Real Analysis

MATH 350

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Chapter 1

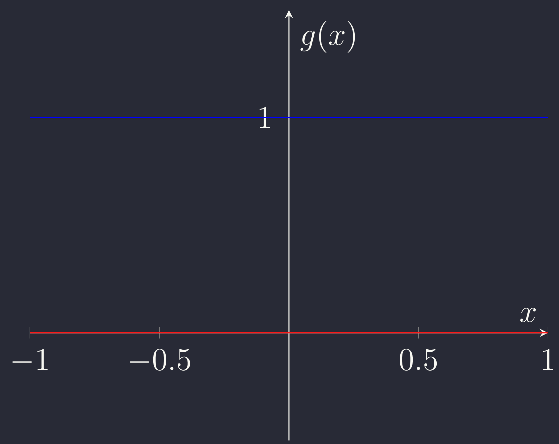
Functional Limits and Continuity

1.1 Discussion: Examples of Dirichlet and Thomae

Definition 1.1.1

The *Dirichlet function* $\lim_{x \to c} g(x)$ does not exist for any $c \in \mathbb{R}$.

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$



Definition 1.1.2

The *Thomae function* is defined as

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is in lowest terms}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Thomae's function, $t(x)$ is continuous at all $x \notin \mathbb{Q}$. It is not continuous at any $x \in \mathbb{Q}$.

1.2 Functional Limits

Recall from calculus I, that a function $f(x)$ is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$.

Definition 1.2.1

Let $f\colon A \to \mathbb{R}$ be a function and let c be a limit point of A. We say $\lim_{x\to c} f(x) = L$, if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Example 1.1: Functional Limit (From book) 1

Let $f(x) = 3x + 1$. Claim: $\lim_{x\to 2} f(x) = 7$.

Proof. Let $\epsilon > 0$. After we have done our scratch work, we can choose $\delta = \epsilon/3$, then $0 < |x - 2| < \delta$ implies $|f(x) - 7| < 3(\epsilon/3) = 3$. $\square$

Scratch Paper. Definition 4.2.1 requires that we produce a $\delta > 0$ so that $0 < |x - 2| < \delta$ leads to the conclusion that $|f(x) - 7| < \epsilon$. Notice that

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3\,|x - 2|\,.$$

Example 1.2: Functional Limit (From book) 2

Let $g(x) = x^2$. Claim: $\lim_{x\to 2} g(x) = 4$.

Proof. Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/5\}$. If $0 < |x - 2| < \delta$, then

$$\begin{aligned} |g(x) - 4| &= \left|x^2 - 4\right| \\ &= |x - 2|\,|x + 2| \\ &< 5\delta \\ &= (5)\frac{\epsilon}{5} \\ &= \epsilon. \end{aligned}$$

$\square$

Scratch Paper. Our goal this time is to make $|g(x) - 4| < \epsilon$ by restricting $|x - 2|$ to be smaller than some carefully chosen δ. As in the previous example, a little algebra reveals

$$|g(x) - 4| = \left|x^2 - 4\right| = |x - 2|\,|x + 2|\,.$$

We can make $|x + 2|$ as small as we like, but we need an upper bound on $|x + 2|$ in order to know how small to choose δ. The presence of the variable x causes some initial confusion, but keep in mind that we are discussing the limt as x approaches 2. If we agree that our $\delta-$neighborhood around $c = 2$ must have radius no bigger than $\delta = 1$, then we get the upper bound $|x + 2| < |3 + 2| = 5$ for all $x \in V_\delta(c)$.

Example 1.3: Functional Limit 1

Let $f(x) = 3x + 1$. Show that $\lim_{x\to 2} f(x) = 7$.

Proof. Let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{3}$. Assume $0 < |x - 2| < \delta$. Since $\delta > 0$, $2 - \delta < x < 2 + \delta$. Then,

$$\begin{aligned} |x-2| &< \delta, \\ |f(x) - 7| &= |3x + 1 - 7| \\ &= |3x - 6| \\ &= 3\,|x - 2| \\ &< 3\delta \\ &= \epsilon. \end{aligned}$$

Therefore, $\lim_{x\to 2} f(x) = 7$. □

Example 1.4: Functional Limit 3

Let $f(x) = x^2$. Claim: $\lim_{x\to 7} f(x) = 49$

Proof. Let $\epsilon > 0$. Set $\delta = \min\{\frac{\epsilon}{8}, 1\}$. If $0 < |x - 7| < \delta$, then

$$\begin{aligned} |f(x) - 49| &= \left|x^2 - 49\right| \\ &= |x-7|\,|x+7| \\ &< 8\delta \\ &< 8\left(\frac{\epsilon}{8}\right) \\ &= \epsilon. \end{aligned}$$

□

Scratch Paper. Always start with the goal statement: $|f(x) - 49| = |x^2 - 49|$. This factors into $|x - 7|\,|x + 7|$. Then, if $\delta < 1$, $|x - 7| < \delta$ and $|x + 7| < 8$. All together, we have $8\delta < \epsilon < \frac{\epsilon}{8}$.

Example 1.5: Functional Limit 4

Claim: $\lim_{x\to 3} \frac{1}{x+1} = \frac{1}{4}$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{12\epsilon, 1\}$. If $0 < |x-3| < \delta$, then

$$\begin{aligned}\left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{4-(x+1)}{4(x+1)}\right| \\ &= \left|\frac{3-x}{4(x+1)}\right| \\ &< \frac{\delta}{4(3)} \\ &= \frac{12\epsilon}{12} \\ &= \epsilon.\end{aligned}$$

Therefore, $\lim_{x\to 3} \frac{1}{x+1} = \frac{1}{4}$ $\square$

Scratch Paper. Goal: $\left|\frac{1}{x+1} - \frac{1}{4}\right|$. Hence,

$$\begin{aligned}\left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{4-(x+1)}{4(x+1)}\right| \\ &= \left|\frac{3-x}{4(x+1)}\right| \\ &< \frac{\delta}{4\,|x+1|} \\ &< \frac{\delta}{4(3)} \\ &= \frac{\delta}{12} \\ &< \epsilon.\end{aligned}$$

Thus, we need a $\delta < 1$, and we can choose $\delta = \min\{12\epsilon, 1\}$. Note: When we are determining the value for $|x+2|$, we solve for $\delta = 3 \pm 1 \Rightarrow x \in (2,4)$. Then, we find $x + 1 = (3,5)$. We choose 3 rather than 5 because of division. We want to be as close as possible.

Example 1.6: Functional Limit 5

Claim: $\lim_{x\to 3}(x^2 + 7x) = 30$.

Proof. Let $\epsilon > 0$ and set $\delta = \min\{\frac{\epsilon}{14}, 1\}$. If $0 < |x-3| < \delta$, then

$$\begin{aligned}\left|x^2 + 7x - 30\right| &= |x-3|\,|x+10| \\ &< 14\delta \\ &= 14\left(\frac{\epsilon}{14}\right) \\ &= \epsilon.\end{aligned}$$

$\square$

Example 1.7: Functional Limit 6

Claim: $\lim_{x\to 3} \frac{2x+3}{4x-9} = 3$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{\frac{\epsilon}{10}, \frac{1}{2}\}$. (Note: We are choosing $\frac{1}{2}$ because we want to avoid having 0 anywhere in the interval.) Assume $0 < |x-3| < \delta$. Since $\delta < \frac{1}{2}$, $\frac{5}{2} < x < \frac{7}{2}$, then $1 < |4x-9| < 5$. (Thus, 0 can not possibly be in the denominator.) $\square$

Scratch Paper.

$$\begin{aligned}\left|\frac{2x+3}{4x+9} - 3\right| &= \left|\frac{2x+3-3(4x+9)}{4x+9}\right| \\ &= \left|\frac{2x+3-12x-27}{4x+9}\right| \\ &= 10\left|\frac{x-3}{4x-4}\right| \\ &< 10\frac{\epsilon/10}{1} \\ &= \epsilon.\end{aligned}$$

Example 1.8: Functional Limit 7

Claim: $\lim_{x\to 4} \sqrt{x} = 2$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{1, 3\epsilon\}$. Assume $0 < |x-4| < \delta$. Then (refer to scratch work). $\square$

Scratch Paper.

$$\begin{aligned}\left|\sqrt{x} - 2\right| &= \left|\sqrt{x} - 2\right| \\ &= \left|\frac{(\sqrt{x}-2)\cdot(\sqrt{x}+2)}{\sqrt{x}+2}\right| \\ &= \left|\frac{x-4}{\sqrt{x}+2}\right| \\ &< \frac{\delta}{3} \\ &< \frac{3\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Notice that we picked $\delta < 1$ such that $3 < x < 4$ so $1 < \sqrt{x} < 2$ and $3 < \sqrt{x} + 2 < 4$.

Theorem 1.2.2: Sequential Criterion for Functional Limits

The following statements are equivalent:

(a) $\lim_{x\to c} f(x) = L$.

(b) For all sequences (x_n) where $x_n \neq c$ and $(x_n) \to c$, we have $\lim_{n\to\infty} f(x_n) = L$.

Proof. (1) $\rightarrow$ (2)

Assume $\lim_{x\to c} f(x) = L$.
Let $(x_n) \to c$ with $x_n \neq c$
Let $\epsilon > 0$.

- Since $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that if $0 < |x-c| < \delta$, then $|f(x) - L| < \epsilon$.
- Since $x_n \to c$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - c| < \delta$.
- Now, for all $n \geq N$, it follows that $x_n - c < \delta$ and thus $|f(x) - L| < \epsilon$.

Thus, $\lim_{n\to\infty} f(x_n) = L$.

(2) $\rightarrow$ (1)

Proof by contrapositive.
Assume (1) is not true. Thus,

$$\lim_{x\to c} f(x) \neq L.$$

There exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists an x with $0 < |x-c| < \delta$ and $|f(x) - L| \geq \epsilon_0$.
For each $n \in \mathbb{N}$, consider $\delta = \frac{1}{n}$. There exists $x_n \in (c - \frac{1}{n}, c + \frac{1}{n})$ with $x_n \neq c$ such that $|f(x) - L| \geq \epsilon_0$.

- Since $|x_n - c| < \frac{1}{n}$, we see that $(x_n) \to c$.
- Since for all $n \in \mathbb{N}$, $|f(x) - L| \geq \epsilon_0$. Then, $\lim_{n\to\infty} f(x) \neq L$.

Thus, $\neg(1) \rightarrow \neg(2)$. So (2) $\rightarrow$ (1) and (1) $\rightarrow$ (2). $\square$

If functional limits and sequential limits are the same thing, then everything we know about sequential limits is also true about functional limits.
Recall Algebraic Limit Theorem. From this, we can write the functional equivalent:
Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then,

- $\lim_{x\to c}(f(x) + g(x)) = L + M$
- $\lim_{x\to c}(f(x) - g(x)) = L - M$
- $\lim_{x\to c}(f(x)g(x)) = LM$
- $\lim_{x\to c}(\frac{f(x)}{g(x)}) = \frac{L}{M}$ unless $M = 0$.

Theorem 1.2.3: Divergence Criterion

Let $f\colon A \to \mathbb{R}$ with c as a limit point of A. If there exists two sequences (x_n) and (y_n) in $A \setminus \{c\}$ (that both converge to c) such that $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, then $\lim_{x\to c} f(x)$ does not exist.

Example 1.9: Divergence Criterion 1

$f(x) = \frac{x}{|x|} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$ Our goal is to show that $\lim_{x\to 0} f(x)$ does not exist.

Proof. Let $(x_n) = (\frac{1}{n})$ and let $(y_n) = (\frac{-1}{n})$. We will see that as $n \to \infty$, $\lim_{n\to\infty} f(x_n) = 1$ and $\lim_{n\to\infty} f(y_n) = -1$. Thus, $\lim_{x\to 0} f(x)$ does not exist. □

Example 1.10: Divergence Criterion 2

$g(x) = \sin(\frac{1}{x})$. Show that $\lim_{x\to 0} g(x)$ does not exist.

Proof. Let $(x_n) = (\frac{1}{2\pi n})$ and let $(y_n) = (\frac{1}{2\pi n + \frac{\pi}{2}})$. We will see that as $n \to \infty$, $\lim_{n\to\infty} g(x_n) = 1$ and $\lim_{n\to\infty} g(y_n) = -1$. Thus, $\lim_{x\to 0} g(x)$ does not exist. □

(section?) Infinite limits

We say $\lim_{n\to\infty} x_n = \infty$ if for all $M > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

We say $\lim_{x\to c} f(x) = \infty$ if for all $M > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $f(x) > M$. Think of vertical asymptotes.

Theorem 1.2.4: Infinite Limits Cauchy Criterion

If $(x_n) \to \infty$, (x_n) will not be Cauchy. It is possible to have $x_{n+1} - x_n$ approach 0, but (x_n) converges to ∞.

1.2.1 Exercises

Exercise: 4.2.10

Right and Left Limits. Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting x approach a from the right-hand side."

Solution.

(a) (a) Give a proper definition in the style of Definition 4.2.1 for the right-hand and left-hand limit statements:

$$\lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = M.$$

(b) (b) Prove that $\lim_{x \to a} f(x) = L$ if and only if both the right and left-hand limits equal L.

Exercise: 4.2.11

Squeeze Theorem. Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} h(x) = L$ at some limit point c of A, show $\lim_{x \to c} g(x) = L$ as well.

Solution.

1.3 Continuous Functions

Definition 1.3.1

We say a function f is *continuous* at c if

$$\lim_{x \to c} f(x) = f(c).$$

Equivalent definition:
For all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then

$$|f(x) - f(c)| < \epsilon.$$

Idea: When x is close to c, $f(x)$ is close to $f(c)$. Then, for the topological definition, we can say if $x \in V_\delta(c)$ then $f(x) \in V_\epsilon(f(c))$.

Definition 1.3.2

We say function f is *continuous* on a set D if f is continuous at every point in D.

The following are equivalent (TFAE):

1. $\lim_{x \to c} f(x) = L$
2. For all sequences (x_n) such that $(x_n) \to c$, we have $\lim_{n \to \infty} f(x_n) = L$.

<u>Continuous Functions</u> (THM 4.3.2 in book)

Claim: Let $a \in \mathbb{R}$. Then $f(x) = a$ is continuous.

Proof. Let $c \in \mathbb{R}$. Let $\epsilon > 0$. Set $\delta = 8$. Now, if $x - c < 8$, then $|f(x) - f(c)| = a - a = 0 < \epsilon$. Thus, constant functions are continuous. □

Claim: $f(x) = x$ is continuous.

Proof. Let $c \in \mathbb{R}$. Let $\epsilon > 0$. Set $\delta = \epsilon$. If $|x - c| < \delta$, then $|f(x) - f(c)| = |x - c| < \delta = \epsilon$. Thus, the identity function is continuous. □

Claim: $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Proof.

- **Case 1:** $c \neq 0$

  Let $c \in [0, \infty)$. Let $\epsilon > 0$. Set $\delta < \epsilon$. If $|x - c| < \delta$, then $|g(x) - g(c)| = |\sqrt{x} - \sqrt{c}| = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} < \frac{\delta}{\sqrt{c}} < \epsilon$. Thus, $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

- **Case 2:** $c = 0$

  Let $c \in [0, \infty)$. Let $\epsilon > 0$ Set $\delta = \epsilon^2$. If $|x - 0| < \delta$, then $|g(x) - g(0)| = |\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \epsilon$. Thus, $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

□

Theorem 1.3.3: Compositions of Continuous Functions

Let f be continuous at c. Let g be continuous at $f(x)$. Then,

$$g \circ f(x) = g(f(x)) \text{ is continuous.}$$

Proof. Let $\epsilon > 0$. Since g is continuous at $f(c)$, there exists $\delta_1 > 0$ such that if $|x - f(c)| < \delta_1$, then $|g(x) - g(f(c))| < \epsilon$. Since f is continuous at c, there exists $\delta_2 > 0$ such that if $|x - c| < \delta_2$, then $|f(x) - f(c)| < \delta_1$. Thus, if $|x - c| < \delta_2$, then $|g(f(x)) - g(f(c))| < \epsilon$. Therefore, $g \circ f(x)$ is continuous. □

Most Common Applications of Continuity
Is with limits.

If f is continuous at c and $(x_n) \to c$, then $\lim_{n\to\infty} f(x_n) = f(c)$. Hence.

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

1.3.1 Exercises

Exercise: 4.3.1

Let $g(x) = \sqrt{3}\, x$.

Solution.

(a) Prove that g is continuous at $c = 0$.

(b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ will be helpful.)

Exercise: 4.3.8

Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbb{R}$.

Solution.

(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.

(b) If $g(r) = 0$ for all $r \in \mathbb{Q}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

(c) If $g(x_0) > 0$ for a single point $x_0 \in \mathbb{R}$, then $g(x)$ is in fact strictly positive for uncountably many points.

Exercise: 4.3.9

Assume $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x : h(x) = 0\}$. Show that K is a closed set.

Solution.

1.4 Continuous Functions on Compact Sets

Theorem 1.4.1: Extreme Value Theorem

If K is compact and $f : K \to \mathbb{R}$ is continuous, then f attains a maximum and minimum value on K.
In other words, there exists $a \in K$ such that $f(a) = \sup\{f(x) \mid x \in K\}$. Also, there exists $b \in K$ such that $f(b) = \inf\{f(x) \mid x \in K\}$.

Proof. We know f is bounded on K from Lemma 4.4.2 below. Hence,

$$S = \sup\{f(x) \mid x \in K\} \text{ exists.}$$

For every natural number, there exists an $x_n \in K$ such that $S - \frac{1}{n} < f(x_n) \leq S$. It follows that $\lim_{n\to\infty} f(x_n) = S$. So, now we have a sequence, (x_n) in the compact set

K. Since K is compact, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence,

$$(x_{n_j}) \text{ with } a = \lim_{j\to\infty} x_{n_j} \in K.$$

Since f is continuous, we have,

$$f(a) = \lim_{j\to\infty} f(x_{n_j}) = S.$$

By a similar method, there exists $b \in K$ such that

$$f(b) = \inf\{f(x) \mid x \in K\}.$$

Note: This proof hinges on the fact that f is bounded! We need to show that f is bounded on K with a proof with subcovers. □

Lemma 1.4.2

How do we know f is bounded on K? That is,

$$f(K) = \{f(x) \mid x \in K\}.$$

Show that $f(K)$ is bounded.

Proof. Let $c \in K$. Since f is continuous at c, there exists $\delta_c > 0$ such that if $|x - c| < \delta_c$, then

$$|f(x) - f(c)| < 1.$$

Do this over every $c \in K$. We get an open cover of K.

$$\mathcal{O} = \{V_{\delta_c}(c) \mid c \in K\}.$$

Since K is compact, the Heine-Borel Theorem says there exists a finite subcover. We get $c_1, c_2 \dots, c_n \in K$ such that $K \subseteq \bigcup_{i=1}^n V_{\delta_{c_i}}(c_i)$. Thus,

$$\begin{aligned} f(K) &\subseteq f\left(\bigcup_{i=1}^n V_{\delta_{c_i}}(c_i)\right) \\ &\subseteq \bigcup_{i=1}^n f(V_{\delta_{c_i}}(c_i)) \\ &\subseteq \bigcup_{i=1}^n (f(c_i) - 1, f(c_i) + 1) \\ &\subseteq [\min f(c_i) - 1, \max f(c_i) + 1]. \end{aligned}$$

Therefore, $f(K)$ is bounded. □

Theorem 1.4.3: Preservation of Compact Sets

If K is compact and $f\colon K \to \mathbb{R}$ is continuous, then $f(K)$ is compact.

Proof. Let (y_n) be a sequence in $f(K)$. We will show (y_n) has a convergent subsequence with its limit in $f(K)$.

For each n there exists $x_n \in K$ such that $f(x_n) = y_n$. So (x_n) is a sequence in a compact set K. There exists a convergent subsequence (x_{n_j}) with

$$a = \lim_{j\to\infty} x_{n_j} \in K.$$

Now consider the corresponding subsequence (y_{n_j}) in $f(K)$. Since f is continuous, we have

$$\begin{aligned}\lim_{j\to\infty} y_{n_j} &= \lim_{j\to\infty} f(x_{n_j}) \\ &= f(a) \in f(K).\end{aligned}$$

So, x_{n_j} is a convergent subsequence with limit in $f(K)$. Therefore, $f(K)$ is compact. $\square$

Definition 1.4.4

A function $f\colon A \to \mathbb{R}$ is *uniformly continuous* on A if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, c \in A$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Compare this definition with Definition 4.3.1. The difference is that the δ is independent of x. That is, we have to find one δ that needs to work for every point x.

Definition 1.4.5

A function $f\colon A \to \mathbb{R}$ is *not uniformly continuous* on A if there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists $x, c \in A$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon_0$.

Theorem 1.4.6

If $K \subseteq \mathbb{R}$ is compact and $f\colon K \to \mathbb{R}$ is continuous, then f is uniformly continuous on K.

Proof. Suppose f is not uniformly continuous on K. Then, there exists $\epsilon_0 > 0$ such that for all $n \in \mathbb{N}$, there exists $x_n, y_n \in K$ such that $\left|x_n - y_n < \frac{1}{n}\right|$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$.

We now have two sequences (x_n) and (y_n) in K. Since K is compact, by the heine-borel theorem, there exists a convergent subsequence (x_{n_i}) which converges to a point $x_0 \in K$.

Since K is compact, (y_{n_i}) has a convergent subsequence $\left(y_{n_{i_j}}\right)$ which converges to a point $y_0 \in K$. Notice that since $\left(x_{n_{i_j}}\right)$ is a subsequence of (x_{n_i}), it converges to x_0. Since f is continuous:

$$\lim_{j\to\infty} f\left(x_{n_{i_j}}\right) = f(x_0) \quad \text{and} \quad \lim_{j\to\infty} f\left(y_{n_{i_j}}\right) = f(y_0).$$

Because

$$\left|f\left(x_{n_{i_j}}\right) - f\left(y_{n_{i_j}}\right)\right| < \frac{1}{n_{i_j}},$$

we can see that

$$\lim_{j\to\infty} \left|x_{n_{i_j}} - y_{n_{i_j}}\right| = 0.$$

It follows that $x_0 = y_0$. But this is a contradiction because $|f(x_0) - f(y_0)| \geq \epsilon_0$. Therefore, f is uniformly continuous on K. $\square$

1.5 The Intermediate Value Theorem

Theorem 1.5.1: Intermediate Value Theorem

Let $f\colon [a,b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there exists $c \in (a,b)$ such that $f(c) = L$.

Note: IVT does not guarantee where, or how many c's are in the interval. It only guarantees that at least one c exists.

Proof. (Using Nested Interval Property) Without the loss of generality, assume $f(a) < f(b)$ and let $y \in (f(a), f(b))$. Let $I_1 = [a_1, b_1]$. Bisect I_1 into two intervals $[a_1, d]$ and d, b_1 where $d = \dfrac{a_1 + b_1}{2}$.

- If $f(d) < y$, set $a_2 = d$, $b_2 = b_1$, and $I_2 = [a_2, b_2]$. Notice that $f(a_2) < y < f(b_2)$.
- If $f(d) > y$, then set $a_2 = a_1$, $b_2 = d$, and $I_2 = [a_2, b_2]$. Notice that $f(a_2) < y < f(b_2)$.

Repeat this process indefinitely. We end up with a sequence of nested intervals $I_n = [a_n, b_n]$, where

- $I_n \subseteq I_{n-1}$
- $f(a_n) < y < f(b_n)$
- $|a_n - b_n| = \dfrac{a_n - b_n}{2^{n-1}}$

By the Nested Interval Property, there exists a point c such that $c \in \bigcap_{n=1}^{\infty} I_n$. In fact, there is a unique point c in the intersection. It follows that

$$c = \lim_{n\to\infty} a_n \quad \text{and} \quad c = \lim_{n\to\infty} b_n.$$

Since f is continuous, we have

$$f(c) = \lim_{n\to\infty} f(a_n) \leq y$$

$$f(c) = \lim_{n\to\infty} f(b_n) \geq y.$$

Therefore, $f(c) = y$. $\square$

What Is Important About Continuous Functions?

If $\lim_{n\to\infty}(x_n = x)$, then $\lim_{n\to\infty} f(x_n) = f(x)$.
Think about $f(x) = 2^x$. Thus, fx makes sense if $x \in \mathbb{Q}$:

$$2^{\frac{p}{q}} = \sqrt[q]{2^p}.$$

But how do we make sense of something like 2^π?
We can find $f\colon \mathbb{Q} \to \mathbb{R}$ is continuous.
We can define $f\colon \mathbb{R} \to \mathbb{R}$ to be continuous
If (q_n) is in $\mathbb{Q}$ and $(a_n \to \pi)$, then we define

$$f(\pi) = \lim_{n\to\infty} f(q_n).$$