Multivariable Calculus Exam I Corrections

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In Class Portion

1. Consider the parametric curve defined by $x(t) = 3t^2 - 8t + 1$, $y(t) = e^{-t^2}$, for $0 \le t \le 2$.

   (a) (4 points) Find the equation, in regular Cartesian coordinates, of the tangent line to this curve at $t = 1$. Please use exact values here!

   Solution. First, we compute the derivatives of $x(t)$ and $y(t)$ with respect to t:

   $$\tfrac{dx}{dt} = 6t - 8 \quad \text{and} \quad \tfrac{dy}{dt} = -2te^{-t^2}.$$

   Plugging this into the formula for slope, we see that:

   $$\frac{dx}{dy} = \frac{dy/dt}{dx/dt} = \frac{-2te^{-t^2}}{6t-8}.$$

   To get our points, we plug in $t = 1$:

   $$x(1) = 3(1)^2 - 8(1) + 1 = -4 \quad \text{and} \quad y(1) = e^{-1}.$$

   Thus, our point is $(-4, e^{-1})$. Plugging in $t = 1$ into the slope formula, we get:

   $$\left.\frac{dy}{dx}\right|_{t=1} = \frac{-2e^{-1}}{6-8} = e^{-1}.$$

   Thus, the equation of the tangent line is:

   $$y = e^{-1}(x+4) + e^{-1}.$$

   (b) (4 points) Is this curve concave up, down, or neither when $t = 1$? Justify this answer.

   Solution. To determine concavity, we must solve the following equation:

   $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt} = \frac{\frac{d}{dt}(-2te^{-t^2})}{6t-8} = \frac{-2e^{-t^2} + 4t^2e^{-t^2}}{6t-8} \Rightarrow t = 1 \Rightarrow -e^{-1}.$$

   Since $-e^{-1} < 0$, the curve is concave down at $t = 1$. (*Correction*: My equation for concavity was incorrect on my test sheet.)

2. (4 points each) Let $\mathbf{u} = 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = -\mathbf{j} + 2\mathbf{k}$.

   (c) Determine $\text{proj}_{\mathbf{v}}\mathbf{u}$. Leave all components as exact values.

Solution. We know that $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$. Thus, we compute:

$$\mathbf{u}\cdot\mathbf{v} = 5(0) + 2(-1) + (-3)(2) = -8 \quad \text{and} \quad \|\mathbf{v}\|^2 = \left(\sqrt{5}\right)^2 = 5.$$

Thus, $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{-8}{5}(-\mathbf{j} + 2\mathbf{k}) = \frac{8}{5}\mathbf{j} - \frac{16}{5}\mathbf{k}$.

3. (4 points) Let $\mathbf{u} = \langle 5, -1, 2\rangle$ and $\mathbf{v} = \langle -2, y, z\rangle$. What is the relationship between y and z which makes $\mathbf{u}$ orthogonal to $\mathbf{v}$?

Solution.

5. (6 points) Find an equation in scalar form of the plane which passes through $(-2, 7, 1)$ and is perpendicular to the planes $3x + y - z = 0$ and $-2x - y + 5z + 1 = 0$ [Hint: Think about what the relationship among the various normal vectors must be.]

Solution.

6. (6 points) Find the exact value of curvature κ for the curve defined by $\mathbf{r}(t) = (t^2 - t)\mathbf{i} + (t^3 - 7t + 1)\mathbf{j} + t^3\mathbf{k}$ at the point $t = 1$. [Hint: Since this is defined in $\mathbb{R}^3$, it is *significantly* easier to use the version of κ which uses a cross product!] Numerical approximations, rounded to 4 decimal places, are appropriate here.

Solution.