Multivariable Calculus Practice Set III

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1. (3 points) Determine the absolute extrema for the function $f(x,y) = x^2 + 3y^2 - 2x - y - xy$ on the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$.

Solution. We first find the critical points of the function:

$$\begin{aligned} \nabla f(x,y) &= \langle 2x - 2 - y,\, 6y - 1 - x \rangle = \mathbf{0} \\ &\implies y = 2x - 2 \quad \text{and} \quad x = 6(2x-2) - 1 - x \\ &\implies y = \tfrac{4}{11} \quad \text{and} \quad x = \tfrac{13}{11} \end{aligned}$$

This gives the critical point $\boxed{\left(\dfrac{13}{11}, \dfrac{4}{11}\right)}$. We also need to check the boundary of the region. Thus:

(ℓ_1): $y = 0$, $0 \le x \le 2 \implies f(x,y) = g(x) = x^2 + 3(0)^2 - 2x - (0) - x(0) = x^2 - 2x \implies g'(x) = 2x - 2$. Therefore, the critical points are $\boxed{(1,0)}$.

(ℓ_2): $x = 0$, $0 \le y \le 1 \implies f(x,y) = h(y) = (0)^2 + 3y^2 - (0) - y - 0 = 3y^2 - y \implies h'(y) = 6y - 1$. Hence, the critical points are $\boxed{\left(0, \dfrac{1}{6}\right)}$.

(ℓ_3): $y = 1 - \frac{1}{2}x$, $0 \le x \le 2 \implies f(x,y) = k(x) = x^2 + 3\left(1 - \frac{1}{2}x\right)^2 - 2x - \left(1 - \frac{1}{2}x\right) - x\left(1 - \frac{1}{2}x\right)$. Solving this equation for x:

$$\begin{aligned} k(x) &= x^2 + 3\left(1 - \tfrac{1}{2}x - \tfrac{1}{2}x + \tfrac{1}{4}x^2\right) - 2x - 1 + \tfrac{1}{2}x - x + \tfrac{1}{2}x^2 \\ &= x^2 + 3\left(1 - x + \tfrac{1}{4}x^2\right) - 2x - 1 + \tfrac{1}{2}x - x + \tfrac{1}{2}x^2 \\ &= \left[x^2 + \tfrac{3}{4}x^2 + \tfrac{1}{2}x^2\right] + \left[-3x - 2x - \tfrac{1}{2}x\right] + [3 - 1] \\ &= \tfrac{9}{4}x^2 - \tfrac{7}{2}x + 2 \\ &= \tfrac{1}{4}(9x^2 - 22x + 8) \\ \implies k'(x) &= \tfrac{1}{4} \cdot \tfrac{d}{dx}[9x^2 - 22x + 8] \\ 0 &= \tfrac{1}{2}(9x - 11) \\ x &= \tfrac{11}{9} \end{aligned}$$

Using this x-value, we plug it back into our equation for y to get the critical point $\boxed{\left(\dfrac{11}{9}, \dfrac{7}{18}\right)}$.

With our function's and lines' critical points found, we also need to find the vertices of the triangle:

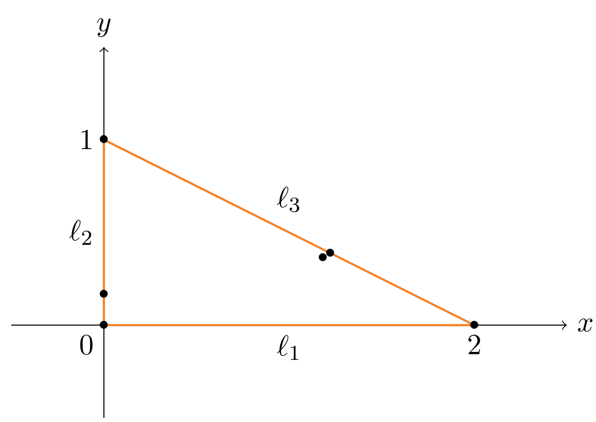
$$\begin{aligned} f(0,0) &= 0^2 + 3(0)^2 - 2(0) - 0 - 0(0) = 0 \\ f(2,0) &= 2^2 + 3(0)^2 - 2(2) - 0 - 2(0) = 0 \\ f(0,1) &= 0^2 + 3(1)^2 - 2(0) - 1 - 0(1) = 2 \end{aligned}$$

Now that we have our critical points, we can evaluate the function at each of these points to determine the absolute extrema:

$$\begin{aligned} f\left(\tfrac{13}{11}, \tfrac{4}{11}\right) &= \left(\tfrac{13}{11}\right)^2 + 3\left(\tfrac{4}{11}\right)^2 - 2\left(\tfrac{13}{11}\right) - \tfrac{4}{11} - \tfrac{13}{11} \\ &\approx -1.363\ldots \\ f(1,0) &= 1^2 + 3(0)^2 - 2(1) - 0 - 1(0) \\ &= -1 \\ f\left(0, \tfrac{1}{6}\right) &= 0^2 + 3\left(\tfrac{1}{6}\right)^2 - 2(0) - \tfrac{1}{6} - 0 \\ &\approx -0.083\ldots \\ f\left(\tfrac{11}{9}, \tfrac{7}{18}\right) &= \left(\tfrac{11}{9}\right)^2 + 3\left(\tfrac{7}{18}\right)^2 - 2\left(\tfrac{11}{9}\right) - \tfrac{7}{18} - \tfrac{11}{9} \\ &\approx -1.361\ldots \end{aligned}$$

Thus, this gives us 7 critical points:

Point	$f(x,y)$	**Type**
$\left(\frac{13}{11}, \frac{4}{11}\right)$	-1.364	Interior CP
$(1,0)$	-1	ℓ_1
$\left(0, \frac{1}{6}\right)$	-0.083	ℓ_2
$\left(\frac{11}{9}, \frac{7}{18}\right)$	-1.361	ℓ_3
$(0,0)$	0	Vertex 1
$(2,0)$	-2	Vertex 2
$(0,1)$	2	Vertex 3



With these values, we can see that the absolute maximum is $\boxed{2}$, which occurs at the vertex $(0,1)$, and the absolute minimum is $\boxed{-1.364}$, which occurs at the critical point $\left(\frac{13}{11}, \frac{4}{11}\right)$.

2. (1 point each) Convert each as indicated; leave each answer as exact:

(a) Convert the rectangular point $(-5, 1)$ to polar coordinates.

Solution.

$$r = \sqrt{(-5)^2 + 1^2} = \sqrt{26}$$
$$\theta = \arctan\left(\frac{1}{-5}\right) = \arctan\left(-\frac{1}{5}\right) = \frac{7\pi}{6}$$

The polar coordinates are $\boxed{\left(\sqrt{26}, \frac{7\pi}{6}\right)}$.

(b) Convert the cylindrical point $(5, \frac{7\pi}{6}, 2)$ to rectangular.

Solution.

$$x = 5\cos\left(\frac{7\pi}{6}\right) = 5\left(-\frac{\sqrt{3}}{2}\right) = -\frac{5\sqrt{3}}{2}$$
$$y = 5\sin\left(\frac{7\pi}{6}\right) = 5\left(-\frac{1}{2}\right) = -\frac{5}{2}$$
$$z = 2$$

The rectangular coordinates are $\boxed{\left(-\frac{5\sqrt{3}}{2}, -\frac{5}{2}, 2\right)}$.

(c) Convert the rectangular point $(-2, 4, -1)$ to spherical.

Solution.

$$\rho = \sqrt{(-2)^2 + 4^2 + (-1)^2} = \sqrt{21}$$
$$\theta = \arctan\left(\frac{4}{-2}\right) = \arctan(-2)$$
$$\phi = \arccos\left(\frac{-1}{\sqrt{21}}\right) = \arccos\left(-\frac{1}{\sqrt{21}}\right)$$

Since the point $(-2, 4)$ is in the second quadrant, we add π to the arctan value. Hence, the spherical coordinates are $\boxed{\left(\sqrt{21}, \pi + \arctan(-2), \arccos\left(-\frac{1}{\sqrt{21}}\right)\right)}$.

(d) Convert the spherical point $(4, \frac{11\pi}{6}, \frac{3\pi}{4})$ to cylindrical.

Solution. The conversion from spherical to cylindrical follows the following equations:

$$r = \rho\sin\phi, \quad \theta = \theta, \quad \text{and} \quad z = \rho\cos\phi.$$

Thus, we have:

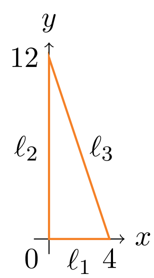
$$\begin{aligned}
r &= 4\sin\left(\tfrac{3\pi}{4}\right) = 4\left(\tfrac{\sqrt{2}}{2}\right) = 2\sqrt{2}\\
\theta &= \tfrac{11\pi}{6}\\
z &= 4\cos\left(\tfrac{3\pi}{4}\right) = 4\left(-\tfrac{\sqrt{2}}{2}\right) = -2\sqrt{2}
\end{aligned}$$

Therefore, we get the cylindrical coordinates $\boxed{\left(2\sqrt{2}, \frac{11\pi}{6}, -2\sqrt{2}\right)}$.

3. (3 points) Determine the value of each given integral. You need to do the work here by hand, but of course can check any answers with technology.

   (a) $\iint_D (x^2 + 6xy)\, dA$ where D is the triangle with vertices $(0,0)$, $(4,0)$, and $(0,12)$

Solution. We can see that this triangle is bounded by three lines:



$$\begin{aligned}
\ell_1 &: y = 0\\
\ell_2 &: x = 0\\
\ell_3 &: y = -3x + 12
\end{aligned}$$

This gives us the limits of integration as follows:

$$\{(x,y)\colon 0 \le x \le 4, \quad 0 \le y \le -3x+12\}.$$

Thus, we can write the double integral as:

$$\begin{aligned}
\iint_D (x^2+6xy)\, dA &= \int_0^4 \int_0^{-3x+12} (x^2+6xy)\, dy\, dx\\
&= \int_0^4 \left[x^2 y + 3xy^2\right]_0^{-3x+12}\, dx\\
&= \int_0^4 \left[x^2(-3x+12) + 3x(-3x+12)^2\right]\, dx\\
&= \int_0^4 \left[-3x^3 + 12x^2 + 3x(9x^2 - 72x + 144)\right]\, dx\\
&= \int_0^4 \left[24x^3 - 204x^2 + 432x\right]\, dx\\
&= 6\int_0^4 \left[4x^3 - 34x^2 + 72x\right]\, dx\\
&= 6\left[x^4 - \tfrac{34}{3}x^3 + 36x^2\right]_0^4\\
&= 48\left[32 - \tfrac{34}{3}(8) + 36(2)\right]\\
&= \boxed{640}
\end{aligned}$$

(b) $\displaystyle\int_0^2 \int_{x^2}^4 4x^3 \cos(y^3)\, dy\, dx$

Solution. To evaluate this integral, we must change the order of integration. The original region is:

$$\{(x, y) \colon 0 \le x \le 2, \quad x^2 \le y \le 4\}.$$

The new region is:

$$\{(x, y) \colon 0 \le y \le 4, \quad 0 \le x \le \sqrt{y}\}.$$

Thus, we can rewrite and solve the double integral:

$$\begin{aligned}
\int_0^2 \int_{x^2}^4 4x^3 \cos(y^3)\, dy\, dx &= \int_0^4 \int_0^{\sqrt{y}} 4x^3 \cos(y^3)\, dx\, dy \\
&= \int_0^4 \left[x^4 \cos(y^3)\right]_0^{\sqrt{y}} dy \\
&= \int_0^4 y^2 \cos(y^3)\, dy \\
&= \left[\tfrac{1}{3} \sin(y^3)\right]_0^4 \\
&= \tfrac{1}{3} \left[\sin(64) - 0\right] \\
&= \boxed{\tfrac{1}{3} \sin(64)}.
\end{aligned}$$

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