Real Analysis

MATH 350

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CHAPTER 1

THE REAL NUMBERS

1.1 Types of Numbers

Definition 1.1.1

The *natural numbers* contain all positive, non-zero, and non-fractional numbers. Expressed as $\mathbb{N} = \{1, 2, 3, 4 \ldots\}$. They do not have an additive inverse, but you can add and multiply them.

Definition 1.1.2

The *integers* contains all non-fractional numbers. Expressed as: $\mathbb{Z} = \{\cdots -2, -1, 0, 1, 2 \ldots\}$—are known as a Group (more specifically, a "ring"). You can add, multiply, and subtract these numbers.

Definition 1.1.3

The *rational numbers* contain all numbers, except irrational numbers. Expressed as: $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$—are known as a "Field." You can add, subtract, multiply, and divide these numbers.

A problem that rational numbers could not explain: The $45, 45, 90$ triangle had a hypotenuse of $\sqrt{2}$. This did not exist at the time, so it was simply $c^2 = 2$. Therefore, new numbers needed to be invented.

Theorem 1.1.4

There does not exist a rational number r such that $r^2 = 2$.

Proof. Suppose there exists a rational number r such that $r^2 = 2$. Since r is rational, there exists $p, q \in \mathbb{Z}$ such that $r = \frac{p}{q}$. We can assume the p and q have no common factors. (If not, we can factor out the common factor.) By our assumption,

$$r^2 = 2$$
$$\frac{p^2}{q^2} = 2$$

It follows that,

$$p^2 = 2q^2$$

Such that p^2 is an even number because if p were odd, then p^2 would be odd. There exists $x \in \mathbb{Z}$ such that $p = 2x$. Recall that $p^2 = 2q^2$. Thus

$$\begin{aligned}(2x)^2 &= 2q^2\\ 4x^2 &= 2q^2\\ 2x^2 &= q\end{aligned}$$

Thus, q^2 is even. Hence q is also even. So p and q are both divisible by 2. This contradicts that p and q have no common factors. Thus, our supposition is false. Therefore, there does not exist a rational number r such that $r^2 = 2$ □

So we are going to work with a larger set called the real numbers, $\mathbb{R}$.

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- You can:
  - Add,
  - Subtract,
  - Multiply,
  - Divide
- In other words, all field axioms apply.
- Totally ordered set for any $x, y \in \mathbb{R}$. Thus, one of these are true:
  1. $x < y$,
  2. $x > y$,
  3. $x = y$
- Think of it as a number line.
- $\mathbb{Q}$ is dense:
  If $a, b \in \mathbb{Q}$ with $a \neq b$, there exists $c \in \mathbb{Q}$ which is between a and b such that $a < c < b$. One example is $\frac{a+b}{2}$.
- $\mathbb{Q}$ is not *complete*, but $\mathbb{R}$ is.
  - *Complete:* Think, "no gaps."

1.2 Preliminaries

Things to remember from Intro and Discrete.

- $\displaystyle\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \ldots.$

Set Notation	Complement
$x \in A$	A^c (not $\overline{A}$)
$A \cup B$	$\mathbb{R} \setminus A$
$A \cap B$	

- $\displaystyle\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \ldots.$

Definition 1.2.1

De Morgan's Laws are defined as $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

1.2.1 Infinite Unions and Intersections

For each $n \in \mathbb{N}$, define $A_n = \{n, n+1, n+2, \ldots\} = \{k \in \mathbb{N} \mid k \geq n\}$. In other words, each subsequent element in the subset will start at n. For example, $A_1 = \{1, 2, \ldots\}$, whereas $A_5 = \{5, 6, \ldots\}$.

$\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$. To show a number $\in \mathbb{N}$ belongs in the set A_n, we can start with that, $k \in \mathbb{N}$. Then $k \in A_k$. Thus, $k \in A_k \subseteq \bigcup_{n=1}^{\infty} A_n$. Therefore, $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$.

$\bigcap_{n=1}^{\infty} A_n = \emptyset$. Obviously, we know that the empty set is a subset of A_n, but to prove that $\bigcap_{n=1}^{\infty} A_n$ is a subset of the empty set, we should suppose a $k \in \mathbb{N}$ such that $k \in \bigcap_{n=1}^{\infty} A_n$. Notice that $k \notin \bigcap_{n=1}^{\infty} A_n$. So, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

1.2.2 Functions and Notation

$f \colon A \to B$ where f is a function, A is a domain, and B is the co-domain. Thus, $f(x) = y$ such that $x \in A$ and $y \in B$.

Some definitions to keep in mind

Definition 1.2.2

The *Dirichlet Function* is defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Definition 1.2.3

Let $f : \mathbb{R} \to \mathbb{R}$. If $E \subseteq \mathbb{R}$, then $f(E) = \{f(x) \mid x \in E\}$.

Example: $g : \mathbb{R} \to \mathbb{R}$, when we say $y \in g(A)$ implies there exists an x such that $g(x) = y$

Definition 1.2.4

The *Triangle Inequality* is defined as: For any $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

The most common application: For any $a, b, c \in \mathbb{R}$, $|a - b| \leq |a - c| + |c - b|$, with the intermediate step of $a - b = (a - c) + (c - b)$.

Definition 1.2.5

A function f is *injective* (or *one-to-one*) if $f(a_1) = f(a_2)$, then $a_1 = a_2$ in B. Note the contrapositive of this definition: If $a_1 \neq a_2$ in A, then $f(a_1) \neq f(a_2)$ in B.

Definition 1.2.6

A function f is *surjective* (or *onto*) if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Note the contrapositive of this definition: If there exists a $b \in B$ such that there is no $a \in A$ such that $f(a) = b$, then the function is not surjective.

1.2.3 Common Strategies for Analysis Proofs

Theorem 1.2.6

Let $a, b \in \mathbb{R}$. Then,

$$a = b \text{ if and only if for all } \epsilon > 0, |a - b| < \epsilon.$$

Proof. We will show this by proving both implications:

($\Rightarrow$) Assume $a = b$. Let $\epsilon > 0$. Then $|a - b| = 0 < \epsilon$

($\Leftarrow$) Assume for all $\epsilon > 0$, $|a - b| < \epsilon$. Suppose $a \neq b$. Then $a - b \neq 0$. So, $|a - b| \neq 0$. Now, Consider $\epsilon_0 = |a - b|$. By our assumption we know that $|a - b| < \epsilon_0$. It is not true that $|a - b| < |a - b|$. Therefore, it must be the case that $a = b$.

Therefore, by showing both sides of the implication accomplish the same thing as the other side, we know that $a = b$ if and only if for all $\epsilon > 0, |a - b| < \epsilon$. $\square$

1.2.4 Mathematical Induction

Inductive Hypothesis: Let $x_1 = 1$. For all $n \in \mathbb{N}$, let $x_{n+1} = \frac{1}{2}x_n + 1$.

Inductive Step: $x_1 = 1, x_2 = 1.5, x_3 = 1.75, x_4 = 1.875.$

Example 1.1: Induction

The sequence (x_n) is increasing. In other words, for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$.

Proof. Suppose the sequence (x_n) is increasing. We will prove this point by using induction.

Base Case: We see that $x_1 = 1$ and $x_2 = 1.5$. Thus, $x_1 \leq x_2$.

Inductive Hypothesis: For $n \in \mathbb{N}$, assume $x_n \leq x_{n+1}$.

Scratch work: We want: $x_{n+1} \leq x_{n+2}$. We know: $x_{n+1} = \frac{1}{2}x_{n+1} + 1$.

Inductive Step: Then $\frac{1}{2}x_n \leq \frac{1}{2}x_{n+1}$. Hence, $\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1$. Therefore we have proven through induction that, $x_{n+1} \leq x_{n+2}$. □

1.3 Axiom of Completeness

Axiom of Completeness

Every nonempty set of real numbers that is bounded has a least upper bound.

Think about $\mathbb{Q}$ and $\mathbb{R}$.

- Both are fields.
  - Both have $+, -, \times, \div$ operations.
- Both are totally ordered
  - $a < b$,
  - $a > b$,
  - or $a = b$
- $\mathbb{R}$ is complete. $\mathbb{Q}$ is not.

1.3.1 Least Upper Bounds and Greatest Lower Bounds

Definition 1.3.1

A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* of A.
Similarly, a set $A \subseteq \mathbb{R}$ is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Note that upper bounds are not unique! For example, consider the line, A, from 0 to 1. There are infinitely many upper bounds past 1 because A is bounded.

Definition 1.3.2

A number s is a *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

We often call the least upper bound the *supremum* of a set.

Example 1.2: Supremum

Imagine a number line from $(1, 8)$. Note that parenthesis mean $<$ and not $\leq$. Hence, the supremum is 8. Wrote simply as $\sup A$.

Example 1.3: Supremum and Infimum 1

Consider a set, $B = [-5, -2] \cup (3, 6) \cup \{13\}$. What is the supremum and the infimum?

Solution. $\sup B = 13$; $\inf B = -5$ because -5 is the greatest lower bound.

Example 1.4: Supremum and Infimum 2

Consider the set, $C = \{\frac{1}{n} : n \in \mathbb{N}\}$. What is the supremum and the infimum?

Solution. $\sup C = 1$, $\inf C = 0$.

Example 1.5: Exam Example

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : \ a \in A\}$$

Then $\sup(c + A) = c + \sup A$.

Solution. To properly verify this we focus separately on each part of Definition 1.3.2. Setting $s = \sup A$, we see that $a \leq s$ for all $a \in A$, which implies $c + a \leq c + s$ for all $a \in A$. Thus, $c + s$ is an upper bound for $c + A$ and condition (i) is verified. For (ii), let b be an arbitrary upper bound for $c + A$; i.e., $c + a \leq b$ for all $a \in A$. This is equivalent to $a \leq b - c$ for all $a \in A$, from which we conclude that $b - c$ is an upper bound for A. Because s is the least upper bound of A, $s \leq b - c$, which can be rewritten as $c + s \leq b$. This verifies part (ii) of Definition 1.3.2, and we conclude $\sup(c + A) = c + \sup A$.

Definition 1.3.4

A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_a \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if a_1 is an element of A and $a_1 \leq a$ for all $a \in A$.

Note that some sets have a maximum and some sets do not. You cannot refer to a maximum without first knowing it exists. This is the same with minimums.

Lemma 1.3.5

Assume s is an upper bound for a set $A \subseteq \mathbb{R}$. Then, s is the supremum of A if and only if for every $\epsilon > 0$, there exists $x \in A$ such that $s - \epsilon < x$.

This lemma allows us to take any positive number and take a "step back." In essence, you can verify something as an upper bound if you continuously back up over and over until you cannot back up any longer.

Proof. We show this by proving both implications:

($\Rightarrow$) Assume $s = \sup A$. Let $\epsilon > 0$. Suppose there are no elements x of A such that $s - \epsilon < x$. Then $s - \epsilon$ would be an upper bound. This contradicts that s is the least upper bound. Therefore, there must exist an element $x \in A$ such that $s - \epsilon < x$.

(⇐) Assume for every $\epsilon > 0$, there exists $x \in A$ such that $s - \epsilon < x$. Let t be an upper bound of A. Suppose $t < s$. Consider $\epsilon_0 = s - t > 0$. By our assumption, there exists $x \in A$ such that $s - \epsilon_0 < x$. So, $t < x$. This contradicts that t is an upper bound of A. So, $t \geq s$. Thus, s is the least upper bound

Therefore, by proving both the right and left implication, we have shown the statement to be true. □

Analogous statement about infimums: Assume z is a lower bound of a set $A \subseteq \mathbb{R}$. Then $z = \inf A \iff$ for all $\epsilon > 0$, there exists $y \in A$ such that $y < z + \epsilon$.

1.4 Consequences of Completeness

Theorem 1.4.1: Nested Interval Property

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n]$. Assume I_n contains I_{n+1}. This results in a nested sequence of intervals, $I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \dots$. Then,

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

tl;dr there has to be something that is common to all sets.

Proof. Notice that the sequence, $a_1, a_2, a_3, \dots$ is increasing. In other words, for each $n \in \mathbb{N}$, since $I_n \supset I_{n+1}$ we have $a_n \leq a_{n+1}$. If we consider the set $A = \{a_n : n \in \mathbb{N}\}$. The element b_1 is an upper bound of A. (Note that b_1 and a_1 corresponds to the endpoints of the first set, I_1. Think of this as a tornado looking structure where the larger the I_n, the smaller the number line.) For each $n \in \mathbb{N}$, $a_n \leq b_n \leq b_1$.

Since A has an upper bound, it must have a least upper bound. Hence, let $\alpha = \sup A$. We claim that $\alpha \in \bigcap_{n=1}^{\infty} I_n$. We said b_1 was an upper bound. In fact, every b_n is an upper bound of A. Choose any $n, m \in \mathbb{N}$. We want to show that $a_n \leq b_m$. Consider the following cases:

Case 1: If $n < m$, then $a_n \leq a_m \leq b_m$. (Think: two number lines stacked on top of each other. The top number line is larger, call it I_n, and it has a_n and b_n as endpoints. Consider a contained line ($I_n \supseteq I_m$) that is smaller, and has endpoints a_m and b_m.)

Case 2: If $n > m$, then $a_n \leq b_n \leq b_m$. So every b_n is an upper bound of A.

Hence,

- Because $\alpha = \sup A$, we have $\alpha \geq a_n$.

- Since b_n is an upper bound of A, we have $\alpha \leq b_n$.

so, $\alpha \in [a_n, b_n] = I_n$. Thus, $\alpha \in \bigcap_{n=1}^{\infty} I_n$. □

Nested, closed, Bounded Intervals $\Rightarrow$ non-empty intersection.

Theorem 1.4.2: Archimedean Principle

1. Given any number $x \in R$, there exists an $n \in N$ satisfying $n > x$.
2. Given any real number $y > 0$, there exists an $n \in N$ satisfying $1/n < y$.

Proof.
1. If $\mathbb{N}$ was bounded, then we can let $s \in \mathbb{N} = \sup \mathbb{N}$. However, we know that there is always a higher number (e.g., $n+1$) for any $n \in \mathbb{N}$ that is given. Thus, by contradiction, there must exist $n \geq x$.
2. For any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

□

Theorem 1.4.3: Density of the Rationals in the Reals

For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Proof. Since $b - a > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. From the Archimedean Principle, since $a \times n \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that $a \times n < m$. Let m be there smallest such natural numbers (by the well ordered principle). Since m is the smallest such natural number, it follows that $m - 1 \leq a \times n < m$. We then see that $a < \frac{m}{n}$. Now, we need to find some $\frac{m}{n} < b$.

$$\begin{aligned} m - 1 &\leq a \times n \\ m &\leq a \times n + 1 \\ \frac{m}{n} &\leq a + \frac{1}{n} \\ \frac{m}{n} &< a + (b - a) \\ \frac{m}{n} &< b \end{aligned}$$

We now have that $a < \frac{m}{n} < b$ so $\frac{m}{n}$ is a rational number in (a, b) □

1.5 Cardinality

Two sets have the same *cardinality* if there exists a bijection between them. Thus, the natural numbers, the integers, and the rational numbers have the same cardinality. A set is *countably infinite* if it has the same cardinality as $\mathbb{N}$. (If it can be put into one-to-one correspondence with $\mathbb{N}$.) A set is *countable* if it is countably infinite or finite.

Theorem 1.5.6

$\mathbb{R}$ is not countable.

Proof. 1 (most common)
Suppose $\mathbb{R}$ is countable. Then we can list them all, or we can enumerate them. $\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\}$. We can write the decimal expansion of each of these. Consider the following table:

$x_1 =$	$\boxed{a_{10}}$	a_{11}	a_{12}	a_{13}	a_{14}	$\dots$
$x_2 =$	a_{20}	$\boxed{a_{21}}$	a_{22}	a_{23}	a_{24}	$\dots$
$x_3 =$	a_{30}	a_{31}	$\boxed{a_{32}}$	a_{33}	a_{34}	$\dots$
$x_4 =$	a_{40}	a_{41}	a_{42}	$\boxed{a_{43}}$	a_{44}	$\dots$
$x_5 =$	a_{50}	a_{51}	a_{52}	a_{53}	$\boxed{a_{54}}$	$\dots$
$x_6 =$	a_{60}	a_{61}	a_{62}	a_{63}	a_{64}	$\dots$

We will now construct a number that is not in this list. Focus on diagonal entries. For each $n \in \mathbb{N}$, let b_n be a digit that is different fron a_{nn}. Now consider the number $y = 0.b_1b_2b_3b_4b_5\dots$. This number y is not in our list. So our list did not include all of $\mathbb{R}$. Avoid repeating 9s. $\square$

Proof. 2 (uses nested interval theorem)
Suppose $\mathbb{R}$ is countable. Then we can enumerate $\mathbb{R}$ $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$. Let I_1 be any closed interval that does not contain x_1. Next, we will find another closed interval I_2 that:

- $I_2 \subseteq I_1$
- $x_2 \notin I_2$

Continue in this fashion creating a sequence of nested closed intervals: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ such that for all $k \in \mathbb{N}$, $x_k \notin I_k$. Now consider:

$$\bigcap_{n=1}^{\infty} I_n$$

- For each $k \in \mathbb{N}$, since $x_k \notin I_k$, we see $x_k \notin \bigcap_{n=1}^{\infty} I_n$.
- By the nested interval theorem, there exists $x \in \mathbb{R}$ such that $x \in \bigcap_{n=1}^{\infty} I_n$. So x is a real number that is not included in our list. $\square$

Theorem 1.5.7

A countable collection of finite sets is *countable*.

Theorem 1.5.8

(i) The union of two countable sets is *countable*.

(ii) A countable union of countable sets is *countable*.

From Theorem 1.5.6, we know that $\mathbb{R}$ is uncountable, but what about $(0, 1)$? It does have the same cardinality of $\mathbb{R}$ because we can make a one-to-one and onto function between both the sets. Similarly, (a, b) also has the same cardinality. What about $[a, b]$?

Recap: $\mathbb{N}$ is countable, and $\mathbb{R}$ is uncountable and has a different cardinality than $\mathbb{N}$. Thus, the question is, do all uncountable sets have the same cardinality as $\mathbb{R}$? The answer is **no**.

Theorem 1.5.9: Canter's Theorem

For any set A, there does not exist an onto map from A into $\mathcal{P}$.

Proof. Suppose there exists an onto function, $f : A \to \mathcal{P}(A)$. So each $a \in A$ is mapped to an element $f(a) \in \mathcal{P}(A)$. Then, $f(a) \subseteq A$. We are going to construct an element of $\mathcal{P}(A)$ which is not mapped to by f.

Consider $B = \{a \in A : a \notin f(a)\}$. Since f is onto there exists $a' \in A$ such that $B = f(a')$. Thus, there are two cases to consider:

- **Case 1:** If $a' \in B = f(a')$, then $a' \notin B$.
- **Case 2:** If $a' \notin B = f(a')$, then $a' \in B$.

As evidenced, both cases lead to contradictions, so B is not the image of any $a \in A$. Therefore f is not onto. □

Example 1.6: Set and Power Set Matching

$A = \{a, b, c\}$.

Solution. $\mathcal{P}(A) = \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$. Note that you can map $\{a\}$, $\{b\}$, $\{c\}$, to elements such as $\emptyset$, $\{a, b\}$, $\{a, b, c\}$, but there are still more elements that are left unmapped. We can extrapolate from our proof a set B such that $B = \{a, c\}$ because those elements are not mapped to.

All of this is to show $\mathcal{P}(\mathbb{R})$ has a larger cardinality than $\mathbb{R}$. Then $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ has a larger cardinality than $\mathcal{P}(\mathbb{R})$.

CHAPTER 2

SEQUENCES AND SERIES

2.1 Discussion: Rearrangement of Infinite Series

Questions:

What is a *sequence*?

A countable, ordered list of elements. An example could be $1, 2, 3, 4, 5, \ldots$. Note that this is *ordered*, therefore distinguishing it from a sequence like $3, 1, 2, 4, 5, 6, \ldots$. Hence, order matters.

A *sequence* is a function whose domain is $\mathbb{N}$. **Note:** The domain $\mathbb{N}$ refers to each element's position in the list. For example, $(a_n) = a_1, a_2, a_3, \ldots$.

We will focus on the *limit* of a sequence. We use sequences to approximate other things.

Example 2.1: Sequence

$3, 3.1, 3.14, 3.141, 3.1415, \cdots \approx \pi$.
$x, x - \frac{x^3}{6}, x - \frac{x^3}{6} + \frac{x^5}{120}, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}, \cdots \approx \sin(x)$

What is a *series*?

An infinite sum. We look at the sequence of partial sums. We ask, do the partial sums approach a limit?

Example 2.2: Alternating Harmonic Series

$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots$

We can rearrange these terms such that we can 'force' the series to converge to a specific number. Therefore, we will need to be careful with our definitions.

2.2 The Limit of a Sequence

Definition 2.2.1

A *sequence* is a function whose domain is $\mathbb{N}$. We write $(a_n) = a_1, a_2, a_3, \ldots$.

Definition 2.2.3

The sequence (a_n) *converges* to L if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$, $|a_n - L| < \epsilon$. In other words, there exists $N \in \mathbb{N}$ such that

- **(In the interval)** $a_N \in (L - \epsilon, L + \epsilon)$.
- **(Stays in the interval)** $\forall n \geq N$, $a_n \in (L - \epsilon, L + \epsilon)$.

Example 2.3: Limit Proof 1

Let $a_n = \frac{1}{n}$. $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$

Proof. Our claim is $\lim_{n\to\infty} \dfrac{1}{n} = 0$. Thus, let $\epsilon = .01$. Does the sequence eventually get inside $(-.01, .01)$? We will set $N = 101$. So, for any $n \geq |0|$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{101} < .01.$$

From A_n and on, the sequence stayed within ϵ of 0. But what about $\epsilon = .001$, $\epsilon = .00001$ and so on?

Actual proof let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Now, for any $n \geq N$,

$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{1/\epsilon}.$$

(Where $\frac{1}{1/\epsilon} = \epsilon$, but is in that form for demonstration purposes.) Therefore $\lim_{n\to\infty} \dfrac{1}{n} = 0$ □

"To get close" means is that we are finding a bigger and bigger N as ϵ gets smaller. Note that the choice of N certainly depends on ϵ. This idea of "getting close" can be seen in the following definition:

Definition 2.2.3B

A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).

2.2.1 Basic Structure of a Limit Proof

Claim: $\lim_{n\to\infty} a_n = L$.
Proof: Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that {something involving ϵ}. Assume $n \geq N$. Then,

$$|a - n - L| \boxed{\dots} < \epsilon$$

(Where $\boxed{\dots}$ is going to be where the majority of the work is going to lie.)

Example 2.4: Limit Proof 2

Claim: $\lim_{n\to\infty} \frac{2n-3}{2n} = 1$

Proof. Let $\epsilon > 0$. *Scratch paper:* Solve for:

$$\left|\frac{2n-3}{2n} - 1\right| = \left|\frac{-3}{2n}\right| = \frac{3}{2n} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n.$$

By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{3}{2\epsilon}$. Assume $n \geq N$, (want to know what happens past this point)

$$\left|\frac{2n-3}{2n} - 1\right| \leq \frac{3}{2N} < \frac{3}{2 \cdot 3/2\epsilon} = \epsilon.$$

Therefore, $\lim_{n\to\infty} \frac{2n-3}{2n} = 1$ □

Example 2.5: Limit Proof 3

Claim: $\lim_{n\to\infty} \frac{2n^2+1}{n^2} = 2$

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that [leave off] *Scratch paper:* Solve for

$$\left|\frac{2n^2+1}{n^2} - 2\right| = \frac{2n^2}{n^2} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n$$

[pick up] there exists $N \in \mathbb{N}$ such that

$$N > \frac{1}{\sqrt{\epsilon}}.$$

Assume $n \geq N$, then

$$\begin{aligned}
\left|\frac{2n^2+1}{n^2} - 2\right| &= \frac{1}{n^2} \\
&\leq \frac{1}{N^2} \\
&< \frac{1}{(1/(\sqrt{\epsilon}))^2} \\
&= \frac{1}{1/\epsilon} \\
&= \epsilon
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{2n^2+1}{n^2} = 2$ □

Example 2.6: Limit Proof 4

Claim: $\displaystyle\lim_{n\to\infty} \frac{7n+8}{3n+6} = \frac{7}{3}$

Proof.

$$\begin{aligned}
\left|\frac{7n+8}{3n+6} - \frac{7}{3}\right| &= \left|\frac{21n+24}{3(3n+6)} - \frac{21n+42}{3(3n+6)}\right| \\
&= \left|\frac{-18}{9n+18}\right| \\
&= \frac{18}{9n+18} < \epsilon ** \\
&= \frac{18}{3} < 9n+18 \\
&= \frac{18}{3} - 18 < 9n \\
&= \frac{18/\epsilon - 18}{9} < n
\end{aligned}$$

$**\ \frac{18}{9n+8} < \frac{18}{9n} < \epsilon \Rightarrow \frac{2}{\epsilon} < N$. $\exists N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Assume $n \geq N$,

$$\begin{aligned} \left|\frac{7n+8}{3n+6} - \frac{7}{3}\right| &= \frac{18}{9n+18} \\ &= \frac{2}{n+2} \\ &< \frac{2}{n} \\ &\leq \frac{2}{N} \\ &< \frac{2}{\epsilon/2} \\ &= \epsilon. \end{aligned}$$

□

Does every sequence have a limit?

Theorem 2.2.4: Uniqueness of Limits

The limit when it exists, is unique.

Proof. Let (x_n) be a convergent sequence. Suppose L and M are limits of this sequence. Without the loss of generality, we are going to assume $M > L$ Let

$$\epsilon = \frac{M-L}{3}.$$

Since n_x converges to L, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N$, $|(x_n) - L| < \epsilon$. Since (x_n) converges to M, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|(x_n) - M| < \epsilon$. Consider $n = \max\{N_1, N_2\}$. Since $n \geq N_1$, $|(x_n) - L| < \epsilon$. Since $n \geq N_2$, $|(x_n) - M| < \epsilon$. Then $L - \epsilon < x_n < L + \epsilon$ and $M - \epsilon < x_n < M + \epsilon$. By our choice of ϵ, we now have

$$(x_n) < L + \epsilon < M - \epsilon < (x_n).$$

This is a contradiction. Thus, (x_n) cannot have two different limits. □

Example 2.7: Limit Proof 5

Let $(x_n) = \dfrac{\cos(n)}{3n}$. Claim: $\lim\limits_{n\to\infty}(x_n) = 0$

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{3\epsilon}$

for all $n \geq N$,

$$\begin{aligned}\left|\frac{\cos(n)}{3n} - 0\right| &= \left|\frac{\cos(n)}{3n}\right| \\ &\leq \frac{1}{3n} \\ &\leq \frac{1}{3N} \\ &< \frac{1}{3(1/3\epsilon)} \\ &= \epsilon\end{aligned}$$

□

Example 2.8: Limit Proof 6

Let $(y_n) = \dfrac{4n-1}{n^2}$. Claim: $\lim\limits_{n\to\infty} y_n = 0$.

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. For all $n \geq N$,

$$\begin{aligned}\left|\frac{4n-1}{n^2} - 0\right| &= \left|\frac{4n-1}{n^2}\right| \\ &= \frac{4n-1}{n} \\ &< \frac{4n}{n^2} \\ &= \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \frac{4}{4/\epsilon} \\ &= \epsilon\end{aligned}$$

□

2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1

A sequence (x_n) is *bounded* if there exists some $M > 0$ such that every term in the sequence belongs to $[-M, M]$.

Theorem 2.3.2

Every convergent sequence is bounded.

Proof. Let (x_n) be a convergent sequence with limit L. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|(x_n) - L| < 1$. Equivalently, $(x_n) \in (L-1, L+1)$. Let

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |L+1|, |L-1|\}.$$

We claim that for all $n \in \mathbb{N}$, $|x_n| \leq M$.

1. This is true for $n < N$.
2. For $n \geq N$, we know $L - 1 < x_n < L + 1$, so $(x_n) \leq \max\{|L-1|, |L+1|\}$

Thus, every term is in $[-M, M]$. □

Theorem 2.3.3: Algebraic Limit Theorem

Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then,

(i) $\lim_{n\to\infty} ca_n = ca$ for all $c \in \mathbb{R}$;

(ii) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(iii) $\lim_{n\to\infty}(a_n b_n) = ab$;

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided $b \neq 0$.

Scratch Paper:

$$\begin{aligned} |ca_n - ca| = |c|\,|a_n - a| &< \epsilon \\ |a_n - a| &< \frac{\epsilon}{|c|} \end{aligned}$$

Leave off and go back to proof[1]

Proof. (i)
Let $\epsilon > 0$.[1] Since (a_n) converges to a, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \frac{\epsilon}{|c|}$. Now, for any $n \geq N$ we have two case because we want to avoid dividing by 0:

- If $c = 0$:
  then each $ca_n = 0$. So (ca_n) converges to 0, which can equal ca.

- If $c > 0$:
  $|ca_n - ca| = |c|\,|a_n - a| < |c|\,\frac{\epsilon}{|c|} = \epsilon$.

(ii)
Scratch paper:

$$\begin{aligned} |(a_n + b_n)| &= |(a_n - a) + (b_n - b)| \quad (2.1) \\ &\leq |a_n - a| + |b_n - b| \quad (2.2) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad (2.3) \end{aligned}$$

Note that (2.2) is from the triangle inequality. Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \frac{\epsilon}{2}$. Since (b_n) converges to b, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon}{2}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \geq N$, (refer back to scratch paper).

(iii)
Scratch paper:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \quad (2.4) \\ &= |a_n(b_n - b) + b(b_n - b)| \quad (2.5) \\ &\leq |a_n|\,|b_n - b| + |b|\,|b_n - b| \quad (2.6) \\ &\leq M\,|b_n - b| + M\,|a_n - a|\,. \quad (2.7) \\ &< M\left(\frac{\epsilon}{2M}\right) + M\left(\frac{\epsilon}{2M}\right) \quad (2.8) \\ &= \epsilon \quad (2.9) \end{aligned}$$

Note that: (2.4) is where we added 0, (2.5) is from the triangle inequality, and (2.6) is just factored. Additionally, we choose N to get the fractions in (2.8) Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since convergent sequences are bounded, then there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$. We can choose M so that $|b_n| \leq M$ as well. Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \frac{\epsilon}{2M}$. Since (b_n) converges to b, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon}{2M}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \geq N$, (refer back to scratch paper, and change (2.4)'s sign from an '$=$' to '$\leq$').

(iv)

Scratch paper:

$$\begin{aligned}\left|\frac{a_n}{b_n}-\frac{a}{b}\right| &= \left|\frac{a_nb-ab_n}{b_nb}\right| \\ &= \left|\frac{a_nb-ab_n+ab_n-ab}{b_nb}\right| \\ &= \left|\frac{a_n(b-b_n)+b(b_n-b)}{b_nb}\right| \\ &= \left|\frac{a_n(b-b_n)}{b_nb}+\frac{b(b_n-b)}{b_nb}\right| \\ &\le \left|\frac{a_n}{b_n}\right||b-b_n|+|b|\left|\frac{b_n-b}{b_nb}\right| \\ &< \epsilon\end{aligned}$$

Let $\epsilon > 0$. Since (b_n) converges to b, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, $|b_n| > \left|\frac{b}{2}\right|$. There also exists $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, $|b_n - b| < \frac{\epsilon|b|^2}{2}$. Now, let $N = \max\{N_1, N_2\}$. Let $n \ge N$, (refer back to scratch paper). $\square$

Lemma 2.3.4

Let (a_n) and $c < a$. There exists $N \in \mathbb{N}$ such that for all $n \ge N$, $a_n > c$. Similarly, if $a < d$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $a_n < d$.

2.3.1 Limits and Order

Theorem 2.3.5: Order Limit Theorem

Let (a_n) and (b_n) be sequences. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

(i) If $a_n \ge c$ for all $n \in \mathbb{N}$, then $a \ge c$.

(ii) If $a_n \le b_n$ for all $n \in \mathbb{N}$, then $a \le b$.

(iii) If there exists $c \in \mathbb{R}$ for which $c \le b_n$ for all $n \in \mathbb{N}$, then $c \le b$. Similarly, if $a_n \le c$ for all $n \in \mathbb{N}$, then $a \le c$.

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Definition 2.4.1

A sequence a_n is *increasing* if $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \ge a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.2: Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges.

Proof. Let (a_n) be an increasing and bounded sequence. Since (a_n) is bounded, the set $A = \{a_n \mid n \in \mathbb{N}\}$ is clearly also bounded. Since A is bounded, $\sup A$ exists. We claim that $\lim_{n\to\infty} a_n = \sup A$. Thus, for all $\epsilon > 0$ and by our definition of supremum, there exists $N \in \mathbb{N}$ such that $\sup A - \epsilon < a_N \leq \sup A$. Since (a_n) is increasing, for all $n \geq N$, $\sup A - \epsilon < a_N \leq a_n \leq \sup A$. It follows that $|a_n - \sup A| < \epsilon$. Therefore, $\lim_{n\to\infty} a_n = \sup A$. □

Example 2.9: MCT

Consider the recursively defined sequence x_n where $x_1 = 3$ and for all $n \in \mathbb{N}$, $x_{n+1} = \frac{1}{4-x_n}$. Show that x_n converges.

Proof. We will show that x_n is monotone and bounded.

- **Part 1: Monotone Decreasing**
  - Base case: $x_1 = 3$, $x_2 = 1$.
  - Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq x_{n+1}$. It follows that

$$\begin{aligned} x_n &\geq x_{n+1} \\ 4 - x_n &\leq 4 - x_{n+1} \\ \frac{1}{4-x_n} &\geq \frac{1}{4-x_{n+1}} \\ x_{n+1} &\geq x_{n+2} \end{aligned}$$

- **Part 2: Bounded Below** Claim: Sequence is bounded below by 0.
  - Base case: $x_1 = 3 > 0$.
  - Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq 0$. It follows that $4 - x_n \leq 4$, and when we take the reciprocal, we get

$$\begin{aligned} \frac{1}{4-x_n} &\leq \frac{1}{4} \\ x_{n+1} &\geq 1/4 \\ &> 0 \end{aligned}$$

  By math induction, x_n is bounded below by 0.

By the Monotone Convergence Theorem, x_n converges.

So, what is the limit? We know (x_n) converges so let $L = \lim_{n\to\infty} x_n$. Then, $\lim_{n\to\infty} x_{n+1} = L$. We also know $x_{n+1} = \frac{1}{4-x_n}$. So $L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{4-x_n} = \frac{1}{4-L}$. It must be true that $L = \frac{1}{4-L}$. Solving for L, we get

$$\begin{aligned} L(4-L) &= 1 \\ 4L - L^2 &= 1 \\ L^2 - 4L + 1 &= 0 \end{aligned}$$

Hence, $L = 2 - \sqrt{3}$ or $L = 2 + \sqrt{3}$. Notice that it cannot be the latter because it is bigger than 3. □

2.4.1 Recap and Summary

We use limits to define multiple things in calculus. This is why we are focusing so heavily upon it. For example,

1. Derivatives: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
2. Integrals: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$
3. Infinite Series: $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ Consider geometric series, C_a such that each term is multiplied by a ratio r. This is represented as $\sum_{n=0}^{\infty} ar^n = 1 + r + r^2 + r^3 \dots$. When we look at partial sums, we get $S_n = 1 + r + r^2 + r^3 + \cdots + r^n$. We can then multiply by r to get $rS_n = r + r^2 + r^3 + r^4 + \cdots + r^{n+1}$. Subtracting the two, we get $(1-r)S_n = 1 - r^{n+1}$. Thus,
$$S_n = \frac{1-r^{n+1}}{1-r}.$$
If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. Thus, $\lim_{n\to\infty} S_n = \frac{1}{1-r}$.

Looking to the future, we are going to use functions and summations together. For example, when we have $f(x) = \sum_{n=0}^{\infty}(a_n)x^n$ such that $f'(x) = \sum_{n=0}^{\infty}(a_n)x^{n-1}$.

Definition 2.4.3

Let (x_n) be a bounded sequence. Then the *limit inferior* is $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k \mid k \geq n\}$. This is the largest a limit can get. The *limit superior* is $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k \mid k \geq n\}$. This is the smallest a limit can get.

See Exercise 2.4.7 in the book for more information.

Example 2.10: Monotone Decreasing Sequence

The following sequence is an example of a monotone decreasing sequence.

$$\begin{aligned} x_1, x_2, x_3, x_4, x_5, x_6, \ldots \sup\{x_k \mid k \geq 1\} = S. \\ x_2, x_3, x_4, x_5, x_6, \ldots \sup\{x_k \mid k \geq 2\} = S. \\ x_3, x_4, x_5, x_6, \ldots \sup\{x_k \mid k \geq 3\} = S. \\ x_4, x_5, x_6, \ldots \sup\{x_k \mid k \geq 4\} = S. \end{aligned}$$

$\limsup_{n\to\infty} x_n$ is guaranteed to exist by the Monotone Convergence Theorem.

Example 2.11: liminf

Let $x_n = (-1)^n(1 + \frac{1}{n})$. Thus, $x_{1,2,3} = -2, 1\frac{1}{2}, -1\frac{1}{3} \ldots$

Example 2.12: Convergence Towards 0

Let $x_n = (-1)^n \frac{1}{n}$. Thus, $x_{1,2,3} = -1, \frac{1}{2}, -\frac{1}{3} \ldots$.

Theorem 2.4.4

A sequence x_n is convergent if, and only if, $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

See Theorem 2.4.6 in the book for another view.

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Definition 2.5.1

Let a_n be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \ldots$ be an increasing sequence of natural numbers. Then, the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is called a *subsequence* of a_n and is denoted by a_{n_k}, where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 2.5.2

Subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let x_{n_k} be a subsequence of x_n, and let $L = \lim_{n\to\infty} x_n$. We want to show that $\lim_{n\to\infty} x_{n_k} = L$. Let $\epsilon > 0$. Since x_n converges to L, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$. Since n_k is increasing, there exists $M \in \mathbb{N}$ such that $n_k \geq N$

for all $k \geq M$. Thus, for all $k \geq M$, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n\to\infty} x_{n_k} = L$.

Let x_{n_k} be a subsequence of x_n. Let $\epsilon > 0$. Since $(x_n) \to L$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$.

Now, looking at x_{n_k}, notice that $n_k \geq k$ for all k. Consider $k = N$. For any $n \geq N$, $n \geq N \geq k$. Thus, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n\to\infty} x_{n_k} = L$. □

Theorem 2.5.3: Divergence Criterion

If x_n has two subsequences that converge to different limits, then x_n diverges.

Building upon this idea of Divergence, we can list some other ways a sequence can diverge:

1. Find one subsequence that diverges.
2. Find tow subsequences that converge to separate limits.
3. Negate the definition of convergence.
   - For example, a sequence converges to L if there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - a| \geq \epsilon$. There exists a subsequence (a_{n_k}) such that for all $k \in \mathbb{N}$, $|a_{n_k} - L| \geq \epsilon$.

Theorem 2.5.4: Bolzano-Weierstrass Theorem

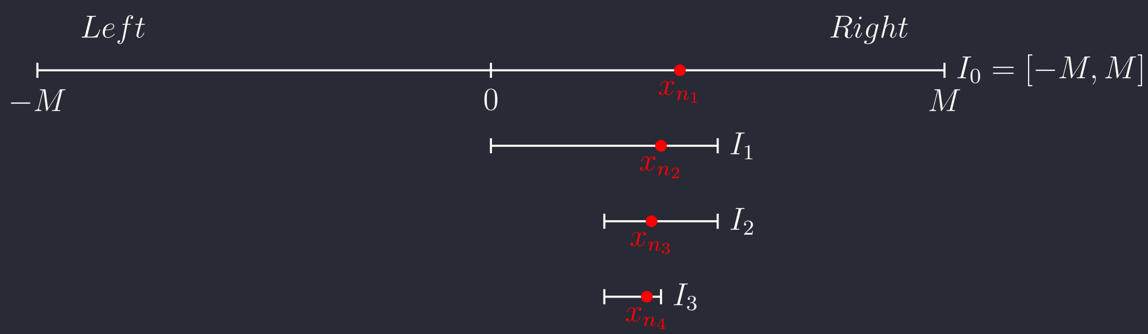
Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. Let x_n be a bounded sequence. There exists an $M > 0$ such that every term x_n belongs to $[-M, M]$. To prove this theorem, we will be utilizing a recursive argument style. Thus, let $I_0 = [-M, M]$. I_0 has length $2M$. Cut I_0 in half with I_1 and I_2 both being half as long as I_0. Since x_n is bounded, there exists an I_L or I_R that contains infinitely many terms of x_n. We will pick one, call it I_1 that is contained in I_L. I_1 has length M. Pick one of those terms inside I_1 and call it x_{n_1}. Now, cut I_1 in half with equal length in intervals. One of them contains infinitely many terms. Call that interval I_2. I_2 has length $\frac{M}{2}$. Pick one of those terms inside I_2 and call it x_{n_2}. Continue this process indefinitely for all $n \geq \mathbb{N}$ with $n_1 > n_2$. Continue this process, and we get

- a sequence of closed intervals I_n.
  - I_n has length $\frac{2M}{2^n}$.
  - They are nested, $I_n \subseteq I_{n-1}$.
- a subsequence x_{n_k}
  - for all k_1, $x_{n_k} \in I_k$.

The Nested Interval Property states that $\bigcup_{n=1}^{\infty} I_n$ is non empty. Let L be a point in $\bigcup_{n=1}^{\infty} I_n$. We claim $\lim_{n\to\infty} x_{n_k} = L$. Let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that

$\frac{2M}{2^n} < \epsilon$. (Since $\lim_{n\to\infty} \frac{2M2^n}{=} 0$. See Theorem 2.5.5) For any $k \geq N$, recall that x_{n_k}, $L \in I_k$. Since I_k has length $\frac{2M}{2^n}$. Thus, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n\to\infty} x_{n_k} = L$ and (x_n) has a convergence subsequence. $\square$



Theorem 2.5.5

Let $b \in (0, 1)$. Then $\lim_{n\to\infty} b^n = 0$.

Proof. The sequence (b^n) is monotone decreasing. This is because $b^{n+1} = b^n b < b^n$. This sequence is also bounded by 0. Hence, by the Monotone Convergence Theorem, (b^n) converges. Now, let $L = \lim_{n\to\infty} b^n$. Consider the subsequence b^{2n}. This sequence also converges to L. Thus,

$$\begin{aligned} L &= \lim_{n\to\infty} b^{2n} \\ &= \lim_{n\to\infty} b^n b^n \\ &= \lim_{n\to\infty} b^n \lim_{n\to\infty} b^n \\ &= L^2. \end{aligned}$$

Thus, $L = 0$ or $L = 1$. The limit cannot be 1 because b^n is decreasing away from 1. Therefore, $L = 0$. $\square$

2.6 The Cauchy Criterion

Recall

How do we prove x_n converges?

1. We know and prove the limit $\to$ claim L, show terms get close to L.
2. Monotone Convergence Theorem.

Definition 2.6.1

A sequence (x_n) is a *Cauchy sequence* if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|x_m - x_n| < \epsilon$.

This says that as terms get close to each other and stay close together, there's some value they're all getting close to.

Geometric Series Review

Remember that geometric series consist of terms that are multiplied by a ratio r. For example, that could look like $1 + r + r^2 + r^3 + \cdots$.

We are most interested in **partial sums**. That is,

$$1 + r + r^2 + \cdots r^{n-1} + r^n = S_n.$$

From here, we we would multiply both sides by r. This gives

$$r + r^2 + \cdots r^n + r^{n+1} = rS_n.$$

When we subtract these two from each other, we get

$$1 - r^{n+1} = S_n - rS_n.$$

This yields the identity

$$1 + r + r^2 + \cdots r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Example 2.13: Cauchy Sequence

Consider the sequence $a_1 = 1, a_2 = 2$, where

$$a_n = \frac{a_{n-1} + a_{n-2}}{2} \text{ for all } n \geq 2.$$

Show this sequence is Cauchy.

Proof. Look at the differences of consequtive terms, $|a_1 - a_2| = 1$, $|a_2 - a_3| = 1/2$, we can see a formula $a_n - a_{n+1} = 1/2^{n-1}$. Assume $|a_n - a_m| = |a_n - a_{n+1} - a_{n+2}| - \cdots -$

$a_{m-1} - a_m$ with $n < m$. From the Triangle Inequality,

$$\begin{aligned}
|a_n - a_m| &\leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| && (2.10)\\
&= \frac{1}{2^{n-1}} + \frac{1}{2}^n + \cdots + \frac{1}{2}^{m-2} && (2.11)\\
&= \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2}^{m-n-1}\right) && (2.12)\\
&= \frac{1}{2^{n-1}}\left(\frac{1 - \frac{1}{2}^{m-n}}{1 - \frac{1}{2}}\right) && (2.13)\\
&= \frac{1}{2^n}\left(1 - \frac{1}{2^{m-n}}\right) && (2.14)\\
&< \frac{1}{2^n}. && (2.15)
\end{aligned}$$

Notice that we were able to pull out the 1/2 and use the geometric series formula at step 2.12. From here we know that $|a_n - a_m| < \frac{1}{2^n}$.

Now, conclude the proof by letting $\epsilon > 0$. We know $(1/2^n) \to 0$. Thus, there exists $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$. For all $n, m \geq N$, (without loss of generality $n < m$) $|a_n - a_m| < \frac{1}{2^n} \leq \frac{1}{2^N} < \epsilon$. Therefore, a_n is Cauchy and it converges. □

Note: To find the limit of this series, a proof strategy is finding subsequences that are odd and even, and show the converge to the same limit.

Theorem 2.6.2: Cauchy Criterion

A sequence x_n converges if, and only if, it is a Cauchy sequence.

Proof. We will show this by proving both implications:

($\Rightarrow$) Assume (x_n) is a convergent seqeuence in $\mathbb{R}$. Given $\epsilon > 0$. Let $L = \lim_{n\to\infty} x_n$. Since $(x_n) \to L$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \frac{\epsilon}{2}$. For all $n, m \geq N$,

$$\begin{aligned} |x_m - x_n| &= |x_m - L + L - x_n| \\ &\leq |x_m - L| + |L - x_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, x_n is a Cauchy sequence.

($\Leftarrow$) Assume x_n is a Cauchy sequence.

- **Step 1:** Show that x_n is bounded.

  Since x_n is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < l$. It follows that for all $n \geq N$, we need to account for $x_1 \ldots, x_{n-1}$. Thus, let $M = \max\{|x_1|, |x_2|, \ldots, |x_{n-1}|, |x_n| + 1\}$. Then for all $n \in \mathbb{N}$, $|x_m| < M$.

- **Step 2:** Since x_n is bounded, there exits a convergent subsequence x_{n_k} by the Bolzano-Weierstrass Theorem. Let L be the limit of the subsequence.

- **Step 3:** Show that x_n converges to L.

  If some get close to L and all get close to each other, they all get close to L. Let $\epsilon > 0$. Since x_{n_k} converges to L, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $|x_{n_k} - L| < \frac{\epsilon}{2}$. Since x_n is Cauchy, there exists $M \in \mathbb{N}$ such that for all $n, m \geq M$, $|x_n - x_m| < \frac{\epsilon}{2}$. Let $M_0 = \max\{N_1, n_k\}$. By the Archimedean Principle, there exists N_0 such that $n_{k_0} \geq M_0$. Then, from the Triangle Inequality, we say that for all $n \geq N_0$,

$$\begin{aligned} |x_n - L| &\leq \left|x_n - x_{n_{k_0}}\right| + \left|x_{n_{k_0}} - L\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

  Therefore, $(x_n) \to L$.

By proving both directions of the inequality, we found that a sequence (x_n) converges if, and only if, it is a Cauchy sequence. $\square$

Definition 2.6.3

A sequence is called *contracting* is there exists $0 < C < 1$ such that for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq C\,|x_n - x_{n-1}|$.

How this works: we take a sequence $a_1, a_2, \ldots$ and subtract $a_1 - a_2$. Then, we have the inequality:

$$\begin{aligned}
|a_2 - a_1| &\leq C\,|a_1 - a_0| \\
|a_3 - a_2| &\leq C\,|a_2 - a_1| \leq C^2\,|a_1 - a_2| \\
|a_4 - a_3| &\leq C\,|a_3 - a_2| \leq C^3\,|a_1 - a_2| \\
&\vdots
\end{aligned}$$

From this, a theorem emerges:

Theorem 2.6.4

If a sequence is contracting, then it is Cauchy, and thus converges.

Proof. Let (a_n) be a contracting sequence; that is, there exists a constant $0 < C < 1$ such that for all $n \in \mathbb{N}$,

$$|a_{n+1} - a_n| \leq C|a_n - a_{n-1}|.$$

We will show that (a_n) is a Cauchy sequence.

First, we observe by induction that for all $k \geq 1$,

$$|a_{n+k} - a_{n+k-1}| \leq C^k|a_n - a_{n-1}|.$$

Proof by induction:

Base case ($k = 1$)**:**

$$|a_{n+1} - a_n| \leq C|a_n - a_{n-1}|.$$

Inductive step: Assume that for some $k \geq 1$,

$$|a_{n+k} - a_{n+k-1}| \leq C^k|a_n - a_{n-1}|.$$

Then,

$$\begin{aligned}
|a_{n+k+1} - a_{n+k}| &\leq C|a_{n+k} - a_{n+k-1}| \\
&\leq C\left(C^k|a_n - a_{n-1}|\right) \\
&= C^{k+1}|a_n - a_{n-1}|.
\end{aligned}$$

Thus, the inequality holds for $k + 1$, completing the induction.

Next, for any integers $m > n$, we have:

$$|a_m - a_n| = \left|\sum_{j=n}^{m-1}(a_{j+1} - a_j)\right| \leq \sum_{j=n}^{m-1}|a_{j+1} - a_j|.$$

Applying the inequality obtained from the induction,

$$|a_{j+1} - a_j| \leq C^{j-n+1}|a_n - a_{n-1}|.$$

Therefore, Since $C^{m-n} \geq 0$, we have:

$$|a_m - a_n| \leq |a_n - a_{n-1}|\left(\frac{C}{1-C}\right).$$

As $n \to \infty$, the term $|a_n - a_{n-1}|$ tends to zero because:

$$|a_n - a_{n-1}| \leq C^{n-1}|a_1 - a_0| \to 0 \quad \text{as } n \to \infty.$$

Therefore, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - a_{n-1}| < \epsilon\left(\frac{1-C}{C}\right).$$

Then, for all $m, n \geq N$ (with $m > n$),

$$|a_m - a_n| \leq |a_n - a_{n-1}|\left(\frac{C}{1-C}\right) < \epsilon.$$

This shows that (a_n) is a Cauchy sequence. Since every Cauchy sequence in $\mathbb{R}$ converges, the sequence (a_n) converges. □

CHAPTER 3

BASIC TOPOLOGY OF REAL NUMBERS

3.1 Discussion: The Cantor Set

We will build this set through an iterative process. Start with a number line C_0 that stretches from 0 to 1. Remove the middle third of the interval, leaving two intervals of length $\frac{1}{3}$. We will call the set of points removed from C_0 C_1. Next, remove the middle third of each of the two intervals, leaving four intervals of length $\frac{1}{9}$. We will call the set of points removed from C_1 C_2. Continue this process indefinitely.

Definition 3.1.1

The *Cantor set*, C, is defined as $C = \bigcap_{n=0}^{\infty} C_n$. This set is

1. non-empty. All end points stay within the interval.
2. uncountable.

The second part of that definition is a bit tricky to prove, but a visual will do for now. We can put all elements of the Cantor set in a one-to-one correspondence with the set of all 0s and 1s. This shows that not only is it uncountable, but it also has the same cardinality as $[0, 1]$.

The total length of removed elements, $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots = \frac{1}{3}(1 + \frac{2}{3} + \frac{4}{9})$. Notice the resemblance to the geometric series? We can write this as

$$\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1.$$

In summary,

1. Start with intercal $[0, 1]$.
2. Remove countably disjoint intervals.
3. Uncountably many points between these intervals, all isolated from each other.
4. The space taken up by the leftover points has "length" 0.

When we review the properties of a fractal, we see that the Cantor set is a fractal. It is self-similar, and the dimension of the Cantor set is $\log_3 2$.

For a cool look at the cantor set as a fractal, check out https://en.wikipedia.org/wiki/File:Cantor_Set_Expansion.gif.

3.2 Open and closed Sets

Definition 3.2.1

For a point $x \in \mathbb{R}$, and $\epsilon > 0$, we define the *epsilon-neighborhood* of x to be $V_\epsilon(x) = \{y \in \mathbb{R} \mid |x - y| < \epsilon\}$.

In other words, $V_\epsilon(x)$ is the open interval $(x - \epsilon, x + \epsilon)$, centered at x with radius ϵ.

3.2.1 Open Sets

Definition 3.2.2

A set $A \subseteq \mathbb{R}$, is called an *open set* if for every $x \in A$, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq A$.

Some Examples of Open Sets

- All open intervals are also open sets.
- $\mathbb{R}$ is open.
- $\emptyset$ is open.
- $\{1\}$ is not open.
- $[0, 2]$ is not open.
- $\mathbb{Q}$ is not open.
- $[4, 6)$ is not open.
- $(0, 1) \cup (1, 3) \cup (5, 10)$ is open.
- $(0, 3] \cap [2, 4)$ is open.
- Cantor set is not open.

Theorem 3.2.3

(i) The union of an arbitrary collection of open sets is open.

(ii) The intersection of a finite collection of open sets is open.

Proof. (i) Let $\{O_\lambda \colon \lambda \in A\}$ be a collection of open sets. Then, let $O = \bigcup_{\lambda \in A} O_\lambda$. Let a be an element of O. To show that O is open, we need to find an ϵ-neighborhood that is completely contained within O to satisfy Definition 3.2.1. But $a \in O$ implies that a is an element of at least one particular $O_{\lambda'}$. Because we are assuming $O_{\lambda'}$ to be open, then we can use Definition 3.2.1 to assert that there exists $V_\epsilon(a) \subseteq O_{\lambda'}$. The fact that $O_{\lambda'} \subseteq O$ confirms that $V_\epsilon(a) \subseteq O$.

(ii) Let $\{O_1, O_2, \ldots, O_N\}$ be a finite collection of open sets. Then, let $a \in \bigcap_{k=1}^{N} O_k$. This means a is an element of every open set. Definition 3.2.1 tells us that for $1 \leq k \leq K$, there exists an $V_\epsilon(a) \subseteq O_k$. From this set, we are in search of one ϵ-neighborhood of a that is contained in every O_k, so the trick is to pick the smallest one. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$, it follows that $V_\epsilon(a) \subseteq V_{\epsilon_k}(a)$ for all k, and hence $V_\epsilon(a) \subseteq \bigcap_{k=1}^{N} O_k$.

□

Note that we cannot use this for cases with infinity. For example, consider $A_n = (-\frac{1}{n}, \frac{1}{n})$. This is open, but $\bigcap_{n=1}^{\infty} A_n = \{0\}$, which is not open.

3.2.2 Closed Sets

Definition 3.2.4

Let $A \subseteq \mathbb{R}$. We say x is a *limit point* of A if for all $\epsilon > 0$, there exists $a \in A$ such that $a \in V_\epsilon(x)$ that is not x. Additionally, a point $x \in \mathbb{R}$ is a *limit point* if, and only if, there exists a sequence (a_n) of points from A that are not x. And $\lim_{n\to\infty}(a_n) = x$.

Definition 3.2.5

A set $B \subseteq \mathbb{R}$ is called a *closed set* if B contains all its limit points.

Important note: Limit points could be outside a set. Consider $(0, 1)$. Even though 0 and 1 do not belong to the set, they are considered limit points that are outside the set.

Some Examples of Closed Sets

- $[0, 1]$ is closed.
- $(0, 1)$ is not closed.
- $\mathbb{R}$ is closed.
- $\emptyset$ is closed.
- $\mathbb{Q}$ is not closed.
- $[3, \infty)$ is closed.
- $\frac{1}{n} \mid n \in \mathbb{N}$ not closed. (Because of 0)
- $[1, 4] \cup \{8\}$ is closed.
- $\{1\}$ is closed.
- $[1, 2)$ is not closed. Note that this set is neither open or closed.

Theorem 3.2.6

A set $B \subseteq \mathbb{R}$ is closed if, and only if, its complement is open. Similarly, a set $A \subseteq \mathbb{R}$ is open if, and only if, its complement is closed.

Proof. We will show this by proving both implications:

($\Rightarrow$) Assume $B \subseteq \mathbb{R}$ is a closed set. We will show that B^c is open. Let $x \in B^c$. So, $x \notin B$. This means x is not a limit point. (From the negated definition of limit point:) There must exist $\epsilon > 0$ such that no elements of B belong to $V_\epsilon(x)$. Then, $V_\epsilon(x) \subseteq B^c$. Therefore, B^c is open.

($\Leftarrow$) Assume B^c is open. We will show that B is closed. Let x be a limit point of B. For all $\epsilon > 0$, there exists a $b \in B$ such that $b \subseteq V_\epsilon(x)$. So, $V_\epsilon(x)$ is not a subset of B^c. This is true for every ϵ. Since B^c is open, it must be that $x \notin B^c$. Thus, $x \in B$. So, B contains all its limit points. Therefore, B is closed. □

Definition 3.2.7

Let $A \subseteq \mathbb{R}$ and let L be the set of limit points of A. The *closure* of A is defined as $\bar{A} = A \cup L$.

Theorem 3.2.8

(i) The intersection of any collection of closed sets is closed.

(ii) The union of finitely many closed sets is closed.

Proof. De Morgan's Laws state that for any collection of sets $\{E_\lambda \colon \lambda \in A\}$ it is true that

$$\left(\bigcup_{\lambda \in A} E_\lambda\right)^c = \bigcap_{\lambda \in A} E_\lambda^c \qquad \text{and} \qquad \left(\bigcap_{\lambda \in A} E_\lambda\right)^c = \bigcup_{\lambda \in A} E_\lambda^c.$$

The result follows directly from these statements and Theorem 3.2.3. □

Theorem 3.2.9

The closure of a set is a closed set.

Note: This theorem may seem trivial, but it answers the question of "Are there limit points in L that are not accounted for?"

Proof. We need to show that $\bar{A}$ contains all the limit points of $\bar{A}$. Let L be the limit points of A. Thus, $\bar{A} = A \cup L$. Let x be a limit point of $\bar{A}$. There exists a sequence of points (x_n) coming from $\bar{A}$ such that $(x_n) \to x$. Then, for all $n \in \mathbb{N}$, either $x_n \in A$ or $x_n \in L$.

- **Case 1:** $x_n \in A$

There exists a subsequence (x_{n_k}) where each $x_{n_k} \in A$. This subsequence also converges to x, and we know the limit belongs to L, so $x \in L \subseteq \bar{A}$.

- **Case 2:** $x_n \in L$
  x_n belongs to A for only finitely many $n \in \mathbb{N}$. Thus, a tail-end of the sequence is comprised entirely of points from L. To simplify things, we will assume the entire sequence (x_n) comes from L. (We know that (x_n) converges to x, but we cannot assume those limit points converge as well.) Let $n \in \mathbb{N}$. Since $x_n \in L$, there exists $a_n \in A$ such that $|x_n - a_n| < \frac{1}{n}$. We now have $(x_n) \to x$ and $(x_n - a_n) \to 0$. Then, $(a_n) \to x$. Thus, $x \in L \subseteq \bar{A}$.

Now that we have shown that either cases leads to the same conclusion, we know that $x \in L \subseteq \bar{A}$, and therefore $\bar{A}$ contains all its limit points. $\square$

Theorem 3.2.10

The closure set $\overline{A}$ is the *smallest* closed set containing A. (Where "smallest" refers to a subset of any other closed set containing A.)

Proof. If B is a closed set containing A, then $A \subseteq B$ and $L \subseteq B$. Thus, $\bar{A} = A \cup L \subseteq B$. $\square$

Example 3.1: Closed Sets 1

Generate countably many closed sets where the union is not closed.

Solution. $B_n = [\frac{1}{n}, 1 - \frac{1}{n}]$. Therefore, $\bigcup_{n=3}^{\infty} B_n = (0,1)$. For example, that would look like: $\{\frac{1}{2}\} \cup \{\frac{1}{3}\} \cup \dots$

Example 3.2: Closed Sets 2

What is the closure of the following sets?

$$\text{(a) } (0,1), \quad \text{(b) } \mathbb{R}, \quad \text{(c) } \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}, \quad \text{(d) } [0,1) \cup (1,3], \quad \text{(e) } \mathbb{Q}$$

Solution. (a) $\bar{A} = [0,1]$, (b) $\bar{A} = \mathbb{R}$, (c) $\bar{A} = A \cup \{0\}$, (d) $\bar{A} = [0,3]$, (e) $\bar{A} = \mathbb{R}$.

3.3 Compact Sets

Definition 3.3.1

A set $K \subseteq \mathbb{R}$ is a *compact set* if every sequence from K has a convergent subsequence where the limit is also K.

Theorem 3.3.2

A set K is compact if, and only if, it is closed and bounded.

Proof. We will show this by proving both implications:

($\Rightarrow$) Assume a set $A \subseteq \mathbb{R}$ is closed and bounded. Thus, there exists a convergent subsequence by Bolzano-Weierstrass Theorem. Because A is closed, the limit is in the set by Definition 3.2.5.

($\Leftarrow$) Assume a set A is compact. If it is not bounded, then there exists an (a_n) that heads toward infinity. This contradicts Definition 3.3.1, so it must be bounded. Then, by the same definition, the limit points belong in the set, so it is closed. □

Definition 3.3.3

Let $A \in \mathbb{R}$. An *open cover* for A is a collection of open sets $\{O_\lambda \mid \lambda \in A\}$ whose union contains the set A; that is $A \subseteq \bigcup_{\lambda \in A} O_\lambda$. Given an open cover for A, a *finite subcover* is a finite sub-collection of open sets from the original open cover whose union still manages to completely contain A.

Theorem 3.3.4: Heine-Borel Theorem

Let K be a subset of $\mathbb{R}$. All the following statements are equivalent in the sense that any of them implies the two others:

(i) K is compact.

(ii) K is closed and bounded.

(iii) Every open cover of K has a finite subcover.

Proof. The first set of "if and only if proofs" will be to prove (i) and (ii) are equivalent. Then, we will prove (ii) and (iii) are equivalent.

($\Rightarrow$) Assume K is compact. We need to show that K is closed and bounded. To show K is bounded, consider the open cover $\mathcal{U} = \{(-n, n) \mid n \in \mathbb{N}\}$. $\mathcal{U}$ covers all of $\mathbb{R}$, so it certainly covers K. Thus, there must exist a finite subcover. Consider the longest interval in the subcover. Then, K is a subset of this interval, so K is bounded. To show K is closed, we need to show every limit point belongs to K. Assume x is a limit point of K. From Definition 3.2.4 for every $a \in K$, let $\epsilon_a = \frac{1}{2}|a - x|$. Consider the open cover $\mathcal{U} = \{V_{\epsilon_a}(a) \mid a \in K\}$. This covers every point on K.

($\Leftarrow$) Because it is closed and bounded, by Theorem 3.3.2, K is compact.

Now for the second part of the proof:

($\Rightarrow$) Let x be a limit point of K. This means there must exist a sequence (x_n) in K with $\lim_{n\to\infty} x_n = x$. Suppose $x \notin K$. For every $y \in K$. Let $\epsilon_y = \frac{1}{2}|y - x|$. Consider the open neighborhood $V_{\epsilon_y}(y)$. Notice $x \in V_{\epsilon_y}(y)$. Now, we will work with the collection of all such neighborhoods $\mathcal{U} = \{V_{\epsilon_y}(y) \mid y \in K\}$. This $\mathcal{U}$ is an open cover of K. By our hypothesis there exists a finite subcover. There are some $y_1, y_2, \ldots, y_m$ such that $K \subset \bigcup_{i=1}^{m} V_{\epsilon_y}(y_i)$. Look at the distance from x to each y_i: $(x - \epsilon_{y_i},\, x + \epsilon_{y_i}) \cap V_{\epsilon_{y_i}}(y_i) = \emptyset$. Similar statements are for every y_i. Let $\epsilon = \min\{\epsilon_{y_1}, \epsilon_{y_2}, \ldots \epsilon_{y_M}\}$. Since there are infinitely many $\epsilon > 0$, we see that $V_\epsilon(x) \cap V_{\epsilon_y}(y_i) = \emptyset$ for every $i \leq M$. So $V_\epsilon(x) \cap K = \emptyset$. This gives us an ϵ-neighborhood around x that does not intersect K. Since (x_n) approaches x, there must be elements from the sequence that are inside of $V_\epsilon(x)$. This creates a contradiction because we said x was a limit point. Therefore $x \in u$ and K must be closed.

($\Leftarrow$) Let $\mathcal{U}$ be an open cover of K. Suppose there is no finite subcover. Since K is bounded there exists a closed interval I_0 that contains K. Bisect I_0 and look at the two sub intervals A and B. My claim is at least one of $A \cap K$ and $B \cap K$ does not have a finite subcover from $\mathcal{U}$. If not, then we would have a finite subcover of all of K. Whichever half does not have a finites of cover will be called I_1. Repeat this process. We get a sequence of nested closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ such that for all $j \in \mathbb{N}$, $I_j \cap K$ does not have a finite subcover from $\mathcal{U}$. Also, as the length of I_j approach 0, by the Nested Interval Property, there exists $x \in \bigcap_{j=1}^{\infty} I_j$. Since each I_j contains an element of K and the interval approaches 0, x must be a limit point of K. Thus, since K is closed, $x \in K$. There must be an open set $U \in \mathcal{U}$ such that $x \in U$. Since U is open and $x \in U$, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq U$. There is an I_j whose length is smaller than ϵ. This means $I_j \subseteq V_\epsilon \subset U$. So $\{U\}$ is a finite subcover of $I_j \cap K$. This contradicts how we defined I_j, therefore there must be a finite subcover from $\mathcal{U}$. $\square$

This allows us to take an infinite amount of ϵ-neighborhoods and turn them into finite subcovers.

Example 3.3: Compactness

Let $A = (0, 1)$. Construct a set $\mathcal{U}$ that is an open cover of $(0, 1)$, but does not have a finite subcover.

Solution. Consider $\mathcal{U} = \{(0, t) \mid 0 < t < 1\}$. Thus, $\mathcal{U}$ is an open cover, but does not contain a finite amount of subcovers because there will always be a point not covered.

CHAPTER 4

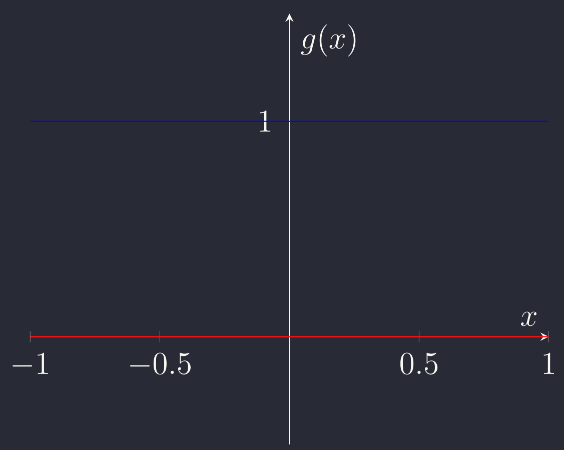
FUNCTIONAL LIMITS AND CONTINUITY

4.1 Discussion: Examples of Dirichlet and Thomae

Definition 4.1.1

The *Dirichlet function* $\lim_{x\to c} g(x)$ does not exist for any $c \in \mathbb{R}$.

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$



Definition 4.1.2

The *Thomae function* is defined as

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is in lowest terms}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Thomae's function, $t(x)$ is continuous at all $x \notin \mathbb{Q}$. It is not continuous at any $x \in \mathbb{Q}$.

4.2 Functional Limits

Recall from calculus I, that a function $f(x)$ is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$.

Definition 4.2.1

Let $f\colon A \to \mathbb{R}$ be a function and let c be a limit point of A. We say $\lim_{x\to c} f(x) = L$, if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Example 4.1: Functional Limit (From book) 1

Let $f(x) = 3x + 1$. Claim: $\lim_{x\to 2} f(x) = 7$.

Proof. Let $\epsilon > 0$. After we have done our scratch work, we can choose $\delta = \epsilon/3$, then $0 < |x - 2| < \delta$ implies $|f(x) - 7| < 3(\epsilon/3) = 3$. □

Scratch Paper. Definition 4.2.1 requires that we produce a $\delta > 0$ so that $0 < |x - 2| < \delta$ leads to the conclusion that $|f(x) - 7| < \epsilon$. Notice that

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3\,|x - 2|\,.$$

Example 4.2: Functional Limit (From book) 2

Let $g(x) = x^2$. Claim: $\lim_{x\to 2} g(x) = 4$.

Proof. Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/5\}$. If $0 < |x - 2| < \delta$, then

$$\begin{aligned} |g(x) - 4| &= \left|x^2 - 4\right| \\ &= |x - 2|\,|x + 2| \\ &< 5\delta \\ &= (5)\frac{\epsilon}{5} \\ &= \epsilon. \end{aligned}$$

□

Scratch Paper. Our goal this time is to make $|g(x) - 4| < \epsilon$ by restricting $|x - 2|$ to be smaller than some carefully chosen δ. As in the previous example, a little algebra reveals

$$|g(x) - 4| = \left|x^2 - 4\right| = |x - 2|\,|x + 2|\,.$$

We can make $|x + 2|$ as small as we like, but we need an upper bound on $|x + 2|$ in order to know how small to choose δ. The presence of the variable x causes some initial confusion, but keep in mind that we are discussing the limt as x approaches 2. If we agree that our $\delta-$neighborhood around $c = 2$ must have radius no bigger than $\delta = 1$, then we get the upper bound $|x + 2| < |3 + 2| = 5$ for all $x \in V_\delta(c)$.

Example 4.3: Functional Limit 1

Let $f(x) = 3x + 1$. Show that $\lim_{x\to 2} f(x) = 7$.

Proof. Let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{3}$. Assume $0 < |x - 2| < \delta$. Since $\delta > 0$, $2 - \delta < x < 2 + \delta$. Then,

$$\begin{aligned} |x-2| &< \delta, \\ |f(x) - 7| &= |3x + 1 - 7| \\ &= |3x - 6| \\ &= 3\,|x - 2| \\ &< 3\delta \\ &= \epsilon. \end{aligned}$$

Therefore, $\lim_{x\to 2} f(x) = 7$. □

Example 4.4: Functional Limit 3

Let $f(x) = x^2$. Claim: $\lim_{x\to 7} f(x) = 49$

Proof. Let $\epsilon > 0$. Set $\delta = \min\{\frac{\epsilon}{8}, 1\}$. If $0 < |x - 7| < \delta$, then

$$\begin{aligned} |f(x) - 49| &= \left|x^2 - 49\right| \\ &= |x - 7|\,|x + 7| \\ &< 8\delta \\ &< 8\left(\frac{\epsilon}{8}\right) \\ &= \epsilon. \end{aligned}$$

□

Scratch Paper. Always start with the goal statement: $|f(x) - 49| = |x^2 - 49|$. This factors into $|x - 7|\,|x + 7|$. Then, if $\delta < 1$, $|x - 7| < \delta$ and $|x + 7| < 8$. All together, we have $8\delta < \epsilon < \frac{\epsilon}{8}$.

Example 4.5: Functional Limit 4

Claim: $\lim_{x\to 3} \frac{1}{x+1} = \frac{1}{4}$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{12\epsilon, 1\}$. If $0 < |x-3| < \delta$, then

$$\begin{aligned}\left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{4-(x+1)}{4(x+1)}\right| \\ &= \left|\frac{3-x}{4(x+1)}\right| \\ &< \frac{\delta}{4(3)} \\ &= \frac{12\epsilon}{12} \\ &= \epsilon.\end{aligned}$$

Therefore, $\lim_{x\to 3} \frac{1}{x+1} = \frac{1}{4}$ □

Scratch Paper. Goal: $\left|\frac{1}{x+1} - \frac{1}{4}\right|$. Hence,

$$\begin{aligned}\left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{4-(x+1)}{4(x+1)}\right| \\ &= \left|\frac{3-x}{4(x+1)}\right| \\ &< \frac{\delta}{4\,|x+1|} \\ &< \frac{\delta}{4(3)} \\ &= \frac{\delta}{12} \\ &< \epsilon.\end{aligned}$$

Thus, we need a $\delta < 1$, and we can choose $\delta = \min\{12\epsilon, 1\}$. Note: When we are determining the value for $|x+2|$, we solve for $\delta = 3 \pm 1 \Rightarrow x \in (2, 4)$. Then, we find $x + 1 = (3, 5)$. We choose 3 rather than 5 because of division. We want to be as close as possible.

Example 4.6: Functional Limit 5

Claim: $\lim_{x\to 3}(x^2 + 7x) = 30$.

Proof. Let $\epsilon > 0$ and set $\delta = \min\{\frac{\epsilon}{14}, 1\}$. If $0 < |x-3| < \delta$, then

$$\begin{aligned}\left|x^2 + 7x - 30\right| &= |x-3|\,|x+10| \\ &< 14\delta \\ &= 14\left(\frac{\epsilon}{14}\right) \\ &= \epsilon.\end{aligned}$$

□

Example 4.7: Functional Limit 6

Claim: $\lim_{x\to 3} \frac{2x+3}{4x-9} = 3$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{\frac{\epsilon}{10}, \frac{1}{2}\}$. (Note: We are choosing $\frac{1}{2}$ because we want to avoid having 0 anywhere in the interval.) Assume $0 < |x-3| < \delta$. Since $\delta < \frac{1}{2}$, $\frac{5}{2} < x < \frac{7}{2}$, then $1 < |4x-9| < 5$. (Thus, 0 can not possibly be in the denominator.) $\square$

Scratch Paper.

$$\begin{aligned}\left|\frac{2x+3}{4x+9} - 3\right| &= \left|\frac{2x+3-3(4x+9)}{4x+9}\right| \\ &= \left|\frac{2x+3-12x-27}{4x+9}\right| \\ &= 10\left|\frac{x-3}{4x-4}\right| \\ &< 10\frac{\epsilon/10}{1} \\ &= \epsilon.\end{aligned}$$

Example 4.8: Functional Limit 7

Claim: $\lim_{x\to 4}\sqrt{x} = 2$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{1, 3\epsilon\}$. Assume $0 < |x-4| < \delta$. Then (refer to scratch work). $\square$

Scratch Paper.

$$\begin{aligned}\left|\sqrt{x}-2\right| &= \left|\sqrt{x}-2\right| \\ &= \left|\frac{(\sqrt{x}-2)\cdot(\sqrt{x}+2)}{\sqrt{x}+2}\right| \\ &= \left|\frac{x-4}{\sqrt{x}+2}\right| \\ &< \frac{\delta}{3} \\ &< \frac{3\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Notice that we picked $\delta < 1$ such that $3 < x < 4$ so $1 < \sqrt{x} < 2$ and $3 < \sqrt{x}+2 < 4$.

Theorem 4.2.2: Sequential Criterion for Functional Limits

The following statements are equivalent:

1. $\lim_{x\to c} f(x) = L$.
2. For all sequences (x_n) where $x_n \neq c$ and $(x_n) \to c$, we have $\lim_{n\to\infty} f(x_n) = L$.

Proof. (1) $\to$ (2)

Assume $\lim_{x\to c} f(x) = L$.
Let $(x_n) \to c$ with $x_n \neq c$
Let $\epsilon > 0$.

- Since $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.
- Since $x_n \to c$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - c| < \delta$.
- Now, for all $n \geq N$, it follows that $x_n - c < \delta$ and thus $|f(x) - L| < \epsilon$.

Thus, $\lim_{n\to\infty} f(x_n) = L$.

(2) $\to$ (1)

Proof by contrapositive.
Assume (1) is not true. Thus,

$$\lim_{x\to c} f(x) \neq L.$$

There exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists an x with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon_0$.
For each $n \in \mathbb{N}$, consider $\delta = \frac{1}{n}$. There exists $x_n \in (c - \frac{1}{n}, c + \frac{1}{n})$ with $x_n \neq c$ such that $|f(x) - L| \geq \epsilon_0$.

- Since $|x_n - c| < \frac{1}{n}$, we see that $(x_n) \to c$.
- Since for all $n \in \mathbb{N}$, $|f(x) - L| \geq \epsilon_0$. Then, $\lim_{n\to\infty} f(x) \neq L$.

Thus, $\neg(1) \to \neg(2)$. So (2) $\to$ (1) and (1) $\to$ (2). $\square$

If functional limits and sequential limits are the same thing, then everything we know about sequential limits is also true about functional limits.
Recall Algebraic Limit Theorem. From this, we can write the functional equivalent:
Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then,

- $\lim_{x\to c}(f(x) + g(x)) = L + M$
- $\lim_{x\to c}(f(x) - g(x)) = L - M$
- $\lim_{x\to c}(f(x)g(x)) = LM$
- $\lim_{x\to c}(\frac{f(x)}{g(x)}) = \frac{L}{M}$ unless $M = 0$.

Theorem 4.2.3: Divergence Criterion

Let $f\colon A \to \mathbb{R}$ with c as a limit point of A. If there exists two sequences (x_n) and (y_n) in $A \setminus \{c\}$ (that both converge to c) such that $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, then $\lim_{x\to c} f(x)$ does not exist.

Example 4.9: Divergence Criterion 1

$f(x) = \frac{x}{|x|} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$ Our goal is to show that $\lim_{x\to 0} f(x)$ does not exist.

Proof. Let $(x_n) = (\frac{1}{n})$ and let $(y_n) = (\frac{-1}{n})$. We will see that as $n \to \infty$, $\lim_{n\to\infty} f(x_n) = 1$ and $\lim_{n\to\infty} f(y_n) = -1$. Thus, $\lim_{x\to 0} f(x)$ does not exist. □

Example 4.10: Divergence Criterion 2

$g(x) = \sin(\frac{1}{x})$. Show that $\lim_{x\to 0} g(x)$ does not exist.

Proof. Let $(x_n) = (\frac{1}{2\pi n})$ and let $(y_n) = (\frac{1}{2\pi n + \frac{\pi}{2}})$. We will see that as $n \to \infty$, $\lim_{n\to\infty} g(x_n) = 1$ and $\lim_{n\to\infty} g(y_n) = -1$. Thus, $\lim_{x\to 0} g(x)$ does not exist. □

We say $\lim_{n\to\infty} x_n = \infty$ if for all $M > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

We say $\lim_{x\to c} f(x) = \infty$ if for all $M > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $f(x) > M$. Think of vertical asymptotes.

Theorem 4.2.4: Infinite Limits Cauchy Criterion

If $(x_n) \to \infty$, (x_n) will not be Cauchy. It is possible to have $x_{n+1} - x_n$ approach 0, but (x_n) converges to ∞.

4.3 Continuous Functions

Definition 4.3.1

We say a function f is *continuous* at c if

$$\lim_{x \to c} f(x) = f(c).$$

Equivalent definition:
For all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then

$$|f(x) - f(c)| < \epsilon.$$

Idea: When x is close to c, $f(x)$ is close to $f(c)$. Then, for the topological definition, we can say if $x \in V_\delta(c)$ then $f(x) \in V_\epsilon(f(c))$.

Definition 4.3.2

We say function f is *continuous on a set* D if f is continuous at every point in D.

The following are equivalent (TFAE):

1. $\lim_{x \to c} f(x) = L$
2. For all sequences (x_n) such that $(x_n) \to c$, we have $\lim_{n \to \infty} f(x_n) = L$.

Continuous Functions (THM 4.3.2 in book)

Claim: Let $a \in \mathbb{R}$. Then $f(x) = a$ is continuous.

Proof. Let $c \in \mathbb{R}$. Let $\epsilon > 0$. Set $\delta = 8$. Now, if $x - c < 8$, then $|f(x) - f(c)| = a - a = 0 < \epsilon$. Thus, constant functions are continuous. $\square$

Claim: $f(x) = x$ is continuous.

Proof. Let $c \in \mathbb{R}$. Let $\epsilon > 0$. Set $\delta = \epsilon$. If $|x - c| < \delta$, then $|f(x) - f(c)| = |x - c| < \delta = \epsilon$. Thus, the identity function is continuous. $\square$

Claim: $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Proof.

- **Case 1:** $c \neq 0$

  Let $c \in [0, \infty)$. Let $\epsilon > 0$. Set $\delta < \epsilon$. If $|x - c| < \delta$, then $|g(x) - g(c)| = |\sqrt{x} - \sqrt{c}| = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} < \frac{\delta}{\sqrt{c}} < \epsilon$. Thus, $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

- **Case 2:** $c = 0$

  Let $c \in [0, \infty)$. Let $\epsilon > 0$ Set $\delta = \epsilon^2$. If $|x - 0| < \delta$, then $|g(x) - g(0)| = |\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \epsilon$. Thus, $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Therefore, $g(x)$ is continuous for all $c \in [0, \infty)$. $\square$

Theorem 4.3.3: Compositions of Continuous Functions

Let f be continuous at c. Let g be continuous at $f(x)$. Then,

$$g \circ f(x) = g(f(x)) \text{ is continuous.}$$

Proof. Let $\epsilon > 0$. Since g is continuous at $f(c)$, there exists $\delta_1 > 0$ such that if $|x - f(c)| < \delta_1$, then $|g(x) - g(f(c))| < \epsilon$. Since f is continuous at c, there exists $\delta_2 > 0$ such that if $|x - c| < \delta_2$, then $|f(x) - f(c)| < \delta_1$. Thus, if $|x - c| < \delta_2$, then $|g(f(x)) - g(f(c))| < \epsilon$. Therefore, $g \circ f(x)$ is continuous. $\square$

$$\frac{\text{Most Common Applications of Continuity}}{\text{Is with limits.}}$$

If f is continuous at c and $(x_n) \to c$, then $\lim_{n\to\infty} f(x_n) = f(c)$. Hence.

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

4.4 Continuous Functions on Compact Sets

Theorem 4.4.1: Extreme Value Theorem

If K is compact and $f : K \to \mathbb{R}$ is continuous, then f attains a maximum and minimum value on K.
In other words, there exists $a \in K$ such that $f(a) = \sup\{f(x) \mid x \in K\}$. Also, there exists $b \in K$ such that $f(b) = \inf\{f(x) \mid x \in K\}$.

Proof. We know f is bounded on K from Lemma 4.4.2 below. Hence,

$$S = \sup\{f(x) \mid x \in K\} \text{ exists.}$$

For every natural number, there exists an $x_n \in K$ such that $S - \frac{1}{n} < f(x_n) \leq S$. It follows that $\lim_{n\to\infty} f(x_n) = S$. So, now we have a sequence, (x_n) in the compact set K. Since K is compact, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence,

$$(x_{n_j}) \text{ with } a = \lim_{j\to\infty} x_{n_j} \in K.$$

Since f is continuous, we have,

$$f(a) = \lim_{j\to\infty} f(x_{n_j}) = S.$$

By a similar method, there exists $b \in K$ such that

$$f(b) = \inf\{f(x) \mid x \in K\}.$$

Note: This proof hinges on the fact that f is bounded! We need to show that f is bounded on K with a proof with subcovers. $\square$

Lemma 4.4.2

How do we know f is bounded on K? That is,

$$f(K) = \{f(x) \mid x \in K\}.$$

Show that $f(K)$ is bounded.

Proof. Let $c \in K$. Since f is continuous at c, there exists $\delta_c > 0$ such that if $|x - c| < \delta_c$, then

$$|f(x) - f(c)| < 1.$$

Do this over every $c \in K$. We get an open cover of K.

$$\mathcal{O} = \{V_{\delta_c}(c) \mid c \in K\}.$$

Since K is compact, the Heine-Borel Theorem says there exists a finite subcover. We get $c_1, c_2 \ldots, c_n \in K$ such that $K \subseteq \bigcup_{i=1}^n V_{\delta_{c_i}}(c_i)$. Thus,

$$\begin{aligned} f(K) &\subseteq f\left(\bigcup_{i=1}^n V_{\delta_{c_i}}(c_i)\right) \\ &\subseteq \bigcup_{i=1}^n f(V_{\delta_{c_i}}(c_i)) \\ &\subseteq \bigcup_{i=1}^n (f(c_i) - 1, f(c_i) + 1) \\ &\subseteq [\min\{f(c_i)\} - 1, \max\{f(c_i)\} + 1]. \end{aligned}$$

Therefore, $f(K)$ is bounded. $\square$

Theorem 4.4.3: Preservation of Compact Sets

If K is compact and $f\colon K \to \mathbb{R}$ is continuous, then $f(K)$ is compact.

Proof. Let (y_n) be a sequence in $f(K)$. We will show (y_n) has a convergent subsequence with its limit in $f(K)$.

For each n there exists $x_n \in K$ such that $f(x_n) = y_n$. So (x_n) is a sequence in a compact set K. There exists a convergent subsequence (x_{n_j}) with

$$a = \lim_{j\to\infty} x_{n_j} \in K.$$

Now consider the corresponding subsequence (y_{n_j}) in $f(K)$. Since f is continuous, we have

$$\begin{aligned}\lim_{j\to\infty} y_{n_j} = \lim_{j\to\infty} f(x_{n_j}) \\ = f(a) \in f(K).\end{aligned}$$

So, x_{n_j} is a convergent subsequence with limit in $f(K)$. Therefore, $f(K)$ is compact. $\square$

Definition 4.4.4

A function $f\colon A \to \mathbb{R}$ is *uniformly continuous* on A if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, c \in A$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Compare this definition with Definition 4.3.1. The difference is that the δ is independent of x. That is, we have to find one δ that needs to work for every point x.

Definition 4.4.5

A function $f\colon A \to \mathbb{R}$ is <u>not</u> *uniformly continuous* on A if there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists $x, c \in A$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon_0$.

Theorem 4.4.6

If $K \subseteq \mathbb{R}$ is compact and $f\colon K \to \mathbb{R}$ is continuous, then f is uniformly continuous on K.

Proof. Suppose f is not uniformly continuous on K. Then, there exists $\epsilon_0 > 0$ such that for all $n \in \mathbb{N}$, there exists $x_n, y_n \in K$ such that $\left|x_n - y_n < \frac{1}{n}\right|$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$.

We now have two sequences (x_n) and (y_n) in K. Since K is compact, by the heine-borel theorem, there exists a convergent subsequence (x_{n_i}) which converges to a point $x_0 \in K$.

Since K is compact, (y_{n_i}) has a convergent subsequence $\left(y_{n_{i_j}}\right)$ which converges to a point $y_0 \in K$. Notice that since $\left(x_{n_{i_j}}\right)$ is a subsequence of (x_{n_i}), it converges to x_0. Since f is continuous:

$$\lim_{j\to\infty} f\left(x_{n_{i_j}}\right) = f(x_0) \quad \text{and} \quad \lim_{j\to\infty} f\left(y_{n_{i_j}}\right) = f(y_0).$$

Because

$$\left|f\left(x_{n_{i_j}}\right) - f\left(y_{n_{i_j}}\right)\right| < \frac{1}{n_{i_j}},$$

we can see that

$$\lim_{j\to\infty}\left|x_{n_{i_j}} - y_{n_{i_j}}\right| = 0.$$

It follows that $x_0 = y_0$. But this is a contradiction because $|f(x_0) - f(y_0)| \geq \epsilon_0$. Therefore, f is uniformly continuous on K. □

4.5 The Intermediate Value Theorem

Theorem 4.5.1: Intermediate Value Theorem

Let $f\colon [a,b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there exists $c \in (a,b)$ such that $f(c) = L$.

Note: IVT does not guarantee where, or how many c's are in the interval. It only guarantees that at least one c exists.

Proof. Without the loss of generality, assume $f(a) < f(b)$ and let $y \in (f(a), f(b))$. Let $I_1 = [a_1, b_1]$. Bisect I_1 into two intervals $[a_1, d]$ and d, b_1 where $d = \frac{a_1+b_1}{2}$.

- If $f(d) < y$, set $a_2 = d$, $b_2 = b_1$, and $I_2 = [a_2, b_2]$. Notice that $f(a_2) < y < f(b_2)$.
- If $f(d) > y$, then set $a_2 = a_1$, $b_2 = d$, and $I_2 = [a_2, b_2]$. Notice that $f(a_2) < y < f(b_2)$.

Repeat this process indefinitely. We end up with a sequence of nested intervals $I_n = [a_n, b_n]$, where

- $I_n \subseteq I_{n-1}$
- $f(a_n) < y < f(b_n)$
- $|a_n - b_n| = \dfrac{a_n - b_n}{2^{n-1}}$

By the Nested Interval Property, there exists a point c such that $c \in \bigcap_{n=1}^{\infty} I_n$. In fact, there is a unique point c in the intersection. It follows that

$$c = \lim_{n\to\infty} a_n \quad \text{and} \quad c = \lim_{n\to\infty} b_n.$$

Since f is continuous, we have

$$f(c) = \lim_{n\to\infty} f(a_n) \leq y \qquad \text{and} \qquad f(c) = \lim_{n\to\infty} f(b_n) \geq y.$$

Therefore, $f(c) = y$. □

What Is Important About Continuous Functions?

If $\lim_{n\to\infty}(x_n = x)$, then $\lim_{n\to\infty} f(x_{n)} = f(x)$.
Think about $f(x) = 2^x$. Thus, $f(x)$ makes sense if $x \in \mathbb{Q}$:

$$2^{\frac{p}{q}} = \sqrt[q]{2^p}.$$

But how do we make sense of something like 2^π?
We can find $f\colon \mathbb{Q} \to \mathbb{R}$ is continuous.
We can define $f\colon \mathbb{R} \to \mathbb{R}$ to be continuous
If (q_n) is in $\mathbb{Q}$ and $(a_n \to \pi)$, then we define

$$f(\pi) = \lim_{n\to\infty} f(q_n).$$

CHAPTER 5

THE DERIVATIVE

5.2 Derivates and the Intermediate Value Property

Definition 5.2.1

Let $g\colon A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, we define the *derivative* of g at c to be

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

provided this limit exists. In this case, we say g is *differentiable* at c. If g' exists for all points $c \in A$, then we say g is *differentiable on* A.

Example 5.1: Differentiation 1

Let $g(x) = x^2$. Use Definition 5.2.1 to find $g'(c)$.

Solution.

$$\begin{aligned}
g'(c) &= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= \lim_{x \to c} \frac{x^2 - c^2}{x - c} \\
&= \lim_{x \to c} \frac{(x - c)(x + c)}{x - c} \\
&= \lim_{x \to c} x + c \\
&= 2c
\end{aligned}$$

Example 5.2: Differentiation 2

Let $g(x) = x^3$. Use Definition 5.2.1 to find $g'(c)$.

Solution.

$$\begin{aligned} g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= \lim_{x\to c} \frac{x^3-c^3}{x-c} \\ &= \lim_{x\to c} \frac{(x-c)(x^2+xc+c^2)}{x-c} \\ &= \lim_{x\to c} x^2+xc+c^2 \\ &= 3c^2 \end{aligned}$$

Example 5.3: Differentiation 3

Let $g(x) = x^4$. Use Definition 5.2.1 to find $g'(c)$.

Solution.

$$\begin{aligned} g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= \lim_{x\to c} \frac{x^4-c^4}{x-c} \\ &= \lim_{x\to c} \frac{(x-c)(x^3+x^2c+xc^2+c^3)}{x-c} \\ &= \lim_{x\to c} x^3+x^2c+xc^2+c^3 \\ &= 4c^3 \end{aligned}$$

Theorem 5.2.2: Power Rule

For any $n \in \mathbb{N}$, if $f(x) = x^n$, then $f'(c) = nc^{n-1}$.

Example 5.4: Differentiation 4

Let $f(x) = |x|$. Use Definition 5.2.1 to find $f'(0)$.

Solution.

$$\lim_{x\to 0} \frac{|x|-|0|}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$$

When we view this from the left and right definitions of the limit, we see that the limit does not exist:

$$\lim_{x\to 0^+} \frac{|x|}{x} = 1 \quad \text{and} \quad \lim_{x\to 0^-} \frac{|x|}{x} = -1.$$

Hence, because these limits are not equal, the limit does not exist, and $f(x) = |x|$ is not differentiable at 0.

Example 5.5: Differentiation 5

Let $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$

Solution. Is this differentiable at 0?

$$\lim_{x\to 0^+} \frac{f(x)}{x} = \infty \quad \text{and} \quad \lim_{x\to 0^-} \frac{f(x)}{x} = 0.$$

Hence, because these limits are not equal, the limit does not exist, and $f(x)$ is not differentiable at 0.

Theorem 5.2.3: Algebraic Differentiation Rules

Let f and g be differentiable at c with $k \in \mathbb{R}$. Then the following functions are differentiable at c:

1. $(f+g)'(c) = f'(c) + g'(c)$
   $\Rightarrow \lim_{x\to c} \frac{f(x)+g(x)-f(c)-g(c)}{x-c}$
2. $(kf)'(c) = kf'(c)$
3. $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$
4. $(f/g)'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{g^2(c)}$

Proof. For (3), we have

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)g(x) - f(c)g(c)}{x-c} &= \lim_{x\to c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x-c} \\
&= \lim_{x\to c} \frac{f(x)(g(x)-g(c)) + g(c)(f(x)-f(c))}{x-c} \\
&= \lim_{x\to c} f(x) \lim_{x\to c} \frac{g(x)-g(c)}{x-c} + g(c) \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \\
&= f(c)g'(c) + g(c)f'(c).
\end{aligned}$$

For (4), we have

$$\begin{aligned}
\lim_{x\to c}\frac{\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}}{x-c} &= \lim_{x\to c}\frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)(x-c)} \\
&= \lim_{x\to c}\frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)} \\
&= \lim_{x\to c}\frac{f(x)-f(c)}{g(x)g(c)(x-c)}+\lim_{x\to c}\frac{f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)} \\
&= \frac{f'(c)g(c)-f(c)g'(c)}{g^2(c)}.
\end{aligned}$$

□

Theorem 5.2.4: Chain Rule

Let f and g be differentiable at c. Then the composition $g\circ f$ is differentiable at c and $(g\circ f)'(c)=g'(f(c))f'(c)$.

Proof.

$$\begin{aligned}
\lim_{x\to c}\frac{g(f(x))-g(f(c))}{x-c} &= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\cdot\frac{f(x)-f(c)}{x-c} \\
&= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \\
&= g'(f(c))f'(c).
\end{aligned}$$

□

Theorem 5.2.5: Interior Extremum Theorem

If f is differentiable at c, and c is a local maximum of f, then $f'(c)=0$.

Proof. We know that $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$. Then,

$$f'(x)=\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}\geq 0,$$

and

$$f'(x)=\lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}\leq 0,$$

so, $f'(c)=0$. □

5.3 The Mean Value Theorems

Theorem 5.3.1: Rolle's Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Since f is continuous on $[a, b]$, the Extreme Value Theorem says f achieves a maximum and minimum in $[a, b]$.

- If either maximum or minimum is in the interior, then the Interior Extremum Theorem says that $f'(c) = 0$.
- If the maximum or minimum is at the endpoints, then f is constant and $f'(c) = 0$ for all $[a, b]$. □

Theorem 5.3.2: Mean Value Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This is saying that there exists a point where the slope of the tangent line is equal to the slope of the secant line. In other words, there is a point where $f'(c)$ happens.

Proof. Let $d(x) = f(x) - [f(a) + \frac{f(b)-f(a)}{b-a}(x - a)]$. Notice $d(x)$ is continuous on $[a, b]$ and differentiable on (a, b). Thus,

$$d(a) = f(a) - (f(a) + 0) = 0 \qquad \text{and} \qquad d(b) = f(b) - (f(a) + f(b) - f(a)) = 0.$$

So, Rolle's Theorem applies to $d(x)$ and there exists a $c \in (a, b)$ where $d'(c) = 0$. Notice

$$d'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

so

$$\begin{aligned} 0 &= f'(c) - \frac{f(b) - f(a)}{b - a} \\ f'(c) &= \frac{f(b) - f(a)}{b - a}. \end{aligned}$$

□

Theorem 5.3.3: Generalized Mean Value

Let f and g be continuous on $[a,b]$ and differentiable on (a,b). There exists $c \in (a,b)$ such that

$$(f(b)-f(a))g(x)-(g(b)-g(a))f(x).$$

Proof. Let $d(x)=(f(b)-f(a))g(x)-(g(b)-g(a))f(x)$. Notice $d(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Thus,

$$\begin{aligned} d(a) &= (f(b)-f(a))g(a)-(g(b)-g(a))f(a) \\ d(b) &= (f(b)-f(a))g(b)-(g(b)-g(a))f(b) \end{aligned}$$

We can show $d(a)=d(b)$, so Rolle's Theorem applies to $d(x)$ and there exists a $c \in (a,b)$ where $d'(c)=0$. Therefore,

$$0 = d'(c) = (f(b)-f(a))g'(c)-(g(b)-g(a))f'(c)$$

□

Theorem 5.3.4: L'Hôspital's Rule

($\frac{0}{0}$ case) Let f and g be continuous on an interval around $a \in \mathbb{R}$. Assume f and g are differentiable on this interval (except possibly at a). Assume $f(a)=g(a)=0$. Also, $g'(x) \neq 0$. If all conditions are met, then the following conditional is valid:

$$\text{If} \quad \lim_{x\to a} \frac{f'(x)}{g'(x)} \quad \text{exists, then} \quad \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

Proof. Choose a number b that is close to a. The Generalized Mean Value Theorem says there exists c between a and b where:

$$\begin{aligned} (f(b)-f(a))g'(c) &= (g(b)-g(a))f'(c) \\ f(b)g'(c) &= g(b)f'(c) \\ \frac{f(b)}{g(b)} &= \frac{f'(c)}{g'(c)}, \end{aligned}$$

because $f(a)=g(a)=0$.

Now, suppose $\lim_{x\to a}\frac{f'(x)}{g'(x)} = L$ exists. Consider any sequence $(b_n) \to a$. By the Generalized Mean Value Theorem, for each b_n there exists c_n between a and b_n such that:

$$\frac{f(b_n)}{g(b_n)} = \frac{f'(c_n)}{g'(c_n)}$$

Since $a < c_n < b_n$ and $b_n \to a$, we have $c_n \to a$ as well. Therefore:

$$\lim_{n\to\infty} \frac{f(b_n)}{g(b_n)} = \lim_{n\to\infty} \frac{f'(c_n)}{g'(c_n)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

Since this holds for any sequence $(b_n) \to a$, we conclude:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

□

Theorem 5.3.5: Darbaux's Theorem

If f is differentiable on $[a, b]$ and $f'(a) < \alpha < f'(b)$, then there exists $c \in (a, b)$ such that $f'(c) = \alpha$.

Exercise: 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$

(d) $A \cap (B \cap C) = (A \cap B) \cap C$

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution.

(a) This is false. Consider the following as a counterexample: If we define A_1 as $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$, we can see why the intersection of these sets of infinite numbers are actually empty. Consider a number m that actually satisfies $m \in A_n$ for every A_n in our collection of sets. Because m is not an element of A_{m+1}, no such m exists and the intersection is empty.

(b) This is true.

(c) False. Consider sets $A = \{1, 2, 3\}$, $B = \{3, 6, 7\}$ and $C = \{5\}$. Note that $A \cap (B \cup C) = \{3\}$ is not equal to $(A \cap B) \cup C = \{3, 5\}$.

(d) This is true.

(e) This is true.

Exercise: 1.2.5 (De Morgan's Laws)

(This definition may prove useful.) Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution.

(a) If $x \in (A \cap B)^c$, and we know that $A^c = \{x \in \mathbb{R} : x \notin A\}$, then we know x must cannot exist in A^c and B^c because $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$. Thus, x is in either A^c or B^c. Put another way $x \in A^c \cup B^c$. Since we have shown that an element that started in $(A \cap B)^c$ ended up in $A^c \cup B^c$, then we know $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Assume $y \in A^c \cup B^c$. Thus, it must be the case that $y \notin A$ or $y \notin B$. Hence, y cannot be exist in both sets at the same time, so $y \in (A \cap B)^c$. Because we have taken an element that started in $A^c \cup B^c$ and have shown that it exists in $(A \cap B)^c$, we have proven $A^c \cup B^c \subseteq (A \cap B)^c$.

Therefore, since both inclusions hold, $(A \cap B)^c = A^c \cup B^c$.

(c) *Proof.* We need to show these expressions are subsets of each other in order to prove they are equivalent.

($\subseteq$) Let $x \in (A \cup B)^c$.
By definition of the complement, this means $x \notin A \cup B$. Therefore, by definition of union, $x \notin A$ and $x \notin B$. Thus, $x \in A^c$ and $x \in B^c$. Hence, $x \in (A^c \cap B^c)$. So, $(A \cup B)^c \subseteq A^c \cap B^c$.

($\supseteq$) Let $x \in (A^c \cap B^c)$.
By definition of intersection, $x \in A^c$ and $x \in B^c$. So, by definition of the complement $x \notin A$ and $x \notin B$. Therefore, $x \notin A \cup B$. Hence, $x \in (A \cup B)^c$. Thus, $A^c \cap B^c \subseteq (A \cup B)^c$.

Since both inclusions hold, we have:

$$(A \cup B)^c = A^c \cap B^c. \qquad \square$$

Exercise: 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(a) = \{f(x) \colon x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} \colon 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution.

(a) Since $f(x) = x^2$, the intervals of $f(A)$ would be $[0, 4]$ and $f(B)$ would be $[1, 16]$. The interval of the intersection of $A \cap B$ is $[1, 2]$. Take this through our function, we get $f(A \cap B) = [1, 4]$. On the other side of the equation, we already know the intervals of $f(A)$ and $f(B)$, and the intersection of theirs would be $[1, 4]$. So they do equal each other. We know $f(A \cup B)$ and $f(A) \cup f(B)$ will be equivalent because $f(A \cup B)$ has an interval of $[0, 16]$, and $f(A) \cup f(B)$ also has an interval of $[0, 16]$ because taking the union of $[0, 4] \cup [1, 16]$ is $[0, 16]$.

(b) Two sets could be $A = [-1, 0]$ and $B = [0, 1]$. For $f(A \cap B)$, we have $\{0\}$ but $f(A) \cap f(B) = [0, 1]$.

(c) *Proof.* Let $x \in g(A \cap B)$. Using the definition of function, we know there exists a $y \in A \cap B$ to which that y is mapped to as $g(y) = x$. From the definition of intersection, we know $y \in A$ and $y \in B$ such that $x = g(y) \in g(A)$ and $x = g(y) \in g(B)$ because $y \in A \cap B$. Putting it together, we have $x \in g(A) \cap g(B)$ thus proving $g(A \cap B) \subseteq g(A) \cap g(B)$ □

(d) Conjecture: For any function g defined as $g : \mathbb{R} \to \mathbb{R}$ and for any subsets $A, B \subseteq \mathbb{R}$, it is always that case that

$$g(A \cup B) = g(A) \cup g(B).$$

Proof. We need to show these expressions are subsets of each other in order to prove they are equivalent.

($\subseteq$) Let $y \in g(A \cup B)$.
By definition of function, there exists $x \in A \cup B$ such that $g(x) = y$. From the definition of union, $y \in A$ or $y \in B$ and thus, $g(x) \in g(A)$ or $g(x) \in g(B)$. Together, we have $g(x) = y \in g(A) \cup g(B)$. Therefore, $g(A \cup B) \subseteq g(A) \cup g(B)$.

($\supseteq$) Let $y \in g(A) \cup g(B)$.
By definition of union, $y \in g(A)$ or $y \in g(B)$. By definition of function, this means there exists an $x \in A$ or $x \in B$ such that $g(x) = y$. Hence, $x \in A \cup B$ by the definition of union, and $g(x) \in g(A \cup B)$. Therefore, $g(A) \cup g(B) \subseteq g(A \cup B)$.

Since we have shown the inclusion for both directions, this proves that $g(A \cup B) = g(A) \cup g(B)$. $\square$

Exercise: 1.2.8

Given a function $f : A \to B$ can be defined as either injective or surjective, give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

Solution.

(a) **Possible.** Define $f\colon \mathbb{N} \to \mathbb{N}$ as $f(a) = a + 1$. This function is injective because when $f(a_1) = f(a_2)$, we have:

$$\begin{aligned} a_1 + 1 &= a_2 + 1 \\ a_1 &= a_2. \end{aligned}$$

However, this function is not surjective because the entire co-domain is not covered; that being 1.

(b) **Possible.** Consider the example:

$$f(1) = 1, f(2) = 1, f(3) = 2, f(4) = 3, \ldots.$$

This function is surjective because for every b in the co-domain there is an a in the domain. However, it is not injective because we have two distinct values, 1 and 2, that map to the same b, 1.

(c) **Possible.** Consider the example:

$$f(1) = 0, f(2) = 1, f(3) = -1, f(4) = 2, f(5) = -2, \ldots.$$

Since every value b in the co-domain is mapped to an element in the domain, this function is surjective. Additionally, because that mapping is unqiue (i.e., $a_1 \neq a_2 \implies f(a_1) \neq f(a_2)$), this function is injective by definition.

Exercise: 1.3.4

Let $A_1, A_2, A_3 \ldots$ be a collection of nonempty sets each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

Solution.

(a) Let A_1 and A_2 be nonempty sets, each bounded above. To find the largest of the two suprema, we can use the following: $\sup(A_1 \cap A_2) = \max\{\sup A_1, \sup A_2\}$. If we extend this notion to $\sup(\bigcup_{k=1}^{n} A_k)$, we can use the same idea from before and write it as $\sup(\bigcup_{k=1}^{n} A_k) = \max\{\sup A_1, \sup A_2, \ldots, \sup A_n\}$.

(b) The formula does not extend to the infinite case. Consider the counterexample $\bigcup_{k=1}^{\infty} A_k$ where $A_k := [k, k+1]$. Even though these sets are bounded above, when we take the union of them, we approach infinity, which is not bounded: $\bigcup_{k=1}^{\infty} A_k = [1,2] \cup [2,3] \cup \cdots = [1, \infty)$.

Exercise: 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution.

(a) Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Define the set $cA := \{ca : a \in A\}$. From the axiom of completeness, because A is bounded above, we know there is a least upper bound, $s = \sup A$. Following from Example 1.3.7, we see that $a \leq s$ for all $a \in A$ which implies $ca \leq cs$ for all $a \in A$. Thus, cs is an upper bound for cA, and the first condition of Definition 1.3.2 is satisfied. For the second condition, we need to look at both $c = 0$ and $c > 0$ to avoid dividing by zero. So, we have two cases:

- **$c = 0$:** If $c = 0$, then $cA = \{0 : a \in A\} = \{0\}$. Since the only element in cA is 0, $\sup(cA) = 0$. Similarly, because $c = 0$, $c \sup A = 0 \cdot \sup A = 0$. Therefore,

$\sup(cA) = c\sup(A)$.

- $c > 0$**:** Let b be an arbitrary upper bound for cA and $c > 0$. In other words, $ca \leq b$ for all $a \in A$. This is equivalent to $a \leq b/c$ where $c \neq 0$, from which we can see that b/c is an upper bound for A. Because s is the least upper bound of A, $s \leq b/c$, which can be rewritten as $cs \leq b$. This verifies the second part of Definition 1.3.2, and we conclude $\sup(cA) = c\sup A$.

(b) Postulate: If $c < 0$, then $\sup(cA) = c\inf(A)$.

Exercise: 1.3.8

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\left\{\frac{m}{n} : m, n \in \mathbb{N} \text{ with } m < n\right\}$.

(b) $\left\{\frac{(-1)^m}{n} : m, n \in \mathbb{N}\right\}$.

(c) $\left\{\frac{n}{3n+1} : n \in \mathbb{N}\right\}$.

(d) $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$.

Solution. To avoid writing out every set definition, I am going to denote each set as A_n where n corresponds to the numerical value of the list from (a) - (d).

(a) $\sup A_1 = 1$, $\inf A_1 = 0$

(b) $\sup A_2 = 1$, $\inf A_2 = -1$

(c) $\sup A_3 = \frac{1}{3}$, $\inf A_3 = \frac{1}{4}$

(d) $\sup A_4 = 1$, $\inf A_3 = 0$

Exercise: 1.4.1

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st? In other words, are there two irrational numbers that can be added and multiplied such that you get a number x such that $x \notin \mathbb{I}$.

Solution.

(a) Let $a, b \in \mathbb{Q}$. This means there exists some $p, q, a, b \in \mathbb{Z}$ such that

$$a = \frac{p}{q}$$

and

$$b = \frac{a}{b}$$

where $q, b \neq 0$. The product of these numbers is

$$ab = \frac{p}{q} \cdot \frac{a}{b} = \frac{pa}{qb}.$$

Since $pa, qb \in \mathbb{Z}$, $ab \in \mathbb{Q}$. The sum of these numbers is

$$a + b = \frac{p}{q} + \frac{a}{b} = \frac{pb + aq}{qb}.$$

Since $pb + aq, qb \in \mathbb{Z}$, $a + b \in \mathbb{Q}$.

(b) Let $a \in \mathbb{Q}$ and $t \in \mathbb{I}$. Assume, for contradiction, that $a + t \in \mathbb{Q}$. This would imply $t = (a + t) - a$ (because we can subtract $t + a$ from the original equation and rearrange terms). Since $a + t, a \in \mathbb{Q}$ their sum would be rational because the rational numbers are closed under addition. However, that would contradict the assumption that $t \in \mathbb{I}$. Hence, $a + t \in \mathbb{I}$.

(c) For $\mathbb{I}$, it is not closed under addition and multiplication. Consider the following counterexample: $\sqrt{2} + (-\sqrt{2}) = 0$ which is not in the irrationals. For multiplication, consider $\sqrt{2} \cdot \sqrt{2} = 2$, which is also not in the irrationals.

Exercise: 2.1.1

What happens if we reverse the order of the quantifiers in Definition 2.2.3?
Definition: A sequence x_n *verconges* to x if there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x - n - x| < \epsilon$.

(a) Give an example of a vercongent sequence.

(b) Is there an example of a vercongent sequence that is divergent?

(c) Can a sequence verconge to two different values?

(d) What exactly is being described in this strange definition?

Solution.

(a) Pick $\epsilon = 2$, $x_n = (-1)^n$ and $x = 0$. This sequence will stay within the bounds of $(-2, 2)$ for all $N \in \mathbb{N}$ and $n \geq N$.

(b) There cannot be a divergent vercongent sequence because vercongence wants us to be bounded, and divergence wants it to grow outside the bounds. These two ideas are mutually exclusive.

(c) Yes. For example, $x_n = 0$ and $x_n = 1$.

(d) This definition is describing a sequence that is bounded. It is a sequence that is not growing outside of a certain range.

Exercise: 2.2.2

Verify, using Definition 2.2.3, that the following sequences converge to the proposed limit.

(a) $\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$

Proof.

(a) Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \dfrac{3}{25\epsilon}$. Then for all $n \geq N$,

$$\begin{aligned}\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \left|\frac{-3}{5(5n+4)}\right| \\ &= \frac{3}{25n+20} \\ &\leq \frac{3}{25n} \\ &\leq \frac{3}{25N} \\ &< \epsilon\end{aligned}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) Let $\epsilon > 0$. By the Archimedean Principle, there exists an $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Then, for $n \geq N$,

$$\begin{aligned}\left|\frac{2n^2}{n^3+3} - 0\right| &= \left|\frac{2n^2}{n^3+3}\right| \\ &= \frac{2n^2}{n^3+3} \\ &< \frac{2n^2}{n^3} \\ &= \frac{2}{n} \\ &\leq \frac{2}{N} \\ &= \frac{2}{2/\epsilon} \\ &= \epsilon.\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$. $\square$

Exercise: 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution.

(a) There is at least one college in the United States where all students are less than seven feet tall.

(b) There is at least one college in the United States where all professors give at least one student a grade of C or lower.

(c) For all colleges in the United States, there exists a student who is less than six feet tall.

Exercise: 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution.

(a) Possible. Consider the sequence $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$ This sequence has infinitely many ones but does not converge to one.

(b) Impossible. Suppose (a_n) is a sequence that converges to a limit $L \neq 1$ and has infinitely many ones. Since (a_n) converges to L, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$. Choose $\epsilon = \frac{|1-L|}{2} > 0$. Then, for $n \geq N$,

$|a_n - L| < \epsilon$, which implies $a_n \neq 1$ beyond this N. This contradicts the existence of infinitely many ones. Therefore, such a sequence is impossible.

(c) Possible. Define a sequence by concatenating increasing blocks of ones separated by zeros: $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \dots)$. Specifically, the sequence consists of n ones followed by a zero for $n = 1, 2, 3, \dots$. For every $n \in \mathbb{N}$, there is a block of n consecutive ones somewhere in the sequence. The sequence does not converge, so it is divergent.

Exercise: 2.2.5

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim_{n\to\infty} a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$

(b) $a_n = [[(12 + 4n)/3n]]$

Reflecting on these examples, comment on the statement following Definition 2.2.3B that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution.

(a) We will show that $\lim\limits_{n\to\infty} a_n = 0$.

Proof. For $n \geq 6$, we have $\frac{5}{n} \leq \frac{5}{6} < 1$, so $a_n = [[5/n]] = 0$. Let $\epsilon > 0$. Choose $N = 6$. Then for all $n \geq N$,

$$|a_n - 0| = |0 - 0| = 0 < \epsilon.$$

Therefore, by the definition of convergence, $\lim\limits_{n\to\infty} a_n = 0$. □

(b) We will show that $\lim\limits_{n\to\infty} a_n = 1$.

Proof. Observe that:

$$a_n = \left\lceil \frac{12+4n}{3n} \right\rceil = \left\lceil \frac{4n+12}{3n} \right\rceil = \left\lceil \frac{4}{3} + \frac{4}{n} \right\rceil.$$

As $n \to \infty$, $\frac{4}{n} \to 0$, so $\frac{4}{3} + \frac{4}{n} \to \frac{4}{3} \approx 1.333$.
For $n \geq 7$, we have:

$$\frac{4}{n} \leq \frac{4}{7} \approx 0.571, \quad \frac{4}{3} + \frac{4}{n} \leq 1.333 + 0.571 = 1.904.$$

Since $1 < \frac{4}{3} + \frac{4}{n} < 2$ for $n \geq 7$, we have:

$$a_n = \left\lceil \frac{4}{3} + \frac{4}{n} \right\rceil = 1.$$

Let $\epsilon > 0$. Choose $N = 7$. Then for all $n \geq N$,

$$|a_n - 1| = |1 - 1| = 0 < \epsilon.$$

Therefore, by the definition of convergence, $\lim_{n\to\infty} a_n = 1$. □

Reflection: In these examples, we see that once the sequence reaches a certain point (i.e., $n \geq N$), the terms remain constant. This means that for any $\epsilon > 0$, we can find a fixed N to satisfy the definition of convergence, regardless of how small ϵ is. However, in general, smaller ϵ-neighborhoods may require larger N because the sequence may not settle into its limit as neatly as it does in these cases.

Exercise: 2.2.6

Prove the Uniqueness of Limits theorem. To get started, assume $(a_n) \to a$ and $(a_n) \to b$. Now argue $a = b$.

Proof. Since $(a_n) \to a$, this means for all $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon/2$. Similarly, since $(a_n) \to b$, this means for all $\epsilon > 0$, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - b| < \epsilon/2$.

Now, let $N = \max\{N_1, N_2\}$ so that

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &\leq |(a_n - a) + (a_n - b)| \\ &< \epsilon/2 + \epsilon/2 \\ &< \epsilon \end{aligned}$$

Then, by Theorem 1.2.6, $a = b$. □

Exercise: 2.2.7

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually, or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution.

(a) The sequence $(-1)^n$ is *frequently* in the set $\{1\}$ because for every $N \in \mathbb{N}$, we can find an $n \geq N$ such that $(-1)^n = 1$.

(b) The definition of *eventually* is stronger because *eventually* implies *frequently*, but *frequently* does not imply *eventually*.

(c) An alternate rephrasing of Definition 2.2.3B using *eventually* is: A sequence (a_n) converges to a if, given any ϵ-neighborhood—$V_\epsilon(a)$ of a—(a_n) is *eventually* in $V_\epsilon(a)$. The term we want is eventually.

(d) If an infinite number of terms of a sequence (x_n) are equal to 2, (x_n) is not *eventually* in $(1.9, 2.1)$ because we can have a sequence (a_n) that will not settle in $(1.9, 2.1)$. For example, $(a_n) = (0, 2, 0, 2, \cdots)$ does not settle in $(1.9, 2.1)$. Whereas, (x_n) is *frequently* in the interval $(1.9, 2.1)$ because for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $x_n \in (1.9, 2.1)$ for all $n \geq N$. We can see an instance of this being true by examining the previous example.

Exercise: 2.3.1

(a) If $\lim_{n\to\infty} x_n = 0$, show that $\lim_{n\to\infty} \sqrt{x_n} = 0$.

(b) If $\lim_{n\to\infty} x_n = x$, show that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

Proof.

(a) *Solution.* Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_n| < \epsilon^2.$$

Then, for all $n \geq N$,

$$|\sqrt{x_n} - 0| = \sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon.$$

Therefore, $\lim_{n\to\infty} \sqrt{x_n} = 0$.

(b) *Solution.* Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|x_n - x| < \delta.$$

We consider two cases:

Case 1: $x > 0$.

Since $x > 0$, choose $\delta = \min\left\{\epsilon\left(2\sqrt{x}\right), \frac{x}{2}\right\}$. Then for all $n \geq N$, we have $x_n > x - \frac{x}{2} = \frac{x}{2} > 0$. Thus,

$$\sqrt{x_n} + \sqrt{x} \geq \sqrt{\frac{x}{2}} + \sqrt{x} > 0.$$

Now,

$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{\delta}{\sqrt{\frac{x}{2}}} \leq \epsilon.$$

Case 2: $x = 0$.

From part (1), we have $\lim_{n\to\infty} \sqrt{x_n} = 0 = \sqrt{0}$.

Therefore, $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

□

Exercise: 2.3.2

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution.

(a) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - 2| < \epsilon$. Now, for any $n \geq N$,

$$\begin{aligned} \left|\frac{2x_n - 1}{3} - 1\right| &= \left|\frac{2x_n - 1 - 3}{3}\right| \\ &= \left|\frac{2x_n - 4}{3}\right| \\ &= \frac{2}{3}\,|x_n - 2| \\ &< |x_n - 2| \\ &< \epsilon \end{aligned}$$

Therefore, $\frac{2x_n - 1}{3} \to 1$ □

(b) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $x_n \geq 1$. Then, we will choose N_2 so that $|x_n - 2| < \epsilon$ for all $n \geq N_2$. Afterwards, we take $N = \max\{N_1, N_2\}$. And note that for $n \geq N$,

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{2}\right| &= \left|\frac{2 - x_n}{2x_n}\right| \\ &< \frac{|2 - x_n|}{2} \\ &< \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

□

Exercise: 2.4.7 (Limit Superior)

Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k \colon k \geq n\}$ converges.

(b) The *limit superior* of (a_n) or $\limsup_{n\to\infty} a_n$, is defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf_{n\to\infty} a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

$$\begin{aligned}
\left|\frac{3n-1}{2n+1} - \frac{3}{2}\right| &= \left|\frac{2(3n-1)-3(2n+1)}{2(2n+1)}\right| \\
&= \left|\frac{6n-2-6n-3}{4n+2}\right| \\
&= \left|\frac{-5}{4n+2}\right| \\
&= \frac{5}{4n+2} \\
&= \frac{5}{4N} \quad \text{Let } N = \frac{5}{4\epsilon} \\
&= \epsilon
\end{aligned}$$

The sequence (x_n) is given by $x_n = \frac{3n-1}{2n+1}$. Determine its limit and give a formal proof. I have: Let $\epsilon > 0$. We want

$$\left|\frac{3n-1}{2n+1} - \frac{3}{2}\right| = \frac{5}{4n+2} < \epsilon.$$

For large n: $\frac{1}{4n+2} < \frac{1}{4n}$ so

$$\frac{5}{4n+2} < \frac{5}{4n}.$$

To ensure $\frac{5}{4n} < \epsilon$, choose N such that $N > \frac{5}{4\epsilon}$. The Archimedean Principle ensures this number.

For $n \geq N$, we have $n > \frac{5}{4\epsilon}$, so $\frac{5}{4n} \leq \frac{5}{4N}$. Since $N > \frac{5}{4\epsilon}$, we know $\frac{5}{4N} < \epsilon$. Therefore,

$$\left|\frac{3n-1}{2n+1} - \frac{3}{2}\right| = \frac{5}{4n+2} < \epsilon.$$

Solution.

(a) We will show that (y_n) converges.

Proof. Since (a_n) is bounded, there exists $M > 0$ such that $|a_n| \leq M$ for all n.
For each n, define $y_n = \sup\{a_k : k \geq n\}$. As n increases, the set $\{a_k : k \geq n\}$ becomes smaller, so the supremum cannot increase. Therefore, the sequence (y_n) is non-increasing:

$$y_{n+1} \leq y_n \quad \text{for all } n.$$

Additionally, since (a_n) is bounded below, so is (y_n). Therefore, (y_n) is a bounded, non-increasing sequence.
By the Monotone Convergence Theorem, every bounded, monotonic sequence converges. Thus, (y_n) converges. □

(b) A reasonable definition for $\liminf_{n\to\infty} a_n$ is to define $z_n = \inf\{a_k : k \geq n\}$ for each n. Then, the *limit inferior* of (a_n) is defined by:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} z_n.$$

Since (a_n) is bounded, each z_n exists and the sequence (z_n) is non-decreasing. As n increases, the set $\{a_k : k \geq n\}$ becomes smaller, so the infimum cannot decrease. Therefore, (z_n) is a bounded, non-decreasing sequence, which converges by the Monotone Convergence Theorem. Hence, $\liminf_{n\to\infty} a_n$ always exists for any bounded sequence.

(c) We will show that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence.

Proof. For each n, we have $z_n = \inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\} = y_n$. This implies:

$$z_n \leq y_n \quad \text{for all } n.$$

Taking limits as $n \to \infty$, we get:

$$\lim_{n\to\infty} z_n \leq \lim_{n\to\infty} y_n,$$

which means:

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

For an example where the inequality is strict, consider the sequence $a_n = (-1)^n$. Then:

$$y_n = \sup\{(-1)^k : k \geq n\} = 1, \quad z_n = \inf\{(-1)^k : k \geq n\} = -1.$$

Therefore:

$$\limsup_{n\to\infty} a_n = 1, \quad \liminf_{n\to\infty} a_n = -1, \quad \liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n.$$

$\square$

(d) We will show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. We will show this by proving both implications:

(⇒) Suppose $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$. We will show that $\lim a_n$ exists and equals L.
Let $\epsilon > 0$. Since $\limsup_{n\to\infty} a_n = L$, there exists N_1 such that for all $n \geq N_1$:

$$y_n = \sup\{a_k : k \geq n\} < L + \epsilon.$$

Similarly, since $\liminf_{n\to\infty} a_n = L$, there exists N_2 such that for all $n \geq N_2$:

$$z_n = \inf\{a_k : k \geq n\} > L - \epsilon.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$:

$$L - \epsilon < z_n \leq a_n \leq y_n < L + \epsilon,$$

which implies:

$$|a_n - L| < \epsilon.$$

Therefore, $\lim a_n = L$.

(⇐) Conversely, suppose $\lim a_n = L$. Then, for every $\epsilon > 0$, there exists N such that for all $n \geq N$:

$$|a_n - L| < \epsilon.$$

This implies that for all $n \geq N$, the set $\{a_k : k \geq n\}$ is contained in $(L - \epsilon, L + \epsilon)$. Therefore:

$$y_n = \sup\{a_k : k \geq n\} \leq L + \epsilon, \quad z_n = \inf\{a_k : k \geq n\} \geq L - \epsilon.$$

Taking limits, we get:

$$\limsup_{n\to\infty} a_n \leq L + \epsilon, \quad \liminf_{n\to\infty} a_n \geq L - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$. □

Exercise: 2.4.10 (Infinite Products)

A close relative of infinite series is the infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots ,$$

which is understood in terms of its sequence of partial products

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\cdots , \quad \text{where} \quad a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = \frac{1}{n}$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = \frac{1}{n^2}$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Solution.

(a) For $a_n = \dfrac{1}{n}$:

The sequence of partial products is:

$$p_m = \prod_{n=1}^{m}\left(1+\frac{1}{n}\right) = \prod_{n=1}^{m}\frac{n+1}{n}$$

This telescopes:

$$p_m = \frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times\cdots\times\frac{m+1}{m} = \frac{m+1}{1} = m+1$$

Therefore, the sequence diverges as $m \to \infty$.

For $a_n = \dfrac{1}{n^2}$:

Compute the first few terms:

$$\begin{aligned}
p_1 &= 1 + \frac{1}{1^2} = 2 \\
p_2 &= \left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right) = 2 \times \frac{5}{4} = \frac{5}{2} \\
p_3 &= p_2 \times \left(1 + \frac{1}{3^2}\right) = \frac{5}{2} \times \frac{10}{9} = \frac{25}{9} \\
p_4 &= p_3 \times \left(1 + \frac{1}{4^2}\right) = \frac{25}{9} \times \frac{17}{16} = \frac{425}{144}
\end{aligned}$$

The sequence increases slowly, suggesting that the infinite product is monotone increasing, and thus it converges.

(b) We will provide an if and only if proof below.

Proof. We will show that the infinite product $\prod_{n=1}^{\infty}(1+a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

($\Rightarrow$) If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. For $a_n \geq 0$, we have $\ln(1 + a_n) \leq a_n$. Thus,

$$\sum_{n=1}^{\infty} \ln(1+a_n) \leq \sum_{n=1}^{\infty} a_n < \infty$$

So the series $\sum_{n=1}^{\infty} \ln(1+a_n)$ converges, which implies that the product $\prod_{n=1}^{\infty}(1+a_n)$ converges.

($\Leftarrow$) Conversely, if $\prod_{n=1}^{\infty}(1+a_n)$ converges, then the partial products are bounded. For $a_n \geq 0$ and $1 + x \geq e^{x/2}$ for small x, we have

$$\ln(1+a_n) \geq \frac{a_n}{2}$$

For sufficiently large n, this gives

$$\sum_{n=1}^{\infty} a_n \leq 2\sum_{n=1}^{\infty} \ln(1+a_n)$$

Since $\sum_{n=1}^{\infty} \ln(1+a_n)$ converges, so does $\sum_{n=1}^{\infty} a_n$. $\square$

Exercise: 2.5.1

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

Solution.

(a) **Impossible.** This violates the Bolzano-Weierstrass Theorem. It assures us that every bounded sequence has a convergent subsequence. If a subsequence is bounded, then it must have a convergent subsequence.

(b) Consider the sequence $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \cdots \frac{1}{n}, \frac{(n-1)}{n})$. From this, you can have a subsequence $(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n})$ which converges to 0, and also a subsequence $(\frac{1}{2}, \frac{2}{3}, \ldots \frac{n-1}{n})$, which converges to 1.

Exercise: 2.5.2

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

Solution.

(a) **True.** If every proper subsequence of (x_n) converges, then (x_n) must converge to the same limit. If (x_n) did not converge, there would exist at least one divergent subsequence or two subsequences converging to different limits, contradicting the assumption.

(b) **True.** If (x_n) contained a divergent subsequence, then (x_n) cannot converge. A convergent sequence has all its subsequences converging to the same limit, so the existence of a divergent subsequence implies that (x_n) diverges (contrapositive).

(c) **True.** Since (x_n) is bounded and diverges, the Bolzano-Weierstrass Theorem guarantees the existence of at least one convergent subsequence. Let this subsequence converge to L_1. Because (x_n) does not converge to L_1, there is an $\epsilon > 0$ and infinitely many terms of (x_n) such that $|x_n - L_1| \geq \epsilon$. Extracting a subsequence from these terms, the Bolzano-Weierstrass Theorem ensures a further subsequence converging to a limit $L_2 \neq L_1$. Thus, (x_n) has two subsequences converging to different limits.

Exercise: 2.5.5

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Proof. Suppose that (a_n) does not converge to $a \in \mathbb{R}$. By the definition of convergence, this means there is a positive real number ϵ_0 such that no matter how large we choose $N \in \mathbb{N}$, there will always exist some $n > N$ where $|a_n - a| \geq \epsilon_0$. In a formal way, this shows that (a_n) does not converge to a within the ϵ_0-neighborhood.

We aim to demonstrate that this leads to a contradiction by constructing a subsequence of (a_n) that stays outside this neighborhood. Begin by selecting n_1 such that $|a_{n_1} - a| \geq \epsilon_0$. Next, since the condition holds for all $N \in \mathbb{N}$, we can find another index $n_2 > n_1$ such that $|a_{n_2} - a| \geq \epsilon_0$. Continuing this process, we generate an increasing sequence of indices $n_1 < n_2 < n_3 < \dots$ such that for each $i \in \mathbb{N}$, $|a_{n_i} - a| \geq \epsilon_0$.

Now consider the subsequence (a_{n_i}) we have built. Since (a_n) is bounded by assumption, its subsequence (a_{n_i}) is also bounded. By the Bolzano-Weierstrass Theorem, every bounded sequence has a convergent subsequence. Let $(a_{n_{i_k}})$ denote a convergent subsequence of (a_{n_i}). According to our assumption, any convergent subsequence of (a_n) must converge to a.

However, each term of $(a_{n_{i_k}})$ remains outside the ϵ_0-neighborhood of a. Thus, it is impossible for $(a_{n_{i_k}})$ to converge to a. This contradiction implies that our initial assumption—that (a_n) does not converge to a—is false. Therefore, the sequence (a_n) must converge to a. □

Exercise: 2.5.6

Use a similar strategy to the one in Theorem 2.5.5 to show

$$\lim b^{1/n} \text{ exists for all } b \geq 0$$

and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Proof. We will show that $\lim_{n\to\infty} b^{1/n}$ exists for all $b \geq 0$ and find its value.

- **Case 1:** $b = 0$.

  When $b = 0$, the sequence becomes $a_n = 0^{1/n} = 0$ for all n. Thus, $\lim_{n\to\infty} b^{1/n} = 0$.

- **Case 2:** $b > 0$.

Suppose, for contradiction, that $\lim_{n\to\infty} b^{1/n} \neq 1$. Then there exists $\epsilon > 0$ and infinitely many n such that $|b^{1/n} - 1| \geq \epsilon$. Extract a subsequence (b^{1/n_k}) where this inequality holds for all k.

Since $b^{1/n} > 0$ and bounded, by the Bolzano-Weierstrass Theorem, the subsequence (b^{1/n_k}) has a further subsequence that converges to a limit L. According to Exercise 2.3.1, any convergent subsequence of $(b^{1/n})$ must have its limit equal to $\lim_{n\to\infty} b^{1/n}$.

Consider $\ln b^{1/n} = \frac{\ln b}{n}$. As $n \to \infty$, $\frac{\ln b}{n} \to 0$, so $\ln b^{1/n} \to 0$, which implies $b^{1/n} \to e^0 = 1$.

This contradicts the assumption that $|b^{1/n_k} - 1| \geq \epsilon$, so $\lim_{n\to\infty} b^{1/n} = 1$.

Conclusion:

$$\lim_{n\to\infty} b^{1/n} = \begin{cases} 0, & \text{if } b = 0, \\ 1, & \text{if } b > 0. \end{cases} \quad \square$$

Exercise: 3.2.4

Let A be a nonempty and bounded above set so that $s = \sup(A)$ exists. (See Definition 1.3.2 and Definition 3.2.7)

(a) Show that $s \in \bar{A}$.

(b) Can an open set contain its supremum?

Solution.

(a) We need to show that $s = \sup(A) \in \bar{A}$, where $\bar{A} = A \cup L$, and L is the set of limit points of A.

Since A is nonempty and bounded above, $s = \sup(A)$ exists.

If $s \in A$, then $s \in \bar{A}$ trivially.

Suppose $s \notin A$. We will show that s is a limit point of A, so $s \in L \subseteq \bar{A}$.

By definition, x is a limit point of A if for all $\epsilon > 0$, there exists $a \in A$ such that $a \in V_\epsilon(x)$ and $a \neq x$.

Fix any $\epsilon > 0$. Since $s = \sup(A)$, for this ϵ, $s - \epsilon$ is not an upper bound of A. Therefore, there exists $a \in A$ such that

$$s - \epsilon < a \leq s.$$

Since $a \leq s$ and $a > s - \epsilon$, we have $|a - s| < \epsilon$, so $a \in V_\epsilon(s)$ and $a \neq s$.

Therefore, s is a limit point of A, and hence $s \in \bar{A}$.

(b) An open set cannot contain its supremum if the supremum is finite.

Assume A is an open set containing its supremum s.

Since A is open and $s \in A$, there exists $\epsilon > 0$ such that

$$V_\epsilon(s) = \{x \in \mathbb{R} \mid |x - s| < \epsilon\} \subseteq A.$$

This means $s + \frac{\epsilon}{2} \in A$.

However, s is an upper bound of A, so no element of A can be greater than s.

This is a contradiction.

Therefore, an open set cannot contain its supremum.

Exercise: 3.2.6

Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.

(b) The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set."

(c) Every nonempty open set contains a rational number.

(d) Every bounded infinite closed set contains a rational number.

(e) The Cantor set is closed.

Solution.

(a) **False.**

Counterexample: Consider the set $U = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n},\ q_n + \frac{1}{n}\right)$, where (q_n) is an enumeration of all rational numbers.

Each interval $\left(q_n - \frac{1}{n},\ q_n + \frac{1}{n}\right)$ is open, and the union U is open. Since every rational number is included in some interval, U contains all rationals. However, $U \neq \mathbb{R}$ because there are irrational numbers not covered by these intervals. Therefore, an open set can contain all rational numbers without being all of $\mathbb{R}$.

(b) **True.**

Proof. The Nested Interval Property holds for any nested sequence of nonempty closed and bounded sets in $\mathbb{R}$. If $\{F_n\}$ is such a sequence with $F_{n+1} \subseteq F_n$ for all n, then the intersection $\bigcap_{n=1}^{\infty} F_n$ is nonempty. This follows from the completeness of $\mathbb{R}$, as every decreasing sequence of nonempty closed and bounded sets has a nonempty intersection. Therefore, replacing "closed interval" with "closed set" does not invalidate the property. □

(c) **True.**

Proof. Let U be a nonempty open set. Then there exists $x \in U$ and $\epsilon > 0$ such that $(x - \epsilon,\ x + \epsilon) \subseteq U$. Since the rationals are dense in $\mathbb{R}$, there exists a rational number $q \in (x - \epsilon,\ x + \epsilon)$. Therefore, U contains a rational number. □

(d) **True.**

Proof. Let F be a bounded infinite closed set. Since F is infinite and bounded, it must contain limit points. As F is closed, it contains its limit points. The real numbers are densely ordered with rationals between any two real numbers. Therefore, F must contain a rational number. □

(e) **True.**

Proof. The Cantor set C is constructed as the intersection of a decreasing sequence of closed sets (finite unions of closed intervals). Since each of these sets is closed and the intersection of closed sets is closed, C is closed. □

Exercise: 3.2.8

Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

(a) $\overline{A \cup B}$

(b) $A \setminus B = \{x \in A \mid x \notin B\}$

(c) $(A^c \cup B)^c$

(d) $(A \cap B) \cup (A^c \cap B)$

(e) $\overline{A}^c \cap \overline{A^c}$

Solution.

(a) $\overline{A \cup B}$

The closure of any set is closed by definition. Therefore, $\overline{A \cup B}$ is definitely closed.

Conclusion: Closed.

(b) $A \setminus B = \{x \in A \mid x \notin B\}$

Since B is closed, its complement B^c is open. Since A is open, the intersection $A \cap B^c = A \setminus B$ is the intersection of two open sets, which is open.

Conclusion: Open.

(c) $(A^c \cup B)^c$

Applying De Morgan's Law:

$$(A^c \cup B)^c = A \cap B^c$$

Since A is open and B^c is open (because B is closed), their intersection $A \cap B^c$ is open.

Conclusion: Open.

(d) $(A \cap B) \cup (A^c \cap B)$

Simplify the expression:

$$(A \cap B) \cup (A^c \cap B) = [A \cup A^c] \cap B = \mathbb{R} \cap B = B$$

Thus, the set equals B, which is closed.

Conclusion: Closed.

(e) $\overline{A}^c \cap \overline{A^c}$

Since A is open, its closure $\overline{A}$ is closed, so $\overline{A}^c$ is open.

Since A^c is closed (being the complement of an open set), $\overline{A^c} = A^c$ is closed.

Therefore, $\overline{A}^c \cap \overline{A^c}$ is the intersection of an open set and a closed set, which is generally open but not necessarily closed.

For example, let $A = (0, 1)$. Then:

$$\overline{A} = [0, 1], \quad \overline{A}^c = (-\infty, 0) \cup (1, \infty)$$

and

$$\overline{A^c} = A^c = (-\infty, 0] \cup [1, \infty)$$

Then:

$$\overline{A}^c \cap \overline{A^c} = [(-\infty, 0) \cup (1, \infty)] \cap [(-\infty, 0] \cup [1, \infty)] = (-\infty, 0) \cup (1, \infty)$$

Which is an open set.

Conclusion: Open.

Exercise: 3.2.11

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Does this result about closures extend to infinite unions of sets?

Solution.

(a) We will prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. Recall that the closure of a set A is defined as $\overline{A} = A \cup L_A$, where L_A is the set of limit points of A.

($\subseteq$) Let $x \in \overline{A \cup B}$. Then $x \in A \cup B$ or x is a limit point of $A \cup B$.

- If $x \in A \cup B$, then $x \in A$ or $x \in B$, so $x \in \overline{A}$ or $x \in \overline{B}$, thus $x \in \overline{A} \cup \overline{B}$.
- If x is a limit point of $A \cup B$, then every neighborhood $V_\epsilon(x)$ contains a point $y \neq x$ such that $y \in A \cup B$. Therefore, $y \in A$ or $y \in B$, so x is a limit point of A or B. Hence, $x \in \overline{A}$ or $x \in \overline{B}$, so $x \in \overline{A} \cup \overline{B}$.

Therefore, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

($\supseteq$) Let $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$.

- If $x \in \overline{A}$, then $x \in A$ or x is a limit point of A. Since $A \subseteq A \cup B$, $x \in A \cup B$ or x is a limit point of $A \cup B$. Thus, $x \in \overline{A \cup B}$.
- Similarly, if $x \in \overline{B}$, then $x \in \overline{A \cup B}$.

Therefore, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Hence, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. $\square$

(b) The result does not necessarily extend to infinite unions of sets.

Consider the sets $A_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Then $\overline{A_n} = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$.

The infinite union is $A = \bigcup_{n=1}^{\infty} A_n = (0, 1)$, so $\overline{A} = [0, 1]$.

The union of the closures is $\bigcup_{n=1}^{\infty} \overline{A_n} = (0, 1)$, since none of the closed intervals $\left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ include the endpoints 0 or 1.

Therefore, $\overline{A} \neq \bigcup_{n=1}^{\infty} \overline{A_n}$.

Hence, the equality does not hold for infinite unions.

Exercise: 3.3.4

Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$

(b) $\overline{F^c \cup K^c}$

(c) $K \setminus F = \{x \in K \mid x \notin F\}$

(d) $\overline{K \cap F^c}$

Solution.

(a) Since K and F are closed, their intersection $K \cap F$ is closed (Theorem 3.2.8).

To show that $K \cap F$ is compact, let $\mathcal{U}$ be any open cover of $K \cap F$. Our goal is to extract a finite subcover from $\mathcal{U}$. We can then use the Bolzano-Weierstrass Theorem (iii) to show that $K \cap F$ is compact.

Since F is closed, its complement F^c is open (Theorem 3.2.6). Then $K \setminus F = K \cap F^c$ is open as the intersection of an open set and K.

Consider the open cover $\mathcal{U}' = \mathcal{U} \cup \{K \setminus F\}$ of K. Every point in K is either in $K \cap F$ (covered by $\mathcal{U}$) or in $K \setminus F$ (covered by $K \setminus F$).

Since K is compact, there exists a finite subcover $\mathcal{U}'' \subseteq \mathcal{U}'$ that covers K.

If $K \setminus F$ is in $\mathcal{U}''$, remove it to obtain a finite subcollection of $\mathcal{U}$ that still covers $K \cap F$. If $K \setminus F$ is not in $\mathcal{U}''$, then $\mathcal{U}'' \subseteq \mathcal{U}$ already covers $K \cap F$.

Therefore, $K \cap F$ is compact.

Conclusion: Both compact and closed.

(b) Since F and K are closed, F^c and K^c are open. The union $F^c \cup K^c$ is open (Theorem 3.2.3), so its closure $\overline{F^c \cup K^c}$ is closed by Theorem 3.2.9.

This set may not be bounded, so it's not necessarily compact.

Conclusion: Definitely closed.

(c) The set $K \setminus F = K \cap F^c$ is the intersection of a compact set K and an open set F^c. This set is open in K but not necessarily open or closed in $\mathbb{R}$.

Since $K \setminus F$ is not necessarily closed, it may not be compact.

Conclusion: Neither compact nor closed.

(d) The set $K \cap F^c$ is open in K, so its closure $\overline{K \cap F^c}$ is closed by Theorem 3.2.9.

To show that $\overline{K \cap F^c}$ is compact, let $\mathcal{U}$ be any open cover of $\overline{K \cap F^c}$.

Since $\overline{K \cap F^c} \subseteq K$ and K is compact, we can consider $\mathcal{U}$ as an open cover of a subset of K.

By the definition of open cover, there exists a finite subcover of $\mathcal{U}$ that covers $\overline{K \cap F^c}$.

Therefore, $\overline{K \cap F^c}$ is compact.

Conclusion: Both compact and closed.

Exercise: 4.2.10 (Right and Left Limits)

Introductory calculus courses typically refer to the right-hand limit of a function as the limit obtained by "letting x approach a from the right-hand side."

(a) Give a proper definition in the style of Definition 4.2.1 for the right-hand and left-hand limit statements:

$$\lim_{x \to a^+} f(x) = L \quad \text{and} \quad \lim_{x \to a^-} f(x) = M.$$

(b) Prove that $\lim_{x \to a} f(x) = L$ if and only if both the right and left-hand limits equal L.

Solution.

(a) **Definition of Right-Hand Limit:**
Let $f : A \to \mathbb{R}$ and let c be a limit point of A. We say that $\lim_{x \to c^+} f(x) = L$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < x - c < \delta$ (and $x \in A$), it follows that $|f(x) - L| < \epsilon$.

Definition of Left-Hand Limit:
Let $f : A \to \mathbb{R}$ and let c be a limit point of A. We say that $\lim_{x \to c^-} f(x) = M$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $-\delta < x - c < 0$ (and $x \in A$), it follows that $|f(x) - M| < \epsilon$.

(b) *Proof.* We will show this by proving both implications:

(⇒) Suppose that $\lim_{x \to c} f(x) = L$. By definition, for every $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$. This inequality holds for all x approaching c from both sides, so both the right-hand and left-hand limits equal L.

(⇐) Conversely, suppose that $\lim_{x \to c^+} f(x) = L$ and $\lim_{x \to c^-} f(x) = L$. Then, for every $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that:

- If $0 < x - c < \delta_1$, then $|f(x) - L| < \epsilon$.
- If $-\delta_2 < x - c < 0$, then $|f(x) - L| < \epsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for all x with $0 < |x - c| < \delta$, we have $|f(x) - L| < \epsilon$.

Therefore, $\lim_{x \to c} f(x) = L$. □

Exercise: 4.2.11 (Squeeze Theorem)

Let f, g, and h satisfy $f(x) \leq g(x) \leq h(x)$ for all x in some common domain A. If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} h(x) = L$ at some limit point c of A, show $\lim_{x\to c} g(x) = L$ as well.

Proof. Let $\epsilon > 0$. Since $\lim_{x\to c} f(x) = L$, there exists $\delta_1 > 0$ such that whenever $0 < |x - c| < \delta_1$, we have $|f(x) - L| < \epsilon$. Similarly, there exists $\delta_2 > 0$ such that whenever $0 < |x - c| < \delta_2$, we have $|h(x) - L| < \epsilon$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for all x with $0 < |x - c| < \delta$, we have $f(x) \leq g(x) \leq h(x)$. Thus,

$$L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon.$$

Therefore, $|g(x) - L| < \epsilon$, and $\lim_{x\to c} g(x) = L$. $\square$

Exercise: 4.3.1

Let $g(x) = \sqrt[3]{x}$.

(a) Prove that g is continuous at $c = 0$.

(b) Prove that g is continuous at a point $c \neq 0$. (The identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ will be helpful.)

Possibily use Example 4.3.8 in the book for (b).

Solution.

(a) *Proof.* To prove that g is continuous at $c = 0$, we need to show that

$$\lim_{x\to 0} g(x) = g(0) = 0.$$

Let $\epsilon > 0$. Choose $\delta = \epsilon^3$. Then, if $|x - 0| = |x| < \delta$, we have

$$|g(x) - g(0)| = |\sqrt[3]{x} - 0| = |\sqrt[3]{x}| = |x|^{1/3} < \delta^{1/3} = (\epsilon^3)^{1/3} = \epsilon.$$

Therefore, g is continuous at $c = 0$. $\square$

(b) *Proof.* Let $c \neq 0$ and $\epsilon > 0$. We need to find $\delta > 0$ such that whenever $|x - c| < \delta$, it follows that $|g(x) - g(c)| < \epsilon$.
Using the identity $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$, set $a = \sqrt[3]{x}$ and $b = \sqrt[3]{c}$. Then,

$$x - c = (\sqrt[3]{x} - \sqrt[3]{c})\left((\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2\right).$$

Therefore,

$$\sqrt[3]{x} - \sqrt[3]{c} = \frac{x - c}{(\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2}.$$

Since $c \neq 0$, we can assume x is sufficiently close to c such that $\sqrt[3]{x}$ is bounded away from 0. Let $M = \max(|\sqrt[3]{c}|, 1)$. If $|x - c| < 1$, then $|\sqrt[3]{x}| \leq M + 1$.
Thus, the denominator can be bounded as:

$$(\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2 \geq (\sqrt[3]{c})^2 > 0,$$

and for x close to c, we have:

$$(\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2 \leq 3(M+1)^2.$$

Now, using the rewritten expression for $|g(x) - g(c)|$:

$$|g(x) - g(c)| = \left|\frac{x - c}{(\sqrt[3]{x})^2 + \sqrt[3]{x}\sqrt[3]{c} + (\sqrt[3]{c})^2}\right|.$$

For $|x - c| < \delta$, we have:

$$|g(x) - g(c)| \leq \frac{|x - c|}{(\sqrt[3]{c})^2}.$$

To ensure $|g(x) - g(c)| < \epsilon$, choose:

$$\delta = \epsilon(\sqrt[3]{c})^2.$$

If $|x - c| < \delta$, then

$$|g(x) - g(c)| \leq \frac{|x - c|}{(\sqrt[3]{c})^2} < \frac{\epsilon(\sqrt[3]{c})^2}{(\sqrt[3]{c})^2} = \epsilon.$$

By choosing $\delta = \epsilon(\sqrt[3]{c})^2$, we ensure that $|x - c| < \delta$ implies $|g(x) - g(c)| < \epsilon$. Therefore, $g(x) = \sqrt[3]{x}$ is continuous at $c \neq 0$. $\square$

Exercise: 4.3.8

Decide if the following claims are true or false, providing either a short proof or counterexample to justify each conclusion. Assume throughout that g is defined and continuous on all of $\mathbb{R}$.

(a) If $g(x) \geq 0$ for all $x < 1$, then $g(1) \geq 0$ as well.

(b) If $g(r) = 0$ for all $r \in \mathbb{Q}$, then $g(x) = 0$ for all $x \in \mathbb{R}$.

Solution.

(a) **True.**

Since g is continuous at $x = 1$, then by Definition 4.3.1, we have:

$$g(1) = \lim_{x \to 1} g(x).$$

Since $g(x) \geq 0$ for all $x < 1$, any sequence (x_n) converging to 1 with $x_n < 1$ will have $g(x_n) \geq 0$. Then, by the Sequential Criterion for Functional Limits (Theorem 4.2.2), we have:

$$\lim_{n \to \infty} g(x_n) \geq 0.$$

By using the definition of continuity again, we have $g(1) = \lim_{x \to 1} g(x)$ so $g(1) \geq 0$.

(b) **True.**

Since g is continuous on $\mathbb{R}$ and $g(r) = 0$ for all rational numbers r, it follows that for any $x \in \mathbb{R}$, we can find a sequence of rational numbers (r_n) converging to x. By definition of continuity,

$$g(x) = \lim_{n \to \infty} g(r_n) = \lim_{n \to \infty} 0 = 0.$$

Therefore, $g(x) = 0$ for all $x \in \mathbb{R}$.

Exercise: 4.3.9

Assume $h \colon \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K = \{x \colon h(x) = 0\}$. Show that K is a closed set.

Proof. To show that K is closed, we will show that K contains all its limit points.

Let (x_n) be a sequence in K that converges to some limit $x \in \mathbb{R}$. Since h is continuous on $\mathbb{R}$ and $h(x_n) = 0$ for all n, it follows that:

$$\lim_{n \to \infty} h(x_n) = h\left(\lim_{n \to \infty} x_n\right) = h(x).$$

Therefore, $h(x) = 0$, which means $x \in K$.

Since every limit point of K is contained in K, the set K is closed. $\square$

Exercise: 4.3.11 (Contraction Mapping Theorem)

Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence (Definition 2.6.1). Hence, we may let $y = \lim y_n$.

(c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

(d) Finally, prove that if x is *any* arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y defined in (b).

Solution.

(a) Let $\epsilon > 0$. Choose $\delta = \dfrac{\epsilon}{c}$. Then, for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq c|x - y| < c\left(\frac{\epsilon}{c}\right) = \epsilon.$$

Thus, f is continuous at every point in $\mathbb{R}$.

(b) *Proof.* Since $y_{n+1} = f(y_n)$, we have $y_2 = f(y_1)$, $y_3 = f(y_2) = f(f(y_1))$, and so on. Thus, the difference between consequent terms in the sequence is:

$$|y_{n+1} - y_n| = |f(y_n) - f(y_{n-1})| \leq c|y_n - y_{n-1}|.$$

Substituting, we see that:

$$|y_{n+1} - y_n| \leq c|y_n - y_{n-1}|.$$

This inequality shows that our sequence is contracting (Definition 2.6.3) with $0 < c < 1$. Thus, by Theorem 2.6.4, the sequence is Cauchy. □

(c) Taking the limit as $n \to \infty$, and using the continuity of f, we have:

$$f(y) = f\left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} y_{n+1} = y.$$

Suppose there is another fixed point z such that $f(z) = z$. Then

$$|y - z| = |f(y) - f(z)| \leq c|y - z|.$$

Since $0 < c < 1$, this implies $y - z = 0$, so $y = z$. Thus, the fixed point is unique.

(d) From the work we did in (b), we know the fixed point must be unique. Therefore, for any $x \in \mathbb{R}$, it must be the case that the sequence $(x, f(x), f(f(x)), \ldots)$ converges to the fixed point y.

Exercise: 4.3.13

Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Solution.

(a) For $f(0)$, we have:

$$\begin{aligned} f(0) &= f(0+0) \\ f(0) &= f(0)+f(0) \\ f(0) &= 2f(0) \\ f(0)-f(0) &= 2f(0)-f(0) \\ 0 &= f(0). \end{aligned}$$

For all $x \in \mathbb{R}$, we have:

$$\begin{aligned} f(0) &= f(x+(-x)) \\ 0 &= f(x)+f(-x) \\ -f(x) &= f(-x). \end{aligned}$$

(b) Let $k = f(1)$.

First, we will show that $f(n) = kn$ for all $n \in \mathbb{N}$ by induction.

Base case: For $n = 1$,

$$f(1) = k = k \cdot 1.$$

Inductive step: Assume $f(n) = kn$ for some arbitrary $n \in \mathbb{N}$. Then,

$$\begin{aligned} f(n+1) &= f(n)+f(1) \\ &= kn+k \\ &= k(n+1). \end{aligned}$$

Thus, by induction, $f(n) = kn$ for all $n \in \mathbb{N}$.

Next, from (a), we have $f(-x) = -f(x)$. Hence, for $z \in \mathbb{Z}$, if $z = -n$ where $n \in \mathbb{N}$,

$$f(z) = f(-n) = -f(n) = -kn = k(-n) = kz. \tag{1}$$

Therefore, $f(z) = kz$ for all integers z.

Now, observe that for $p, q \in \mathbb{Z}$ with $q \neq 0$, we have:

$$f\left(\underbrace{\frac{p}{q}+\frac{p}{q}+\cdots+\frac{p}{q}}_{q \text{ times}}\right) = f(p).$$

Similarly, by the additive condition (q times on each side):

$$f\left(\frac{p}{q}+\frac{p}{q}+\cdots+\frac{p}{q}\right) = f\left(\frac{p}{q}\right)+f\left(\frac{p}{q}\right)+\cdots+f\left(\frac{p}{q}\right).$$

This is equivalent to:
$$q \cdot f\left(\frac{p}{q}\right) = f(p).$$
Therefore,
$$f\left(\frac{p}{q}\right) = \frac{1}{q} f(p)$$
Putting everything together, we let $r = \frac{p}{q}$. Then,
$$\begin{aligned} f\left(\frac{p}{q}\right) &= \frac{1}{q} f(p) \\ &= \frac{1}{q}(kp) \quad \text{(from (1))} \\ &= k\left(\frac{p}{q}\right) \\ f(r) &= kr. \end{aligned}$$
Thus, $f(r) = kr$ for any rational number r.

(c) Since f is continuous at $x = 0$, we will show f is continuous everywhere.

Let $c \in \mathbb{R}$ and let $\epsilon > 0$. Since f is continuous at 0, there exists $\delta > 0$ such that if $|h| < \delta$, then $|f(h)| < \epsilon$.

For $x = c + h$ with $|h| < \delta$, we have:
$$\begin{aligned} |f(x) - f(c)| &= |f(c+h) - f(c)| \\ &= |f(c) + f(h) - f(c)| \\ &= |f(h)| \\ &< \epsilon. \end{aligned}$$
Thus, f is continuous at c.

Since f agrees with the continuous function kx on all rational numbers and f is continuous on $\mathbb{R}$, it follows that $f(x) = kx$ for all $x \in \mathbb{R}$.

Therefore, any additive function that is continuous at $x = 0$ must be linear, $f(x) = kx$.

Exercise: 4.4.3

Show that $f(x) = \dfrac{1}{x^2}$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Solution.

(a) **Uniform Continuity on $[1, \infty)$:**

Observe that for all $x, y \in [1, \infty)$,

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = |y - x| \left(\frac{y + x}{x^2y^2}\right).$$

Since $x, y \geq 1$, we have $x^2y \geq 1$ and $xy^2 \geq 1$. Therefore,

$$\frac{y + x}{x^2y^2} = \frac{y}{x^2y^2} + \frac{x}{x^2y^2} \leq \frac{1}{x^2y} + \frac{1}{xy^2} \leq 1 + 1 = 2.$$

Now, let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Then, when $|x - y| < \delta$ for $x, y \in [1, \infty)$, we have:

$$|f(x) - f(y)| = |y - x| \left(\frac{y + x}{x^2y^2}\right) < \delta \cdot 2 = \epsilon.$$

(b) **Not Uniformly Continuous on $(0, 1]$:**

We will show that f is not uniformly continuous on $(0, 1]$ by demonstrating that no matter how small we choose $\delta > 0$, there exist $x, y \in (0, 1]$ such that $|x - y| < \delta$ but $|f(x) - f(y)|$ is arbitrarily large.

Let $\epsilon = 1$. For the sake of contradiction, assume that f is uniformly continuous on $(0, 1]$; then, there exists $\delta > 0$ such that for all $x, y \in (0, 1]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Choose $x = \frac{\delta}{2}$ and $y = \frac{\delta}{2} + h$ for some h with $0 < h < \frac{\delta}{2}$. Then $|x - y| = h < \delta$. Compute $|f(x) - f(y)|$:

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| = \left|\frac{y^2 - x^2}{x^2y^2}\right| = \frac{|y^2 - x^2|}{x^2y^2}.$$

As $x \to 0^+$, $x^2y^2 \to 0$, making the denominator very small and the entire expression very large.

Specifically, take $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{1}{n+1}$ for $n \in \mathbb{N}$. Then,

$$|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \delta \quad \text{for large } n.$$

However,

$$|f(x_n) - f(y_n)| = \left|n^2 - (n+1)^2\right| = \left|n^2 - n^2 - 2n - 1\right| = 2n + 1.$$

Thus, as $|x_n - y_n| \to 0$, $|f(x_n) - f(y_n) \to \infty$. Therefore, by the Sequential Criterion for Absence of Uniform Continuity theorem, f is not uniformly continuous on $(0, 1]$.

Exercise: 4.4.8

Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0, 1]$ with range $(0, 1)$.

(b) A continuous function defined on $(0, 1)$ with range $[0, 1]$.

(c) A continuous function defined on $(0, 1]$ with range $(0, 1)$.

Solution.

(a) **Impossible**. There does not exist a continuous function defined on $[0, 1]$ with range $(0, 1)$. This is because $[0, 1]$ is a compact set, and the continuous image of a compact set is also compact by the Preservation of Compact Sets theorem. However, $(0, 1)$ is not a compact set since it is not closed in $\mathbb{R}$. Therefore, a continuous function cannot map $[0, 1]$ onto $(0, 1)$.

(b) **Possible**. An example of a continuous function defined on $(0, 1)$ with range $[0, 1]$ is:

$$f(x) = \frac{1}{2}\sin(4\pi x) + \frac{1}{2}.$$

This is because $\sin(4\pi x)$ oscillates between -1 and 1. Therefore, $\frac{1}{2}\sin(4\pi x)$ oscillates between $-\frac{1}{2}$ and $\frac{1}{2}$. Adding $\frac{1}{2}$ shifts the range to $[0, 1]$.

(c) **Impossible**. There does not exist a continuous function defined on $(0, 1]$ with range $(0, 1)$. Since $x = 1$ is in the domain $(0, 1]$ and f is continuous at $x = 1$, the value $f(1)$ exists and must be in the range $(0, 1)$. However, because f approaches $f(1)$ as $x \to 1^-$, the function attains its supremum at $x = 1$, meaning $f(1)$ should be included in the range. But the range $(0, 1)$ excludes its endpoints, leading to a contradiction. Therefore, such a function cannot exist.

Exercise: 1.2.13

For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where

$$\bigcap_{i=1}^{n} B_i \neq \emptyset \quad \text{is true for every } n \in \mathbb{N},$$

but

$$\bigcap_{i=1}^{\infty} B_i = \emptyset$$

fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Proof. In this proof, we plan to prove (c). Thus, we need to show that:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$$

and

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

(⊆) Let $x \in (\bigcup_{i=1}^{\infty} A_i)^c$. This means x is in the union set of A_i for all $i \in \mathbb{N}$. Then, because we are taking the complement of $(\bigcup_{i=1}^{\infty} A_i)$, that means $x \notin A_i$ for all $i \in \mathbb{N}$. Hence, x is in the complement of each A_i. Thus, we can use the definition of intersection to assert $x \in \bigcap_{i=1}^{\infty} A_i^c$.
Therefore, we have shown:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c.$$

(⊇) Similar to before, let $x \in \bigcap_{i=1}^{\infty} A_i^c$. Because $x \in A_i^c$ for all $i \in \mathbb{N}$ we know $x \notin A_i$. Hence, $x \notin (\bigcup_{i=1}^{\infty} A_i)$, which means $x \in (\bigcup_{i=1}^{\infty} A_i)^c$.
Therefore, we have shown:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

By showing both inclusions, we see that:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c. \qquad \square$$

Exercise: 2.2.7

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually, or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution.

(a) The sequence $(-1)^n$ is *frequently* in the set $\{1\}$ because for every $N \in \mathbb{N}$, we can find an $n \geq N$ such that $(-1)^n = 1$.

(b) The definition of *eventually* is stronger because *eventually* implies *frequently*, but *frequently* does not imply *eventually*.

(c) An alternate rephrasing of Definition 2.2.3B using *eventually* is: A sequence (a_n) converges to a if, given any ϵ-neighborhood—$V_\epsilon(a)$ of a—(a_n) is *eventually* in $V_\epsilon(a)$. The term we want is eventually.

(d) If an infinite number of terms of a sequence (x_n) are equal to 2, (x_n) is not *eventually* in $(1.9, 2.1)$ because we can have a sequence (a_n) that will not settle in $(1.9, 2.1)$. For example, $(a_n) = (0, 2, 0, 2, \cdots)$ does not settle in $(1.9, 2.1)$. Whereas, (x_n) is *frequently* in the interval $(1.9, 2.1)$ because for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $x_n \in (1.9, 2.1)$ for all $n \geq N$. We can see an instance of this being true by examining the previous example.

Definition 2.2.3B

A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).

Exercise: 4.5.4

Let g be continuous on an interval A and let F be the set of points where g fails to be one-to-one; that is,

$$F = \{x_1 \in A : f(x_1) = f(x_2) \text{ for some } x_1 \neq x_2 \text{ and } x_2 \in A\}.$$

Show F is either empty or uncountable.

Proof. Assume F is not empty; that is, there exists distinct points $x_1, x_2 \in A$ such that $g(x_1) = g(x_2)$. (Note that if F were empty, then we would be done.)

Without loss of generality, assume $x_1 < x_2$.

From here, we consider two cases. Either g is constant, or g is not constant on $[x_1, x_2]$. For the first case:

Case 1: g is constant on $[x_1, x_2]$.

Since $g(x) = y$ for all $x \in [x_1, x_2]$, every point in this interval belongs to F. Then, because $[x_1, x_2]$ is uncountable (intervals in $\mathbb{R}$ contain uncountably many points), it follows that F is uncountable.

Case 2: g is not constant on $[x_1, x_2]$.

Because g is not constant, there exists some $c \in (x_1, x_2)$ such that $g(c) \neq y$. By the continuity of g, we can apply the Intermediate Value Theorem (IVT) to analyze g on the intervals $[x_1, c]$ and $[c, x_2]$:

(i) On $[x_1, c]$: $g(x_1) = y$ and $g(c) \neq y$. By the IVT, $g(x)$ attains every value between y and $g(c)$ on $[x_1, c]$.

(ii) On $[c, x_2]$: $g(c) \neq y$ and $g(x_2) = y$. By the IVT, $g(x)$ attains every value between $g(c)$ and y on $[c, x_2]$.

For any value k between y and $g(c)$, there exist at least two points $x_1' \in [x_1, c]$ and $x_2' \in [c, x_2]$ such that $g(x_1') = g(x_2') = k$. These points x_1' and x_2' are distinct because they belong to different subintervals. Thus, every such k corresponds to multiple $x \in [x_1, x_2]$ where $g(x) = g(x')$, and these x belong to F.

Since there are uncountably many such k (as the range of g over $[x_1, x_2]$ is uncountable), it follows that there are uncountably many $x \in [x_1, x_2]$ in F.

Therefore, because there are uncountably many such x, it follows that F is uncountable. □

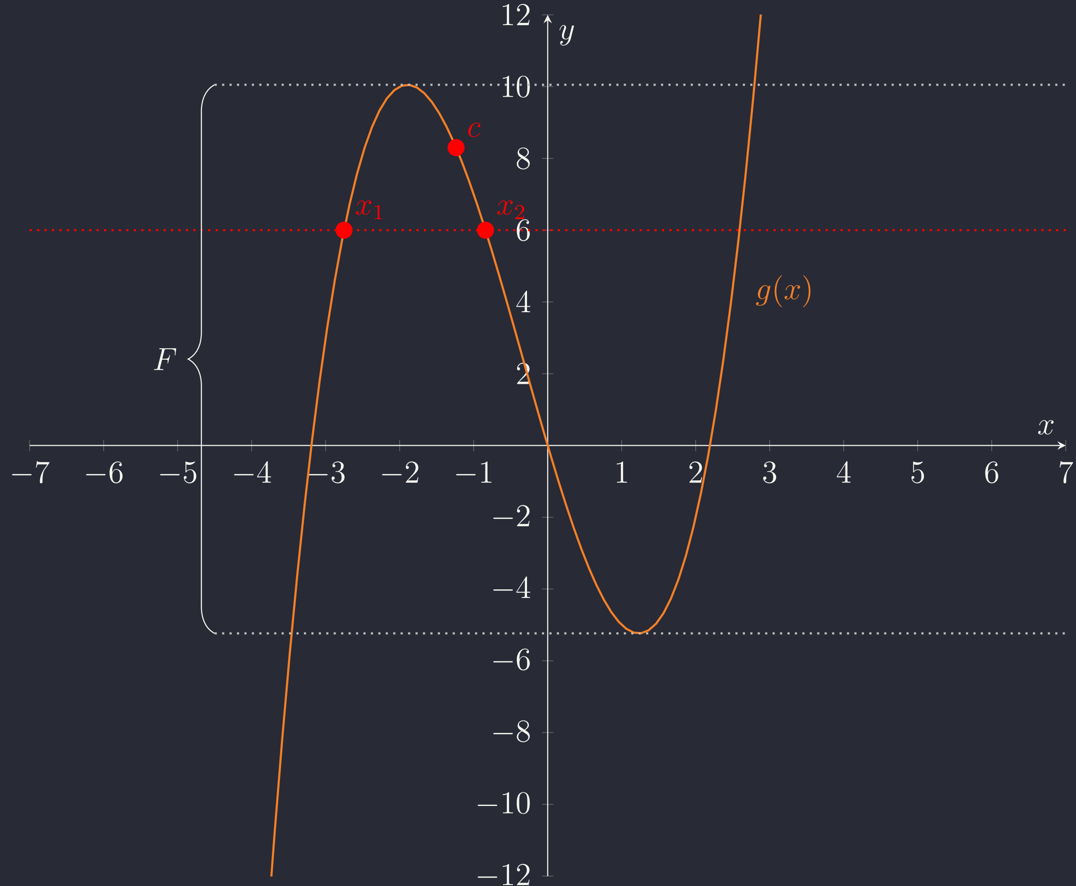


Figure 5.1: Graph of $g(x) = x^3 + x^2 - 7x$ (Non-empty)