1. Consider the vector field $\mathbf{F} = \langle 2x - 2y,\ 2x + 2y,\ 0\rangle$

(a) (2 points) Show that $\mathbf{F}$ is not conservative.

Solution. For a vector field to be conservative, its curl must be zero. Computing the curl of $\mathbf{F}$:

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} \\ &= (0-0)\mathbf{i} + (0-0)\mathbf{j} + (2-(-2))\mathbf{k} \\ &= 4\mathbf{k}.\end{aligned}$$

Since $\nabla \times \mathbf{F} = \langle 0, 0, 4\rangle \neq \langle 0, 0, 0\rangle$, $\mathbf{F}$ is not conservative.

(b) (2 points) Show that $\mathbf{F}$ is not solenoidal.

Solution. A vector field is solenoidal if its divergence is zero. Thus:

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(2x - 2y) + \frac{\partial}{\partial y}(2x + 2y) + \frac{\partial}{\partial z}(0) \\ &= 2 + 2 + 0 \\ &= 4.\end{aligned}$$

Since $\nabla \cdot \mathbf{F} = 4 \neq 0$, $\mathbf{F}$ is not solenoidal.

(c) (2 points each) Let $\mathbf{G} = \langle 2x,\ 2y,\ 0\rangle$ and $\mathbf{H} = \langle -2y,\ 2x,\ 0\rangle$. Notice that $\mathbf{F} = \mathbf{G} + \mathbf{H}$.

i. Show that $\mathbf{G}$ *is* conservative, and find a potential function g.

Solution.

$$\nabla \times \mathbf{G} = (0-0)\mathbf{i} - (0-0)\mathbf{j} + \left(\frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(2x)\right)\mathbf{k}.$$

Since $\nabla \times \mathbf{G} = \langle 0, 0, 0\rangle$, $\mathbf{G}$ is conservative.

A potential function is one where $\nabla g = \mathbf{G}$. Thus, since we have a (relatively) straightforward vector field, the following will work for a potential function:

$$g(x, y, z) = x^2 + y^2 + K. \quad \text{(Setting } K = 0 \text{ for simplicity.)}$$

We can check this by finding ∇g:

$$\nabla g(x, y, z) = \left\langle \frac{\partial}{\partial x}(x^2 + y^2), \frac{\partial}{\partial y}(x^2 + y^2), \frac{\partial}{\partial z}(x^2 + y^2) \right\rangle = \langle 2x, 2y, 0\rangle = \mathbf{G}.$$

ii. Show that **H** *is* solenoidal – so that it is the curl of some other vector field **C**. Find such a **C**. [Hint: You might want to choose the z-component to be 0.]

Solution.

$$\begin{aligned}\nabla \cdot \mathbf{H} &= \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial z}(0) \\ &= 0 + 0 + 0 \\ &= 0.\end{aligned}$$

Since $\nabla \cdot \mathbf{H} = 0$, the vector field **H** is solenoidal.

For finding the vector field **C**, we can look at the formula for the curl of a vector field from (a). We see that we need to have our x-component and y-component to be equal to **H**. Thus, for the x-component, we need to get $-2y$. The only part of the formula that allows us that is the $\frac{\partial R}{\partial y}$ part. The same argument can be said for the y-component. We need to get a positive $2x$, and we can get that through $(-\frac{\partial R}{\partial x})$. Thus, this leads me to believe that we need to keep our function in R of **C**, and ensure P and Q are both 0. This leaves us with:

$$\mathbf{C} = \langle 0, 0, -x^2 - y^2 \rangle.$$

Taking the curl of this vector field, we can confirm our choice:

$$\begin{aligned}\nabla \times \mathbf{C} &= \left\langle \frac{\partial}{\partial y}(-x^2 - y^2) - 0, 0 - \frac{\partial}{\partial x}(-x^2 - y^2), 0 - 0 \right\rangle \\ &= \langle -2y, -(-2x), 0 \rangle \\ &= \langle -2y, 2x, 0 \rangle.\end{aligned}$$

(d) (2 points) Conclude that we have decomposed **F** into a purely conservative (i.e., irrotational) part and a purely solenoidal (i.e., divergence-free) part, so that $\mathbf{F} = \nabla g + \nabla \times \mathbf{C}$. [This can always be done, so long as everything is continuous enough.]

Solution. We have shown that:

- $\mathbf{G} = \langle 2x, 2y, 0 \rangle$ is conservative with potential function $g(x, y, z) = x^2 + y^2$, so $\mathbf{G} = \nabla g$.
- $\mathbf{H} = \langle -2y, 2x, 0 \rangle$ is solenoidal and can be expressed as $\mathbf{H} = \nabla \times \mathbf{C}$ where $\mathbf{C} = \langle 0, 0, -x^2 - y^2 \rangle$.
- $\mathbf{F} = \mathbf{G} + \mathbf{H}$.

Therefore, we have successfully decomposed **F** as follows:

$$\begin{aligned}\mathbf{F} &= \mathbf{G} + \mathbf{H} \\ &= \nabla g + \nabla \times \mathbf{C} \\ &= \nabla(x^2 + y^2) + \nabla \times \langle 0, 0, -x^2 - y^2 \rangle \\ &= \langle 2x, 2y, 0 \rangle + \langle -2y, 2x, 0 \rangle \\ &= \langle 2x - 2y,\ 2x + 2y,\ 0 \rangle .\end{aligned}$$

2. (8 points) Let $\mathbf{F} = \left\langle 6x^2y,\, -6x - 4y \right\rangle$. Let C be the rectangle with endpoints $(0,0)$, $(4,0)$, $(4,1)$, and $(0,1)$, with positive orientation. Determine the exact value of the flux of $\mathbf{F}$ over C: $\oint \mathbf{F} \cdot \mathbf{N}\, ds$.

Solution. Since the rectangle is closed, we can use Green's Theorem for flux:

$$\oint \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D (\nabla \cdot \mathbf{F})\, dA = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA.$$

Finding the integrand:

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(6x^2y) = 12xy, \quad \frac{\partial Q}{\partial y} = \frac{\partial}{\partial y}(-6x - 4y) = -4 \quad \Rightarrow \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 12xy - 4.$$

Leaving us with:

$$\int_0^4 \int_0^1 (12xy - 4)\, dy\, dx.$$

Integrating with respect to y:

$$\int_0^1 (12xy - 4)\, dy = \left[6xy^2 - 4y\right]_0^1 = 6x - 4.$$

Then, integrating with respect to x:

$$\int_0^4 (6x - 4)\, dx = \left[3x^2 - 4x\right]_0^4 = (3 \cdot 16 - 4 \cdot 4) = \boxed{32}.$$

3. (8 points) Determine $\iint_S z\,dS$ where S is the surface $y = 3x + z^2$, where $0 \le x \le 1$ and $0 \le z \le 2$.

Solution. Parameterize S with:

$$r(x,z) = (x,\, 3x + z^2,\, z), \quad 0 \le x \le 1, \quad 0 \le z \le 2.$$

Then, we need to find dS:

$$dS = \|r_x \times r_z\|\, dx\, dy,$$

where

$$r_x = (1,3,0) \quad \text{and} \quad r_z = (0,2z,1)$$

gives

$$r_x \times r_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 0 \\ 0 & 2z & 1 \end{vmatrix} = \langle 3, -1, 2z \rangle \quad \Rightarrow \quad \|r_x \times r_z\| = \sqrt{10 + 4z^2}.$$

Setting up the integral:

$$\iint_S z\,dS = \int_0^1 \int_0^2 z\sqrt{10 + 4z^2}\, dz\, dx.$$

Since the integrand does not depend on x, we simply find:

$$\int_0^1 dx = 1.$$

This leaves us with the z-integral:

$$\int_0^2 z\sqrt{10 + 4z^2}\, dz.$$

Using u-substitution:

$$u = 10 + 4z^2 \quad \Rightarrow \quad du = 8z\, dz,$$

with new bounds 10 and 26:

$$\int_{10}^{26} \frac{1}{8}\sqrt{u}\, du = \frac{1}{8}\left[\frac{2}{3}u^{3/2}\right]_{10}^{26} = \boxed{\frac{1}{12}(26^{3/2} - 10^{3/2})}.$$

4. (8 points) Let $\mathbf{F}(x, y, z) = \langle y,\, x,\, xz \rangle$ and the surface S be the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above $0 \le x \le 1$ and $0 \le y \le 1$, where positive orientation is directed upward. Determine

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

[Hint: The surface, projected into the xy-plane, is *not* a quarter of the unit circle, and therefore, it is likely easiest to parameterize S in Cartesian coordinates.]

Solution. Parameterize S with:

$$\mathbf{r}(x, y) = (x, y, 4 - x^2 - y^2), \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Then, finding $d\mathbf{S}$:

$$d\mathbf{S} = r_x \times r_y \, dx \, dy,$$

where

$$r_x = (1, 0, -2x) \quad \text{and} \quad r_y = (0, 1, -2y)$$

gives

$$r_x \times r_y = (2x, 2y, 1).$$

Therefore:

$$d\mathbf{S} = (2x, 2y, 1) \, dx \, dy.$$

This leaves us with the following:

$$\begin{aligned}
\iint_S \mathbf{F}(\mathbf{r}(x, y)) \cdot (r_x \times r_y) \, dx \, dy &= \int_0^1 \int_0^1 (y, x, x(4 - x^2 - y^2)) \cdot (2x, 2y, 1) \, dx \, dy \\
&= \int_0^1 \int_0^1 (4xy + 4x - x^3 - xy^2) \, dx \, dy \\
&= \int_0^1 \left[2xy^2 + 4xy - x^3 y - \frac{1}{3} x y^3 \right]_0^1 dx \\
&= \int_0^1 \left(2x + 4x - \frac{1}{3} x - x^3 \right) dx \\
&= \left[x^2 + 2x^2 - \frac{1}{6} x^2 - \frac{1}{4} x^4 \right]_0^1 \\
&= \boxed{\frac{31}{12}}.
\end{aligned}$$

5. (8 points) Calculate the flux of $\mathbf{F}(x,y,z) = \left\langle x^3+y,\, y^3+z^2,\, z^3+x^3 \right\rangle$ across the surface of the sphere centered at the origin with radius 2, with positive orientation. [Since the surface is closed, there are two distinct ways to do this – though both are things you can work out, one is *significantly* easier than the other.]

Solution. The equation of a sphere with radius 2 gives:

$$x^2+y^2+z^2=4.$$

Then, by the divergence theorem:

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V (\nabla\cdot\mathbf{F})\,dV, \quad \text{where} \quad V=\{x^2+y^2+z^2\le 4\}.$$

Solving for $\nabla\cdot\mathbf{F}$:

$$\frac{\partial}{\partial x}(x^3+y)+\frac{\partial}{\partial y}(y^3+z^2)+\frac{\partial}{\partial z}(z^3+x^3)=3x^2+3y^2+3z^2=3(x^2+y^2+z^2).$$

Switching to spherical coordinates:

$$3(x^2+y^2+z^2)=3r^2, \quad \text{with} \quad dV=r^2\sin(\phi)\,dr\,d\phi\,d\theta.$$

This leaves us with the integral:

$$\iiint_V 3r^2\,dV = 3\int_0^2\int_0^\pi\int_0^{2\pi} r^2(r^2\sin(\phi))\,d\theta\,d\phi\,dr = 3(2\pi)(2)\int_0^2 r^4\,dr = 12\pi\left[\frac{1}{5}r^5\right]_0^2 = \boxed{\frac{384\pi}{5}}.$$