Probability and Statistics: Practice Set 1

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September 21, 2025

1. (2 points) Four people are choosing an integer from $\{1, 2, \ldots, 10\}$ at random—assume that each choice is equally likely. What is the probability that all four choose different numbers?

   Solution. For all 4 people, each integer has a $\frac{1}{10}$ chance of being picked. That means there are $10^4 = 10000$ possible combinations. To find how many permutations, we calculate ${}_{10}P_4 = 5040$. Using this number, we find a $\frac{5040}{10000} = 50.4\%$ chance that all four choose different numbers.

2. (2 points) A pair of fair, 6-sided dice are thrown. Find the probability that the sum has a total above 9.

   Solution. Each two pair outcome has a $\frac{1}{36}$ chance of being picked. There are one, two, and three ways to get a 12, 11, and 10, respectively. Therefore, there is a $\frac{1+2+3}{36} = \frac{1}{6}$ chance of getting a sum above 9.

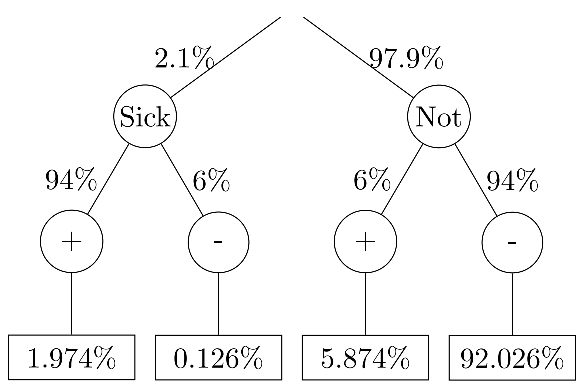
3. (2 points each) Equals-sign-itis is a disease where some people's face break out into little red equals signs if they take too many mathematics courses. Currently, 2.1% of Hendrix students are infected. A test can be taken before the break to determine if you are infected—it is 94% reliable.

   (a) What is the probability that you have the disease if your test comes back positive?

   Solution. The probability of having the disease and testing positive can be calculated by finding the following:

$$
\begin{aligned}
P(\text{sick} \mid +) &= \frac{P(\text{sick} \cap +)}{P(+)} \\
&= \frac{0.01974}{0.01974 + 0.05874} = 0.2515.
\end{aligned}
$$

   Therefore, you have a $\boxed{25.15\%}$ chance of having the disease if your test comes back positive.



   (b) Determine the probability that you have the disease if you test comes back negative?

   Solution. Similarly to the previous problem:

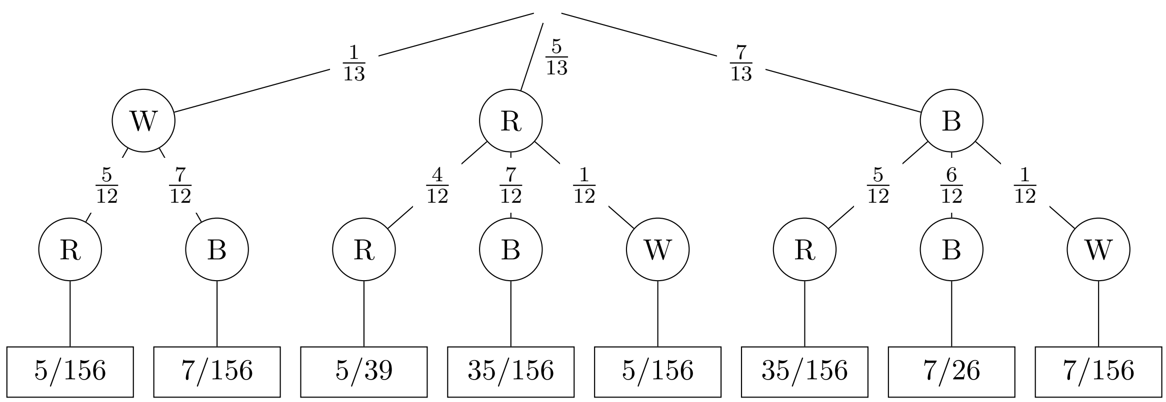
$$
P(\text{sick} \mid -) = \frac{P(\text{sick} \cap -)}{P(-)} = \frac{0.00126}{0.00126 + 0.92026} = 0.00137.
$$

   Therefore, you have a $\boxed{0.14\%}$ chance of having the disease if your test comes back negative.

4. An urn contains 5 Red, 7 Blue, and 1 White marbles. Two balls will be selected, without replacement.

(a) (2 points) Assuming that order matters, write the sample space for this experiment, and each outcome's probability. [Hint: If you draw a Red, followed by a White, you might write $(R, W), 0.12345$. A probability tree might be useful here.]

Solution. For the top level, there are $\frac{1}{13}$, $\frac{5}{13}$, and $\frac{7}{13}$ chances for White, Red, and Blue, respectively. After we remove a marble, then the next layer has 12 total marbles, and 1 less colored marble respective to the color in the above layer, as we can see below:



Thus, the sample space is

$$S = \left\{ RW = \frac{5}{156}, WB = \frac{7}{156}, RR = \frac{5}{39}, RB = \frac{35}{156}, RW = \frac{5}{156}, BR = \frac{35}{156}, BB = \frac{7}{26}, BW = \frac{7}{156} \right\}$$

(b) (2 points) Let event A be getting at least one Red marble. Find $P(A)$.

Solution. The probability of getting at least one marble entails adding each combination of Red and another marble. Given the values from the tree, we can calculate the following equation:

$$P(A) = \frac{5}{156} + \frac{5}{39} + \frac{35}{156} + \frac{5}{156} + \frac{35}{156} = \frac{25}{39}.$$

(c) (2 points) Let event B be getting at a White marble on the second draw. Find $P(B)$.

Solution. Since it's only possible to get a White marble on the second draw after pulling a Red or Blue, we have to take that into account with our calculation:

$$P(B) = P(R \cap W) + P(B \cap W) = \frac{5}{156} + \frac{7}{156} = \frac{1}{13}.$$

(d) (2 points) Find $P(A \mid B)$, using the events in the previous two parts.

Solution. $\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{5/156}{1/13} = \frac{5}{12}}$

5. (2 points) An urn contains 10 marbles: 4 Red and 6 Blue. A second urn contains 16 Red marbles and an unknown number of Blue marbles. A single marble is drawn from each urn. The probability that both marbles are the same color is 0.44. How many blue marbles are there in the second urn?

Solution. We have two urns: Urn_1: 4 Red, 6 Blue: 10 total, and Urn_2: 16 Red, b Blue: $16 + b$ total. Let event A be getting both red. Thus, we find the probability to be:

$$P(A) = \frac{4}{10} \cdot \frac{16}{16+b}.$$

Similarly, let event B be getting both blue. Hence:

$$P(B) = \frac{6}{10} \cdot \frac{b}{16+b}.$$

Thus, the total probability is

$$P(A \cup B) = \frac{4}{10} \cdot \frac{16}{16+b} + \frac{6}{10} \cdot \frac{b}{16+b} = \frac{(64+6b)}{10(16+b)}.$$

Setting this equal to 0.44, we can solve for b:

$$\begin{aligned} \frac{(64+6b)}{10(16+b)} &= 0.44 \\ 64 + 6b &= 4.4(16+b) \\ 64 + 6b &= 70.4 + 4.4b \\ 1.6b &= 6.4 \\ b &= 4. \end{aligned}$$

Therefore, the total amount of Blue marbles in Urn_2 is $\boxed{4}$.

6. (2 points) A fair coin is flipped three times. Given that you have at least one Head, find the probability that you have at least two Heads.

Solution. Let event A be the probability of getting at least 2 Heads, and event B being at least 1 Head. The sample space consists of 8 outcomes, only one of which does not contain at least one head: $\{TTT\}$. Hence, $P(B) = \frac{7}{8}$. Then, there are 4 outcomes that contain at least one head, so $P(A) = \frac{4}{8} = \frac{1}{2}$. It follows that $P(A \cap B) = P(A)$ given that having at least two heads implies having at least one. Hence:

$$P(\text{At least } 2H \mid \text{At least } 1H) = \frac{P(A \cap B)}{P(B)} = \frac{1/2}{7/8} = \frac{4}{7}.$$

Therefore, the probability of at least 2 Heads given 1 Head is $\boxed{\frac{4}{7}}$.

7. (2 points) In a certain state, car license plates have the format of three letters followed by three numbers. How many possible license plates are there?

Solution. Since we can use duplicate letters and numbers in the license plates, we find the total possible license plates with the following equation: $\boxed{26^3 \cdot 10^3 = 17576000}$.

8. (2 points) Three cards are drawn from a standard 52-card deck. Find the probability that at least one card is an Ace.

Solution. The probability of getting 3 cards that have no Aces in them is

$$P(OOO) = \frac{48}{52} \cdot \frac{47}{51} \cdot \frac{46}{50} = \frac{4324}{5525}.$$

We can find the probability of at least one Ace by finding the complement of this probability:

$$P(\text{at least 1 Ace}) = P'(OOO) = 1 - \frac{4324}{5525} = \frac{1201}{5525}.$$

Therefore, we have approximately a $\boxed{21.73\%}$ chance of getting at least one Ace.

9. (3 points) Show that $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

Solution. Expanding the right side we get the following:

$$\begin{aligned}\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!(n-1-(r-1))!} + \frac{(n-1)!}{r!(n-1-r)!} \\ &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!}.\end{aligned}$$

To continue, we need to find a common denominator, and to do so, we need to adjust two denominator terms $(r-1)!$ and $(n-r-1)!$. Multiplying these terms by a common term $r(n-r)$, we get the following:

$$= \frac{r \cdot (n-1)!}{r \cdot (r-1)!(n-r)!} + \frac{(n-r)(n-1)!}{r!(n-r) \cdot (n-r-1)!}.$$

Now, when we multiply those terms $r \cdot (r-1)! = r!$ and $(n-r) \cdot (n-r-1) = (n-r)!$, we are "going back" one term to get matching denominators.

$$\begin{aligned}&= \frac{r(n-1)! + (n-r)(n-1)!}{r!(n-r)!} \\ &= \frac{(n-1)!(r+(n-r))}{r!(n-r)!} \\ &= \frac{(n-1)!n}{r!(n-r)!} \\ &= \frac{n!}{r!(n-r)!}.\end{aligned}$$

Therefore, by definition of the binomial coefficient, the two sides of the equation are equal.

10. (3 points) Show that $\displaystyle\sum_{r=0}^{n} \binom{n}{r} = 2^n$.

Solution. Starting with the right side, we can factor 2^n, use the Binomial Theorem, and then simplify to show the specified property:

$$\begin{aligned} 2^n &= (1+1)^n \\ (1+1)^n &= \sum_{r=0}^{n} \binom{n}{r} (1)^{n-r}(1)^r \\ 2^n &= \sum_{r=0}^{n} \binom{n}{r}. \end{aligned}$$