Multivariable Calculus Exam III Corrections

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In-Class Portion

1. (6 points) Set up, but do not evaluate the integral, including appropriate limits, to find the circulation of the vector field $\mathbf{F} = \cos(x\mathbf{i}) + xe^y\mathbf{j}$ along the curve $\mathbf{r}(t) = t^2\mathbf{i} + \sqrt{t}\mathbf{j}$, for $0 \leq t \leq 4$. [You should write down a definite integral with only t as a variable and with dt as the differential.]

   Solution. Given the formula:
   $$\int_C \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r},$$
   we can write the integral as:
   $$\int_0^4 \left\langle \cos(t^2), t^2e^{\sqrt{t}} \right\rangle \cdot \left\langle 2t, \frac{1}{2\sqrt{t}} \right\rangle dt = \int_0^4 \left(2t\cos(t^2) + t^2e^{\sqrt{t}}\frac{1}{2\sqrt{t}} \right) dt.$$

2. (6 points) Set up, but do not evaluate the integral, including appropriate limits, to find the flux of the vector field $\mathbf{F} = \cos(x\mathbf{i}) + xe^y\mathbf{j}$ across the curve $\mathbf{r}(t) = t^2\mathbf{i} + \sqrt{t}\mathbf{j}$, for $0 \leq t \leq 4$. [You should write down a definite integral with only t as a variable and with dt as the differential.]

   Solution. Given the formula:
   $$\int_C \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle \, dt,$$
   we can write the integral as:
   $$\int_0^4 \left\langle \cos(t^2), t^2e^{\sqrt{t}} \right\rangle \cdot \left\langle \frac{1}{2\sqrt{t}}, -2t \right\rangle dt = \int_0^4 \left(\cos(t^2)\frac{1}{2\sqrt{t}} - t^2e^{\sqrt{t}}(2t) \right) dt$$

6. (10 points) Set up the **line integral** that Stokes' Theorem would use to evaluate

$$\iint_S \nabla \times (x^2 z\mathbf{i} + xy^2\mathbf{j} + xy\mathbf{k}) \cdot d\mathbf{S},$$

where S is the part of the paraboloid $z = 1 - x^2 - y^2$ that lies above the xy-plane, oriented upward. **DO NOT** worry about working the integral - but you should write the eventual integral as a dt (or maybe $d\theta$) integral.

Solution. This paraboloid is bounded by the circle $x^2 + y^2 = 1$ in the xy-plane. This leaves us with the following bounding circle:

$$z = 0, \quad x^2 + y^2 = 1,$$

parameterized using polar coordinates:

$$\mathbf{r}(\theta) = \langle \cos(\theta), \sin(\theta), 0 \rangle \quad \text{for } 0 \leq \theta \leq 2\pi.$$

Then, for our integral, we need to find:

$$d\mathbf{r} = \langle -\sin(\theta), \cos(\theta), 0 \rangle \; d\theta.$$

We can then write the integral as:

$$\begin{aligned}\oint_S \mathbf{F}(\mathbf{r}(\theta)) \cdot d\mathbf{r} &= \int_0^{2\pi} \big\langle 0, \cos(\theta)\sin^2(\theta), \cos(\theta)\sin(\theta) \big\rangle \cdot \langle (-\sin(\theta), \cos(\theta), 0) \rangle \; d\theta \\ &= \int_0^{2\pi} \cos^2(\theta)\sin^2(\theta)\, d\theta.\end{aligned}$$

Half-Point Redo

5. (6 points) Consider the vector field $\mathbf{F}(x,y) = (2x + y\cos(xy))\mathbf{i} + (-3 + x\cos(xy))\mathbf{j}$.

(a) Show that $\mathbf{F}$ is conservative.

Solution. A vector field is conservative if the mixed partial derivatives are equal. That is:

$$\frac{\partial}{\partial y}(2x + y\cos(xy)) = \cos(xy) - xy\sin(xy) = \frac{\partial}{\partial x}(-3 + x\cos(xy)).$$

Therefore, $\mathbf{F}$ is conservative.

(b) Find a potential function f for $\mathbf{F}$.

Solution. We can find a potential function by following the following algorithm:

- Integrate P with respect to x to get $g(x,y) + h(y)$:

$$\begin{aligned} g(x,y) &= \int (2x + y\cos(xy))\,dx \\ &= \int 2x\,dx + \int y\cos(xy)\,dx \end{aligned}$$

  Make a substitution $u = xy$, then $du = y\,dx$ and $dx = \frac{du}{y}$:

$$g(x,y) = x^2 + y\int \cos(u)\,\frac{du}{y}$$

  The y's cancel out, and we substitute back $u = xy$:

$$g(x,y) = x^2 + \sin(xy)$$

  Therefore,

$$g(x,y) + h(y) = x^2 + \sin(xy) + h(y).$$

- Take the partial derivative of $g(x,y)+h(y)$ with respect to y, which results in function $g_y(x,y)+h'(y)$:

$$g_y(x,y) = \frac{\partial}{\partial y}\left(x^2 + \sin(xy) + h(y)\right) = x\cos(xy) + h'(y).$$

- Use the equation $g_y(x,y) = Q$ to find $h'(y)$:

$$x\cos(xy) + h'(y) = -3 + x\cos(xy) \implies h'(y) = -3.$$

- Integrate $h'(y)$ with respect to y to find $h(y)$:

$$h(y) = \int -3\,dy = -3y + C.$$

- Substitute $h(y)$ back into $g(x,y)$ to find the potential function:

$$f(x,y) = g(x,y) + h(y) = x^2 + \sin(xy) - 3y + C.$$

Therefore, the potential function is:

$$f(x,y) = x^2 + \sin(xy) - 3y + C.$$

We can check this work by taking the gradient of f and checking that it is equal to $\mathbf{F}$:

$$\nabla f(x,y) = \left\langle \frac{\partial}{\partial x} f(x,y), \frac{\partial}{\partial y} f(x,y) \right\rangle = \langle 2x + y\cos(xy), -3 + x\cos(xy) \rangle = \mathbf{F}(x,y).$$

(c) Determine the value of $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is a plane curve which starts at $(0,0)$ and ends at $(2,1)$.

Solution. Since $\mathbf{F}$ is conservative, we can use the Fundamental Theorem of Line Integrals to evaluate the integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(2,1) - f(0,0).$$

We can find $f(2,1)$ and $f(0,0)$:

$$\begin{aligned} f(2,1) &= 2^2 + \sin(2) - 3(1) + C = 4 + \sin(2) - 3 + C = 1 + \sin(2) + C \\ f(0,0) &= 0^2 + \sin(0) - 3(0) + C = 0 + 0 - 0 + C = C. \end{aligned}$$

Therefore:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = (1 + \sin(2) + C) - C = 1 + \sin(2).$$