact that in models of the form $L(A,\mathbb{R})$, every set is ordinal definable from A and a member of ω^ω, which implies that every set has an elementary submodel which is the surjective image of ω^ω. This means that there is a minimal (relative to a parameter from ω^ω) witness to each true $\utilde{\Sigma}^2_1(A)$ statement, and that that minimal witness is in loc-$\utilde{\Delta}^2_1(A)^{L(A,\mathbb{R})}$. In particular, membership in loc-$\utilde{\Sigma}^2_1(A)$ sets is witnessed by loc-$\utilde{\Delta}^2_1(A)$ sets. This gives another proof that pointclasses of the form loc-$\utilde{\Sigma}^2_1(A)$ are $\forall^{\omega^\omega}$-closed.

Theorem 5.1.6 (Solovay Basis Theorem)**.** *Let A be a subset of ω^ω and let x and y be an element of ω^ω. Let ϕ be a formula whose quantifiers range over the hereditarily countable sets. If there exists a set B in $\mathcal{P}(\omega^\omega) \cap L(A, \mathbb{R})$ such that $\phi(A, B, x, y)$ holds, then there exists such a set B in loc-$\utilde{\Delta}^2_1(A)$.*

5.1.7 Remark. To see that the pointclasses loc-$\utilde{\Sigma}^2_1(A)$ satisfy the prewellordering property (without assuming $\mathsf{DC}_{\mathbb{R}}$), fix a $\Sigma^2_1(A)$ set X of the form

$$\{x \in \omega^\omega : L(A, \omega^\omega) \models \exists B \subseteq \omega^\omega\, \psi(A, B, x)\}.$$

Orders $\leq^*$ and $<^*$ witnessing that X has the prewellordering property can be defined using the least ordinal α such that $L_\alpha(A, \omega^\omega)$ contains a set B as in the definition of X. Elementary submodel arguments similar to the proof of the Solovay Basis Theorem show that this gives a loc-$\utilde{\Sigma}^2_1(A)$-norm on X.

The following gives Theorem 5.1.4.

Theorem 5.1.8. *If AD holds then loc-$\utilde{\delta}^2_1(A)$ is a strong partition cardinal, for each $A \subseteq \omega^\omega$.*

Our proofs of Theorems 5.1.5 and 5.1.8 diverge at just one point, establishing that each of the relevant pointclasses satisfies the prewellordering property (see the discussion just before the statement of Theorem 5.2.3).

5.2 The Martin Conditions

We complete the proof of Theorem 5.1.8 (so also Theorem 5.1.4) by presenting an argument of Martin connecting closure properties of pointclasses with partition properties of their associated ordinals. The following definition (a slightly modified version of Definition 2.33 on page 1783 of [6]) is due to Martin; the properties that it lists are known as the Martin conditions.

5.2.1 Definition. Let $\lambda \leq \kappa$ be ordinals. We say that κ is λ-*reasonable* if there exist a nonselfdual boldface pointclass Γ closed under $\forall^{\omega^\omega}$ and a function

$$\phi\colon \omega^\omega \to \mathcal{P}(\lambda \times \kappa)$$

such that, letting $\Delta = \Gamma \cap \check{\Gamma}$,

1. for each function $F\colon \lambda \to \kappa$ there is an $x \in \omega^\omega$ such that $\phi(x) = F$;
2. for each pair $(\beta, \gamma) \in \lambda \times \kappa$, the set

$$R_{\beta,\gamma} = \{x \in \omega^\omega : \{\eta < \kappa : (\beta, \eta) \in \phi(x)\} = \{\gamma\}\}$$

is in Δ;

3. for each $\beta < \lambda$ and each A in $\exists^{\omega^\omega}\Delta \cap \mathcal{P}(R_\beta)$ there exists a $\gamma_0 < \kappa$ such that
$$A \subseteq \bigcup_{\gamma<\gamma_0} R_{\beta,\gamma},$$
where $R_\beta = \bigcup\{R_{\beta,\gamma} : \gamma < \kappa\}$.

Given a function ϕ and sets R_β $(\beta < \lambda)$ as in Definition 5.2.1, $\phi(x)$ is a function from λ to κ for each $x \in \bigcap_{\beta<\lambda} R_\beta$.

We first show how reasonableness gives partition properties, and then how the established properties of the pointclasses loc-$\utilde{\Sigma}^2_1(A)$ give reasonableness. Given a set C of ordinals, we let $C(\omega)$ denote the set of $\gamma \in C$ for which the ordertype of $C \cap \gamma$ is $\eta + \omega$, for some ordinal η.

Theorem 5.2.2 ($\mathsf{ZF} + \mathsf{AD}$; Martin)**.** *Let Γ be a nonselfdual boldface pointclass such that $\delta(\Gamma)$ is a regular cardinal and $\Gamma \cap \check{\Gamma}$ is a projective algebra closed under intersections and unions of length less than $\delta(\Gamma)$. Let λ be an ordinal such that $\omega\cdot\lambda \leq \delta(\Gamma)$ and Γ witnesses that $\delta(\Gamma)$ is $\omega\cdot\lambda$-reasonable. Then for each function $P\colon [\kappa]^\lambda \to 2$ there is a club $C_0 \subseteq \kappa$ such that, for any cofinal $C \subseteq C_0$, P is constant on $[C(\omega)]^\lambda$.*

Proof. Let κ be $\delta(\Gamma)$ and let Δ be $\Gamma \cap \check{\Gamma}$. Let ϕ, $R_{\beta,\gamma}$ $(\beta < \omega\cdot\lambda, \gamma < \kappa)$ and R_β $(\beta < \omega\cdot\lambda)$ be as in Definition 5.2.1, with respect to $\omega\cdot\lambda$, κ and Γ. Fix $P\colon [\kappa]^\lambda \to 2$.

Consider the following game in which player I plays the values of $x \in \omega^\omega$ and player II plays the values of $y \in \omega^\omega$.

I	$x(0)$		$x(1)$		$x(2)$		$\ldots$
II		$y(0)$		$y(1)$		$\ldots$	

The winner of the game is determined as follows. If there is a least ordinal $\beta < \omega\cdot\lambda$ such that $\{x, y\} \not\subseteq R_\beta$, then II wins provided that $x \notin R_\beta$. Otherwise, letting $f_{x,y} : \lambda \to \kappa$ be defined by setting $f_{x,y}(\beta)$ to be
$$\sup\{\max(\phi(x)(\eta), \phi(y)(\eta)) : \eta < \omega\cdot(\beta+1)\},$$
II wins if and only if $P(f_{x,y}) = 1$.

Suppose first that τ is a winning strategy for player II. Fix for this paragraph a pair (β, γ) in $(\omega\cdot\lambda) \times \kappa$. Let
$$S_{\beta,\gamma} = \bigcap_{\delta\leq\beta}\bigcup_{\eta\leq\gamma} R_{\delta,\eta}.$$
Then $S_{\beta,\gamma}$ is in Δ, since Δ is closed under intersections and unions of length less than κ. Therefore, the set
$$S_{\beta,\gamma} \circ \tau = \{x \circ \tau : x \in S_{\beta,\gamma}\}$$

is in Δ, as Δ is a projective algebra. We have that $S_{\beta,\gamma} \subseteq \bigcap_{\delta \leq \beta} R_\delta$, which implies that $S_{\beta,\gamma} \circ \tau \subseteq R_\beta$. Letting $\theta(\beta,\gamma)$ be

$$\sup\{\phi(y)(\beta) : y \in S_{\beta,\gamma} \circ \tau\},$$

we have by part (3) of Definition 5.2.1 that $\theta(\beta,\gamma) < \kappa$.

Let C_0 be the set of $\eta < \kappa$ such that $\theta(\beta,\gamma) < \eta$ for all $\beta \in \eta \cap (\omega \times \lambda)$ and $\gamma < \eta$. Since κ is regular, C_0 is club in κ. Let C be any cofinal subset of C_0. Suppose now that $F \colon \lambda \to C(\omega)$ is increasing. We want to see that $P(F) = 1$. Applying part 1 of Definition 5.2.1, let x be such that $\phi(x)$ is an increasing function, with range contained in C, such that

$$F(\beta) = \sup\{\phi(x)(\delta) : \delta < \omega \cdot (\beta + 1)\}$$

for all $\beta < \omega \cdot \lambda$. Let $y = x \circ \tau$. Then y is in $\bigcap_{\beta < \omega\cdot\lambda} R_\beta$. It suffices to see that $\phi(y)(\beta) \leq \phi(x)(\beta + 1)$ for all $\beta < \omega \cdot \lambda$, since then $F = f_{x,y}$, so $P(F) = 1$.

Fix $\beta < \omega \cdot \lambda$. We have that x is in $S_{\beta,\phi(x)(\beta)}$, so y is in $S_{\beta,\phi(x)(\beta)} \circ \tau$. Then

$$\phi(y)(\beta) \leq \theta(\beta, \phi(x)(\beta)) < \phi(x)(\beta + 1),$$

since $\phi(x)$ is increasing and $\phi(x)(\beta + 1)$, being in C, is closed under θ.

A winning strategy for player I can be used as a winning strategy for player II in the game where the values of P are reversed. It follows that if there is a winning strategy for player I, then there is homogeneous set for value 0. $\square$

Remarks 4.2.9 and 5.1.7 show that the pointclasses loc-$\underset{\sim}{\Sigma}^2_1(A)$ are nonselfdual boldface pointclass with the prewellordering property, closed under $\forall^{\omega^\omega}$ and unions, and that the corresponding selfdual pointclasses Δ are projective algebras. In addition, Remark 4.2.10 shows that (aside from closure under $\forall^{\omega^\omega}$) the same holds the pointclasses $\underset{\sim}{\Sigma}^2_1(A)$, if the Wadge hierarchy is assumed to be wellfounded. Theorems 4.3.3, 4.3.4 and 4.3.5 together show that the ordinals loc-$\underset{\sim}{\delta}^2_1(A)$ are regular cardinals (and the same holds for the ordinals $\underset{\sim}{\delta}^2_1(A)$, if $\underset{\sim}{\Sigma}^2_1$ is $\forall^{\omega^\omega}$-closed and $\underset{\sim}{\Sigma}^2_1(A)$ has the prewellordering property). Theorem 5.2.3 then gives the versions of Theorems 5.1.5 and 5.1.8 for the weaker notion of strong partition cardinal discussed in Remark 5.1.1.

Theorem 5.2.3 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Let Γ be a nonselfdual boldface pointclass with the prewellordering property, closed under $\forall^{\omega^\omega}$ and unions, such that $\Gamma \cap \check{\Gamma}$ is a projective algebra. Let κ be $\delta(\Gamma)$, and suppose that κ is a regular cardinal. Then $\kappa \to (\kappa)^\kappa$.*

Proof. Let Δ be $\Gamma \cap \check{\Gamma}$. By Theorem 4.3.2, Δ is closed under intersections and unions of length less than $\delta(\Gamma)$. By Theorem 5.2.2, it suffices to see that κ is $\omega \cdot \kappa$-reasonable. We will find a function $\phi \colon \omega^\omega \to \mathcal{P}(\kappa \times \kappa)$ such that Γ and ψ witness this. By Theorems 2.4.9 and 4.2.14 there exist a set $X \subseteq \omega^\omega$ in $\Gamma \setminus \check{\Gamma}$ and a surjective Γ-norm $f \colon X \to \kappa$. By Remark 4.2.4, for each $x \in X$, the set

$$\{y \in X : f(y) \leq f(x)\}$$

is in Δ. Let, for each $x \in X$,

$$[x]_f = \{y \in X : f(y) = f(x)\},$$

$$[<x]_f = \{y \in X : f(y) < f(x)\}$$

and

$$C_x = \omega^\omega \setminus ([x]_f \cup [<x]_f).$$

Since Δ is a projective algebra closed under unions and intersections of length less than κ, each of these sets is in Δ.

Applying Theorem 2.4.9, fix a sequence $\bar{U} = \langle U_n : n < \omega \setminus \{0\}\rangle$ of $\underset{\sim}{\Sigma}^1_1$-universal sets with the s-m-n property. Define $\phi\colon \omega^\omega \to \mathcal{P}(\kappa \times \kappa)$ by setting, for each $z \in \omega^\omega$ and $(\beta, \gamma) \in \kappa \times \kappa$, $(\beta, \gamma) \in \phi(z)$ if and only if there exist x, y in X with

- $f(x) = \beta$,
- $f(y) = \gamma$ and
- $(x, y) \in U_{2,z}([x]_f, C_x)$.
- for all $(v, w) \in U_{2,z}([x]_f, C_x)$, $f(w) = \gamma$.

The Uniform Coding Lemma (Theorem 3.0.3) implies that part (1) of Definition 5.2.1 is satisfied by Γ and ϕ. To see this, given $F\colon \kappa \to \kappa$, let Z_F be

$$\{(x, y) \in X \times X : F(f(x)) = f(y)\}.$$

By the Uniform Coding Lemma, there is a $z \in \omega^\omega$ such that, for all $x \in X$,

1. $U_{2,z}([x]_f, C_x) \subseteq Z_F \cap ([x]_f \times \omega^\omega)$,
2. $U_{2,z}([x]_f, C_x) \neq \emptyset$ if and only if $Z_F \cap ([x]_f \times \omega^\omega) \neq \emptyset$.

It follows from this that $\phi(z) = F$.

To see that part (2) of Definition 5.2.1 is satisfied, fix $(\beta, \gamma) \in \kappa \times \kappa$. By the definition of $R_{\beta,\gamma}$, for each $z \in \omega^\omega$, $z \in R_{\beta,\gamma}$ implies $(\beta, \gamma) \in \phi(z)$. The reverse implication follows from last condition on the definition of ϕ. It suffices then to see that $\{z \in \omega^\omega : (\beta, \gamma) \in \phi(z)\}$ is in Δ. This can be seen by fixing $x_*, y_* \in X$ such that $f(x_*) = \beta$ and $f(y_*) = \gamma$, and applying the fact that Δ is a projective algebra.

For part (3), fix $\beta < \kappa$ and $A \subseteq R_\beta$ with A in $\exists^{\omega^\omega}\Delta$ (which is the same as Δ). Fix $x_* \in X$ with $f(x_*) = \beta$. Let D be the set of $y \in \omega^\omega$ for which there exist $z \in A$ and $x \in [x_*]_f$ with $(x, y) \in U_{2,z}([x_*]_f, C_{x_*})$. Then $D \subseteq X$ and $D \in \Delta$, so, by Lemma 4.2.5, there is a $\gamma_0 < \kappa$ such that $f[D] \subseteq \gamma_0$. Fixing $z \in A$ then, we have that $z \in R_{\beta,\gamma}$ for some $\gamma < \kappa$. By the previous paragraph, $(\beta, \gamma) \in \phi(z)$. If x and y witness this, then $x \in [x_*]_f$ and $y \in D$, which implies that $\gamma < \gamma_0$. □

Given ordinals μ, κ, we define (μ, κ)-*club directedness* to be the statement that if $\langle \mathcal{C}_\alpha : \alpha < \mu \rangle$ is a sequence of nonempty sets of club subsets of κ, then there is a club subset of κ contained in at least one member of each $\mathcal{C}_\alpha$. Given $\mu < \kappa$ and a function $P\colon [\kappa]^\kappa \to \mu$, for each $\alpha < \mu$, let $P_\alpha\colon [\kappa]^\kappa \to 2$ be defined by letting $P_\alpha(f)$ be 0 if $P(f) = \alpha$ and 1 otherwise. Applying Theorem 5.2.2, we get for each $\alpha < \mu$ a nonempty set $\mathcal{C}_\alpha$ of club subsets of κ with the properties of C_0 as stated there, with respect to P_α. If C is a club subset of κ contained in at least one member of each $\mathcal{C}_\alpha$, then P is constant on $[C(\omega)]^\kappa$.

To finish the proofs of Theorems 5.1.5 and 5.1.8 then it suffices to show that established properties of the pointclasses loc-$\undertilde{\Sigma}^2_1(A)$ and $\undertilde{\Sigma}^2_1(A)$ imply (μ, κ)-club directedness, where κ is $\delta(\Gamma)$ for the given pointclass Γ, and $\mu < \kappa$. We do this using the Martin conditions and the Moschovakis Coding Lemma.

Theorem 5.2.4 $(\mathsf{ZF} + \mathsf{AD})$**.** *Let Γ be a nonselfdual boldface pointclass closed under $\forall^{\omega^\omega}$ and unions, and assume that Γ has the prewellordering property. Let κ a regular uncountable cardinal. If Γ witnesses that κ is $\omega \cdot \kappa$-reasonable, then, for all $\mu < \kappa$, (μ, κ)-club directedness holds.*

Proof. Let ϕ, $R_{\beta,\gamma}$ $(\beta < \lambda, \gamma < \kappa)$ and R_β $(\beta < \lambda)$ be as in Definition 5.2.1, with respect to $\omega \cdot \kappa$, κ and Γ. Fix $\mu < \kappa$, and let $\langle \mathcal{C}_\alpha : \alpha < \mu \rangle$ be a sequence of nonempty sets of club subsets of κ. Let X be $\bigcup_{\alpha<\mu} R_{0,\alpha}$, and let $f\colon X \to \mu$ be such that $x \in R_{0,f(x)}$ for all $x \in X$. Then f is onto. Let $<_f$ be the set $\{(x, y) \in X \times X : f(x) < f(y)\}$ is in Δ. By Theorem 4.3.2, $<_f$ is in Δ. Let Z be the set of $(x, y) \in X \times \omega^\omega$ such that $\phi(y)$ is an increasing function whose range is in $\mathcal{C}_{f(x)}$. Let A be as in the conclusion of the Moschovakis Coding Lemma (Theorem 3.0.1), with respect to X, Z, and f. Then A is in $\exists^{\omega^\omega}\Delta$, and so is $A_1 = \{y : \exists x(x, y) \in A\}$. Furthermore, $A_1 \subseteq \bigcap_{\beta<\kappa} R_\beta$. If follows from condition (3) of Definition 5.2.1 that there is a function $g\colon \kappa \to \kappa$ such that

$$A_1 \subseteq \bigcap_{\beta<\kappa} \bigcup_{\gamma<g(\beta)} R_{\beta,\gamma}.$$

Then the closure points of g form a club subset of κ contained in at least one member of each $\mathcal{C}_\alpha$. $\square$

5.2.5 Remark. The version of Martin's theorem presented in this section may be less than optimal. In Remark 2.32 of [6] it is stated that if Γ and ϕ witness that κ is λ-reasonable, then $\Gamma \cap \check{\Gamma}$ is closed under unions of length less than κ, Γ is closed under intersections and countable unions, and $\mathsf{PWO}(\check{\Gamma})$ holds.

It is an open question whether AD implies that there are no strong partition cardinals greater than (or equal to) Θ. Remark 9.2.10 discusses a connection between this question and the question of whether $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ implies that all sets of reals are ∞-Borel.

Chapter 6

Suslin sets and Uniformization

6.1 Suslin sets

Recall that a *tree* T on a set X is a set of finite sequences from X which is closed under initial segments. We write $[T]$ for the set of infinite sequences whose finite initial segments are all in T. A tree is *wellfounded* if $[T] = \emptyset$; otherwise, it is *illfounded*.

If T is a tree on a product of two sets $X \times Y$, and s is a finite sequence of elements of X, then T_s denotes the set of $t \in Y^{<\omega}$ for which $(s,t) \in T$. For each $f \in X^\omega$, we write T_f for $\bigcup_{n\in\omega} T_{f\restriction n}$. The *projection* of T, $\mathrm{p}[T]$, is the set of $f \in x^\omega$ for which T_f is illfounded, i.e., for which there exists a $g \in Y^\omega$ such that $(f,g) \in [T]$ (identifying, for notational simplicity, pairs of sequences with sequences of pairs).

If T is an illfounded tree on a set X, and $\leq$ is a wellordering of X, the *leftmost branch* of T *relative to* $\leq$ ($\mathrm{lb}_{\leq}(T)$) is the unique $f \in X^\omega$ such that, for each $n \in \omega$, $f(n)$ is the $\leq$-least $x \in X$ for which the tree

$$\{\sigma \in X^{<\omega} : (f\restriction n)^\frown\langle x\rangle^\frown\sigma \in T\}$$

is illfounded. When X is an ordinal or a finite product of ordinals we use the ordinal ordering or the corresponding lexicographical ordering for $\leq$ and write $\mathrm{lb}(T)$.

6.1.1 Definition. Given ordinals γ and η, a set $A \subseteq \eta^\omega$ is γ-*Suslin* if there exists a tree $T \subseteq (\eta \times \gamma)^{<\omega}$ such that $A = \mathrm{p}[T]$. A set $A \subseteq \eta^\omega$ is *Suslin* if it is γ-Suslin for some ordinal γ.

6.1.2 Remark. Every Suslin subset of ω^ω is γ-Suslin for some $\gamma < \Theta$. To see this, suppose that α is an ordinal and T is tree on $\omega \times \alpha$, and observe that the set $\{\mathrm{lb}(T_x)(n) : x \in \mathrm{p}[T], n \in \omega\}$ is a surjective image of ω^ω, and therefore has ordertype less than Θ. Alternately one can use the fact that (assuming only ZF) if β is an ordinal with $T \in L_\beta(T,\mathbb{R})$, then $L_\beta(T,\mathbb{R})$ has an elementary submodel which contains ω^ω and is a surjective image of ω^ω.

Given an ordinal γ, we write $\mathcal{S}_\gamma$ for the smallest boldface pointclass containing each γ-Suslin subsets of ω^ω and $\mathcal{S}_{<\gamma}$ for $\bigcup_{\alpha<\gamma}\mathcal{S}_\alpha$. Then $\mathcal{S}_\gamma$ and $\mathcal{S}_{<\gamma}$ are boldface pointclasses closed under $\exists^{\omega^\omega}$, intersections and unions.

6.1.3 Remark. If γ is less than Θ, and AD holds, then, by the Coding Lemma, $\mathsf{CC}_{\mathcal{P}(\gamma)}$ holds, so $\mathcal{S}_\gamma$ is closed under countable intersections and countable unions.

The following definition is due to Kechris.

6.1.4 Definition. A cardinal κ is *Suslin* if it is infinite and there exists $A \subseteq \omega^\omega$ which is κ-Suslin but not γ-Suslin for any $\gamma < \kappa$.

In symbols, κ is Suslin if and only if $\mathcal{S}_\kappa \setminus \mathcal{S}_{<\kappa} \neq \emptyset$. By Remark 6.1.2, every Suslin cardinal is less than Θ. By Remark 6.1.3, AD implies that a limit of Suslin cardinals of countable cofinality is a Suslin cardinal if it is below Θ (more is true, as well shall see in Section 11.4).

6.1.5 Remark. If $\mathsf{Lipschitz\ Determinacy}$ holds, and $\kappa < \lambda$ are Suslin cardinals, then, by Theorem 2.1.4, $\check{\mathcal{S}}_\kappa \subseteq \mathcal{S}_\lambda$. It follows (under the same assumption) that if κ is a limit of Suslin cardinals, then $\mathcal{S}_{<\kappa}$ is a projective algebra. If AD holds, λ is a Suslin cardinal and Δ is a projective algebra contained in S_λ, then $\mathcal{S}_\lambda$ contains $\bigcup_\omega \Delta$ (see Remark 6.1.3). If in addition $\mathsf{DC}_\mathbb{R}$ holds and Δ is not closed under countable unions then, by Theorem 4.4.4, every wellordered union of sets in Δ is in $\mathcal{S}_\lambda$.

6.1.6 Remark. Moschovakis's Second Periodicity Theorem (see Section 6C of [32]) implies that under $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$, the pointclass of Suslin sets is closed under $\forall^{\omega^\omega}$ and $\exists^{\omega^\omega}$. If κ is the largest Suslin cardinal (which can happen), it follows that $\mathcal{S}_\kappa$ is closed under $\forall^{\omega^\omega}$ and $\exists^{\omega^\omega}$.

6.1.7 Remark. By the Moschovakis Coding Lemma, if AD holds then for each $\kappa < \Theta$ is there is a subset of ω^ω which is not κ-Suslin. It follows that if every subset of ω^ω is Suslin, then the Suslin cardinals are cofinal in Θ. The converse is shown in Corollary 6.1.18 below.

The proof of the following theorem uses the Coding Lemma to code trees projecting to sets of reals. The theorem will be used in the proof of Theorem 6.1.11 below.

Theorem 6.1.8 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Let Γ be a boldface pointclass closed under $\exists^{\omega^\omega}$, and let κ be an infinite cardinal such that there exists a prewellordering $\preceq$ of ω^ω of length κ such that both $\preceq$ and its corresponding strict prewellordering are in Γ. Then $\mathcal{S}_\kappa \subseteq \Gamma$.*

Proof. Fix $A \in \mathcal{S}_\kappa \cap \mathcal{P}(\omega^\omega)$. Let $T \subseteq (\omega \times \kappa)^{<\omega}$ be a tree of cardinality κ projecting to A. Let $\preceq$ be as in the statement of the theorem, let $\prec$ be the strict part of $\preceq$ and let X be the domain of $\preceq$. Let $r\colon X \to \kappa$ be the function which sends each member of X to its $\preceq$-rank, and let $b\colon \kappa \to T$ be a bijection. Let $\pi\colon \omega^\omega \to (\omega^{<\omega} \times (\omega^\omega)^{<\omega})$ be a recursive bijection. Let Z be the set of

$(x, y) \in X \times \omega^\omega$ such that $\pi(y)$ a tuple $(s, z_0, \ldots, z_{|s|-1})$ such that s is the first coordinate of $b(r(x))$ and for all $k < |s|$, $b(r(z_k))$ is the initial segment of $b(r(x))$ of length k.

By the Coding Lemma, there is a set $R \subseteq Z$ in Γ such that for all x, if there exists a y with $(x, y) \in Z$, then there exist $(x', y') \in R$ with $r(x) = r(x')$. Then A is the set of $w \in \omega^\omega$ for which there exist $\langle q_i : i \in \omega \rangle \in (\omega^\omega)^\omega$ such that for all $n \in \omega$ there exist $(x, y) \in R$ such that, letting $\pi(y) = (s, z_0, \ldots, z_{|s|})$, $s = w \restriction n$ and, for each $m < n$, $r(z_m) = r(q_m)$. This shows that A is in Γ. $\square$

6.1.9 Remark. Remarks 6.1.5 and 6.1.6 and Theorem 6.1.8 give the two following facts under AD.

- If κ is a limit of Suslin cardinals, then all prewellorderings in $\mathcal{S}_{<\kappa}$ have length less than κ.
- If $\mathsf{DC}_{\mathbb{R}}$ holds and κ is the largest Suslin cardinal, then all prewellorderings in $\mathcal{S}_\kappa \cap \check{\mathcal{S}}_\kappa$ have length less than κ.

6.1.10 Definition. Let γ be an ordinal and let T be a tree on $\omega \times \gamma$. The *minimization* of T is the tree

$$\{(x \restriction n, \mathrm{lb}(T_x) \restriction n) : x \in \mathrm{p}[T], n \in \omega\}.$$

We say that T is *minimal* if it is equal to its minimization.

We note that the minimization of a tree T as in Definition 6.1.10 is minimal and has the same projection as T.

Theorem 6.1.11 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$; Kechris [11])**.** *If κ is a Suslin cardinal, then $\mathcal{S}_\kappa$ is nonselfdual.*

Proof. Assume toward a contradiction that $\mathcal{S}_\kappa$ is closed under complements. Since $\mathcal{S}_\kappa$ is (always) $\exists^{\omega^\omega}$-closed, it follows from this assumption that it is also $\forall^{\omega^\omega}$-closed. Since $\mathcal{S}_\kappa$ is closed under countable unions, it follows from Corollary 4.4.6 that every member of $\mathcal{S}_\kappa$ is an element of a selfdual pointclass contained in $\mathcal{S}_\kappa$ and closed under wellordered unions.

Let A be a subset of ω^ω in $\mathcal{S}_\kappa \setminus \mathcal{S}_{<\kappa}$. Let Γ be a selfdual pointclass with $A \in \Gamma$ such that Γ is closed under wellorderd unions. Then every element of $\mathcal{S}_{<\kappa}$ is in Γ. If the cofinality of κ were uncountable, it would follow that $\mathcal{S}_\kappa = \Gamma$, giving a contradiction. To see this, let T be a tree on $\omega \times \kappa$, and for each $\alpha < \kappa$, let $T \restriction \alpha$ denote $T \cap (\omega \times \alpha)^{<\omega}$. If κ has uncountable cofinality, then $\mathrm{p}[T] = \bigcup_{\alpha<\kappa} \mathrm{p}[T \restriction \alpha]$, and the latter is a wellordered union of sets in $\mathcal{S}_{<\kappa}$.

To complete the proof, we show that κ has uncountable cofinality. Let δ be the supremum of the lengths of the prewellorderings in $\mathcal{S}_\kappa$. Since $\mathcal{S}_\kappa$ is a projective algebra, $o(\mathcal{S}_\kappa) \leq \delta$. Since $\mathcal{S}_\kappa$ is closed under countable unions, δ has uncountable cofinality. We show that $\kappa = \delta$. By Theorem 6.1.8, there is no prewellordering of ω^ω in $\mathcal{S}_\kappa$ of length κ, so $\delta \leq \kappa$. To show that $\kappa \leq \delta$, let T be a minimal tree on $\omega \times \kappa$ projecting to A. Then T has cardinality κ. For each $\sigma \in T$, let A_σ be the set of $x \in A$ for which σ is an initial segment of $(x, \mathrm{lb}(T_x))$.

By the minimality of T, each A_σ is nonempty, and for distinct σ and τ in T of the same length, A_σ and A_τ are disjoint. For each $\rho < o(\mathcal{S}_\kappa)$, let $T\restriction\rho$ be the set of $\sigma \in T$ for which the Wadge rank of A_σ is less than ρ. For each $n \in \omega$, let $<_{n,\rho}$ be the lexicographical order on the members of $T\restriction\rho$ of length n. We claim that each $<_{n,\rho}$ has ordertype less than δ. The claim completes the proof of the theorem, since it implies that T is a union of $|o(\mathcal{S}_\kappa)|$ many sets of cardinality at most $|\delta|$. To prove the claim, fix $n \in \omega$ and $\rho < o(\mathcal{S}_\kappa)$. The union of all the sets $A_\tau \times A_\sigma$ for $\tau <_{n,\rho} \sigma$ is a wellordered union of sets of Wadge rank bounded below $o(\mathcal{S}_\kappa)$, and is therefore in $\mathcal{S}_\kappa$. This is a prewellordering with the same length as $<_{n,\rho}$, which must therefore be less than δ. $\square$

Lemma 6.1.14 below shows that if λ is a Suslin cardinal, then there exists a strictly $\subseteq$-increasing λ-sequence of Suslin sets, induced by a single tree on $\omega \times \lambda$.

Let $<^2_\ell$ be the following version of the lexicographical order on finite sequences of pairs of ordinals : $(s_0, t_0) <^2_\ell (s_1, t_1)$ if and only if there exists an $i \in \omega$ such that the following hold:

- $(s_0\restriction i, t_0\restriction i) = (s_1\restriction i, t_1\restriction i)$;
- $s_0(i) < s_1(i) \vee (s_0(i) = s_1(i) \wedge t_0(i) < t_1(i))$.

If A is a set of pairwise incompatible sequences of pairs (i.e., such that no sequence in the set extends any other one), and $n \in \omega$ is such that each member of A has length at most n, then the restriction of $<^2_\ell$ to A wellorders A.

6.1.12 Definition. Let $\eta \leq \gamma$ be ordinals, and let T be a tree on $\omega \times \gamma$. We say that T is *η-full* if

$$\{\langle(0, \alpha)\rangle : \alpha < \eta\} \subseteq T.$$

Lemma 6.1.13. *Let γ and η be ordinals, let T be a minimal tree on $\omega \times \gamma$ such that every member of* $\mathrm{p}[T]$ *takes value* 0 *at* 0*, and let $A \subseteq T$ be an antichain such that*

- *every element of $[T]$ intersects A;*
- *the restriction of $<^2_\ell$ to A has ordertype η.*

Then there is a minimal η-full tree T' on $\omega \times \max\{\gamma, \eta\}$, definable from A and T, such that $\mathrm{p}[T] = \mathrm{p}[T']$.

Proof. Enumerate A in $<^2_\ell$-order as $\langle(s_\alpha, t_\alpha) : \alpha < \eta\rangle$. For each $\alpha < \eta$, let

- T_α be the set of $(s, t) \in T$ which are compatible with (s_α, t_α);
- $f_\alpha\colon \gamma^{<\omega} \to \max\{\gamma, \eta\}^{<\omega}$ be defined by setting
  - $|f_\alpha(t)| = |t|$;
  - for all $i \in |t| \cap |t_\alpha|$, $f_\alpha(t)(i) = \alpha$;
  - for all $i \in |t| \setminus |t_\alpha|$, $f_\alpha(t)(i) = t(i)$;

- T'_α be $\{(s, f_\alpha(t)) : (s,t) \in T_\alpha\}$.

Let $T' = \bigcup_{\alpha<\eta} T'_\alpha$. Then T' is η-full, and since $\mathrm{p}[T_\alpha] = \mathrm{p}[T'_\alpha]$ for all $\alpha < \eta$, and every element of $[T]$ intersects A, $\mathrm{p}[T'] = \mathrm{p}[T]$. To see that T' is minimal, fix $(s', t') \in T'$ and $\alpha < \eta$ such that $(s', t') \in T'_\alpha$. It suffices to consider the case where $|s'| \geq |s_\alpha|$. Let $(s,t) \in T_\alpha$ be such that $s' = s$ and $t' = f_\alpha(t)$. Since T is minimal, there exists an $x \in \mathrm{p}[T_\alpha]$ such that (s,t) is an initial segment of $(x, \mathrm{lb}(T_x))$. Applying the definition of $<^2_\ell$ we have that for all $\beta < \alpha$, $x \notin \mathrm{p}[T_\beta]$, so $x \notin \mathrm{p}[T'_\beta]$. Then $\mathrm{lb}(T'_x) = \mathrm{lb}(T'_{\alpha,x})$, and t' is an initial segment of $\mathrm{lb}(T'_x)$ as desired. ☐

Lemma 6.1.14. *If λ is a Suslin cardinal, there is a minimal λ-full tree on $\omega \times \lambda$.*

Proof. Let T_0 be a tree on $\omega \times \lambda$ projecting to a subset of ω^ω which is not γ-Suslin for any $\gamma < \lambda$, and such that each member of $\mathrm{p}[T_0]$ takes value 0 at 0. Let T_1 be the minimization of T_0. Since T_0 witnesses that λ is Suslin, $|T_1| = \lambda$.

$$\{(x\restriction n, \mathrm{lb}(T_{0,x})\restriction n) : x \in \mathrm{p}[T_0], n \in \omega\}.$$

Then $|T_1| = \lambda$ and T_1 is minimal. If some level of T_1 has $<^2_\ell$-ordertype at least λ, let A be the length-λ $<^2_\ell$-initial segment of such a level, and let T'_1 be the set of nodes of T_1 compatible with a member of A. Then T'_1 is minimal, and the lemma follows by applying Lemma 6.1.13 with T'_1 and A.

Suppose then that T_1 does not contain such an antichain. For each $n \in \omega$, let κ_n be the cardinality of the set of nodes of T_1 of length n. Then each κ_n is less than λ, and $\sup_{n\in\omega} \kappa_n = \lambda$. We can then complete the proof by using (for each $n \in \omega$) the nth level to produce a minimal κ_n-full tree on $\omega \times \kappa_n$, modifying the trees to make their domains disjoint, and combining these trees to make a minimal λ-full tree on $\omega \times \lambda$. We give the details below.

For each $n \in \omega$, let A_n be the length-κ_n $<^2_\ell$-initial segment of the nth level of T_1, and let $T_{1,n}$ be the set of nodes of T_1 compatible with a member of A_n. Then $T_{1,n}$ is minimal; let $T_{2,n}$ be the result of applying Lemma 6.1.13 to $T_{1,n}$ and A_n.

We now modify the trees $T_{2,n}$, achieving minimality (via the function $\pi^*_{0,n}$ defined below) by making the members of their left coordinates disjoint (aside from 0) and fullness (via $\pi^*_{1,n}$). Let $\pi\colon \omega \times \omega \to \omega$ be an injection, and for each $n \in \omega$ let $\pi^*_{0,n}$ be the length-preserving function on $\omega^{<\omega}$ defined by setting $\pi^*_{0,n}(s)(0)$ to be $s(0)$ (in the case where s is nonempty) and $\pi^*_{0,n}(s)(i) = \pi(n,i)$ for all nonzero $i \in \mathrm{dom}(s)$. Again for each $n \in \omega$, let $\gamma_n = \sum_{m<n} \kappa_m$ (using ordinal addition) and let $\pi^*_{1,n}$ be the length-preserving function on $\gamma^{<\omega}$ defined by setting $\pi^*_{1,n}(t)(0)$ to be $\gamma_n + t(0)$ (in the case where s is nonempty) and $\pi^*_{1,n}(t)(i) = t(i)$ for all nonzero $i \in \mathrm{dom}(t)$. Finally, let T_2 be

$$\{(\pi^*_{0,n}(s), \pi^*_{1,n}(t)) : n \in \omega, (s,t) \in T_{2,n}\}.$$

Then T_2 is as desired. ☐

Corollary 6.1.15. *Let λ is a Suslin cardinal and let Δ be the smallest pointclass containing $\mathcal{S}_\lambda \cup \check{\mathcal{S}}_\lambda$. Then there is a prewellordering of ω^ω of length λ in $\bigcup_\lambda \Delta$.*

Proof. Let T be a minimal λ-full tree on $\omega \times \lambda$. Define $\leq$ on ω^ω by setting $x \leq y$ to hold if either $x \notin \mathrm{p}[T]$ or $x, y \in \mathrm{p}[T]$ and $\mathrm{lb}(T_x)(0) \leq \mathrm{lb}(T_y)(0)$. Then $\leq$ is a prewellordering of length λ. $\square$

Every $\underset{\sim}{\Sigma}^1_1$ prewellordering of ω^ω has countable length (see [32, 20], for instance). Applying this fact in a forcing extension by $\mathrm{Col}(\omega, \kappa)$, we get the following.

Theorem 6.1.16 (Kunen-Martin)**.** *If κ is an infinite cardinal and $\leq$ is a κ-Suslin prewellordering of ω^ω, then $\leq$ has length less than κ^+.*

Theorem 6.1.17 (ZF+AD)**.** *If κ is a limit of Suslin cardinals, then $\delta(\mathcal{S}_{<\kappa}) = \kappa$.*

Proof. That $\delta(\mathcal{S}_{<\kappa}) \leq \kappa$ follows from Theorem 6.1.16 or Remark 6.1.9. The reverse inequality follows from Corollaries 4.4.6 and 6.1.15, noting that $\mathcal{S}_{<\kappa}$ is a projective algebra, by Remark 6.1.5. $\square$

Corollary 6.1.18 (ZF + AD)**.** *The Suslin cardinals are cofinal in Θ if and only if every subset of ω^ω is Suslin.*

Proof. For the forward direction, note that otherwise, by Theorem 6.1.17, there is a surjection from ω^ω onto Θ. The reverse direction is shown in Remark 6.1.7. $\square$

Given a cardinal κ, we say that a subset of topological space X is *weakly κ-Borel* if it is in the smallest collection of subsets of X containing the closed sets and closed under complements and unions and intersections of cardinality at most κ. For instance, the weakly $\aleph_0$-Borel sets are just the Borel sets in the usual sense. A set is *weakly $<\kappa$-Borel* if it is in the smallest collection of subsets of X containing the closed sets and closed under complements and unions and intersections of cardinality less than κ. A set is *weakly ∞-Borel* if it is weakly κ-Borel for some cardinal κ.

Some authors call these sets "∞-Borel" and use "effectively ∞-Borel" for the notion of ∞-Borel defined in Chapter 9. Example 9.1.5 shows that the two notions are not in general the same.

6.1.19 Remark. Suppose that $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$ holds, and that Δ is a projective algebra which is not closed under countable unions. By Theorem 4.4.4, $\bigcup_\omega \Delta$ has the prewellordering property. By the First Periodicity Theorem (see Remark 4.2.7), $\forall^{\omega^\omega} \bigcup_\omega \Delta$ does as well. Let $\Gamma = \forall^{\omega^\omega} \bigcup_\omega \Delta$ and let $\Lambda = \Gamma \cap \check{\Gamma}$. By Theorem 4.3.2, every weakly $<\delta(\Gamma)$-Borel subset of ω^ω is in Λ.

The converse of the following theorem is also true (see Theorem 9.1.15).

Theorem 6.1.20 (ZF+AD+DC$_\mathbb{R}$)**.** *Suppose that κ is a limit of Suslin cardinals and that κ has uncountable cofinality. Then every subset of ω^ω which is weakly $<\kappa$-Borel is also $<\kappa$-Suslin.*

Proof. We make use of Remark 6.1.3 and the comments immediately before. Let $\gamma < \kappa$ be a limit of Suslin cardinals of countable cofinality. By Theorem 6.1.17, $\delta(\mathcal{S}_{<\gamma}) = \gamma$. By Remark 6.1.5, $\mathcal{S}_{<\gamma}$ is a projective algebra which is not closed under countable unions. Letting $\Gamma = \forall^{\omega^\omega} \bigcup_\omega \mathcal{S}_{<\gamma}$ and $\Lambda = \Gamma \cap \check{\Gamma}$, we have by Remark 6.1.19 that every $<\gamma$-Borel set is in Λ. Since $\mathcal{S}_\gamma$ is closed under countable unions, it contains $\bigcup_\omega S_{<\gamma}$. If $\lambda > \gamma$ is a Suslin cardinal greater than γ, then $\mathcal{S}_\lambda$ is $\exists^{\omega^\omega}$-closed and contains the complement of every set in $\mathcal{S}_\gamma$, so $\Lambda \subseteq \check{\Gamma} \subseteq \mathcal{S}_\lambda$. $\square$

The following theorem will be used to show that if AD holds then AD^+ holds in $L(\mathbb{R})$ (see Remark 7.0.5 and Corollary 9.2.6).

Theorem 6.1.21 (Martin-Steel [30])**.** *Assuming* $\mathsf{AD} + V{=}L(\mathbb{R})$*, the Suslin sets are exactly the* $\underset{\sim}{\Sigma}^2_1$ *sets.*

6.2 Uniformization

Given sets X, Y and $A \subseteq X \times Y$, we say that a function $f \colon X \to Y$ *uniformizes* A if $\mathrm{dom}(f) = \{x \in X \mid A_x \neq \emptyset\}$, and $(x, f(x)) \in A$ for all $x \in \mathrm{dom}(f)$. We say then that A is *uniformized.* If $A \subseteq \alpha^\omega \times \beta^\omega$ is the projection of a tree T on $\alpha \times \beta \times \gamma$, for some ordinals α, β and γ, then a uniformizing function for A can be constructed from T by considering T as a tree on $\alpha \times (\beta \times \gamma)$ and using the function $x \mapsto \mathrm{lb}(T_x)$ for $x \in \mathrm{p}[T]$. This gives the following fact.

Theorem 6.2.1. *Every Suslin subset of* $(\omega^\omega)^2$ *is uniformized.*

We let Uniformization be the statement that every subset of $(\omega^\omega)^2$ has a uniformizing function.

6.2.2 Remark. Using a coding a finite sequences of reals by individual reals, it is easy to see that Uniformization implies $\mathsf{DC}_\mathbb{R}$. Similarly, $\mathsf{AD}_\mathbb{R}$ implies Uniformization, via the one-round game where player I plays an $x \in \omega^\omega$, and player II wins by playing a $y \in \omega^\omega$ such that the pair (x, y) is in the given payoff set.

6.2.3 Definition. Given a set X, a set $A \subseteq X^\omega$ is *quasi-determined* (as a subset of X^ω) if there is a function $\pi \colon X^{<\omega} \to \mathcal{P}(X) \setminus \{\emptyset\}$ such that one of the two following statements holds.

1. For every $x \in X^\omega$, if $x(2n) \in \pi(x{\restriction}2n)$ holds for all $n \in \omega$, then $x \in A$.
2. For every $x \in X^\omega$, if $x(2n+1) \in \pi(x{\restriction}(2n+1))$ hold for all $n \in \omega$, then $x \notin A$.

A function $\pi \colon X^{<\omega} \to \mathcal{P}(X) \setminus \{0\}$ is then said to be a *quasi-strategy* in the game G_A. Such a function π as in case (1) above is a *winning quasi-strategy* for player I; in case (2) it is a winning quasi-strategy for player II. We let quasi-AD_X denote the statement that every subset of X^ω is quasi-determined, as a subset of X^ω.

If X is wellorderable, then quasi-AD_X and AD_X are equivalent. Remark 6.2.4 shows that a form of uniformization suffices to prove this.

6.2.4 Remark. If X is a set, and every subset of $X^{<\omega} \times X$ is uniformized, then every quasi-determined game on X is determined. As quasi-$\mathsf{AD}_{\mathbb{R}}$ implies Uniformization (via a game where I plays a real, and II plays the infinite string of coordinates of a partner real), quasi-$\mathsf{AD}_{\mathbb{R}}$ and $\mathsf{AD}_{\mathbb{R}}$ are equivalent.

Given a set X, we let OD_X denote the class of sets which are definable from an ordinals and finite sequence from X (i.e., which are ordinal definable from a finite sequence from X; see page 194 of [8] or page 145 of [24]). If there is a wellordering of X in OD_X, then there is a class-length wellordering of OD_X which is ordinal definable from a finite sequence from X.

The following theorem is the fundamental tool for showing the failure of Uniformization in certain models of AD. In particular, it can be used to show that, in $L(\mathbb{R})$, there are $\utilde{\Pi}^2_1$ sets which cannot be uniformized, and are therefore not Suslin.

Theorem 6.2.5 (Kechris-Solovay)**.** *Suppose that there is a set Z such that every subset of ω^ω is ordinal definable from Z and a member of ω^ω, and that for all $x \in \omega^\omega$, $\omega^\omega \not\subseteq \mathrm{OD}_{\{Z,x\}}$. Then the set $\{(x,y) \in (\omega^\omega)^2 \mid y \notin \mathrm{OD}_{\{Z,x\}}\}$ is not uniformizable.*

Proof. Any uniformizing function would be ordinal definable from Z and some $x \in \omega^\omega$, which would mean that $f(x) \in \mathrm{OD}_{\{Z,x\}}$. $\square$

Theorem 6.2.6. *Assume that AD holds and let Δ be a boldface selfdual pointclass closed under $\exists^{\omega^\omega}$, and let κ be a cardinal less than $\mathrm{cof}(o(\Delta))$. Then $\mathcal{S}_\kappa \subseteq \Delta$.*

Proof. Let A be a κ-Suslin subset of ω^ω. We may assume that A is not λ-Suslin for any $\lambda < \kappa$. By Proposition 2.5.8, we may fix a prewellordering $\preceq$ in Δ of length κ on a set X. Using a bijection between $\kappa \times \omega$ and ω if necessary, we may fix a tree $T \subseteq (\omega \times \kappa)^{<\omega}$ such that $A = p[T]$ and such that there is a function $u\colon \kappa \to T$ such that for each $\alpha \in \kappa$, $u(\alpha)$ is the unique node (s,t) of T for which α is the last value taken by t. Let $f\colon X \to \kappa$ be the function which sends each member of X to its $\preceq$-rank. Let $\pi\colon \omega^\omega \to (\omega^{<\omega} \times (\omega^\omega)^{<\omega})$ be a recursive bijection. Let Z be the set of $(x,y) \in X \times \omega^\omega$ such that $\pi(y)$ is a tuple $(s, z_0, \ldots, z_{|s|-1})$ for some nonempty $s \in \omega^{<\omega}$, such that

- s is the first coordinate of $u(f(x)))$,
- for all positive $n < |s|$, $s \restriction n$ is the first coordinate of $u(f(z_{n-1})))$.

By the Coding Lemma, there is a set $R \subseteq Z$ such that for all x, if there exists a y with $(x,y) \in Z$, then there exist $(x',y') \in R$ with $f(x) = f(y)$. Then A is the set of $w \in \omega^\omega$ for which there exist $\langle q_i : i \in \omega\rangle \in (\omega^\omega)^\omega$ such that for all $n \in \omega$ there exist $(x,y) \in R$ such that, letting $\pi(y) = (s, z_0, \ldots, z_{|s|-1})$, $s = w\restriction n$ and, for each $m < n$, $f(z_m) = f(q_m)$. This shows that A is in Δ. $\square$

6.3 The Solovay sequence

The *Solovay sequence* [36] is the unique continuous sequence $\langle\theta_\alpha : \alpha \leq \beta\rangle$ such that

- θ_0 is the least ordinal which is not the surjective image of ω^ω under an OD function;
- for every ordinal α such that $\alpha + 1 \leq \beta$, $\theta_{\alpha+1}$ is the least ordinal which is not the surjective image of ω^ω under an $\mathrm{OD}_{\{A\}}$ function for any $A \subseteq \omega^\omega$ of Wadge rank θ_α;
- $\theta_\beta = \Theta$.

We call β the *length* of the Solovay sequence. In $L(\mathbb{R})$, $\theta_0 = \Theta$, so $\beta = 0$. For each $\alpha \leq \beta$, if we let Γ_α be the set subsets of ω^ω of Wadge rank less than θ_α, then $\mathrm{HOD}_{\Gamma_\alpha}$ is a model of ZF containing ω^ω whose subsets of ω^ω are exactly the members of Γ_α.

6.3.1 Remark. If $\mathsf{AD}_\mathbb{R}$ holds, and the length of the Solovay sequence has uncountable cofinality, then $L(\Gamma_\alpha) \models \mathsf{AD}_\mathbb{R}$ for club many α on the Solovay sequence. To see this, suppose that $\mathsf{AD}_\mathbb{R}$ holds, and consider any family $\{A_x : x \in \omega^\omega\}$ of subsets of $(\omega^\omega)^\omega$. Consider the real game in which player I plays x and the two players play the real game with payoff set A_x, with player II choosing first which player to be in the game with payoff set A_x. Player I cannot have a winning strategy. This shows for instance that, under $\mathsf{AD}_\mathbb{R}$, for each $A \subseteq \omega^\omega$ there is a $B \subseteq \omega^\omega$ such that every real game with payoff set Wadge below A has a winning strategy with payoff set below B.

Theorems 2.5.9 and 6.2.5 give the following. Recall from Remark 2.1.8 that $\mathsf{Lipschitz\ Determinacy}$ implies $\aleph_1 \not\leq 2^{\aleph_0}$.

Theorem 6.3.2. *If* $\mathsf{Lipschitz\ Determinacy}$ *and* $\mathsf{Uniformization}$ *hold, and the Wadge rank of each subset of ω^ω exists, then the length of the Solovay sequence is a limit ordinal.*

Proof. Suppose that the length of the Solovay sequence is not a limit ordinal. Then there is a set $A \subseteq \omega^\omega$ such that every ordinal below Θ is a surjective image of ω^ω via a function which is ordinal definable from A. By Theorem 2.5.9, for each $\gamma < \Theta$ there is a γ-sequence of subsets of ω^ω of increasing Wadge rank, ordinal definable from A. Since every set of reals has Wadge rank less than Θ, it follows from $\mathsf{Lipschitz\ Determinacy}$ that every set of reals is definable from A, a real and an ordinal. On the other hand, since $\mathsf{Uniformization}$ and $\aleph_1 \not\leq 2^{\aleph_0}$ hold (see Remark 2.1.8), Theorem 6.2.5 implies that there is a set $B \subseteq \omega^\omega$ which is not ordinal definable from A and a real. $\square$

6.3.3 Remark. While it is possible for nonlimit elements of the Solovay sequence to be singular (and in fact, under AD^+, nonlimit members of the Solovay sequence below Θ have cofinality ω by Theorem 13.1.6) if AD holds and the length of the Solovay sequence is not a limit ordinal, then Θ is regular, since in this case there is an $A \subset \omega^\omega$ such that every element of $\mathcal{P}(\omega^\omega)$ is definable from A, an element of ω^ω and an ordinal below Θ (for instance, via an elementary submodel argument as in Remark 6.1.2).

Solovay [36] showed that, assuming $\mathsf{AD}_{\mathbb{R}}$ + V=$L(\mathcal{P}(\omega^\omega))$, DC holds if and only if the length of the Solovay sequence has uncountable cofinality. For the easier direction, note that $\mathsf{Lipschitz\ Determinacy}$ + $\mathsf{DC}_{\mathbb{R}}$ + $\mathsf{CC}_{\mathcal{P}(\omega^\omega)}$ implies that $\mathrm{cof}(\Theta)$ is uncountable, since otherwise there exists an ω-sequence of sets of reals whose Wadge ranks are unbounded in Θ, and thereby a set of reals of Wadge rank at least Θ, contracting Corollary 2.5.11. The following theorem gives the harder direction. Part of the proof is reused in the proof of Theorem 8.0.2, which shows that $\mathsf{DC}_{\mathbb{R}}$ follows from the assumption that the ultrapower of the ordinals by the Turing measure is wellfounded.

A ranking function on a tree T is a function $\rho\colon T \to \mathrm{Ord}$ such that $\rho(s) > \rho(t)$ whenever t is an extension of S. The existence of a ranking function for T implies that T is wellfounded; the reverse implication follows from DC_T. The canonical ranking function rank_T of a wellfounded tree T sends each node t of T to the least ordinal greater than all the values $\mathrm{rank}_T(s)$ for s a proper extension of t in T.

Theorem 6.3.4 (Solovay [36])**.** *If* AD + $\mathsf{DC}_{\mathbb{R}}$ *holds,* V=$\mathrm{HOD}_{\mathcal{P}(\omega^\omega)}$ *and* $\mathrm{cof}(\Theta)$ *is uncountable, then* DC *holds.*

Proof. By Remark 0.4.2, it suffices to show that $\mathsf{DC}_{\mathcal{P}(\omega^\omega)}$ holds. Let T be a tree on $\mathcal{P}(\omega^\omega)$ without terminal nodes. We want to see that T has an infinite path. For each $\xi < \Theta$, let T_ξ be the set of $t \in T$ such that every element of the range of t has Wadge rank less than ξ. Then T_ξ is a surjective image of ω^ω, and, since $\mathsf{DC}_{\mathbb{R}}$ holds, either T_ξ has an infinite path or T_ξ has a ranking function. We assume that the latter case holds for all ξ, since otherwise we are done.

For each $t \in T$, let $w(t)$ be the least Wadge rank of a set $A \subseteq \omega^\omega$ such that $t^\frown\langle A\rangle$ is in T. Suppose that for each $\xi < \Theta$, $w[T_\xi]$ is bounded in Θ. Define $s\colon \Theta \to \Theta$ by setting $s(\xi)$ to be the supremum of $w[T_\xi]$. Then since $\mathrm{cof}(\Theta) > \aleph_0$ there is a $\zeta < \Theta$ closed under w. Then T_ζ is a subtree of T without terminal nodes, contradicting the asumption from the previous paragraph.

We continue the proof under the assumption that for some $\xi_* < \Theta$, $w[T_{\xi_*}]$ is cofinal in Θ. Since T_{ξ_*} is a surjective image of ω^ω, the ordertype of $w[T_{\xi_*}]$ must be less than Θ. It follows that Θ is singular. Let λ be its cofinality and let $c\colon \lambda \to \Theta$ be increasing, continuous and cofinal.

Toward a contradiction, suppose that T does not contain an infinite path. Then each T_ξ is also wellfounded. Let $s\colon \omega^\omega \to \lambda$ be a surjection. For each $x \in \omega^\omega$, let $s'(x) = \sup\{s(y) : y \in \omega^\omega,\ y \leq_{\mathrm{Tu}} x\}$. Then s' is Turing invariant (i.e., Turing-equivalent inputs give the same output). Since $\lambda > \aleph_0$, $s'(x) < \lambda$ for each $x \in \omega^\omega$. For each $t \in T$, let r_t be the function on ω^ω defined by setting $r_t(x)$ to be 0 if some member of t has Wadge rank at least $s'(x)$, and the rank of $\mathrm{rank}_{T_{s'(x)}}(t)$ otherwise. Then each r_t is Turing invariant, and $r_t(x) < r_{t'}(x)$

for a Turing cone of x whenever $t < t'$ in T. Let $R = \{r_t : t \in T\}$ and for each $\alpha < \lambda$ let $R_\alpha = \{r_t : t \in T_{c(\alpha)}\}$.

Define the strict linear order $\prec$ on R by setting $r_t \prec r_{t'}$ to hold when $r_t < r_{t'}$ on a Turing code. Since every element of T has a proper extension in T, $\prec$ has no smallest element. Since $\mathsf{DC}_\mathbb{R}$ holds and each T_ξ is wellfounded, the restriction of $\prec$ to each set R_α is wellfounded, and in particular does have a smallest element. Let $\alpha_0 = 0$, and for each $n \in \omega$ let α_{n+1} be the least $\alpha < \lambda$ such that $R_{\alpha_{n+1}}$ contains a strict $\prec$ lower bound for R_{α_n}. Letting $\xi = \sup\{\alpha_n : n \in \omega\}$, $\xi < \lambda$ and $(R_\xi, \prec)$ is illfounded, giving a contradiction. $\square$

Part II

AD^+

Chapter 7

$<\Theta$-determinacy

Let λ be an ordinal, and let $\pi\colon \lambda^{<\omega} \to \omega^{<\omega}$ be such that $s \subseteq t$ implies $\pi(s) \subseteq \pi(t)$ and $\mathrm{dom}(s) = \mathrm{dom}(\pi(s))$ (we say that a function with this property is *extension-preserving*). Given $A \subseteq \omega^\omega$, let $\mathcal{G}_{\pi,A}$ be the game in which players I and II alternate choosing ordinals $\alpha_i \in \lambda$ ($i \in \omega$), where I wins if and only if $\bigcup_{n\in\omega} \pi(\langle \alpha_0, \ldots, \alpha_n\rangle) \in A$.

$$\begin{array}{c|ccccccc}
\mathrm{I} & \alpha_0 & & \alpha_2 & & \alpha_4 & & \ldots \\
\hline
\mathrm{II} & & \alpha_1 & & \alpha_3 & & \ldots &
\end{array}$$

The game $\mathcal{G}_{\pi,A}$

We let λ-$\mathsf{Determinacy}$ denote the statement that for all $\pi\colon \lambda^{<\omega} \to \omega^{<\omega}$ and $A \subseteq \omega^\omega$ as above, $\mathcal{G}_{\pi,A}$ is determined. Given an ordinal γ, we let $<\gamma$-$\mathsf{Determinacy}$ denote the statement that λ-$\mathsf{Determinacy}$ holds for all $\lambda < \gamma$.

7.0.1 Remark. We are primarily interested in $<\Theta$-$\mathsf{Determinacy}$, which is usually called $\mathsf{Ordinal\ Determinacy}$. There is an alternate definition of ordinal determinacy in which one has an ordinal λ, a continuous function $f\colon \lambda^\omega \to \omega^\omega$ (with respect to the discrete topology on λ and the corresponding product topology on λ^ω) and a payoff set $A \subseteq \omega^\omega$. The corresponding game is a defined as above, except that I wins if and only if $f(\langle \alpha_i : i < \omega\rangle) \in A$. The difference is that the usual definition allows continuous functions, whereas the function mapping runs of the game into ω^ω in our definition is Lipschitz. These two versions of ordinal determinacy are equivalent. To see this, given a continuous $f\colon \lambda^\omega \to \omega$ and an $A \subseteq \omega^\omega$, define $\rho\colon \lambda^{<\omega} \to \omega^{<\omega}$ by letting $\rho(s) = \bigcap\{f(x) : x \in [s]\}$. Add a dummy symbol $*$ to ω, and let μ be the function on (finite or infinite) sequences from $\omega \cup \{*\}$ that removes the $*$ terms. Let $\pi\colon \lambda^{<\omega} \to \omega^{<\omega}$ be such that $\pi(s)$ codes the longest initial segment of $\rho(s)$ of length at most $|s|$ (so $\pi(s^\frown\langle n\rangle)$ is $\pi(s)^\frown\langle k\rangle$ if $\mu(\pi(s)^\frown\langle k\rangle)$ is an initial segment of $\rho(s^\frown\langle n\rangle)$ and $\pi(s)^\frown\langle *\rangle$ if there is no such k). Let A' be the set of $x \in (\omega \cup \{*\})^\omega$ for which $\mu(x) \in A$. Then a winning strategy for either player in $\mathcal{G}_{\pi,A'}$ is a winning strategy for the same player in the game corresponding to f and A.

7.0.2 Remark. Consider the game of length ω in which player I plays $\alpha \in \omega_1$ (and then makes no other moves) and then player II plays (digit by digit) an element x of 2^ω, with II winning if and only if x codes (under some fixed coding) a wellordering of ω of ordertype α. Player I can never have a winning strategy in this game, and if the Axiom of Choice holds, then player II has a winning strategy. By part (2) of Remark 1.1.2, if AD holds then there is no injection from ω_1 into ω^ω, so this game is not determined.

The *Lusin-Sierpiński order* (sometimes called the Brouwer-Kleene order) is the ordering on finite sequences of ordinals defined by setting $s <_{\mathrm{LS}} t$ if either s properly extends t or $s(n) < t(n)$ for the least n such that $s(n) \neq t(n)$ (identifying a finite sequence s with the corresponding function with domain $|s|$). Then $<_{\mathrm{LS}}$ is a (class-sized) strict linear order. Given a tree T consisting of finite sequences of ordinals, $<_{\mathrm{LS}}$ is wellfounded if and only if T is (i.e., if and only if T has no infinite branches). To see this, note that any node of T such that the restriction of $<_{\mathrm{LS}}$ is illfounded must have a property extension with the same property.

As noted in Remark 5.1.1, strong partition cardinals satisfy the conditions on δ in the statement of Theorem 7.0.3

Theorem 7.0.3 ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$; Moschovakis, Woodin)**.** *Suppose that $\kappa < \delta$ are infinite cardinals, and that δ is regular cardinal such that*

$$\forall \mu < \delta(\delta \to (\delta)^\mu_\mu)$$

holds. Let $A \subseteq \omega^\omega$ be such that A and $\omega^\omega \setminus A$ are both κ-Suslin. Let $\lambda < \delta$ be an ordinal, and let $\pi\colon \lambda^{<\omega} \to \omega^{<\omega}$ be extension-preserving. Then $\mathcal{G}_{\pi,A}$ is determined.

Proof. Fix trees T and S on $\omega \times \kappa$ such that $A = \mathrm{p}[T]$ and $\omega^\omega \setminus A = \mathrm{p}[S]$.

For each $u \in \omega^{<\omega}$, recalling that T_u denotes the set $\{s : (u,s) \in T\}$, let

$$T^*_u = \bigcup\{T_{u\restriction n} : n \leq |u|\}$$

and let

$$S^*_u = \bigcup\{S_{u\restriction n} : n \leq |u|\}.$$

For each such u, $<_{\mathrm{LS}} \restriction T^*_u$ and $<_{\mathrm{LS}} \restriction S^*_u$ are wellorderings; let ζ_u and ξ_u denote their respective ordertypes. Then $\max\{\zeta_u, \xi_u\} < \kappa^+ < \delta$. For all $x \in \omega^\omega$, $x \in A$ if and only if the restriction of $<_{\mathrm{LS}}$ to

$$T_x = (\bigcup_{n\in\omega} T_{x\restriction n})$$

is illfounded, and $x \notin A$ if and only if the restriction of $<_{\mathrm{LS}}$ to

$$S_x = (\bigcup_{n\in\omega} S_{x\restriction n})$$

is illfounded.

Let OP^T_u denote the set of functions from $T^*_u$to δ for which, for all $s, t \in T^*_u$, if $s <_{\mathrm{LS}} t$ then $f(s) < f(t)$. Let OP^S_u denote the set of functions from S^*_u to δ for which, for all $s, t \in S^*_u$, if $s <_{\mathrm{LS}} t$ then $f(s) < f(t)$.

We define two games of length ω, $\mathcal{G}^T$ and $\mathcal{G}^S$. In $\mathcal{G}^T$, player I plays ordinals $\alpha_i \in \lambda$ (for $i \in \omega$ even) and player II plays pairs (α_i, f_i) (for $i \in \omega$ odd) such that each α_i is in λ and each f_i is a function in $OP^T_{\pi(\langle\alpha_0,\ldots,\alpha_i\rangle)}$. Player II wins a run $\mathcal{G}^T$ if and only if $f_i \subseteq f_j$ for all $i < j$ in ω.

$$\begin{array}{c|ccccccc}
\mathrm{I} & \alpha_0 & & \alpha_2 & & \alpha_4 & & \ldots \\
\hline
\mathrm{II} & & \alpha_1, f_1 & & \alpha_3, f_3 & & \ldots &
\end{array}$$

The game $\mathcal{G}^T$

In $\mathcal{G}^S$, player II plays ordinals $\alpha_i \in \lambda$ (for $i \in \omega$ odd) and player I plays pairs (α_i, f_i) (for $i \in \omega$ even) such that each α_i is in λ and each f_i is a function in $OP^S_{\pi(\langle\alpha_0,\ldots,\alpha_i\rangle)}$. Player I wins a run of $\mathcal{G}^S$ if and only if $f_i \subseteq f_j$ for all $i < j$ in ω.

$$\begin{array}{c|ccccccc}
\mathrm{I} & \alpha_0, f_0 & & \alpha_2, f_2 & & \alpha_4, f_4 & & \ldots \\
\hline
\mathrm{II} & & \alpha_1 & & \alpha_3 & & \ldots &
\end{array}$$

The game $\mathcal{G}^S$

The games $\mathcal{G}^T$ and $\mathcal{G}^S$ are each closed. Therefore, in $\mathcal{G}^T$, either player I has a winning strategy, or player II has a winning quasi-strategy (not necessarily a strategy, as the set of possible moves for II may not be wellorderable), and in $\mathcal{G}^S$, either player II has a winning strategy, or player I has a winning quasi-strategy. It cannot be, however, that player II has a winning quasi-strategy τ_{II} in $\mathcal{G}^T$ and player I has a winning quasi-strategy τ_I in $\mathcal{G}^S$. Supposing that such quasi-strategies existed, using $\mathsf{DC}_{\mathbb{R}}$, the fact that $\delta < \Theta$, the Moschovakis Coding Lemma (to code subsets of δ by reals), and the fact that each function f_i as above can be coded by a subset of δ, we could find $\langle(\alpha_i, f_i) : i \in \omega\rangle$ such that

$$\langle\alpha_0, (\alpha_1, f_1), \alpha_2, (\alpha_3, f_3), \ldots\rangle$$

is according to τ_{II} and

$$\langle(\alpha_0, f_0), \alpha_1, (\alpha_2, f_2), \alpha_3, \ldots\rangle$$

is according to τ_I. Then $\bigcup_{i\in\omega} f_{2i+1}$ would witness that $\pi(\langle\alpha_i : i\in\omega\rangle)\notin A$, and $\bigcup_{i\in\omega} f_{2i}$ would witness that $\pi(\langle\alpha_i : i\in\omega\rangle)\in A$, giving a contradiction.

The proof is then completed by proving the following claims.

Claim 1. *If I has a winning strategy in $\mathcal{G}^T$, then I has a winning strategy in $\mathcal{G}_{\pi,A}$.*

Claim 2. *If II has a winning strategy in $\mathcal{G}^S$, then II has a winning strategy in $\mathcal{G}_{\pi,A}$.*

Recall that for an ordinal α and a set of ordinals X, $[X]^\alpha$ denotes the collection of subsets of X of ordertype α, and that we identify each element of $[X]^\alpha$ with the corresponding order-preserving function on α which enumerates it. Recalling notation from Chapter 5, given an ordinal-valued function g on $\omega\cdot\alpha$, for some ordinal α, let g^* be the function on α defined by setting $g^*(\xi)$ to be the supremum of $\{g(\omega\cdot\xi+n) : n\in\omega\}$. For each set X of ordinals, let $X^*(\alpha)$ be $\{g^* : g\in[X]^{\omega\cdot\alpha}\}$. Let μ_α be the set of $A\subseteq[\delta]^\alpha$ for which there exists a club $C\subseteq\delta$ with $C^*(\alpha)\subseteq A$. By Theorem 5.1.3, each μ_α is a δ-complete (so λ-complete) measure on the corresponding set $[\delta]^\alpha$.

Proof of Claim 1 : For each $u\in\omega^{<\omega}$, let Q_u be the bijection from $[\delta]^{\zeta_u}$ to OP^T_u with the property that $Q_u(g)$ and g have the same range, for all $g\in[\delta]^{\zeta_u}$. Let ν_u be $\{Q_u[A] : A\in\mu_{\zeta_u}\}$. Then ν_u a δ-complete ultrafilter on OP^T_u.

Fix a winning strategy Σ for I in $\mathcal{G}^T$. We define a strategy σ for I in $\mathcal{G}_{\pi,A}$. We let $\sigma(\langle\rangle)=\Sigma(\langle\rangle)$. For each positive $n\in\omega$, we let $\sigma(\langle\alpha_0,\ldots,\alpha_{2n-1}\rangle)$, be the unique value α such that, letting $u=\pi(\langle\alpha_0,\ldots,\alpha_{2n-1}\rangle)$, for ν_u-many $f\in OP^T_u$,

$$\Sigma(\alpha_0,(\alpha_1,f{\restriction}T^*_{u\restriction 2}),\alpha_2,(\alpha_3,f{\restriction}T^*_{u\restriction 4}),\ldots,\alpha_{2n-2},(\alpha_{2n-1},f))=\alpha.$$

Now suppose that $\langle\alpha_i : i\in\omega\rangle$ is the result of a run of $\mathcal{G}_{\pi,A}$ according to σ, and that I has lost, so that $\pi(\langle\alpha_i : i\in\omega\rangle)$ is not in A. Let $x=\pi(\langle\alpha_i : i\in\omega\rangle)$. Then $x\notin p[T]$, so $<_{\mathrm{LS}}{\restriction}T_x$ is wellfounded and there exists a function $f\colon T_x\to\delta$ preserving $<_{\mathrm{LS}}$, i.e., such that $f{\restriction}T^*_{x\restriction n}\in OP^T_{x\restriction n}$ for all $n\in\omega$.

For each $n\in\omega$, let A_{2n+2} be the set in $\nu_{x\restriction 2n+2}$ used to choose α_{2n+2}. Using $\mathsf{CC}_{\mathbb{R}}$, the Moschovakis Coding Lemma and the fact that $\delta<\Theta$, we can find a sequence $\langle C_{2n+2} : n\in\omega\rangle$ of club subsets of δ such that each C_{2n+2} witnesses (via the corresponding function $Q_{x\restriction 2n+2}$) that the corresponding set A_{2n+2} is in $\nu_{x\restriction 2n+2}$ (so, for all $g\in[C_{2n+2}]^{\omega\cdot\zeta_{x\restriction 2n+2}}$, $Q_{x\restriction 2n+2}(g^*)$ is in A_{2n+2}). Let C be $\bigcap_{n\in\omega} C_{2n+2}$, and let C' be the set of $\beta\in C$ such that the ordertype of $C\cap\beta$ is $\gamma+\omega$, for some ordinal γ.

Let g be an order-preserving function from the range of f into C', and for each $n\in\omega$, let f_{2n+1} be $(g\circ f){\restriction}T^*_{x\restriction 2n+2}$. Then $Q^{-1}_{x\restriction 2n+2}(f_{2n+1})$ is equal to h^* for some $h\in[C]^{\omega\cdot\zeta_{x\restriction 2n+2}}$, so f_{2n+1} is in A_{2n+2}. We have then that the run of $\mathcal{G}^T$ given by

$$\alpha_0,(\alpha_1,f_1),\alpha_2,(\alpha_3,f_3),\ldots$$

is according to Σ yet losing for player I, giving a contradiction.

Proof of Claim 2: For each $u \in \omega^{<\omega}$, let Q_u be the bijection from $[\delta]^{\xi_u}$ to OP^S_u with the property that $Q_u(g)$ and g have the same range, for all $g \in [\delta]^{\xi_u}$. Let ν_u be $\{Q_u[A] : A \in \mu_{\xi_u}\}$. Then ν_u a δ-complete ultrafilter on OP^S_u.

Fix a winning strategy Σ for II in $\mathcal{G}^S$. We define a strategy σ for II in $\mathcal{G}_{\pi,A}$. For each $n \in \omega$, we let $\sigma(\langle \alpha_0, \dots, \alpha_{2n}\rangle)$, be the unique value α such that, letting $u = \pi(\langle \alpha_0, \dots, \alpha_{2n}\rangle)$, for ν_u-many $f \in OP^S_u$,

$$\Sigma((\alpha_0, f{\restriction}S^*_{u\restriction 1}), \alpha_1, (\alpha_2, f{\restriction}S^*_{u\restriction 1}), \dots, \alpha_{2n-1}, (\alpha_{2n}, f)) = \alpha.$$

Now suppose that $\langle \alpha_i : i \in \omega\rangle$ is the result of a run of $\mathcal{G}_{\pi,A}$ according to σ, and that II has lost, so that $\pi(\langle \alpha_i : i \in \omega\rangle)$ is in A. Let $x = \pi(\langle \alpha_i : i \in \omega\rangle)$. Then $x \notin p[S]$, so $<_{\mathrm{LS}} {\restriction}S_x$ is wellfounded and there exists a function $f \colon S_x \to \delta$ preserving $<_{\mathrm{LS}}$, i.e., such that $f{\restriction}S^*_{x\restriction n} \in OP^S_{x\restriction n}$ for all $n \in \omega$.

For each $n \in \omega$, let A_{2n+1} be the set in $\nu_{x\restriction 2n+1}$ used to choose α_{2n+1}. Using $\mathsf{CC}_{\mathbb{R}}$, the Moschovakis Coding Lemma and the fact that $\delta < \Theta$, we can find a sequence $\langle C_{2n+1} : n \in \omega\rangle$ of club subsets of δ such that each C_{2n+1} witnesses (via the corresponding function $Q_{x\restriction 2n+1}$) that the corresponding set A_{2n+1} is in $\nu_{x\restriction 2n+1}$ (so, for all $g \in [C_{2n+1}]^{\omega\cdot\xi_{x\restriction 2n+1}}$, $Q_{x\restriction 2n+1}(g^*)$ is in A_{2n+1}). Let C be $\bigcap_{n\in\omega} C_{2n+1}$, and let C' be the set of $\beta \in C$ such that the ordertype of $C \cap \beta$ is $\gamma + \omega$, for some ordinal γ.

Let g be an order-preserving function from the range of f into C', and for each $n \in \omega$, let f_{2n} be $(g \circ f){\restriction}S^*_{x\restriction 2n+1}$. Then $Q^{-1}_{x\restriction 2n+1}(f_{2n})$ is equal to h^* for some $h \in [C]^{\omega\cdot\xi_{x\restriction 2n+1}}$, so f_{2n} is in A_{2n+1}. We have then that the run of $\mathcal{G}^S$ given by

$$(\alpha_0, f_0), \alpha_1, (\alpha_2, f_2), \alpha_3, \dots$$

is according to Σ yet losing for player II, giving a contradiction. $\square$

Putting together Theorems 5.1.4 and 7.0.3, we get the following.

Corollary 7.0.4 ($\mathsf{ZF} + \mathsf{AD}$). *For every $\lambda < \Theta$ and every extension-preserving function $\pi \colon \lambda^{<\omega} \to \omega^{<\omega}$, if $A \subseteq \omega^\omega$ is Suslin and co-Suslin, then the game $\mathcal{G}_{\pi,A}$ is determined.*

7.0.5 Remark. Suppose that $\pi \colon \lambda^{<\omega} \to \omega^{<\omega}$ is an extension-preserving function, for some cardinal $\lambda < \Theta$, and that $A \subseteq \omega^\omega$. Let $\prec$ be a strict prewellordering of a subset of ω^ω of length λ. By the Moschovakis Coding Lemma, every strategy in $\mathcal{G}_{\pi,A}$ is coded by a subset of ω^ω in pos-$\underset{\sim}{\Sigma}^1_1(\prec)$. In particular, if $\mathcal{G}_{\pi,A}$ is determined, then it is determined in $L(\pi, A, \prec, \mathbb{R})$. Similarly, since the negation of $<\Theta$-$\mathsf{Determinacy}$ is a $\underset{\sim}{\Sigma}^2_1$ statement (using the Coding Lemma), Theorem 6.1.21 implies that, assuming $\mathsf{AD} + V = L(\mathbb{R})$, the failure of $<\Theta$-determinacy would imply the nondeterminacy of a game of the form $\mathcal{G}_{\pi,A}$ for some Suslin, co-Suslin set A, contradicting Theorem 7.0.3.

Using the fact that, under AD, every subset of ω_1 is constructible from a real, and the sharp of each real exists, it is not hard to see that AD implies ω_1-$\mathsf{Determinacy}$ (this observation is due to the author). It is apparently an open question whether AD implies ω_2-$\mathsf{Determinacy}$.

Chapter 8

Cone measure ultrapowers

In this chapter we analyze ultrapowers by the cone measures introduced in Section 1.2. Given an equivalence relation E on a set X, a function f with domain X is said to be *E-invariant* if $f(x) = f(y)$ whenever xEy. Given an ordered equivalence relation $(E, \leq_E)$ (on some set X) such that the cone measure μ_E is an ultrafilter on the set $\mathcal{C}_E$ consisting of the E-equivalence classes, we let WF_E denote the assertion that the ultrapower $\mathrm{Ord}^{\mathcal{C}_E}/\mu_E$ (formed by taking all E-invariant functions on X, modulo μ_E) is wellfounded. Note that in this situation $\mathrm{Ord}^{\mathcal{C}_E}/\mu_E$ is wellfounded if and only if $V^{\mathcal{C}_E}/\mu_E$ is. Given a function f with domain $\mathcal{C}_E$, or an E-invariant function f with domain X, we write $[f]_{\mu_E}$ for the element of $V^{\mathcal{C}_E}/\mu_E$ represented by f. Given two ordered equivalence relations $(E, \leq_E)$ and $(F, \leq_F)$ on the same set X, if $\leq_F \subseteq \leq_E$ then $\mathrm{Ord}^{\mathcal{C}_E}/\mu_E$ embeds into $\mathrm{Ord}^{\mathcal{C}_F}/\mu_F$, so WF_F implies WF_E.

The wellfoundedness of the ultrapower $\mathrm{Ord}^{\mathcal{C}_E}/\mu_E$ follows from DC plus the assumption that μ_E is a countably complete ultrafilter on $\mathcal{C}_E$. Recall that DC holds in models of the form $L(A, \mathbb{R})$ satisfying $\mathsf{DC}_{\mathbb{R}}$, if A is a set contained in $L(\mathbb{R})$.

Woodin has shown that AD^+ implies $\mathsf{WF}_{\mathrm{Tu}}$, where Tu is Turing equivalence. We will not use this result in this book, however, and instead will assume statements of the form WF_E when they are needed.

8.0.1 Remark. By Remark 1.2.7, if $(E, \leq_E)$ is an ordered equivalence relation such that all downward cones are countable and the corresponding cone measure μ_E is a countably complete ultrafilter, then E induces a countably complete ultrafilter on ω_1, which gives the statement $\aleph_1 \not\leq 2^{\aleph_0}$. Suppose that M_x $(x \in \omega^\omega)$ is an E-invariant association of elements of ω^ω to models of ZFC such that the corresponding μ_E-ultrapower M is wellfounded. Since $\aleph_1 \not\leq 2^{\aleph_0}$, ω_1^V is strongly inaccessible in M. It follows then that for each ordinal $\gamma < \omega_1^V$, $V_\gamma^{M_x} = V_\gamma^M$ for an E-cone of x.

Theorem 8.0.2 below shows that $\mathsf{DC}_{\mathbb{R}}$ follows from certain instances of WF_E (and in fact one needs only that the ultrapower of ω_1 is wellfounded). We say that an ordered equivalence relation $(E, \leq_E)$ (or the corresponding order $\leq_E$) is *locally countable* if for all $x \in \omega^\omega$, $\bigcup \mathcal{D}_E(x)$ is countable.

Theorem 8.0.2 (Solovay [36])**.** *Let $(E, \leq_E)$ be a locally countable ordered equivalence relation on ω^ω. If the E-cone measure is an ultrafilter, and the image of ω_1 in the corresponding ultrapower is wellfounded, then $\mathsf{DC}_{\mathbb{R}}$ holds.*

Proof. Let T be a tree of finite sequences from ω^ω without an infinite path. We will find a ranking function for T. For each $\bar{x} \in T$ and each $z \in \omega^\omega$, let $f_{\bar{x}}([z]_E)$ be 0 if the members of $\bar{x}$ are not all in $\mathcal{D}_E(z)$, and the rank of $T \restriction \mathcal{D}_E(z)$ below $\bar{x}$ otherwise (which exists, since, as $\mathcal{D}_E(z)$ is countable, $\mathsf{DC}_{\mathcal{D}_E(z)}$ holds). Then if $\bar{x}$ properly extends $\bar{y}$ in T, $f_{\bar{x}}([z]_E) < f_{\bar{y}}([z]_E)$ for all $[z]_E$ containing both $\bar{x}$ and $\bar{y}$. The map $\bar{x} \mapsto [f_{\bar{x}}]_{\mu_E}$ then ranks T. □

A similar argument gives Theorem 8.0.3 below, which is similar to Theorem 4.3.5 and will be used in Section 8.2. Recall that, given a set $A \subseteq \omega^\omega$, $\Delta_\omega(A)$ be the smallest projective algebra with A as an element, and we say that the members of $\Delta_\omega(A)$ are *projective in* A. We let $\delta_\omega(A)$ be the supremum of the lengths of the prewellorderings in $\Delta_\omega(A)$.

Theorem 8.0.3. *Suppose that there exists a locally countable ordered equivalence relation $(E, \leq_E)$ on ω^ω such that μ_E is an ultrafilter and WF_E holds. Then for each $A \subseteq \omega^\omega$, $\delta_\omega(A)$ is the supremum of the ranks of the wellfounded preorders on ω^ω in $\Delta_\omega(A)$.*

Proof. It suffices to fix a wellfounded transitive relation R on ω^ω in $\Delta_\omega(A)$ and show that its rank is less than $\delta_\omega(A)$. Given $x \in \omega^\omega$, define a function $\rho_x : \mathcal{C}_E \to \omega_1$ by setting $\rho_x(e)$ to be the rank of $R_x \restriction e$, where R_x is

$$\{(y, z) \in \omega^\omega \times \omega^\omega : xRyRz\}.$$

Define the relation $\leq^*$ on ω^ω by setting $x \leq^* y$ if and only if

$$\{e \in \mathcal{C}_E : \rho_x(e) \leq \rho_y(e)\} \in \mu_E.$$

Since μ_E is an ultrafilter and WF_E holds, $\leq^*$ is a prewellordering. Furthermore, $\leq^*$ is projective in A. Since R is transitive, if xRy, then for all $e \in \mathcal{C}_E$, $\rho_y(e) \leq \rho_x(e)$. It follows that the rank of R is less than or equal to the length of $\leq^*$. □

8.1 S-cones

We are interested in a special class of ordered equivalence relations, for which we establish the following notation.

8.1.1 Definition. Let S be a set of ordinals.

- We write $\leq_S$ for the binary relation on ω^ω defined by setting $x \leq_S y$ if and only if $x \in L[S, y]$.

- We write $\equiv_S$ for the equivalence relation $\leq_S \cap \geq_S$ on ω^ω.
- For each $x \in \omega^\omega$, $[x]_S$ denotes the $\equiv_S$-equivalence class of x.
- We write $\mathcal{D}_S$ for the set of $\equiv_S$-classes.
- An S-*cone* is either a set of the form $\{y \in \omega^\omega : x \in L[S,y]\}$ or a set of the form $\{[y]_S : y \in \omega^\omega, x \in L[S,y]\}$, for some $x \in \omega^\omega$.
- We write μ_S for the set of subsets of $\mathcal{D}_S$ which contain an S-cone.
- A function f on ω^ω is S-invariant if $f(x) = f(y)$ whenever $x \equiv_S y$.
- Given an S-invariant function f with domain ω^ω, $[f]_{\mu_S}$ denotes the element of $V^{\mathcal{D}_S}/\mu_S$ represented by f.

Recall from Corollary 1.2.6 that TD implies that μ_S is an ultrafilter on $\mathcal{D}_S$, for each set $S \subseteq \mathrm{Ord}$. We will be primarily interested in ultrapowers of the form $\prod_{d \in \mathcal{D}_S} f(d)/\mu_S$, for S-invariant functions f on ω^ω. We start by showing that the function $f(x) = \omega_1^{L[S,x]}$ represents ω_1 in these ultrapowers (this fact appears as Lemma 3.3 of [21]).

Theorem 8.1.2 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Let S be a set of ordinals and let f be the function on ω^ω defined by setting $f(x)$ to be $\omega_1^{L[S,x]}$. Then f represents ω_1 in $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$.*

Proof. Since μ_S is a countably complete ultrafilter (by Corollary 1.2.6), each $\alpha < \omega_1$ is represented by the function with constant value α. For each $\alpha < \omega_1$, $\{[x]_S : x \in \omega^\omega,\ f(x) > \alpha\}$ is in μ_S, so $[f]_{\mu_S} > \alpha$.

Suppose now that g is a $\equiv_S$-invariant function on ω^ω such that

$$\{[x]_S : x \in \omega^\omega,\ f(x) > g(x)\} \in \mu_S.$$

We want to see that there is an $\alpha < \omega_1$ such that $\{[x]_S : x \in \omega^\omega,\ g(x) = \alpha\} \in \mu_S$. By the countable completeness of μ_S, this follows from the existence of an $\alpha < \omega_1$ such that $\{[x]_S : x \in \omega^\omega,\ g(x) < \alpha\} \in \mu_S$, which we will show.

Let WO be the set of wellorderings of ω, and for each $z \in \mathrm{WO}$, let $|z|$ denote the ordinal ordertype of z. Let $\mathcal{G}$ be the game in which player I produces $x \in \omega^\omega$, II produces $y \in \omega^\omega$ and $z \subseteq \omega \times \omega$, and II wins if and only if $x \leq_S y$, $z \in \mathrm{WO}$ and $|z| > g(y)$.

Given a strategy σ for player I, let $y \in \omega^\omega$ be such that $\sigma \in L[S,y]$, and let $z \in \mathrm{WO} \cap L[S,y]$ be such that $|z| > g(y)$ (such a z exists since $g(y) < f(y) = \omega_1^{L[S,y]}$). If x is the response given by σ to y and z then $x \in L[S,y]$ and $g(y) < |z|$, so II wins the corresponding run of the game. So I cannot have a winning strategy.

Then AD implies that there is a winning strategy σ for II. Denote by $\sigma_y(x)$ and $\sigma_z(x)$ the values y and z respectively given by σ in response to a play x for I. Then $\{\sigma_z(x) : x \in \omega^\omega\}$ is an analytic subset of WO, so by $\utilde{\Sigma}^1_1$-boundedness (mentioned just before and generalized by Theorem 6.1.16) there is a countable ordinal α greater than all the members of $\{|\sigma_z(x)| : x \in \omega^\omega\}$. Then for all $x \in \omega^\omega$ with $\sigma \in L[S,x]$, $\sigma_y(x) \equiv_S x$, so $\alpha > |\sigma_z(x)| > g(\sigma_y(x)) = g(x)$.

□

Theorem 8.1.3 shows that if μ_S is an ultrafilter then for an S-cone of $y \in \omega^\omega$, $L[S,y]$ satisfies GCH below ω_1^V. This fact was first proved by Steel in [39] assuming AD and later by Woodin from the hypothesis below. It will be used in the proof of Theorem 11.2.1. The (standard) generalization of the theorem from CH from $\diamondsuit_{\omega_1}$ was observed by the author.

Theorem 8.1.3 ($\mathsf{ZF} + \mathsf{CC}_{\mathbb{R}}$). *Let S be a set of ordinals such that $\leq_S$ is locally countable and μ_S is an ultrafilter on $\mathcal{D}_S$. Then there is an $x \in \omega^\omega$ such that for all $y \in \omega^\omega$ with $x \leq_S y$, and all $\gamma < \omega_1^V$ which are infinite cardinals in $L[S,y]$,*

$$L[S,y] \models \diamondsuit_{\gamma^+}.$$

Proof. By Remark 1.2.7, μ_S being an ultrafilter implies that ω_1^V is measurable, which in turn implies that $\aleph_1 \not\leq 2^{\aleph_0}$. Since μ_S is an ultrafilter, there is an $x_0 \in \omega^\omega$ such that either

- $L[S,y] \models \diamondsuit_{\omega_1}$ for all $y \in \omega^\omega$ with $x_0 \in L[S,y]$ or
- $L[S,y] \models \neg\diamondsuit_{\omega_1}$ for all $y \in \omega^\omega$ with $x_0 \in L[S,y]$.

We show that the former holds, by finding one such $y \geq_S x_0$.

Let P be the set of ordered pairs from $\omega^{<\omega}$, and let $\pi\colon P \to \omega$ be a recursive bijection. Let

$$p\colon \omega^\omega \times \omega^\omega \to \mathcal{P}(\omega)$$

be defined by setting $p(x,y)$ to be $\pi[\{(x{\restriction}n, y{\restriction}n) : n \in \omega\}]$. Then p sends each pair from ω^ω to an infinite subset of ω, and distinct pairs from ω^ω to sets having finite intersection.

Recall that the partial order $\mathrm{Col}^*(\omega_1, \omega^\omega)$ consists of injections from countable ordinals to ω^ω, ordered by extension, and adds a bijection $g\colon \omega_1 \to \omega^\omega$. Let $\dot{X}$ be a $\mathrm{Col}^*(\omega_1, \omega^\omega)$-name for the p-image of the set of $(x,y) \in \omega^\omega \times \omega^\omega$ such that $g^{-1}(x) < g^{-1}(y)$. Let $Q_{\dot{X}}$ be a $\mathrm{Col}^*(\omega_1, \omega^\omega)$-name for the Jensen-Solovay almost disjoint coding forcing for the realization of $\dot{X}$ (see Section 0.5 for more on both of these partial orders).

Suppose that M is a transitive model of ZFC, and that (g,z) is M-generic for $(\mathrm{Col}^*(\omega_1, \omega^\omega) * Q_{\dot{X}})^M$. Then z is a subset of ω such that, for each pair (x,y) from $(\omega^\omega)^V$, $z \cap p(x,y)$ is finite if and only if $g^{-1}(x) < g^{-1}(y)$. It follows that g is an element of $M[z]$ (see Remark A.0.5 of [27] for a discussion of why $M[z]$ is a generic extension of M). Moreover, $M[g] \models \diamondsuit_{\omega_1}$, and since $Q_{\dot{X}_g}$ is c.c.c. and has cardinality $\aleph_1$ in $M[g]$, $\diamondsuit_{\omega_1}$ holds in $M[g][z]$ (which is the same as $M[z]$).

Since ω_1^V is measurable, and since $L[S,x_0] \models \mathsf{AC}$, the set

$$\mathcal{P}(\mathrm{Col}^*(\omega_1, \omega^\omega) * Q_{\dot{X}})^{L[S,x_0]}$$

is countable, so there exists a pair (g, z) which is $L[S, x_0]$-generic for this partial order. The argument just given shows that $L[S, x_0, z] \models \diamondsuit_{\omega_1}$, so any real in $L[S, x_0, z]$ coding the pair (x_0, z) works as our desired y.

Again using the measurability of ω_1^V, $\mathcal{P}(\alpha) \cap L[S, x_0]$ is countable for each $\alpha < \omega_1^V$ so there exist $L[S, x_0]$-generic filters for each corresponding partial order $\mathrm{Col}(\omega, \alpha)$ (see Section 0.5 again). Since these extensions must satisfy $\diamondsuit_{\omega_1}$, we have that $L[S, x_0] \models \diamondsuit_{\gamma^+}$ for all $\gamma < \omega_1^V$ which are infinite cardinals in $L[S, x_0]$. To see this, fix such a γ, and recall that $\diamondsuit_{\gamma^+}$ is equivalent to the version in which the witnessing sequence is allowed γ many guesses at each $\alpha < \gamma^+$ (see [23], for instance). Working in $L[S, x_0]$, suppose that $\langle \sigma_\alpha : \alpha < \gamma^+ \rangle$ is such that each σ_α is a $\mathrm{Col}(\omega, \gamma)$-name for a subset of α, and the realizations of the σ_α's are forced to be witness to $\diamondsuit_{\omega_1}$. For each $\alpha < \gamma^+$, let A_α be the set of $a \in \mathcal{P}(\alpha) \cap L[S, x_0]$ such that the statement $\check{a} = \sigma_\alpha$ is forced by some condition in $\mathrm{Col}(\omega, \gamma)$. Since $\mathrm{Col}(\omega, \gamma)$ has cardinality γ, each A_α has cardinality at most γ. Applying the definition of $\diamondsuit_{\omega_1}$ to the elements of $\mathcal{P}(\gamma^+) \cap L[S, x_0]$, we see that $\langle A_\alpha : \alpha < \gamma^+ \rangle$ witnesses the generalized form of $\diamondsuit_{\gamma^+}$ in $L[S, x_0]$. The same considerations apply to each $y \in \omega^\omega$ such that $y \geq_S x_0$. □

8.2 Measurable cardinals from cone measures

We give here an application of Theorem 8.0.3 demonstrating the existence of measurable cardinals.

Theorem 8.2.1 (ZF + AD + $\mathsf{DC}_{\mathbb{R}}$). *For each $A \subseteq \omega^\omega$, $\delta_\omega(A)$ is a limit of measurable cardinals.*

Proof. Since $\delta_\omega(A) < \Theta$, there is a set $C \subseteq \omega^\omega$ of Wadge rank greater than $\delta_\omega(A)$. By the Moschovakis Coding Lemma, it suffices to establish the theorem in $L(C, \mathbb{R})$, where DC holds and therefore the ultrapower of the ordinals by the Turing measure is wellfounded, so Theorem 8.0.3 applies. Given $B \subseteq \omega^\omega$, we let ρ_B denote the supremum of the ranks of the wellfounded relations in $\utilde{\Sigma}^1_1(B)$, and we let δ^1_B be the supremum of the ranks of the $\utilde{\Delta}^1_1(B)$ prewellorderings. By Theorem 8.0.3, $\delta_\omega(A) = \sup\{\rho_B : B \in \Delta_\omega(A)\}$.

Let $B \subseteq \omega^\omega$ be projective in A, let $U \subseteq (\omega^\omega)^3$ be a complete $\utilde{\Sigma}^1_1(B)$-set, and let B^* be the set of x for which U_x is a wellfounded preorder. Define a prewellordering $\leq^*$ on B^* by setting $x \leq^* y$ if and only if the rank of $\mathrm{rk}(U_x) \leq \mathrm{rk}(U_y)$, where $\mathrm{rk}(U_x)$ denotes the rank of U_x.

Claim 1. *If $Z \subseteq B^*$ is in $\utilde{\Sigma}^1_1(B)$, then $\{\mathrm{rk}(U_x) : x \in Z\}$ is bounded in ρ_B.*

Proof. The supremum of this set is the rank of the transitive wellfounded relation $\{((x, y), (x, z)) : x \in Z, (x, y, z) \in U\}$, which is in $\utilde{\Sigma}^1_1(B)$. □

We adapt Solovay's original proof of the measurability of ω_1 under AD to find a measure on ρ_B. For each $X \subseteq \rho_B$ we consider the game $\mathcal{G}_X$, where player I builds $x_i \in \omega^\omega$ for even $i \in \omega$ and player II builds $x_i \in \omega^\omega$ for odd $i \in \omega$. We use the way of doing this illustrated in the following diagram.

I	$x_0(0)$		$x_0(1), x_2(0)$		$x_0(2), x_2(1), x_4(0)$		...
II		$x_1(0)$		$x_1(1), x_3(0)$		...	

The game $\mathcal{G}_X$.

Given x_i $(i \in \omega)$ produced by a run of this game, the winner is decided as follows.

- If there is an $i \in \omega$ such that $x_i \notin B^*$, then I wins if the least such i is odd, and II wins if the least such i is even.
- If $x_i \in B^*$ for all $i \in \omega$, and there is an $i \in \omega$ such that $\neg(x_i \leq^* x_{i+1})$, then I wins if the least such i is even, and II wins if the least such i is odd.
- If $x_i \in B^*$ and $x_i \leq^* x_{i+1}$ holds for all $i \in \omega$, then I wins if and only if $\sup\{\mathrm{rk}(U_{x_i}) : i \in \omega\} \in X$.

A winning strategy for player II in $\mathcal{G}_X$ (for some $X \subseteq \rho_B$) can be converted into a winning strategy for player I in $\mathcal{G}_{\rho_B \setminus X}$, playing any fixed member of B^* for x_0. It follows that, for each $X \subseteq \rho_B$, player I has a winning strategy in at least one of the games $\mathcal{G}_X$ and $\mathcal{G}_{\rho_B \setminus X}$. Let W be the set of $X \subseteq \rho_B$ for which I has a winning strategy.

Claim 2 shows that W is a countably complete ultrafilter. Its completeness, which by the claim is at least δ^1_B, is a measurable cardinal. The theorem them follows from Claim 2 and the fact that $\{\delta^1_B : B \in \Delta_\omega(A)\}$ is cofinal in $\delta_\omega(A)$ (by the definition of $\delta_\omega(A)$).

Claim 2. *W is closed under intersections of cardinal less than δ^1_B.*

We finish by proving the claim, using the Moschovakis Coding Lemma plus the boundedness fact established in Claim 1. Fix $\gamma < \delta^1_B$ and a prewellordering $\leq_R$ in $\underset{\sim}{\Delta}^1_1(B)$ of a set $R \subseteq \omega^\omega$ such that $\leq_R$ has length γ. Let $=_R$ be $\leq_R \cap \geq_R$, and let $<_R$ be $\leq_R \setminus =_R$. For each $a \in R$, let $\mathrm{rk}_R(a)$ denote the rank of a in $\leq_R$. Suppose that $\langle X_\alpha : \alpha < \gamma\rangle$ is a sequence of sets in W. It suffices to show that the intersection $\bigcap\{X_\alpha : \alpha < \gamma\}$ is nonempty. Let Y be the set of pairs (a, b) such that $a \in R$ and b is a winning strategy for player I in $\mathcal{G}_{X_{\mathrm{rk}_R(a)}}$ (more formally, b is a code for a strategy using some fixed bijection between ω and $\omega^{<\omega}$). By the Coding Lemma, there is a pos-$\underset{\sim}{\Sigma}^1_1(<_R)$ set $Y^* \subseteq Y$ such that for each $\alpha < \gamma$ there is an $(a, b) \in Y^*$ with $\mathrm{rk}_R(a) = \alpha$. Let $Z = \{b : \exists a\, (a, b) \in Y^*\}$. Then Z is a collection of (codes for) winning strategies and Z is in $\underset{\sim}{\Sigma}^1_1(B)$.

Let Σ be the set of finite sequences $\langle y_i : j < j_*\rangle$ from B^* such that, whenever

- $\langle x_i : i < \omega\rangle$ is a run of $\mathcal{G}_\emptyset$ (the payoff set is irrelevant here) where I plays according to a strategy from Z, and

- $x_{2j+1} = y_j$ for all $j < j_*$,

$\mathrm{rk}(U_{x_{2j}}) < \mathrm{rk}(U_{x_{2j+1}})$ for all $j < j_*$. By Claim 1, each sequence in Σ has a proper extension in Σ. By $\mathsf{DC}_{\mathbb{R}}$, there is a a sequence $\langle y_i : i \in \omega \rangle$ whose finite initial segments are all in Σ. Since $\langle y_j : j < \omega \rangle$ forms a losing play for player II against any of the strategies from Z, it follows that $\sup\{\mathrm{rk}(U_{y_j}) : j \in \omega\}$ is in X_α for all $\alpha < \gamma$, as desired. $\square$

8.3 The degree order on sets of ordinals

Given $S, T \subseteq \mathrm{Ord}$, we write $S \leq_{\mathcal{D}} T$ ($\mathcal{D}$ for "degree") to mean that

$$\{[x]_{\mathrm{Tu}} : \omega^\omega \cap L[S,x] \subseteq L[T,x]\} \in \mu_{\mathrm{Tu}}.$$

We write $S \equiv_{\mathcal{D}} T$ for

$$(S \leq_{\mathcal{D}} T) \wedge (T \leq_{\mathcal{D}} S)$$

and $S <_{\mathcal{D}} T$ for

$$(S \leq_{\mathcal{D}} T) \wedge \neg(T \leq_{\mathcal{D}} S).$$

8.3.1 Remark. A reflection argument shows that for any set S of ordinals there is a bounded $T \subseteq \Theta$ with $S \leq_{\mathcal{D}} T$. Ultimately we will be concerned only with the restriction of $\leq_{\mathcal{D}}$ to bounded subsets of Θ.

Theorem 8.3.2 shows that $\leq_{\mathcal{D}}$ is a total order when μ_{Tu} is a countably complete ultrafilter. As with Theorem 8.1.3, Theorem 8.3.2 was first proved by Steel in [39] assuming AD and later by Woodin from $\mathsf{TD} + \mathsf{CC}_{\mathbb{R}}$. Recall that TD was later shown to imply $\mathsf{CC}_{\mathbb{R}}$ [33].

Theorem 8.3.2 (ZF+TD). *If S and T are sets of ordinals such that $\neg(T \leq_{\mathcal{D}} S)$ then for a Turing cone of $x \in \omega^\omega$, for all $\gamma < \omega_1^V$, $\mathcal{P}(\gamma) \cap L[S,x]$ is a set of cardinality $|\gamma|^{L[T,x]}$ in $L[T,x]$.*

To prove Theorem 8.3.2, we use a variation of Mathias forcing relative to a nonprincipal ultrafilter U on ω. Our partial order is a slight simplification of the one used originally by Woodin, and is forcing-equivalent to it. Let $\mathcal{S}$ be the set of nonempty finite subsets of ω and let $\mathcal{F}$ be the set of functions from ω to U. We let $\max(s)$ denote the largest element of a set s in $\mathcal{S}$. The domain of $\mathbb{P}_U$ is $\mathcal{S} \times \mathcal{F}$. The order on $\mathbb{P}_U$ is defined as follows : $(s', F') \leq (s, F)$ if the following hold:

- $s' \cap (\max(s) + 1) = s$;
- for each $k \in s' \setminus s$, $k \in F(\max(s \cap k))$;
- for each $i \in \omega$, $F'(i) \subseteq F(i)$.

The second condition above distinguishes $\mathbb{P}_U$ from the usual Mathias forcing relative to an ultrafilter. This condition enables Lemmas 8.3.3 and 8.3.4 below. The corresponding facts hold for Mathias forcing, but only for Ramsey ultrafilters; the ultrafilters we use below are not Ramsey.

Note that if $(s', F') \leq (s, F)$ and i is the least member of $s' \setminus s$, then $(s', F') \leq (s \cup \{i\}, F) \leq (s, F)$. That is, (s', F') can be reached from (s, F) by one-point extensions.

Let us say that a function $F \in \mathcal{F}$ is *refining* if $F(i) \subseteq F(j) \subseteq (\omega \setminus (j+1))$ whenever $j < i \in \omega$. For densely many $\mathbb{P}_U$-conditions (s, F), F is refining. Given $F \in \mathcal{F}$, say that an infinite $a \subseteq \omega$ is *F-fast* if, for all $i \in a$, $a \setminus (i+1) \subseteq F(i)$. Every infinite subset of an F-fast set is F-fast. Every infinite $a \subseteq \omega$ (in the ground model or any outer model) generates a filter $G_a \subseteq \mathbb{P}_U$ consisting of those $(s, F) \in \mathbb{P}_U$ such that s is an initial segment of a and $a \setminus (\max(s) + 1)$ is F-fast.

Lemma 8.3.3 (The Genericity Condition)**.** *Assume that $\mathsf{CC}_{\mathbb{R}}$ holds. Let M be a transitive model of ZF, and let U be a nonprincipal ultrafilter on ω in M. Let a be an infinite subset of ω. Then G_a is M-generic for $\mathbb{P}_U$ if and only if, for each $F \in \mathcal{F}$ in M, there is an $i \in a$ such that $a \setminus i$ is F-fast.*

Proof. The forward direction follows immediately from the fact that for every $s \in \mathcal{S}$, and any two functions $F, F' \in \mathcal{F}$, (s, F) and (s, F') are compatible. For the reverse direction, fix a with the given property and let D be a dense open subset of $\mathbb{P}_U$ in M. Working in M, we recursively define a ranking function ρ_D on $\mathcal{S}$, letting $\rho_D(s)$ be

- 0 if there exists an $F \in \mathcal{F}$ such that $(s, F) \in D$;
- α if $\neg(\rho_D(s) < \alpha)$ and $\{i \in \omega : \rho_D(s \cup \{i\}) < \alpha\} \in U$;
- ω_1 if $\neg(\rho_D(s) < \omega_1)$.

Observe that, for each $s \in \mathcal{S}$, if $\{i \in \omega : \rho_D(s \cup \{i\}) < \omega_1\} \in U$, then $\rho_D(s)$ is at most

$$\sup\{\rho_D(s \cup \{i\}) : i \in \omega,\ \rho_D(s \cup \{i\}) < \omega_1\},$$

which is less than ω_1. It follows that, by $\mathsf{CC}_{\mathbb{R}}$, we can choose, for each $s \in \mathcal{S}$ a function $F_s \colon \omega \to U$ such that

- if $\rho_D(s) = 0$, then $(s, F_s) \in D$;
- if $\rho_D(s) < \omega_1$, then for all $i \in F_s(\max(s))$, $i > \max(s)$ and
$$\rho_D(s \cup \{i\}) < \rho_D(s);$$
- if $\rho_D(s) = \omega_1$, then for all $i \in F_s(\max(s))$, $i > \max(s)$ and
$$\rho_D(s \cup \{i\}) = \omega_1.$$

Define $F_*\colon \omega \to U$ by setting $F_*(i)$ to be $\bigcap\{F_s(i) : s \in \mathcal{S} \cap \mathcal{P}(i+1)\}$.

We claim that $\rho_D(s) < \omega_1$ for all $s \in \mathcal{S}$. To see this, suppose that $s \in \mathcal{S}$ is such that $\rho_D(s) = \omega_1$. By the choice of F_s, and the definition of F_*, we have (passing from (s, F_*) to (s', F') by one-point extensions) that $\rho_D(s') = \omega_1$ whenever $(s', F') \in \mathbb{P}_U$ is such that $(s', F') \leq (s, F_*)$. This contradicts our assumption that D is dense.

Finally, fix an $i \in a$ such that $a \setminus i$ is F^*-fast. Then the sequence of values $\langle \rho_D(a \cap (j+1)) : j \in a \setminus i \rangle$ must decrease until reaching 0. If $j \in a \setminus i$ is such that $\rho_D(a \cap (j+1)) = 0$, then $(a \cap (j+1), F_*)$ is in $G_a \cap D$. □

The Genericity Condition gives the following.

Lemma 8.3.4. *Let M be a transitive model of ZF, let U be a nonprincipal ultrafilter on ω in U, and suppose that $a \in [\omega]^\omega$ is such that G_a is an M-generic filter for $\mathbb{P}_U$. Then for any infinite $a' \subseteq a$, $G_{a'}$ is an M-generic filter for $\mathbb{P}_U$.*

Recall that a subset of HF is *recursive* if it is Δ_1 over HF. Let us say that a function $\pi\colon 2^{<\omega} \to [\omega]^{<\omega}$ is *diffuse* if, letting $\pi_*(x)$ denote $\bigcup\{\pi(t{\restriction}i) : i < \omega\}$ for each $x \in 2^\omega$, for all distinct $x_0, \ldots, x_{n-1} \in 2^\omega$, if $a_i \in \{\pi_*(x_i), \omega \setminus \pi_*(x_i)\}$ for each $i < n$ then $\bigcap_{i<n} a_i$ is infinite. We will use a fixed recursive diffuse function below. One such function can be built by assigning disjoint infinite subsets $a_{\bar{s},f}$ of ω to each pair $(\bar{s}, f)$ such that, for some $n, k \in \omega$, $\bar{s} = \langle s_m : m < n \rangle$ is an n-tuple of distinct functions from k to 2 and f is a function from n to 2, and letting, for each $\ell \in a_{\bar{s},f}$, $m < n$ and $t \in 2^{<\omega}$ for which s_m is an initial segment of t, $\ell \in \pi(t)$ if and only if $f(m) = 1$.

Given a function $\pi\colon 2^{<\omega} \to [\omega]^{<\omega}$, say that a set $b \subseteq \omega$ is *π-weak* if there exist infinitely many $x \in 2^\omega$ such that at least one of $b \cap \pi^*(x)$ and $b \setminus \pi_*(x)$ is finite.

Lemma 8.3.5. *If $\pi\colon 2^{<\omega} \to [\omega]^{<\omega}$ is diffuse, n is a positive integer and b_i $(i < n)$ are π-weak subsets of ω, then $\bigcup_{i<n} b_i \neq \omega$.*

Proof. Since each b_i is π-weak, there exist distinct $x_i \in 2^\omega$ $(i < n)$ and $a_i \in \{\pi_*(x_i), \omega \setminus \pi_*(x_i)\}$ $(i < n)$ such that, for each $i < n$, $b_i \cap a_i$ is finite. No member of the set

$$(\bigcap_{i<n} a_i) \setminus \bigcup_{i<n} (b_i \cap a_i)$$

is in $\bigcup_{i<n} b_i$. Since π is diffuse, this set is infinite. □

We separate out one lemma which we be used again in the proof of Theorem 10.2.6.

Lemma 8.3.6. *Let $\pi\colon 2^{<\omega} \to [\omega]^{<\omega}$ be a recursive diffuse function in a model M of ZF. Let U be, in M, an ultrafilter on ω disjoint from the collection of π-weak sets, and suppose that $\mathcal{P}(\mathbb{P}_U)^M$ is countable. Let y be in $2^\omega \setminus M$, and let a be a subset of ω. Then there exists a $c \subseteq \omega$ which is M-generic for $\mathbb{P}_U^M$ such that $a \in L[c, y]$.*

Proof. By $\underset{\sim}{\Sigma}^1_1$ absoluteness, for each $b \in U$, every $z \in 2^\omega$ such that at least one of $b \cap \pi^*(z)$ and $b \setminus \pi^*(z)$ is finite is in M. It follows that $\pi^*(y)$ and $\omega \setminus \pi^*(y)$ have infinite intersection with each member of U.

By Lemma 8.3.3, there exists an M-generic $b \subseteq \omega$ such that $b \cap \pi^*(y)$ and $b \setminus \pi^*(y)$ are both infinite. To see this, enumerate all functions $F\colon \omega \to U$ in M as $\langle F_i : i \in \omega\rangle$. Let b be $\{n_i : i \in \omega\}$ where, for each $i \in \omega$, $n_i < n_{i+1}$, $n_i \in \bigcap_{j<i} F_j(\{n_j\})$ and $n_i \in \pi^*(y)$ if and only if i is even. Let c be an infinite subset of b, enumerated in increasing order by $\langle m_i : i < \omega\rangle$, such that, for each $i \in \omega$, $m_i \in \pi^*(y)$ if and only if $i \in a$. Then c is M-generic, by Lemma 8.3.4. Since π is recursive, $\pi^*(y) \in L[y]$, so $a \in L[c, y]$. $\square$

The end of the proof of Theorem 8.3.2 uses following theorem of Solovay, which will be used again in Section 8.6. We will use the theorem only for the case where x is a set of ordinals, in which case the proof is slightly easier.

Theorem 8.3.7 (Solovay)**.** *Suppose that $M \subseteq N$ are transitive models of* ZF*, $\mathbb{P}$ is a partial order in M and $\mathcal{P}(\mathbb{P} \times \mathbb{P}) \cap N$ is countable. Suppose that $x \in M[G]$ for every N-generic $G \subseteq \mathbb{P}$. Then $x \in M$.*

Proof. It suffices to prove the theorem in the case where x is transitive, replacing x with $\mathrm{TC}(\{x\})$ if necessary. Since $\mathcal{P}(\mathbb{P} \times \mathbb{P}) \cap N$ is countable, there exists an N-generic (G, H) for $\mathbb{P} \times \mathbb{P}$. Arguing by induction on the rank of x, we may assume that the members of x are all in M. Then if τ and σ are $\mathbb{P}$-names in M such that $\tau_G = \sigma_H = x$, it follows by genericity that there is some condition $(p, q) \in G \times H$ such that, for all $y \in M$ p decides the statement $\check{y} \in \tau$ and q decides the statement $\check{y} \in \sigma$, from which it follows that $x \in M$. $\square$

Proof of Theorem 8.3.2. Let $x \in \omega^\omega$, $y \in 2^\omega$ be such that $y \in L[T, x] \setminus L[S, x]$. Let $\pi\colon 2^{<\omega} \to [\omega]^{<\omega}$ be a recursive diffuse function. Let U be, in $L[S, x]$, an ultrafilter on ω not containing any π-weak subset of ω, and let $\mathbb{P}$ be $\mathbb{P}^{L[S,x]}_U$. Let $a \subseteq \omega$ HC-code an enumeration in ordertype ω of the $\mathbb{P}$-names in $L[S, x]$ for elements of ω^ω. By Lemma 8.3.6, there is an $L[S, x]$-generic set c for $\mathbb{P}$ such that $a \in L[c, y]$. Then $\omega^\omega \cap L[S, x][c]$ is a countable set in $L[T, x, c]$.

It follows by Turing Determinacy that there is an $x_* \in \omega^\omega$ such that, for all $y \geq_{\mathrm{Tu}} x_*$, $\omega^\omega \cap L[S, y]$ is a countable set in $L[T, y]$. Applying this fact and Theorem 8.3.7 (with $M = N = L[T, y]$) to the $\mathrm{Col}(\omega, \gamma)$-extension of any such model $L[T, y]$, we see that for all such y, $\mathcal{P}(\gamma) \cap L[S, y]$ is a set of cardinality $|\gamma|^{L[T,y]}$ in $L[T, y]$. $\square$

Theorem 8.3.8 lists some consequences of Theorem 8.3.2. Note that neither part of the theorem assumes (or implies) that S is in $L[T, y]$. A natural application of part (1b) is when S is an ∞-Borel code.

Part (1b) follows from part (1a) and Theorem 8.3.7, by another collapsing argument.

Theorem 8.3.8 $(\mathsf{ZF} + \mathsf{TD})$**.** *Let S and T be sets of ordinals.*

1. *If x is a base of a cone witnessing that $S \leq_{\mathcal{D}} T$, then the following hold for all $y \geq_{\mathrm{Tu}} x$.*
   (a) *For every $\gamma < \omega_1^V$, $\mathcal{P}(\gamma) \cap L[S, y] \subseteq L[T, y]$.*
   (b) *For every formula ϕ,*
   $$\{z \in \omega^\omega : L[S, z] \models \phi(S, z)\} \cap L[T, y] \in L[T, y].$$
2. *If $x \in \omega^\omega$ is a base of a Turing cone witnessing that $\neg(T \leq_{\mathcal{D}} S)$, then the following hold for all $y \geq_{\mathrm{Tu}} x$.*
   (a) *For every $\gamma < \omega_1^V$,*
   $$\mathcal{P}(\gamma) \cap L[S, y] \in H(|\gamma|^+)^{L[T,y]}.$$
   (b) *Every regular cardinal of $L[T, y]$ below ω_1^V is a strongly inaccessible cardinal in $L[S, y]$.*

Proof. Part (1a) follows a collapse argument using Theorem 8.3.7, as in the last paragraph of the proof of Theorem 8.3.2. For part (1b), it is enough by Theorem 8.3.7 to see that the set $A = \{z \in \omega^\omega : L[S, z] \models \phi(S, z)\} \cap L[T, y]$ is in $L[T, y][g]$ whenever g is $L[T, y]$-generic for the partial order $\mathrm{Col}(\omega, \mathbb{R})$ (as defined in $L[T, y]$). Fixing such a g, we have that A is in $L[B]$, for some bounded subset B of ω_1^V in $L[S, y, g]$. It follows from part (1a) then that B and A are in $L[T, y]$. Part (2a) is Theorem 8.3.2, and part (2b) follows directly from part (2a). □

8.4 Pointed trees

In this section we prove a result due to Martin which will be used in Sections 8.5 and 8.6. We continue to use the notation from Definition 8.1.1, and add the following. Recall that a tree is perfect if each node has an incompatible pair of extensions, and, given a tree a, that $[a]$ denotes the set of infinite branches through a.

8.4.1 Definition. Given a set of ordinals S, we say that

- a set $A \subseteq \omega^\omega$ is *S-positive* if it intersects every S-cone;
- a tree $a \subseteq \omega^{<\omega}$ is *S-pointed* if a is perfect and, for every $x \in [a]$, $a \in L[S, x]$;

The following theorem, which appears in [13], shows that, assuming AD, the sets of the form $[a]$, for a an S-pointed tree, are dense in the containment order on the S-positive subsets of ω^ω.

Theorem 8.4.2 ($\mathsf{ZF} + \mathsf{AD}$; Martin)**.** *Let S be a set of ordinals and let A be a subset of ω^ω. Then $A \subseteq \omega^\omega$ is S-positive if and only if there is an S-pointed perfect tree a such that $[a] \subseteq A$.*

Proof. The reverse implication (which does not use the assumption of AD) follows from the fact that for any perfect tree $a \subseteq \omega^{<\omega}$ and any $x \in \omega^\omega$ there is a $y \in [a]$ such that $x \in L[a, y]$.

For the forward implication, consider the game $\mathcal{G}$ where I and II respectively build x and y in ω^ω, and II wins if and only if $x \leq_S y$ and $y \in A$. If σ is a strategy for I and $y \in A$ is such that σ is in $L[S, y]$, then y is a winning play for II against σ. This shows that if A is S-positive then I cannot have a winning strategy. Suppose then that σ is a winning strategy for II. For each $s \in \omega^{<\omega}$, let $\sigma(s)$ denote the sequence of ($|s|$-many) moves made according to σ in response to s. Let $z \in 2^\omega$ be such that $\sigma \in L[S, z]$, and let Z be the set of $t \in \omega^{<\omega} \cup \omega^\omega$ such that $t(2n) = z(n)$ whenever $n \in \omega$ and $2n \in \mathrm{dom}(t)$. Working in $L[S, z]$ we can choose for each $s \in 2^{<\omega}$, recursively in $|s|$, a t_s in $Z \cap \omega^{<\omega}$, in such a way that, for all $s, s' \in 2^{<\omega}$,

- if $|s| = |s'|$ then $|t_s| = |t_{s'}|$;
- if $s \perp s'$ then $\sigma(t_s) \neq \sigma(t_{s'})$.

The achievability of the second condition above follows from the fact that σ is a winning strategy for II, which implies that, for any finite $t \in Z$, the set

$$\{x \circ \sigma : x \in \omega^\omega \cap Z, t \subseteq x\}$$

is S-positive, and therefore has size at least 2. Let a be the tree of initial segments of the members of $\{t_s : s \in 2^{<\omega}\}$. We claim that a is as desired. To see this, fix $y \in [a]$ and $x \in \omega^\omega \cap Z$ such that $y = x \circ \sigma$. Since σ is a winning strategy for II, y is in A, and x is in $L[S, y]$. This implies that z and σ are in $L[S, y]$ and therefore that a is in $L[S, y]$ as desired. □

8.5 Coding ultrapowers

This section continues to use the notation from Definitions 8.1.1 and 8.4.1, for a set of ordinals S. We work under the assumption that μ_S is an ultrafilter (which AD implies, by Corollary 1.2.6) on $\mathcal{D}_S$, and let δ^∞_S be $\prod \omega_2^{L[S,x]}/\mu_S$, assuming that this ultraproduct is wellfounded (as is it, assuming DC). In Remark 8.6.7 we will show (assuming AD plus the wellfoundedness of the ultrapower $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$) that $\delta^\infty_S \leq \Theta$. In this section we will show that $\delta^\infty_S < \Theta$, under the additional assumption that, for some set of ordinals T, $S <_{\mathcal{D}} T$.

Given a set of ordinals T, we define the relation $<^{\mathrm{c}}_T$ on $(\omega^\omega)^2$ by setting $(x, y) <^{\mathrm{c}}_T (z, w)$ to hold if and only if

- $x = z$;
- y and w are in $L[T, x]$;
- y comes before w in the constructibility order in $L[T, x]$ using T and x as predicates.

In the statement of Theorem 8.5.1 the set $\leq_S \times <^{\mathrm{c}}_T$ is used for conveneince as a set coding both $\leq_S$ and $<^{\mathrm{c}}_T$.

Theorem 8.5.1 (ZF+AD)**.** *Let S and T be sets of ordinals. Suppose that f is an S-invariant function on ω^ω such that, for an S-cone of x, $f(x)$ is a transitive set in $H(\aleph_1)^{L[T,x]}$. Then $\prod f(x)/\mu_S$ is isomorphic to a relation which is projective in $\leq_S \times <^{\mathrm{c}}_T$.*

Proof. Let R_f be the set of (x, y) such that $y \in \omega^\omega \cap L[T, x]$ and y HC-codes $f(x)$, and let R^*_f be the set of $<^{\mathrm{c}}_T$-minimal pairs in R_f. Then R^*_f is a uniformizing function for R_f. Note however that R^*_f need not be S-invariant.

Let E be the set of $y \in \omega^\omega$ which HC-code a transitive set. Then E is coanalytic. If $y \in E$ HC-codes a transitive set M, then, since M is rigid, for each $k \in \omega$ there is a unique p such that there is an isomorphism between

$$(\omega, \{(n, m) \in \omega \times \omega : y(2^n 3^m) = 0\})$$

and $(\mathrm{TC}(\{M\}), \in)$ sending k to p. We call this element $p_{y,k}$. The following sets are projective:

- $\{(y, y', k, k') \in E^2 \times \omega^2 : p_{y,k} = p_{y',k'}\}$;
- $\{(y, y', k, k') \in E^2 \times \omega^2 : p_{y,k} \in p_{y',k'}\}$;

Let H be the set of S-invariant functions h such that $h(x) \in f(x)$ for an S-cone of $x \in \omega^\omega$. For each $h \in H$ there is a $k \in \omega$ such that the set $A_{h,k} = \{x \in \omega^\omega : p_{R^*_f(x),k} = h(x)\}$ is S-positive. Since R^*_f may not be S-invariant, there may be more than one such k. Any such pair $(k, A_{h,k})$ codes h on an S-cone, however, in the sense that, for an S-cone of $z \in \omega^\omega$,

$$h(z) = h(x) = p_{R^*_f(x),k}$$

for any $x \in [z]_S \cap A_{h,k}$.

Let $\langle \sigma_i : i < \omega \rangle$ be a recursive listing of $\omega^{<\omega}$, and let C_f be the set of pairs $(k, b) \in \omega \times 2^\omega$ such that $a_b = \{\sigma_i : b(i) = 1\}$ is an S-pointed perfect tree and, for all $x, y \in [a_b]$, if $x \equiv_S y$ then $p_{R^*_f(x),k} = p_{R^*_f(y),k}$. By Theorem 8.4.2, every S-positive set contains the (S-positive) set of paths through some S-pointed tree. It follows that for each $h \in H$, given $k \in \omega$ such that $A_{h,k}$ is S-positive, there is a $b \in 2^\omega$ such that $[a_b] \subseteq A_{h,k}$ and $(k, b) \in C_f$.

For each $(k, b) \in C_f$ and $z \in \omega^\omega$, set $h_{(k,b)}([z]_S)$ to be the common value of $p_{R^*_f(x),k}$ for all $x \in [z]_S \cap [a_b]$ (if there is such a value, and $\emptyset$ otherwise). By the two previous paragraphs, for each $h \in H$ there exists a $(k, b) \in C_f$ such that $h(z) = h_{(k,b)}([z]_S)$ for an S-cone of $z \in \omega^\omega$. Define an equivalence relation $\sim$ on C_f by setting

$$(k, b) \sim (k', b')$$

if $h_{(k,b)}$ and $h_{(k',b')}$ agree on an S-cone, and let $[k, b]_f$ denote the $\sim$-class of (k, b). Define $[k, b]_f \in_f [k', b']_f$ to hold whenever

$$h_{(k,b)}([z]_S) \in h_{(k',b')}([z]_S)$$

for an S-cone of $z \in \omega^\omega$. Then $(C_f/\sim, \in_f)$ is isomorphic to $\prod f(x)/\mu_S$. Since C_f, $\sim$ and the relation on C_f induced by $\in_f$ are projective in $\leq_S \times <^c_T$ we are done. $\square$

8.6 Forcing with positive sets

Given a set S consisting of ordinals, let $\mathbb{P}_S$ be the partial order of S-positive subsets of ω^ω and let $\mathbb{S}_S$ be the partial order of S-pointed perfect trees, each ordered by $\subseteq$. By Theorem 8.4.2, each $\mathbb{P}_S$ condition contains the set of infinite paths through some S-pointed tree, which is in turn S-positive. It follows that a V-generic filter for either partial order generates one for the other.

A V-generic filter $G \subseteq \mathbb{P}_S$ induces an ultrapower $\prod_{x\in\omega^\omega} V/G$, whose elements are represented by (not necessarily S-invariant) functions in V with domain ω^ω. For each $a \in HC$, the constant function from ω^ω to $\{a\}$ represents a in this ultrapower. The identity function represents the generic $x_G \in \omega^\omega$ added by $\mathbb{P}_S$. The following theorem shows that the corresponding ultrapower of any set of ordinals, in particular the ultrapower of S itself, is the same in the ultrapowers with respect to μ_S and G, where the μ_S ultrapower is computed in V and the G ultrapower is computed in the $\mathbb{P}_S$-extension.

Theorem 8.6.1. *Let S be a set of ordinals such that $\leq_S$ is locally countable and μ_S is a countably complete ultrafilter on $\mathcal{D}_S$. Let $G \subseteq \mathbb{P}_S$ be a V-generic filter. Let k be the map sending $[f]_{\mu_S}$ to $[f]_G$, for each S-invariant function f in V from ω^ω to* Ord. *Then k maps $(\prod_S \mathrm{Ord}/\mu_S)^V$ isomorphically to $(\prod \mathrm{Ord}/G)^{V[G]}$.*

Proof. Since each S-positive set has S-positive intersection with each S-cone, k is injective (mod μ_S) and order preserving. We show that it is onto. Let $g\colon A \to \mathrm{Ord}$ be a function in V with S-positive domain. Define f on

$$\bigcup\{[x]_S : x \in A\}$$

by letting $f(x)$ be $\min(g[[x]_S \cap A])$. Then f is S-invariant. Let C be

$$\{x \in A : f(x) = g(x)\}.$$

It suffices (by the genericity of G) to show that C is S-positive, since C forces that $[g]_G = [f]_G = k([f]_{\mu_S})$. Supposing otherwise, let $x \in \omega^\omega$ be such that $\{y \in \omega^\omega : y \geq_S x\}$ is disjoint from C. Since A is S-positive, there is a $y \geq_S x$ in A. Then there is a $z \in [y]_S \cap A$ such that $g(z) = f(y)$. Then $z \in C$ and $z \geq_S x$, giving a contradiction. $\square$

We let j_G denote the embedding of V into $\prod_{x\in\omega^\omega} V/G$ sending each element of V to the class represented by its corresponding constant function, assuming that G is a V-generic filter for $\mathbb{P}_S$. We let j_S be the corresponding embedding of V into $\prod_{x\in\omega^\omega} V/\mu_S$.

Corollary 8.6.2. *Let S be a set of ordinals such that $\leq_S$ is locally countable and μ_S is a countably complete ultrafilter on $\mathcal{D}_S$. Let $G \subseteq \mathbb{P}_S$ be a V-generic filter. Then for every set of ordinals T, $j_G(T) = j_S(T)$.*

Proof. By Theorem 8.6.1, for each ordinal γ there is an S-invariant function $f\colon \omega^\omega \to \mathrm{Ord}$ representing γ in both ultrapowers. Given f, either $f(x) \in T$ for an S-cone of x or $f(x) \notin T$ for an S-cone of x. In either case the same must hold on a set in G. $\square$

8.6.3 Remark. We are primarily interested here in the ultrapower

$$\prod_{x\in\omega^\omega} L[S,x]/G,$$

which by Theorem 8.6.1 and Corollary 8.6.2 is isomorphic to $\prod L[S,x]/\mu_S$. Since the functions used to from $\prod_{x\in\omega^\omega} L[S,x]/G$ are not required to be S-invariant, the usual proof of Łoś's Theorem goes through, showing that whenever $f_1,\ldots,f_n$ are functions on ω^ω in V such that each value $f_i(x)$ is in the corresponding model $L[S,x]$, and ϕ is an n-ary formula,

$$\prod_{x\in\omega^\omega} L[S,x]/G \models \phi([f_1]_G,\ldots,[f_n]_G)$$

if and only if the set of x for which $L[S,x] \models \phi(f_1(x),\ldots,f_n(x))$ is in G. In particular, for each $n \in \omega$ the function $x \mapsto \omega_n^{L[S,x]}$ represents the value of ω_n in the ultrapower.

Assuming that the ultrapower $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$ is wellfounded, we let S^∞ be the set of ordinals represented by the function with constant value S. Given $A \subseteq \omega^\omega$, we say that S *computes* A *on a cone* if $A \cap L[S,x] \in L[S,x]$ for an S-cone of $x \in \omega^\omega$. Part (1b) of Theorem 8.3.8 shows that if S and T are sets of ordinals such that $S \leq_{\mathcal{D}} T$, then T computes $\{x \in \omega^\omega : L[S,x] \models \phi(S,x)\}$ on a cone, for any binary formula ϕ.

Corollary 8.6.4. *Let S be a set of ordinals such that $\leq_S$ is locally countable, μ_S is a countably complete ultrafilter on $\mathcal{D}_S$ and $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$ is wellfounded. Let $G \subseteq \mathbb{P}_S$ be a V-generic filter and let x_G be the corresponding generic element of ω^ω. Then the following hold.*

1. *The ultraproduct $\prod_{x\in\omega^\omega} L[S,x]/G$ is isomorphic to $L[S^\infty,x_G]$.*
2. *$L[S^\infty,x_G] \models \Diamond_{\omega_1}$*
3. *For all $\gamma < j_G(\omega_1^V)$ which are infinite cardinals of $L[S^\infty,x_G]$,*
$$L[S^\infty,x_G] \models 2^\gamma = \gamma^+.$$
4. *$\omega_1^V = \omega_1^{L[S^\infty,x_G]}$*
5. *$\delta_S^\infty = \omega_2^{L[S^\infty,x_G]}$*
6. *$(\omega^\omega)^V \subseteq L[S^\infty,x_G]$*

7. *If $A \subseteq \omega^\omega$ is such that S computes A on a cone, then the function sending each $x \in \omega^\omega$ to $A \cap L[S,x]$ represents a set $A^* \subseteq (\omega^\omega)^{V[G]}$ in $L[S^\infty, x_G]$ such that $A^* \cap (\omega^\omega)^V = A$.*

Proof. For part (1), the fact that the identity function on ω^ω represents x_G shows that $\prod_{x\in\omega^\omega} L[S,x]/G$ is isomorphic to $L[j_G(S), x_G]$. Corollary 8.6.2 implies that this is the same as $L[S^\infty, x_G]$. Parts (2) and (3) follow from Remark 8.6.3 and Theorem 8.1.3. Part (4) follows from Theorems 8.6.1 and 8.1.2, and Remark 8.6.3. Part (5) follows from Theorem 8.6.1 and Remark 8.6.3. Parts (6) and (7) follow from Theorem 8.6.1. □

8.6.5 Remark. Let S be a set of ordinals such that μ_S is a countably complete ultrafilter on $\mathcal{D}_S$ and $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$ is wellfounded. For all $x, y \in \omega^\omega$, $y \in L[S^\infty, x]$ if and only if $y \in L[S,x]$ for an S-cone of z, i.e., if and only if $y \in L[S,x]$. So $\leq_S$ and $\leq_{S^\infty}$ are the same relation.

The following theorem will be used in Sections 11.2 and 11.4. The last part of the theorem uses the notion of ∞-Borel code as defined in Section 9.1.

Theorem 8.6.6 ($\mathsf{ZF} + \mathsf{AD}$). *Let S be a set of ordinals such that $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$ is wellfounded. Suppose that $A \subseteq \omega^\omega$ is such that S computes A on a cone. Then $A \in L(S^\infty, \mathbb{R})$. In particular, if S is a $\leq_{\mathcal{D}}$-maximal set of ordinals, then every ∞-Borel set is in $L(S^\infty, \mathbb{R})$.*

Proof. We show that $A \in L(S^\infty, \mathbb{R})[x_G]$ whenever x_G is V-generic for $\mathbb{S}_{S^\infty}$; the theorem then follows by Theorem 8.3.7. By Remark 8.6.5, the partial orders $\mathbb{S}_S$ and $\mathbb{S}_{S^\infty}$ are the same, and by Theorem 8.4.2 they are forcing-equivalent to $\mathbb{P}_S$. It follows that if x_G is V-generic for $\mathbb{S}_{S^\infty}$ then it is V-generic for $\mathbb{P}_S$, and therefore that there is an $A^* \in L[S^\infty, x_G]$ such that $A^* \cap (\omega^\omega)^V = A$. Since $(\omega^\omega)^V \in L(S^\infty, \mathbb{R})[x_G]$ we are done. The second part of the theorem follows from the first part, and part (1b) of Theorem 8.3.8.

□

By Theorem 9.2.4, the ∞-Borel sets are all ∞-Borel in $L(S^\infty, \mathbb{R})$, assuming the hypotheses of Theorem 8.6.6.[1]

We say that S is *strongly maximal* if it computes each subset of ω^ω on a cone.

8.6.7 Remark. Suppose that AD holds. Let S be a set of ordinals such that the corresponding ultrapower $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$ is wellfounded. Let x_G be V-generic for $\mathbb{S}_S$. The partial order $\mathbb{S}_S$ is a surjective image of ω^ω (in V), so forcing with $\mathbb{S}_S$ preserves Θ^V as a cardinal. Since $L[S^\infty, x_G]$ is an inner model of $V[G]$, Θ^V is a cardinal in $L[S^\infty, x_G]$. Since $\omega_1^{L[S^\infty, x_G]} = \omega_1^V < \Theta^V$, it must be that $\Theta^V \geq \omega_2^{L[S^\infty, x_G]}$, which is equal to δ_S^∞, by Corollary 8.6.4. Similarly, it

[1] Do we need this? : Similarly, if AD holds and $A \subseteq \omega^\omega$ is not ∞-Borel, then every ∞-Borel set is ∞-Borel in $L(A, \mathbb{R})$ by Theorem 9.2.4, and one can apply the theorem in $L(S, A, \mathbb{R})$ assuming (in addition) only AD. In this case, however, the S^∞ in the statement of the theorem is as computed in $L(S, A, \mathbb{R})$.

follows from Corollary 8.6.4 that if $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$ is wellfounded and S is strongly maximal then each prewellordering of ω^ω in V is a subset of a prewellordering of $(\omega^\omega)^{L[S^\infty, x_G]}$ in $L[S^\infty, x_G]$. Since $L[S^\infty, x_G]$ satisfies CH, this implies that $\delta_S^\infty \geq \Theta$.

Chapter 9

∞-Borel sets

In Section 9.1 we introduce two equivalent notions of ∞-Borel code. In Section 9.2 we show that every ∞-Borel set A has an ∞-Borel code which is projective in A (Theorem 9.2.4).

9.1 ∞-Borel codes

Given infinite ordinals γ andδ, we let $\mathcal{L}^0_{\gamma,\delta}$ be the language with propositional variables P_α ($\alpha \in \delta$), closed under negation as well as conjunctions and disjunctions indexed by members of γ. We associate to each sentence in $\mathcal{L}^0_{\gamma,\delta}$ a unique code (in $\mathcal{P}(\gamma)$, if $\delta \leq \gamma$ and γ is a cardinal) via a definable (injective) pairing function $\prec \cdot,\cdot \succ$ on the ordinals (as defined in Section 0.2), and write ϕ_S for the sentence coded by the set S. Fixing one such coding, we let

- $\phi_{\{\prec 0,\alpha\succ\}}$ be P_α for each $\alpha \in \delta$,
- $\phi_{\{\prec 1,\alpha\succ:\alpha\in S\}}$ be $\neg\phi_S$, and
- $\phi_{\{\prec 2+\alpha,\beta\succ:\alpha<\eta,\beta\in S_\alpha\}}$ be $\bigwedge_{\alpha<\eta}\phi_{S_\alpha}$.

We let $\mathcal{L}^0_{\infty,\delta}$ be the class of sentences which are in $\mathcal{L}^0_{\gamma,\delta}$ for some infinite ordinal γ. Each $x \in 2^\delta$ can be thought of as an $\mathcal{L}^0_{\infty,\delta}$-structure, where, for each $\alpha \in \delta$, P_α is interpreted as true if and only if $x(\alpha) = 1$. For a sentence ϕ of $\mathcal{L}^0_{\infty,\delta}$, we let A_ϕ be the set of $x \in 2^\delta$ such that $x \models \phi$, and say that ϕ is an ∞-*Borel code* (or a κ-Borel code, if $\phi \in \mathcal{L}^0_{\kappa,\delta}$) for A_ϕ. We also say that a set of ordinals S is an ∞-Borel code or κ-Borel code if the corresponding formula ϕ_S is.

9.1.1 Definition. Given an infinite cardinal κ, a subset of 2^δ is κ-*Borel* if it is equal to A_ϕ for some $\phi \in \mathcal{L}^0_{\kappa,\delta}$, and $<\kappa$-*Borel* if it is γ-Borel for some cardinal $\gamma < \kappa$. A subset of 2^δ is ∞-*Borel* if it is κ-Borel for some infinite ordinal κ.

9.1.2 Remark. For each infinite cardinal κ, the class of κ-Borel subsets of 2^ω is closed under continuous preimages, and thus Wadge reducibility. To see this, fix $\phi \in \mathcal{L}^0_{\kappa,\omega}$ and a continuous function $f\colon 2^\omega \to 2^\omega$. To express "$f(x) \models \phi$" it suffices to find, for each $n \in \omega$, an expression for "$f(x)(n) = 1$", since then one can form a ψ with $A_\psi = f^{-1}[A_\phi]$ by replacing each instance of each P_n in ϕ with the corresponding expression. The existence of these expressions follows from the continuity of f (and, in the case $\kappa = \omega$, the compactness of 2^ω).

The following example shows that Suslin subsets of 2^ω are ∞-Borel (this also follows easily from item (2) of Theorem 9.1.6 below). Given a tree T and a node $t \in T$, we let T_t (for this example) be the set of s such that the concatenation $t^\frown s$ is in T. We define $\mathrm{rank}(T)$ to be 0 if T is empty or $T = \{\emptyset\}$. Otherwise, $\mathrm{rank}(T)$ is the strict supremum of the set $\{\mathrm{rank}(T_t) : t \in T,\ |t| = 1\}$.

9.1.3 Remark. Let T be a tree on $(2 \times \gamma)$, for some ordinal γ. Given a node $(s,t) \in T$ and an $x \in 2^\omega$, let $T_{x,s,t}$ denote the set of $(s',t') \in 2^{<\omega} \times \gamma^{<\omega}$ for which $s^\frown s'$ is an initial segment of x and $(s^\frown s', t^\frown t') \in T$. For such a T one can define recursively on $\alpha \leq |\gamma|^+$ formulas $\phi^\alpha_{(s,t)} \in \mathcal{L}^0_{|\gamma|^+,\omega}$, simultaneously for each $(s,t) \in 2^{<\omega} \times \gamma^{<\omega}$, such that $A_{\phi^\alpha_{(s,t)}}$ is the set of $x \in 2^\omega$ for which $\mathrm{rank}(T_{x,s,t}) = \alpha$. The projection of T is then defined by $\bigwedge_{\alpha<|\gamma|^+} \neg\phi^\alpha_{\langle\emptyset,\emptyset\rangle}$.

9.1.4 Remark. There is in general no method for picking an ∞-Borel code for each ∞-Borel set. To see this, let $\mathbb{E}_0$ be the equivalence relation of mod-finite agreement on 2^ω, and suppose that every subset of 2^ω has the property of Baire. If F were a function picking for each $\mathbb{E}_0$-class a code for the class, then by the Kuratowski-Ulam theorem (which among other things implies that if every set of reals has the property of Baire then every wellordered union of meager sets is meager) comeagerly many reals would have the same code for their class.

9.1.5 Remark. The collection of ∞-Borel subsets of 2^ω is not necessarily closed under wellordered unions, even assuming DC. For example, let $\{x_\alpha : \alpha < \omega_1\}$ be mutually generic Cohen reals over L, and consider the model $L(\omega^\omega, F)$, where ω^ω is the set of all reals in the forcing extension, and F is the function taking $\alpha < \omega_1$ to the $\mathbb{E}_0$-class of x_α. This model satisfies $\mathsf{DC}_\mathbb{R}$ sicne it contains all the reals of the forcing extension, which satisfies AC. It then satisfies DC, as in Remark 0.4.2. The set $\bigcup_{\alpha<\omega_1} F(\alpha)$ cannot be ∞-Borel in this model, even though it is a wellordered union of countable sets. To see this, note first that any ∞-Borel code for this set would be ordinal definable from F and finitely many reals. These finitely many reals would appear in $L[\{x_\alpha : \alpha < \beta\}]$ for some $\beta < \omega_1$. The homogeneity of the rest of the forcing would then imply that the ∞-Borel code for the union would be in this model. This is impossible, since for instance if ω_1^L is countable it is possible to build two generic filters for the remainder of the forcing for which the resulting model $L(\omega^\omega, F)$ is the same, and such that some real x is equal to x_β in one extension and not a member of any $F(\alpha)$ in the other. For instance, starting with one such generic, replace x_β with $\{(2n, x_\beta(n) : n \in \omega\} \cup \{(2n+1, x_{\beta+1}(n) : n \in \omega\}$ and each $x_{\beta+1+\alpha}$ with $x_{\beta+2+\alpha}$. (This example was constructed by the author in collaboration with

Trevor Wilson, although it was very likely noticed earlier. A similar example can be constructed adding countably many Cohen reals; in this case $\mathsf{DC}_{\mathbb{R}}$ fails.)

Theorem 9.1.6 gives a number of convenient reformulations of the notion of ∞-Borel set. Item (2) below is especially useful.

Given ordinals α and δ, $x \in 2^{\delta}$ and $C \subseteq \alpha^{\omega} \times 2^{\delta}$, let $G(\alpha, x, C)$ be the length-ω game on α such that player I wins if $(\vec{\beta}, x) \in C$, where $\vec{\beta}$ is the sequence produced by the run of the game. The notion of closure in parts (3) and (4) below refers to the product topology on both α^{ω} and 2^{δ}.

Theorem 9.1.6 (ZF). *Let δ be an ordinals. The following are equivalent, for a set $A \subseteq 2^{\delta}$.*

1. *A is ∞-Borel.*
2. *For some set of ordinals S and some first-order formula θ,*
$$A = \{x \in 2^{\omega} : L[S, x] \models \theta(S, x)\}.$$
3. *For some ordinal α and some clopen $C \subseteq \alpha^{\omega} \times 2^{\delta}$, A is the set of $x \in 2^{\omega}$ for which Player I has a winning strategy in $G(\alpha, x, C)$.*
4. *For some ordinal α and some closed $C \subseteq \alpha^{\omega} \times 2^{\delta}$, A is the set of $x \in 2^{\omega}$ for which Player I has a winning strategy in $G(\alpha, x, C)$.*

Proof. The implication from (1) to (2) follows from the fact that sentences in $\mathcal{L}^{0}_{\infty,\delta}$ can be coded by sets of ordinals, and the definability of the relation $x \models \phi$. For the reverse direction, for each ordinal α, let

- $\mathcal{L}^{*}_{\alpha}$ be the extension of the first-order language of set theory given by adding predicate symbols $\dot{S}$, $\dot{x}$ and constant symbols $\dot{\beta}$ for each $\beta \in \alpha$ and
- T_{α} denote the collection of $\mathcal{L}^{*}_{\alpha}$ sentences of the form
$$\forall x_1 \ldots \forall x_n((x_1 = X^{\beta_1}_{\phi_1} \wedge \ldots \wedge x_n = X^{\beta_n}_{\phi_n}) \rightarrow \psi)$$
  where
  - each β_i is in α;
  - each ϕ_i is a unary formula in the corresponding $\mathcal{L}^{*}_{\beta_i}$;
  - each expression of the form $x_i = X^{\beta_i}_{\phi_i}$ denotes the formula saying that x_i is the set of sets satisfying ϕ_i in the structure
  $$(L_{\dot{\beta}_i}[\dot{S}, \dot{x}]; \dot{S}, \dot{x}, \dot{\delta}\,(\delta \in d_i), \in),$$
    where d_i is the set of symbols from $\{\dot{\delta} : \delta < \beta_i\}$ appearing in ϕ_i;
  - ψ is a $\mathcal{L}^{*}_{\alpha}$ formula with free variables $x_1, \ldots, x_n$ (so each sentence of $\mathcal{L}^{*}_{\alpha}$ is itself an element of T_{α}).

Since each expression of the form $x_i = X_{\phi_i}$ above asserts that x_i is defined by ϕ_i, each sentence in T_α holds in the same structures as the corresponding sentence

$$\exists x_1 \ldots \exists x_n (x_1 = X^{\beta_1}_{\phi_1} \wedge \ldots \wedge x_n = X^{\beta_n}_{\phi_n} \wedge \psi).$$

We refer to this below as the *existential form* of the sentence.

Working (in $L[S]$) by recursion on α, we describe a procedure which associates to each sentence $\theta \in T_\alpha$ a sentence $\rho_{\theta,\alpha}$ of $\mathcal{L}^0_{|\omega\cup\alpha|,\delta}$ such that for all $x \in 2^\delta$, $L_\alpha[S,x] \models \theta$ (with the symbols $\dot\beta$ ($\beta < \alpha$), $\dot S$ and $\dot x$ given their natural interpretations) if and only if $x \models \rho_{\theta,\alpha}$. For finite α this follows from the fact that the $\mathcal{L}^*_\alpha$-theory of $(L_\alpha[S,x]; S, x, \beta\,(\beta \in \alpha), \in)$ depends only on $x\restriction\alpha$ (as S is fixed) so the desired $\rho_{\theta,\alpha}$ can consist of a disjunction of conjunctions describing $x\restriction\alpha$ exactly (and since there is a canonical finite set of such sentences our procedure can pick the least one in a suitable ordering).

For the limit and successor cases, we induct on the complexity of ψ in the representation of the members of T_α given above. For limit α, the induction hypothesis gives us the desired sentences $\rho_{\theta,\alpha}$ when ψ is a Δ_0 formula, and the steps corresponding to $\wedge$, $\vee$ and $\neg$ are handled by combining the formulas $\rho_{\theta,\alpha}$ in the same manner (to see that this works in the case of $\vee$ and $\neg$, use the fact that each sentence in T_α has an equivalent existential form). If ψ has the form $Qx\psi'$, for Q either $\forall$ or $\exists$, we let $\rho_{\theta,\alpha}$ be the conjunction (when $Q = \forall$) or the disjunction (when $Q = \exists$) of all the sentences $\rho_{\theta^\beta_\phi,\alpha}$, where $\beta < \alpha$, ϕ is a unary formula in $\mathcal{L}^*_\beta$ and θ^β_ϕ is formed from θ by adding $\forall x$ to the beginning of θ and $x = X^\beta_\phi$ at the appropriate place. The point here is that $\alpha \times \mathcal{L}^*_\alpha$ is wellorderable, and each set in the range of the quantifier Q is defined by some formula in $\bigcup_{\beta<\alpha} \mathcal{L}^*_\beta$.

The successor step from α to $\alpha + 1$ is similar, except that, given a sentence $\theta \in T_{\alpha+1}$ as above, with ψ a Δ_0 formula, we must deal with the possibility that some of the formulas ϕ_i define subsets of $L_\alpha[S,x]$, so that we cannot simply apply the induction hypothesis. Each such $T_{\alpha+1}$ sentence θ is equivalent to a sentence θ' in T_α, formed by moving the introduction of the corresponding variables and their definitions into the ψ part of θ'. For instance, if $x_i = X^\alpha_{\phi_i}$ appears in θ, then it does not appear and θ', and ψ' (the ψ part of θ') asserts about the set of things satisfying ϕ_i in

$$(L_{\dot\alpha}[\dot S, \dot x]; \dot S, \dot x, \dot\delta\,(\delta \in d_i), \in)$$

what ψ says about x_i (this involves rewriting the atomic formulas of ψ). The formula ψ may also need to be rewritten to deal with the predicates for S and x, resulting (if $\alpha < \delta$) in formulas ψ_0 and ψ_1 to accommodate the two possible values of $x(\alpha)$. Letting θ_0 and θ_1 be the two rewritten forms of θ, we can let $\rho_{\theta,\alpha+1}$ be $(P_\alpha \wedge \rho_{\theta_0,\alpha}) \vee ((\neg P_\alpha) \wedge \rho_{\theta_1,\alpha})$. The rest of this step is the same as the limit case.

To complete the proof, fix an ordinal $\gamma > \delta \cup \sup(S)$ such that for all $x \in 2^\delta$, $L_\gamma[S,x] \models \theta(S,x)$ if and only if $L[S,x] \models \theta(S,x)$, and note that there is an element of T_γ which, for each $x \in 2^\delta$, holds in the expanded version of $L_\gamma[S,x]$ if and only if $L[S,x] \models \theta(S,x)$.

That (3) implies (4) is immediate. The implication from (4) to (2) follows from the absoluteness of the existence of winning strategies in closed games (via a ranking function, as in [3]). To get from (1) to (3), consider a witness ϕ to (1) as a tree. Players I and II choose a path from the root of the tree to a terminal node, with I choosing at disjunctions, II choosing at conjunctions, and I winning if and only if $x(\gamma) = 1$, for P_γ the terminal node reached by the run of the game. If $x \models \phi$, then I can play to maintain that x satisfies each sub-sentence visited during the run of the game, and conversely for II. $\square$

9.1.7 Remark. The construction of the $\mathcal{L}^*_{\infty,\delta}$-sentence in the implication from (2) to (1) in the proof of Theorem 9.1.6 is carried out in $L[S]$, aside from the choice of γ.

9.1.8 Remark. Part (2) of Theorem 9.1.6 makes sense for subsets of spaces of the form ω^δ and their products (or sets of subsets of any set in L), so we will say that a set $A \subseteq (\omega^\delta)^n$ (for some $n \in \omega$ and some ordinal δ) is ∞-*Borel* is there exists a set S of ordinals such that $A = \{x \in (\omega^\delta)^n : L[S,x] \models \phi(S,x)\}$, for some binary formula ϕ. One could also revise the infinitary language $\mathcal{L}^0_{\infty,\delta}$ above to describe (for instance) subsets of ω^δ, starting with variables $P_{\alpha,m}$ ($\alpha \in \delta, m \in \omega$) interpreted so that $x \models P_{\alpha,m}$ if and only if $x(\alpha) = m$. Under this formulation everything goes through as above (trivially, as we are merely relabeling the set of propositional variables). In this sense, however, ω^ω is $\aleph_1$-Borel as a subset of $\mathcal{P}(\omega \times \omega)$, but not $\aleph_0$-Borel. When we describe subsets of $(\omega^\omega)^n$ in this revised sense (i.e., when we say that some set of ordinals is an ∞-Borel code for a subset of $(\omega^\omega)^n$) then we restrict our attention to κ-Borel sets for uncountable κ.

9.1.9 Definition. Given a set of ordinals S and a binary formula ϕ from the first order language of set theory, we say that the pair (S,ϕ) is an ∞-*Borel** code for the set

$$\{x \in (\omega^\delta)^n : L[S,x] \models \phi(S,x)\}.$$

If in addition S is a subset of an ordinal γ, we say that the pair (S,ϕ) is a γ-*Borel** code for the set

$$\{x \in (\omega^\delta)^n : L_\gamma[S,x] \models \phi(S,x)\}.$$

9.1.10 Remark. For any infinite cardinal γ, every γ-Borel subset of 2^ω has a γ-Borel* code, using the coding of sentences in $\mathcal{L}^0_{\gamma,\omega}$ by sets of ordinals introduced at the beginning of this section. The proof of (1) from (2) in Theorem 9.1.6 contains a reflection argument, so it does not immediately give a bound in the reverse direction.

9.1.11 Remark. If $A \subseteq \omega^\omega$ is ∞-Borel, then this is witnessed by a pair (S,ϕ) with S a bounded subset of Θ. This can be shown by taking a continuum-sized elementary submodel (containing ω^ω) of a suitable model of the form $L_\alpha(T,\mathbb{R})$, where (T,ϕ) is a given ∞-Borel code for A. Since every element of $L_\alpha(T,\mathbb{R})$ is definable in $L_\alpha(T,\mathbb{R})$ from T, an ordinal and an element of ω^ω, such a submodel can be built assuming only ZF.

9.1.12 Remark. If $A \subseteq \omega^\omega$ is ∞-Borel, and $\aleph_1 \not\leq 2^{\aleph_0}$ holds, then A has the property of Baire and is Lebesgue measurable. To see this for the Baire property, let $\mathbb{P}$ be Cohen forcing and let $\dot{c}$ be the canonical $\mathbb{P}$-name for the generic Cohen real. Suppose that (S,ϕ) is an ∞-Borel* code for A. Since $\aleph_1 \not\leq 2^{\aleph_0}$ and $L[S] \models \mathsf{AC}$, $\omega^\omega \cap L[S]$ is countable, so the set of reals which are Cohen-generic over $L[S]$ is comeager. For each such real x, let g_x be the filter for which $\dot{c}_{g_x} = x$. Let A' be the set of x which are Cohen-generic over $L[S]$ for which there is a condition $p \in g_x$ forcing that $L[S,\dot{c}] \models \phi(S,\dot{c})$. Then A' is Borel, and the symmetric difference $A \triangle A'$ is contained in the set of reals which are not Cohen-generic over $L[S]$. For Lebesgue measurability the argument is the same, using random forcing instead of Cohen forcing. One can use the same argument with Mathias forcing to show that ∞-Borel subsets of $\mathcal{P}(\omega)$ satisfies the Ramsey property. Unlike the cases of the Baire property and Lebesgue measurability, it is still unknown whether AD alone implies that every subset of $\mathcal{P}(\omega)$ has the Ramsey property.

9.1.13 Example. One can define, by recursion on α, $\mathcal{L}^0_{\infty,\omega}$-sentences $\phi_{T,\alpha}$, for all (set-sized) trees T on $\omega \times \mathrm{Ord}$, in such a way that each $A_{\phi_{T,\alpha}}$ is the set of $x \in \omega^\omega$ for which the tree $T_x = \{t \in \mathrm{Ord}^{<\omega} : (x\restriction|t|, t) \in T\}$ has rank at most α. Starting with the standard tree on $\omega \times \omega$ whose projection is the set of (codes for, under some bijection $\pi \colon \omega \to \omega \times \omega$) illfounded linear orders on ω, this operation gives a sequence of $\mathcal{L}^0_{\aleph_1,0}$-sentences $\langle \phi_\alpha : \alpha < \omega_1 \rangle$ such that each A_{ϕ_α} is the set of codes for wellorders of ω of ordertype α.

9.1.14 Remark. It follows from Theorem 9.1.6 that Suslin subsets of 2^ω are ∞-Borel. Theorem 6.2.1 says that Suslin sets can be uniformized. The proof of that theorem shows that if γ is an ordinal and T is a tree on $\omega \times \gamma$ such that $\mathrm{p}[T]$ is nonempty, then $\mathrm{p}[T]$ has a member in $L[T]$. However, it is possible to have a $\phi \in \mathcal{L}^0_{\infty,0}$ such that A_ϕ is nonempty in V but empty in $L[\phi]$. For instance, there is a ϕ in L such that A_ϕ is the set of Cohen reals over L. Another way to see this is to note that consistently there are ∞-Borel subsets of $(\omega^\omega)^2$ which are not uniformized (for instance, in $L(\mathbb{R})$, assuming AD, by Corollary 9.2.6 and Theorem 6.2.5) whereas if S were an ∞-Borel code for a set $A \subseteq (\omega^\omega)^2$ and $L[S,x] \cap A$ were nonempty for each x, then a uniformizing function could be found by picking the constructibly least member of each set $L[S,x] \cap A$ relative to S and x.

We will make use of the following lemma in Section 11.4.

Theorem 9.1.15 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}_\mathbb{R}$). *Suppose that κ is a limit of Suslin cardinals and that κ has uncountable cofinality. Then the following are equivalent for each $A \subseteq \omega^\omega$.*

- *A is weakly-$<\kappa$-Borel.*
- *A is $<\kappa$-Borel.*

- *A is $<\kappa$-Suslin.*

Proof. For any cardinal λ, λ-Borel sets are weakly λ-Borel, and, by either Remark 9.1.3, λ-Suslin sets are λ^+-Borel. Theorem 6.1.20 says that, under $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, if κ is a limit of Suslin cardinals of uncountable cofinality, then all weakly $<\kappa$-Borel sets are $<\kappa$-Suslin. □

In conjunction with Remark 2.5.9 (on Solovay's diagonalization method for building sets of large Wadge rank), Theorems 6.1.17 and 6.1.20 give the following.

Theorem 9.1.16 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *If κ is a limit of Suslin cardinals of uncountable cofinality, then $o(\mathcal{S}_{<\kappa}) = \delta(\mathcal{S}_{<\kappa}) = \kappa$.*

Proof. That $\delta(\mathcal{S}_{<\kappa}) = \kappa$ follows from Theorem 6.1.17. Since $\mathcal{S}_{<\kappa}$ is a projective algebra, $o(\mathcal{S}_{<\kappa}) \leq \delta(\mathcal{S}_{<\kappa})$, by Proposition 2.5.8. Since in addition (by Theorem 9.1.15) it is closed under unions and intersections of length less than κ, Remark 2.5.9 gives that $o(\mathcal{S}_{<\kappa}) \geq \kappa$. □

9.2 Local ∞-Borel codes

Given a set $A \subseteq \omega^\omega$, let δ_A denote the supremum of the lengths of the prewellorderings P of ω^ω such that either $P \leq_{\mathrm{Wa}} A$ or $P \leq_{\mathrm{Wa}} \check{A}$. One can natually form a wellfounded relation isomorphic to the disjoint union of the set of all such prewellordering. For instance, recalling from Remark 2.5.2 that $\mathcal{F}^{\mathrm{c},2,1}$ denotes the set of $x \in \omega^\omega$ such that $f^{\mathrm{c},2,1}_x$ is a continuous function from $(\omega^\omega)^2$ to ω^ω (relative to a fixed universal closed subset of $(\omega^\omega)^4$), let $\leq_A$ be the relation on $(\omega^\omega)^2$ defined by setting $(x_0, y_0) \leq_A (x_1, y_1)$ to hold if $x_0 = x_1$, $(f^{\mathrm{c},2,1}_{x_0})^{-1}[A]$ is a prewellordering and

$$(y_0, y_1) \in (f^{\mathrm{c},2,1}_{x_0})^{-1}[A].$$

Assuming $\mathsf{Lipschitz\ Determinacy} + \mathsf{DC}_{\mathbb{R}}$, $\leq_A$ is a wellfounded preorder projective in A, and the rank of $\leq_A$ is δ_A. If μ_{Tu} is an ultrafilter on the Turing degrees, then, by Theorem 8.0.3, $\delta_A < \delta_\omega(A)$. In this section we show (assuming $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$) that if $A \subseteq 2^\omega$ is ∞-Borel then it is δ_A-Borel.

We define the *rank* $\mathrm{rk}(\phi)$ of a sentence in $\mathcal{L}^0_{\infty,\delta}$ (for some ordinal δ) by setting

- $\mathrm{rk}(P_\alpha) = 0$ for each propositional variable P_α;
- $\mathrm{rk}(\neg\phi) = \mathrm{rk}(\phi)$ for all $\phi \in \mathcal{L}^0_{\infty,\delta}$;
- $\mathrm{rk}(\bigvee_{\alpha\in Y} \phi_\alpha) = \mathrm{rk}(\bigwedge_{\alpha\in Y} \phi_\alpha) = \sup\{\mathrm{rk}(\phi_\alpha) + 1 : \alpha \in Y\}$ for all sets $Y \subseteq \mathrm{Ord}$ and all formulas ϕ_α ($\alpha \in Y$) in $\mathcal{L}^0_{\infty,\delta}$,

For each $\phi \in \mathcal{L}^0_{\infty,\delta}$ we define the formula $\phi^* \in \mathcal{L}^0_{\infty,\delta}$ by recursively on the rank of ϕ as follows.

- $P^*_\alpha = P_\alpha$, for each propositional term P_α;

- $(\neg\phi)^* = \neg\phi^*$, for all $\phi \in \mathcal{L}^0_{\infty,\delta}$;
- for all sets $Y \subseteq \mathrm{Ord}$, and all $\mathcal{L}^0_{\infty,\delta}$-sentences ϕ_α ($\alpha \in Y$),
  - $(\bigvee_{\alpha\in Y}\phi_\alpha)^* = \bigvee_{\alpha\in X}\phi^*_\alpha$, where $X = \{\alpha \in Y : A_{\phi_\alpha} \not\subseteq \bigcup_{\beta\in\alpha\cap Y} A_{\phi_\beta}\}$;
  - $(\bigwedge_{\alpha\in Y}\phi_\alpha)^* = \bigwedge_{\alpha\in X}\phi^*_\alpha$, where $X = \{\alpha \in Y : A_{\phi_\alpha} \not\supseteq \bigcap_{\beta\in\alpha\cap Y} A_{\phi_\beta}\}$,

Then for all $\phi \in \mathcal{L}^0_{\infty,\delta}$, $A_\phi = A_{\phi^*}$, $\mathrm{rk}(\phi) \geq \mathrm{rk}(\phi^*)$ and $\phi^{**} = \phi^*$. Note that the operation $\phi \mapsto \phi^*$ merely removes from each conjunction and disjunction in ϕ those terms which have no effect on the corresponding intersection or union.

We define the following ordinals, all of which are at most Θ (and which, by Theorem 9.2.11, which in turn uses a deeper theorem of Woodin to be proved later, are all the same under $\mathsf{AD} + \mathsf{DC}_\mathbb{R}$):

- χ_B, the least χ such that all ∞-Borel subsets of 2^ω are χ-Borel;
- κ_B, the supremum of the lengths of wellordered sequences of ∞-Borel codes for nonempty disjoint subsets of 2^ω;
- λ_B, the supremum of the lengths of the ∞-Borel prewellorderings of 2^ω;
- ρ_B, the supremum of the Wadge ranks of ∞-Borel subsets of 2^ω.

The existence of analytic (and therefore ∞-Borel) sets which are not Borel shows that $\chi_\mathrm{B} > \aleph_0$.

Theorem 9.2.1 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_\mathbb{R}$). $\chi_\mathrm{B} \leq \kappa_\mathrm{B} \leq \lambda_\mathrm{B}$

Proof. To see that $\chi_\mathrm{B} \leq \kappa_\mathrm{B}$, let ϕ be any sentence in $\mathcal{L}^0_{\kappa,\omega}$. For any disjunction $\bigvee_{\alpha<\gamma}\psi_\alpha$ in ϕ^* the sentences $\psi_\alpha \wedge \neg(\bigvee_{\beta<\alpha}\psi_\beta)$ are ∞-Borel codes for nonempty disjoint sets. A similar remark applies to the conjunctions. It follows that ϕ^* is in $\mathcal{L}^0_{\kappa_\mathrm{B},\omega}$.

To see that $\kappa_\mathrm{B} \leq \lambda_\mathrm{B}$, note that one can convert a sequence of ∞-Borel codes for nonempty disjoint sets to an ∞-Borel code for a prewellordering in a uniform fashion by means of uniformly coding the product. Consider $\langle\phi_\alpha : \alpha < \gamma\rangle$ with the sets A_{ϕ_α} disjoint and nonempty. The set

$$\bigcup_{\alpha\leq\beta<\gamma} A_{\phi_\alpha} \times A_{\phi_\beta}$$

is a prewellordering of length γ. If each ϕ_α is in $\mathcal{L}^0_{\xi,\omega}$, the resulting code is in $\mathcal{L}^0_{\max\{\xi,|\gamma|^+\},\omega}$. □

Since the collection of ∞-Borel subsets of ω^ω is closed under continuous preimages, $\rho_\mathrm{B} = \Theta$ if and only if all subsets of ω^ω are ∞-Borel. The same is true for λ_B (and, as well shall see, χ_B and κ_B).

Theorem 9.2.2 (ZF + $\mathsf{Lipschitz\ Determinacy}$ + $\mathsf{DC}_\mathbb{R}$). *The equation $\lambda_\mathrm{B} = \Theta$ holds if and only if every subset of ω^ω is ∞-Borel.*

Proof. By definition, Θ is the supremum of the lengths of the prewellorderings of ω^ω, so if every subset of ω^ω is ∞-Borel then $\lambda_{\mathrm{B}} = \Theta$. For the other direction, suppose that $A \subseteq \omega^\omega$ is not ∞-Borel and that P is an ∞-Borel prewellorder. Then $A \not\leq_W P$ as the ∞-Borel sets form an initial segment of the Wadge hierarchy. Thus either $P \leq_W A$ or $P \leq_W \omega^\omega \setminus A$, and in either case the length of P is less than δ_A, which is less than Θ. $\square$

Theorem 9.2.3 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *The following statements are equivalent.*

1. *All subsets of* ω^ω *are* ∞*-Borel.*
2. *At least one of* χ_{B}, κ_{B}, λ_{B} *and* ρ_{B} *is* Θ*.*
3. $\chi_{\mathrm{B}} = \kappa_{\mathrm{B}} = \lambda_{\mathrm{B}} = \rho_{\mathrm{B}} = \Theta$

Proof. Statement (2) follows immediately from statement (3). Theorem 9.2.2 implies that (1) implies (2) and (3) implies (1). By Theorems 9.2.1 and 9.2.2, and the remark on ρ_{B} before the statement of Theorem 9.2.2, (2) implies (1).

All that remains to be shown is that if all subsets of ω^ω are ∞-Borel then $\chi_{\mathrm{B}} = \Theta$. This follows from the Moschovakis Coding Lemma, which implies that for each $\chi < \Theta$ there is a surjection from ω^ω to $\mathcal{P}(\chi)$, and therefore (if χ is a cardinal, which χ_B is) one from ω^ω to the χ-Borel sets. $\square$

The main theorem in this section is the following.

Theorem 9.2.4 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *If* $A \subseteq 2^\omega$ *is* ∞*-Borel then* A *is* δ_A*-Borel.*

Proof. If $\delta_A \geq \chi_{\mathrm{B}}$ we are done. Supposing otherwise, using a sentence ϕ in $\mathcal{L}^0_{\infty,\omega}$ of minimal rank such that $\phi^* \notin \mathcal{L}^0_{\delta_A,\omega}$, we may find a sequence $\langle \phi_\alpha : \alpha \leq \delta_A \rangle$, consisting of $\mathcal{L}^0_{\delta_A,\omega}$ sentences, such that the corresponding sets A_{ϕ_α} are nonempty and pairwise disjoint. Then the prewellordering (call it $\leq$) built as in the second half of the proof of Theorem 9.2.1 (and then relabeled using a bijection between δ_A and $\delta_A + 1$) is δ_A-Borel, and, as it has length $\delta_A + 1$ (so greater than δ_A), it is not Wadge below either A or $\breve{A}$. By Wadge Determinacy, then, A is Wadge below $\leq$, which means that A is δ_A-Borel. $\square$

Suppose that $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ holds, and let $A \subseteq 2^\omega$ be ∞-Borel. By Proposition 2.5.8, there is a prewellordering of $\mathcal{F}^{\mathrm{c}}$ of length δ_A which is projective in A. By the Moschovakis Coding Lemma, every subset of δ_A is coded (using this prewellordering) by a set of reals projective in A. Putting this all together with Theorem 9.2.4 gives the following corollaries.

Corollary 9.2.5 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *If* $A \subseteq 2^\omega$ *is* ∞*-Borel then* $A = A_\phi$ *for some* $\phi \in \mathcal{L}^0_{\delta_A,\omega} \cap L(A, \mathbb{R})$.

Corollary 10.3.7 below is a stronger version of the following corollary, modulo the additional assumption of $\mathsf{DC}_{\mathbb{R}}$.

Corollary 9.2.6 ($\mathsf{ZF} + \mathsf{AD}$)**.** *In* $L(\mathbb{R})$*, every subset of* 2^ω *is* ∞*-Borel.*

Proof. We work in $L(\mathbb{R})$, noting that AD implies that $\mathsf{DC}_{\mathbb{R}}$ holds there by Theorem 0.4.1. By the Moschovakis Coding Lemma, the assertion that there is a subset of 2^ω which is not ∞-Borel is $\utilde{\Sigma}^2_1$. By the Solovay Basis Theorem (Theorem 5.1.6), this assertion, if true, is witnessed by a $\utilde{\Delta}^2_1$ set. By Theorem 6.1.21, in $L(\mathbb{R})$, all $\utilde{\Delta}^2_1$ sets are Suslin, and therefore ∞-Borel. $\square$

Combining Theorem 0.4.1 (for $\mathsf{DC}_{\mathbb{R}}$) and Remark 7.0.5 (for $<\Theta$-$\mathsf{Determinacy}$) with Corollary 9.2.6, we get that the implication from AD to AD^+ holds in $L(\mathbb{R})$.

Corollary 9.2.7 ($\mathsf{ZF} + \mathsf{AD}$)**.** $L(\mathbb{R}) \models \mathsf{AD}^+$.

A similar argument shows that AD^+ follows from the conjunction of AD with Σ^2_1-reflection into the Suslin, co-Suslin sets. Woodin has shown that the converse holds.

Corollary 9.2.8 ($\mathsf{ZF} + \mathsf{AD}$)**.** *If every true Σ^2_1 sentence is witnessed by a set which is Suslin and co-Suslin, then AD^+ holds.*

Proof. No counterexample to any of the three parts of AD^+ can be witnessed by a set which is Suslin and co-Suslin. For $\mathsf{DC}_{\mathbb{R}}$ this follows from the fact that Suslin sets can be uniformized, and for $<\Theta$-$\mathsf{Determinacy}$ it follows from Theorem 7.0.3. For the ∞-Borel property it follows from Theorem 9.1.6. The assertion that any of the three parts of AD^+ fails is a Σ^2_1 sentence. For $\mathsf{DC}_{\mathbb{R}}$ this is immediate. For $<\Theta$-$\mathsf{Determinacy}$ this follows from the Coding Lemma. For the ∞-Borel property this follows from the Coding Lemma and Theorem 9.2.4. $\square$

Remark 7.0.5 and Theorem 9.2.4 show that AD^+ reflects down to inner models containing the reals.

Theorem 9.2.9 ($\mathsf{ZF} + \mathsf{AD}^+$)**.** *If M is an inner model of ZF containing ω^ω, then $M \models \mathsf{AD}^+$.*

Proof. Since $\omega^\omega \subseteq M$, $M \models \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$. By Remark 7.0.5, M satisfies $<\Theta$-$\mathsf{Determinacy}$. By Theorem 9.2.4 (noting that the ordinals δ_A have the same value whether computed in M or V), every subset of ω^ω in M is ∞-Borel in M. $\square$

9.2.10 Remark. Woodin has shown that assuming $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$, for each ordinal $\gamma < \Theta$, all sets of reals in $L(\mathcal{P}(\gamma))$ are ∞-Borel. While we do not use this result in this book, we note the following consequence. Suppose that $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ holds, and that some set of reals is not ∞-Borel. Let λ be a strong partition cardinal above the Wadge rank of some set which is not ∞-Borel. Then, in $L(\mathcal{P}(\lambda))$, λ is a strong partition cardinal greater than Θ.

We conclude this section by showing that, assuming that every set of reals in $L(\mathcal{P}(\chi_{\mathrm{B}}))$ is ∞-Borel, the four ordinals defined above are all the same under $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$.

Theorem 9.2.11 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *If every subset of ω^ω in $L(\mathcal{P}(\chi_{\mathrm{B}}))$ is ∞-Borel, then*
$$\chi_{\mathrm{B}} = \kappa_{\mathrm{B}} = \lambda_{\mathrm{B}} = \rho_{\mathrm{B}} = \Theta^{L(\mathcal{P}(\chi_B))}.$$

Proof. It follows from the assumption that every subset of ω^ω in $L(\mathcal{P}(\chi_{\mathrm{B}}))$ is ∞-Borel that the ∞-Borel subsets of ω^ω are exactly the subsets of ω^ω in $L(\mathcal{P}(\chi_{\mathrm{B}}))$, all of which are ∞-Borel in $L(\mathcal{P}(\chi_{\mathrm{B}}))$. It follows that $\chi_{\mathrm{B}} = \chi_{\mathrm{B}}^{L(\mathcal{P}(\chi_{\mathrm{B}}))}$, $\lambda_{\mathrm{B}} = \lambda_{\mathrm{B}}^{L(\mathcal{P}(\chi_{\mathrm{B}}))}$ and $\rho_{\mathrm{B}} = \rho_{\mathrm{B}}^{L(\mathcal{P}(\chi_{\mathrm{B}}))}$. By Theorem 9.2.3,
$$L(\mathcal{P}(\chi_{\mathrm{B}})) \models \chi_{\mathrm{B}} = \kappa_{\mathrm{B}} = \lambda_{\mathrm{B}} = \rho_{\mathrm{B}} = \Theta.$$

The theorem (i.e., the remaining case κ_{B}) then follows from Theorem 9.2.1, applied in V. $\square$

Chapter 10

Vopěnka algebras

Recall that for a set X, OD_X is the class of all sets which are definable from a finite sequence from $X \cup \mathrm{Ord}$. We let HOD_X denote the class of all sets in OD_X whose transitive closures are contained in OD_X.

A proof of Theorem 10.0.1 below is given in Section 5.2 of [35]. Note that the members of X are not necessarily in $\mathrm{HOD}_{\{X\}}$. This can happen for instance if X is an uncountable subset of ω^ω.

Theorem 10.0.1 (Gödel)**.** *If X is a set such that $X \in \mathrm{OD}_X$, then HOD_X is a model of* ZF*. If in addition there is a wellordering of X in OD_X, then HOD_X is a model of* ZFC*.*

Vopěnka's theorem (see page 249 of [8]) says that every set of ordinals is set-generic over HOD. Theorem 10.1.2 below is a relativized version of this fact. Relativizing to the sets $\bar{x}$ and Y there requires only minor modifications of the original proof. The rest of this chapter contains variations of Vopěnka's theorem due to Woodin.

10.1 The Vopěnka algebra

Suppose that $\bar{x}$ is a finite sequence, and Y is a set in $\mathrm{OD}_{\bar{x}}$. Let κ be the least cardinal λ such that $\bar{x} \in V_\lambda$ and every subset of $\mathcal{P}(Y)$ in $\mathrm{OD}_{\bar{x}}$ is in $\mathrm{OD}_{\bar{x}}^{V_\lambda}$. Let $P_{\bar{x},Y}$ be the set of pairs $(n, \bar{\alpha})$ such that

- n is the Gödel number of a ternary formula ϕ,
- $\bar{\alpha}$ is a finite sequence of elements of κ and
- the set $B_{n,\bar{\alpha}} = \{C \subseteq Y : V_\kappa \models \phi(C, \bar{x}, \bar{\alpha})\}$ is nonempty.

Then $P_{\bar{x},Y}$ is in $\mathrm{HOD}_{\bar{x}}$, as is the reflexive and transitive ordering $\leq_{\bar{x},Y}$ on $P_{\bar{x},Y}$ defined by setting $(n, \bar{\alpha}) \leq_{\bar{x},Y} (m, \bar{\beta})$ to hold when $B_{n,\bar{\alpha}} \subseteq B_{m,\bar{\beta}}$. We let $\equiv_{\bar{x},Y}$ be the induced equivalence relation on $P_{\bar{x},Y}$, and for each $(n, \bar{\alpha}) \in P_{\bar{x},Y}$, let $[n, \bar{\alpha}]_{\bar{x},Y}$ denote the corresponding equivalence class. We let $\mathbb{V}_{\bar{x},Y}$ be the partial order whose domain is the set of $\equiv_{\bar{x},Y}$-classes of $P_{\bar{x},Y}$, with the order inherited from $\leq_{\bar{x},Y}$. We write $\mathbb{V}_Y$ (etc.) when $X = \emptyset$ and $\mathbb{V}_{\{S\},Y}$ (etc.) for $\mathbb{V}_{\langle S\rangle,Y}$.

10.1.1 Remark. There are natural modifications of the definition of $P_{\bar{x},Y}$ and $\mathbb{V}_{\bar{x},Y}$ just given to allow for the case where X is an arbitrary subset of HOD_X (which does not generalize the case under consideration here, since we are not requiring $\bar{x}$ to be in $\mathrm{HOD}_{\bar{x}}$). In this case the members of $P_{X,Y}$ can be triples $(n, \bar{x}, \bar{\alpha})$ with $\bar{x}$ a finite sequence from X, and the sets $B_{n,\bar{\alpha}}$ are replaced with sets $B_{n,\bar{x},\bar{\alpha}}$ of the form $\{C \subseteq Z : V_\kappa \models \phi(C, \bar{x}, \bar{\alpha})\}$. The analysis then goes through largely as we have here, except that the last paragraph of the proof of Theorem 10.1.2 is slightly more involved, since the corresponding model $\mathrm{HOD}_{X\cup\{E\}}$ may not satisfy Choice. This form of the Vopěnka algebra can be used to force over models of the form HOD_Γ, where Γ is an initial segment of the Wadge hierarchy.

Theorem 10.1.2. *Let $\bar{x}$ be a finite sequence, and let Y be a set in $\mathrm{HOD}_{\bar{x}}$. For each $E \subseteq Y$, there is a $\mathrm{HOD}_{\bar{x}}$-generic filter $G \subseteq \mathbb{V}_{\bar{x},Y}$ such that*

$$\mathrm{HOD}_{\bar{x}^\frown\langle E\rangle} = \mathrm{HOD}_{\bar{x}}[G].$$

Proof. Fix $E \subseteq Y$. If D is a subset of $P_{\bar{x},Y}$ and an element of $\mathrm{HOD}_{\bar{x}}$, then

$$\bigcup\{B_{n,\bar{\alpha}} : (n, \bar{\alpha}) \in D\}$$

is in $\mathrm{OD}_{\bar{x}}$. It follows that if this union is not all of $\mathcal{P}(Y)$, then there is a pair $(m, \bar{\beta})$ in $P_{\bar{x},Y}$ such that

$$B_{m,\bar{\beta}} = \mathcal{P}(Y) \setminus \bigcup\{B_{n,\bar{\alpha}} : (n, \bar{\alpha}) \in D\}.$$

From this it follows that the set of equivalences classes of pairs $(n, \bar{\alpha}) \in P_{\bar{x},Y}$ for which $E \in B_{n,\bar{\alpha}}$ is a $\mathrm{HOD}_{\bar{x}}$-generic filter for $\mathbb{V}_{\bar{x},Y}$. Let G_E denote this filter. The definition just given shows that G_E is in $\mathrm{HOD}_{\bar{x}^\frown\langle E\rangle}$ and therefore that $\mathrm{HOD}_{\bar{X}^\frown\langle G_E\rangle}$ is contained in $\mathrm{HOD}_{\bar{x}^\frown\langle E\rangle}$.

The set K consisting of those pairs $(y, (n, \bar{\alpha})) \in Y \times P_{\bar{x},Y}$ for which

$$B_{n,\bar{\alpha}} = \{C \subseteq Y : y \in C\}$$

is also a member of $\mathrm{HOD}_{\bar{x}}$. Since E is equal to

$$\{y \in Y : \exists\, [n, \bar{\alpha}]_{\bar{x},Y} \in G_E\, (\delta, (n, \bar{\alpha})) \in K\},$$

it follows that E is in $\mathrm{HOD}_{\bar{x}}[G_E]$. Since $\mathrm{HOD}_{\bar{x}}[G_E] \subseteq \mathrm{HOD}_{\bar{x}^\frown\langle G_E\rangle}$, it follows also that E is in $\mathrm{HOD}_{\bar{x}^\frown\langle G_E\rangle}$, $\mathrm{HOD}_{\bar{x}^\frown\langle E\rangle} = \mathrm{HOD}_{\bar{x}^\frown\langle G_E\rangle}$ and that each of these models contains $\mathrm{HOD}_{\bar{x}}[G_E]$.

To see that $\mathrm{HOD}_{\bar{x}^\frown\langle E\rangle} \subseteq \mathrm{HOD}_{\bar{x}}[G_E]$, it suffices to see that every set of ordinals in $\mathrm{HOD}_{\bar{x}^\frown\langle E\rangle}$ is in $\mathrm{HOD}_{\bar{x}}[G_E]$, since $\mathrm{HOD}_{\bar{x}^\frown\langle E\rangle}$ satisfies Choice. Fix then an ordinal ζ and a set $Q \subseteq \zeta$ in $\mathrm{HOD}_{\bar{x}^\frown\langle E\rangle}$. Fix a quaternary formula ϕ and a finite set of ordinals $\bar{\alpha}$ such that $Q = \{\xi \in \zeta : \phi(\xi, \bar{x}, \bar{\alpha}, E)\}$. Let T be

$$\{(\xi, (n, \bar{\beta})) \in \zeta \times P_{\bar{x},Y} : B_{n,\bar{\beta}} = \{C \subseteq Y : \phi(\xi, \bar{x}, \bar{\alpha}, C)\}\}.$$

Then T is in $\mathrm{HOD}_{\bar{x}}$, and Q, which is

$$\{\xi \in \zeta : \exists\,[n,\bar{\beta}]_{\bar{x},Y} \in G_E\,(\xi,(n,\bar{\beta})) \in T\},$$

is in $\mathrm{HOD}_{\bar{x}}[G_E]$. □

10.1.3 Remark. Note that Theorem 10.1.2 does not say that G_E is in $\mathrm{HOD}_{\bar{x}}[E]$.

10.1.4 Remark. Theorem 8.1.3 shows that, assuming $\mathsf{CC}_{\mathbb{R}}$ and that S is a set of ordinals for which $\leq_S$ is locally countable and μ_S is an ultrafilter on $\mathcal{D}_S$,

$$L[S,x] \models \mathsf{CH} + 2^{\aleph_1} = \aleph_2$$

for an S-cone of $x \in \omega^\omega$. Given such an S and x, $\mathbb{V}^{L[S,x]}_{\{S\},\omega}$ has cardinality at most $\aleph_2$ in $L[S,x]$. Since x is generic over $\mathrm{HOD}^{L[S,x]}_{\{S\}}$ via $\mathbb{V}^{L[S,x]}_{\{S\},\omega}$ by Theorem 10.1.2, the models $\mathrm{HOD}^{L[S,x]}$ and $L[S,x]$ have the same cardinals above (and including) $\omega_2^{L[S,x]}$. This also shows that $\mathbb{V}^{L[S,x]}_{\{S\},\omega}$ has cardinality exactly $\aleph_2$ in $L[S,x]$.

The following theorem, due to Woodin, shows that certain models of the form $\mathrm{HOD}_{\bar{x}}$ have the form $L[B]$, for B a set of ordinals. The case where $F = \omega^\omega$ and $S \subseteq \mathrm{Ord}$ was done first by Kechris and Woodin. The proof the theorem uses parts of the proof of Theorem 10.1.2.

Theorem 10.1.5. *Suppose that $V = L(F)$, for some transitive set F, and let S be any set in $L(F)$. Then $\mathrm{HOD}_{\{S\}} = L[B]$, for some set B of ordinals.*

Before proving Theorem 10.1.5, we prove a lemma (a variation of Theorem 8.3.7), and then prove the theorem in the case $V = L[F]$, for F a (not necessarily transitive) set of ordinals.

Lemma 10.1.6. *Let M_1 and M_2 be transitive models of ZFC, with $M_1 \subseteq M_2$. Suppose that $\mathbb{P}$ is a partial order in M_1, $G \subseteq \mathbb{P}$ is an M_2-generic filter and $M_2 \subseteq M_1[G]$. Then $M_1 = M_2$.*

Proof. Since M_1 and M_2 are models of ZFC, it is enough to see that every set of ordinals in M_2 is in M_1. Letting X be a set of ordinals in M_2, we have that $X = \tau_G$, for some $\mathbb{P}$-name τ in M_1. Since G is M_2-generic, this means that some condition in $\mathbb{P}$ decides all of τ, so X is in M_1. □

Now, suppose that $V = L[F]$, where F is a subset of an ordinal γ. Since Choice holds in $\mathrm{HOD}_{\{S\}}$, there exist in $\mathrm{HOD}_{\{S\}}$ an ordinal η and bijection $\pi\colon \mathbb{V}_{\{S\},\gamma} \to \eta$. Let $\leq_\gamma$ be the partial order on η induced by π. Let K be the set of pairs

$$(\delta,(n,\bar{\alpha})) \in \gamma \times P_{\{S\},\gamma}$$

such that $B_{n,\bar{\alpha}} = \{C \subseteq \gamma : \delta \in C\}$ (i.e., the corresponding set K from the proof of Theorem 10.1.2 with $\bar{S}$ as X and γ as Y), and let K_γ be the π-image of K, that is,

$$\{(\delta,\pi([(n,\bar{\alpha})]_{\{S\},\gamma})) : (\delta,(n,\bar{\alpha})) \in K\}.$$

Then $\leq_\gamma$ and K_γ are in $\mathrm{HOD}_{\{S\}}$. By Theorem 10.1.2, F is $\mathbb{V}_{\{S\},\gamma}$-generic over $\mathrm{HOD}_{\{S\}}$, via the generic filter G_F. Then $\pi[G_F]$ is $\mathrm{HOD}_{\{S\}}$-generic for $\leq_\gamma$, and F is in $L[\leq_\gamma, K_\gamma][G_F]$. Then

$$V = L[F] = \mathrm{HOD}_{\{S\}}[G_F] = L[\leq_\gamma, K_\gamma][\pi[G_F]],$$

which implies that $\mathrm{HOD}_{\{S\}} = L[\leq_\gamma, K_\gamma]$, by Lemma 10.1.6, with $L[\leq_\gamma, K_\gamma]$ as M_1 and $\mathrm{HOD}_{\{S\}}$ as M_2.

We introduce a new partial order to deal with the general case. Recall that, given an infinite set Y, $\mathrm{Col}^*(\omega, Y)$ is the partial order of finite partial injections from ω to Y (each with domain some $n \in \omega$), ordered by extension. Given a filter $G \subseteq \mathrm{Col}^*(\omega, Y)$, let $a_G = \{(i,j) \in \omega \times \omega : g(i) \in g(j)\}$, where $g = \bigcup G$. Then a_G is a subset of $\omega \times \omega$, and, if Y is transitive, $V[G] = V[a_G]$.

As above, suppose that $\bar{x}$ is a finite sequence and Y is an (infinite) set such that Y is in $\mathrm{OD}_{\bar{x}}$. Let κ be the least cardinal λ such that $\bar{x} \in V_\lambda$, Y is in $\mathrm{OD}_{\bar{x}}^{V_\lambda}$ and every subset of $\mathcal{P}(\mathrm{Col}^*(\omega, Y))$ in $\mathrm{OD}_{\bar{x}}$ is in $\mathrm{OD}_{\bar{x}}^{V_\lambda}$. Let $P^\omega_{\bar{x},Y}$ be the set of triples $(n, m, \bar{\alpha})$ such that

- $n \in \omega$,
- m is the Gödel number of a ternary formula ϕ,
- $\bar{\alpha}$ is a finite sequence of elements of κ and
- the set $B_{n,m,\bar{\alpha}} = \{p \in \mathrm{Col}^*(\omega, Y) : \mathrm{dom}(p) = n, V_\kappa \models \phi(p, \bar{x}, \bar{\alpha})\}$ is nonempty.

Given $(n, m, \bar{\alpha}) \in P^\omega_{\bar{x},Y}$ and $k \leq n$, let $B_{n,m,\bar{\alpha}} \restriction k$ denote the set

$$\{p \restriction k : p \in B_{n,m,\bar{\alpha}}\}.$$

This set is evidently also in $\mathrm{HOD}_{\bar{x}}$. The set $P^\omega_{\bar{x},Y}$ is in $\mathrm{HOD}_{\bar{x}}$, as is the reflexive and transitive ordering $\leq^\omega_{\bar{x},Y}$ on $P^\omega_{\bar{x},Y}$ defined by setting

$$(n, m, \bar{\alpha}) \leq^\omega_{Z,X} (k, j, \bar{\beta})$$

if $n \geq k$ and

$$B_{n,m,\bar{\beta}} \restriction k \subseteq B_{k,j,\bar{\beta}}.$$

Let $\equiv^\omega_{\bar{x},Y}$ be the induced equivalence relation on $P^\omega_{\bar{x},Y}$, and let $[n, m, \bar{\alpha}]^\omega_{\bar{x},Y}$ denote the $\equiv^\omega_{\bar{x},Y}$-class of a tuple $(n, m, \bar{\alpha})$. Let $\mathbb{V}^\omega_{\bar{x},Y}$ be the partial order whose domain is the set of $\equiv^\omega_{\bar{x},Y}$-classes of $P^\omega_{\bar{x},Y}$, with the order inherited from $\leq^\omega_{\bar{x},Y}$. Each injection $f \colon \omega \to Y$ defines a filter H_f on $\mathbb{V}^\omega_{\bar{x},Y}$, consisting of the $\equiv^\omega_{\bar{x},Y}$-classes of those $(n, m, \bar{\alpha}) \in P^\omega_{\bar{x},Y}$ for which $f \restriction n$ is in $B_{n,m,\bar{\alpha}}$.

Lemma 10.1.7. *For any set finite sequence $\bar{x}$ with $\bar{x} \in \mathrm{OD}_{\bar{x}}$, and any infinite set Y in $\mathrm{OD}_{\bar{x}}$, if $G \subseteq \mathrm{Col}^*(\omega, Y)$ is a V-generic filter, and $g = \bigcup G$, then H_g is $\mathbb{V}^\omega_{\bar{x},Y}$-generic over $\mathrm{HOD}_{\bar{x}}$, and a_g is in $\mathrm{HOD}_{\bar{x}}[H_g]$.*

Proof. To see that H_g is $\mathbb{V}^{\omega}_{\bar{x},Y}$-generic over $\mathrm{HOD}_{\bar{x}}$, let D be a dense open subset of $\mathbb{V}^{\omega}_{\bar{x},Y}$ in $\mathrm{HOD}_{\bar{x}}$, and let p be a condition in $\mathrm{Col}^*(\omega, Y)$. Let D^0 be the set of $(n, m, \bar{\alpha}) \in P^{\omega}_{\bar{x},Y}$ with $[(n, m, \bar{\alpha})]^{\omega}_{\bar{x},Y} \in D$. Applying the genericity of G, it suffices to find a condition $p' \leq p$ and a tuple $(n, m, \bar{\alpha})$ in D^0 such that p' is in $B_{n,m,\bar{\alpha}}$. Let n_p be the domain of p. The set

$$\bigcup\{B_{n,m,\bar{\alpha}} \restriction n_p : (n, m, \bar{\alpha}) \in D^0, n \geq n_p\},$$

being in $\mathrm{OD}_{\bar{x}}$, must be the set of injections from n_p to Y, since otherwise the complement of this set has the form $B_{n_p,j,\bar{\beta}}$ for some j and $\bar{\beta}$, and we get a contradiction by considering a $(n, m, \bar{\alpha}) \in D^0$ with $B_{n,m,\bar{\alpha}} \restriction n_p \subseteq B_{n_p,j,\bar{\beta}}$. It follows that there exist $p' \leq p$ and $(n, m, \bar{\alpha})$ as desired.

As in the proof of Theorem 10.1.2, let K be the set of pairs

$$((i, j), (n, m, \bar{\alpha})) \in (\omega \times \omega) \times P^{\omega}_{\bar{x},Y}$$

such that $\{i, j\} \subseteq n$ and $p(i) \in p(j)$ for all $p \in B_{n,m,\bar{\alpha}}$. Then again K is in $\mathrm{HOD}_{\bar{x}}$, and, as

$$a_g = \{(i, j) \in \omega \times \omega : \exists\, [n, m, \bar{\alpha}]^{\omega}_{\bar{x},Y} \in H_g\, ((i, j), (n, m, \bar{\alpha})) \in K\},$$

a_g is in $\mathrm{HOD}_{\bar{x}}[H_g]$. $\square$

Theorem 10.1.5 now follows.

Proof of Theorem 10.1.5. We may assume that $F = V_\alpha$, for some infinite ordinal α (so that A is ordinal definable). In $\mathrm{HOD}_{\{S\}}$, there exist an ordinal γ and bijection $\pi \colon \mathbb{V}^{\omega}_{\{S\},F} \to \gamma$. Let $\leq_\gamma$ be the partial order on γ induced by π. Let K be the set of pairs

$$((i, j), (n, m, \bar{\alpha})) \in (\omega \times \omega) \times P^{\omega}_{\{S\},F}$$

such that $\{i, j\} \subseteq n$ and $p(i) \in p(j)$ for all $p \in B_{n,m,\bar{\alpha}}$ (i.e., the set K from the proof of Lemma 10.1.7 with F as Z and $\langle S \rangle$ as $\bar{x}$), and let K_γ be the π-image of K, that is,

$$\{((i, j), \pi([(n, m, \bar{\alpha})]^{\omega}_{A,\{S\}})) : ((i, j), (n, m, \bar{\alpha})) \in K\}.$$

Let $G_0 \subseteq \mathrm{Col}^*(\omega, F)$ be a V-generic filter and let $g = \bigcup G_0$. Then $\pi[H_g]$ is a $\mathrm{HOD}_{\{S\}}$-generic filter for $\leq_\gamma$. As in the proof of Lemma 10.1.7, a_g is in the model $L[\leq_\gamma, K_\gamma][\pi[H_g]]$, and therefore so is F. Applying Lemma 10.1.6 with $L[\leq_\gamma, K_\gamma]$ as M_1, $\mathrm{HOD}_{\{S\}}$ as M_2, $\leq_\gamma$ as $\mathbb{P}$ and $\pi[H_g]$ as G, we get that $L[\leq_\gamma, K_\gamma] = \mathrm{HOD}_{\{S\}}$. As the model $L[\leq_\gamma, K_\gamma]$ has the form $L[B]$, for some set B of ordinals, we are done. $\square$

The proof of the following theorem uses the following standard facts about sharps : (1) if κ is a measurable cardinal then the sharp of each bounded subset of κ exists; (2) if X is a set such that $X^{\#}$ exists, and Y exists in a set-generic forcing extension of $L[X]$, then $Y^{\#}$ exists.

Theorem 10.1.8 (ZF + AD)**.** *Suppose that $A \subseteq \omega^\omega$ is such that $\mathcal{P}(\omega^\omega) \not\subseteq L(A,\mathbb{R})$. Then $A^\#$ exists.*

Proof. Every $\mathrm{OD}_{\{A\}}$ subset of ω^ω in $L(A,\mathbb{R})$ is definable in $L(A,\mathbb{R})$ from A and an element of Θ. It follows that the partial order $\mathbb{V}^\omega_{\langle A\rangle, H(\aleph_1)\cup\{A\}}$ as computed in the model $L(A,\mathbb{R})$ has cardinality at most $\Theta^{L(A,\mathbb{R})}$ in $L(A,\mathbb{R})$, and moreover from the proof of Theorem 10.1.5 that $\mathrm{HOD}^{L(A,\mathbb{R})}_{\{A\}}$ is equal to $L[B]$ for some $B \subseteq \Theta^{L(A,\mathbb{R})}$ such that (by Lemma 10.1.7) A exists in a extension of $L[B]$ via a generic filter for $\mathbb{V}^\omega_{\langle A\rangle, H(\aleph_1)\cup\{A\}}$ induced by any V-generic filter for $\mathrm{Col}^*(\omega, H(\aleph_1)\cup\{A\})$. Since Θ is both greater than $\Theta^{L(A,\mathbb{R})}$ and a limit of measurable cardinals (by Remark 5.1.2 or Theorem 8.2.1, for instance), $B^\#$ exists. It follows that $A^\#$ exists in any forcing extension of V by $\mathrm{Col}^*(\omega, H(\aleph_1)\cup\{A\})$. Since it must exist as the same set in any such extension, $A^\#$ exists already in V. □

10.2 Codes for projections, and Uniformization

This section uses the results of Chapter 8, using the notation $\leq_S$, μ_S, $\mathcal{D}_S$, etc. introduced in Definition 8.1.1. Theorem 1.2.2 shows that AD implies that, for each set of ordinals S, μ_S is an ultrafilter on the corresponding set $\mathcal{D}_S$; if $\mathsf{CC}_\mathbb{R}$ holds (as it does under AD) then μ_S is countably complete. If there exists any set of ordinals S for which μ_S is a countably complete ultrafilter on $\mathcal{D}_S$, then ω_1^V is measurable (see Remark 1.2.7), so $\aleph_1 \not\leq 2^{\aleph_0}$.

Since Lemma 10.2.1 below seems to hold only for ∞-Borel subsets of ω^ω (as opposed to ω^δ for larger δ), this section concentrates on ∞-Borel sets of reals. For each set S of ordinals, and each $x \in \omega^\omega$, let Q^S_x denote the definably least poset on the ordinals isomorphic in $\mathrm{HOD}^{L[S,x]}_S$ to the poset $\mathbb{V}^{L[S,x]}_{\{S\},\omega}$ (via the definability order on $\mathrm{HOD}^{L[S,x]}_S$, say), and let K^S_x (also a set of pairs of ordinals) denote the corresponding version of the set K from the proof of Theorem 10.1.2 (i.e., relabeled as in the proof of Theorem 10.1.5 to refer to Q^S_x). Note that Q^S_x depends only on $[x]_S$. We will use this notation throughout this section.

Theorem 10.2.2 below shows that if $\mathsf{DC}_\mathbb{R}$ holds and μ_S is an ultrafilter for each set of ordinals S then the collection of ∞-Borel sets of reals is projectively closed. We start with a lemma whose hypothesis is weaker (and appeal to the fact that $\aleph_1 \not\leq 2^{\aleph_0}$ is equivalent to the assertion that ω_1^V is strongly inaccessible in every inner model satisfying Choice). Background material on some of the forcing claims (e.g., involving regular embeddings) made in the last paragraph of the proof can be found in the appendix to [27].

Lemma 10.2.1 ($\mathsf{ZF} + \aleph_1 \not\leq 2^{\aleph_0}$)**.** *Let S be a set of ordinals, let ϕ be a ternary formula and let B denote the set*

$$\{x \in \omega^\omega : \exists y \in \omega^\omega\ L[S,x,y] \models \phi(S,x,y)\}.$$

Then the following are equivalent, for each $x \in \omega^\omega$.

1. $x \in B$

2. *For some $w \in \omega^\omega$, for all $z \geq_S w$, $L[S, Q^S_z, K^S_z, x] \models$ "there is some $p \in \mathrm{Col}(\omega, 2^{Q^S_z})$ forcing that there exists a $y \in \omega^\omega$ such that $L[S, x, y] \models \phi(S, x, y)$."*

3. *For some $z \in \omega^\omega$, $L[S, Q^S_z, K^S_z, x] \models$ "there is some $p \in \mathrm{Col}(\omega, 2^{Q^S_z})$ forcing that there exists a $y \in \omega^\omega$ such that $L[S, x, y] \models \phi(S, x, y)$."*

Proof. That (2) implies (3) is immediate. To see that (3) $\Rightarrow$ (1), fix one such z. The cardinality of Q^S_z in $L[S, z]$ is at most $(2^{2^{\aleph_0}})^{L[S,z]}$, which is below ω_1^V, as ω_1^V is strongly inaccessible in $L[S, z]$. Since ω_1^V is strongly inaccessible in $L[S, Q^S_z, K^S_z, x]$,

$$\mathcal{P}(Q^S_z) \cap L[S, Q^S_z, K^S_z, x]$$

is countable, and $L[S, Q^S_z, K^S_z, x]$-generic filters for Q^S_z exist below each condition. Since forcing with $\mathrm{Col}(\omega, 2^{Q^S_z})$ adds a generic filter for Q^S_z, statement (1) follows.

For (1) $\Rightarrow$ (2), fix $y_0 \in \omega^\omega$ such that $L[S, x, y_0] \models \phi(S, x, y_0)$ and $z \in \omega^\omega$ such that $z \geq_S y_0$ and $z \geq_S x$. Letting $G_z \subseteq Q^S_z$ be the filter induced by z as in the proof of Theorem 10.1.2, G_z is $\mathrm{HOD}^{L[S,z]}_{\{S\}}$-generic. It follows that G_z is Q^S_z-generic over $L[S, Q^S_z, K^S_z]$ and (using K^S_z) that z is a member of $L[S, Q^S_z, K^S_z][G_z]$. Since x is in $L[S, Q^S_z, K^S_z][G_z]$, $L[S, Q^S_z, K^S_z, x]$ is a generic extension of $L[S, Q^S_z, K_z]$, and $L[S, Q^S_z, K^S_z][G_z]$ is a generic extension of $L[S, Q^S_z, K^S_z, x]$ by a partial order of cardinality at most $(2^{|Q^S_z|})^{L[S, Q^S_z, K^S_z]}$ in $L[S, Q^S_z, K^S_z, x]$. This partial order regularly embeds into $\mathrm{Col}(\omega, 2^{Q^S_z})$ in $L[S, Q^S_z, K^S_z, x]$. This gives (2). $\square$

If DC holds in $L(S, \mathbb{R})$ (as it does if $\mathsf{DC}_\mathbb{R}$ holds), then for each set S of ordinals the ultrapower

$$V^{\mathcal{D}_S}/\mu_S$$

(formed using all S-invariant functions in $L(S, \mathbb{R})$) is wellfounded in $L(S, \mathbb{R})$. We let S^∞, Q^∞_S and K^∞_S be the sets in $\prod_{z \in \omega^\omega} V/\mu_S$ represented by the functions $z \mapsto S$ and $z \mapsto Q^S_z$ and $z \mapsto K^S_z$, respectively.

Theorem 10.2.2 ($\mathsf{ZF} + \mathsf{DC}_\mathbb{R}$)**.** *Let S be a set of ordinals such that $\leq_S$ is locally countable and μ_S is an ultrafilter. Let ϕ be a ternary formula, let S be a set of ordinals, and let B be*

$$\{x \in \omega^\omega : \exists y \in \omega^\omega\, L[S, x, y] \models \phi(S, x, y)\}.$$

Then there exist a set of ordinals T in $\mathrm{OD}_{\{S\}}$ and a binary formula ψ such that

$$B = \{x \in \omega^\omega : L[T, x] \models \psi(T, x)\}.$$

Proof. By $\mathsf{DC}_\mathbb{R}$, μ_S is countably complete. Since μ_S is an ultrafilter, ω_1^V is a measurable cardinal, which implies that $\aleph_1 \not\leq 2^{\aleph_0}$ (as in Remark 1.2.7). We work in $L(S, \mathbb{R})$, which satisfies DC by the assumption that V satisfies $\mathsf{DC}_\mathbb{R}$.

Since DC holds in $L(S, \mathbb{R})$, for each $x \in \omega^\omega$ the ultraproduct

$$\prod_{z \in \omega^\omega} L[S, Q^S_z, K^S_z, x]/\mu_S$$

is wellfounded in $L(S, \mathbb{R})$. Furthermore, for each $x \in \omega^\omega$,

$$L[S^\infty, Q^\infty_S, K^\infty_S, x] = \prod_{z \in \omega^\omega} L[S, Q^S_z, K^S_z, x]/\mu_S$$

and, by Lemma 10.2.1, x is in B if and only if $L[S^\infty, Q^\infty_S, K^\infty_S, x]$ satisfies the statement "there is some $p \in \mathrm{Col}(\omega, 2^{Q^\infty_S})$ forcing that

$$\exists y \in \omega^\omega\, L[S^\infty, x, y] \models \phi(S^\infty, x, y)."$$

Then we can let T be set of the ordinals coding the triple $(S^\infty, Q^\infty_S, K^\infty_S)$ under some fixed coding in L of triples of ordinals by ordinals, and let ψ be the corresponding version of the statement just given. $\square$

The following is an immediate consequence of Theorem 10.2.2.

Theorem 10.2.3 $(\mathsf{ZF} + \mathsf{TD} + \mathsf{DC}_\mathbb{R})$. *The set of ∞-Borel subsets of $(\omega^\omega)^{<\omega}$ is projectively-closed.*

10.2.4 Remark. Since T can be taken to be in $\mathrm{OD}_{\{S\}}$ in the conclusion of Theorem 10.2.2, $\mathsf{TD} + \mathsf{DC}_\mathbb{R}$ implies that there is an ordinal definable class-sized function (minimizing in the ordinal definability order from S) taking in ∞-Borel* codes (S, ϕ) for subsets of $(\omega^\omega)^2$ and returning the corresponding ∞-Borel* codes for $\{x \in \omega^\omega : \exists y \in \omega^\omega\, L[S, x, y] \models \phi(S, x, y)\}$.

Theorem 10.2.6 below is a uniformization result derived from Lemma 10.2.1 and Theorem 10.2.2. Given sets B and x, we let $\sigma_{B,x}$ denote $\omega^\omega \cap \mathrm{OD}_{\{B,x\}}$. Recall that for a set $A \subseteq (\omega^\omega)^2$ and $y \in \omega^\omega$, A_y denotes the set $\{z \in \omega^\omega : (y, z) \in A\}$. As defined in Section 6.2, such a set A can be uniformized if there is a function $f\colon \{y \in \omega^\omega : A_y \neq \emptyset\} \to \omega^\omega$ such that $(y, f(y)) \in A$ for all $y \in \mathrm{dom}(f)$. Recall also that, for $x, y \in \omega^\omega$, $x \oplus y$ denotes the element z of ω^ω such that, for all $n \in \omega$, $z(2n) = x(n)$ and $z(2n+1) = y(n)$.

The proof of Theorem 10.2.6 uses the following observation.

Lemma 10.2.5. *Suppose that A and B are sets, with $A \subseteq (\omega^\omega)^2$. Suppose that for a Turing cone of $x \in \omega^\omega$, for all $y \in L[x]$, if $A_y \neq \emptyset$ then $A_y \cap \sigma_{B,x} \neq \emptyset$. Then A can be uniformized.*

Proof. Let x_0 be a base for a cone witnessing the relevant assumption of the lemma. Given y such that $A_y \neq \emptyset$, define $f(y)$ to be the $\mathrm{OD}_{\{B,x_0,y\}}$-least $z \in \sigma_{B,x_0 \oplus y}$ with $(y, z) \in A$. $\square$

Theorem 10.2.6 $(\mathsf{ZF} + \mathsf{TD} + \mathsf{DC}_\mathbb{R})$. *Suppose that (S, ϕ) is an ∞-Borel* code for a set $A \subseteq (\omega^\omega)^2$. Let B be a set such that $S \in \mathrm{OD}_{\{B\}}$ and suppose that the μ_S-ultrapower of the ordinals is wellfounded. Suppose that, for a Turing cone of $x \in \omega^\omega$, $\sigma_{B,x} \not\subseteq L[S^\infty, Q^\infty_S, K^\infty_S, x]$. Then A can be uniformized.*

Proof. Again, TD implies that $\aleph_1 \not\leq 2^{\aleph_0}$. Let A^* be the set of $y \in \omega^\omega$ with $A_y \neq \emptyset$. We prove that for a Turing cone of $x \in \omega^\omega$, for all $y \in L[x] \cap A^*$, $A_y \cap \sigma_{B,x} \neq \emptyset$. The theorem them follows from Lemma 10.2.5. Let $x_0 \in \omega^\omega$ be such that for all $x \geq_{\mathrm{T}} x_0$,

$$\sigma_{B,x} \not\subseteq L[S^\infty, Q_S^\infty, K_S^\infty, x].$$

Applying TD, suppose toward a contradiction that for a Turing cone of x there exists a $y \in L[S,x] \cap A^*$ with $A_y \cap \sigma_{B,x} = \emptyset$. Let $x_1 \geq_{\mathrm{Tu}} x_0$ be a base for a Turing cone witnessing this.

By Lemma 10.2.1 (and the homogeneity of $\mathrm{Col}(\omega, Q_S^\infty)$ to remove the reliance on the condition p given there), for every $y \in A^*$,

$$L[S^\infty, Q_S^\infty, K_S^\infty, y] \models \text{“}V^{\mathrm{Col}(\omega, Q_S^\infty)} \models \exists z\, L[S,y,z] \models \phi(S,y,z)\text{”}.$$

Applying Lemma 8.3.5 and the discussion just before it, let π be a recursive diffuse function and let U be, in the model $L[S^\infty, Q_S^\infty, K_S^\infty, x_1]$, an ultrafilter on ω not containing any π-weak set. Let $\mathbb{P}$ denote the partial order $\mathbb{P}_U^{L[S^\infty, Q_S^\infty, K_S^\infty, x_1]}$, again as in Section 8.3. Fix

$$t \in \sigma_{B,x_1} \setminus L[S^\infty, Q_S^\infty, K_S^\infty, x_1].$$

For any $L[S^\infty, Q_S^\infty, K_S^\infty, x_1]$-generic

$$g \subseteq \mathbb{P},$$

and any

$$y \in L[S^\infty, Q_S^\infty, K_S^\infty, x_1][g] \cap A^*,$$

$$L[S^\infty, Q_S^\infty, K_S^\infty, x_1][g] \models \text{“}V^{\mathrm{Col}(\omega, Q_S^\infty)} \models \exists z \in \omega^\omega\, L[S,y,z] \models \phi(S,y,z)\text{”},$$

since $L[S^\infty, Q_S^\infty, K_S^\infty, x_1][g]$ is an outer model of $L[S^\infty, Q_S^\infty, K_S^\infty, y]$.

We have that

$$L[S^\infty, Q_S^\infty, K_S^\infty, x_1] = \prod L[S, Q_x^S, K_x^S, x_1]/\mu_S.$$

Since $\aleph_1 \not\leq 2^{\aleph_0}$, $L[S^\infty, Q_S^\infty, K_S^\infty, x_1] \cap \mathcal{P}(\mathcal{P}(\omega^\omega))$ is countable, and (by Remark 8.0.1) there is an $x_2 \geq_{\mathrm{Tu}} x_1$ in ω^ω such that, for all $x \geq_{\mathrm{Tu}} x_2$,

1. $L[S^\infty, Q_S^\infty, K_S^\infty, x_1] \cap \mathcal{P}(\mathcal{P}(\omega^\omega)) = L[S, Q_x^S, K_x^S, x_1] \cap \mathcal{P}(\mathcal{P}(\omega^\omega))$,
2. $\mathbb{P} \in L[S, Q_x^S, K_x^S, x_1]$,
3. all nice $\mathbb{P}$-names in $L[S^\infty, Q_S^\infty, K_S^\infty, x_1]$ for elements of ω^ω are in $L[S, Q_x^S, K_x^S, x_1]$,
4. for any $L[S, Q_x^S, K_x^S, x_1]$-generic $g \subseteq \mathbb{P}$ and any
$$y \in L[S, Q_x^S, K_x^S, x_1][g] \cap A^*,$$
$$L[S, Q_x^S, K_x^S, x_1][g] \models \text{“}V^{\mathrm{Col}(\omega, Q_x^S)} \models \exists z \in \omega^\omega\, L[S,y,z] \models \phi(S,y,z)\text{”}.$$

By Lemma 8.3.6, there exists an $L[S^\infty, Q_S^\infty, K_S^\infty, x_1]$-generic filter $g^* \subseteq \mathbb{P}$, with corresponding generic $a^* \subseteq \omega$, such that there is a $w \in \omega^\omega \cap L[a^*, t]$ which HC-codes an ω-sequence listing all the members of $\mathcal{P}(\mathbb{P} \times Q_{x_2}^S \times \omega)$ in $L[S, Q_{x_2}^S, K_{x_2}^S, x_1]$.

Let $y \in A^*$ be an element of

$$L[x_1 \oplus a^*].$$

It suffices to show that $A_y \cap \sigma_{B,x_1 \oplus a^*} \neq \emptyset$, which we now do.

Since $t \in \sigma_{B,x_1 \oplus a^*}$ and $w \in L[a^*, t]$, there exists an $L[S, Q_{x_2}^S, K_{x_2}^S, x_1][a^*]$-generic filter $G \subseteq \mathrm{Col}(\omega, Q_{x_2}^S)$ in $\mathrm{OD}_{\{B, x_1, a^*\}}$. By item (4) from the choice of x_2, there is a

$$z \in \omega^\omega \cap L[S, Q_{x_2}^S, K_{x_2}^S, x_1][a^*][G]$$

such that $L[S, y, z] \models \phi(S, y, z)$. Since

$$\omega^\omega \cap L[S, Q_{x_2}^S, K_{x_2}^S, x_1][a^*][G] \subseteq \sigma_{B,x_1 \oplus a^*},$$

we are done. □

In Section 8.3 we defined, for sets S, T of ordinals, $S \leq_{\mathcal{D}} T$ to mean that $\omega^\omega \cap L[S, x] \subseteq L[T, x]$ for a Turing cone of x. Theorem 10.2.6 shows that if $\mathsf{TD} + \mathsf{DC}_\mathbb{R}$ holds, and A is an ∞-Borel subset of $(\omega^\omega)^2$ with an ∞-Borel code S such that $(S, Q_S^\infty, K_S^\infty)$ is not $\leq_{\mathcal{D}}$-maximal, then A can be uniformized. The following corollary gives a version of the converse. Its proof shows that if S is a set of ordinals such that the set $\{(x, y) \in (\omega^\omega)^2 : y \notin L[S, x]\}$ can be uniformized, then S is not $\leq_{\mathcal{D}}$-maximal.

Corollary 10.2.7 ($\mathsf{ZF} + \mathsf{TD} + \mathsf{DC}_\mathbb{R}$). *Suppose that every subset of ω^ω is ∞-Borel. Then* Uniformization *is equivalent to the assertion that $\leq_{\mathcal{D}}$ has no greatest element.*

Proof. The reverse direction follows from Theorem 10.2.6, letting B be any set of ordinals which is strictly $\leq_{\mathcal{D}}$-above any set of ordinals constructing S, Q_S^∞ and K_S^∞. The forward direction requires only that $\aleph_1 \not\leq 2^{\aleph_0}$ and every subset of ω^ω is ∞-Borel. Let S be a set of ordinals, and let $f \colon \omega^\omega \to \omega^\omega$ uniformize the set $\{(x, y) \in (\omega^\omega)^2 : y \notin L[S, x]\}$. Let (T, ϕ) be an ∞-Borel* code for $\{(x, i, j) : (i, j) \in f(x)\}$. Then for all (x, i, j), $(i, j) \in f(x)$ if and only if $L[T, x] \models \phi(T, x, i, j)$, which shows that $L[T, x]$ is closed under f for all x. □

10.3 The Vopěnka algebra for ∞-Borel sets

Given a set Y consisting of ordinals, the Vopěnka algebra can be modified to use ∞-Borel codes in $\mathrm{OD}_{\{Y\}}$ to refer to sets. As in Section 10.1, we define one-step and ω-sequence versions. As we will be using Theorem 10.2.2, we restrict to the version for subsets of ω^ω.

We fix a cardinal κ such that for each $n \in \omega$ and each $A \subseteq (\omega^\omega)^n$, if there exist a set of ordinals S in $\mathrm{OD}_{\{Y\}}$ and a formula ϕ such that

$$A = \{\bar{x} \in (\omega^\omega)^n : L[S, \bar{x}] \models \phi(S, \bar{x})\}$$

then there exist a set of ordinals S in $V_\kappa \cap \mathrm{OD}_{\{Y\}}$ and a formula ψ such that

$$A = \{\bar{x} \in (\omega^\omega)^n : L_\kappa[S, \bar{x}] \models \psi(S, \bar{x})\}.$$

Let $P_{\infty,Y}$ be the set of pairs (m, S) such that

- m is the Gödel number of a binary formula ϕ,
- $S \in V_\kappa \cap \mathrm{OD}_{\{Y\}}$ is a set of ordinals,
- the set $B_{m,S} = \{x \in \omega^\omega : L_\kappa[S, x] \models \phi(S, x)\}$ is nonempty.

Then again we have an induced ordering $\leq_{\infty,Y}$ on $P_{\infty,Y}$ defined by setting

$$(m, S) \leq_{\infty,Y} (j, T)$$

if $B_{m,S} \subseteq B_{j,T}$. Let $\equiv_{\infty,Y}$ be the induced equivalence relation on $P_{\infty,Y}$, and let $[(m, S)]_{\infty,Y}$ denote the $\equiv_{\infty,Y}$-class of each $(m, S) \in P_{\infty,Y}$. Let $\mathbb{V}_{\infty,Y}$ be the partial order whose domain is the set of $\equiv_{\infty,Y}$-classes of $P_{\infty,Y}$, with the order inherited from $\leq_{\infty,Y}$.

For the sequence version, let $P^\omega_{\infty,Y}$ be the set of triples (n, m, S) such that

- $n \in \omega$,
- m is the Gödel number of a binary formula ϕ,
- $S \in V_\kappa \cap \mathrm{OD}_{\{Y\}}$ is a set of ordinals,
- the set $B^\omega_{n,m,S} = \{\bar{x} \in (\omega^\omega)^n : L[S, \bar{x}] \models \phi(S, \bar{x})\}$ is nonempty.

Then again we have an induced ordering $\leq^\omega_{\infty,Y}$ on $P^\omega_{\infty,Y}$ defined by setting

$$(n, m, S) \leq^\omega_{\infty,Y} (k, j, T)$$

if $n \geq k$ and $\{\bar{x}\restriction k : \bar{x} \in B^\omega_{n,m,S}\} \subseteq B^\omega_{k,j,T}$. Let $\equiv^\omega_{\infty,Y}$ be the induced equivalence relation on $P^\omega_{\infty,Y}$, and let $[(n, m, S)]^\omega_{\infty,Y}$ denote the $\equiv^\omega_{\infty,Y}$-class of each $(n, m, S) \in P^\omega_{\infty,Y}$. Let $\mathbb{V}^\omega_{\infty,Y}$ be the partial order whose domain is the set of $\equiv^\omega_{\infty,Y}$-classes of $P^\omega_{\infty,Y}$, with the order inherited from $\leq^\omega_{\infty,Y}$.

10.3.1 Remark. By Theorem 10.2.2, for each bounded $S \subseteq \kappa$ in $\mathrm{HOD}_{\{Y\}}$ and each ternary formula ϕ, there exist a bounded $T \subseteq \kappa$ and a binary formula ϕ such that $\{x \in \omega^\omega : \exists y \in \omega^\omega\ L[S, x, y] \models \phi(S, x, y)\} = \{x \in \omega^\omega : L[T, x] \models \psi(T, x)\}$.

10.3.2 Remark. Let (n, m, S) be an element of $P^\omega_{\infty,Y}$, and let ϕ be the formula with Gödel number m. Let n' be an element of $\omega \setminus n$. Then

$$\{\bar{x} \in (\omega^\omega)^{n'} : \bar{x}\restriction n \in B_{n,m,S}\} = \{\bar{x} \in (\omega^\omega)^{n'} : L_\kappa(S, \bar{x}) \models \phi(S, \bar{x}\restriction n)\},$$

so this set has the form $B_{n',k,S}$ for some $k \in \omega$, and $B_{n',k,S} \leq^\omega_{\infty,Y} B_{n,m,S}$. Moreover, there is a recursive function (not depending on S) sending each such tuple (n, m, n') to a corresponding value k.

10.3.3 Remark. Any set in $\mathrm{OD}_{\{Y\}}$ consisting of sets of ordinals can be coded by a single set of ordinals in $\mathrm{OD}_{\{Y\}}$, from which it follows that for any subset D of $P_{\infty,Y}$ in $\mathrm{OD}_{\{Y\}}$ the set $\bigcup\{B_{m,S} : (m,S) \in D\}$ has the form $B_{j,T}$ for some $(j,T) \in P_{\infty,Y}$. Using this fact, the first paragraph of the proof of Theorem 10.1.2 adapts to prove that for each $x \in \omega^\omega$, the set G_x consisting of those $[(m,S)]_{\infty,Y} \in P_{\infty,Y}$ such that $x \in B_{m,S}$ is a $\mathrm{HOD}_{\{Y\}}$-generic filter. This fact is implicit in Lemma 10.3.5 below, using the fact that the first coordinate of a $\mathbb{V}^\omega_{\infty,Y}$-generic sequence is generic for $\mathbb{V}_{\infty,Y}$. For each $i \in \omega$, $x(i)$ is the unique $j \in \omega$ such that $[(\emptyset, m_{i,j})]_{\infty,Y} \in G_x$, where $m_{i,j}$ is the Gödel number of the formula $\phi(v)$ expressing $(i,j) \in v$. It follows then that $\mathrm{HOD}_{\{Y\}}[x] = \mathrm{HOD}_{\{Y\}}[G_x]$, since G_x is the set of $[(m,S)]_{\infty,Y} \in \mathbb{V}_{\infty,Y}$ such that $L_\kappa[S,x] \models \phi(S,x)$ (where ϕ is the formula with Gödel number m), and this can be computed in $\mathrm{HOD}_{\{Y\}}[x]$. Since every condition in $\mathbb{V}_{\infty,Y}$ is a member of G_x for some $x \in \omega^\omega$, every $\mathrm{HOD}_{\{Y\}}$-generic filter $G \subseteq \mathbb{V}_{\infty,Y}$ is equal to G_x, for $x = \{(i,j) : [(\emptyset, m_{i,j})]_{\infty,Y} \in G\}$.

10.3.4 Remark. Since the first coordinate of a $\mathbb{V}^\omega_{\infty,Y}$-generic sequence is generic for $\mathbb{V}_{\infty,Y}$, there is a $\mathbb{V}_{\infty,Y}$-name $\dot{Q}$ such that $\mathbb{V}^\omega_{\infty,Y}$ is forcing-equivalent to $\mathbb{V}_{\infty,Y} * \dot{Q}$, via a map which carries the generic real for $\mathbb{V}_{\infty,Y}$ to the first coordinate of the generic sequence produced by forcing with $\mathbb{V}^\omega_{\infty,Y}$.

As with the partial order $\mathbb{V}^\omega_{\bar{x},Y}$ in Section 10.1, each injection $f\colon \omega \to \omega^\omega$ defines a filter H_f on $\mathbb{V}^\omega_{\infty,Y}$, consisting of the $\equiv^\omega_{\infty,Y}$-classes of those $(n,m,S) \in P^\omega_{\infty,Y}$ for which $f{\restriction}n$ is in $B_{n,m,S}$. Lemma 10.3.5 is the version of Lemma 10.1.7 for $\mathbb{V}^\omega_{\infty,Y}$. Theorem 10.2.2 and Remark 10.3.2 are used in the proof of the lemma.

The remainder of this section uses the notation μ_S and $\mathcal{D}_S$ introduced in Definition 8.1.1, for S a set of ordinals. Recall from Remark 1.2.5 that if $\mu_\emptyset$ is a countably complete ultrafilter on $\mathcal{D}_\emptyset$ (i.e., the constructibility degrees), then, for each set $S \subseteq \mathrm{Ord}$, μ_S is a countably complete ultrafilter on $\mathcal{D}_S$. The statement $\aleph_1 \not\leq 2^{\aleph_0}$ implies that the relation $\leq_S$ is locally countable for each such S.

Lemma 10.3.5 ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \aleph_1 \not\leq 2^{\aleph_0}$)**.** *Suppose that $\mu_\emptyset$ is an ultrafilter on $\mathcal{D}_\emptyset$, and let Y be a set of ordinals. If $G \subseteq \mathrm{Col}^*(\omega, \omega^\omega)$ is a V-generic filter, and $g = \bigcup G$, then H_g is $\mathbb{V}^\omega_{\infty,Y}$-generic over $\mathrm{HOD}_{\{Y\}}$, and g is in $\mathrm{HOD}_{\{Y\}}[H_g]$.*

Proof. Let $D \subseteq \mathbb{V}^\omega_{\infty,Y}$ be a dense set in $\mathrm{HOD}_{\{Y\}}$, and let p be a condition in $\mathrm{Col}^*(\omega, \omega^\omega)$. Let n_p be the domain of p. By strengthening the conditions in D if necessary, we may assume, applying Remark 10.3.2, that each element of D has the form $[(n,m,S)]^\omega_{\infty,Y}$ for some $(n,m,S) \in P^\omega_{\infty,Y}$ with $n \geq n_p$. Let D_0 be the set of $(n,m,S) \in P^\omega_{\infty,Y}$ with $[(n,m,S)]^\omega_{\infty,Y} \in D$. To show that H_g is $\mathrm{HOD}_{\{Y\}}$-generic (applying the genericity of G), we want to find a condition $p' \leq p$ and a (n,m,S) in D_0 such that p' is in $B_{n,m,S}$. By Remark 10.3.1 there is an $\mathrm{OD}_{\{Y\}}$ function associating each $(n,m,S) \in D_0$ to a tuple $(n_p,k,T) \in P^\omega_{\infty,Y}$ such that $B_{n,m,S}{\restriction}n_p = B_{n_p,k,T}$. Applying the first sentence of Remark 10.3.3, there exist $T \in V_\kappa \cap \mathrm{OD}_{\{Y\}}$ and a binary formula ψ (with Gödel number k) such that

$$B_{n_p,k,T} = \{\bar{x} \in (\omega^\omega)^{n_p} : \exists (n,m,S) \in D_0 \ \bar{x} \in B_{n,m,S}{\restriction}n_p\}.$$

Furthermore, $B_{n_p,k,T}$ must be the set of all injections from n_p to ω^ω, since otherwise the complement of this set has the form $B_{n_p,j,R}$ for some $j \in \omega$ and $R \in V_\kappa \cap \mathrm{OD}_{\{Y\}}$, and we get a contradiction by considering a $(n, m, S) \in D_0$ with

$$\{\bar{x}\restriction n_p : \bar{x} \in B_{n,m,S}\} \subseteq B_{n_p,j,R}.$$

There exists then a triple $(n, m, S) \in D_0$ such that $p \in B_{n,m,S}\restriction n_p$, as desired. This gives the genericity of H_g.

To see that g is in $\mathrm{HOD}_{\{Y\}}[H_g]$, as in the proof of Theorem 10.1.2, let K be the set of pairs

$$((i, j, k), (n, m, S)) \in (\omega \times \omega \times \omega) \times P^\omega_{\infty,Y}$$

such that $i < n$ and $p(i)(j) = k$ for all $p \in B_{n,m,S}$. Then again K is in $\mathrm{HOD}_{\{Y\}}$, and K can be used to define a $\mathbb{V}^\omega_{\infty,Y}$-name $\tau \in \mathrm{HOD}_{\{Y\}}$ such that for every $\mathrm{Col}^*(\omega, \omega^\omega)$-generic function g over V, $\tau_{H_g} = g$. □

The following is the main theorem of this section. The theorem shows that, in models of the form $L(Y, \mathbb{R})$ satisfying $\mathsf{TD} + \mathsf{DC}_\mathbb{R}$, where Y is a set of ordinals, the orders $\mathbb{V}_{\langle Y\rangle,\omega}$ and $\mathbb{V}_{\infty,Y}$ (defined in Sections 10.1 and 10.3 respectively) are isomorphic.

Theorem 10.3.6 ($\mathsf{ZF} + \mathsf{DC}_\mathbb{R} + \aleph_1 \not\leq 2^{\aleph_0}$)**.** *Suppose that $\mu_\emptyset$ is an ultrafilter on $\mathcal{D}_\emptyset$, and let Y be a set of ordinals such that $V{=}L(Y, \mathbb{R})$. Then for each $\mathrm{OD}_{\{Y\}}$ set $A \subseteq \omega^\omega$ there exist an $\mathrm{OD}_{\{Y\}}$ set S of ordinals and a binary formula ϕ such that $A = \{x \in \omega^\omega : L[S, x] \models \phi(S, x)\}$.*

Proof. Let $\bar{a}$ be a finite set of ordinals, let ψ be a ternary formula, and let A be the set of $x \in \omega^\omega$ for which $\psi(\bar{a}, x, Y)$ holds. By Theorem 10.1.5, we may fix a set $B \subseteq \mathrm{Ord}$ such that $\mathrm{HOD}_{\{Y\}} = L[B]$.

By Remark 10.3.4, $\mathbb{V}^\omega_{\infty,Y}$ is forcing-equivalent to an iteration of the form $\mathbb{V}_{\infty,Y} * \dot{Q}$, for some $\mathbb{V}_{\infty,Y}$-name $\dot{Q}$. By Remark 10.3.3, for every $x \in \omega^\omega$ there is a $\mathrm{HOD}_{\{Y\}}$-generic filter $G_x \subseteq \mathbb{V}_{\infty,Y}$ such that $\mathrm{HOD}_{\{Y\}}[G_x] = \mathrm{HOD}_{\{Y\}}[x]$.

Let $\tau \in \mathrm{HOD}_{\{Y\}}$ be the $\mathbb{V}^\omega_{\infty,Y}$-name for the generic function g from the end of the proof of Lemma 10.3.5. Let $\dot{R}$ be a $\mathbb{V}^\omega_{\infty,Y}$-name in $\mathrm{HOD}_{\{Y\}}$ for the range of the realization of τ. Let $\dot{R}_*$ be the $\mathbb{V}_{\infty,Y} * \dot{Q}$-name induced by $\dot{R}$ and a map in $\mathrm{HOD}_{\{Y\}}$ witnessing the forcing-equivalence of $\mathbb{V}_{\infty,Y} * \dot{Q}$ with $\mathbb{V}^\omega_{\infty,Y}$ which carries the generic real for $\mathbb{V}_{\infty,\{Y\}}$ to the first coordinate of the generic sequence produced by forcing with $\mathbb{V}^\omega_{\infty,Y}$, as in Remark 10.3.4. We want to see that for each $x \in \omega^\omega$, all generic $\dot{Q}_{G_x}$-extensions of $\mathrm{HOD}_{\{Y\}}[x]$ agree about whether or not $\psi(\bar{\alpha}, x, Y)$ holds in $L(Y, \dot{R}_*)$. By Lemma 10.3.5, $(\omega^\omega)^V$ is the realization of $\dot{R}_*$ in one of these extensions. The theorem will then follow, with S a set of ordinals coding B, Y, $\bar{\alpha}$ and $\dot{R}_*$.

If not all such extensions agreed, there would be $x \in \omega^\omega$ and (n, m, S), (n', m', S') in $P^\omega_{\infty,Y}$ such that

- $x \in B_{n,m,S} \cap B_{n',m',S'}$;

- $[(n, m, S)]^{\omega}_{\infty,Y} \Vdash$ "$L(\check{Y}, \dot{R}) \models \phi(\check{\bar{\alpha}}, \tau(0), \check{Y})$";
- $[(n', m', S')]^{\omega}_{\infty,Y} \Vdash$ "$L(\check{Y}, \dot{R}) \models \neg\phi(\check{\bar{\alpha}}, \tau(0), \check{Y})$";

Let $p \colon n + n' \to \omega^\omega$ and $\pi \colon \omega \to \omega$ be such that

- $p(0) = x$;
- $\pi(0) = 0$;
- $\pi \restriction (n + n')$ is a permutation of $n + n'$;
- $\pi(i) = i$ for all $i \geq n + n'$;
- $p \restriction n \in B_{n,m,S}$;
- the function $p' \colon n' \to \omega^\omega$ defined by setting $p'(i) = p(\pi(i))$ is in $B_{n',m',S'}$.

Let $g \colon \omega \to \omega^\omega$ be the union of a V-generic filter for $\mathrm{Col}^*(\omega, \omega^\omega)$ containing p, and let $g' \colon \omega \to \omega^\omega$ be defined by setting $g'(i) = g(\pi(i))$ for all $i \in \omega$. Then g' is also the union of a V-generic filter for $\mathrm{Col}^*(\omega, \omega^\omega)$, $\dot{R}_{H_g} = \dot{R}_{H_{g'}}$ and $g(0) = g'(0)$, giving a contradiction. $\square$

Part (1) of the following corollary appears as Theorem 1.9 in [1].

Corollary 10.3.7 ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}} + \aleph_1 \not\leq 2^{\aleph_0}$). *Suppose that $\mu_\emptyset$ is an ultrafilter on $\mathcal{D}_\emptyset$. Let Y be a set of ordinals such that $V = L(Y, \mathbb{R})$. Then the following hold.*

1. *Every subset of ω^ω is ∞-Borel.*
2. $\mathrm{HOD}_{\{Y,x\}} = \mathrm{HOD}_{\{Y\}}[x]$ *for all $x \in \omega^\omega$.*

Proof. For part (1), every set in $L(Y, \mathbb{R})$ is definable from Y, a finite set of ordinals and a finite subset of ω^ω. It follows then that for each $A \subseteq \omega^\omega$ in $L(Y, \mathbb{R})$, there exist an $\mathrm{OD}_{\{Y\}}$-set $B \subseteq \omega^\omega \times \omega^\omega$ and an $x \in \omega^\omega$ such that $A = \{y \in \omega^\omega : (x, y) \in B\}$. By Theorem 10.3.6, there exist an $\mathrm{OD}_{\{Y\}}$ set S of ordinals and a ternary formula ϕ such that $B = \{(x, y) \in \omega^\omega : L[S, x, y] \models \phi(S, x, y)\}$. Letting T be a set of ordinals coding S and x, there is a binary formula ϕ' such that $A = \{y \in \omega^\omega : L[T, y] \models \phi'(T, y)\}$.

For part (2), fix $x \in \omega^\omega$. Clearly, $\mathrm{HOD}_{\{Y\}}[x]$ is contained in $\mathrm{HOD}_{\{Y,x\}}$. To show the reverse inclusion, let $G_x \subseteq \mathbb{V}_{\{Y\},\omega}$ (from Section 10.1) be the $\mathrm{HOD}_{\{Y\}}$-generic filter induced by x. By Theorem 10.1.2 it suffices to show that G_x is in $\mathrm{HOD}_{\{Y\}}[x]$.

Let κ_0 be the cardinal κ used in the definition of $P_{\{Y\},\omega}$ and, for each $(n, \bar{\alpha}) \in P_{\{Y\},\omega}$ let $B^0_{n,\bar{\alpha}}$ be the set $B_{n,\bar{\alpha}}$ from this definition. Similarly, let κ_1 be the cardinal κ used in the definition of $P_{\infty,Y}$, and, for each $(m, S) \in P_{\infty,Y}$ let $B^1_{m,S}$ be the set $B_{m,S}$ from that definition. By Theorem 10.3.6, for each $(n, \bar{\alpha}) \in P_{\{Y\},\omega}$ there is a pair $(m, S) \in P_{\infty,Y}$ such that $B^0_{n,\bar{\alpha}} = B^1_{m,S}$. Furthermore, the set of pairs $((n, \bar{\alpha}), (m, S))$ in $P_{\{Y\},\omega} \times P_{\infty,Y}$ such that $B^0_{n,\bar{\alpha}} = B^1_{m,S}$ is in $HOD_{\{Y\}}$. The set of $(m, S) \in P_{\infty,Y}$ such that $x \in B^1_{m,S}$ is in $\mathrm{HOD}_{\{Y\}}[x]$. It follows that G_x, which is the set of $(n, \bar{\alpha})$ such that $x \in B^0_{n,\bar{\alpha}}$, is in $\mathrm{HOD}_{\{Y\}}[x]$. $\square$

10.3.8 Remark. Suppose that $\mathsf{DC}_{\mathbb{R}}$ and $\aleph_1 \not\leq 2^{\aleph_0}$ hold, and that $\mu_\emptyset$ is an ultrafilter on $\mathcal{D}_\emptyset$. Let κ be an infinite cardinal, and suppose that $A \subseteq \omega^\omega$ is κ-Borel. By Theorem 9.2.4, there is a κ-Borel code S for A such that $L(S, \mathbb{R}) = L(A, \mathbb{R})$. By Theorem 10.3.6, in $L(S, \mathbb{R})$ every subset of ω^ω is ∞-Borel. By the Moschovakis Coding Lemma, not every subset of ω^ω in $L(S, \mathbb{R})$ is κ-Borel. It follows then that there is some $B \subseteq \omega^\omega$ in $L(S, \mathbb{R})$ which is λ-Borel but not $<\lambda$-Borel, for some $\lambda > \kappa$.

Chapter 11

Suslin sets and strong codes

This chapter presents three applications of the material in Chapters 7-10. In Section 11.2 we show that, assuming $<\Theta$-$\mathsf{Determinacy}$ + $\mathsf{DC}_{\mathbb{R}}$, every subset of ω^ω which has an ∞-Borel code which is not $\leq_{\mathcal{D}}$-maximal is Suslin (Theorem 11.2.3). In Section 11.3 we show that AD + $\mathsf{Uniformization}$ implies that every subset of ω^ω is ∞-Borel. In Section 11.4 we show that, under AD^+, the set of Suslin cardinals is closed below Θ.

11.1 Generic codes

Given a nice $\mathbb{P}$-name τ for an element of ω^ω, we let, for each $n \in \omega$, $D_{\tau,n}$ be the set
$$\{p \in \mathbb{P} : \exists m \in \omega\, (p, (n, m)) \in \tau\}.$$
When $\{D_{\tau,n} : n \in \omega\} \subseteq B$, we write $A_{\mathbb{P},B,\tau}$ for the set of values τ_g, where g ranges over the set of all B-generic filters contained in $\mathbb{P}$ (see Section 0.2 for the definitions of nice name and B-generic filter, and our corresponding notational conventions). When $\mathbb{P}$ is a partial order, we write $\mathrm{dom}(\mathbb{P})$ for the underlying set. Similarly, if B is a sequence, we write $\mathrm{dom}(B)$ for the corresponding index set, and $\mathrm{range}(B)$ for the set indexed by B.

11.1.1 Definition. Given $A \subseteq \omega^\omega$, an ordinal α, a partial order $\mathbb{P}$, a sequence B of dense open subsets of $\mathbb{P}$ and a nice $\mathbb{P}$-name τ for an element of ω^ω, we say that $(\mathbb{P}, B, \tau)$ is a *generic code* for A if $\{D_{\tau,n} : n \in \omega\} \subseteq \mathrm{range}(B)$ and $A_{\mathbb{P},B,\tau} = A$. If we say that C is a generic code, then we mean that C is a triple of the form $(\mathbb{P}_C, B_C, \tau_C)$, and write A_C for $A_{\mathbb{P}_C,B_C,\tau_C}$. If in addition α is an ordinal containing $\mathrm{dom}(\mathbb{P}) \cup \mathrm{dom}(B)$, then we say that $(\mathbb{P}, B, \tau)$ is a *generic α-code* for A. We say that $(\mathbb{P}, B, \tau)$ is a *generic ∞-code* if it is a generic α-code for some ordinal α.

Theorem 10.1.2 shows that if (S, ψ) is an ∞-Borel* code for a set $A \subseteq \omega^\omega$, then $(\mathbb{V}_{\{S\},\omega^2}\restriction p, B, \tau\restriction p)$ is a generic code for A, where, letting τ be the $\mathbb{V}_{\{S\},\omega^2}$-

name for the associated generic real (given by the set K in the proof of Theorem 10.1.2),

- p is the $\mathbb{V}_{\{S\},\omega}$-condition corresponding to the set
$$\{x \in \omega^\omega : L[S,x] \models \psi(S,x)\},$$
- $\mathbb{V}_{\{S\},\omega^2}\restriction p$ and $\tau\restriction p$ are the corresponding restrictions of $\mathbb{V}_{\{S\},\omega^2}$ and τ below p and
- B is the set of dense open subsets of $\mathbb{V}_{\{S\},\omega^2}\restriction p$ in $\mathrm{HOD}_{\{S\}}$.

Using the definability order on $\mathrm{HOD}_{\{S\}}$, one can convert this generic code to a generic ∞-code. In the proof of Theorem 11.1.6 we use the fact that, by Theorem 8.1.3, for a Turing cone of $x \in \omega^\omega$ there exists in $L[S,x]$ a generic $\omega_2^{L[S,x]}$-code for $A \cap L[S,x]$.

11.1.2 Definition. Suppose that $(\mathbb{P}, B, \tau)$ is a generic ∞-code. For any set X of ordinals, we let $\mathbb{P}_X$ be $\mathbb{P}\restriction\mathrm{dom}(\mathbb{P}) \cap X$, B_X be $\langle b_\beta : \beta \in X \cap \mathrm{dom}(B)\rangle$ and τ_X be $((\mathrm{dom}(\mathbb{P}) \cap X) \times \omega^2) \cap \tau$. Similarly, if $C = (\mathbb{P}, B, \tau)$, we write C_X for $(\mathbb{P}_X, B_X, \tau_X)$.

11.1.3 Definition. Suppose that $C = (\mathbb{P}, B, \tau)$ is a generic α-code for a set A, for some ordinal α. We associate to $(\mathbb{P}, B, \tau)$ a game on $\mathrm{dom}(\mathbb{P}) \cup \mathrm{dom}(B)$, called $\mathcal{G}_{\mathbb{P},B,\tau}$ or $\mathcal{G}_C$, where I and II collaborate to build a countable subset $\sigma \subseteq \mathrm{dom}(\mathbb{P}) \cup \mathrm{dom}(B)$, and I wins if $C_\sigma = (\mathbb{P}_\sigma, B_\sigma, \tau_\sigma)$ is a generic code for a subset of A. We say that $(\mathbb{P}, B, \tau)$ is a *strong generic code* (or *strong α-generic code*) for A_ϕ if II does not have a winning strategy in $\mathcal{G}_{\mathbb{P},B,\tau}$.

Note that a strong generic α-code is also a strong generic β-code for any ordinal $\beta \geq \alpha$.

11.1.4 Remark. If $(\mathbb{P}, B, \tau)$ is an α-Borel code for a set A which is Suslin and co-Suslin, then $\mathcal{G}_{\mathbb{P},B,\tau}$ is determined, by Theorem 7.0.3.

11.1.5 Remark. For any infinite cardinal κ and κ-generic code $(\mathbb{P}, B, \tau)$, κ-Determinacy implies the determinacy of the game $\mathcal{G}_{\mathbb{P},B,\tau}$, since a run of the game continuously builds a subset of ω coding a generic ω-code.

Unlike ∞-Borel codes, strong generic codes witness Suslinity, in the context of $<\Theta$-Determinacy.

Theorem 11.1.6. *Let κ be an infinite cardinal and let C be a generic κ-code for a set $A \subseteq \omega^\omega$. If player I has a winning strategy in the game $\mathcal{G}_C$, then A is κ-Suslin. Moreover, if Σ is a winning strategy for player I in $\mathcal{G}_C$, then $A = \mathrm{p}[T]$, for some tree $T \subseteq (\omega \times \kappa \times 2)^{<\omega}$ which is definable from C and Σ.*

Proof. Let C be $(\mathbb{P}, B, \tau)$, and fix a winning strategy Σ for player I in $\mathcal{G}_C$. Then for all $x \in \omega^\omega$, $x \in A$ if and only if there is exist a countable $\sigma \subseteq \mathrm{dom}(\mathbb{P}) \cup \mathrm{dom}(B)$ produced by a run of $\mathcal{G}_C$ where I plays according to Σ and a B_σ-generic filter $g \subseteq \mathbb{P}_\sigma$ such that $x = \tau_{\sigma,g}$. To see this, note first of all that, since a Σ is winning strategy for player I, if there is a play against Σ producing σ with x equal to $\tau_{\sigma,g}$ for some B_σ-generic $g \subseteq \mathbb{P}_\sigma$, then $x \in A$. For the other direction, fix $x \in A$ and a B-generic filter $G \subseteq \mathbb{P}$ with $\tau_G = x$. Let σ be a run of $\mathcal{G}_C$ where player I plays according to Σ and, for each $D \in B_\sigma$, $D \cap \mathbb{P}_\sigma$ is nonempty. Then $G \cap \mathbb{P}_\sigma$ is B_σ-generic, and $\tau_{\sigma,G\cap\mathbb{P}_\sigma} = x$.

The set of (x, y, z) such that

- $x \in \omega^\omega$,
- $y \in \kappa^\omega$ is a run of $\mathcal{G}_C$ where I plays according to Σ,
- $z \in 2^\omega$ and
- $y[z^{-1}[\{1\}]]$ is a filter on $y[\omega] \cap \mathbb{P}$ for which the realization of
$$\tau_{y[\omega],y[z^{-1}[\{1\}]]}$$
is x

is the set of paths through a tree on $\omega \times \kappa \times 2$ whose projection is A. □

There is a natural join operation on wellordered sequences of generic ∞-codes, which we now define. Suppose that ζ is an ordinal and $\langle C_\alpha : \alpha < \zeta\rangle$ is a sequence of generic ∞-codes. Let C_α be $(\mathbb{P}_\alpha, B_\alpha, \tau_\alpha)$, for each $\alpha < \zeta$. For each $\alpha < \zeta$, let $\leq_\alpha$ be the order on $\mathrm{dom}(\mathbb{P}_\alpha)$ given by $\mathbb{P}_\alpha$. Let X be the set $\{(\alpha, \gamma) : \gamma \in \mathrm{dom}(\mathbb{P}_\alpha)\}$. Let $\leq_X$ the partial order on X defined by setting $(\alpha, \gamma) \leq_X (\alpha', \gamma')$ to hold if $\alpha = \alpha'$ and $\gamma \leq_\alpha \gamma'$. Let $\mathbb{P}_X$ denote the corresponding partial order on X.

For each $\alpha < \zeta$, let B_α be the sequence $\langle b_{\alpha,\delta} : \delta \in \mathrm{dom}(B_\alpha)\rangle$. Let Y be the set $\{(\alpha, \delta) : \delta \in \mathrm{dom}(B_\alpha)\}$. For each pair (α, δ) in Y, let $e_{\alpha,\delta}$ be the set of $(\beta, \gamma) \in X$ such that either $\beta \neq \alpha$ or $\gamma \in b_{\alpha,\delta}$.

Let Z be the set of pairs $((\alpha, \gamma), (n, m))$ such that $(\alpha, \gamma) \in X$ and the pair $(\gamma, (n, m))$ is in τ_α. Then Z is a nice $\mathbb{P}_X$-name for an element of ω^ω. Let B_* be
$$\{e_{\alpha,\delta} : (\alpha, \delta) \in Y\} \cup \{D_{Z,n} : n \in \omega\}.$$
Then $(\mathbb{P}_X, B_*, Z)$ is a generic code for $\bigcup_{\alpha<\zeta} A_{C_\alpha}$.

Next we map $(\mathbb{P}_X, B_*, Z)$ over to a generic ∞-code. Let $f \colon \eta \to X$ and $g \colon \xi \to Y$ (for some ordinals η, ξ) be the bijections induced by the Gödel ordering on pairs of ordinals. Let $\mathbb{P}$ be the partial order $f(\gamma) \leq_X f(\gamma')$ on η. Let $B = \langle b_\delta : \delta < \omega + \xi\rangle$ be defined by setting each b_n $(n \in \omega)$ to be $f^{-1}[D_{Z,n}]$ for each $n \in \omega$, and each $b_{\omega+\nu}$ to be $f^{-1}[b_{\alpha,\delta}]$, where $g(\nu) = (\alpha, \delta)$. Let τ be the set of pairs $(\gamma, (n, m))$ for which $(f(\gamma), (n, m)) \in Z$. Let C be $(\mathbb{P}, B, \tau)$. Then C is a generic $(\eta \cup \xi)$-code for $\bigcup_{\alpha<\kappa} A_{C_\alpha}$. We call C the *join* of $\langle C_\alpha : \alpha < \zeta\rangle$. Note that if ζ is an infinite cardinal then $(\eta \cup \xi) = \zeta$.

Theorem 11.1.7 below shows that the join operation just defined preserves strong generic codes, under the appropriate form of ordinal determinacy.

Theorem 11.1.7 ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$). *Let κ be an infinite cardinal below Θ such that κ-$\mathsf{Determinacy}$ holds, and let $\bar{C} = \langle C_\alpha : \alpha < \kappa \rangle$ be a sequence of strong κ-codes. Then the join of $\bar{C}$ is a strong generic κ-code for $\bigcup_{\alpha<\kappa} A_{C_\alpha}$.*

Proof. Let $C = (\mathbb{P}, B, \tau)$ be the join of $\bar{C}$. To see that C is a strong generic code, fix a set $A \subseteq \omega^\omega$ of Wadge rank greater than κ. It suffices to show that player II does not have a winning strategy in $\mathcal{G}_C$ in $L(A, \mathbb{R})$, since $L(A, \mathbb{R})$ contains all subsets of κ, by the Coding Lemma. Since $\mathsf{DC}_{\mathbb{R}}$ holds, DC holds in $L(A, \mathbb{R})$. Fixing a strategy Σ for player II, we can find by DC a countable $\sigma \subseteq \eta \cup \xi$ and winning strategies ρ_α for player I in the games $\mathcal{G}_{C_\alpha}$ ($\alpha \in \sigma \cap \eta$) such that

- $\omega \subseteq g[\sigma]$,
- σ is the result of a run of $\mathcal{G}_C$ where II has played by Σ and
- for each $\alpha \in \sigma \cap \eta$, the set

  $$\sigma_\alpha = \{\gamma \in \mathrm{dom}(\mathbb{P}_\alpha) : (\alpha, \gamma) \in f[\sigma]\} \cup \{\delta \in \mathrm{dom}(B_\alpha) : (\alpha, \delta) \in g[\sigma]\}$$

  is the result of a run of $\mathcal{G}_{C_\alpha}$ where player I has played by ρ_α.

Then
$$A_{C_\sigma} = \bigcup_{\alpha\in\sigma} A_{C_{\alpha,\sigma_\alpha}} \subseteq \bigcup_{\alpha\in\sigma} A_{C_\alpha},$$
showing that I wins this run of the game. □

Examining the proof of Theorem 7.0.3, we get an alternate version of Theorem 11.1.7 where instead of assuming κ-$\mathsf{Determinacy}$ we assume that the union is Suslin, and that there is a strong partition cardinal above κ.

Theorem 11.1.8 ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$). *Suppose that $\kappa < \delta$ are infinite cardinals and that δ is a regular cardinal such that*

$$\forall \mu < \delta(\delta \to (\delta)^\mu_\mu)$$

holds. Let $\bar{C} = \langle C_\alpha : \alpha < \kappa \rangle$ be a sequence of strong κ-codes. Let A be $\bigcup_{\alpha<\kappa} A_{C_\alpha}$, and suppose that A is Suslin. Then there is a strong κ-code for A which is definable from $\bar{C}$.

Proof. (Sketch) Let C_α be $(\mathbb{P}_\alpha, B_\alpha, \tau_\alpha)$, for each $\alpha < \kappa$. Let $C = (\mathbb{P}, B, \tau)$ be the join of $\bar{C}$. Then A_C is a generic κ-code for A. We want to see that player I as winning strategy in $\mathcal{G}_C$. As discussed in Remark 11.1.5, $\mathcal{G}_C$ is the same as $\mathcal{G}_{\rho,D}$ for some extension-preserving function $\rho \colon \kappa^{<\omega} \to \omega^{<\omega}$ and some $D \subseteq \omega^\omega$. The Second Periodicity Theorem (see Remark 6.1.6) implies that under $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ the pointclass of Suslin sets is closed under both universal and existential real quantification. It follows that since A_C is Suslin, then the payoff set D is also Suslin. Let T be a tree on $\omega \times \kappa$ projecting to D. To show that $\mathcal{G}_{\rho,D}$ is determined, run the proof of Theorem 7.0.3, omitting all mentions of S there, since there is no S in our context. Since the proof of Claim 1 of Theorem 7.0.3

does not mention S, it suffices to show that player I has a winning strategy in the augmented game $\mathcal{G}^T$ defined there. Since the game $\mathcal{G}^T$ is closed, it suffices to see that player II does not have a winning strategy. However, a winning strategy for player II in $\mathcal{G}^T$ gives one for player II in $\mathcal{G}_C$, and the second half of the proof of Theorem 11.1.8 shows that there can be no such strategy. □

In Section 11.2 we see how to convert ∞-Borel codes into strong generic codes.

11.2 Producing strong generic codes

In this section we use the terms μ_S and $\mathcal{D}_S$ from Definition 8.1.1, for S a set of ordinals. Recall from Section 8.5 that δ_S^∞ denotes $\prod \omega_2^{L[S,x]}/\mu_S$. Recall also from Theorem 11.1.6 that subsets of ω^ω with strong generic codes are Suslin. The following theorem produces strong generic codes.

Theorem 11.2.1. *Let S be a set of ordinals. Suppose that the following hold.*

- *The ultrapower* $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$ *is wellfounded, and* $\delta_S^\infty < \Theta$.
- δ_S^∞-Determinacy.

Then if S is an ∞-Borel code for a set $A \subseteq \omega^\omega$, then A has a strong generic δ_S^∞-code which is contained definable from S.

Proof. Since ω-Determinacy (i.e., AD) holds, $\aleph_1 \not\leq 2^{\aleph_0}$, by Remark 1.1.2. Furthermore, $\leq_S$ is locally countable and μ_S is a countably complete ultrafilter, by the results of Section 1.2. Then by Theorem 8.1.3 we have that for some $x_0 \in \omega^\omega$ and all $y \in \omega^\omega$ such that $[y]_S \geq_S [x]_S$, GCH holds in $L[S, y]$ below ω_1^V (which is strongly inaccessible in $L[S, y]$, since $\aleph_1 \not\leq 2^{\aleph_0}$). For each such y, it follows that the partial order $\mathbb{V}_{\{S\},\omega^2}$ (defined at the beginning of Section 10.1) as computed in $L[S, y]$, being isomorphic to the subset relation restricted to a subset of $\mathcal{P}(\omega^\omega)$, has cardinality at most $\aleph_2$ in $L[S, y]$. Furthermore, since antichains in this partial order correspond to sequences of pairwise disjoint subsets of $\mathcal{P}(\omega^\omega)$, $\mathbb{V}^{L[S,y]}_{\{S\},\omega^2}$ is $\aleph_2$-c.c. in $L[S, y]$, and the set of all antichains of $\mathbb{V}^{L[S,y]}_{\{S\},\omega^2}$ has cardinality $\aleph_2$ in $L[S, y]$.

Since $\omega_3^{L[S,y]}$ is the cardinal successor of $\omega_2^{L[S,y]}$ in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$ (see Remark 10.1.4), it follows, using Theorem 10.1.2 and the definability order in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$ that there is an S-invariant function f on ω^ω such that, for an S-cone of y, $f(y)$ is, in $L[S, y]$, an $\omega_2^{L[S,y]}$-generic code $C_y = (\mathbb{P}_y, B_y, \tau_y)$ for A_{ϕ_S}. Moreover, $\mathbb{P}_y$ can be taken to be a partial order on $\omega_2^{L[S,y]}$ which is isomorphic to $(\mathbb{V}_{\{S\},\omega^2}\restriction p)^{L[S,y]}$, where p is the $(\mathbb{V}_{\{S\},\omega})^{L[S,y]}$-condition corresponding to the set $\{x \in \omega^\omega : L[S, x] \models \psi(S, x)\}$, and B_y and τ_y are the corresponding images, respectively, of the set of all dense open subsets of $(\mathbb{V}_{\{S\},\omega^2}\restriction p)^{L[S,y]}$ in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$ and the canonical name for the generic real added by $(\mathbb{V}_{\{S\},\omega^2}\restriction p)^{L[S,y]}$ (corresponding to the set K from the proof of Theorem 10.1.2).

One key point is for that each such y, A_{C_y} is contained in A_{ϕ_S} as computed in V. To see this, note that if some $\mathrm{HOD}^{L[S,y]}_{\{S\}}$-generic filter gave a counterexample, then, since S is in $\mathrm{HOD}^{L[S,y]}_{\{S\}}$, this would be forced by some condition in $(\mathbb{V}_{\{S\},\omega^2}\lceil p)^{L[S,y]}$. Since $L[S,y]$ contains a $(\mathbb{V}_{\{S\},\omega^2})^{L[S,y]}$-generic real below each condition, such a counterexample would then exist in $L[S,y]$.

Now we consider the ultrapower $V^{\mathcal{D}_S}/\mu_S$, which we have assumed to be wellfounded.[1] Let $C = (\mathbb{P}, B, \tau)$ be the triple represented by the function f. We want to see that this is a strong generic code for A. We have that $\mathrm{dom}(\mathbb{P}) \cup \mathrm{dom}(B)$ is contained in δ^∞_S, which by assumption is less than Θ.

Since δ^∞_S-Determinacy holds, the game $\mathcal{G}_{\mathbb{P},B,\tau}$ from Definition 11.1.3 is determined (see Remark 11.1.5). We want to see then that player II does not have a winning strategy. Suppose towards a contradiction that Σ is such a strategy. For each $y \in \omega^\omega$ such that $y \geq x_0$, let R_y be the tree of attempts to find a countable $\sigma \subseteq \delta^\infty_S$ resulting from a play of $\mathcal{G}_C$ according to Σ, and an isomorphism between $C_y = (\mathbb{P}_y, B_y, \tau_y)$ and $C_\sigma = (\mathbb{P}_\sigma, B_\sigma, \tau_\sigma)$. If any of these trees is illfounded, then we have a contradiction. If not, then (since there is in each case a wellordering of R_y definable from Σ, C and C_y) we can find (in an S-invariant fashion) ranking functions ρ_y on each such tree R_y. Let j be the map sending each element of V to the element of $V^{\mathcal{D}_S}/\mu_S$ represented by the corresponding constant function. In addition, let R^* and ρ^* be the elements of $V^{\mathcal{D}_S}/\mu_S$ represented by the maps $y \mapsto R_y$ and $y \mapsto \rho_y$. Then in the wellfounded model

$$\prod \mathrm{HOD}^{L[S,C,\Sigma,y]}_{\{S,C,\Sigma\}}/\mu_S,$$

ρ^* is a ranking function on the tree R^* of attempts to a find countable $\sigma \subseteq j(\delta^\infty_S)$ resulting from a play of $j(\mathcal{G}_C)$ according to $j(\Sigma)$, and an isomorphism between C and $j(C)_{j[\sigma]}$. Since $j[\delta^\infty_S]$ is closed under $j(\Sigma)$, and (letting B be the sequence $\langle b_\alpha : \alpha \in \mathrm{dom}(B)\rangle$) C is isomorphic to

$$(j[\mathbb{P}], \langle j[b_{j(\alpha)}] : \alpha \in \mathrm{dom}(B)\rangle, j[\tau]),$$

which is

$$j(C)_{j[\delta^\infty_S]} = (j(\mathbb{P})_{j[\delta^\infty_S]}, j(B)_{j[\delta^\infty_S]}, j(\tau)_{j[\delta^\infty_S]}),$$

there is a nonempty subset of R^* without terminal nodes, corresponding to those nodes where the isomorphism is contained in j, and the run of the game is contained in $j[\delta^\infty_S]$. It follows that R^* is illfounded, giving the desired contradiction. $\square$

11.2.2 Remark. Aside from the use of AD at the beginning of the proof, the only use of δ^∞_S-Determinacy in the proof of Theorem 11.2.1 was to show the determinacy of the game $\mathcal{G}_C$. Since each C_y in the proof is a generic code in $L[S,y]$ for a subset of A containing $A \cap L[S,y]$, C is a generic code for A. If A

[1] I think we need only $\mathsf{DC}_\mathbb{R}$ in this case, since we can assume that we are working in a model of the form $L(S, \omega^\omega, B)$ for some $B \subseteq \omega^\omega$. Check! Recall that by Solovay's theorem we get $\mathsf{DC}_\mathbb{R}$ from this assumption.

happens to be Suslin, then, $\mathcal{G}_C$ is determined, then, by Remark 11.1.4. It follows that the assumption of δ^∞_S-Determinacy can be removed from the statement of Theorem 11.2.1, if one assumes instead that A is Suslin. The point then is that the theorem gives a way of picking Suslin representations using ∞-Borel representations, for Suslin sets.

Theorem 11.1.6 gives the following.

Theorem 11.2.3 (ZF + $\mathsf{DC}_{\mathbb{R}}$ + $<\Theta$-Determinacy)**.** *Every subset of ω^ω which has an ∞-Borel code which is not $\leq_{\mathcal{D}}$-maximal is Suslin.*

Proof. Suppose that $A \subseteq \omega^\omega$ has an ∞-Borel code which is not $\leq_{\mathcal{D}}$-maximal. A reflection argument shows that there exist bounded subsets S and T of Θ such that $S <_{\mathcal{D}} T$ and S is an ∞-Borel code for A. By Theorem 8.5.1, and part (2b) of Theorem 8.3.8, $\delta^\infty_S < \Theta$. Letting $B \subseteq$ have Wadge rank greater than δ^∞_S, the hypothesis of Theorem 11.2.1 (with respect to S) are satisfied in $L(B, \mathbb{R})$. It follows from Theorem 11.1.6 that A is Suslin. $\square$

In addition, one gets the following equivalences under AD^+.

Theorem 11.2.4 (ZF + AD^+)**.** *The following are equivalent.*

1. *There is no $\leq_{\mathcal{D}}$-maximal set of ordinals.*
2. *The Suslin cardinals are cofinal in Θ.*
3. *Every subset of ω^ω is Suslin.*
4. Uniformization

Proof. The equivalence of (2) and (3) is Corollary 6.1.18, which requires only AD. By Theorem 11.2.3, (1) implies (3). That (3) implies (4) follows from Theorem 6.2.1. The equivalence of (4) and (1) follows from Corollary 10.2.7. $\square$

Recall from Theorem 6.3.2 that if every subset of ω^ω has a Wadge rank, and Lipschitz Determinacy and Uniformization hold, then the length of the Solovay sequence is a limit ordinal. A version of the converse holds assuming AD^+.

Theorem 11.2.5. *If AD^+ holds and the Solovay sequence has limit length, then every subset of ω^ω is Suslin.*

Proof. By Theorem 11.2.3 it suffices to show every subset of ω^ω has an ∞-Borel code which is not $\leq_{\mathcal{D}}$-maximal. Suppose that $A \subseteq \omega^\omega$ is a counterexample. Applying Remark 9.1.11, fix an ∞-Borel code S for A which is a bounded subset of Θ. The set S_∞ from Section 8.6 is definable from S, and it follows from Theorem 8.6.6 that every subset of ω^ω is definable from S_∞ and a real. If $B \subseteq \omega^\omega$ has Wadge rank on the Solovay sequence, and greater than the supremum of S, it follows by the Moschovakis Coding Lemma that S, and thus every set of reals, is definable from B and a real. This implies that Θ is the least member of the Solovay sequence above the Wadge rank of B, contradicting our hypothesis that the Solovay sequence has limit length. $\square$

11.3 ∞-Borel representations from Uniformization

The main theorem of this section is Theorem 11.3.3, which says that if AD and Uniformization hold, then all sets of reals are ∞-Borel. Theorem 11.3.2 is a slightly stronger version, in which the assumption of AD is replaced with some of its consequences. Again we use the notation from Definition 8.1.1, with S as $\emptyset$. Note that by Theorems 12.3.1 and 13.0.1, $\mathsf{AD}_{\mathbb{R}}$ follows from $\mathsf{AD} + \mathsf{DC} +$ Uniformization. Theorem 11.3.2 is Theorem 5.10 of [21], whose presentation we follow closely.

Before beginning the proof of Theorem 11.3.2 we introduce some terminology for treating generic filters over countable structures in terms of descriptive set theory, using terms introduced in see Section 0.2. Given a nice $\mathbb{P}$-name τ and a set $g \subseteq \tau$, we let τ_g be the set of pairs (n, m) for which there exists a $p \in g$ with $(p, (n, m)) \in \tau$. We say that a set Y consisting of filters on $\mathcal{P}(\mathbb{P})$ is *comeager* if there is a countable set X such that Y contains every X-generic filter contained in $\mathbb{P}$. We write $p \Vdash^* \tau \in A$ to mean that for a comeager set of filters $g \subseteq \mathbb{P}$, τ_g is in A, and define $p \Vdash^* \tau \notin A$ similarly. We let $D^A_{\mathbb{P},\tau}$ be the set of $p \in \mathbb{P}$ such that either $p \Vdash^* \tau \in A$ or $p \Vdash^* \tau \notin A$. If $\mathbb{P}$ is countable, and every subset of ω^ω has the property of Baire, then each set of the form $D^A_{\mathbb{P},\tau}$ is dense in $\mathbb{P}$.

Given $A \subseteq \omega^\omega$, the *term relation* for A is the class $\mathrm{TR}(A)$ consisting of those triples $(\mathbb{P}, p, \tau)$ such that $\mathbb{P}$ is a poset, τ is a nice $\mathbb{P}$-name for an element of ω^ω and $p \Vdash^* \tau \in A$. Note that $\mathrm{TR}(A)$ is definable from A. We write $\dot{t}_{A,\mathbb{P}}$ for the set of (p, τ) for which $(\mathbb{P}, p, \tau) \in \mathrm{TR}(A)$. Note that $\dot{t}_{A,\mathbb{P}}$ is a $\mathbb{P}$-name, as is every subset of it. If $\mathbb{P}$ is a countable partial order, T is a countable set of nice $\mathbb{P}$-names for reals and each set $D^A_{\mathbb{P},\tau}$ $(\tau \in T)$ is dense in $\mathbb{P}$, then, for comeagerly many filters $g \subseteq \mathbb{P}$, for all $\tau \in T$, $\tau_g \in A$ if and only if, for some $p \in g$, $(p, \tau) \in \dot{t}_{A,\mathbb{P}}$.

We say that a transitive set M is said to be *weakly-*$(A, \mathbb{P})$*-closed* if $\dot{t}_{A,\mathbb{P}} \cap M \in M$. Applying the definition to the set of names of the form $\check{x}$ $(x \in \omega^\omega \cap M)$ we see that this implies that $A \cap M \in M$, if M is a model of a certain weak fragment of ZF. We say that M is *strongly-*$(A, \mathbb{P})$*-closed* if it is weakly $(A, \mathbb{P})$-closed and, for all M-generic filters $g \subseteq \mathbb{P}$, $(\dot{t}_{A,\mathbb{P}} \cap M)_g$ (that is, $\{\tau_g : \exists p \in g\, (p, \tau) \in \dot{t}_{A,\mathbb{P}} \cap M\}$) is equal to $A \cap M[g]$ (here were are referring to existing M-generic filters, not filters existing in a forcing extension of V). We say that M is weakly (or strongly) A-closed if it is weakly (or strongly) $(A, \mathbb{P})$-closed for all partial orders $\mathbb{P} \in M$.

11.3.1 Remark. Suppose that M is an inner model of ZFC, and that $\mathbb{P}$ is a partial order in M such that $\mathcal{P}(\mathbb{P}) \cap M$ is countable. Let $A \subseteq \omega^\omega$ be such that M is strongly $(A, \mathbb{P})$-closed. Let S be a set of ordinals in M such that $\mathcal{P}(\mathbb{P}) \cap M$ and $\dot{t}_{A,\mathbb{P}} \cap M$ are both in $L[S]$. Suppose that N is an inner model with $S \in N$, such that $A \cap N \in N$ and, for every $x \in \omega^\omega \cap N$, there is an $L[S]$-generic filter $g \subseteq \mathbb{P}$ such that $L[S, x] = L[S][g]$. Then, in N, $A \cap N$ is the set of x such that $L[S, x] \models x \in (\dot{t}_{A,\mathbb{P}} \cap M)_g$, where $g \subseteq \mathbb{P}$ any $L[S]$ generic filter such that $L[S, x] = L[S][g]$. It follows that $A \cap N$ is ∞-Borel in N.

The appendix to [27] contains a proof of the general forcing fact that whenever $\mathbb{P}$ is a partial order and τ is a $\mathbb{P}$-name for a subset of the ground model, there is a partial order $\mathbb{P}'$ such that, whenever $g \subseteq \mathbb{P}$ is V-generic, there is a

V-generic filter $h \subseteq \mathbb{P}'$ such that $V[h] = V[\tau_g]$.

Theorem 11.3.2 ($\mathsf{ZF} + \mathsf{Uniformization} + \aleph_1 \not\leq 2^{\aleph_0}$). *Suppose that every subset of ω^ω has the Baire property, and that $\mu_\emptyset$ is an ultrafilter on $\mathcal{D}_\emptyset$. Then every subset of ω^ω is ∞-Borel.*

Proof. Fix $A \subseteq \omega^\omega$. Let B be the set of $(x, y) \in (\omega^\omega)^2$ such that

- x HC-codes a pair $(\mathbb{P}, \tau)$ where $\mathbb{P}$ is a countable poset and τ is a nice $\mathbb{P}$-name for an element of ω^ω;
- y HC-codes a countable set $\mathcal{D}$ consisting of dense open subsets of $\mathbb{P}$, with $D^A_{\mathbb{P},\tau} \in \mathcal{D}$, such that for any $\mathcal{D}$-generic filter $g \subseteq \mathbb{P}$,
  - $\tau_g \in A$ if and only if $\exists p \in g\ p \Vdash^* \tau \in A$, and
  - $\tau_g \notin A$ if and only if $\exists p \in g\ p \Vdash^* \tau \notin A$.

The conclusion of the condition on y is equivalent to : for all $\mathcal{D}$-generic filters $g \subseteq \mathbb{P}$, $\tau_g \in \dot{t}_{A,\mathbb{P},g}$ if and only if $\tau_g \in A$. By the density of the sets $D^A_{\mathbb{P},\tau}$ (mentioned above), the assumption that every set of reals has the property of Baire implies that for each $x \in \omega^\omega$ coding a pair $(\mathbb{P}, \tau)$ as above, there is a $y \in \omega^\omega$ such that $(x, y) \in B$.

Let f be a function uniformizing B, and $F \subseteq \omega^\omega$ code f as follows : $x \in F$ if and only if, letting $x' \in \omega^\omega$ be such that $x'(n) = x(n+2)$ for all $n \in \omega$, $x' \in \mathrm{dom}(f)$ and $f(x')(x(0)) = x(1)$. For each $x \in \omega^\omega$, let N_x be

$$L_{\omega_1}[\mathrm{TR}(A), \mathrm{TR}(F), x],$$

and let M_x be

$$\mathrm{HOD}^{N_x}_{\{\mathrm{TR}(A),\mathrm{TR}(F)\}}.$$

Then the models N_x and M_x are weakly A-closed and weakly F-closed, and since $\aleph_1 \not\leq 2^{\aleph_0}$ they are models of ZFC.

Claim 1. *For each $x \in \omega^\omega$, M_x is strongly A-closed.*

Proof of Claim 1. Fix $x \in \omega^\omega$ and a partial order $\mathbb{P} \in M_x$. Let C be the set of nice $\mathbb{P}$-names in M_x for elements of ω^ω. Let $\mathbb{Q}$ be $\mathrm{Col}(\omega, \mathbb{P})$, and for each $\tau \in C$, let $\dot{z}_\tau$ be a nice $\mathbb{Q}$-name for an element of ω^ω HC-coding the pair $(\mathbb{P}, \tau)$.

Since M_x is weakly F-closed, $\dot{t}_{F,\mathbb{Q}} \cap M_x$ is in M_x. Let $\mathcal{E}$ be a countable set of dense open subsets of $\mathbb{Q}$, containing each dense open subset of $\mathbb{Q}$ from M_x and having the property that for each $\mathcal{E}$-generic $h \subseteq \mathbb{Q}$ and each nice name $\rho \in M_x$ for an element of ω^ω, $\rho_h \in F$ if and only if, for some $p \in h$, $(p, \rho) \in \dot{t}_{F,\mathbb{Q}}$. Applying this property to the collection of nice $\mathbb{Q}$-names ρ for which it is forced by $1_\mathbb{Q}$ that (for some $\tau \in C$) $\rho(n+2) = \dot{z}_\tau(n)$ for all $n \in \omega$, we get that for any $\mathcal{E}$-generic filter $h \subseteq \mathbb{Q}$, and for each $\tau \in C$, $\dot{z}_{\tau,h} \in \omega^\omega$ and $f(\dot{z}_{\tau,h})$ is in $M_\sigma[h]$, being the set of pairs (i, j) for which there exists a pair $(p, \rho) \in \dot{t}_{F,\mathbb{Q}} \cap M_x$ with $p \in h$, $(\rho_h(0), \rho_h(1)) = (i, j)$ and $\rho_h(n+2) = \dot{z}_{\tau,h}(n)$ for all $n \in \omega$. Fix such a h, and let $\mathcal{D}_h$ be the collection of dense open subsets of $\mathbb{P}$ which are HC-coded by the members of $\{f(\dot{z}_{\tau,h}) : \tau \in C\}$.

Let $g \subseteq \mathbb{P}$ be an $M_x[h]$-generic filter. Then g is $\mathcal{D}_h$-generic. It follows then that $(\dot{t}_{A,\mathbb{P}} \cap M_x)_g = A \cap M_\sigma[g]$. Since $M_x[h][g] = M_x[g][h]$ (i.e., (g,h) is generic for $\mathbb{P} * \mathbb{Q}$), it follows that $(\dot{t}_{A,\mathbb{P}} \cap M_x)_g = A \cap M_x[g]$ holds for any M_x-generic filter $g \subseteq \mathbb{P}$, as desired. This ends the proof of Claim 1. $\square$

Claim 2. *There is a $\emptyset$-invariant function c on ω^ω such that, for each $x \in \omega^\omega$, $c(x)$ is a pair $(S_x, \phi_x) \in M_x$ such that, for all $y \in \omega^\omega \cap N_x$, $y \in A$ if and only if $L[S_x, y] \models \phi_x(S_x, y)$.*

Proof of Claim 2. This follows from Remark 11.3.1, with M_x as M and N_x as N, using Theorem 10.1.2 and the fact, shown in the proof of Theorem 10.1.2, that there is a $\mathbb{V}_{\bar{x},Y}$-name τ (using K) such that $\tau_{G_E} = E$, for all $E \subseteq Y$ (here $\bar{x}$ is $\langle \mathrm{TR}(A), \mathrm{TR}(F)\rangle$ and Y is $\omega \times \omega$). $\square$

We complete the proof working in the inner model $L(\mathrm{TR}(A), \mathrm{TR}(F), \omega^\omega)$. Since $\mathsf{Uniformization}$ holds, $\mathsf{DC}_\mathbb{R}$ holds. It follows that $L(\mathrm{TR}(A), \mathrm{TR}(F), \omega^\omega)$ satisfies DC, and, since $\mu_\emptyset$ is a countably complete ultrafilter, that $\mathrm{Ord}^{\mathcal{D}_\emptyset}/\mu_\emptyset$ is wellfounded when computed in $L(\mathrm{TR}(A), \mathrm{TR}(F), \omega^\omega)$. The function $x \mapsto \phi_x$ from Claim 2 is constant on a $\mu_\emptyset$-cone; let ϕ be the constant value. The function $x \mapsto S_x$ represents a set of ordinals S. The function $x \mapsto A \cap L[x]$ represents A. For each $y \in \omega^\omega$, since the function $x \mapsto S_x$ is $\emptyset$-invariant and each model of the form $L[S_x, A, y]$ satisfies Choice, whenever

- $f_0, \ldots, f_{n-1}$ are $\emptyset$-invariant functions on ω^ω in $L(\mathrm{TR}(A), \mathrm{TR}(F), \mu, \omega^\omega)$ such that $f_i(x) \in L[S_x, A, y]$ for all $x \in \omega^\omega$ and $i < n$, and
- ψ is an n-ary formula,

$L[S, A, y] \models \psi([f_0]_\emptyset, \ldots, [f_{n-1}]_\emptyset)$ if and only if

$$\{x \in \omega^\omega : L[S_x, A, y] \models \psi(f_0(x), \ldots, f_{n-1}(x))\} \in \mu_\emptyset.$$

For each $y \in \omega^\omega$, $y \in N_x$ for $\mu_\emptyset$-cone of x. By Claim 2 then, letting ψ be the formula $y \in A \Leftrightarrow L[S, y] \models \phi(S, y)$, this implies that the pair (S, ϕ) witnesses that A is ∞-Borel. $\square$

Putting together Remark 1.1.2 (for $\aleph_1 \not\leq 2^{\aleph_0}$ and $\mathsf{Baire}(\mathcal{P}(\omega^\omega))$) and Corollary 1.2.6 (for $\mu_\emptyset$) with Theorem 11.3.2, we have the following.

Theorem 11.3.3. *If* AD + $\mathsf{Uniformization}$ *holds then every subset of ω^ω is ∞-Borel.*

11.4 Closure of the Suslin cardinals

Recall from Chapter 6 that a cardinal κ is *Suslin* if there is a set $A \subseteq \omega^\omega$ which is κ-Suslin but not γ-Suslin for any $\gamma < \kappa$. In this section we prove Theorem 11.4.1 below, on the closure of the set of Suslin cardinals. Corollary 11.4.4 gives that under AD^+ this set is closed below Θ. Again, we follow the presentation in [21]. [2]

Theorem 11.4.1 ($\mathsf{ZF} + \mathsf{DC}_{\mathbb{R}}$; Steel-Woodin). *Suppose that*

1. $\kappa < \Theta$ *is a limit of Suslin cardinals,*
2. κ-Determinacy *holds and that*
3. $\langle S_\alpha : \alpha < \kappa\rangle$ *is a sequence of bounded subsets of* κ *which are* ∞*-Borel codes for pairwise disjoint subsets of* 2^ω*.*

Then κ *is a Suslin cardinal.*

It follows from AD alone (or, more directly part (1) of Corollary 3.0.2 and the fact that AD implies $\mathsf{CC}_{\mathbb{R}}$; see part (1) of Remark 1.1.2) that if $\kappa < \Theta$ is a limit of an ω-sequence of Suslin cardinals, then κ is a Suslin cardinal. We assume then for the rest of this section that $\mathrm{cof}(\kappa)$ is uncountable, in which case (since we are assuming $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$) the $<\kappa$-Borel sets and the $<\kappa$-Suslin sets coincide, by Theorem 9.1.15.

11.4.2 Remark. The map $\phi \mapsto \phi^*$ from Section 9.2 shows that a sequence $\langle S_\alpha : \alpha < \kappa\rangle$ as in hypothesis (3) of Theorem 11.4.1 exists if there is an ∞-Borel set which is not κ-Borel. By Remark 10.3.8 (assuming $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$) if there is an ∞-Borel set which is not $<\kappa$-Borel, then there is one which is not κ-Borel. If $\kappa < \Theta$, then the existence of a ∞-Borel set which is not κ-Borel then follows from the assumption that every subset of 2^ω is ∞-Borel (and the Coding Lemma). In the case where κ is a limit of Suslin cardinals with $\mathrm{cof}(\kappa) > \omega$ (and assuming $\mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ but not necessarily that every sets of reals is ∞-Borel), Theorem 9.1.15 implies that if there is a Suslin cardinal greater than or equal to κ then there is an ∞-Borel set which is not $<\kappa$-Borel, and therefore one which is not κ-Borel.

Let μ_S (for $y \in \omega^\omega$ and S a set of ordinals) be as defined in Section 8.5. Assuming AD, μ_S is an ultrafilter on the set of S-degrees, for each such S, by Corollary 1.2.6. In this section we will use only the case $S = \emptyset$. While it is not necessary for the present argument, we note that the two following lemmas imply the corresponding version where $\emptyset$ is replaced with an arbitrary set of ordinals.

Lemma 11.4.3 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$). *If* κ *is a limit of Suslin cardinals of uncountable cofinality, then for all bounded* $S \subseteq \kappa$, $\prod \omega_2^{L[S,x]}/\mu_\emptyset < \kappa$*.*

Proof. Fix a bounded $S \subseteq \kappa$. By Theorem 9.1.15, every $<\kappa$-Borel set is in $\mathcal{S}_{<\kappa}$. In particular the set $\{(x, y) : y \notin L[S, x]\}$ is $<\kappa$-Suslin, so (by the leftmost

[2] Given AD, AD^+ is equivalent to the Suslin cardinals being closed below Θ. Θ-Determinacy would follow from the existence of a strong partition cardinal above Θ, which may not be possible. What is the relationship between the Wadge rank of a set of reals, and the least cardinal for which it is Suslin, if any?

branch construction for uniformizing Suslin sets) there is a bounded $T \subseteq \kappa$ such that $S <_{\mathcal{D}} T$. It follows from part (2b) of Theorem 8.3.8 that the set of $[x]_\emptyset$ for which $\omega_2^{L[S,x]} < \omega_1^{L[T,x]}$ is in $\mu_\emptyset$. The relation $\leq_\emptyset \times <_T^{\mathrm{c}}$ from the Theorem 8.5.1) is $<\kappa$-Borel, so also in $\mathcal{S}_{<\kappa}$. By Remark 6.1.5, $\mathcal{S}_{<\kappa}$ is a projective algebra, so every relation projective in $\leq_\emptyset \times <_T^{\mathrm{c}}$ is also in $\mathcal{S}_{<\kappa}$. By the Kunen-Martin Theorem (Theorem 6.1.16) every prewellordering in $\mathcal{S}_{<\kappa}$ has length less than κ. The lemma then follows from Theorem 8.5.1. □

The proof of Theorem 11.4.1 is completed by appealing to Theorems 11.1.7 and 11.2.1.

Proof of Theorem 11.4.1. We assume that κ has uncountable cofinality, as the countable cofinality case was dealt with just after the statement of the theorem. For each $\alpha < \kappa$, let A_α be the subset of 2^ω coded by S_α. Using the coding at the beginning of Chapter 8, we may find $S_{\alpha,\beta}$ ($\alpha \leq \beta < \kappa$), bounded subsets of κ, such that each $S_{\alpha,\beta}$ is an ∞-Borel code for the set $A_\alpha \times A_\beta$. Lemma 11.4.3 and Theorem 11.2.1 give a sequence $\langle C_{\alpha,\beta} : \alpha \leq \beta < \kappa \rangle$, where each $C_{\alpha,\beta}$ is a generic $\gamma_{\alpha,\beta}$-code for the corresponding set $A_\alpha \times A_\beta$, for some $\gamma_{\alpha,\beta} < \kappa$. Let $\leq$ be $\bigcup_{\alpha\leq\beta<\kappa} A_\alpha \times A_\beta$. Then $\leq$ is a prewellording of length κ. By the Kunen-Martin Theorem (Theorem 6.1.16), $\leq$ cannot be $<\kappa$-Suslin. It suffices to see then that $\leq$ is κ-Suslin.

By Theorem 11.1.6, there exists then a sequence $\langle T_{\alpha,\beta} : \alpha \leq \beta < \kappa \rangle$ of trees on $\omega \times \omega \times \kappa$ such that each $T_{\alpha,\beta}$ projects to the corresponding set $A_\alpha \times A_\beta$. This sequence of trees induces a κ-Suslin representation for $\leq$. Alternately, the set $\bigcup_{\alpha\leq\beta<\kappa} A_\alpha \times A_\beta$ has a strong κ-generic code by Theorem 11.1.7, and, since all indices appearing in this sentence are in κ, it induces a κ-Suslin representation for $\leq$ by Theorem 11.1.6.

□

By Remark 11.4.2, Theorem 11.4.1 has the following consequence.

Corollary 11.4.4 ($\mathsf{ZF} + \mathsf{AD}^+$)**.** *The set of Suslin cardinals is closed below* Θ*.*

Replacing the use of Theorem 11.2.1 in the proof of Theorem 11.4.1 with the alternate version described in Remark 11.2.2, we get the following weaker conclusion under AD.

Theorem 11.4.5 ($\mathsf{ZF} + \mathsf{AD}$)**.** *The set of Suslin cardinals is closed below its supremum.*

Chapter 12

Scales from Uniformization

In this chapter we will prove Theorem 12.3.1, which says that $\mathsf{AD} + \mathsf{DC} + \mathsf{Uniformization}$ implies that every subset of ω^ω is Suslin. The proof is a combination of arguments of Becker, Harrington, Kechris, Steel and Woodin.

12.1 Ordinal determinacy in the codes

The proof of Theorem 12.3.1 uses a weaker notion of strong generic code (introduced in Section 12.2) where the players play real number codes for ordinals. The determinacy of the associated game follows from AD, by Theorem 12.1.1, which is a variation of the Harrington-Kechris theorem [4] on the determinacy (under AD) of certain real games where the payoff set depends only on the rank of the reals played in a fixed prewellordering.

In this chapter, we will say that a sequence $\bar{P} = \langle P_i : i \in \omega \rangle$ of subsets of ω^ω is *good* if, for each $i \in \omega$, every subset of $\omega^\omega \times \omega^\omega$ which is projective in P_i is uniformized by a function which is Σ^1_1 in P_{i+1} (without parameters). Recall that for any set $P \subseteq \omega^\omega$, there is a universal $\Sigma^1_1(P)$ set $U \subseteq \omega \times \omega^\omega$, in the sense that every $\Sigma^1_1(P)$ subset of ω^ω equal to $\{x : (n, x) \in U\}$ for some $n \in \omega$. Moreover, such a set can be defined from P, so in particular for any sequence $\langle P_i : i \in \omega \rangle$ of subsets of ω^ω there is a sequence $\langle U_i : i \in \omega \rangle$ such that each U_i is a universal $\Sigma^1_1(P_i)$ subset of $\omega \times \omega^\omega$. We will use a special case of this fact below.

We say that a sequence $\langle \leq_i : i < \omega \rangle$ of prewellorderings of 2^ω is *suitable* if there exists a good sequence $\langle P_i : i \in \omega \rangle$ of subsets of ω^ω such that, for all i, $\leq_i$ is projective in P_i.

Now suppose that $Q = \langle \leq_i : i \in \omega \rangle$ is a sequence of prewellorderings of 2^ω, and, for each $i \in \omega$, let γ_i be the length of $\leq_i$ and ρ_i be the associated rank function. Given $A \subseteq \prod_{i \in \omega} \gamma_i$, we let $\mathcal{G}(Q, A)$ be the game where players I and II successively pick $u_i \in 2^\omega$ and player I wins if and only if $\langle \rho_i(u_i) : i \in \omega \rangle \in A$.

Theorem 12.1.1 is a version of the Harrington-Kechris theorem from [4]. The proof we give is the Harrington-Kechris proof as modified by Becker and

Woodin.

Theorem 12.1.1 (ZF + AD). *Let $Q = \langle \leq_i : i < \omega \rangle$ be a suitable sequence of prewellorderings of 2^ω, and for each $i \in \omega$ let γ_i be the length of $\leq_i$. Then for each $A \subseteq \prod_{i \in \omega} \gamma_i$ the game $\mathcal{G}(Q, A)$ is determined.*

The rest of this section is a proof of Theorem 12.1.1, so we assume AD throughout. We fix a suitable sequence $Q = \langle \leq_i : i < \omega \rangle$ of prewellorderings of 2^ω, as witnessed by sets $P_i \subseteq \omega^\omega$ ($i \in \omega$). For each $i \in \omega$ we let γ_i be the length of $\leq_i$ and ρ_i be the corresponding rank function. Fix $A \subseteq \prod_{i \in \omega} \gamma_i$.

We say that a subtree T of $2^{<\omega}$ is 2-*perfect* if each node of T has a unique shortest pair of incompatible extensions. There is then a natural bijection between $[T]$ and 2^ω. Let $\mathcal{T}_{\mathrm{P}}$ be the set of 2-perfect subtrees of $2^{<\omega}$.

Using the uniform existence of universal $\Sigma^1_1(P)$ sets (discussed above), we can fix a sequence $\langle U_i : i \in \omega \rangle$ such that each U_i is a $\Sigma^1_1(P_i)$-universal subset of $\omega \times (2^\omega)^2$. Fix a sequence $\langle T_i : i \in \omega \rangle$ such that each T_i is a function from $\omega \times 2^\omega$ to $(\mathcal{T}_{\mathrm{P}})^\omega$, and, for all $(i, n, x, y) \in (\omega)^2 \times (2^\omega)^2$, $T_i(n, x)$ is the element of $(\mathcal{T}_{\mathrm{P}})^\omega$ which is HC-coded by the unique y with $(n, x, y) \in U_i$, if there exists such a y. Then, for each $i \in \omega$, every $\Sigma^1_1(P_i)$ function from 2^ω to $(\mathcal{T}_{\mathrm{P}})^\omega$ is equal to the function $x \mapsto T_i(n, x)$, for some $n \in \omega$

The Kuratowski-Ulam theorem (see 5A.9 of [32]) implies that if every subset of ω^ω has the property of Baire then every wellordered union of meager sets is meager. The proof of Lemma 12.1.2 uses the Kuratowski-Ulam theorem and a diagonalization along the lines of the Recursion Theorem (Theorem 2.4.8), which results in the double use of m in the formula $T_{i+1}(m, y)_m$.

Lemma 12.1.2. *For each $i \in \omega$ and each $H : \omega \times (2^\omega)^2 \to 2^\omega$ projective in P_i there exists an $m \in \omega$ such that for each $y \in 2^\omega$, $\rho_i(H(m, y, z))$ is the same for all $z \in [T_{i+1}(m, y)_m]$.*

Proof. Fix $i \in \omega$. Let B be the set of (m, y, T) such that $(m, y) \in \omega \times 2^\omega$, $T \in \mathcal{T}_{\mathrm{P}}$ and $\rho_i(H(m, y, z))$ is the same for all $z \in [T]$. Then B is projective in P_i, and, by Kuratowksi-Ulam, for each (m, y) there is a T such that $(m, y, T) \in B$.

Since $\langle P_j : j \in \omega \rangle$ is good, there is a $\Sigma^1_1(P_{i+1})$ function g on $\omega \times 2^\omega$ such that, for all $(m, y) \in \omega \times 2^\omega$, $(m, y, g(m, y)) \in B$. Moreover, the function sending each $y \in 2^\omega$ to the sequence $\langle g(n, y) : n \in \omega \rangle$ is $\Sigma^1_1(P_{i+1})$, so for some $m \in \omega$ we have that $g(n, y) = T_{i+1}(m, y)_n$ for all $(n, y) \in \omega \times 2^\omega$. Then for all $y \in 2^\omega$, $T_{i+1}(m, y)_m = g(m, y)$, so $\rho_i(H(m, y, z))$ is the same for all paths $z \in [T_{i+1}(m, y)_m]$. □

We next define an integer game whose determinacy (which follows from AD) implies the determinacy of $\mathcal{G}(Q, A)$.

Fix a recursive homeomorphism χ between 2^ω and $2^\omega \times \omega \times (2^\omega)^2$. Given $i \in \omega$, say that $(n, w, x) \in \omega \times 2^\omega \times \omega^\omega$ is *i-good* if $x \in [T_i(n, w)_n]$. For each $i \in \omega$, let

$$F_i : \omega \times (2^\omega)^2 \to 2^\omega \times \omega \times (2^\omega)^2$$

be the function which on input (n, w, x) returns the χ-value of the image of x under the natural bijection between $[T_i(n, w)_n]$ and 2^ω if $x \in [T_i(n, w)_n]$ (i.e., if

(n, w, x) is i-good), and some arbitrary fixed value otherwise. For all $i, n \in \omega$ and $w \in 2^\omega$, then, the function sending x to $F_i(n, w, x)$ is a bijection between $[T_i(n, w)_n]$ and $2^\omega \times \omega \times (2^\omega)^2$. Given

$$(n, w, u, m, y, z) \in \omega \times (2^\omega)^2 \times \omega \times (2^\omega)^2,$$

we let $F_i^*(n, w, u, m, y, z)$ be the unique $x \in [T_i(n, w)_n]$ such that $F_i(n, w, x) = (u, m, y, z)$.

12.1.3 Remark. For each $i \in \omega$, the functions F_i and F_i^* are defined using T_i, which was defined from U_i, which is a universal $\Sigma^1_1(P_i)$ set. In particular, F_i and F_i^* are projective in P_i.

We let $\mathcal{G}'(Q, A)$ be the integer game in which player I plays $(n_0, w_0, x_0) \in \omega \times (2^\omega)^2$ and player II plays $(m_0, y_0, z_0) \in \omega \times (2^\omega)^2$, both one coordinate per move, as in the following diagram.

$$\begin{array}{c|cccccccc}
\mathrm{I} & n_0 & & w_0(0) & & x_0(0) & & w_0(1) & & \dots \\
\hline
\mathrm{II} & & m_0 & & y_0(0) & & z_0(0) & & \dots &
\end{array}$$

The game $\mathcal{G}'_{Q,A}$.

To determine the winner of the game, we then let, for each $i \in \omega$,

$$(u_{2i}, n_{i+1}, w_{i+1}, x_{i+1})$$

be $F_{2i}(n_i, w_i, x_i)$ and

$$(u_{2i+1}, m_{i+1}, y_{i+1}, z_{i+1})$$

be $F_{2i+1}(m_i, y_i, z_i)$. Player I wins this run of $\mathcal{G}'(Q, A)$ if and only if either

- there is an $i \in \omega$ such that (m_i, y_i, z_i) is not $(2i+1)$-good, and such that, for all $j \leq i$, (n_j, w_j, x_j) is $(2j)$-good, or
- (n_i, w_i, x_i) is $(2i)$-good for all $i \in \omega$, and $\langle \rho_i(u_i) : i < \omega \rangle \in A$.

Since $\mathcal{G}'(Q, A)$ is an integer game, AD implies that it is determined. We will show that if either player I has a winning strategy in $\mathcal{G}'(Q, A)$ then the same player has one in $\mathcal{G}(Q, A)$. This gives the determinacy of each game $\mathcal{G}(Q, A)$ under AD.

We first fix a winning strategy σ for player I in $\mathcal{G}'(Q, A)$, and write $(n, w, x) = \sigma(m, y, z)$ to mean that (n, w, x) is the result of player I's moves when he plays according to σ and player II plays (m, y, z). Our desired strategy for $\mathcal{G}(Q, A)$ will be induced by the function E given in the following lemma.

Lemma 12.1.4. *There is a function* $E\colon (2^\omega)^{<\omega} \to (\omega \times 2^\omega \times 2^\omega)^{<\omega}$ *such that*

1. *for all* $t, t' \in (2^\omega)^{<\omega}$, *if* $t \subseteq t'$ *then* $E(t) \subseteq E(t')$;
2. *for all* $t \in (2^\omega)^{<\omega}$, $\text{length}(E(t)) = \text{length}(t) + 1$;
3. *for all* $u_1, u_3 \ldots, u_{2k-1} \in 2^\omega$, *if*

$$E(u_1, u_3, \ldots, u_{2k-1}) = ((m_0, y_0, v_0), (m_1, y_1, v_2), \ldots, (m_k, y_k, v_{2k})),$$

then for each $z_k \in [T_{2k+1}(m_k, y_k)_{m_k}]$, *if*

- *for all* $i < k$, $z_i = F^*_{2i+1}(m_i, y_i, u_{2i+1}, m_{i+1}, y_{i+1}, z_{i+1})$;
- $(n_0, w_0, x_0) = \sigma(m_0, y_0, z_0)$;
- *for all* $i \leq k$, $(u_{2i}, n_{i+1}, w_{i+1}, x_{i+1}) = F_{2i}(n_i, w_i, x_i)$;

then $\rho_{2i}(u_{2i}) = \rho_{2i}(v_{2i})$ *for all* $i \leq k$.

Granting the claim, we get a strategy for I in $\mathcal{G}(Q, A)$ as follows: When II plays $u_1, u_3 \ldots$, I answers with $v_0, v_2, \ldots$, where

$$E(u_1, u_3, \ldots, u_{2k-1}) = ((m_0, y_0, v_0), (m_1, y_1, u_2), \ldots, (m_k, y_k, v_{2k}))$$

for each $k \in \omega$.

We will show that this is indeed a winning strategy for player I. Suppose that II has played u_{2i+1} ($i \in \omega$) in a run of the game and I produced v_{2i} ($i \in \omega$) following this strategy. Then we have m_i and y_i ($i \in \omega$) as produced by E. For each $k \in \omega$ define $C^k_0, C^k_1, \ldots, C^k_k$ as follows :

- C^k_k is the set of $z \in [T_{2k+1}(m_k, y_k)_{m_k}]$ such that

$$F_{2k+1}(m_k, y_k, z) = (u_{2k+1}, m_{k+1}, y_{k+1}, z')$$

  for some $z' \in 2^\omega$;

- for each $i < k$, C^k_i is the set of $z \in [T_{2i+1}(m_i, y_i)_{m_i}]$ such that

$$F_{2i+1}(m_i, y_i, z) = (u_{2i+1}, m_{i+1}, y_{i+1}, z')$$

  for some $z' \in C^k_{i+1}$.

Then for all $i \leq k$ in ω, $C^{k+1}_i \subseteq C^k_i$. In particular $\langle C^k_0 : k \in \omega \rangle$ is a $\subseteq$-decreasing sequence of nonempty closed sets, so we may pick a z_0 in their intersection. Now suppose that player II plays (m_0, y_0, z_0) in $\mathcal{G}'(Q, A)$ against σ. Then (since $z_0 \in \bigcap_{k\in\omega} C^k_0$) the induced values of each m_i and y_i (momentarily treating them as variables and forgetting that we have fixed values for them) according to the rules of $\mathcal{G}'(Q, A)$ agree with the (fixed) values given by E. Moreover, the triples (m_i, y_i, z_i) are each respectively $(2i+1)$-good, and the corresponding values of u_{2i+1} are also as in the given run of the game.

Furthermore, if $(n_0, w_0, x_0) = \sigma(m_0, y_0, z_0)$, then the induced values (n_i, w_i, x_i) are each respectively $(2i)$-good, since σ is a winning strategy for I. For each

$i \in \omega$, the triple (n_i, w_i, x_i) induces a value for u_{2i} via the function F_{2i}. For each $k \in \omega$, since $z_0 \in C_0^k$, there exist $z_1, \ldots, z_k$ satisfying the conditions in part (3) of Lemma 12.1.4, so $\rho_{2i}(u_{2i}) = \rho_{2i}(v_{2i})$ for all $i \leq k$. The same is then true for all $i \in \omega$. Since I won the run of $\mathcal{G}'(Q, A)$ by playing (n_0, w_0, x_0), he wins the run of $\mathcal{G}(Q, A)$ by playing v_{2i} $(i \in \omega)$.

It remains then to prove Lemma 12.1.4.

Proof of Lemma 12.1.4. We construct $f(t)$ recursively on the length of t. We first consider the case where $t = \emptyset$, where we have to produce m_0, y_0 and v_0 such that, for any $z \in [T_1(m_0, y_0)_{m_0}]$, if $(n_0, w_0, x_0) = \sigma(m_0, y_0, z)$, and $F_0(n_0, w_0, x_0) = (u, n', w', x')$, then $\rho_0(u) = \rho_0(v_0)$.

For each $y \in 2^\omega$, let σ_y be the strategy for player I in $\mathcal{G}'(Q, A)$ which is HC-coded by y if there is one, and some fixed recursive strategy otherwise. Let $y_0 \in \mathit{creals}$ be such that $\sigma = \sigma_{y_0}$.

Let H be the function sending (a candidate) $m \in \omega$ and $y, z \in 2^\omega$ to the first coordinate of $F_0(\sigma_y(m, y, z))$. Then H is projective in P_0, so Lemma 12.1.2 gives an $m_0 \in \omega$ such that $\rho_0(H(m_0, y_0, z))$ is the same for all $z \in [T_1(m_0, y_0)_{m_0}]$. Then we can let v_0 be any such value $H(m_0, y_0, z)$, for instance, the value produced when z is the leftmost branch of $T_0(m_0, y_0)_{m_0}$.

In the case where $t \neq \emptyset$, let $t = (u_1, \ldots, u_{2k+1})$. We may assume that

$$E(u_1, \ldots, u_{2k-1}) = ((m_0, y_0, v_0), \ldots, (m_k, y_k, v_{2k}))$$

is known. We need to find m_{k+1}, y_{k+1} and v_{2k+2} such that for each $z_{k+1} \in [T_{2k+3}(m_{k+1}, y_{k+1})_{m_{k+1}}]$, if

- for all $i \leq k$, $z_i = F^*_{2i+1}(m_i, y_i, u_{2i+1}, m_{i+1}, y_{i+1}, z_{i+1})$;
- $(n_0, w_0, x_0) = \sigma(m_0, y_0, z_0)$;
- for all $i \leq k+1$, $(u_{2i}, n_{i+1}, w_{i+1}, x_{i+1}) = F_{2i}(n_i, w_i, x_i)$;

then $\rho_{2k+2}(u_{2k+2}) = \rho_{2k+2}(v_{2k+2})$.

Consider the function H which takes in

- m (a candidate for m_{k+1}),
- $y \in 2^\omega$ (HC-coding a tuple consisting of a strategy $\hat{\sigma}$ for player I in $\mathcal{G}'(Q, A)$ along with values $(\hat{m}_0, \hat{y}_0)$,...,$(\hat{m}_k, \hat{y}_k)$ in $\omega \times 2^\omega$), and
- $z \in 2^\omega$

and returns the corresponding u_{2k+2} as above, with z in the role of z_{k+1}, $\hat{\sigma}$ in the role of σ and each $\hat{m}_i$ and $\hat{y}_i$ in the role of the corresponding m_i or y_i. By Remark 12.1.3, this function is projective in P_{2k+2}. Applying Lemma 12.1.2 (with $y = y_{k+1}$ HC-coding the tuples consisting of σ and the values (m_0, y_0),...,(m_k, y_k)) again we can find an m_{k+1} as desired. Again we can let v_{k+1} be the value of $H(m_{k+1}, y_{k+1}, z)$, where z is the leftmost branch of $T_{2k+3}(m_{k+1}, y_{k+1})_{m_{k+1}}$. □

We now show that if player II has a winning strategy in $\mathcal{G}'(Q, A)$ then he also has one in $\mathcal{G}(Q, A)$. Fix a winning strategy σ for player II in $\mathcal{G}'(Q, A)$, and write $(m, y, z) = \sigma(n, w, x)$ to mean that (m, y, z) is the result of player II's moves when he plays according to σ and player I plays (n, w, x). Our desired strategy for $\mathcal{G}(Q, A)$ will be induced by the function E given in the following lemma.

Lemma 12.1.5. *There is a function* $E\colon (2^\omega)^{<\omega} \to (\omega \times 2^\omega \times 2^\omega)^{<\omega}$ *such that*

1. *for all* $t, t' \in (2^\omega)^{<\omega}$, *if* $t \subseteq t'$ *then* $E(t) \subseteq E(t')$;
2. *for all* $t \in (2^\omega)^{<\omega}$, $\mathrm{length}(E(t)) = \mathrm{length}(t)$;
3. *for all* $u_0, u_2 \ldots, u_{2k} \in 2^\omega$, *if*

$$E(u_0, u_2, \ldots, u_{2k}) = ((n_1, w_1, v_1), (n_2, w_2, v_3), \ldots, (n_{k+1}, w_{k+1}, v_{2k+1})),$$

then for each $x_{k+1} \in [T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}]$, *if*

- *for all* $i \leq k$, $x_i = F^*_{2i}(n_i, w_i, u_{2i}, n_{i+1}, w_{i+1}, z_{i+1})$;
- $(m_0, y_0, z_0) = \sigma(n_0, w_0, x_0)$;
- *for all* $i \leq k$, $(u_{2i+1}, m_{i+1}, y_{i+1}, z_{i+1}) = F_{2i+1}(m_i, y_i, z_i)$;

then $\rho_{2i+1}(u_{2i+1}) = \rho_{2i+1}(v_{2i+1})$ *for all* $i \leq k$.

Granting the claim, we get a strategy for II in $\mathcal{G}(Q, A)$ as follows: When I plays $u_0, u_2 \ldots$, II answers with $v_1, v_3, \ldots$, where

$$E(u_0, u_2, \ldots, u_{2k}) = ((n_1, w_1, v_1), (n_2, w_2, v_3), \ldots, (n_{k+1}, w_{k+1}, v_{2k+1}))$$

for each $k \in \omega$.

We show that this is a winning strategy for player II. Suppose that I has played u_{2i} ($i \in \omega$) in a run of the game and II produced v_{2i+1} ($i \in \omega$) following this strategy. Then we have m_i and y_i ($i \in \omega$) as produced by E. For each $k \in \omega$ define $C^k_0, C^k_1, \ldots, C^k_k$ as follows :

- C^k_k is the set of $x \in [T_{2k}(n_k, w_k)_{n_k}]$ such that
$$F_{2k}(n_k, w_k, x) = (u_{2k}, n_{k+1}, w_{k+1}, x')$$
for some $x' \in 2^\omega$;
- for each $i < k$, C^k_i is the set of $x \in [T_{2i}(n_i, w_i)_{n_i}]$ such that
$$F_{2i}(n_i, w_i, x) = (u_{2i}, n_{i+1}, w_{i+1}, x')$$
for some $x' \in C^k_{i+1}$.

Then for all $i \leq k$ in ω, $C_i^{k+1} \subseteq C_i^k$. In particular $\langle C_0^k : k \in \omega \rangle$ is a $\subseteq$-decreasing sequence of nonempty closed sets, so we may pick a x_0 in their intersection. Now suppose that player I plays (n_0, w_0, x_0) in $\mathcal{G}'(Q, A)$ against σ. Then (since $x_0 \in \bigcap_{k\in\omega} C_0^k$) the induced values of each n_i and w_i (momentarily treating them as variables and forgetting that we have fixed values for them) according to the rules of $\mathcal{G}'(Q, A)$ agree with the (fixed) values given by E. Moreover, the triples (n_i, w_i, w_i) are each respectively $(2i)$-good, and the corresponding values of u_{2i} are also as in the given run of the game.

Furthermore, if $(m_0, y_0, z_0) = \sigma(n_0, w_0, x_0)$, then, since σ is a winning strategy for II, the induced values (m_i, y_i, z_i) are each respectively $(2i + 1)$-good. For each $i \in \omega$, the triple (m_i, y_i, z_i) induces a value for u_{2i+1} via the function F_{2i+1}. For each $k \in \omega$, since $x_0 \in C_0^k$, there exist $x_1, \ldots, x_{k+1}$ satisfying the conditions in part (3) of Lemma 12.1.5, so $\rho_{2i+1}(u_{2i+1}) = \rho_{2i+1}(v_{2i+1})$ for all $i \leq k$. The same is then true for all $i \in \omega$. Since II won the run of $\mathcal{G}'(Q, A)$ by playing (m_0, y_0, z_0), he wins the run of $\mathcal{G}(Q, A)$ by playing v_{2i+1} $(i \in \omega)$.

It remains then to prove Lemma 12.1.5.

Proof of Lemma 12.1.5. We construct $f(t)$ recursively on the length of t. We can let $E(\emptyset) = \emptyset$. Let $t = (u_0, \ldots, u_{2k})$. We may assume that

$$E(u_1, \ldots, u_{2k-2}) = ((n_1, w_1, v_0), \ldots, (n_k, w_k, v_{2k-1}))$$

is known. We need to find n_{k+1}, w_{k+1} and v_{2k+1} such that for each $x_{k+1} \in [T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}]$, if

- for all $i \leq k$, $x_i = F_{2i}^*(n_i, w_i, u_{2i}, n_{i+1}, w_{i+1}, x_{i+1})$;
- $(m_0, y_0, z_0) = \sigma(n_0, w_0, x_0)$;
- for all $i \leq k$, $(u_{2i+1}, m_{i+1}, y_{i+1}, z_{i+1}) = F_{2i+1}(m_i, y_i, z_i)$;

then $\rho_{2k+1}(u_{2k+1}) = \rho_{2k+1}(v_{2k+1})$.

Consider the function H which takes in

- n (a candidate for n_{k+1}),
- $w \in 2^\omega$ (HC-coding a tuple consisting of a strategy $\hat{\sigma}$ for player II in $\mathcal{G}'(Q, A)$ along with values $(\hat{n}_1, \hat{w}_1)$,. . .,$(\hat{n}_k, \hat{w}_k)$ in $\omega \times 2^\omega$), and
- $x \in 2^\omega$

and returns the corresponding u_{2k+1} as above, with x in the role of x_{k+1}, $\hat{\sigma}$ in the role of σ and each $\hat{n}_i$ and $\hat{w}_i$ in the role of the corresponding n_i or w_i. By Remark 12.1.3, this function is projective in P_{2k+1}. Applying Lemma 12.1.2 (with $w = w_k$ HC-coding the tuple consisting of σ and the values (n_1, w_1),. . .,(n_k, w_k)) again we can find an n_{k+1} as desired. We can let v_{2k+1} be the value of $H(n_{k+1}, w_{k+1}, x)$, where x is the leftmost branch of $T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}$. $\square$

12.2 Boundedness for ∞-Borel relations

In Section 12.1 we considered a version of coded ordinal determinacy, where the players play elements of 2^ω representing ordinals via some fixed prewellorderings. The following definition gives a version of the notion of strong generic ∞-code relative to such a game.

12.2.1 Definition. Suppose that δ is an infinite cardinal, $\pi \colon 2^\omega \to \delta$ is a surjection and C is a generic δ-code. Associate to π and C a game $\mathcal{G}_{\pi,C}$ on 2^ω, where I and II collaborate to build a countable set $\sigma \subseteq 2^\omega$, and I wins if $A_{C_{\rho[\sigma]}} \subseteq A_C$. We say that ϕ is a *ρ-strong generic ∞-code* (or *π-strong generic δ-code*) for A_ϕ if player I has a winning strategy in $\mathcal{G}_{\pi,C}$.

Replacing the use of δ^∞_S-Determinacy in the proof of Theorem 11.2.1 with AD, and using Theorem 12.1.1, we get the following. Recall that AD implies that μ_S is an ultrafilter on $\mathcal{D}_S$, for each set S of ordinals (Corollary 1.2.6).

Theorem 12.2.2 (ZF + AD)**.** *Let S be a set of ordinals such that the ultrapower $\mathrm{Ord}^{\mathcal{D}_S}/\mu_S$ is wellfounded, and $\delta^\infty_S < \Theta$. If S is an ∞-Borel code for a set $A \subseteq \omega^\omega$, then A has a π-strong generic δ^∞_S-code which is definable from S.*

We use Theorem 12.2.2 to prove Theorem 12.2.3 below, which is a boundedness result for the ranks of wellfounded relations with a suitable ∞-Borel representation. The proof uses Woodin's collapse embedding technique.

Theorem 8.5.1 and Corollary 10.2.7 together give that (under AD + $\mathsf{DC}_\mathbb{R}$) if Uniformization holds then for each bounded subset S of Θ, $\delta^\infty_S < \Theta$. If in addition DC holds, then there are cofinally many $\kappa < \Theta$ with the property that every subset of ω^ω with Wadge rank less κ has an ∞-Borel code S with $\delta^\infty_S < \kappa$.

Theorem 12.2.3 (ZF + Lipschitz Determinacy + DC)**.** *Let $\gamma < \Theta$ be an ordinal. Suppose that, for each $i \in \omega$,*

- *π_i is a surjection from 2^ω onto some ordinal $\gamma_i < \gamma$;*
- *C_i is a π_i-strong generic γ_i-code for a subset of $\omega^\omega \times \omega^\omega$.*

Suppose also that $R = \bigcup_{i\in\omega} A_{\phi_{S_i}}$ is a wellfounded relation. Then the rank of R is less than γ^+.

Proof. For each $i \in \omega$, let Σ_i be a strategy witnessing that C_i is a π_i-strong generic γ_i-code. Let $C = (\mathbb{P}, B, \tau)$ be the join of $\langle C_i : i \in \omega\rangle$ Since DC holds, we may fix a suitable sequence $\langle \leq_i : i \in \omega\rangle$ of prewellorderings of 2^ω such that the length of each $\leq_i$ is γ. Let $B \subseteq \omega^\omega$ be of Wadge rank greater than γ, and such that C, $\langle \pi_i : i \in \omega\rangle$, $\langle \Sigma_i : i \in \omega\rangle$ and $\langle \leq_i : i \in \omega\rangle$ are in $L(B, \mathbb{R})$. Let τ be a $\mathrm{Col}(\omega,\gamma)$-name for an element of ω^ω HC-coding the pair (γ, C).

Let $c \colon \gamma \to \mathrm{Col}(\omega,\gamma)$ be a bijection, and for each $i \in \omega$ let ρ_i be the rank function for $\leq_i$. Given a condition $p \in \mathrm{Col}(\omega,\gamma)$ and a set $X \subseteq \omega^\omega$, let $\mathcal{G}_{\tau,p}(X)$ be the game where players I and II alternately play $x_i \in 2^\omega$ in such a way that $\langle c(\rho_i(x_i))$ is a descending sequence of conditions in $\mathrm{Col}(\omega,\gamma)$ below p, with the

nth condition deciding at least the first n digits of the realization of τ, with player II winning if the induced realization of τ is in X (or if player I violates the rules before player II does). By Theorem 12.1.1, each game $\mathcal{G}_{\tau,p}(X)$ is determined.

For each condition $p \in \mathrm{Col}(\omega,\gamma)$, let I^p_τ be the set of $X \subseteq \omega^\omega$ for which player I has a winning strategy in $\mathcal{G}_{\tau,p}(X)$. Then each I^p_τ is an ideal, and $I^p_\tau \subseteq I^q_\tau$ whenever $p \leq q$. Furthermore, for any $p \in \mathrm{Col}(\omega,\gamma)$ ad $X \subseteq \omega^\omega$, either player II has a winning strategy in $\mathcal{G}_{\tau,p}(X)$ or there exists a $q \leq p$ such that player II has a winning strategy in $\mathcal{G}_{\tau,p}(\omega^\omega \setminus X)$. Let $g \subset \mathrm{Col}(\omega,\gamma)$ be $L(B,\mathbb{R})$-generic, and let U_g be the set of $X \in \mathcal{P}(\omega^\omega) \cap V$ for which $\omega^\omega \setminus X$ is in I^p_τ, for some $p \in g$. Then U_g an $L(B,\mathbb{R})$-ultrafilter

Since DC holds, the ultrapower $M_g = \mathrm{Ult}(L(B,\mathbb{R}),U_g)$ (using functions in V) is wellfounded, and, since $\mathsf{Uniformization}$ holds, the induced embedding j from $L(B,\mathbb{R})$ to M_g is elementary. Moreover, $j(R)$ is wellfounded. The identity function represents τ_g, so γ and C are both in $H(\aleph_1)^{M_g}$. Since γ is countable in M_g, ϕ codes a Borel relation R' in M_g which agrees with R on $\omega^\omega \cap V$. Since $\mathrm{Col}(\omega,\gamma)$ has cardinality γ, $(\gamma^+)^V \geq \omega_1^{M_g}$. If R' is wellfounded in M_g, then we have a contradiction to $\utilde{\Sigma}^1_1$-boundedness.

To show that R' is wellfounded, we show that R' is contained in $j(R)$. It suffices to fix $i \in \omega$ and show that, in M_g, $A_{C_i} \subseteq A_{j(C_i)}$. Fixing such an i, consider the tree T of attempts to find a run σ of $\mathcal{G}_{j(\pi_i),j(C_i)}$ according to $j(\Sigma_i)$ where the induced $j(C_i)_{j(\pi_i)[\sigma]}$ is isomorphic to C_i. It suffices to see that this tree has an infinite path in M_g. If there is no such path, then T has a ranking function in M_g. Since $j(\Sigma_i)$ maps finite sequences from $2^\omega \cap V$ to elements of $2^\omega \cap V$, each element of $T \cap j[(2^\omega \cap V)^{<\omega}]$ has an extension in $T \cap j[(2^\omega \cap V)^{<\omega}]$. Consideration of the least rank of an element of $T \cap j[(2^\omega \cap V)^{<\omega}]$ shows that this is impossible. $\square$

12.3 Becker's argument

In this section we complete the proof of the following theorem

Theorem 12.3.1 ($\mathsf{ZF}+\mathsf{AD}+\mathsf{DC}$). *If* $\mathsf{Uniformization}$ *holds then every subset of* ω^ω *is Suslin.*

The rest of this section is a proof of Theorem 12.3.1. We start with an arbitrary $A \subseteq \omega^\omega$, which we will show to be Suslin. Since AD and $\mathsf{Uniformization}$ hold, we have that all subsets of ω^ω are ∞-Borel. Since AD, DC and $\mathsf{Uniformization}$ hold, there exist a suitable sequence $\langle \leq_i : i \in \omega\rangle$ of prewellorderings of 2^ω (with lengths γ_i and rank functions ρ_i for $i \in \omega$), as witnessed by a good sequence $\langle P_i : i \in \omega\rangle$, and $\gamma < \Theta$ such that, applying Theorem 12.2.2 and the remark just before Theorem 12.2.3,

1. $A \leq_{\mathrm{W}} P_0$;
2. $\mathrm{length}(\leq_0) > \omega$;

3. the sequence $\langle \mathrm{length}(\leq_i) : i \in \omega\rangle$ is strictly increasing;

4. for all $n \in \omega$ there is an $a \in 2^\omega$ such that, for all $i \in \omega$, a is the unique $u \in 2^\omega$ with $\rho_{2i}(u) = n$;

5. γ is equal to both the $\sup_{i\in\omega} \gamma_i$ the supremum of the Wadge ranks of the sets P_i;

6. for every subset A of ω^ω projective in any P_i there exist a bounded $S \subseteq \gamma$ and a surjection π from 2^ω onto some ordinal below γ such that S is a π-strong ∞-Borel code for A.

Let $\mathcal{A}$ be the set of $a \in 2^\omega$ satisfying condition (4) for any $n \in \omega$.

Say that a tree on γ is *good* if it is a subset of $\bigcup_{n\in\omega} \prod_{i<n} \gamma_i$. Using a surjection $f \colon \gamma \to \eta$ (and pauses) one can find a good wellfounded tree on γ of rank η, for any $\eta < \gamma^+$, using, for instance the tree of sequences of the form $\langle \alpha_0, \ldots, \alpha_k\rangle$ such that

- each α_i is in γ_i;
- if $i < k$ and $\alpha_i \in \omega$, then either $\alpha_{i+1} = \alpha_i - 1$ or $\alpha_i = 0$ and $\alpha_{i+1} \notin \omega$;
- if $i < j \leq k$, $\alpha_i = \omega + \beta$ and $\alpha_j = \omega + \delta$, then $f(\beta) > f(\delta)$.

Let $b \colon 2^\omega \to (2^\omega)^\omega$ be a recursive bijection, with component functions b_i ($i \in \omega$). For each $i \in \omega$, let $<_i$ be the strict part of $\leq_i$. For each $n \in \omega$, let $\prec^0_n$ be the strict prewellordering of $(2^\omega)^n$ induced by the lexicographical order and $<_0, \ldots, <_n$, and let $\prec_n$ be the strict prewellordering on 2^ω defined by setting $x \prec_n n$ to hold if and only if $(b_0(x), \ldots, b_n(x)) \prec^0_n (b_0(y), \ldots, b_n(y))$.

Fixing a universal $\utilde{\Sigma}^1_1$-set $U \subseteq 2^\omega \times \omega^\omega \times (2^\omega)^2$ and applying the Moschovakis Coding Lemma, we can let, for each $n \in \omega$ and $u \in 2^\omega$, $S_n(u)$ be the subset of $\prod_{i<n} \gamma_i$ coded by $b_n(u)$ relative to $\prec_n$. That is, letting $U(\prec_n)$ be the universal pos-$\utilde{\Sigma}^1_1(\prec_n)$ subset of $(2^\omega)^3$ induced by U, $S_n(u)$ is the set of $\langle \alpha_i : i < n\rangle$ such that, for some $(v, w) \in U(\prec_n)_{b_n(u)}$, for each $i < n$ the $\leq_i$-rank of $b_i(v)$ is α_i.

Given $m \leq n$ in ω and $u \in 2^\omega$, we let $S_n(u)\restriction m$ denote the set $\{t\restriction m : t \in S_n(u)\}$. Given $m \leq n$ in ω and $u, v \in 2^\omega$, we say that $S_n(u)$ *extends* $S_m(v)$ if $S_n(u)\restriction m \subseteq S_m(v)$. We let Tr be the set of $\langle u_n : n < m\rangle \in (2^\omega)^{<\omega}$ such that, for all positive $n < m$, $S_n(u_n)$ extends $S_{n-1}(u_{n-1})$. Given $\bar{u} = \langle u_n : n \in \omega\rangle \in [\mathrm{Tr}]$, we write $\mathrm{Tr}(\bar{u})$ for $\bigcup_{n\in\omega} S_n(u_n)$.

Let $\mathcal{G}_1$ be the game where players I and II alternate picking $u_i \in 2^\omega$, with player II winning if and only if either some u_{2i} is not in $\mathcal{A}$ or

- $\langle u_{2n+1} : n \in \omega\rangle \in [\mathrm{Tr}]$ and
- $\langle \rho_{2i}(u_{2i})\rangle \in A$ if and only if $\mathrm{Tr}(\langle u_{2n+1} : n \in \omega\rangle)$ is illfounded.

If II has a winning strategy in this game, then A is Suslin.

Let $\Lambda = \bigcup_{n\in\omega} \utilde{\Sigma}^1_1(P_n)$, and say that a strategy σ for player I is *simple* if, for each n, the fragment of σ for the first n moves of the game is in Λ.

The results of Section 12.2 gives the following claim.

Claim 1. *Player I does not have a simple winning strategy in $\mathcal{G}_1$.*

Proof. Suppose that Σ is a simple winning strategy for player I in $\mathcal{G}_1$. Noting that the supremum of the ranks of the good trees on γ is γ^+, we aim to contradict Theorem 12.2.3. For each $i \in \omega$, let R_i be the set of triples

$$(\langle u_{2i} : i \in \omega\rangle, \langle x_{2n+1} : n < m\rangle, \langle z_j : j < m\rangle)$$

such that

- $\langle \rho_{2i}(u_{2i}) : i \in \omega\rangle \in A$;
- $\langle u_{2n+1} : n < m\rangle \in \mathrm{Tr}$;
- $\langle u_{2i} : i \leq m+1\rangle$ is the sequence of moves played by Σ against the sequence $\langle u_{2n+1} : n < m\rangle$;
- for all $n < m$, $\langle \rho_{2j+1}(z_j) : i < n\rangle \in S_n(u_{2n+1})$.

Order R_i by identity on the first coordinate, and extension in the other two. Then $\bigcup_{i\in\omega} R_i$ is a wellfounded relation of rank γ^+. Item (6) above shows that the hypotheses of Theorem 12.2.3 apply to the R_i's, giving the desired contradiction. $\square$

We now code $\mathcal{G}_1$ with a game $\mathcal{G}_2$ in the same way that $\mathcal{G}'(Q, A)$ coded $\mathcal{G}(Q, A)$ in Section 12.1. We reuse the objects $\langle T_i : i < \omega\rangle$, χ, $\langle F_i : i <\in \omega\rangle$ and $\langle F_i^* : i \in \omega\rangle$ defined there. Let Q be $\langle \leq_i : i \in \omega\rangle$.

We let $\mathcal{G}_2$ be the integer game in which player I plays $(n_0, w_0, x_0) \in \omega \times 2^\omega \times 2^\omega$ and player II plays $(m_0, y_0, z_0) \in \omega \times 2^\omega \times 2^\omega$, both one coordinate per move. To determine the winner of the game, we then let, for each $i \in \omega$,

$$(u_{2i}, n_{i+1}, w_{i+1}, x_{i+1})$$

be $F_{2i}(n_i, w_i, x_i)$ and

$$(u_{2i+1}, m_{i+1}, y_{i+1}, z_{i+1})$$

be $F_{2i+1}(m_i, y_i, z_i)$. Player I wins this run of $\mathcal{G}_2(Q, A)$ if and only if either

- there is an $i \in \omega$ such that (m_i, y_i, z_i) is not $(2i + 1)$-good, and such that, for all $j \leq i$, (n_j, w_j, x_j) is $(2j)$-good, or
- (n_j, w_j, x_j) is $(2j)$-good for all $i \in \omega$ and $\langle u_i : i \in \omega\rangle$ is a winning run of $\mathcal{G}_1$ for player I.

Since $\mathcal{G}_2$ is an integer game, AD implies that it is determined.

The games $\mathcal{G}_1$ and $\mathcal{G}_2$ are different from the games considered in Section 12.1, in that the winning condition is not invariant under the $\leq_{2i+1}$-ranks of the plays made by player II. It is, however, invariant under the $\leq_{2i}$-ranks of the moves made by player I. Only this condition was needed in the proof of Theorem 12.1.1 to show that if player I had a winning strategy in $\mathcal{G}'(Q, A)$ then he had one in $\mathcal{G}(Q, A)$. Moreover, for each $n \in \omega$ the strategy produced for player I in the proof of Theorem 12.1.1 used only finitely many functions of the form F^*_{2i+1} or F_{2i}. This gives the following, showing that player I does not have a winning strategy in $\mathcal{G}_1$, by Claim 1.

Claim 2. *If player I has a winning strategy in $\mathcal{G}_2$, then he has a simple winning strategy in $\mathcal{G}_1$.*

Finally, a modification of the proof in Section 12.1 gives the following claim, which completes the proof of Theorem 12.3.1.

Claim 3. *If player II has a winning strategy in $\mathcal{G}_2$, then she has one in $\mathcal{G}_1$.*

The rest of this section is a proof of Claim 3. For notational convenience, we will label player I's moves as a_i $(i \in \omega)$, so a_i and u_{2i} refer to the same object. Suppose that σ is a winning strategy for player II. Given

- $\bar{a} = \langle a_0, \ldots, a_k \rangle \in \omega^{<\omega}$,
- $\bar{s} = \langle (n_1, w_1), \ldots, (n_{k+1}, w_{k+1}) \rangle \in (\omega \times 2^\omega)^{<\omega}$ and
- $x_{k+1} \in [T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}]$,

let $B(\bar{a}, \bar{s}, x_{k+1})$ be u_{2k+1}, where

- for all $i \leq k$, $x_i = F^*_{2i}(n_i, w_i, a_i, n_{i+1}, w_{i+1}, x_{i+1})$;
- $(m_0, y_0, z_0) = \sigma(n_0, w_0, x_0)$;
- for all $i \leq k$, $(u_{2i+1}, m_{i+1}, y_{i+1}, z_{i+1}) = F_{2i+1}(m_i, y_i, z_i)$;

In addition, say that $(\bar{a}, \bar{s}, x_k)$ is coherent if the resulting sequence $\langle u_{2i+1} : i \leq k \rangle$ is in Tr.

We define the analogous version of the function E from the proof of Theorem 12.1.1, giving us our strategy for player II in $\mathcal{G}_1$.

Lemma 12.3.2. *There is a function $E\colon \mathcal{A}^{<\omega} \to (\omega \times 2^\omega)^{<\omega}$ such that*

1. *for all $t, t' \in \mathcal{A}^{<\omega}$, if $t \subseteq t'$ then $E(t) \subseteq E(t')$;*
2. *for all $t \in \mathcal{A}^{<\omega}$, $\mathrm{length}(E(t)) = \mathrm{length}(t)$;*
3. *for all $\bar{a} = \langle a_0, \ldots, a_k \rangle \in \mathcal{A}^{<\omega}$, if*

$$E(a_0, \ldots, a_k) = \bar{s} = \langle (n_1, w_1), \ldots, (n_{k+1}, w_{k+1}) \rangle,$$

then

(a) *for each $x_{k+1} \in [T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}]$ and each $\bar{\xi} \in \prod_{i<k} \gamma_i$ there exists an $\ell \in \omega$ such that for any*

$$x \in [T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}]$$

with $x \restriction \ell = x'_k \restriction \ell$, $\bar{\xi} \in S_k(B(\bar{a}, \bar{s}, x_{k+1}))$ and only if $\bar{\xi} \in S_k(B(\bar{a}, \bar{s}, x))$;

(b) *there exist* $s_1, \ldots, s_k$ *such that for each*

$$x_{k+1} \in [T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}],$$

if for all $i \leq k$, $x_i = F^*_{2i}(n_i, w_i, a_i, n_{i+1}, w_{i+1}, x_{i+1})$, *then* $x_i \restriction k = s_k$ *for all* $i \leq k+1$.

Given a function E as in Lemma 12.3.2, we define a strategy for player II in $\mathcal{G}_1$ as follows. If at any time player I plays an element of $2^\omega \setminus \mathcal{A}$, then player II has won, so we can let her response be any element of 2^ω. In response to $\bar{a} = \langle a_0, \ldots, a_k \rangle \in \mathcal{A}^{<\omega}$, letting

$$\bar{s} = E(\bar{a}) = \langle (n_1, w_1), \ldots, (n_{k+1}, w_{k+1}) \rangle,$$

II plays a v such that $S_k(v)$ is the union of all sets of the form $S_k(B(\bar{a}, E(\bar{a}), x)$ for which $x \in [T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}]$ and $(\bar{a}, \bar{s}, x)$ is coherent. Since $\mathcal{A}$ is countable, we can choose such a v for each such sequence from $\mathcal{A}$.

To see that this is a winning strategy for player II, fix $\bar{a} = \langle a_i : i \in \omega \rangle$ and let $\langle v_{2i+1} : i \in \omega \rangle$ be the sequence of moves played by player II in response, according to this strategy. Let

$$S_* = \bigcup_{i \in \omega} S_i(v_{2i+1}).$$

Let $\bar{s} = \langle (n_i, w_i) : i \in \omega \setminus \{0\} \rangle$ be as given by $\bar{a}$ and E, and for each $k \in \omega$ let $\bar{s} \restriction k$ denote $\langle (n_i, w_i) i \in k \setminus \{0\} \rangle$.

As in the proof of the Theorem 12.1.1, for each $k \in \omega$ define $C^k_0, C^k_1, \ldots, C^k_k$ as follows :

- C^k_k is the set of $x \in [T_{2k}(n_k, w_k)_{n_k}]$ such that

  $$F_{2k}(n_k, w_k, z) = \langle a_k, n_{k+1}, w_{k+1}, x' \rangle$$

  for some $z' \in 2^\omega$;

- for each $i < k$, C^k_i is the set of $x \in [T_{2i}(n_i, w_i)_{n_i}]$ such that

  $$F_{2i}(n_i, w_i, x) = \langle a_i, n_{i+1}, w_{i+1}, x' \rangle$$

  for some $x' \in C^k_{i+1}$.

Then $\langle C^k_0 : k \in \omega \rangle$ is a $\subseteq$-decreasing sequence of nonempty closed sets, so we may pick a x_0 in their intersection and let player I play (n_0, w_0, x_0). Then the induced values (n_i, w_i, x_i) according to the rules of the game agree with the values (n_i, w_i) given by E, are each respectively $(2i)$-good.

Let (m_0, y_0, z_0) be $\sigma(n_0, w_0, x_0)$, and let

$$(u_{2i+1}, m_{i+1}, y_{i+1}, z_{i+1}) = F_{2i+1}(m_i, y_i, z_i)$$

for each $i \in \omega$. Let $S = \bigcup_{i\in\omega} S_i(u_{2i+1})$. For each $i \in \omega$, x_{i+1} is in

$$[T_{2i+2}(n_{i+1}, w_{i+1})_{n_{i+1}}]$$

and u_{2i+1} is equal to $B(\bar{a}\restriction(i+1), \bar{s}\restriction(i+2), x_{i+1})$. It follows that $S \subseteq S_*$.

Since σ is a winning strategy for player II, $\langle \rho_{2i}(a_i) : i \in \omega\rangle$ is in A if and only if S is wellfounded. It suffices then to see that S is wellfounded if and only if S_* is. Since $S \subseteq S_*$, one direction of this is immediate.

For the other direction, we show that $[S_*] \subseteq [S]$. Suppose toward a contradiction that $\langle \xi_i : i \in \omega\rangle$ is an infinite path through S_*, and that $k \in \omega$ is such that $\langle \xi_i : i < k\rangle$ is not in S. Then $\langle \xi_i : i < k\rangle$ is not in

$$S_k(B(\bar{a}\restriction(k+1), \bar{s}\restriction(k+2), x_{k+1})).$$

Let $\ell \geq k+1$ be as in conclusion (3a) of Lemma 12.3.2, with respect to $\bar{\xi}$ and x_{k+1}. Let $x \in [T_{2\ell}(n_\ell, w_\ell)_{n_\ell}]$ be such that $(\bar{a}\restriction(\ell+1), \bar{s}\restriction(\ell+2), x)$ is coherent, and let x' be the member of $[T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}]$ induced by the functions $F^*_{2\ell-2}, \ldots, F^*_{2k+2}$. Then $x_{k+1}\restriction\ell = x'\restriction\ell$ by conclusion (3b) of Lemma 12.3.2, so $\langle \xi_i : i < k\rangle$ is not in

$$S_k(B(\bar{a}\restriction(k+1), \bar{s}\restriction(k+2), x')),$$

by conclusion (3a). Since $(\bar{a}\restriction(\ell+1), \bar{s}\restriction(\ell+2), x)$ is coherent $\langle \xi_i : i < k\rangle$ is not in

$$S_\ell(B(\bar{a}\restriction(\ell+1), \bar{s}\restriction(\ell+2), x)\restriction k.$$

It follows then that $\bar{\xi}$ is not in $S_\ell(v_{2\ell+1})$. This completes the proof that S_* is wellfounded, and the proof that the strategy defined above is winning for player II.

It remains to prove Lemma 12.3.2.

Proof of Lemma 12.3.2. We can let $E(\emptyset) = \emptyset$. Let $a_0, \ldots, a_k \in \omega$ be given and suppose, in the case where $k > 1$, that $E(a_0, \ldots, a_{k-1})$ has been chosen. Fix a recursive coding $w \mapsto e_w$ of elements of $(\omega \times 2^\omega)^{k-1}$ by elements of 2^ω, and let w_k be such that $E(a_0, \ldots, a_{k-1}) = e_{w_k}$. Let $e_{w,i}$ refer to the ith pair from e_w, for each $i < k$.

We need to choose n_{k+1} in such a way that

$$(n_{k+1}, w_{k+1}, T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}) \in D,$$

where D is the set of (n, w, T) such that $(n, w) \in \omega \times 2^\omega$, $T \subseteq \omega^{<\omega}$ is a 2-perfect tree and

- for all $x \in [T]$ and all $\bar{\xi} \in \prod_{i<k} \gamma_i$, there exists an $\ell \in \omega$ such that, for any $x' \in [T]$ with $x\restriction\ell = x'\restriction\ell$,

  $$\bar{\xi} \in S_k(B(\langle a_0, \ldots, a_k\rangle, e_w^\frown(n, w), x))$$

  if and only if

  $$\bar{\xi} \in S_k(B(\langle a_0, \ldots, a_k\rangle, e_w^\frown(n, w), x'));$$

- there exist $s_1, \ldots, s_k$ such that for each $x_k \in [T]$, if for all $i < k$, $x_i = F^*_{2i}(e_{w,i}, a_i, e_{w,i+1}, x_{i+1})$, then $x_i \restriction k = s_k$ for all $i \leq k$ (letting $e_{w,k}$ denote the pair (n, w)).

Note that D is projective in P_{2k+1}.

For each (n, w) there is a T such that $(n, w, T) \in D$. To see this, note first that (since every subset of ω^ω has the Baire Property) for each $\bar{\xi} \in \prod_{i<k} \gamma_i$, the following set is comeager : the set of $x \in \omega^\omega$ for which there is some $\ell \in \omega$ such that, for comeagerly many $x' \in \omega^\omega$, if $x \restriction \ell = x' \restriction \ell$, then

$$\bar{\xi} \in S_k(B(\langle a_0, \ldots, a_k \rangle, E(\langle a_0, \ldots, a_{k-1} \rangle^\frown(n, w), x))$$

if and only if

$$\bar{\xi} \in S_k(B(\langle a_0, \ldots, a_k \rangle, E(\langle a_0, \ldots, a_{k-1} \rangle^\frown(n, w), x')).$$

By the Kuratowski-Ulam theorem, the intersection of these sets over all $\bar{\xi}$ (fixing n and w) is also comeager. Again applying the Baire Property, we can shrink this set to a somewhere comeager set of x for which the values $x_i \restriction k$ are all the same, where, for each $i < k$, $x_i = F^*_{2i}(n_i, w_i, a_i, n_{i+1}, w_{i+1}, x_{i+1})$. Any T such that $[T]$ is contained in this somewhere comeager set suffices.

The rest of the proof is the same as the end of the proof of Lemma 12.1.2. Since $\langle P_i : i \in \omega \rangle$ is good, there is a $\Sigma^1_1(P_{2k+2})$ function g on $\omega \times 2^\omega$ such that, for all $(n, w) \in \omega \times \omega^\omega$, $(n, w, g(n, w)) \in D$. Moreover, the function sending each $w \in 2^\omega$ to the sequence $\langle g(n, w) : n \in \omega \rangle$ is $\Sigma^1_1(P_{2k+2})$, so for some $n_{k+1} \in \omega$ we have that $g(n, w) = T_{2k+2}(n_{k+1}, w)_n$ for all $(n, w) \in \omega \times 2^\omega$. Then

$$T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}} = g(n_{k+1}, w_{k+1}),$$

so $(n_{k+1}, w_{k+1}, T_{2k+2}(n_{k+1}, w_{k+1})_{n_{k+1}}) \in D$.

$\square$

Chapter 13

$\mathsf{AD}_{\mathbb{R}}$

In this chapter we will prove the following theorem, which is due to Martin and Woodin independently.

Theorem 13.0.1 (Martin, Woodin)**.** *If* AD *holds and every subset of* ω^ω *is Suslin, then* $\mathsf{AD}_{\mathbb{R}}$ *holds.*

We will give two versions of one of the key steps in our proof of this theorem. One is a theorem of Martin (Theorem 13.0.2 below) which we will not prove. The other is a lemma which we will prove in Section 13.1, which can be used in place of Martin's theorem. Our proof of this lemma will not be entirely self-contained, either. We prove another key lemma in Section 13.2 and finish the proof of Theorem 13.0.1 in Section 13.3.

We will be using material on towers of measures, some of which we briefly review here. More thorough treatments can be found in [29, 27, 37], among other places. Given a set X and an $i \in \omega$, an ultrafilter μ on X^{i+1} *projects* to the ultrafilter μ' on X^i whose members are the set of $A \subseteq X^i$ for which $\{s^\frown\langle x\rangle : s \in A, x \in X\}$ is in μ (we sometimes refer to ultrafilters as *measures*). Each function on X^i has a natural reinterpretation as function on X^{i+1} (ignoring the last coordinate of the input) and this reinterpretation induces a factor map $j_{\mu',\mu}$ from a μ'-ultrapower to a μ-ultrapower. A *tower* of ultrafilters on X is a sequence $\vec{\mu} = \langle \mu_i : i \in \omega\rangle$ such that each μ_i is an ultrafilter on X^i, and each μ_{i+1} projects to the corresponding μ_i. There is a natural direct limit embedding associated to a tower, and the tower is said to be *wellfounded* if the corresponding image model is wellfounded. Equivalently, $\vec{\mu}$ is wellfounded if whenever $A_i \in \mu_i$ for each $i \in \omega$, there is an $x \in X^\omega$ such that $x{\restriction}i \in A_i$ for all $i \in \omega$.

Given a tree T on a product $X \times Y$, and an $s \in X^{<\omega}$, T_s denotes the set $\{t \in Y^{|s|} : (s,t) \in T\}$. A tree T on $\omega \times Y$ (for some set Y) is *homogeneous* if there exists a collection of measures $\{\mu_s : s \in \omega^{<\omega}\}$ such that,

- for each $s \in \omega^{<\omega}$, $T_s \in \mu_s$, and

- for each $x \in \omega^\omega$, $x \in \mathrm{p}[T]$ if and only if $\langle \mu_{x\restriction n} : n \in \omega\rangle$ is a wellfounded tower.

A tree T on $\omega \times Y$ (for some set Y) is *weakly homogeneous* if there exists a countable set σ such that, for each $x \in \omega^\omega$, $x \in \mathrm{p}[T]$ if and only if there exists a wellfounded tower $\langle \mu_n : n \in \omega\rangle$ such that, for each $n \in \omega$, $\mu_n \in \sigma$ and $T_{x\restriction n} \in \mu_n$. We refer the reader to [29, 27, 37] for more on homogeneous trees and weakly homogeneous trees. A subset of ω^ω is *homogeneously Suslin* if it is the projection of a homogeneous tree, and *weakly homgeneously Suslin* if it is the projection of a weakly homogeneous tree (equivalently, if it is the projection, in a different sense, of a homogeneously Suslin subset of $\omega^\omega \times \omega^\omega$).

The following theorem of Martin appears as Theorem 1.1 of [31].

Theorem 13.0.2 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$; Martin)**.** *For any $A \subseteq \omega^\omega$, A is homogeneously Suslin if and only if A and its complement are Suslin.*

13.0.3 Remark. The construction of the Martin-Solovay tree (in [28]; see also Section 1.3 of [27]) implies that the complement of a weakly homogeneously Suslin set is Suslin. It is the other direction of Theorem 13.0.2 which is used in the proof of Theorem 13.0.1.

13.1 Weakly homogeneous trees

We start by proving a theorem of Kunen on definability of measures. Given a set X, we let $\mathrm{meas}(X)$ denote the set of countable complety ultrafilters on X. Recall (from part (3) of Remark 1.1.2) that AD implies that every set of reals has the property of Baire, which implies that every nonprincipal ultrafilter (on any set) is countably complete. The following theorem shows that, assuming AD, every nonprincipal ultrafilter on an ordinal below Θ is ordinal definable. Our proof is taken from [37].

We use the ordered equivalence relation $(\equiv_{\mathrm{Ma}}, \leq_{\mathrm{Ma}})$ from Chapter 8 (although any ordered equivalence relation as thick as $(\equiv_{\mathrm{Ma}}, \leq_{\mathrm{Ma}})$, such as the Turing degrees or the constructibility degrees) would work just as well). By Theorem 1.2.2, AD implies that the $\equiv_{\mathrm{Ma}}$-cone measure μ_{Ma} is an ultrafilter on the set $\mathcal{C}_{\mathrm{Ma}}$ of $\equiv_{\mathrm{Ma}}$ equivalence classes.

Theorem 13.1.1 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$; Kunen)**.** *For every $\kappa < \Theta$, every element of* $\mathrm{meas}(\kappa)$ *is ordinal definable.*

Proof. Fix $\kappa < \Theta$, $\mu \in \mathrm{meas}(\kappa)$ and $A \subseteq \omega^\omega$ of Wadge rank κ. By Proposition2.5.8 and Corollary 3.0.2, there is a surjection $F\colon \omega^\omega \to \mathcal{P}(\kappa)$ in $L(A, \mathbb{R})$. Let γ be the supremum of κ and the Wadge rank of $F^{-1}[\mu]$. The model $L(B, \mathbb{R})$ is the same for all $B \subseteq \omega^\omega$ of Wadge rank γ, which means that this model is ordinal definable. It suffices then to fix such a B and prove that μ is ordinal definable in $L(B, \mathbb{R})$. Note that F and μ are elements of $L(B, \mathbb{R})$.

Since $\mathsf{DC}_{\mathbb{R}}$ holds, DC holds in $L(B, \mathbb{R})$, which means that $(\mathrm{Ord}^{\mathcal{C}_{\mathrm{Ma}}}/\mu_{\mathrm{Ma}})^{L(B,\mathbb{R})}$ is wellfounded. Let

- F_μ be the function on ω^ω defined by setting $F_\mu(x)$ to be $F(x)$ if $F(x) \in \mu$, and $\kappa \setminus F(x)$ otherwise;
- f_μ be the function on $\mathcal{C}_{\mathrm{Ma}}$ defined by setting $f_\mu(d)$ to be the least member of $\bigcap\{F_\mu(x) : x \leq_{\mathrm{Ma}} y\}$, for any $y \in d$ (this does not depend on the choice of y);
- γ_μ be the ordinal represented by f_μ in the ultrapower $(\mathrm{Ord}^{\mathcal{C}_{\mathrm{Ma}}}/\mu_{\mathrm{Ma}})^{L(B,\mathbb{R})}$.

Then μ is definable from γ_μ in $L(B,\mathbb{R})$, as it is the set of $C \subseteq \kappa$ such that, for any function g representing γ_μ in

$$(\mathrm{Ord}^{\mathcal{C}_{\mathrm{Ma}}}/\mu_{\mathrm{Ma}})^{L(B,\mathbb{R})},$$

$\{d \in \mathcal{C}_{\mathrm{Ma}} : g(d) \in C\} \in \mu_{\mathrm{Ma}}$. □

13.1.2 Remark. Let κ be an ordinal below Θ, and let $F\colon \omega^\omega \to \mathcal{P}(\kappa)$ be a surjection. If the Wadge ranks of the members of $\{F^{-1}[\mu] : \mu \in \mathrm{meas}(\kappa)\}$ are not cofinal in Θ, then there is a surjection from ω^ω onto $\mathrm{meas}(\kappa)$. By Theorem 13.1.1, each set $F^{-1}[\mu]$ is ordinal definable from F. By Theorem 6.2.5, if the Wadge ranks of the members of $\{F^{-1}[\mu] : \mu \in \mathrm{meas}(\kappa)\}$ are cofinal in Θ, then there is a counterexample to $\mathsf{Uniformization}$.

The following theorem is our alternative to Theorem 13.0.2. By Remark 13.1.2, then theorem implies, under $\mathsf{AD} + \mathsf{Uniformization}$, that for every $\kappa < \Theta$, every tree on $\omega \times \kappa$ is weakly homogeneous.
[1]

Lemma 13.1.3 ($\mathsf{ZF} + \mathsf{AD}$). *If $\kappa < \Theta$ and there is a surjection from ω^ω to $\mathrm{meas}(\kappa^{<\omega})$, then every tree on $\omega \times \kappa$ is weakly homogeneous.*

Proof. Let $F\colon \omega^\omega \to \mathrm{meas}(\kappa^{<\omega})$ be a surjection, and define

$$F_*\colon \mathcal{C}_{\mathrm{Ma}} \to \mathcal{P}_{\aleph_1}(\mathrm{meas}(\kappa^{<\omega}))$$

by setting $F_*(d)$ to be $F[d]$. Let U be the set of $C \subseteq \mathcal{P}_{\aleph_1}(\mathrm{meas}(\kappa^{<\omega}))$ such that $F_*^{-1}[C] \in \mu_{\mathrm{M}}$. Then U is a fine, countably complete measure on $\mathcal{P}_{\aleph_1}(\mathrm{meas}(\kappa^{<\omega}))$.

Fix a tree T on $\omega \times \kappa$. As usual, given $x \in \omega^\omega$, we let T_x be the set $\{s \in \kappa^{<\omega} : (x{\restriction}|s|, s) \in T\}$. We will show that the set of $\sigma \in \mathcal{P}_{\aleph_1}(\mathrm{meas}(\kappa^{<\omega}))$ witnessing the weak homogeneity of T is in U.

To do this, we define for each such σ a game G_σ. This game will be closed for player I, and player I will have a winning strategy if σ does not witness the weak homogeneity of T. For each $i \in \omega$, in round i of the game $\mathcal{G}_\sigma$, player I plays $k_i \in \omega$, $\alpha_i \in \kappa$ and $\beta_i \in \mathrm{Ord}$, and player II plays a measure $\mu_i \in \sigma$ concentrating on κ^i. Player II is required to play so that for each $i \in \omega$, μ_{i+1} projects to μ_i.

[1] There is an alternate hypothesis for the following lemma, saying that κ is less than the sup of the Suslin cardinals. By the coding of measures theorem of Kunen, assuming AD and that there is a Suslin cardinal above κ, there are fewer than Θ many measures on κ (or $\kappa^{<\omega}$ for that matter.) I didn't quite say this in the previous remark.

Player I is required to play so that for each $i \in \omega$, $(\langle k_0, \ldots, k_i\rangle, \langle \alpha_0, \ldots, \alpha_i\rangle)$ is in T and $j_{\mu_i,\mu_{i+1}}(\beta_i) > \beta_{i+1}$, where $j_{\mu_i,\mu_{i+1}}$ is the factor map corresponding to the projection of μ_{i+1} to μ_i. The first player to break these rules loses; if there is no such player then player I wins.

I	k_0, α_0, β_0		k_1, α_1, β_1		k_2, α_2, β_2		$\ldots$
II		μ_0		μ_1		$\ldots$	

The game G_σ

If σ does not witness that T is weakly homogeneous then there is an $x \in \mathrm{p}[T]$ for which there is no wellfounded tower $\langle \mu_i : i < \omega\rangle$ consisting of measures from σ, with $\{s \in \kappa^i : (x{\restriction}i, s) \in T\} \in \mu_i$ for each i. Lemma 1.3.8 of [27] shows that in this case there is a continuous witness to the illfoundedness of towers of measures in σ concentrating on T_x. This gives a winning strategy for player I in G_σ, letting $x = \langle k_i : i < \omega\rangle$, $\langle \alpha_i : i < \omega\rangle$ be any fixed witness to $x \in \mathrm{p}[T]$, and the values β_i come from this continuous witness.

It suffices now to show that for U-many σ, I does not have a winning strategy for G_σ. Assume toward a contradiction that for U-almost every σ, player I does has a winning strategy in G_σ. Since each game G_σ is closed, and since the set of possible moves is for player I wellorderable (this is also true for player II, since all measures on $\kappa^{<\omega}$ are ordinal-definable, but not needed for the argument that follows), we can let, for each σ for which I has a winning strategy in in G_σ, $H(\sigma)$ be the winning strategy for I where I plays the least move leading to a subgame where she still has a winning strategy.

Now:

- let k_0 be the element of ω played by $H(\sigma)$ in round 0 for U-many σ;
- let μ_0 be the set of $A \subseteq \kappa$ such that, for U-many σ, the ordinal α_0^σ played by $H(\sigma)$ in round 0 is in A (then μ_0 is a countably complete measure on κ);
- let k_1 be the element of ω played by $H(\sigma)$ in round 1 in response to μ_0, for U-many σ;
- let μ_1 be the set of $A \subseteq \kappa^2$ such that, for U-many σ, letting $\langle \alpha_0^\sigma, \alpha_1^\sigma\rangle$ be as played by $H(\sigma)$ in rounds 0 and 1 in response to μ_0, $\langle \alpha_0^\sigma, \alpha_1^\sigma\rangle \in A$ (then μ_1 is a countably complete measure on κ^2 projecting to μ_0).

Continuing in this way, we get $x = \langle k_i : i \in \omega\rangle \in \omega^\omega$ and a tower of measures $\vec{\mu}$. Each μ_i concentrates on T_x because $(\alpha_0^\sigma, \ldots, \alpha_i^\sigma)$ is in T_x for U-many σ. The tower $\vec{\mu}$ is wellfounded since, if $A_i \in \mu_i$ for all $i < \omega$ then by the countable completeness of U there is a σ such that $(\alpha_0^\sigma, \ldots, \alpha_i^\sigma) \in A_i$ for all $i \in \omega$. However, by the countable completeness of U there is a σ containing $\{\mu_i : i \in \omega\}$, so that $\vec{\mu}$ is a legal play by player II against player I's winning strategy $H(\sigma)$.

Then player I's moves β_i induced by playing $H(\sigma)$ against $\vec{\mu}$ continuously witness the illfoundedness of $\vec{\mu}$, giving a contradiction. $\square$

13.1.4 Remark. If Lipschitz Determinacy holds, and all sets of reals are weakly homogeneously Suslin, then all set of reals are homogeneously Suslin. Otherwise, the homogeneously Suslin sets would form a proper initial segment of the Wadge hierarchy, and every subset of ω^ω would be the continuous image of a member of this initial segment. This would induce a surjection from ω^ω onto $\mathcal{P}(\omega^\omega)$.

13.1.5 Remark. Suppose that AD holds and that κ is the largest Suslin cardinal below some member θ_* of the Solovay sequence. Let Γ be the set of subsets of ω^ω of Wadge rank less than θ_*. Then $\Theta^{\mathrm{HOD}_\Gamma} = \theta_*$, and, by the Coding Lemma, $\mathcal{P}(\kappa)$ is contained in HOD_Γ. By Theorem 6.1.11 there exists in HOD_Γ a tree T on $\omega \times \kappa$ projecting to a set whose complement is not Suslin in HOD_Γ. There is no surjection in HOD_Γ from ω^ω onto the measures on κ or even onto any countable set of measures witnessing the weak homogeneity of T, since otherwise, the Martin-Solovay construction (see Remark 13.0.3) would produce Suslin representations for the complement of the projection of T (note that since $\omega^\omega \subseteq \mathrm{HOD}_\Gamma$, any countable subset of a surjective image of ω^ω in HOD_Γ is also in HOD_Γ). By Theorem 13.1.1, each measure on κ is coded by a subset of ω^ω in HOD_Γ. It follows that if $\theta_* < \Theta$, then there is a countable set of measures witnessing the T is weakly homogeneously Suslin by Lemma 13.1.3, and the minimal Wadge ranks of subsets of ω^ω in HOD_Γ coding these measures must be cofinal in θ_*. The Martin-Solovay construction also shows in this case that the complement of the projection of T is θ_*-Suslin. Briefly, this follows from the fact that, each $x \in \omega^\omega \setminus \mathrm{p}[T]$, the rank function on T_x maps into κ^+, so, for each $n \in \omega$, the restriction of the rank function to the nth level of T_x represents an ordinal below Θ in any ultrapower by a measure on κ^n. This shows that θ_* is the least Suslin cardinal above κ.

Theorem 13.1.6. *If AD^+ holds then every nonlimit member of the Solovay sequence below Θ has cofinality ω.*

Proof. Let θ_* be a nonlimit member of the Solovay sequence, and let Γ be the set of subsets of ω^ω of Wadge rank less than θ_*. There exists an $A \in \Gamma$ such that every element of Γ is ordinal definable from A and some element of ω^ω. The proof of Theorem 6.2.5 shows that Uniformization fails in HOD_Γ. By Theorem 9.2.9, $\mathrm{HOD}_\Gamma \models \mathsf{AD}^+$. By Corollaries 6.1.18 and 11.4.4, there is largest Suslin cardinal below θ_*. If $\theta_* < \Theta$, then Remark 13.1.5 shows that θ_* has countable cofinality. $\square$

13.2 Normal measures on $\mathcal{P}_{\aleph_1}(\lambda)$

In this section we show (assuming $\mathsf{DC}_\mathbb{R}$) that if λ is a Suslin cardinal and λ-Determinacy holds, then there is a normal fine ultrafilter on $\mathcal{P}_{\aleph_1}(\lambda)$.

13.2.1 Definition. Given a set X, we say that an ultrafilter μ on $\mathcal{P}(\mathcal{P}_{\aleph_1}(X))$ is

- *fine* if for all $x \in X$, $\{\sigma \in \mathcal{P}_{\aleph_1}(X) : x \in \sigma\}$ is in μ;
- *normal* if whenever $f\colon \mathcal{P}_{\aleph_1}(X) \to \mathcal{P}_{\aleph_1}(X)$ is such that $f(\sigma)$ is a nonempty subset of σ for each nonempty $\sigma \in \mathcal{P}_{\aleph_1}(X)$, there exists an $x \in X$ such that
$$\{\sigma \in \mathcal{P}_{\aleph_1}(X) : \alpha \in f(\sigma)\}$$
is in μ.

13.2.2 Remark. Given a set $A \subseteq \mathcal{P}_{\aleph_1}(\omega^\omega)$, consider the game $\mathcal{G}_A$ where players I and II alternate picking finite subsets of ω^ω, with player I winning if the unions of all the sets chosen is in A. This game is determined under $\mathsf{AD}_{\mathbb{R}}$. Solovay [36] showed that if all such games are determined, then the set of A for which player I has a winning strategy in $\mathcal{G}_A$ is a normal fine ultrafilter on $\mathcal{P}_{\aleph_1}(\omega^\omega)$.

Theorem 13.2.3. *If λ is a Suslin cardinal, and λ-$\mathsf{Determinacy}$ + $\mathsf{DC}_{\mathbb{R}}$ holds, then there is a normal fine measure on $\mathcal{P}_{\aleph_1}(\lambda)$.*

Proof. Fix a minimal λ-full tree T on $\omega \times \lambda$, as given by Lemma 6.1.14. For each $\sigma \subseteq \lambda$, let $T\restriction\sigma$ be the set of nodes (s,t) of T for which the range of t is contained in σ. We say that σ is *full* if, for each node (s,t) of $T\restriction\sigma$, there exists an $x \in p[T\restriction\sigma]$ whose leftmost branch in T contains (s,t).

Given an injection $g\colon \omega \to \lambda$, we let T_g be the tree on $\omega \times \omega$ consisting of those nodes $(s,t) \in \omega^{<\omega} \times \omega^{<\omega}$ such that $(s, g \circ t)$ is in T. Let us say that a function $g\colon \omega \to \lambda$ is *full* if it is injective and its range is full. We call the function $(s,t) \mapsto (s, g\circ t)$ embedding T_g into T the *g-induced map.*

Claim. *If g and g' are distinct full functions from ω to λ, then $T_g \neq T_{g'}$.*

Proof of Claim. Since T is λ-full, whenever g and g' are distinct functions from ω to λ, their induced maps are also distinct. It suffices then to see that, given a full function $g\colon \omega \to \lambda$, the g-induced map can be recovered from T_g without using g.

Let T^1_g be $\{t : \exists s\,(s,t) \in T_g\}$. We work recursively through T^1_g (using some wellordering of T^1_g in ordertype ω which lists each node before its successors), building a $\subseteq$-increasing sequence of finite partial length-preserving embeddings $e_k\colon \omega^{<\omega} \to \lambda^{<\omega}$ ($k \in \omega$) such that the corresponding function $(s,t) \mapsto (s, e_k(t))$ (which we will call e^*_k) maps T_g (partially) into T (and possibly halting if our construction breaks down, although we will eventually see that it doesn't). We let e_0 be the map which sends the empty sequence to itself. Now suppose that we have built e_k (for some $k \in \omega$), and that t is the least node of T^1_g in our wellordering outside the domain of e_k. Then t is of length $n+1$, for some $n \in \omega$, $t\restriction n$ is in the domain of e_k and, for all s such that $(s, t\restriction n) \in T_g$, $(s, e_k(t\restriction n))$ is in T. Let E_k be the set of $\gamma < \lambda$ for which there exists a length-preserving embedding $e\colon T^1_g \to \lambda^{<\omega}$ extending $e_k \cup \{(t, (e_k \circ (t\restriction n))^\frown\langle\gamma\rangle)\}$ such that the map $(s,r) \mapsto (s, e\circ r)$ embeds T_g into T. If E_k is empty, then we stop the construction. Otherwise, letting γ_* be the least element of E_k, we let e_{k+1} be $e_k \cup \{(t, (e_k \circ (t\restriction n))^\frown\langle\gamma_*\rangle)\}$. This completes the construction.

We now verify inductively that our maps e^*_k agree with the g-induced map. For e^*_0 this is clear. Supposing that this is true for a given $k \in \omega$, and letting t be the least node of T^1_g outside domain of e_k, we need to see that $g(t(n))$ is the least element of E_k (where t has length $n+1$). Since e^*_k agrees with the g-induced map, $g(t(n))$ is in E_k, which means that e_{k+1} was defined. Let $s \in \omega^{<\omega}$ be such that $(s,t) \in T_g$. Since g is full, there is an $x \in \omega^\omega$ whose T-leftmost branch (x,f) goes through $(s, g \circ t)$. This shows that $g(t(n))$ is the least element of E_k, as desired, since the image of f under a length-preserving embedding of T^1_g into $\lambda^{<\omega}$ witnessing the contrary would contradict the assumption that f is the leftmost branch of T_x. $\square$

We define now a function F whose domain is the set of trees on $\omega \times \omega$:

- for trees of the form T_g, for g a full function from ω to λ, $F(T_g)$ is the range of g;
- for all other trees on $\omega \times \omega$, F takes the value $\emptyset$.

By the claim, this function is well defined. We now apply λ-$\mathsf{Determinacy}$.

Given a set $A \subseteq \mathcal{P}_{\aleph_1}(\lambda)$, let G_A be the game where

- players I and II build a $\subseteq$-increasing sequence $\langle p_n : n \in \omega\rangle$ of finite partial injections from ω to λ, with n contained in the domain of p_n for each $n \in \omega$;
- letting $g = \bigcup_{n\in\omega} p_n$, player II wins if g is full and $F(T_g)$ is in A.

This game is determined by λ-$\mathsf{Determinacy}$. Let μ be the set of $A \subseteq \mathcal{P}_{\aleph_1}(\lambda)$ for which II has a winning strategy. We wish to see that μ is a normal fine ultrafilter.

To check that μ is fine, fix an $\alpha < \lambda$ and a strategy Σ for player I in the game G_A, where $A = \{\sigma \in \mathcal{P}_{\aleph_1}(\lambda) : \alpha \in \sigma\}$. Using $\mathsf{DC}_\mathbb{R}$ and the assumption that every node of T is part of the leftmost branch of some element of $p[T]$, we can find a winning play for II against Σ (i.e., such that the function g produced is full and has α in its range).

To see that μ is an ultrafilter, fix $A \subseteq \mathcal{P}_{\aleph_1}(\lambda)$. We show first that at least one of A and $\mathcal{P}_{\aleph_1}(\lambda) \setminus A$ is in μ. Fix strategies Σ_A and $\Sigma_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ for player I in G_A and $G_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$, respectively. Using $\mathsf{DC}_\mathbb{R}$ we can find a countable full set $\sigma \subseteq \lambda$ which is closed under both Σ_A and $\Sigma_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ (in the sense that a response by either strategy to a sequence of moves whose ranges are contained in σ will a function with range contained in σ). Then, playing as II, we can produce runs of G_A and $G_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ where I plays with Σ_A and $\Sigma_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ respectively, and the range of the resulting function g is σ in each case. This shows that Σ_A and $\Sigma_{\mathcal{P}_{\aleph_1}(\lambda)\setminus A}$ cannot both be winning strategies for I. The same argument, with I and II reversed, shows that A and $\mathcal{P}_{\aleph_1}(\lambda) \setminus A$ cannot both be in μ.

To see that μ is normal, fix a function f which picks a nonempty subset of each $\sigma \in \mathcal{P}_{\aleph_1}(\lambda)$. For each $\alpha \in \lambda$, let A_α be the set of $\sigma \in \mathcal{P}_{\aleph_1}(\lambda)$ for which α is in $f(\sigma)$. We want to see that player II has a winning strategy in some A_α.

Let G be the game in which player II goes first, picking $\alpha < \lambda$, after which the players play G_{A_α}, with I going first as usual. We want to see that player II has a winning strategy in this game. If she doesn't, then I does, by λ-$\mathsf{Determinacy}$. Player I's winning strategy in the game G induces a sequence $\langle \Sigma_\alpha : \alpha < \lambda \rangle$, where each Σ_α is a winning strategy for I in the corresponding game G_{A_α}. Using $\mathsf{DC}_{\mathbb{R}}$ again we can find a countable full set $\sigma \subseteq \lambda$ which is closed under Σ_α for each α in σ. Then we can find runs of the games G_{A_α} $(\alpha \in \sigma)$, where in each case I plays according to Σ_γ and the range of the resulting function g is σ. Since $f(\sigma)$ is a nonempty subset of σ, this gives a contradiction. □

We note the following theorem, whose proof is essentially the same as the proof of the claim in the proof of Theorem 13.2.3.

Theorem 13.2.4. *Let λ be an ordinal, and let T be a tree on $\omega \times \lambda$. Suppose that M is a transitive set such that, in M, T is minimal. Let N be a transitive set and let $j \colon M \to N$ be an elementary embedding. Then $j[T]$ is in $L[T, j(T)]$.*

Proof. The map $j{\restriction}T$ embeds T into $j(T)$. We want to find this map in $L[T, j(T)]$. We use the fact that if

- P is a wellfounded model of ZF (in particular, a forcing extension of $L[T, j(T)]$),
- S and R are trees in P such that, in P, S is countable and T is wellorderable,
- g is, in P, a length-preserving partial embedding of S into R, and
- there is a length-preserving embedding of S into R extending g (in V),

then there is, in P, a length-preserving embedding of S into R extending g.

Let T_1 be $\{t \in \exists s\,(s,t) \in T\}$. We work recursively through T_1, using some wellordering $\langle t_\alpha : \alpha < \delta \rangle$ of T_1 which lists each node before its successors, building a continuous $\subseteq$-increasing sequence of partial length-preserving functions $e_\beta \colon T_1 \to j(\lambda)^{<\omega}$ $(\beta < \delta)$ such that

- the domain of each e_β is $\{t_\alpha : \alpha < \beta\}$;
- for all $\alpha < \beta$ and all $s \in \omega^{<\omega}$, if $(s, t_\alpha) \in T$ then $(s, e_\beta(t_\alpha)) \in j(T)$.

Each e_α will have $\{(s_\beta, t_\beta) : \beta < \alpha\}$ as its domain. Then e_0 is the empty function. We let e_1 be the map which sends the empty sequence to itself. Now suppose that we have built e_β, for some $\alpha \in [1, \delta)$. Then t_β is of length $n+1$, for some $n \in \omega$, and $t_\beta{\restriction}n$ is in the domain of e_β. Let E_β be the set of $\gamma < \lambda$ for which, in any generic extension of $L[T, j(T)]$ in which T is countable, there exists a length-preserving function $e \colon T_1 \to j(\lambda)^{<\omega}$ extending e_β such that,

- $e(t_\beta) = e_\beta(t_\beta{\restriction}n)^\frown\langle\gamma\rangle$;
- for all $(s,t) \in T$, $(s, e(t)) \in j(T)$.

If E_β is empty, then we stop the construction. Otherwise, letting γ_* be the least element of E_β, we let $e_{\beta+1}$ be $e_\beta \cup \{(t_\beta, e_\beta(t_\beta \restriction n)^\frown\langle\gamma_*\rangle)\}$. This completes the construction.

We now verify inductively that our maps e_β agree with $j \restriction T_1$. For e_0 and e_1 this is clear. The limit case of the induction also follows immediately. Supposing that it is true for a given $\beta < \delta$, we need to see that $j(t_\beta(n))$ is the least element of E_β (where t_β has length $n+1$). Our induction hypothesis implies that $j(t_\beta(n))$ is in E_β, which means that $e_{\beta+1}$ was defined. Fix an $x \in \omega^\omega \cap M$ such that t_β is an initial segment of $\mathrm{lb}(T_x)$. Then $j(t_\beta)$ is an initial segment of $\mathrm{lb}(j(T)_x)$ (as computed in any wellfounded model of ZF). This shows that $j(t_\alpha(n))$ is the least element of E_α, as desired, since a length-preserving embedding of T into $j(T)$ witnessing the contrary would produce a path though $j(T)_x$ contradicting the fact that $j(t_\beta)$ is an initial segment of $\mathrm{lb}(j(T)_x)$. $\square$

13.3 AD implies $\mathsf{AD}_{\mathbb{R}}$ if all sets of reals are Suslin

In this section we complete our proofs of Theorem 13.0.1. Our proofs will show that quasi-$\mathsf{AD}_{\mathbb{R}}$ holds. Then we will be done by Remark 6.2.4. Recall that if every subset of ω^ω is Suslin, then $\mathsf{Uniformization}$ holds, and that $\mathsf{Uniformization}$ implies $\mathsf{DC}_{\mathbb{R}}$.

We fix a bijection $\pi\colon \omega \times \omega \to \omega$ such that $\pi(i,j) < \pi(m,n)$ whenever $i+j < m+n$. It follows that for all odd $n \in \omega$, $\pi^{-1}[(n+1)/2] \subseteq (n+1)\times\omega$. Let π_* be the induced bijection from $(\omega^\omega)^\omega \to \omega^\omega$ (so $\pi_*(\langle x_n : n \in \omega\rangle)(\pi(i,j)) = x_i(j)$ for all $i, j \in \omega$). We fix a real game $\mathcal{G}$, and let A be the π_* image of its payoff set. By either Theorem 13.0.2 or Lemma 13.1.3 and Remark 13.1.4, we may fix κ-homogeneous trees S and T (on $\omega\times\kappa$, for some cardinal κ) such that $p[S] = A$ and $p[T] = \omega^\omega \setminus A$.

The key remaining step is given in Lemma 13.3.1 below. Note that in the statement of the lemma the trees S and T are homogeneous in V, but not necessarily in M.

Lemma 13.3.1 ($\mathsf{ZF} + \mathsf{AD}$)**.** *Let κ be an ordinal. Let $\mathcal{G}$ be a real game with payoff set $A \subseteq (\omega^\omega)^\omega$, and let S and T be homogeneous trees on $\omega\times\kappa$ such that $\mathrm{p}[S] = \pi_*[A]$ and $\mathrm{p}[T] = \omega^\omega \setminus \pi_*[A]$. Let M be an inner model of ZF containing S and T, such that $\omega^\omega \cap M$ is countable. Then, in M, the real game with payoff set $\pi_*^{-1}[\mathrm{p}[S]]$ is quasi-determined.*

Proof. Let $\mathcal{G}_M$ be the real game with payoff set $\pi_*^{-1}[\mathrm{p}[S]]$, as computed in M. We define games $\mathcal{G}^*_I$ and $\mathcal{G}^*_{II}$, both in M. In $\mathcal{G}^*_I$, players I and II alternate playing $x_i \in \omega^\omega \cap M$. In each turn, I also plays $\alpha_{i/2} \in \kappa$. Player I wins if and only if $(\pi_*(\langle x_i : i < \omega\rangle), \langle\alpha_i : i < \omega\rangle)$ is in $[S]$.

I	x_0, α_0		x_2, α_1		x_4, α_2		$\ldots$
II		x_1		x_3		$\ldots$	

The game $\mathcal{G}^*_I$.

In the game $\mathcal{G}^*_{II}$, players I and II alternate playing $x_i \in \omega^\omega \cap M$. In each turn, II also plays $\alpha_{(i-1)/2} \in \kappa$. Player II wins if and only if

$$(\pi_*(\langle x_i : i < \omega \rangle), \langle \alpha_i : i < \omega \rangle)$$

is in $[T]$.

I	x_0		$x_2 x$		x_4		$\ldots$
II		x_1, α_0		x_3, α_1		$\ldots$	

The game $\mathcal{G}^*_{II}$.

The first of these games is closed, and the second is open, so they are both quasi-determined. Moreover, for each game there is a quasi-strategy in M which is a winning quasi-strategy in V, since M and V compute the same ranking function when applying the proof of open determinacy to these games. Since $\omega^\omega \cap M$ is countable, such a winning quasi-strategy can be converted in V into a winning strategy.

We want to see that, in M, either I has a winning quasi-strategy in $\mathcal{G}^*_I$ or II has a winning quasi-strategy in $\mathcal{G}^*_{II}$, since the corresponding player could use his strategy to win the game $\mathcal{G}_M$. It suffices then to derive a contradiction from the assumption that, in V, player II has a winning strategy Σ_{II} in $\mathcal{G}^*_I$ and player I has a winning strategy Σ_I in $\mathcal{G}^*_{II}$. This contradiction follows from the fact that, in V, if player II has a winning strategy in $\mathcal{G}^*_I$, then player II has a winning strategy in $\mathcal{G}_M$ and, similarly that if player I has a winning strategy in $\mathcal{G}^*_{II}$, then player II has a winning strategy in $\mathcal{G}_M$. These facts are essentially the same, and essentially the same as Martin's theorem that homogeneously Suslin sets are determined (see Exercise 32.2 of [10], for instance). We give a proof for the case where player I has a winning strategy in $\mathcal{G}^*_{II}$.

Let Σ_I be such a strategy. Let $\{\nu_s : s \in \omega^{<\omega}\}$ be a set of measures witnessing that T is homogeneously Suslin. Then for each $s \in \omega^{<\omega}$, ν_s is a countably complete ultrafilter on $\kappa^{|s|}$, and $\{t \in \kappa^{|s|} : (s,t) \in T\}$ is in ν_s. Define a strategy Σ_* for player I in $\mathcal{G}_M$ as follows. Let $\Sigma_*(\langle\rangle) = \Sigma_I(\langle\rangle)$. Now fix a sequence $\bar{x} = \langle x_0, \ldots, x_n \rangle \in (\omega^\omega \cap M)^{<\omega}$ with n odd, and let $s_{\bar{x}} \in \omega^{(n+1)/2}$ be the common initial segment of length $(n+1)/2$ of the π_*-values of the elements of $(\omega^\omega)^\omega$ extending $\bar{x}$. Let $\Sigma_*(\bar{x})$ be the unique $y \in \omega^\omega \cap M$ such that the set

$$R^y_{\bar{x}} = \{t \in \kappa^{(n+1)/2} : \Sigma_I(\langle x_0, (x_1, t(0)), \ldots, (x_n, t((n-1)/2)) \rangle = y\}$$

is in $\nu_{s_{\bar{x}}}$. Now suppose that $\langle x_i : i < \omega \rangle$ is a run of G_M where I has played according to Σ_* and lost. Since the measures ν_s witness the homogeneity of T, there is an $f \in \kappa^\omega$ such that $f \restriction ((n+1)/2)$ is in $R^{x_{n+1}}_{\langle x_i : i < n \rangle}$ for each odd $n \in \omega$. Then $\langle x_0, (x_1, f(0)), x_2, (x_3, f(1)), \ldots \rangle$ is a run of G^*_{II} where I plays according to Σ_I and loses, giving a contradiction. $\square$

Proof of Theorem 13.0.1. Fix $A \subseteq (\omega^\omega)^\omega$ and let S and T be homogeneous trees projecting to $\pi_*[A]$ and its complement respectively, on $\omega \times \kappa$, for some $\kappa < \Theta$. Let λ be a Suslin cardinal greater than $\Theta^{\mathrm{HOD}_{\{S,T\}}}$ (which exists by Remark 6.1.7 if all subsets of ω^ω are Suslin), and let $B \subseteq \omega^\omega$ have Wadge rank greater than λ. We work in $L(B, \mathbb{R})$, which satisfies DC and λ-Determinacy (by Corollary 7.0.4 and Remark 7.0.5).

By the proof of Lemma 10.1.7, we may fix a partial order P on $\Theta^{\mathrm{HOD}_{\{S,T\}}}$ and a set $K \subseteq \Theta^{\mathrm{HOD}_{\{S,T\}}}$, both in $\mathrm{HOD}_{\{S,T\}}$, such that, in any forcing extension of V by $\mathrm{Col}^*(\omega, \mathbb{R})$, there is an $L[P, K]$-generic filter $G \subseteq P$ such that $\mathbb{R}^V$ is contained in $L[P, K][G]$.

Let μ_λ be a normal fine measure on $\mathcal{P}_{\aleph_1}(\lambda)$, as given by Theorem 13.2.3. For each countable $\sigma \subseteq \lambda$, let P_σ and K_σ be the restrictions of P to $\sigma \cap \Theta^{\mathrm{HOD}_{\{S,T\}}}$. By Lemma 13.3.1, for any such σ, in any inner model of any forcing extension of $L[S, T, P_\sigma, K_\sigma]$ by P_σ, the real game with payoff set $\pi_*^{-1}[\mathrm{p}[S]]$ is determined.

Since μ_λ is a normal fine measure, the ultraproduct

$$\prod_{\sigma \in \mathcal{P}_{\aleph_1}(\lambda)} L[S, T, P_\sigma, K_\sigma]/\mu_\lambda$$

is a wellfounded model $L[S', T', P, K]$, and by elementarity, in any inner model of any forcing extension of $L[S', T', P, K]$ by P, the real game with payoff set $\pi_*^{-1}[p[S']]$ is determined. However, $p[S'] = p[S] = \pi_*[A]$, and $L[S', T', P, K](\mathbb{R})$ is such an inner model, which shows that the real game with payoff set $\pi_*^{-1}[\mathrm{p}[S]]$ is determined. $\square$

Combining Theorem 11.2.4 and 13.0.1 with Remark 6.2.2 (i.e., $\mathsf{AD}_{\mathbb{R}}$ implies Uniformization) gives the following.

Theorem 13.3.2 (ZF + AD^+)**.** *The following are equivalent.*

1. *There is no $\leq_{\mathcal{D}}$-maximal set of ordinals.*
2. *The Suslin cardinals are cofinal in Θ.*
3. *Every subset of ω^ω is Suslin.*
4. $\mathsf{AD}_{\mathbb{R}}$
5. Uniformization

By Theorems 11.4.1 and 13.3.2, the context of the following theorem is exactly the case where AD^+ holds and $\mathsf{AD}_{\mathbb{R}}$ fails.

Theorem 13.3.3 (ZF + AD^+)**.** *If there is a largest Suslin cardinal then there is a set of ordinals S such that $\mathcal{P}(\omega^\omega)$ is contained in $L(S, \mathbb{R})$.*

Proof. By the Moschovakis Coding Lemma, if there is a largest Suslin cardinal, then some subset of ω^ω is not Suslin. It follows by Theorem 11.2.4 that there is a $\leq_{\mathcal{D}}$-maximal set of ordinals. From this and Theorem 8.6.6 it follows that there is a set S of ordinals such that $\mathcal{P}(\omega^\omega)$ is contained in $L(S, \mathbb{R})$. $\square$

Our proof of Theorem 13.0.1 (using Theorem 13.0.2) gives the following local version.

Theorem 13.3.4 ($\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$)**.** *Suppose that*

- $A \subseteq (\omega^\omega)^\omega$ *has the property that* $\pi_*^{-1}[A]$ *and* $\omega^\omega \setminus \pi_*^{-1}[A]$ *are both* κ*-homogeneously Suslin as witnessed by trees* S *and* T *on* $\omega \times \kappa$*, and*
- $\lambda \geq \Theta^{\mathrm{HOD}_{\{S,T\}}}$ *is a Suslin cardial such that* λ-Determinacy *holds.*

Then the real game with payoff set A *is determined.*

[2]

[2]The rest of this section is from an email from Howard. ZF + AD does not imply $\mathsf{AD}_{\mathbb{R}}$ for Suslin-co-Suslin sets. In fact, $\mathsf{ZF} + \mathsf{AD}$ does not imply that Π^1_1 games on reals are determined. The real theorem is this.

Theorem 13.3.5 ($\mathsf{ZF} + \mathsf{DC}$)**.** *All* Π^1_1 *games on reals are determined if and only if the sharp of* $L(\mathbb{R})$ *exists.*

Therefore, in $L(\mathbb{R})$, there is a non-determined Π^1_1 game. (AD is irrelevant.) This theorem was proved sometime in the 20th century by somebody (but not Howard). The theorem may not be published, and the proof may not even be available anywhere. What follows is a weaker theorem which is enough to answer Question 12.

Theorem 13.3.6 ($\mathsf{ZF} + \mathsf{DC} + V = L(\mathbb{R})$)**.** *There exists a* Π^1_1 *subset of* $(\omega^\omega)^\omega$ *such that the corresponding real game is not determined.*

Proof. Suppose the theorem is false. Let M be the minimal model of $ZF^- + \mathsf{DC} + V = L(R)$ + "all Π^1_1 games on reals are determined" containing all reals.

Work in V (= L(R)). A Lowenheim-Skolem argument, using DC, shows that given any strategy σ for either player, for a game on reals:

(*) There exists a countable elementary substructure M' of M and there exists a countable set of reals, S, such that $\mathbb{R} \cap M' = S$ and S is closed under σ.

Let $\mathcal{G}$ be the following game. On each move both players play a real and Player II also plays two elements of 2. After ω moves, the two players together have played a countable set of reals, S, and Player II has played w, x in 2^ω. Player II wins the run of $\mathcal{G}$ if

- w is a code for a countable ordinal α,
- $(\mathbb{R} \cap L_\alpha(S)) = S$,
- $L_\alpha(S)$ is a model of "V is the minimal model of $\mathsf{ZF}^- + \mathsf{DC} + V = L(\mathbb{R})$ + all Π^1_1 games on reals are determined containing all reals" and
- x is the set of true sentences of N.

Now (*) implies that Player I cannot have a winning strategy. And (*) implies that if σ is ANY winning strategy for Player II, then the x played by σ is constant, i.e., independent of Player I's moves, and is the set of true sentences of M.

A winning strategy in M is a winning strategy in V. Since there exists a winning strategy for $\mathcal{G}$ in M, truth in M is definable in M. Contradiction. $\square$

Chapter 14

Questions

1. Does TD follow from $\aleph_1 \not\leq 2^{\aleph_0}$ plus the assumption that the cone measure on the constructibility degrees is an ultrafilter?
2. Does $\mathsf{ZF} + \mathsf{AD}$ imply $\mathsf{DC}_{\mathbb{R}}$?
3. Does $\mathsf{ZF}+\mathsf{AD}$ imply that the Wadge hierarchy is wellfounded? (yes if $\mathsf{DC}_{\mathbb{R}}$ is also assumed)
4. Does $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ imply that all subsets of ω^ω are ∞-Borel?
5. Does $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ imply $<\Theta$-$\mathsf{Determinacy}$?
6. Does $\mathsf{ZF} + \mathsf{AD} + \mathsf{DC}_{\mathbb{R}}$ imply ω_2-$\mathsf{Determinacy}$?
7. Does $\mathsf{ZF} + \mathsf{AD}$ imply $\mathsf{WF}_{\mathrm{Tu}}$?
8. Does $\mathsf{ZF} + \mathsf{AD}$ imply that $\chi_{\mathrm{B}} = \kappa_{\mathrm{B}}$? (yes if $\mathsf{DC}_{\mathbb{R}}$ is also assumed)
9. Does $\mathsf{ZF} + \mathsf{AD}$ imply that $\kappa_{\mathrm{B}} = \lambda_{\mathrm{B}}$? (yes if $\mathsf{DC}_{\mathbb{R}}$ is also assumed)
10. Does $\mathsf{Turing\ Determinacy} + \mathsf{CC}_{\mathbb{R}}$ imply AD?
11. Does $\mathsf{Turing\ Determinacy}$ imply $\mathsf{DC}_{\mathbb{R}}$?
12. Does $\mathsf{ZF} + \mathsf{AD}_{\mathbb{R}}$ imply $<\Theta$-Determinacy?
13. Does AD imply that there are no strong partition cardinals greater than (or equal to) Θ?
14. Does AD^+ imply that for each set of ordinals S there is a bounded $T \subseteq \Theta$ such that $S \equiv_{\mathcal{D}} T$?
15. Does AD imply that each pointclass $\Sigma^2_1(A)$ is $\forall^{\omega^\omega}$-closed? (AD^+ does.)

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Index