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Chapter 1

The Real Numbers

1.1 Types of Numbers

Definition 1.1.1

The *natural numbers* contain all positive, non-zero, and non-fractional numbers. Expressed as $\mathbb{N} = \{1, 2, 3, 4, \dots\}$. They do not have an additive inverse, but you can add and multiply them.

Definition 1.1.2

The *integers* contains all non-fractional numbers. Expressed as: $\mathbb{Z} = \{\dots - 2, -1, 0, 1, 2, \dots\}$ —are known as a Group (more specifically, a “ring”). You can add, multiply, and subtract these numbers.

Definition 1.1.3

The *rational numbers* contain all numbers, except irrational numbers. Expressed as: $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$ —are known as a “Field.” You can add, subtract, multiply, and divide these numbers.

A problem that rational numbers could not explain: The 45,45,90 triangle had a hypotenuse of $\sqrt{2}$. This did not exist at the time, so it was simply $c^2 = 2$. Therefore, new numbers needed to be invented.

Theorem 1.1.4

There does not exist a rational number r such that $r^2 = 2$.

Proof. Suppose there exists a rational number r such that $r^2 = 2$. Since r is rational, there exists $p, q \in \mathbb{Z}$ such that $r = \frac{p}{q}$. We can assume the p and q have no common



factors. (If not, we can factor out the common factor.) By our assumption,

$$r^2 = 2$$

$$\frac{p^2}{q^2} = 2$$

It follows that,

$$p^2 = 2q^2$$

Such that p^2 is an even number because if p were odd, then p^2 would be odd. There exists $x \in \mathbb{Z}$ such that $p = 2x$. Recall that $p^2 = 2q^2$. Thus

$$(2x)^2 = 2q^2$$

$$4x^2 = 2q^2$$

$$2x^2 = q$$

Thus, q^2 is even. Hence q is also even. So p and q are both divisible by 2. This contradicts that p and q have no common factors. Thus, our supposition is false. Therefore, there does not exist a rational number r such that $r^2 = 2$ \square

So we are going to work with a larger set called the real numbers, \mathbb{R} .

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- You can:
 - Add,
 - Subtract,
 - Multiply,
 - Divide
- In other words, all field axioms apply.
- Totally ordered set for any $x, y \in \mathbb{R}$. Thus, one of these are true:
 - (a) $x < y$,
 - (b) $x > y$,
 - (c) $x = y$
- Think of it as a number line.
- \mathbb{Q} is dense:

If $a, b \in \mathbb{Q}$ with $a \neq b$, there exists $c \in \mathbb{Q}$ which is between a and b such that $a < c < b$. One example is $\frac{a+b}{2}$.
- \mathbb{Q} is not *complete*, but \mathbb{R} is.
 - *Complete*: Think, “no gaps.”



1.2 Preliminaries

Things to remember from Intro and Discrete.

Set Notation	Complement
$x \in A$	A^c (not \bar{A})
$A \cup B$	$\mathbb{R} \setminus A$
$A \cap B$	

- $\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \dots$
- $\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \dots$

Definition 1.2.1

De Morgan's Laws are defined as $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

1.2.1 Infinite Unions and Intersections

For each $n \in \mathbb{N}$, define $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$. In other words, each subsequent element in the subset will start at n . For example, $A_1 = \{1, 2, \dots\}$, whereas $A_5 = \{5, 6, \dots\}$.

$\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$. To show a number $\in \mathbb{N}$ belongs in the set A_n , we can start with that, $k \in \mathbb{N}$. Then $k \in A_k$. Thus, $k \in A_k \subseteq \bigcup_{n=1}^{\infty} A_n$. Therefore, $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$.

$\bigcap_{n=1}^{\infty} A_n = \emptyset$. Obviously, we know that the empty set is a subset of A_n , but to prove that $\bigcap_{n=1}^{\infty} A_n$ is a subset of the empty set, we should suppose a $k \in \mathbb{N}$ such that $k \in \bigcap_{n=1}^{\infty} A_n$. Notice that $k \notin \bigcap_{n=1}^{\infty} A_n$. So, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

1.2.2 Functions and Notation

$f: A \rightarrow B$ where f is a function, A is a domain, and B is the co-domain. Thus, $f(x) = y$ such that $x \in A$ and $y \in B$.



Some definitions to keep in mind

Definition 1.2.2

The *Dirichlet Function* is defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Definition 1.2.3

Let $f : \mathbb{R} \rightarrow \mathbb{R}$. If $E \subseteq \mathbb{R}$, then $f(E) = \{f(x) \mid x \in E\}$.

Example: $g : \mathbb{R} \rightarrow \mathbb{R}$, when we say $y \in g(A)$ implies there exists an x such that $g(x) = y$

Definition 1.2.4

The *Triangle Inequality* is defined as: For any $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

The most common application: For any $a, b, c \in \mathbb{R}$, $|a - b| \leq |a - c| + |c - b|$, with the intermediate step of $a - b = (a - c) + (c - b)$.

Definition 1.2.5

A function f is *injective* (or *one-to-one*) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B . Note the contrapositive of this definition: If $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Definition 1.2.6

A function f is *surjective* (or *onto*) if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Note the contrapositive of this definition: If there exists a $b \in B$ such that there is no $a \in A$ such that $f(a) = b$, then the function is not surjective.

1.2.3 Common Strategies for Analysis Proofs

Theorem 1.2.6

Let $a, b \in \mathbb{R}$. Then,

$$a = b \text{ if and only if for all } \epsilon > 0, |a - b| < \epsilon.$$

Proof. We show this by proving both implications:

(\Rightarrow) Assume $a = b$. Let $\epsilon > 0$. Then $|a - b| = 0 < \epsilon$



(\Leftarrow) Assume for all $\epsilon > 0$, $|a - b| < \epsilon$. Suppose $a \neq b$. Then $a - b \neq 0$. So, $|a - b| \neq 0$. Now, Consider $\epsilon_0 = |a - b|$. By our assumption we know that $|a - b| < \epsilon_0$. It is not true that $|a - b| < |a - b|$. Therefore, it must be the case that $a = b$.

Therefore, by showing both sides of the implication accomplish the same thing as the other side, we know that $a = b$ if and only if for all $\epsilon > 0$, $|a - b| < \epsilon$. \square

1.2.4 Mathematical Induction

Inductive Hypothesis: Let $x_1 = 1$. For all $n \in \mathbb{N}$, let $x_{n+1} = \frac{1}{2}x_n + 1$.

Inductive Step: $x_1 = 1, x_2 = 1.5, x_3 = 1.75, x_4 = 1.875$.

Example 1.1: Induction

The sequence (x_n) is increasing. In other words, for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$.

Proof. Suppose the sequence (x_n) is increasing. We will prove this point by using induction.

Base Case: We see that $x_1 = 1$ and $x_2 = 1.5$. Thus, $x_1 \leq x_2$.

Inductive Hypothesis: For $n \in \mathbb{N}$, assume $x_n \leq x_{n+1}$.

Scratch work: We want: $x_{n+1} \leq x_{n+2}$. We know: $x_{n+1} = \frac{1}{2}x_n + 1$.

Inductive Step: Then $\frac{1}{2}x_n \leq \frac{1}{2}x_{n+1}$. Hence, $\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1$. Therefore we have proven through induction that, $x_{n+1} \leq x_{n+2}$. \square



1.2.5 Exercises

Exercise: 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

- (a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.
- (b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.
- (c) $A \cap (B \cup C) = (A \cap B) \cup C$
- (d) $A \cap (B \cap C) = (A \cap B) \cap C$
- (e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution.

- (a) This is false. Consider the following as a counterexample: If we define A_1 as $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$, we can see why the intersection of these sets of infinite numbers are actually empty. Consider a number m that actually satisfies $m \in A_n$ for every A_n in our collection of sets. Because m is not an element of A_{m+1} , no such m exists and the intersection is empty.
- (b) This is true.
- (c) False. Consider sets $A = \{1, 2, 3\}$, $B = \{3, 6, 7\}$ and $C = \{5\}$. Note that $A \cap (B \cup C) = \{3\}$ is not equal to $(A \cap B) \cup C = \{3, 5\}$.
- (d) This is true. A proof would start with $x \in A \cap (B \cap C)$.
- (e) This is true. A proof would start with $x \in A \cap (B \cup C)$.

**Exercise: 1.2.5**

De Morgan's Laws Let A and B be subsets of \mathbb{R} .

- (a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.
- (b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.
- (c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution.

- (a) If $x \in (A \cap B)^c$, and we know that $A^c = \{x \in \mathbb{R} : x \notin A\}$, then we know x must not exist in A and B because $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$. Thus, x is in either A^c or B^c . Put another way $x \in A^c \cup B^c$. Since we have shown that an element that started in $(A \cap B)^c$ ended up in $A^c \cup B^c$, then we know $(A \cap B)^c \subseteq A^c \cup B^c$.
- (b) Assume there exists a $y \in A^c \cup B^c$. Thus, it must be the case that $y \notin A$ or $y \notin B$. Hence, y cannot exist in both sets at the same time, so $y \in (A \cap B)^c$. Because we have taken an element that started in $A^c \cup B^c$ and have shown that it exists in $(A \cap B)^c$, we have proven $A^c \cup B^c \subseteq (A \cap B)^c$.
- (c)



Proof. We need to show these expressions are subsets of each other in order to prove they are equivalent.

(\subseteq) Assume there exists $x \in (A \cap B)^c$, and we know that $A^c = \{x \in \mathbb{R} : x \notin A\}$, then we know x must not exist in A and B because $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$. Thus, x is in either A^c or B^c . Put another way $x \in A^c \cup B^c$. Since we have shown that an element that started in $(A \cap B)^c$ ended up in $A^c \cup B^c$, then we know $(A \cap B)^c \subseteq A^c \cup B^c$.

(\supseteq) Now assume there exists a $y \in A^c \cup B^c$. Thus, it must be the case that $y \notin A$ or $y \notin B$. Hence, y cannot exist in both sets at the same time, so $y \in (A \cap B)^c$. Because we have taken an element that started in $A^c \cup B^c$ and have shown that it exists in $(A \cap B)^c$, we have proven $A^c \cup B^c \subseteq (A \cap B)^c$.

Therefore, we have shown through proving both sides of the implication, that these two statements are logically equivalent. In that, all elements of $A^c \cup B^c$ are the same elements that are in $(A \cap B)^c$ \square

Exercise: 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A ; that is, $f(A) = \{f(x) : x \in A\}$.

- Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?
- Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.
- Show that, for an arbitrary function $g : \mathbb{R} \rightarrow \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.
- Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g .

Solution.

- Since $f(x) = x^2$, the intervals of $f(A)$ would be $[0, 4]$ and $f(B)$ would be $[1, 16]$. The interval of the intersection of $A \cap B$ is $[1, 2]$. Take this through our function, we get $f(A \cap B) = [1, 4]$. On the other side of the equation, we already know the intervals of $f(A)$ and $f(B)$, and the intersection of theirs would be $[1, 4]$. So



they do equal each other. We know $f(A \cup B)$ and $f(A) \cup f(B)$ will be equivalent because $f(A \cup B)$ has an interval of $[0, 16]$, and $f(A) \cup f(B)$ also has an interval of $[0, 16]$ because taking the union of $[0, 4] \cup [1, 16]$ is $[0, 16]$.

- (b) Two sets could be $A = [5, 6]$ and $B = [0, 0]$. Because the sets have nothing in common even after taking their function, they do not equal each other.

(c)

Proof. Let $x \in g(A \cap B)$. Using the definition of function, we know there exists a $y \in A \cap B$ to which that y is mapped to as $g(y) = x$. From the definition of intersection, we know $y \in A$ and $y \in B$ such that $x = g(y) \in g(A)$ and $x = g(y) \in g(B)$ because $y \in A \cap B$. Putting it together, we have $x \in g(A) \cap g(B)$ thus proving $g(A \cap B) \subseteq g(A) \cap g(B)$ \square

- (d) Conjecture: For any function g defined as $g : \mathbb{R} \rightarrow \mathbb{R}$ and for any subsets $A, B \subseteq \mathbb{R}$, the following holds:

$$g(A \cup B) = g(A) \cup g(B)$$

Proof. We need to show these expressions are subsets of each other in order to prove they are equivalent.

(\subseteq) Take any element $x \in g(A \cup B)$. By definition of function, we know there exists some $y \in A \cup B$ such that $g(y) = x$. From the definition of union, we know $y \in A$ or $y \in B$ such that $x = g(y) \in g(A)$ or $x = g(y) \in g(B)$ or both. Putting it together, we have $x \in g(A) \cup g(B)$ thus proving $g(A \cup B) \subseteq g(A) \cup g(B)$.

(\supseteq) Take any element $p \in g(A) \cap g(B)$. By definition of union, we know p is either in $g(A)$ or $g(B)$ or both. From the definition of function, we know that if $p \in g(A)$ or $p \in g(B)$ then there exists some $q \in A$ or $q \in B$ such that $g(q) = p$. Putting it together, we have $q \in A \cup B$. Moreover, this means $p = g(q) \in g(A \cup B)$. And since $p \in g(A) \cup g(B)$ implies $p \in g(A \cup B)$, we know $g(A) \cup g(B) \subseteq g(A \cup B)$.

Therefore, since we have proven that both expressions are functions of each other, we have proved that they are equal. \square

**Exercise: 1.2.8**

Given a function $f : A \rightarrow B$ can be defined as either **injective** or **surjective**, give an example of each or state that the request is impossible:

- (a) $f : \mathbb{N} \rightarrow \mathbb{N}$ that is 1-1 but not onto.
- (b) $f : \mathbb{N} \rightarrow \mathbb{N}$ that is onto but not 1-1.
- (c) $f : \mathbb{N} \rightarrow \mathbb{Z}$ that is 1-1 and onto.

Solution.

- (a) The function $f(a) = a + 1$ is 1-1 because when

$$\begin{aligned} f(a_1) &= f(a_2) \\ a_1 + 1 &= a_2 + 1 \\ a_1 &= a_2 \end{aligned}$$

However, the function is not onto because the entire co-domain is not covered. That being 1.

- (b) *We need to find a function that will cover every entry in the co-domain, while also avoiding a scenario where $a_1 = a_2$...* Consider the function,

$$f(a) = \begin{cases} a & \text{if } a \text{ is odd,} \\ a - 1 & \text{if } a \text{ is even} \end{cases}$$

This function is onto because every natural number is covered, but it is not 1-1 because $a_1 \neq a_2 - 1$.

- (c) This request is not possible. There is no way to map every natural number to every integer because we are simply missing 0! (Not 0 factorial, we do have the number 1, I just mean the number 0 in an exclamatory sense.)

**Exercise: 1.2.13**

For this exercise, assume [Exercise 1.2.5](#) has been successfully completed.

- (a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

- (b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets B_1, B_2, B_3, \dots where

$$\bigcap_{i=1}^n B_i \neq \emptyset \quad \text{is true for every } n \in \mathbb{N},$$

but

$$\bigcap_{i=1}^{\infty} B_i = \emptyset$$

fails.

- (c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.



Proof. In this proof, we plan to prove (c). Thus, we need to show that:

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$$

and

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

(\subseteq) Let $x \in \left(\bigcup_{i=1}^{\infty} A_i \right)^c$. This means x is in the union set of A_i for all $i \in \mathbb{N}$. Then, because we are taking the complement of $\left(\bigcup_{i=1}^{\infty} A_i \right)$, that means $x \notin A_i$ for all $i \in \mathbb{N}$. Hence, x is in the complement of each A_i . Thus, we can use the definition of intersection to assert $x \in \bigcap_{i=1}^{\infty} A_i^c$. Therefore, we have shown:

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c.$$

(\supseteq) Similar to before, let $x \in \bigcap_{i=1}^{\infty} A_i^c$. Because $x \in A_i^c$ for all $i \in \mathbb{N}$ we know $x \notin A_i$. Hence, $x \notin \left(\bigcup_{i=1}^{\infty} A_i \right)$, which means $x \in \left(\bigcup_{i=1}^{\infty} A_i \right)^c$. Therefore, we have shown:

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

By showing both inclusions, we see that:

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c.$$

□

1.3 Axiom of Completeness

Axiom of Completeness

Every nonempty set of real numbers that is **bounded** has a **least upper bound**.

Think about \mathbb{Q} and \mathbb{R} .

- Both are fields.
 - Both have $+$, $-$, \times , \div operations.



- Both are totally ordered
 - $a < b$,
 - $a > b$,
 - or $a = b$
- \mathbb{R} is complete. \mathbb{Q} is not.

1.3.1 Least Upper Bounds and Greatest Lower Bounds

Definition 1.3.1

A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* of A .

Similarly, a set $A \subseteq \mathbb{R}$ is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Note that upper bounds are not unique! For example, consider the line, A , from 0 to 1. There are infinitely many upper bounds past 1 because A is bounded.

Definition 1.3.2

A number s is a *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

- s is an upper bound for A ;
- if b is any upper bound for A , then $s \leq b$.

We often call the least upper bound the *supremum* of a set.

Example 1.2: Supremum

Imagine a number line from $(1, 8)$. Note that parenthesis mean $<$ and not \leq . Hence, the supremum is 8. Wrote simply as $\sup A$.

Example 1.3: Supremum and Infimum 1

Consider a set, $B = [-5, -2] \cup (3, 6) \cup \{13\}$. What is the supremum and the infimum?

Solution. $\sup B = 13$; $\inf B = -5$ because -5 is the greatest lower bound.

Example 1.4: Supremum and Infimum 2

Consider the set, $\mathbb{C} = \{\frac{1}{n} : n \in \mathbb{N}\}$. What is the supremum and the infimum?



Solution. $\sup \mathbb{C} = 1, \inf \mathbb{C} = 0.$

Example 1.5: L

Let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}$$

Then $\sup(c + A) = c + \sup A$.

Solution. To properly verify this we focus separately on each part of [Definition 1.3.2](#). Setting $s = \sup A$, we see that $a \leq s$ for all $a \in A$, which implies $c + a \leq c + s$ for all $a \in A$. Thus, $c + s$ is an upper bound for $c + A$ and condition (i) is verified. For (ii), let b be an arbitrary upper bound for $c + A$; i.e., $c + a \leq b$ for all $a \in A$. This is equivalent to $a \leq b - c$ for all $a \in A$, from which we conclude that $b - c$ is an upper bound for A . Because s is the least upper bound of A , $s \leq b - c$, which can be rewritten as $c + s \leq b$. This verifies part (ii) of [Definition 1.3.2](#), and we conclude $\sup(c + A) = c + \sup A$.

Definition 1.3.4

A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_0 \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if a_1 is an element of A and $a_1 \leq a$ for all $a \in A$.

Note that some sets have a maximum and some sets do not. You cannot refer to a maximum without first knowing it exists. This is the same with minimums.

Lemma 1.3.5

1.3.8 Assume s is an [upper bound](#) for a set $A \subseteq \mathbb{R}$. Then, s is the supremum of A if and only if for every $\epsilon > 0$, there exists $x \in A$ such that $s - \epsilon < x$.

This lemma allows us to take any positive number and take a “step back.” In essence, you can verify something as an upper bound if you continuously back up over and over until you cannot back up any longer.

Proof. We show this by proving both implications:

(\Rightarrow) Assume $s = \sup A$. Let $\epsilon > 0$. Suppose there are no elements x of A such that $s - \epsilon < x$. Then $s - \epsilon$ would be an upper bound. This contradicts that s is the least upper bound. Therefore, there must exist an element $x \in A$ such that $s - \epsilon < x$.



(\Leftarrow) Assume for every $\epsilon > 0$, there exists $x \in A$ such that $s - \epsilon < x$. Let t be an upper bound of A . Suppose $t < s$. Consider $\epsilon_0 = s - t > 0$. By our assumption, there exists $x \in A$ such that $s - \epsilon_0 < x$. So, $t < x$. This contradicts that t is an upper bound of A . So, $t \geq s$. Thus, s is the least upper bound

Therefore, by proving both the right and left implication, we have shown the statement to be true. \square

Analogous statement about infimums: Assume z is a lower bound of a set $A \subseteq \mathbb{R}$. Then $z = \inf A \iff$ for all $\epsilon > 0$, there exists $y \in A$ such that $y < z + \epsilon$.

1.3.2 Exercises

Exercise: 1.3.4

Let $A_1, A_2, A_3 \dots$ be a collection of nonempty sets each of which is bounded above.

- Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^n A_k)$.
- Consider $\sup(\bigcup_{k=1}^\infty A_k)$. Does the formula in (a) extend to the infinite case?

Solution.

- Let A_1 and A_2 be nonempty sets, each bounded above. To find the largest of the two suprema, we can use the following: $\sup(A_1 \cap A_2) = \max\{\sup A_1, \sup A_2\}$. If we extend this notion to $\sup(\bigcup_{k=1}^n A_k)$, we can use the same idea from before and write it as $\sup(\bigcup_{k=1}^n A_k) = \max\{\sup A_1, \sup A_2, \dots, \sup A_n\}$.
- The formula does not extend to the infinite case. Consider the counterexample $\bigcup_{k=1}^\infty A_k$ where $A_k := [k, k + 1]$. Even though these sets are bounded above, when we take the union of them, we approach infinity, which is not bounded: $\bigcup_{k=1}^\infty A_k = [1, 2] \cup [2, 3] \cup \dots = [1, \infty)$.

Exercise: 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

- If $c \geq 0$, show that $\sup(cA) = c \sup A$.
- Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution.

- Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Define the set $cA := \{ca : a \in A\}$. From the axiom of completeness, because A is bounded above, we know there is a



least upper bound, $s = \sup A$. Following from Example 1.3.7, we see that $a \leq s$ for all $a \in A$ which implies $ca \leq cs$ for all $a \in A$. Thus, cs is an upper bound for cA , and the first condition of Definition 1.3.2 is satisfied. For the second condition, we need to look at both $c = 0$ and $c > 0$ to avoid dividing by zero. So, we have two cases:

- $c = 0$: If $c = 0$, then $cA = \{0 : a \in A\} = \{0\}$. Since the only element in cA is 0, $\sup(cA) = 0$. Similarly, because $c = 0$, $c\sup A = 0 \cdot \sup A = 0$. Therefore, $\sup(cA) = c\sup(A)$.
- $c > 0$: Let b be an arbitrary upper bound for cA and $c > 0$. In other words, $ca \leq b$ for all $a \in A$. This is equivalent to $a \leq b/c$ where $c \neq 0$, from which we can see that b/c is an upper bound for A . Because s is the least upper bound of A , $s \leq b/c$, which can be rewritten as $cs \leq b$. This verifies the second part of Definition 1.3.2, and we conclude $\sup(cA) = c\sup A$.

(b) Postulate: If $c < 0$, then $\sup(cA) = c\inf(A)$.

Exercise: 1.3.8

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

- (a) $\left\{ \frac{m}{n} : m, n \in \mathbb{N} \text{ with } m < n \right\}$.
- (b) $\left\{ \frac{(-1)^m}{n} : m, n \in \mathbb{N} \right\}$.
- (c) $\left\{ \frac{n}{3n+1} : n \in \mathbb{N} \right\}$.
- (d) $\left\{ \frac{m}{m+n} : m, n \in \mathbb{N} \right\}$.

Solution. To avoid writing out every set definition, I am going to denote each set as A_n where n corresponds to the numerical value of the list from (a) - (d).

- (a) $\sup A_1 = 1, \inf A_1 = 0$
- (b) $\sup A_2 = 1, \inf A_2 = -1$
- (c) $\sup A_3 = \frac{1}{3}, \inf A_3 = \frac{1}{4}$
- (d) $\sup A_4 = 1, \inf A_4 = 0$



1.4 Consequences of Completeness

Theorem 1.4.1: Nested Interval Property

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = (a_n, b_n)$. Assume I_n contains I_{n+1} . This results in a nested sequence of intervals.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \dots$$

Then, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

tl;dr there has to be something that is common to all of the sets.

Proof. Notice that the sequence, a_1, a_2, a_3, \dots is increasing. In other words, for each $n \in \mathbb{N}$, since $I_n \supset I_{n+1}$ we have $a_n \leq a_{n+1}$. If we consider the set $A = \{a_n : n \in \mathbb{N}\}$. The element b_1 is an upper bound of A . (Note that b_1 and a_1 corresponds to the endpoints of the first set, I_1 . Think of this as a tornado looking structure where the larger the I_n , the smaller the number line.) For each $n \in \mathbb{N}$, $a_n \leq b_n \leq b_1$.

Since A has an upper bound, it must have a least upper bound. Hence, let $\alpha = \sup A$. We claim that $\alpha \in \bigcap_{n=1}^{\infty} I_n$. We said b_1 was an upper bound. In fact, every b_n is an upper bound of A . Choose any $n, m \in \mathbb{N}$. We want to show that $a_n \leq b_m$. Consider the following cases:

Case 1: If $n < m$, then $a_n \leq a_m \leq b_m$. (Think: two number lines stacked on top of each other. The top number line is larger, call it I_n and it has a_n and b_n as endpoints. Consider a contained line ($I_n \supseteq I_m$) that is smaller, and has endpoints a_m and b_m .)

Case 2: If $n > m$, then $a_n \leq b_n \leq b_m$. So every b_n is an upper bound of A .

Hence,

- Because $\alpha = \sup A$, we have $\alpha \geq a_n$.
- Since b_n is an upper bound of A , we have $\alpha \leq b_n$.

so, $\alpha \in [a_n, b_n] = I_n$. Thus, $\alpha \in \bigcap_{n=1}^{\infty} I_n$. □

Nested, closed, Bounded Intervals \Rightarrow non-empty intersection.

Theorem 1.4.2: Archimedean Principle

- (a) Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.
- (b) Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.



Proof. (a) If \mathbb{N} was bounded, then we can let $s \in \mathbb{N} = \sup \mathbb{N}$. However, we know that there is always a higher number (e.g., $n + 1$) for any $n \in \mathbb{N}$ that is given. Thus, by contradiction, there must exist $n \geq x$.

(b) For any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

□

Theorem 1.4.3: Density of the Rationals in the Reals

For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Proof. Since $b - a > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. From the **Archimedean Principle**, since $a \times n \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that $a \times n < m$. Let m be the smallest such natural numbers (by the well ordered principle). Since m is the smallest such natural number, it follows that $m - 1 \leq a \times n < m$. We then see that $a < \frac{m}{n}$. Now, we need to find some $\frac{m}{n} < b$.

$$\begin{aligned} m - 1 &\leq a \times n \\ m &\leq a \times n + 1 \\ \frac{m}{n} &\leq a + \frac{1}{n} \\ \frac{m}{n} &< a + (b - a) \\ \frac{m}{n} &< b \end{aligned}$$

We now have that $a < \frac{m}{n} < b$ so $\frac{m}{n}$ is a rational number in (a, b)

□

Exercise: 1.4.1

Recall that \mathbb{I} stands for the set of irrational numbers.

- (a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of \mathbb{Q} as well.
- (b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.
- (c) Part (a) can be summarized by saying that \mathbb{Q} is closed under addition and multiplication. Is \mathbb{I} closed under addition and multiplication? Given two irrational numbers s and t , what can we say about $s + t$ and st ? In other words, are there two irrational numbers that can be added and multiplied such that you get a number x such that $x \notin \mathbb{I}$.

Solution.



- (a) Let $a, b \in \mathbb{Q}$. This means there exists some $p, q, a, b \in \mathbb{Z}$ such that

$$a = \frac{p}{q}$$

and

$$b = \frac{a}{b}$$

where $q, b \neq 0$. The product of these numbers is

$$ab = \frac{p}{q} \cdot \frac{a}{b} = \frac{pa}{qb}.$$

Since $pa, qb \in \mathbb{Z}$, $ab \in \mathbb{Q}$. The sum of these numbers is

$$a + b = \frac{p}{q} + \frac{a}{b} = \frac{pb + aq}{qb}.$$

Since $pb + aq, qb \in \mathbb{Z}$, $a + b \in \mathbb{Q}$.

- (b) Let $a \in \mathbb{Q}$ and $t \in \mathbb{I}$. Assume, for contradiction, that $a + t \in \mathbb{Q}$. This would imply $t = (a + t) - a$ (because we can subtract $t + a$ from the original equation and rearrange terms). Since $a + t, a \in \mathbb{Q}$ their sum would be rational because the rational numbers are closed under addition. However, that would contradict the assumption that $t \in \mathbb{I}$. Hence, $a + t \in \mathbb{I}$.
- (c) For \mathbb{I} , it is not closed under addition and multiplication. Consider the following counterexample: $\sqrt{2} + (-\sqrt{2}) = 0$ which is not in the irrationals. For multiplication, consider $\sqrt{2} \cdot \sqrt{2} = 2$, which is also not in the irrationals.

1.5 Cardinality

Two sets have the same *cardinality* if there exists a bijection between them. Thus, the natural numbers, the integers, and the rational numbers have the same cardinality. A set is *countably infinite* if it has the same cardinality as \mathbb{N} . (If it can be put into one-to-one correspondence with \mathbb{N} .) A set is *countable* if it is countably infinite or finite.

Theorem 1.5.6

\mathbb{R} is not countable.

Proof. 1 (most common)

Suppose \mathbb{R} is countable. Then we can list them all, or we can enumerate them. $\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\}$. We can write the decimal expansion of each of these. Consider the following table:



$x_1 =$	$\boxed{a_{10}}$	a_{11}	a_{12}	a_{13}	a_{14}	\dots
$x_2 =$	a_{20}	$\boxed{a_{21}}$	a_{22}	a_{23}	a_{24}	\dots
$x_3 =$	a_{30}	a_{31}	$\boxed{a_{32}}$	a_{33}	a_{34}	\dots
$x_4 =$	a_{40}	a_{41}	a_{42}	$\boxed{a_{43}}$	a_{44}	\dots
$x_5 =$	a_{50}	a_{51}	a_{52}	a_{53}	$\boxed{a_{54}}$	\dots
$x_6 =$	a_{60}	a_{61}	a_{62}	a_{63}	a_{64}	\dots

We will now construct a number that is not in this list. Focus on diagonal entries. For each $n \in \mathbb{N}$, let b_n be a digit that is different from a_{nn} . Now consider the number $y = 0.b_1b_2b_3b_4b_5\dots$. This number y is not in our list. So our list did not include all of \mathbb{R} . Avoid repeating 9s. \square

Proof. 2 (uses nested interval theorem)

Suppose \mathbb{R} is countable. Then we can enumerate \mathbb{R} $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$. Let I_1 be any closed interval that does not contain x_1 . Next, we will find another closed interval I_2 that:

- $I_2 \subseteq I_1$
- $x_2 \notin I_2$

Continue in this fashion creating a sequence of nested closed intervals: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ such that for all $k \in \mathbb{N}$, $x_k \notin I_k$. Now consider:

$$\bigcap_{n=1}^{\infty} I_n$$

- For each $k \in \mathbb{N}$, since $x_k \notin I_k$, we see $x_k \notin \bigcap_{n=1}^{\infty} I_n$.
- By the nested interval theorem, there exists $x \in \mathbb{R}$ such that $x \in \bigcap_{n=1}^{\infty} I_n$. So x is a real number that is not included in our list.

\square

Theorem 1.5.7

A countable collection of finite sets is *countable*.

Theorem 1.5.8

- The union of two countable sets is *countable*.
- A countable union of countable sets is *countable*.

From Theorem 1.5.6, we know that \mathbb{R} is uncountable, but what about $(0, 1)$? It does



have the same cardinality of \mathbb{R} because we can make a one-to-one and onto function between both the sets. Similarly, (a, b) also has the same cardinality. What about $[a, b]$?

Recap: \mathbb{N} is countable, and \mathbb{R} is uncountable and has a different cardinality than \mathbb{N} . Thus, the question is, do all uncountable sets have the same cardinality as \mathbb{R} ? The answer is **no**.

Theorem 1.5.9: Canter's Theorem

For any set A , there does not exist an onto map from A into \mathcal{P} .

Proof. Suppose there exists an onto function, $f : A \rightarrow \mathcal{P}(A)$. So each $a \in A$ is mapped to an element $f(a) \in \mathcal{P}(A)$. Then, $f(a) \subseteq A$. We are going to construct an element of $\mathcal{P}(A)$ which is not mapped to by f .

Consider $B = \{a \in A : a \notin f(a)\}$. Since f is onto there exists $a' \in A$ such that $B = f(a')$. Thus, there are two cases to consider:

- **Case 1:** If $a' \in B = f(a')$, then $a' \notin B$.
- **Case 2:** If $a' \notin B = f(a')$, then $a' \in B$.

As evidenced, both cases lead to contradictions, so B is not the image of any $a \in A$. Therefore f is not onto. \square

Example 1.6: Set and Power Set Matching

$A = \{a, b, c\}$.

Solution. $\mathcal{P}(A) = \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$. Note that you can map $\{a\}, \{b\}, \{c\}$, to elements such as $\emptyset, \{a, b\}, \{a, b, c\}$, but there are still more elements that are left unmapped. We can extrapolate from our proof a set B such that $B = \{a, c\}$ because those elements are not mapped to.

All of this is to show $\mathcal{P}(\mathbb{R})$ has a larger cardinality than \mathbb{R} . Then $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ has a larger cardinality than $\mathcal{P}(\mathbb{R})$.

Chapter 2

Sequences and Series

2.1 Discussion: Rearrangement of Infinite Series

Questions:

What is a *sequence*?

A countable, ordered list of elements. An example could be $1, 2, 3, 4, 5, \dots$. Note that this is *ordered*, therefore distinguishing it from a sequence like $3, 1, 2, 4, 5, 6, \dots$. Hence, order matters.

A *sequence* is a function whose domain is \mathbb{N} . **Note:** The domain \mathbb{N} refers to each element's position in the list. For example, $(a_n) = a_1, a_2, a_3, \dots$.

We will focus on the *limit* of a sequence. We use sequences to approximate other things.

Example 2.1: Sequence

$3, 3.1, 3.14, 3.141, 3.1415, \dots \approx \pi$.

$x, x - \frac{x^3}{6}, x - \frac{x^3}{6} + \frac{x^5}{120}, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}, \dots \approx \sin(x)$

What is a *series*?

An infinite sum. We look at the sequence of partial sums. We ask, do the partial sums approach a limit?

Example 2.2: Alternating Harmonic Series

$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \dots$

We can rearrange these terms such that we can ‘force’ the series to converge to a specific number. Therefore, we will need to be careful with our definitions.



2.2 The Limit of a Sequence

Definition 2.2.1

A *sequence* is a function whose domain is \mathbb{N} . We write $(a_n) = a_1, a_2, a_3, \dots$

Definition 2.2.3

The sequence (a_n) *converges* to L if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$. In other words, there exists $N \in \mathbb{N}$ such that

- **(In the interval)** $a_N \in (L - \epsilon, L + \epsilon)$.
- **(Stays in the interval)** $\forall n \geq N, a_n \in (L - \epsilon, L + \epsilon)$.

Example 2.3: In-class

Let $a_n = \frac{1}{n}$. $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

Proof. Our claim is $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$. Thus, let $\epsilon = .01$. Does the sequence eventually get inside $(-.01, .01)$? We will set $N = 101$. So, for any $n \geq |0|$,

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{101} < .01.$$

From A_n and on, the sequence stayed within ϵ of 0. But what about $\epsilon = .001$, $\epsilon = .00001$ and so on?

Actual proof let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Now, for any $n \geq N$,

$$\left| \frac{1}{n} \right| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{1/\epsilon}.$$

(Where $\frac{1}{1/\epsilon} = \epsilon$, but is in that form for demonstration purposes.) Therefore $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ \square

“To get close” means is that we are finding a bigger and bigger N as ϵ gets smaller. Note that the choice of N certainly depends on ϵ . This idea of “getting close” can be seen in the following definition:



Definition 2.2.3B

A sequence (a_n) converges to a if, given any ϵ -neighborhood $V_\epsilon(a)$ of a , there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ -neighborhood contains all but a finite number of the terms of (a_n) .

2.2.1 Basic Structure of a Limit Proof

Claim: $\lim_{n \rightarrow \infty} a_n = L$.

Proof: Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that {something involving ϵ }. Assume $n \geq N$. Then,

$$|a_n - L| \boxed{\dots} < \epsilon$$

(Where $\boxed{\dots}$ is going to be where the majority of the work is going to lie.)

Example 2.4: In-class

Claim: $\lim_{n \rightarrow \infty} \frac{2n-3}{2n} = 1$

Proof. Let $\epsilon > 0$. *Scratch paper:* Solve for:

$$\left| \frac{2n-3}{2n} - 1 \right| = \left| \frac{-3}{2n} \right| = \frac{3}{2n} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n.$$

By the **Archimedean Principle**, there exists $N \in \mathbb{N}$ such that $N > \frac{3}{2\epsilon}$. Assume $n \geq N$, (want to know what happens past this point)

$$\left| \frac{2n-3}{2n} - 1 \right| \leq \frac{3}{2N} < \frac{3}{2 \cdot 3/2\epsilon} = \epsilon.$$

Therefore, $\lim_{n \rightarrow \infty} \frac{2n-3}{2n} = 1$ □

Example 2.5: C

Claim: $\lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2} = 2$

Proof. Let $\epsilon > 0$. By the **Archimedean Principle**, there exists $N \in \mathbb{N}$ such that [leave off] *Scratch paper:* Solve for

$$\left| \frac{2n^2+1}{n^2} - 2 \right| = \frac{1}{n^2} < \epsilon \Rightarrow \frac{1}{\epsilon} < n^2$$



[pick up] there exists $N \in \mathbb{N}$ such that

$$N > \frac{1}{\sqrt{\epsilon}}.$$

Assume $n \geq N$, then

$$\begin{aligned} \left| \frac{2n^2 + 1}{n^2} - 2 \right| &= \frac{1}{n^2} \\ &\leq \frac{1}{N^2} \\ &< \frac{1}{(1/(\sqrt{\epsilon}))^2} \\ &= \frac{1}{1/\epsilon} \\ &= \epsilon \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2} = 2$

□

Example 2.6: In-class

Claim: $\lim_{n \rightarrow \infty} \frac{7n+8}{3n+6} = \frac{7}{3}$

Proof.

$$\begin{aligned} \left| \frac{7n+8}{3n+6} - \frac{7}{3} \right| &= \left| \frac{21n+24}{3(3n+6)} - \frac{21n+42}{3(3n+6)} \right| \\ &= \left| \frac{-18}{9n+18} \right| \\ &= \frac{18}{9n+18} < \epsilon * * \\ &= \frac{18}{3} < 9n+18 \\ &= \frac{18}{3} - 18 < 9n \\ &= \frac{18/\epsilon - 18}{9} < n \end{aligned}$$



* * $\frac{18}{9n+8} < \frac{18}{9n} < \epsilon \Rightarrow \frac{2}{\epsilon} < N$. $\exists N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Assume $n \geq N$,

$$\begin{aligned} \left| \frac{7n+8}{3n+6} - \frac{7}{3} \right| &= \frac{18}{9n+18} \\ &= \frac{2}{n+2} \\ &< \frac{2}{n} \\ &\leq \frac{2}{N} \\ &< \frac{2}{\epsilon/2} \\ &= \epsilon \end{aligned}$$

□

Does every sequence have a limit?

Theorem 2.2.4: Uniqueness of Limits

The limit when it exists, is unique.

Proof. Let (x_n) be a convergent sequence. Suppose L and M are limits of this sequence. Without the loss of generality, we are going to assume $M > L$. Let

$$\epsilon = \frac{M - L}{3}.$$

Since x_n converges to L , there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|x_n - L| < \epsilon$. Since (x_n) converges to M , there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|x_n - M| < \epsilon$. Consider $n = \max\{N_1, N_2\}$. Since $n \geq N_1$, $|x_n - L| < \epsilon$. Since $n \geq N_2$, $|x_n - M| < \epsilon$. Then $L - \epsilon < x_n < L + \epsilon$ and $M - \epsilon < x_n < M + \epsilon$. By our choice of ϵ , we now have

$$(x_n) < L + \epsilon < M - \epsilon < (x_n).$$

This is a contradiction. Thus, (x_n) cannot have two different limits.

□

Example 2.7:

Let $(x_n) = \frac{\cos(n)}{3n}$. Claim: $\lim_{n \rightarrow \infty} (x_n) = 0$

Proof. Let $\epsilon > 0$. By the **Archimedean Principle**, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{3\epsilon}$



for all $n \geq N$,

$$\begin{aligned}
 \left| \frac{\cos(n)}{3n} - 0 \right| &= \left| \frac{\cos(n)}{3n} \right| \\
 &\leq \frac{1}{3n} \\
 &\leq \frac{1}{3N} \\
 &< \frac{1}{3(1/3\epsilon)} \\
 &= \epsilon
 \end{aligned}$$

□

Example 2.8:

Let $(y_n) = \frac{4n-1}{n^2}$. Claim: $\lim_{n \rightarrow \infty} y_n = 0$.

Proof. Let $\epsilon > 0$. By the [Archimedean Principle](#), there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. For all $n \geq N$,

$$\begin{aligned}
 \left| \frac{4n-1}{n^2} - 0 \right| &= \left| \frac{4n-1}{n^2} \right| \\
 &= \frac{4n-1}{n} \\
 &< \frac{4n}{n^2} \\
 &= \frac{4}{n} \\
 &\leq \frac{4}{N} \\
 &< \frac{4}{4/\epsilon} \\
 &= \epsilon
 \end{aligned}$$

□

2.2.2 Exercises

Exercise: 2.2.2(b)

Verify, using [Definition 2.2.3](#), that the following sequences converge to the proposed limit.

(b) $\lim_{n \rightarrow \infty} \frac{2n^2}{n^3+3} = 0$



Proof.

- (b) Let $\epsilon > 0$. By the **Archimedean Principle**, there exists an $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Then, for $n \geq N$,

$$\begin{aligned} \left| \frac{2n^2}{n^3 + 3} - 0 \right| &= \left| \frac{2n^2}{n^3 + 3} \right| \\ &= \frac{2n^2}{n^3 + 3} \\ &< \frac{2n^2}{n^3} \\ &= \frac{2}{n} \\ &\leq \frac{2}{N} \\ &= \frac{2}{2/\epsilon} \\ &= \epsilon. \end{aligned}$$

Therefore, $\lim_{n \rightarrow \infty} \frac{2n^2}{n^3 + 3} = 0$. □

Exercise: 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

- (a) At every college in the United States, there is a student who is at least seven feet tall.
- (b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
- (c) There exists a college in the United States where every student is at least six feet tall.

Solution.

- (a) There is at least one college in the United States where all students are less than seven feet tall.
- (b) There is at least one college in the United States where all professors give at least one student a grade of C or lower.
- (c) For all colleges in the United States, there exists a student who is less than six feet tall.



Exercise: 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

- (a) A sequence with an infinite number of ones that does not converge to one.
- (b) A sequence with an infinite number of ones that converges to a limit not equal to one.
- (c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution.

- (a) Possible. Consider the piecewise function: $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$
- (b) Impossible. A sequence that converges must have its terms approach a specific value (the limit). If the sequence has an infinite number of ones, it must have subsequences of ones arbitrarily far out. For the sequence to converge to a limit different from 1, the terms would have to approach that different limit, say $L \neq 1$, meaning the ones must become rare or eventually stop appearing, contradicting the infinite number of ones. Therefore, such a sequence is impossible.
- (c) Possible. $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots)$

Exercise: 2.2.7

Here are two useful definitions:

- (i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.
- (ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.
 - (a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?
 - (b) Which definition is stronger? Does frequently imply eventually, or does eventually imply frequently?
 - (c) Give an alternate rephrasing of [Definition 2.2.3B](#) using either frequently or eventually. Which is the term we want?
 - (d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?



Solution.

- (a) The sequence $(-1)^n$ is *frequently* in the set $\{1\}$ because for every $N \in \mathbb{N}$, we can find an $n \geq N$ such that $(-1)^n = 1$.
- (b) The definition of *eventually* is stronger because *eventually* implies *frequently*, but *frequently* does not imply *eventually*.
- (c) An alternate rephrasing of Definition 2.2.3B using *eventually* is: A sequence (a_n) converges to a if, given any ϵ -neighborhood— $V_\epsilon(a)$ of a — (a_n) is *eventually* in $V_\epsilon(a)$. The term we want is *eventually*.
- (d) If an infinite number of terms of a sequence (x_n) are equal to 2, (x_n) is not *eventually* in $(1.9, 2.1)$ because we can have a sequence (a_n) that will not settle in $(1.9, 2.1)$. For example, $(a_n) = (0, 2, 0, 2, \dots)$ does not settle in $(1.9, 2.1)$. Whereas, (x_n) is *frequently* in the interval $(1.9, 2.1)$ because for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $x_n \in (1.9, 2.1)$ for all $n \geq N$. We can see an instance of this being true by examining the previous example.

2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1

A sequence (x_n) is *bounded* if there exists some $M > 0$ such that every term in the sequence belongs to $[-M, M]$.

Theorem 2.3.2

Every convergent sequence is bounded.

Proof. Let (x_n) be a convergent sequence with limit L . There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|(x_n) - L| < 1$. Equivalently, $(x_n) \in (L - 1, L + 1)$. Let

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |L + 1|, |L - 1|\}.$$

We claim that for all $n \in \mathbb{N}$, $|x_n| \leq M$.

- (a) This is true for $n < N$.
- (b) For $n \geq N$, we know $L - 1 < x_n < L + 1$, so $(x_n) \leq \max\{|L - 1|, |L + 1|\}$

Thus, every term is in $[-M, M]$. □



Theorem 2.3.3: Algebraic Limit Theorem

Let $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$. Then,

- (i) $\lim_{n \rightarrow \infty} ca_n = ca$ for all $c \in \mathbb{R}$;
- (ii) $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$;
- (iii) $\lim_{n \rightarrow \infty} (a_n b_n) = ab$;
- (iv) $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided $b \neq 0$.

Scratch Paper:

$$|ca_n - ca| = |c| |a_n - a| < \epsilon$$

$$|a_n - a| < \frac{\epsilon}{|c|}$$

Leave off and go back to proof¹

Proof. (i)

Let $\epsilon > 0$.¹ Since (a_n) converges to a , there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \frac{\epsilon}{|c|}$. Now, for any $n \geq N$ we have two case because we want to avoid dividing by 0:

- If $c = 0$:
then each $ca_n = 0$. So (ca_n) converges to 0, which can equal ca .

- If $c \neq 0$:
 $|ca_n - ca| = |c| |a_n - a| < |c| \frac{\epsilon}{|c|} = \epsilon.$

□

(ii)

Scratch paper:

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \tag{2.1}$$

$$\leq |a_n - a| + |b_n - b| \tag{2.2}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \tag{2.3}$$

Note that (2.2) is from the triangle inequality. Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since (a_n) converges to a , there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \frac{\epsilon}{2}$. Since (b_n) converges to b , there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon}{2}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \geq N$, (refer back to scratch paper).

(iii)



Scratch paper:

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \quad (2.4)$$

$$= |a_n(b_n - b) + b(b_n - b)| \quad (2.5)$$

$$\leq |a_n| |b_n - b| + |b| |b_n - b| \quad (2.6)$$

$$\leq M |b_n - b| + M |a_n - a|. \quad (2.7)$$

$$< M \left(\frac{\epsilon}{2M} \right) + M \left(\frac{\epsilon}{2M} \right) \quad (2.8)$$

$$= \epsilon \quad (2.9)$$

Note that: (2.4) is where we added 0, (2.5) is from the triangle inequality, and (2.6) is just factored. Additionally, we choose N to get the fractions in (2.8) Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since convergent sequences are bounded, then there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$. We can choose M so that $|b_n| \leq M$ as well. Since (a_n) converges to a , there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \frac{\epsilon}{2M}$. Since (b_n) converges to b , there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon}{2M}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \geq N$, (refer back to scratch paper, and change (2.4)'s sign from an '=' to ' \leq ').

(iv)

Scratch paper:

$$\begin{aligned} \left| \frac{a_n}{b_n} - \frac{a}{b} \right| &= \left| \frac{a_n b - ab_n}{b_n b} \right| \\ &= \left| \frac{a_n b - ab_n + ab_n - ab}{b_n b} \right| \\ &= \left| \frac{a_n(b - b_n) + b(b_n - b)}{b_n b} \right| \\ &= \left| \frac{a_n(b - b_n)}{b_n b} + \frac{b(b_n - b)}{b_n b} \right| \\ &\leq \left| \frac{a_n}{b_n} \right| |b - b_n| + |b| \left| \frac{b_n - b}{b_n b} \right| \\ &< \epsilon \end{aligned}$$

Let $\epsilon > 0$. Since (b_n) converges to b , there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|b_n| > \left| \frac{b}{2} \right|$. There also exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon |b|^2}{2}$. Now, let $N = \max\{N_1, N_2\}$. Let $n \geq N$, (refer back to scratch paper).



Lemma 2.3.4

Let (a_n) and $c < a$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > c$. Similarly, if $a < d$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n < d$.

2.3.1 Limits and Order

Theorem 2.3.5: Order Limit Theorem

Let (a_n) and (b_n) be sequences. If $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$, then

- (i) If $a_n \geq c$ for all $n \in \mathbb{N}$, then $a \geq c$.
- (ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.
- (iii) If there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

2.3.2 Exercises

Exercise: 2.3.1

- (a) Assume $\lim_{n \rightarrow \infty} x_n = 0$ with $x_n \geq 0$. Show that $\lim_{n \rightarrow \infty} \sqrt{x_n} = 0$.
- (b) Assume $\lim_{n \rightarrow \infty} x_n = 49$ with $x_n \geq 0$. Show that $\lim_{n \rightarrow \infty} \sqrt{x_n} = 7$.

Proof.

- (a) Let $\epsilon > 0$. Since (x_n) converges to 0, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - 0| < \epsilon^2$. Now, for any $n \geq N$, $|\sqrt{x_n} - 0| = \sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon$. Therefore, $\lim_{n \rightarrow \infty} \sqrt{x_n} = 0$.
- (b) Let $\epsilon > 0$. Since (x_n) converges to 49, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\begin{aligned} |x_n - 49| &= \left| \frac{(x_n - 49)(x_n + 49)}{\sqrt{x_n} + 7} \right| \\ &= \left| \frac{x_n - 49}{\sqrt{x_n} + 7} \right| \\ &\leq \frac{|x_n - 49|}{7} \end{aligned}$$

□



Exercise: 2.3.2

Using only [Definition 2.2.3](#), prove that if $(x_n) \rightarrow 2$, then

(a) $\left(\frac{2x_n - 1}{3}\right) \rightarrow 1;$

(b) $(1/x_n) \rightarrow 1/2.$

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution.

- (a) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - 2| < \epsilon$. Now, for any $n \geq N$,

$$\begin{aligned} \left| \frac{2x_n - 1}{3} - 1 \right| &= \left| \frac{2x_n - 1 - 3}{3} \right| \\ &= \left| \frac{2x_n - 4}{3} \right| \\ &= \frac{2}{3} |x_n - 2| \\ &< |x_n - 2| \\ &< \epsilon \end{aligned}$$

Therefore, $\frac{2x_n - 1}{3} \rightarrow 1$ □

- (b) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $x_n \geq 1$. Then, we will choose N_2 so that $|x_n - 2| < \epsilon$ for all $n \geq N_2$. Afterwards, we take $N = \max\{N_1, N_2\}$. And note that for $n \geq N$,

$$\begin{aligned} \left| \frac{1}{x_n} - \frac{1}{2} \right| &= \left| \frac{2 - x_n}{2x_n} \right| \\ &< \frac{|2 - x_n|}{2} \\ &< \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

□



2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Definition 2.4.1

A sequence a_n is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.2: Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges.

Proof. Let (a_n) be an increasing and bounded sequence. Since (a_n) is bounded, the set $A = \{a_n \mid n \in \mathbb{N}\}$ is clearly also bounded. Since A is bounded, $\sup A$ exists. We claim that $\lim_{n \rightarrow \infty} a_n = \sup A$. Thus, for all $\epsilon > 0$ and by our definition of supremum, there exists $N \in \mathbb{N}$ such that $\sup A - \epsilon < a_N \leq \sup A$. Since (a_n) is increasing, for all $n \geq N$, $\sup A - \epsilon < a_N \leq a_n \leq \sup A$. It follows that $|a_n - \sup A| < \epsilon$. Therefore, $\lim_{n \rightarrow \infty} a_n = \sup A$. \square

Example 2.9: MCT

Consider the recursively defined sequence x_n where $x_1 = 3$ and for all $n \in \mathbb{N}$, $x_{n+1} = \frac{1}{4-x_n}$. Show that x_n converges.

Proof. We will show that x_n is monotone and bounded.

- **Part 1: Monotone Decreasing**

- Base case: $x_1 = 3, x_2 = 1$.
- Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq x_{n+1}$. It follows that

$$\begin{aligned} x_n &\geq x_{n+1} \\ 4 - x_n &\leq 4 - x_{n+1} \\ \frac{1}{4 - x_n} &\geq \frac{1}{4 - x_{n+1}} \\ x_{n+1} &\geq x_{n+2} \end{aligned}$$

- **Part 2: Bounded Below Claim:** Sequence is bounded below by 0.

- Base case: $x_1 = 3 > 0$.
- Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq 0$. It follows that $4 - x_n \leq 4$,



and when we take the reciprocal, we get

$$\begin{aligned}\frac{1}{4-x_n} &\leq \frac{1}{4} \\ x_{n+1} &\geq 1/4 \\ &> 0\end{aligned}$$

By math induction, x_n is bounded below by 0.

By the Monotone Convergence Theorem, x_n converges.

So, what is the limit? We know (x_n) converges so let $L = \lim_{n \rightarrow \infty} x_n$. Then, $\lim_{n \rightarrow \infty} x_{n+1} = L$. We also know $x_{n+1} = \frac{1}{4-x_n}$. So $L = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{4-x_n} = \frac{1}{4-L}$. It must be true that $L = \frac{1}{4-L}$. Solving for L , we get

$$\begin{aligned}L(4-L) &= 1 \\ 4L - L^2 &= 1 \\ L^2 - 4L + 1 &= 0\end{aligned}$$

Hence, $L = 2 - \sqrt{3}$ or $L = 2 + \sqrt{3}$. Notice that it cannot be the latter because it is bigger than 3. \square

2.4.1 Recap and Summary

We use limits to define multiple things in calculus. This is why we are focusing so heavily upon it. For example,

- (a) Derivatives: $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$
- (b) Integrals: $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$
- (c) Infinite Series: $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$ Consider geometric series, C_a such that each term is multiplied by a ratio r . This is represented as $\sum_{n=0}^{\infty} ar^n = 1 + r + r^2 + r^3 \dots$. When we look at partial sums, we get $S_n = 1 + r + r^2 + r^3 + \dots + r^n$. We can then multiply by r to get $rS_n = r + r^2 + r^3 + r^4 + \dots + r^{n+1}$. Subtracting the two, we get $(1-r)S_n = 1 - r^{n+1}$. Thus,

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

If $|r| < 1$, then $\lim_{n \rightarrow \infty} r^n = 0$. Thus, $\lim_{n \rightarrow \infty} S_n = \frac{1}{1-r}$.

Looking to the future, we are going to use functions and summations together. For example, when we have $f(x) = \sum_{n=0}^{\infty} (a_n)x^n$ such that $f'(x) = \sum_{n=0}^{\infty} (a_n)x^{n-1}$.

Definition 2.4.3

Let x_n be a bounded sequence. Then the *limit inferior* is $\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \inf\{x_k \mid k \geq n\}$. This is the largest a limit can get. The *limit superior* is $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sup\{x_k \mid k \geq n\}$. This is the smallest a limit can get.



See Exercise 2.4.7 in the book for more information.

Example 2.10: *Monotone Decreasing Sequence*

$$x_1, x_2, x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 1\} = S.$$

$$x_2, x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 2\} = S.$$

$$x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 3\} = S.$$

$$x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 4\} = S.$$

$\limsup_{n \rightarrow \infty} x_n$ is guaranteed to exist by the **Monotone Convergence Theorem**.

Example 2.11: \liminf

Let $x_n = (-1)^n(1 + \frac{1}{n})$. Thus, $x_{1,2,3} = -2, 1\frac{1}{2}, -1\frac{1}{3} \dots$

Example 2.12: Convergence Towards 0

Let $x_n = (-1)^n \frac{1}{n}$. Thus, $x_{1,2,3} = -1, \frac{1}{2}, -\frac{1}{3} \dots$

Theorem 2.4.4

A sequence x_n is convergent if, and only if, $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n$.

See Theorem 2.4.6 in the book for another view.

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Definition 2.5.1

Let a_n be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \dots$ be an increasing sequence of natural numbers. Then, the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \dots$ is called a *subsequence* of a_n and is denoted by a_{n_k} , where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 2.5.2

Subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let x_{n_k} be a subsequence of x_n , and let $L = \lim_{n \rightarrow \infty} x_n$. We want to show that $\lim_{n \rightarrow \infty} x_{n_k} = L$. Let $\epsilon > 0$. Since x_n converges to L , there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$. Since n_k is increasing, there exists $M \in \mathbb{N}$ such that $n_k \geq N$



for all $k \geq M$. Thus, for all $k \geq M$, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n \rightarrow \infty} x_{n_k} = L$.

Let x_{n_k} be a subsequence of x_n . Let $\epsilon > 0$. Since $(x_n) \rightarrow L$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$.

Now, looking at x_{n_k} , notice that $n_k \geq k$ for all k . Consider $k = N$. For any $n \geq N$, $n \geq N \geq k$. Thus, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n \rightarrow \infty} x_{n_k} = L$. \square

Theorem 2.5.3: Divergence Criterion

If x_n has two subsequences that converge to different limits, then x_n diverges.

Building upon this idea of Divergence, we can list some other ways a sequence can diverge:

- (a) Find one subsequence that diverges.
- (b) Find two subsequences that converge to separate limits.
- (c) Negate the **definition of convergence**.
 - For example, a sequence converges to L if there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - L| \geq \epsilon$. There exists a subsequence (a_{n_k}) such that for all $k \in \mathbb{N}$, $|a_{n_k} - L| \geq \epsilon$.

Theorem 2.5.4: Bolzano-Weierstrass Theorem

Every bounded sequence in \mathbb{R} has a convergent subsequence.

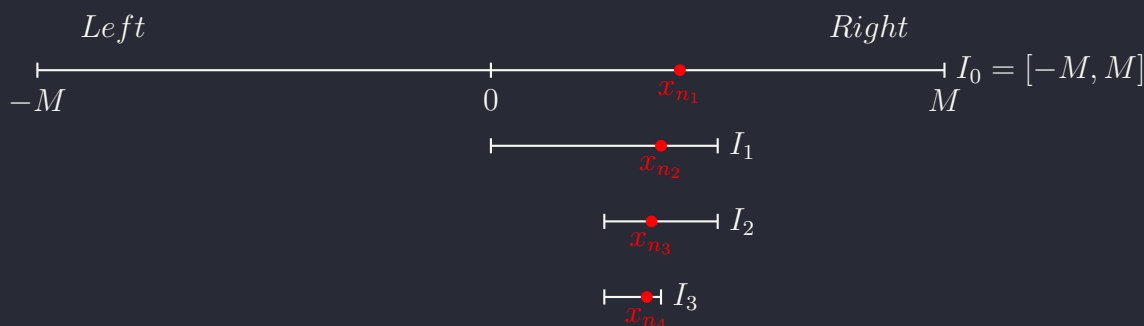
Proof. Let x_n be a bounded sequence. There exists an $M > 0$ such that every term x_n belongs to $[-M, M]$. To prove this theorem, we will be utilizing a recursive argument style. Thus, let $I_0 = [-M, M]$. I_0 has length $2M$. Cut I_0 in half with I_1 and I_2 both being half as long as I_0 . Since x_n is bounded, there exists an I_L or I_R that contains infinitely many terms of x_n . We will pick one, call it I_1 that is contained in I_L . I_1 has length M . Pick one of those terms inside I_1 and call it x_{n_1} . Now, cut I_1 in half with equal length in intervals. One of them contains infinitely many terms. Call that interval I_2 . I_2 has length $\frac{M}{2}$. Pick one of those terms inside I_2 and call it x_{n_2} . Continue this process indefinitely for all $n \geq \mathbb{N}$ with $n_1 > n_2$. Continue this process, and we get

- a sequence of closed intervals I_n .
 - I_n has length $\frac{2M}{2^n}$.
 - They are nested, $I_n \subseteq I_{n-1}$.
- a subsequence x_{n_k}
 - for all k_1 , $x_{n_k} \in I_k$.

The **Nested Interval Property** states that $\bigcup_{n=1}^{\infty} I_n$ is non empty. Let L be a point in $\bigcup_{n=1}^{\infty} I_n$. We claim $\lim_{n \rightarrow \infty} x_{n_k} = L$. Let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that $\frac{2M}{2^N} < \epsilon$. (Since $\lim_{n \rightarrow \infty} \frac{2M}{2^n} = 0$. See **Theorem 2.5.5**) For any $k \geq N$, recall that x_{n_k} ,



$L \in I_k$. Since I_k has length $\frac{2M}{2^n}$. Thus, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n \rightarrow \infty} x_{n_k} = L$ and (x_n) has a convergence subsequence. \square



Theorem 2.5.5

Let $b \in (0, 1)$. Then $\lim_{n \rightarrow \infty} b^n = 0$.

Proof. The sequence (b^n) is monotone decreasing. This is because $b^{n+1} = b^n b < b^n$. This sequence is also bounded by 0. Hence, by the **Monotone Convergence Theorem**, (b^n) converges. Now, let $L = \lim_{n \rightarrow \infty} b^n$. Consider the subsequence b^{2n} . This sequence also converges to L . Thus,

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} b^{2n} \\ &= \lim_{n \rightarrow \infty} b^n b^n \\ &= \lim_{n \rightarrow \infty} b^n \lim_{n \rightarrow \infty} b^n \\ &= L^2. \end{aligned}$$

Thus, $L = 0$ or $L = 1$. The limit cannot be 1 because b^n is decreasing away from 1. Therefore, $L = 0$. \square

2.5.1 Exercises

Exercise: 2.5.1

Give an example of each of the following, or argue that such a request is impossible.

- A sequence that has a subsequence that is bounded but contains no subsequence that converges.
- A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.



Solution.

- (a) Impossible. This violates the **Bolzano-Weierstrass Theorem**. It assures us that every bounded sequence has a convergent subsequence. If a subsequence is bounded, then it must have a convergent subsequence.
- (b) Consider the sequence $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots, \frac{1}{n}, \frac{(n-1)}{n})$. From this, you can have a subsequence $(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n})$ which converges to 0, and also a subsequence $(\frac{1}{2}, \frac{2}{3}, \dots, \frac{n-1}{n})$, which converges to 1.

Exercise: 2.5.2

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

- (a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.
- (c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

Solution.

- (a) False. As shown in Example 2.5.4, if we have a sequence like $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, \dots)$, we see that it is divergent, and has two proper subsequences $(\frac{1}{5}, \frac{1}{5}, \dots)$ and $(-\frac{1}{5}, -\frac{1}{5}, \dots)$ that both converge to values $\frac{1}{5}$ and $-\frac{1}{5}$, respectively.
- (c) True. If we assume that (x_n) is bounded and diverges, then by the **Bolzano-Weierstrass Theorem**, every bounded sequence has a convergent subsequence. Let (x_{n_k}) be that convergent subsequence of (x_n) , and let L_1 be its limit. Since (x_n) diverges, it cannot converge to L_1 . Thus, there exists an $\epsilon_0 > 0$ such that for all terms of (x_n) , they stay outside the ϵ_0 -neighborhood of L_1 . Assume this sub-subsequence (x_{m_k}) contains these terms. When we apply the same logic to this sub-subsequence, we see that by the **Bolzano-Weierstrass Theorem**, (x_{m_k}) has a convergent subsequence with limit L_2 , where $L_2 \neq L_1$ because the terms of (x_{m_k}) stay outside the ϵ_0 -neighborhood of L_1 . Thus (x_n) contains two subsequences that converge to different limits, L_1 and L_2 .

Exercise: 2.5.5

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a .

Proof. Suppose that (a_n) does not converge to $a \in \mathbb{R}$. By the definition of convergence, this means there is a positive real number ϵ_0 such that no matter how large we choose $N \in \mathbb{N}$, there will always exist some $n > N$ where $|a_n - a| \geq \epsilon_0$. In a formal way, this



shows that (a_n) does not converge to a within the ϵ_0 -neighborhood.

We aim to demonstrate that this leads to a contradiction by constructing a subsequence of (a_n) that stays outside this neighborhood. Begin by selecting n_1 such that $|a_{n_1} - a| \geq \epsilon_0$. Next, since the condition holds for all $N \in \mathbb{N}$, we can find another index $n_2 > n_1$ such that $|a_{n_2} - a| \geq \epsilon_0$. Continuing this process, we generate an increasing sequence of indices $n_1 < n_2 < n_3 < \dots$ such that for each $i \in \mathbb{N}$, $|a_{n_i} - a| \geq \epsilon_0$.

Now consider the subsequence (a_{n_i}) we have built. Since (a_n) is bounded by assumption, its subsequence (a_{n_i}) is also bounded. By the **Bolzano-Weierstrass Theorem**, every bounded sequence has a convergent subsequence. Let $(a_{n_{i_k}})$ denote a convergent subsequence of (a_{n_i}) . According to our assumption, any convergent subsequence of (a_n) must converge to a .

However, each term of $(a_{n_{i_k}})$ remains outside the ϵ_0 -neighborhood of a . Thus, it is impossible for $(a_{n_{i_k}})$ to converge to a . This contradiction implies that our initial assumption—that (a_n) does not converge to a —is false. Therefore, the sequence (a_n) must converge to a . \square

Exercise: 2.5.6

Use a similar strategy to the one in **Theorem 2.5.5** to show

$$\lim b^{1/n} \text{ exists for all } b \geq 0$$

and find the value of the limit. (The results in **Exercise 2.3.1** may be assumed.)

Proof. For this proof, we will be examining different cases that b may fall under. These cases are $b = 0$ and $b > 0$.

1. If $b = 0$, then the sequence $(b^{1/n})$ is simply $(0, 0, 0, \dots)$. Thus, the limit of this sequence is 0.
2. For $b > 0$, it is more complicated. We see that this sequence is decreasing because as n increases, the exponent $\frac{1}{n}$ trends toward a higher denominator, so $b^{1/n}$ decreases. We can see this by observing the property that for any $m < n$, $\frac{1}{n} < \frac{1}{m}$. This implies that $b^{1/n} < b^{1/m}$.

Further, we see that this sequence is bounded below by 0 when $b > 1$ and by 1 when $b = 1$ (because 1 to any exponent is just 1). Therefore, this sequence is monotone and bounded below, so it must converge to some $L \geq 0$ by the **Monotone Convergence Theorem**.

To find L , we need to consider another set of cases for b .

- i. If $b = 1$, then $b^{1/n} = 1$ for all n , so its limit is 1.
- ii. If $b > 1$, then as n increases, the exponent $\frac{1}{n}$ approaches 0, so $b^{1/n}$ approaches $b^0 = 1$. So, its limit is also 1.



- iii. If $0 < b < 1$, then we know $\lim_{n \rightarrow \infty} b^{1/n} = 1$ because $b^{1/n}$ is a decreasing sequence bounded by 0, and when we have smaller and smaller powers of a number less than 1, it pushes it closer to 1.

Through these cases, we see that for all $b \geq 0$, $\lim_{n \rightarrow \infty} b^{1/n} = 1$. \square

2.6 The Cauchy Criterion

Recall

How do we prove x_n converges?

- (a) We know and prove the limit \rightarrow claim L , show terms get close to L .
- (b) **Monotone Convergence Theorem.**

Definition 2.6.1

A sequence x_n is a *Cauchy sequence* if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|x_m - x_n| < \epsilon$.

This says that as terms get close to each other and stay close together, there's some value they're all getting close to.

Example 2.13: Cauchy Sequence

Consider the sequence $a_1 = 1, a_2 = 2$, where

$$a_n = \frac{a_{n-1} + a_{n-2}}{2} \text{ for all } n \geq 2.$$

Show this sequence is Cauchy.

Proof. Look at the differences of consecutive terms, $|a_1 - a_2| = 1$, $|a_2 - a_3| = 1/2$, we can see a formula $a_n - a_{n+1} = 1/2^{n-1}$. Assume $|a_n - a_m| = |a_n - a_{n+1} - a_{n+2}| - \dots -$



$a_{m-1} - a_m$ with $n < m$. From the **Triangle Inequality**,

$$|a_n - a_m| \leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \quad (2.10)$$

$$= \frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{m-2}} \quad (2.11)$$

$$= \frac{1}{2^{n-1}} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}} \right) \quad (2.12)$$

$$= \frac{1}{2^{n-1}} \left(\frac{1 - \frac{1}{2^{m-n}}}{1 - \frac{1}{2}} \right) \quad (2.13)$$

$$= \frac{1}{2^n} \left(1 - \frac{1}{2^{m-n}} \right) \quad (2.14)$$

$$< \frac{1}{2^n}. \quad (2.15)$$

Notice that we were able to pull out the $1/2$ and use the geometric series formula at step 2.12. From here we know that $|a_n - a_m| < \frac{1}{2^n}$.

Now, conclude the proof by letting $\epsilon > 0$. We know $(1/2^n) \rightarrow 0$. Thus, there exists $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$. For all $n, m \geq N$, (without loss of generality $n < m$) $|a_n - a_m| < \frac{1}{2^n} \leq \frac{1}{2^N} < \epsilon$. Therefore, a_n is **Cauchy** and it converges. \square

Note: To find the limit of this series, a proof strategy is finding subsequences that are odd and even, and show the converge to the same limit.

Theorem 2.6.2: Cauchy Criterion

A sequence x_n converges if, and only if, it is a Cauchy sequence.

Proof. We show this by proving both implications:

(\Rightarrow) Assume (x_n) is a convergent sequence in \mathbb{R} . Given $\epsilon > 0$. Let $L = \lim_{n \rightarrow \infty} x_n$. Since $(x_n) \rightarrow L$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \frac{\epsilon}{2}$. For all $n, m \geq N$,

$$\begin{aligned} |x_m - x_n| &= |x_m - L + L - x_n| \\ &\leq |x_m - L| + |L - x_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, x_n is a Cauchy sequence.



(\Leftarrow) Assume x_n is a Cauchy sequence.

- **Step 1:** Show that x_n is bounded.

Since x_n is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < 1$. It follows that for all $n \geq N$, we need to account for x_1, \dots, x_{n-1} . Thus, let $M = \max\{|x_1|, |x_2|, \dots, |x_{n-1}|, |x_n| + 1\}$. Then for all $n \in \mathbb{N}$, $|x_n| < M$.

- **Step 2:** Since x_n is bounded, there exists a convergent subsequence x_{n_k} by the **Bolzano-Weierstrass Theorem**. Let L be the limit of the subsequence.
- **Step 3:** Show that x_n converges to L .

If some get close to L and all get close to each other, they all get close to L . Let $\epsilon > 0$. Since x_{n_k} converges to L , there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $|x_{n_k} - L| < \frac{\epsilon}{2}$. Since x_n is Cauchy, there exists $M \in \mathbb{N}$ such that for all $n, m \geq M$, $|x_n - x_m| < \frac{\epsilon}{2}$. Let $M_0 = \max\{N, n_k\}$. By the **Archimedean Principle**, there exists N_0 such that $n_{k_0} \geq M_0$. Then, from the **Triangle Inequality**, we say that for all $n \geq N_0$,

$$\begin{aligned} |x_n - L| &\leq |x_n - x_{n_{k_0}}| + |x_{n_{k_0}} - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $(x_n) \rightarrow L$.

By proving both directions of the inequality, we found that a sequence (x_n) converges if, and only if, it is a Cauchy sequence. \square

Definition 2.6.3

A sequence is called *contracting* if there exists $0 < C < 1$ such that for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq C |x_n - x_{n-1}|$.

How this works: we take a sequence a_1, a_2, \dots and subtract $a_1 - a_2$. Then, we have the inequality:

$$\begin{aligned} |a_2 - a_3| &\leq C |a_1 - a_2| \\ |a_3 - a_4| &\leq C |a_2 - a_3| \leq C^2 |a_1 - a_2| \\ |a_4 - a_5| &\leq C |a_3 - a_4| \leq C^3 |a_1 - a_2| \\ &\vdots \end{aligned}$$

From this, a theorem emerges:

**Theorem 2.6.4**

If a sequence is contracting, then it is Cauchy, and thus converges.

Proof. No formal proof for this one. Just use the triangle inequality along with the geometric sum identity to show that the sequence is Cauchy. Something like $|a_n - a_m| \leq \frac{C^n}{1-C} |a_1 - a_2|$. Then, use the **Cauchy Criterion** to show that the sequence converges. \square

Chapter 3

Basic Topology of Real Numbers

3.1 The Cantor Set

We will build this set through an iterative process. Start with a number line C_0 that stretches from 0 to 1. Remove the middle third of the interval, leaving two intervals of length $\frac{1}{3}$. We will call the set of points removed from C_0 C_1 . Next, remove the middle third of each of the two intervals, leaving four intervals of length $\frac{1}{9}$. We will call the set of points removed from C_1 C_2 . Continue this process indefinitely.

Definition 3.1.1

The *Cantor set*, C , is defined as $C = \bigcap_{n=0}^{\infty} C_n$. This set is

- (a) Non-empty. All end points stay within the interval.
- (b) C is uncountable.

The second part of that definition is a bit tricky to prove, but a visual will do for now. We can put all elements of the Cantor set in a one-to-one correspondence with the set of all 0s and 1s. This shows that not only is it uncountable, but it also has the same cardinality as $[0, 1]$.

The total length of removed elements, $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots = \frac{1}{3}(1 + \frac{2}{3} + \frac{4}{9})$. Notice the resemblance to the geometric series? We can write this as

$$\frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{3} \left(\frac{1}{1 - \frac{2}{3}}\right) = 1.$$

In summary,

- (a) Start with interval $[0, 1]$.
- (b) Remove countably disjoint intervals.
- (c) Uncountably many points between these intervals, all isolated from each other.
- (d) The space taken up by the leftover points has “length” 0.



When we review the properties of a fractal, we see that the Cantor set is a fractal. It is self-similar, and the dimension of the Cantor set is $\log_3 2$.

For a cool look at the cantor set as a fractal, check out https://en.wikipedia.org/wiki/File:Cantor_Set_Expansion.gif.

3.2 Open and closed Sets

Definition 3.2.1

For a point $x \in \mathbb{R}$, and $\epsilon > 0$, we define the *epsilon-neighborhood* of x to be $V_\epsilon(x) = \{y \in \mathbb{R} \mid |x - y| < \epsilon\}$.

3.2.1 Open Sets

Definition 3.2.2

A set $A \subseteq \mathbb{R}$, is called an *open set* if for every $x \in A$, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subset A$.

Some Examples of Open Sets

- All open intervals are also open sets.
- \mathbb{R} is open.
- \emptyset is open.
- $\{1\}$ is not open.
- $[0, 2]$ is not open.
- \mathbb{Q} is not open.
- $[4, 6)$ is not open.
- $(0, 1) \cup (1, 3) \cup (5, 10)$ is open.
- $(0, 3] \cap [2, 4)$ is open.
- Cantor set is not open.

Theorem 3.2.3

- (i) The union of an arbitrary collection of open sets is open.
- (ii) The intersection of a finite collection of open sets is open.

Proof. (ii). Let A_1, A_2, \dots, A_n be open sets. Let $x \in \bigcap_{i=1}^n A_i$. Then for each $i \leq n$, $x \in A_i$. Since each A_i is open, there exists $\epsilon_i > 0$ such that $V_{\epsilon_i}(x) \subset A_i$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_n\} > 0$. Then, for all $i \leq n$, $V_\epsilon(x) \subseteq V_{\epsilon_i}(x) \subseteq A_i$. \square

Note that we cannot use this for cases with infinity. For example, consider $A_n = (-\frac{1}{n}, \frac{1}{n})$. This is open, but $\bigcap_{n=1}^{\infty} A_n = \{0\}$, which is not open.



3.2.2 Closed Sets

Definition 3.2.4

Let $A \subseteq \mathbb{R}$. We say x is a *limit point* of A if for all $\epsilon > 0$, there exists $a \in A$ such that $a \in V_\epsilon(x)$ that is not x . Additionally, a point $x \in \mathbb{R}$ is a *limit point* if, and only if, there exists a sequence (a_n) of points from A that are not x . But $\lim_{n \rightarrow \infty}(a_n) = x$.

Definition 3.2.5

Let $B \subseteq \mathbb{R}$ is called a *closed set* if B contains all its limit points.

Important note: Limit points could be outside a set. Consider $(0, 1)$. Even though 0 and 1 do not belong to the set, they are considered limit points that are outside the set.

Some Examples of Closed Sets

- $[0, 1]$ is closed.
- $(0, 1)$ is not closed.
- \mathbb{R} is closed.
- \emptyset is closed.
- \mathbb{Q} is not closed.
- $[3, \text{infy})$ is closed.
- $\frac{1}{n} \mid n \in \mathbb{N}$ not closed. (Because of 0)
- $[1, 4] \cup \{8\}$ is closed.
- $\{1\}$ is closed.
- $[1, 2)$ is not closed. Note that this set is neither open or closed.

Theorem 3.2.6

A set $B \subseteq \mathbb{R}$ is closed if, and only if, its complement is open. Similarly, a set $A \in \mathbb{R}$ is open if, and only if, its complement is closed.

Proof. We show this by proving both implications:

- (\Rightarrow) Assume $B \subseteq \mathbb{R}$ is a closed set. We will show that B^c is open. Let $x \in B^c$. So, $x \notin B$. This means x is not a limit point. (From the negated definition of limit point:) There must exist $\epsilon > 0$ such that no elements of B belong to $V_\epsilon(x)$. Then, $V_\epsilon(x) \subseteq B^c$. Therefore, B^c is open.



(\Leftarrow) Assume B^c is open. We will show that B is closed. Let x be a limit point of B . For all $\epsilon > 0$, there exists a $b \in B$ such that $b \in V_\epsilon(x)$. So, $V_\epsilon(x)$ is not a subset of B^c . This is true for every ϵ . Since B^c is open, it must be that $x \notin B^c$. Thus, $x \in B$. So, B contains all its limit points. Therefore, B is closed.

□

Definition 3.2.7

Let $A \subseteq \mathbb{R}$ and let L be the set of limit points of A . The *closure* of A is defined as $\bar{A} = \bar{A} = A \cup L$.

Theorem 3.2.8

- (i) The intersection of any collection of closed sets is closed.
- (ii) The union of finitely many closed sets is closed.

Theorem 3.2.9

The closure of a set is a closed set.

Note: This theorem may seem trivial, but it answers the question of “Are there limit points in L that are not accounted for?”

Theorem 3.2.10

The closure set A is the *smallest* closed set containing A . (Where “smallest” refers to a subset of any other closed set containing A .)

Example 3.1: Closed Sets

Generate countably many closed sets where the union is not closed.

Solution. $B_n = [\frac{1}{n}, 1 - \frac{1}{n}]$. Therefore, $\bigcup_{n=3}^{\infty} B_n = (0, 1)$. For example, that would look like: $\{\frac{1}{2}\} \cup \{\frac{1}{3}\} \cup \dots$