

2. (2 points) Show that $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^4}$ does not exist.

Solution.

$$\begin{aligned} \bullet \ x = 0 \text{ path: } \lim_{(x,y) \rightarrow (0,0)} \frac{0 \cdot y^2}{0 + y^4} &= \frac{0}{y^2} = 0. & \bullet \ y = 0 \text{ path: } \lim_{(x,y) \rightarrow (0,0)} \frac{x \cdot 0}{x^2 + 0} &= \frac{0}{x^2} = 0. \\ \bullet \ x = y^2 \text{ path: } \lim_{(x,y) \rightarrow (0,0)} \frac{y^2 \cdot y^2}{y^4 + y^4} &= \frac{y^4}{2y^4} = \frac{1}{2}. \end{aligned}$$

Since the limit is not the same along all paths, the limit does not exist.

3. (2 points each) Find each indicated partial derivative:

(a) $\frac{\partial}{\partial x} (xy^2 \cos(x + y^3) - e^{xy})$

Solution.

$$\begin{aligned} \frac{\partial}{\partial x} (xy^2 \cos(x + y^3) - e^{xy}) &= y^2 \frac{\partial}{\partial x} [x \cos(x + y^3)] - \frac{\partial}{\partial x} [e^{xy}] \\ &= y^2 (\cos(x + y^3) + x(-\sin(x + y^3))) - ye^{xy} \\ &= y^2 (\cos(x + y^3) - x \sin(x + y^3)) - ye^{xy}. \end{aligned}$$

(b) $\frac{\partial}{\partial y} (\ln(x + y + z) - y^2 z^3 + x)$

Solution.

$$\begin{aligned} \frac{\partial}{\partial y} (\ln(x + y + z) - y^2 z^3 + x) &= \frac{\partial}{\partial y} [\ln(x + y + z)] - z^3 \frac{\partial}{\partial y} [y^2] + \frac{\partial}{\partial y} [x] \\ &= \frac{1}{x+y+z} - 2yz^3. \end{aligned}$$

(c) $\frac{\partial^2}{\partial x \partial y} (x^3 y - y^3 \tan(xy))$

Solution.

$$\begin{aligned} \frac{\partial^2}{\partial x \partial y} (x^3 y - y^3 \tan(xy)) &= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} [x^3 y] - \frac{\partial}{\partial y} [y^3 \tan(xy)] \right] \\ &= \frac{\partial}{\partial x} [x^3 - (3y^2 \tan(xy) + xy^3 \sec^2(xy))] \\ &= \frac{\partial}{\partial x} [x^3] - \frac{\partial}{\partial x} [3y^2 \tan(xy)] - \frac{\partial}{\partial x} [xy^3 \sec^2(xy)]. \end{aligned}$$

Splitting this into 3 partial derivatives:

$$\frac{\partial}{\partial x} [x^3] = 3x^2, \quad -\frac{\partial}{\partial x} [3y^2 \tan(xy)] = -3y^3 \sec^2(xy),$$

with the final derivative worked out:

$$\begin{aligned} -\frac{\partial}{\partial x} [xy^3 \sec^2(xy)] &= y^3 \sec^2(xy) + [(xy^3) \cdot 2y \sec^2(xy) \tan(xy)] \\ &= -y^3 \sec^2(xy) - 2xy^4 \sec^2(xy) \tan(xy). \end{aligned}$$

Combining these results, we have:

$$3x^2 - 3y^3 \sec^2(xy) - y^3 \sec^2(xy) - 2xy^4 \sec^2(xy) \tan(xy).$$

Since three terms contain a factor of $y^3 \sec^2(xy)$, we can factor this out to get:

$$3x^2 - y^3 \sec^2(xy)(3 + 1 + 2xy \tan(xy)).$$

Adding and simplifying further, we get:

$$\boxed{3x^2 - 2y^3 \sec^2(xy)(2 + xy \tan(xy))}.$$

4. (3 points) Complete each of the following steps to prove that $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2} = 0$.

Let $\epsilon > 0$. Choose $\delta = \epsilon/3$. Suppose that (x, y) is chosen so that $\|(x, y) - (0, 0)\| < \delta$ and $(x, y) \neq (0, 0)$.

- (a) Explain why $\sqrt{x^2 + y^2} < \delta$.

Solution. Since we have that $\|(x, y) - (0, 0)\| < \delta$, when we find the magnitude of this difference, we get:

$$\boxed{\sqrt{x^2 + y^2} < \delta}.$$

- (b) Explain why $x^2 \leq x^2 + y^2$, and thus $\frac{x^2}{(x^2 + y^2)} \leq 1$.

Solution. Because y^2 will always be positive and $x^2 = x^2$, it must be the case that $x^2 \leq x^2 + y^2$. Hence, when we divide both sides by $x^2 + y^2$, we get:

$$\boxed{\frac{x^2}{(x^2 + y^2)} \leq 1}.$$

- (c) Explain why $\frac{3x^2}{(x^2 + y^2)} \leq 3$.

Solution. Since $\frac{x^2}{(x^2 + y^2)} \leq 1$, multiplying both sides by 3 gives us:

$$\boxed{\frac{3x^2}{(x^2 + y^2)} \leq 3}.$$

- (d) Explain why $\frac{3x^2|y|}{(x^2 + y^2)} \leq 3|y|$.

Solution. Similarly to the previous step, we know that since $|y| \geq 0$, so when we multiply both sides of the inequality by $|y|$, the inequality is unchanged.

(e) Now, show that $\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leq 3\sqrt{x^2 + y^2}$.

Solution. First, note that $|y| \leq \sqrt{x^2 + y^2}$ (since $y^2 \leq x^2 + y^2$ by the same logic in (b)). Then, recall that $\sqrt{x^2 + y^2} < \delta$. We can multiply both sides by 3 to get $3\sqrt{x^2 + y^2} < 3\delta$. Since $\delta = \epsilon/3$, when we substitute this δ for ϵ in our equation, we get:

$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| \leq 3|y| \leq 3\sqrt{x^2 + y^2} < 3\delta = 3 \cdot \frac{\epsilon}{3} = \epsilon.$$

(f) Conclude that whenever (x, y) is in the δ -disk centered at $(0, 0)$, then $\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \epsilon$.

Solution. Combining the previous steps, we have shown that when (x, y) is in the δ -disk centered at $(0, 0)$ (i.e., $\|(x, y) - (0, 0)\|$), we have:

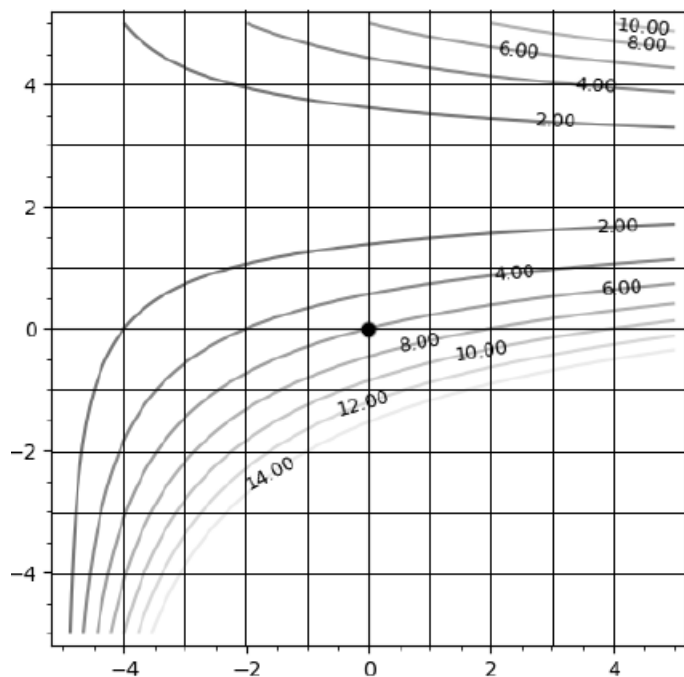
$$\left| \frac{3x^2y}{x^2 + y^2} - 0 \right| < \epsilon.$$

This shows that for every $\epsilon > 0$, we can choose a $\delta = \frac{\epsilon}{3}$ so that whenever $\|(x, y) - (0, 0)\| < \delta$, the inequality holds. Hence, this proves that:

$$\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y}{x^2 + y^2} = 0.$$

□

5. (1 point each) Consider the contour plot of the function $f(x, y)$ shown below.



Determine the sign (+, −, or 0) of each of the following partial derivatives, including a *brief* justification.

(a) $f_x(0,0)$

Solution. We see that $f(0,0) = 6$, and when we move in the x -direction, the function increases slightly. Thus, $f_x(0,0)$ is positive.

(b) $f_y(0,0)$

Solution. If we move in the y -direction, the function decreases slightly towards 4. Thus, $f_y(0,0)$ is negative.

(c) $f_{xx}(0,0)$

Solution. We can see in the plot that as we continue in the x -direction, the lines get closer together, indicating that the function is increasing. Thus, $f_{xx}(0,0)$ is positive.

(d) $f_{yy}(0,0)$

Solution. As we move in the y -direction, the lines get further apart, indicating that the function is decreasing. Hence, $f_{yy}(0,0)$ is negative.

(e) $f_{xy}(0,0)$

Solution. The function increasing in the x -direction, but when we start to move up, we can see that the lines begin to get further apart from each other. Therefore, $f_{xy}(0,0)$ is negative.

6. (2 points) Find an equation of the tangent plane to $f(x,y) = x^2y - \sqrt{x} + y$ at the point $(3,1)$.

Solution. To find the equation of the tangent plane, we need to fill out the following equation:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

First, we find z_0 by plugging in the point $(3,1)$ into the function f :

$$z_0 = f(3,1) = 3^2 \cdot 1 - \sqrt{3} + 1 = 9 - \sqrt{3} + 1 = 10 - \sqrt{3}.$$

Thus, we must find the partial derivatives of f :

$$f_x(x,y) = 2xy - \frac{1}{2}x^{-1/2}, \quad f_y(x,y) = x^2 + 1.$$

Plugging in the point $(3,1)$ into these partial derivatives, we get:

$$f_x(3,1) = \frac{12\sqrt{3}-1}{2\sqrt{3}}, \quad f_y(3,1) = 10.$$

Finally, we can plug in these values into the equation of the tangent plane to get:

$$z = 10 - \sqrt{3} + \frac{12\sqrt{3}-1}{2\sqrt{3}}(x-3) + 10(y-1).$$

7. (2 points) For the function $f(x, y, z) = \frac{x + \sin(xy)}{x^2 + y^2 + z^2 + 1}$, find $\nabla f(x, y, z)$.

Solution. From the handout, we know that:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z)\mathbf{i} + \frac{\partial f}{\partial y}(x, y, z)\mathbf{j} + \frac{\partial f}{\partial z}(x, y, z)\mathbf{k}.$$

Thus, we must find the partial derivatives of f :

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} \left[\frac{x + \sin(xy)}{x^2 + y^2 + z^2 + 1} \right] \\ &= \frac{(x^2 + y^2 + z^2 + 1)(1 + y \cos(xy)) - 2x(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2} \\ \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} \left[\frac{x + \sin(xy)}{x^2 + y^2 + z^2 + 1} \right] \\ &= \frac{(x^2 + y^2 + z^2 + 1)(x \cos(xy)) - 2y(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2} \\ \frac{\partial f}{\partial z} &= \frac{\partial}{\partial z} \left[\frac{x + \sin(xy)}{x^2 + y^2 + z^2 + 1} \right] \\ &= -\frac{2z(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2}\end{aligned}$$

Therefore, the gradient of f is:

$$\nabla f(x, y, z) = \left\langle \frac{(x^2 + y^2 + z^2 + 1)(1 + y \cos(xy)) - 2x(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2}, \frac{(x^2 + y^2 + z^2 + 1)(x \cos(xy)) - 2y(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2}, \frac{-2z(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2} \right\rangle.$$

8. (2 points) Consider the function $f(x, y) = x^2y - y^3$. Find the directional derivative for f , at $(3, 4)$, in the direction of $\mathbf{u} = 5\mathbf{i} - 2\mathbf{j}$.

Solution. We first find the partial derivative of f with respect to x and y :

$$f_x(x, y) = 2xy, \quad f_y(x, y) = x^2 - 3y^2.$$

Then, at the point $(3, 4)$:

$$f_x(3, 4) = 2 \cdot 3 \cdot 4 = 24, \quad f_y(3, 4) = 3^2 - 3 \cdot 4^2 = -39.$$

Then, we find the unit vector in the direction of $\langle 5, -2 \rangle$:

$$\frac{\langle 24, -39 \rangle \cdot \langle 5, -2 \rangle}{\sqrt{29}} = \frac{198}{\sqrt{29}}.$$