HENDRIX
COLLEGE

Homework 5: Sections 10 & 11

Algebra

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Section 10

9. Find all left cosets of the subgroup $\{\rho_0, \rho_2\}$ of the group D_4 given by Table 8.12.

Solution. There are only 4 distinct left cosets:

$$\{\rho_0, \rho_2\}, \{\rho_1, \rho_3\}, \{\mu_1, \mu_2\}, \{\delta_1, \delta_2\}.$$

11. Rewrite Table 8.12 in the order exhibited by the left cosets in Exercise 9. Do you seem to get a coset group of order 4? If so, is it isomorphic to $\mathbb{Z}_4$ or the Klein 4-group V?

Solution. The multiplication table on the left is made up of left cosets for D_4. The subgroup $H = \{\rho_0, \rho_2\}$ partitions D_4 into the four left cosets from Exercise 9. Each colored block corresponds to one of these cosets. The table on the right is the multiplication table for the Klein 4-group. Notice that

$$\{\rho_0, \rho_2\} = e, \ \{\rho_1, \rho_3\} = a, \ \{\mu_1, \mu_2\} = b, \ \{\delta_1, \delta_2\} = c.$$

Hence, $D_4/H \simeq V$.

	ρ_0	ρ_2	ρ_1	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_0	ρ_0	ρ_2	ρ_1	ρ_3	μ_1	μ_2	δ_1	δ_2
ρ_2	ρ_2	ρ_0	ρ_3	ρ_1	μ_2	μ_1	δ_2	δ_1
ρ_1	ρ_1	ρ_3	ρ_2	ρ_0	δ_1	δ_2	μ_2	μ_1
ρ_3	ρ_3	ρ_1	ρ_0	ρ_2	δ_2	δ_1	μ_1	μ_2
μ_1	μ_1	μ_2	δ_2	δ_1	ρ_0	ρ_2	ρ_3	ρ_1
μ_2	μ_2	μ_1	δ_1	δ_2	ρ_2	ρ_0	ρ_1	ρ_3
δ_1	δ_1	δ_2	μ_1	μ_2	ρ_1	ρ_3	ρ_0	ρ_2
δ_2	δ_2	δ_1	μ_2	μ_1	ρ_3	ρ_1	ρ_2	ρ_0

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

(I had a lot of fun making these tables. Thanks for the assigning this awesome problem!)

15. Let $\sigma = (1,2,5,4)(2,3)$ in S_5. Find the index of $\langle\sigma\rangle$ in S_5.

> *Solution.* We can rewrite σ as $(1,2,3,5,4)$ to help with composing. Following that, we get $\sigma^2 = (1,3,4,2,5)$, $\sigma^3 = (1,5,2,4,3)$, $\sigma^4 = (1,4,5,3,2)$, and $\sigma^5 = e$. Therefore, $|\langle\sigma\rangle| = 5$, and the index of $\langle\sigma\rangle$ in S_5 is $|S_5|/|\langle\sigma\rangle| = 120/5 = 24$

16. Let $\mu = (1,2,4,5)(3,6)$ in S_6. Find the index of $\langle\mu\rangle$ in S_6.

> *Solution.* Because we are using disjoint cycles, the transposition cycle is the identity element when the power of μ is even (from Example 9.14). We can make the same connection for the 4-element cycle: when the power of μ is a multiple of 4, it is equal to the identity element. Thus, we know that $\mu^4 = e$, $|\langle\mu\rangle| = 4$, and the index of $\langle\mu\rangle$ in S_6 is $720/4 = 180$.

29. Let H be a subgroup of a group G. Prove that if the partition of G into left cosets of H is the same as the partition into right cosets of H, then $g^{-1}hg \in H$ for all $g \in G$ and all $h \in H$. (Note that this is the converse of Exercise 28.)

> *Solution.* The statement "If the partition of G into left cosets of H is the same as the partition into right cosets of H," implies $gH = Hg$ for all $g \in G$. Now, let $g \in G$ and $h \in H$. Our goal is to show that $g^{-1}hg \in H$. Consider the element hg. Since $h \in H$, $hg \in Hg$. Because $Hg = gH$, it must be that $hg \in gH$. By definition of left coset, $hg \in gH$ means that $hg = gh'$ for some $h' \in H$. So, we just solve for this h'. We start by multiplying both sides on the left by g^{-1}:
>
> $$\begin{aligned} g^{-1}(hg) &= g^{-1}(gh') \\ g^{-1}hg &= (g^{-1}g)h' \\ g^{-1}hg &= h'. \end{aligned}$$
>
> Since $h' \in H$, we have shown that $g^{-1}hg \in H$.

44. Let S_A be the group of all permutations of the set A, and let c be one particular element of A.

(a) Show that $\{\sigma \in S_A \mid \sigma(c) = c\}$ is a subgroup $S_{c,c}$ of S_A.

Solution.

- **Identity:** The identity permutation e maps every element to itself, so $e(c) = c$. Thus, $e \in S_{c,c}$.
- **Closure:** Let $\sigma, \tau \in S_{c,c}$. This means $\sigma(c) = c$ and $\tau(c) = c$. We must check their product:

$$(\sigma\tau)(c) = \sigma(\tau(c)) = \sigma(c) = c.$$

- **Inverses:** Let $\sigma \in S_{c,c}$. As before, this means $\sigma(c) = c$, and we also have to check σ^{-1}. We will do this by multiplying σ^{-1} to both sides of the equation:

$$\begin{aligned} \sigma^{-1}(\sigma(c)) &= \sigma^{-1}(c) \\ (\sigma^{-1}\sigma)(c) &= \sigma^{-1}(c) \\ e(c) &= \sigma^{-1}(c) \\ c &= \sigma^{-1}(c). \end{aligned}$$

Because $\sigma^{-1}(c) = c$, the inverse σ^{-1} is in $S_{c,c}$.

Since $S_{c,c}$ contains the identity, is closed under the binary operation, and is closed under inverses, it is a subgroup of S_A.

(b) Let $d \neq c$ be another particular element of A. Is $S_{c,d} = \{\sigma \in S_A \mid \sigma(c) = d\}$ a subgroup of S_A? Why or why not?

Solution. No, it is not a subgroup because it does not contain the identity element. The condition to be in $S_{c,d}$ is that $\sigma(c) = d$. Since we are given $c \neq d$, we have $e(c) \neq d$. Therefore, $e \neq S_{c,d}$, and the set cannot be a subgroup.

Section 11

In Exercises 6 and 7, find the order of the given element of the direct product.

6. $(3, 10, 9)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{15}$.

Solution. For each pair, we find

$$\frac{4}{\gcd(3,4)} = 4, \quad \frac{12}{\gcd(10,12)} = 6, \quad \frac{15}{\gcd(9,15)} = 5.$$

Thus, $\operatorname{lcm}(4, 6, 5) = 60$.

7. $(3, 6, 12, 16)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{20} \times \mathbb{Z}_{24}$.

Solution. For each pair, we find

$$\frac{4}{\gcd(3,4)} = 4, \quad \frac{12}{6,12} = 2, \quad \frac{20}{\gcd(12,20)} = 5, \quad \frac{24}{\gcd(16,24)} = 3.$$

Thus, $\operatorname{lcm}(4, 2, 5, 3) = 60$.

16. Are the groups $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ isomorphic? Why or why not?

Solution. When we split these into two separate decompositions, we will see that they are isomorphic. First, we know that $\mathbb{Z}_{mn} \simeq \mathbb{Z}_n \times \mathbb{Z}_m \iff \gcd(n, m) = 1$. Since 12 is not a prime number and $\gcd(3, 4) = 1$, then

$$\mathbb{Z}_{12} \simeq \mathbb{Z}_3 \times \mathbb{Z}_4, \quad \text{and} \quad \mathbb{Z}_2 \times \mathbb{Z}_{12} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4.$$

The same thing applies for $\mathbb{Z}_6$. It decomposes into $\mathbb{Z}_2 \times \mathbb{Z}_3$ because $\gcd(2, 3) = 1$. Thus,

$$\mathbb{Z}_4 \times \mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4.$$

Since both $\mathbb{Z}_2 \times \mathbb{Z}_{12}$ and $\mathbb{Z}_4 \times \mathbb{Z}_6$ decompose into the same product, they are isomorphic.

In exercises 23 and 25, proceed as in Example 11.13 to find all abelian groups, up to isomorphism, of the given order.

23. Order 32.

Solution. Order 32 can be expressed as the product of 2^5. Then, by using Theorem 11.12, we get possibilities

(1) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

(2) $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

(3) $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$

(4) $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

(5) $\mathbb{Z}_8 \times \mathbb{Z}_4$

(6) $\mathbb{Z}_{16} \times \mathbb{Z}_2$

(7) $\mathbb{Z}_{32}$

Therefore, there are 7 abelian groups.

25. Order 1089.

Solution. Order 1089 can be expressed as the product of prime powers $3^2 11^2$. Thus,

(1) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$

(2) $\mathbb{Z}_9 \times \mathbb{Z}_{11} \times \mathbb{Z}_{11}$

(3) $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{121}$

(4) $\mathbb{Z}_9 \times \mathbb{Z}_{121}$

Therefore, there are 4 abelian groups.