Probability and Statistics: Practice Set 3

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October 2, 2025

1. (2 points each) Consider the function $f(x) = cx^5$ on $[0, 1]$.

   (a) Find the value of c for which this is a probability density function.

   Solution.

   $$\int_0^1 cx^5\,dx = 1 \implies c\int_0^1 x^5\,dx = 1 \implies c\left[\frac{x^6}{6}\right]_0^1 = 1 \implies c \cdot \frac{1}{6} = 1 \implies c = 6.$$

   (b) Using this value of c, determine $E(X)$.

   Solution.

   $$E(X) = \int_0^1 x \cdot 6x^5\,dx = 6\left[\frac{x^7}{7}\right]_0^1 = \frac{6}{7}.$$

   (c) Again, using this c, determine $\mathrm{Var}(X)$.

   Solution.

   $$\begin{aligned}
   \mathrm{Var}(X) &= \int_0^1 \left(x - \frac{6}{7}\right)^2 6x^5\,dx \\
   &= \int_0^1 \left(x^2 - \frac{12}{7}x + \frac{36}{49}\right) 6x^5\,dx \\
   &= \int_0^1 \left(6x^7 - \frac{72}{7}x^6 + \frac{216}{49}x^5\right) dx \\
   &= \frac{6}{8} - \frac{72}{49} + \frac{36}{49} \\
   &= \frac{3}{196}.
   \end{aligned}$$

   (d) Find $\pi_{0.25}$.

   Solution.

   $$\int_0^b 6x^5\,dx = 0.25 \implies 6\left[\frac{x^6}{6}\right]_0^b = 0.25 \implies b^6 = 0.25 \implies b = 0.25^{\frac{1}{6}} \implies b \approx 0.7937.$$

   Therefore, $\pi_{0.25} \approx 0.7937$.

2. (2 points) Consider the function $f(x) = \dfrac{\ln(x)}{x^2}$ on the interval $[1, \infty)$.

(a) Show that this is a probability distribution. [Hint: You will likely need integration by parts here.]

Solution. We choose $u = \ln(x)$ so that $du = \frac{1}{x}$, and $dv = \frac{1}{x^2}$ so that $v = -\frac{1}{x}$. This leaves us with the following:

$$
\begin{aligned}
[uv]_1^\infty - \int_1^\infty v\,du &= \left[-\frac{\ln(x)}{x}\right]_1^\infty + \int_1^\infty \frac{1}{x^2}\,dx \\
&= \left(\frac{\ln(1)}{1} - \lim_{x\to\infty} \frac{\ln(x)}{x}\right) - \left(\lim_{x\to\infty} \frac{1}{x} - \frac{1}{1}\right) \\
&= (0-0) - (0-1) \\
&= 1
\end{aligned}
$$

Since $f(x) \geq 0$ for all $x \in [1, \infty)$ and the integral over this interval is 1, this is a valid probability density function.

(b) Show that $E(X)$ is undefined.

Solution.

$$E(X) = \int_1^\infty x \cdot \frac{\ln(x)}{x^2}\,dx = \int_1^\infty \frac{\ln(x)}{x}\,dx.$$

For this problem, we can use u-substitution. Let $u = \ln(x)$, then $du = \frac{1}{x}\,dx$, which transforms the limits from 1 to ∞ into 0 to ∞:

$$E(X) = \int_0^\infty u\,du = \left[\frac{u^2}{2}\right]_0^\infty = \infty.$$

Therefore, $E(X)$ is undefined.

3. (2 points) A pdf is given by $f(x) = \dfrac{1}{2}$ for $x \in [0,1] \cup [2,3]$, and 0 otherwise. [i.e., this is like a uniform distribution, but with a "gap" from 1 to 2.] Determine $E(X)$ and $\mathrm{Var}(X)$.

Solution.

$$E(X) = \int_0^1 x\frac{1}{2}\,dx + \int_2^3 x\frac{1}{2}\,dx = \left[\frac{x^2}{4}\right]_0^1 + \left[\frac{x^2}{4}\right]_2^3 = \left[\frac{1}{4} - 0\right] + \left[\frac{9}{4} - 1\right] = \frac{3}{2}$$

$$\mathrm{Var}(x) = \int_0^1 x^2\frac{1}{2}\,dx + \int_2^3 x^2\frac{1}{2}\,dx - \frac{9}{4} = \left[\frac{x^3}{6}\right]_0^1 + \left[\frac{x^3}{6}\right]_2^3 - \frac{9}{4} = \left[\frac{1}{6} - 0\right] + \left[\frac{27}{6} - \frac{8}{6}\right] - \frac{9}{4} = \frac{13}{12}$$

4. (2 points each) We have shown in class that the gamma function Γ has the two properties:

- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
- $\Gamma(x) = (x-1)\Gamma(x-1)$.

Use these to find the exact values of:

(a) $\Gamma\left(\frac{5}{2}\right)$.

Solution.

$$\Gamma\left(\frac{5}{2}\right) = \left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi} = \frac{3\sqrt{\pi}}{4}$$

(b) $\Gamma\left(-\frac{7}{2}\right)$.

Solution.

$$\Gamma(x) = (x-1)\Gamma(x-1)$$

$$\frac{\Gamma(x)}{(x-1)} = \Gamma(x-1)$$

Let $z = x - 1$:

$$\frac{\Gamma(z+1)}{z} = \Gamma(z)$$

With this formula:

$$\Gamma\left(-\frac{2}{7}\right) = \left(-\frac{2}{7}\right)\left(-\frac{2}{5}\right)\left(-\frac{2}{3}\right)\left(-\frac{2}{1}\right)\sqrt{\pi} = \frac{16\sqrt{\pi}}{105}$$

5. (2 points each) Customers arrive at a certain bank according to an approximate Poisson process at a mean rate of 15 per hour.

   (a) What is the probability that between 10 and 13 customers come in a particular hour?

   Solution. Let X count the number of customers in 1 particular hour. Then,

$$P(10 \le X \le 13) = P(X \le 13) - P(X \le 9) = 0.2934.$$

   Therefore, there is a 29.3% chance that between 10 and 13 customers come in a particular hour.

   (b) What is the probability that the first customer comes between 5 and 10 minutes into the bank's opening?

   Solution. Let X be the waiting time for the first customer to arrive. Using the following theta:

$$\theta = \frac{1 \text{ hour}}{15 \text{ customers}} \cdot \frac{60 \text{ minutes}}{1 \text{ hour}} = 4 \text{ minutes per customer},$$

   we can plug this into the equation:

$$P(5 \le X \le 10) = P(X \le 10) - P(X \le 5) = 1 - e^{-10/4} - 1 + e^{-5/4} = 0.2044$$

   Therefore, there is a 20.4% chance that the first customer comes between 5 and 10 minutes into the bank's opening.

   (c) What is the probability that the *third* customer arrives between 5 and 10 minutes into the bank's opening?

   Solution. Let $\alpha = 3$ with same theta from the previous part. Then:

$$\int_5^{10} \frac{1}{\Gamma(\alpha)\theta^\alpha} \cdot x^{\alpha-1} e^{-x/\theta}\, dx = \int_5^{10} \frac{x^2 e^{-x/4}}{32}\, dx = 0.3247.$$

   Therefore, there is a 32.5% chance that the third customer comes between 5 and 10 minutes into the bank's opening.

6. (1 point each) The mean airspeed of an unladen swallow is 30 ft/s, with a standard deviation of 4.3 ft/s. Assuming the distribution of speed is normal:

(a) What is the probability that a randomly selected swallow will have a speed between 28 and 38 ft/s?

Solution.

$$P(28 \le X \le 38) = \frac{1}{4.3\sqrt{2\pi}} \int_{28}^{38} \exp\left(-\frac{(t-30)^2}{2(4.3)^2}\right)\, dt = 0.6477.$$

(b) Suppose that 12 swallows are selected at random. What is the probability that exactly 8 have their speed between 28 and 38 ft/s?

Solution. Let Y count the birds that have their speed between 28 and 38 ft/s. This implies $Y \sim Binomial(12, 0.6477)$. Hence,

$$P(Y = 8) = \binom{12}{8} 0.6477^8 \,(1 - 0.6477)^4 = 0.2361.$$

(c) Determine $\pi_{0.90}$ for the swallow speed.

Solution.

$$\pi_{0.90} = \text{InvNorm}(.9, 30, 4.3, \text{LEFT}) = 35.5107.$$

7. (3 points) An auto insurance company insures an automobile worth \$15,000 for one year under a policy with a \$1,000 deductible. During the policy year there is a 0.04 chance of partial damage to the car and a 0.02 chance of a total loss of the car. If there is partial damage to the car, the amount X of damage (in thousands) follows a distribution with density function $f(x) = 0.5003e^{-x/2}$, for $0 \leq x \leq 15$. Find the expected value of the payment the insurance company makes in a year.

Solution. Let E be the expected payment for the insurance company. We have three cases to consider:

1. **No Damage:** The probability of no damage is 0.94. In this case, the insurance company pays \$0.
2. **Total Loss:** The probability of total loss is 0.02. The payment is the value of the car minus the deductible, which is \$15,000 − \$1,000 = \$14,000.
3. **Partial Damage:** The probability of partial damage is 0.04. The damage amount X, in thousands of dollars, follows the probability density function $f(x) = 0.5003e^{-x/2}$ for $0 \leq x \leq 15$. The deductible is \$1,000. The insurance payment for partial damage is therefore $1000(X - 1)$ if $X > 1$ and 0 otherwise.

The expected payment is the sum of the probabilities of each event multiplied by the corresponding payment amount. Let Y correspond to the payment for partial damage:

$$E = (0.94 \cdot \$0) + (0.02 \cdot \$14{,}000) + (0.04 \cdot E[Y]).$$

To find the expected payment for partial damage, we calculate the following integral:

$$\begin{aligned} E[Y] &= \int_1^{15} 1000(x-1)(0.5003e^{-x/2})\,dx \\ &= 500.3 \int_1^{15} (x-1)e^{-x/2}\,dx \\ &\approx \$1211.96. \end{aligned}$$

Now, we can find the total expected payment:

$$E = \$0 + \$280 + 0.04 \cdot \$1211.96 = \$280 + \$48.48 = \$328.48$$

Therefore, the expected value of the payment the insurance company makes in a year is approximately **\$328.48**.