2. (3 points) Determine $\int_C y^2\,dx + z\,dy + x\,dz$, where C is the line segment which connects $(2,0,0)$ to $(3,4,5)$.

Solution. We can parameterize this line segment as follows:

$$\mathbf{r}(t) = (2+t)\mathbf{i} + (4t)\mathbf{j} + (5t)\mathbf{k}, \quad 0 \le t \le 1$$

We get this from taking each component of the vector function and setting $t = 0$, to get the point $(2,0,0)$. Then, by setting $t = 1$, we get the point $(3,4,5)$. Now, we can differentiate $\mathbf{r}(t)$ to get $\mathbf{r}'(t)$:

$$\mathbf{r}'(t) = \left\langle \tfrac{d}{dt}\left[(2+t)\right] + \tfrac{d}{dt}\left[(4t)\right] + \tfrac{d}{dt}\left[(5t)\right] \right\rangle = \langle 1,4,5 \rangle\,.$$

This gives us the following:

$$dx = dt, \quad dy = 4dt, \quad dz = 5dt.$$

Expressing the integral in terms of t, we can solve the integral as follows:

$$\begin{aligned}
\int_C y^2\,dx + z\,dy + x\,dz &= \int_0^1 (4t)^2 dt + (5t)(4dt) + (2+t)(5dt) \\
&= \int_0^1 (16t^2 + 20t + 10 + 5t)dt \\
&= \int_0^1 (16t^2 + 25t + 10)dt \\
&= \left[\frac{16}{3}t^3 + \frac{25}{2}t^2 + 10t\right]_0^1 \\
&= \boxed{\frac{167}{6}}.
\end{aligned}$$

3. (3 points) Determine $\int_C \frac{1}{x^2+y^2+z^2}\,ds$, where C is given by $\langle \cos t, \sin t, t\rangle$, $0 \le t \le \pi$.

Solution. First, we need to find ds, which is given by finding the derivative of the vector function and taking the norm:

$$ds = ||\mathbf{r}'(t)||\,dt = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\,dt = \sqrt{1+1}\,dt = \sqrt{2}dt.$$

Rewriting the integral in terms of t, we have:

$$\int_C \frac{1}{x^2+y^2+z^2}\,ds = \int_0^\pi \frac{1}{\cos^2 t + \sin^2 t + t^2}\sqrt{2}\,dt = \sqrt{2}\int_0^\pi \frac{1}{1+t^2}\,dt.$$

We know that $\int \frac{1}{1+t^2} dt = \tan^{-1}(t)$, so we can evaluate the integral as follows:

$$\begin{aligned}
\sqrt{2}\int_0^\pi \frac{1}{1+t^2}\,dt &= \sqrt{2}\left[\tan^{-1}(t)\right]_0^\pi \\
&= \sqrt{2}\left(\tan^{-1}(\pi) - \tan^{-1}(0)\right) \\
&= \boxed{\sqrt{2}\tan^{-1}(\pi).}
\end{aligned}$$

4. (3 points) Let $\mathbf{F}(x,y) = 3x^2y^2\mathbf{i} + (2x^3y+5)\mathbf{j}$. Find a scalar function f such that $\nabla f = \mathbf{F}$ and use this to determine $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is given by $\mathbf{r}(t) = (t^3-2t)\mathbf{i} + (t^3+2t)\mathbf{j}$ for $0 \le t \le 1$.

Solution. First, we need to check if $\mathbf{F}$ is conservative. We can do this by checking if the mixed partials are equal:

$$\frac{\partial}{\partial y}(3x^2y^2) = 6x^2y, \quad \frac{\partial}{\partial x}(2x^3y+5) = 6x^2y.$$

Since these are equal, we can conclude that $\mathbf{F}$ is conservative. Now, we need to find a scalar function f such that $\nabla f = \mathbf{F}$. We can do this by integrating the components of $\mathbf{F}$:

$$\begin{aligned} f(x,y) &= \int 3x^2y^2\,dx + h(y) \\ &= x^3y^2 + h(y). \end{aligned}$$

Now, we can differentiate f with respect to y and set it equal to the second component of $\mathbf{F}$:

$$\begin{aligned} \frac{\partial}{\partial y}(x^3y^2 + h(y)) &= 2x^3y + h'(y) \\ &= 2x^3y + 5. \end{aligned}$$

This gives us $h'(y) = 5$, so we can integrate to find $h(y)$:

$$h(y) = 5y + K.$$

Thus, we have:

$$f(x,y) = x^3y^2 + 5y + K.$$

Now, we can use the Fundamental Theorem of Line Integrals to evaluate the integral:

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} &= f(\mathbf{r}(1)) - f(\mathbf{r}(0)) \\ &= f(-1,3) - f(0,0) \\ &= \left[(-1)^3(3)^2 + 5(3) + K\right] - \left[(0)^3(0)^2 + 5(0) + K\right] \\ &= [-9 + 15 + K] - [0 + 0 + K] \\ &= -9 + 15 + K - K \\ &= \boxed{6}. \end{aligned}$$

5. (2 points) Set up, but do not evaluate the "direct" integral for the previous problem. Then, use your calculator to determine a numerical approximation for the integral. Did you get the same answer?

Solution. We can set up the integral as follows:

6. (3 points) Find the work done by the force field $\mathbf{F} = x^2\mathbf{i} + y^3\mathbf{j}$ in moving an object from $(1,0)$ to $(2,2)$.

Solution.

7. (3 points) For what value(s), if any, of a is $(3x^2y + az)\mathbf{i} + x^3j + (3x+3z^2)\mathbf{k}$ conservative?

Solution.

8. (3 points) Find the circulation of $\mathbf{F} = xy\mathbf{i} + x^2y^3\mathbf{j}$ along C, where C is the counter-clockwise oriented triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$. Determine the value of this integral by working three separate line integrals.

   Solution.

9. (3 points) Find the flux of $\mathbf{F} = xy\mathbf{i} + x^2y^3\mathbf{j}$ over C, the same counter-clockwise oriented triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 2)$ as in the previous problem (notice that the vector field is the same as well). Determine this by working three separate line integrals.

   Solution.