Multivariable Calculus Notes

MATH 230

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CHAPTER 1

PARAMETRIC EQS AND POLAR COORDS

1.1 Parametric Equations

1.1.1 Introduction

Most of your calculus experience has been single variable, so that the functions under consideration were typically $f : \mathbb{R} \to \mathbb{R}$. Our course is divided into roughly 3 sections:

- Parametric Equations/Functions: Functions of the form $f : \mathbb{R} \to \mathbb{R}^n$ (Chapters 1 - 3)
- Scalar Functions: Functions of the form $f : \mathbb{R}^n \to \mathbb{R}$ (Chapters 4 - 5)
- Vector Fields: Functions of the form $f : \mathbb{R}^n \to \mathbb{R}^n$ (Chapter 6)

1.1.2 Parametric Equations

A *parametric equation* (or, *sometimes parametric function* or *vector-valued function*) is a function of the form $f\colon \mathbb{R} \to \mathbb{R}^n$. We will typically consider $n = 2$ or $n = 3$ and call the input variable the parameter, usually denoted by t. We write them as

$$f(t) = \begin{cases} x(t) \\ y(t) \end{cases} \qquad \text{or} \qquad f(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases} .$$

A *parametric curve* is the set of points $(x(t), y(t))$ in $\mathbb{R}^2$ or $(x(t), y(t), z(t))$ in $\mathbb{R}^3$ traced out. Note that in general, the curve may not be a function for y in terms of x, but is a function of the parameter t.

1.1.3 Graphing Parametric Curves in the Second Dimension

Elimination of the Parameter

In some cases, we can explicitly solve for t in terms of one of x or y. When this is possible, you can write $y(x)$ or $x(y)$ and use your "regular" algebraic knowledge. We call this process *eliminating the parameter*.

Using Technology

- Your TI-84 can graph this if you switch to `par` mode.
- Likewise, GeoGebra can do this, using the `curve` function.
  - In general, the syntax is: `curve(x(t), y(t), t, min, max)`

1.1.4 The Cycloid

A wheel of radius a is rolling along a flat road at a constant velocity. The curve generated by a point along the edge of the wheel traces out a shape called a *cycloid*. Let t represent the angle - in radians!!!! - rotated through, and that the point of interest starts at the origin. Before we find the equations for the point, let's find the location of the center of the circle:

$$f_{\text{center}}(t) = \begin{cases} x(t) = at \\ y(t) = a \end{cases}$$

Then, relative to the center, our point along the edge has equations

$$f(t) = \begin{cases} x(t) = -a\sin t \\ y(t) = -a\cos t \end{cases}$$

Thus, our point has parametric equations

$$f(t) = \begin{cases} x(t) = a(t - \sin t) \\ y(t) = a(1 - \cos t) \end{cases}$$

1.1.5 Final Notes

Next time, we'll start asking Calculus-y questions: What are the velocities in the x, y, and total directions? What total distance does it travel? What is the area of the region under one period of the cycloid?

- The syllabus has a number of practice problems to work on. These are not required, and not to be turned in, but are for you to work before class next time.
- We will talk about them at the start of the next class. You should try them beforehand.
- The most common reason for a lack of success in this class is not spending time working problems on your own.

1.2 Calculus of Parametric Curves

For this section, we will have a parametric curve in R2, defined by $f(t) = \begin{cases} x(t) \\ y(t) \end{cases}$. In many cases, the curve does not describe y as a function of x. However, we can still carry over many ideas from single variable calculus.

1.2.1 Slope for a Parametric Curve

Given a point t_0, the *slope of the curve* in the xy-plane is given by

$$\left.\frac{dy}{dx}\right|_{t=t_0} = \left.\frac{dy/dt}{dx/dt}\right|_{t=t_0}.$$

Note that this is undefined when $x'(t_0) = 0$.

The *tangent line* at t_0 is given by

$$y = \left(\left.\frac{dy}{dx}\right|_{t=t_0}\right)(x - x(t_0)) + y(t_0).$$

1.2.2 Second Derivative

The value of the second derivative for the curve at t_0 is given by

$$\left.\frac{d^2y}{dx^2}\right|_{t=t_0} = \left.\frac{d}{dt}\left(\frac{dy}{dx}\right)\right|_{t=t_0} = \left.\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\right|_{t=t_0}.$$

Note the benefit of Leibnitz notation for each of these two derivatives!

1.2.3 Area Under a Curve

Suppose that a parametric curve is non-self intersecting. Then, the signed area of the region between the curve and the x-axis on the t interval $[t_a, t_b]$ is given by

$$A = \int_{t_a}^{t_b} y(t)\frac{dx}{dt}\,dt.$$

1.2.4 Arc Length

The *arc length* of a parametric curve over the t interval $[t_a, t_b]$ is given by

$$s = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

1.2.5 Surface Area

The *surface area* of the region obtained by rotating a non-self intersecting parametric curve is given by

$$S = \int_{t_a}^{t_b} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

1.2.6 The Cycloid

We can apply each of the above to the cycloid:

- *Derivative*: $\frac{dy}{dx} = \frac{dy}{dx} = \frac{\sin t}{1-\cos t}$. Note that the slope is then independent of the radius of the wheel and that the slope is undefined at each of $t = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$.

- *Cartesian Equation*: With radius of 3 and when $t = \frac{\pi}{3}$, the point is found by solving for $x(\frac{\pi}{3})$ and $y(\frac{\pi}{3})$:

$$\begin{aligned} x\left(\frac{\pi}{3}\right) &= 3\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right) = \pi - \frac{3\sqrt{3}}{2} \\ y\left(\frac{\pi}{3}\right) &= 3\left(1 - \cos\left(\frac{\pi}{3}\right)\right) = \frac{3}{2} \\ (x, y) &= \left(\pi - \frac{3\sqrt{3}}{2}, \frac{3}{2}\right) \end{aligned}$$

  Plugging in our t value into our derivative, we get a slope of

$$\frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

  Now, we can write the equation of the tangent line as

$$y = \sqrt{3}\left(x - \pi + \frac{3\sqrt{3}}{2}\right) + \frac{3}{2}.$$

- *Concavity*: $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)$.

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d/dt(dy/dx)}{dx/dt} \\
&= \frac{\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)}{a - a\cos t} \\
&= \frac{\frac{\cos t(1-\cos t)-\sin t\sin t}{(1-\cos t)^2}}{a - a\cos t} \\
&= \frac{\cos t - \cos^2 - \sin^2(t)}{(1-\cos t)^2 a(1-\cos t)} \\
&= \frac{\cos t - 1}{a(1-\cos t)^2} \\
&= -\frac{1}{a(1-\cos t)^2} \\
&= -\frac{a}{a^2(1-\cos t)^2} \\
&= -\frac{a}{y^2}
\end{aligned}
$$

After some work, we find that $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$, which shows that the cycloid is always concave down.

- *Area*: The area of one period of the cycloid $A = 3\pi a^2$, after some work:

$$
\begin{aligned}
A &= \int_{t_a}^{t_b} y(t)x'(t)dt \\
&= \int_0^{2\pi} (a - a\cos t)(a - a\cos t)dt \\
&= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)dt \\
&= a^2 \left(t + \frac{t}{2} + \frac{1}{4}\sin(2t)\right)\Bigg|_0^{2\pi} \\
&= a^2 \left[\left(2\pi + \frac{2\pi}{2} + \frac{1}{4}\sin(2\pi)\right) - \left(0 + \frac{0}{2} + \frac{1}{4}\sin(0)\right)\right] \\
&= a^2[2\pi + \pi] \\
&= 3\pi a^2.
\end{aligned}
$$

- *Arc Length*: The arc length of one period of the cycloid is $s = 8a$, again after some work:

$$
\begin{aligned}
s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{(a - a\cos t)^2 + (a\sin t)^2}\, dt \\
&= a\int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt \\
&= a\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt \\
&= \sqrt{2}a\int_0^{2\pi} \sqrt{1 - \cos t}\, dt \\
&= \sqrt{2}a\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{t}{2}\right)}\, dt \\
&= \sqrt{2}a \cdot \sqrt{2}\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\, dt \\
&= 2a\left(-2\cos\left(\frac{t}{2}\right)\right)\Big|_0^{2\pi} \\
&= 8a.
\end{aligned}
$$

- *Surface Area*: The surface area of the solid obtained by rotating one period of the cycloid around the x-axis is $S = \frac{64\pi a^2}{3}$, after a lot of tedious work.

$$
S = \int_0^{2\pi} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt
$$

CHAPTER 2

VECTORS IN SPACE

2.1 Vectors in the Plane

2.1.1 Notation

In print, we write vectors in bold like: $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$, ... In handwriting, we often write vectors with an arrow over the top: $\vec{v}$, $\vec{w}$, $\vec{u}$, ...

2.1.2 Vectors

A *vector* is a quantity with both *magnitude* (size, length, strength, ...) and *direction*. Given two points in the plane $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, the vector from P to Q, denoted $\overrightarrow{PQ} = \mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

We can also simply state components (known as *component form*): $\mathbf{v} = \langle x, y \rangle$.

The *zero vector*, denoted $\mathbf{0}$, is $\mathbf{0} = \langle 0, 0 \rangle$. Note that $\mathbf{0} \neq 0$.

A *scalar* is a real number (or a magnitude), without direction.

If c is a scalar and $\mathbf{v} = \langle x, y \rangle$, then

$$c\mathbf{v} = c\langle x, y \rangle = \langle cx, cy \rangle.$$

This operation is called *scalar multiplication*. Scalar multiplication changes the magnitude of a vector, but not its direction.

Note that the individual components of a vector are themselves *scalars*. You need to keep track of which is which.

If $\mathbf{v} = \langle x_1, y_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2 \rangle$, then the *vector sum*

$$\mathbf{v} + \mathbf{w} = \langle x_1 + x_2, y_1 + y_2 \rangle.$$

That is, we add component wise.

If $\mathbf{v} = \langle x_1, y_1 \rangle$, then the *magnitude* of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2}.$$

This is really just the Pythagorean theorem.

2.2 Vectors in Space

In $\mathbb{R}^3$, we have three axes, x, y, and z, which follow the *right-hand rule*: point the fingers of the right hand in the direction of the positive x-axis, curl them towards the positive y-axis, and the thumb points in the direction of the positive z-axis.

Since the distance formula in $\mathbb{R}^3$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$, then $\mathbf{u} = \langle x, y, z \rangle$ we have $\|\mathbf{u}\| = \sqrt{x^2 + y^2 + z^2}$.

To *normalize* a vector, we divide by its magnitude: $\mathbf{v} = \langle x, y, z \rangle$, then $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left\langle \frac{x}{\|\mathbf{v}\|}, \frac{y}{\|\mathbf{v}\|}, \frac{z}{\|\mathbf{v}\|} \right\rangle$. This gives us a *unit vector* in the direction of $\mathbf{v}$.

Everything else is basically the same.

2.2.1 Vector Properties

Suppose that each of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and r and s are scalars. Then the following properties hold:

- *Additive Commutativity*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
- *Additive Associativity*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- *Additive Identity*: $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- *Additive Inverse*: $-\mathbf{v} = (-1)\mathbf{v}$ and $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- *Scalar Associativity*: $r(s\mathbf{u}) = (rs)\mathbf{u}$.
- *Scalars Distributive over Vectors*: $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.
- *Vectors Distributive over Scalars*: $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$.
- *Multiplicative Identity*: $1\mathbf{u} = \mathbf{u}$.
- *Zero Scalar*: $0\mathbf{u} = \mathbf{0}$.

2.2.2 Special Vectors

A *unit vector* is a vector $\mathbf{u}$ such that $\|\mathbf{u}\| = 1$.

In $\mathbb{R}^2$ the *standard unit vectors* are $\hat{\imath} = \mathbf{i} = \langle 1, 0 \rangle$ and $\hat{\jmath} = \mathbf{j} = \langle 0, 1 \rangle$. This allows us to write $\mathbf{v} = \langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j}$, for example.

In $\mathbb{R}^3$, we have three stand unit vectors, $\hat{\imath} = \mathbf{i} = \langle 1, 0, 0 \rangle$, $\hat{\jmath} = \mathbf{j} = \langle 0, 1, 0 \rangle$, and $\hat{k} = \mathbf{k} = \langle 0, 0, 1 \rangle$.

It is a picky detail, but $\mathbf{i} \in \mathbb{R}^2 \neq \mathbf{i} \in \mathbb{R}^3$.

2.3 The Dot Product

Suppose $\mathbf{u} = \langle u_1, u_2, \ldots u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots v_n \rangle$ are vectors in $\mathbb{R}^n$. Then the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

That is, we multiply the corresponding components and sum the results.

It should be clear that $\mathbf{u} \cdot \mathbf{v}$ results in a scalar. The dot product is a special type of inner product.

Think of the dot product as a way to measure how much of one vector points in the same direction as another.

2.3.1 Properties of the Dot Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and c be a scalar. Then the following properties hold:

- *Commutativity*: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- *Distributive Property*: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
- *Scalar Associativity*: $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.
- *Self-Product*: $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.
- *Magnitude*: $\|\mathbf{v}\| = \sqrt{v} \cdot v$
- *Angle*: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos(\theta)$, where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. (Law of Cosines.)
- *Orthogonality*: $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

2.3.2 Projections

The *projection* of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\mathrm{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

This is a vector parallel to $\mathbf{v}$, which has length equal to the amount of $\mathbf{u}$ which points in the same direction as $\mathbf{v}$.

Think of a projection as a measure of how much of one vector points in the same direction as another.

2.3.3 Work

If a constant force **F** moved an object from P to Q, the *work* done is given by

$$W = \mathbf{F} \cdot \overrightarrow{PQ}.$$

Thus, if that force acts at an angle θ to the line of motion, the work is:

$$W = (\|\mathbf{F}\|)\,\|\mathbf{PQ}\|\cos(\theta).$$

Later this semester, we will learn how to compensate for a non-constant force, and over a non-linear path.

2.4 The Cross Product

Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Then, the *cross product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is the unique right-hand rule vector orthogonal to each of $\mathbf{u}$ and v whose magnitude is equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then,

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 \rangle .$$

NOTE: You will never multiply an v_1-coordinate by an u_1-coordinate. This is true for all v_n and u_n coordinates.

You can show by working the algebra that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

With determinants, you can do this in one step:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Oddly, we can only define a cross-product in $\mathbb{R}$, $\mathbb{R}^3$, and $\mathbb{R}^7$, while the dot product is *always* defined.

Example

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle 2, 1, 4 \rangle \cdot \langle 1, -3, 1 \rangle \\ &= \langle (1)(1) - 4(-3), 4(1) - 2(1), 2(-3) - 1(1) \rangle \\ &= \langle 13, 2, -7 \rangle . \end{aligned}$$

2.4.1 Properties of the Cross Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and c be a scalar. Then the following properties hold:

- *Anticommutativity*: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- *Distributive Property*: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
- *Scalar Associativity*: $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v})$.
- *Zero*: $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
- *Nilpotence*: $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- *Scalar Triple Product*: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.
- *Angle*: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin(\theta)$, where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

2.4.2 Standard Unit Vectors and the Cross Product

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
- $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
- $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
- $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
- $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

2.4.3 Torque

Torque, denoted by τ, measures the tendency to produce a rotation about an axis.

If $\mathbf{r}$ is a radial vector from an axis to a force and $\mathbf{F}$ is the force, then the torque induced on the axis by the force is given by:

$$\tau = \mathbf{r} \times \mathbf{F} \qquad \text{or} \qquad \|\tau\| = \|\mathbf{r}\| \, \|\mathbf{F}\| \sin(\theta),$$

2.5 Equations of Lines and Planes

For the vector equation, parametric equation, and the symmetric equation, use these points for the examples: $(3, 5, 1) + (9, 1, 2)$.

2.5.1 Lines

Lines in Two Dimensions

A line in $\mathbb{R}^2$ which contains the point (x_0, y_0) and is parallel to the vector $\mathbf{v} = \langle a, b \rangle$ has parametric form

$$f(t) = \begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \end{cases}.$$

Lines in Three Dimensions

In $\mathbb{R}^3$, we have more options for the form of a line. Suppose that our line contains the point $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and is parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$. Then:

Vector Equation

The *vector equation* of a line is given by $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

Example: Find all 3 equations of lines

From our example, $\mathbf{v} = \langle 6, -4, 1 \rangle$ and $\mathbf{r}_0 = \langle 3, 5, 1 \rangle$.

Vector equation: $\mathbf{r}(t) = \langle 3, 5, 1 \rangle + t \langle 6, -4, 1 \rangle$.

Parametric Equation

The *parametric equation* of a line is given by

$$f(t) = \begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \\ z(t) = z_0 + tc \end{cases}.$$

From our example, we would get $x(t) = 3 + 6t$, $y(t) = 5 - 4t$, and $z(t) = 1 + t$.

Symmetric Equation

For the following formula, we get a, b and c from subtracting the x, y, and z components of the direction vector from the point vector.
As long as each of a, b, $c \neq 0$, the symmetric equation is

$$\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}.$$

(Notice that in two dimensions, this is just the equation of the line: $(\frac{b}{a})(x-x_0)+y_0=y$, when solved for y.)
From our example, we would get

$$\frac{x-3}{9-3}=\frac{y-5}{1-5}=\frac{z-1}{1-2}\implies\frac{x-3}{6}=\frac{y-5}{-4}=-z+1.$$

Line Segment

Suppose that $P=(x_0,y_0,z_0)$ and $Q=(x_1,y_1,z_1)$. The line segment from P to Q is given by

$$\mathbf{r}(t)=(1-t)\mathbf{p}+t\mathbf{q},$$

where $\mathbf{p}=\langle x_0,y_0,z_0\rangle$, $\mathbf{q}=\langle x_1,y_1,z_1\rangle$, and $0\le t\le 1$.

The parametric equations for this segment are

$$f(t)=\begin{cases}x(t)=x_0+t(x_1-x_0)\\ y(t)=y_0+t(y_1-y_0)\\ z(t)=z_0+t(z_1-z_0)\end{cases}.$$

Distance Between Point and Line

The distance from a point M to a line which contains the point P and has direction vector $\mathbf{v}$ is given by

$$d=\left|\frac{\overrightarrow{PM}\times\mathbf{v}}{\|\mathbf{v}\|}\right|.$$

Notice that you are free to choose any point on the line you'd like!

Relationships Between Lines

- *Equal*: Same direction vector, share a point.
- *Parallel*: Same direction vector, do not share a point.
- *Intersecting*: Different direction vectors, share a point.
- *Skew*: Different direction vectors, do not share a point.

2.5.2 Planes

A plane can be defined by:

- any three non-colinear points,
- any two intersection points,
- a line and a point not on the line, or
- given two orthogonal vectors with a common starting point: "spin" one vector in place; notice the other sweeps out a circle, which can be extended to a plane. * In notes *

Of particular importance for a plane is a *normal vector*. A vector **n** is a normal vector provided it is orthogonal to $\overrightarrow{PQ}$ for any two points P and Q which are in the plane.

2.5.3 Equations of a Plane

Like lines, we have three equations of a plane. Let P and Q be points in the plane and $n = \langle a, b, c \rangle$.

Vector Equation

The *vector equation* of a plane is $n \cdot \overrightarrow{PQ} = 0$. Note that this is an implicit definition (i.e. it is not useful for directly writing down an equation, but is the fundamental idea of why this all works)!

Scalar Equation

If (x_0, y_0, z_0) is any point in the plane, the *scalar equation* of the plane is given by

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

General Form

The *general form* of the equation of a plane is given by $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$.

Distance Between Point and Plane

- Equal: Share a common point, have parallel normal vectors
- Parallel: Do not share a common point, do have parallel normal vectors
- Intersecting: If their normal vectors are not parallel, the two planes intersect in a line.
  - You can use algebra to find a point in common – i.e. solve both equations for the planes
  - Find the line's direction vector by taking the cross product of the planes' normal vectors.

2.5.4 Examples

Use the points $P(3,5,1)$, $Q(9,1,2)$, and $R(0,2,5)$.

Example 1

Find the scalar equation of the plane containing P, Q, and R.

We know $\mathbf{PQ} = \langle 6,-4,1\rangle$ and $\mathbf{PR} = \langle -3,-3,4\rangle$. Then, $\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \langle 13,-27,-30\rangle$.

Thus, the equation of the plane is $13(x-3) - 27(y-5) - 30(z-1) = 0$.

To check, plug in the points: $13(3) - 27(5) - 30(1) = 0$, $13(9) - 27(1) - 30(2) = 0$, and $13(0) - 27(2) - 30(5) = 0$.

Distance from point to line:

Suppose M is a point and P is any point on some line l. Refer to notes for graph. Let $d = \|\mathbf{PM}\| \sin\theta$. Let $\mathbf{v}$ be a directional vector of l.

$\|\mathbf{PM} \times \mathbf{v}\| = \|\mathbf{PM}\| \, \|\mathbf{v}\| \sin\theta$. This gives us $d = \frac{\|\mathbf{PM}\times\mathbf{v}\|}{\|\mathbf{v}\|}$.

Distance from point to plane:

Let P be the point with $\mathbf{n}$ as the norm vector. We start with $d = \|\mathrm{proj}_{\mathbf{n}}\mathbf{PQ}\|$. Then, $d = \frac{\|\mathbf{PQ}\times\mathbf{n}\|}{\|\mathbf{n}\|\cdot\|\mathbf{n}\|} \, \|\mathbf{n}\|$. Look in the book for the rest of this.

2.6 Quadratic Surfaces

2.6.1 Spheres

A *sphere*, centered at (x_0, y_0, z_0) with radius r, is given by the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

2.6.2 Cylinder

A *cylinder* is a surface in $\mathbb{R}^3$ which consists of all lines that are parallel to a given line and pass through a given plane curve. The lines that make up a cylinder are called *rulings*. The *trace* of a cylinder is the cross section generated by intersecting the cylinder with a coordinate plane.

2.6.3 Quadratic Surfaces

A *quadratic surface* is a surface in $\mathbb{R}^3$ whose equation can be written as

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

By a change of axes (rotations) and origin (translations), we can rewrite these always as one of

$$Ax^2 + By^2 + Cz^2 = 1, \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

While cylinders have rulings made of parallel lines, quadratic surfaces do not (at least, not in general). However, their traces are always conic sections: lines, parabolas, circles, ellipses, or hyperbolas.

CHAPTER 3

VECTOR-VALUED FUNCTIONS

3.1 Vector Valued Functions and Space Curves

3.1.1 Vector-Valued Functions

Recall that a *function* f from a domain D to codomain E is a rule which assigns a single element of E to each element of D.

If each of $f_1, f_2, \ldots, f_n\colon \mathbb{R} \to \mathbb{R}$ is a function we can then define the *vector-valued function* $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^n$ by

$$\mathbf{r}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t)\rangle$$

- When $n = 2$, we might write $\mathbf{r} = \langle f(t), g(t)\rangle = f(t)\hat{\imath} + g(t)\hat{\jmath}$,
- and when $n = 3$, we might write $\mathbf{r} = \langle f(t), g(t), h(t)\rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$.

3.1.2 Curves

A *plane curve* is the set of points satisfying the parameterized curve $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^2$ over a given domain and for a given function $\mathbf{r}$.

A *space curve* is the set of points satisfying the parameterized curve $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^3$ over a given domain and for a given function $\mathbf{r}$.

We will call these *vector parameterizations* of the curve.

3.1.3 Limits

Formal Definition of a Limit

Suppose that $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function and that $a \in \mathbb{R}$, though perhaps nto in the domain of $\mathbf{r}$.

If there exists a vector $\mathbf{L} \in \mathbb{R}^n$ such that for each choice of $\epsilon > 0$ there exists $\delta > 0$ such that whenever $t \in \mathbb{R}$, $t \neq a$ and $|t - a| < \delta$, then

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon,$$

then we say that $\mathbf{r}$ has *limit* $\mathbf{L}$ as t approaches a, and we write

$$\lim_{t\to a} \mathbf{r}(t) = \mathbf{L}.$$

That is, if we want the output from $\mathbf{r}$ to be close to $\mathbf{L}$, we can always choose inputs close to a to make that occur.

Limit Properties

Suppose that $f, g, h\colon \mathbb{R} \to \mathbb{R}$ are functions and that each of $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$, and $\lim_{t\to a} h(t)$ exist. Then, if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then $\lim_{t\to a} \mathbf{r}(t)$ exists as well, and has value

$$\lim_{t \to a} \mathbf{r}(t) = \left(\lim_{t \to a} f(t)\right) \mathbf{i} + \left(\lim_{t \to a} g(t)\right) \mathbf{j} + \left(\lim_{t \to a} h(t)\right) \mathbf{k}.$$

That is, we can essentially do limits component-wise.
Also, there is nothing special here about $\mathbb{R}^3$. This works equally well for $\mathbb{R}^n$ for any n.

3.1.4 Continuity

Suppose that $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function and that $a \in \mathbb{R}$. The statement that $\mathbf{r}$ is *continuous* at $t = a$ means that

- $\lim_{t \to a} \mathbf{r}(t)$ exists,
- $\mathbf{r}(a)$ is defined, and
- $\mathbf{r}(a) = \lim_{t \to a} \mathbf{r}(t)$.

This is of course essentially the same as our Calculus I definition of continuity, but now in $\mathbb{R}^n$.

Like that definition, informally, it means that the curve generated by the function is in "one piece."

3.2 Calculus of Vector Valued Functions

Last time, we saw that limits and continuity for vector valued functions work component-wise. This is mostly the case for derivatives and integrals as well.

3.2.1 Derivatives

Suppose that $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^n$ is a vector valued function. We define the *derivative* of $\mathbf{r}$ as

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

when this limit exists. We can of course also define the left- or right- hand derivatives as needed.

3.2.2 Properties of Derivatives

Suppose that $f, \mathbf{r}, \mathbf{u}$ are differentiable and c is a scalar. Then, we have the following properties:

- $\frac{d}{dt}(c\mathbf{r}) = c\frac{d}{dt}\mathbf{r}$
- $\frac{d}{dt}(\mathbf{r} \pm \mathbf{u}) = \frac{d}{dt}\mathbf{r} \pm \frac{d}{dt}\mathbf{u}$
- $\frac{d}{dt}(f(t)\mathbf{r}(t)) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$
- $\frac{d}{dt}(\mathbf{r} \cdot \mathbf{u}) = \mathbf{r}' \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{u}'$
- $\frac{d}{dt}(\mathbf{r} \times \mathbf{u}) = \mathbf{r}' \times \mathbf{u} + \mathbf{r} \times \mathbf{u}'$
- $\frac{d}{dt}(\mathbf{r}(f(t))) = \mathbf{r}'(f(t))f'(t)$

From the fourth above, note that if $\mathbf{r} \cdot \mathbf{r}$ is constant, then $\mathbf{r}$ is orthogonal to $\mathbf{r}'$.

3.2.3 Tangent Vectors

Suppose that $\mathbf{r}(t)$ is a differentiable vector-valued function and t_0 is the domain. This means that $\mathbf{r}(t_0)$ is tangenet to the curve generated by $\mathbf{r}(t)$ at $t = t_0$. In particular, it is a tangent vector in the sense that if we interpret $\mathbf{r}(t)$ to refer to the position of osme particle as a function of time, then $\mathbf{r}'(t_0)$ is the velocity vector, and $||\mathbf{r}'(t_0)||$ describes the speed of the particle.

Unit Tangent Vector

Of importance is the *principle unit tangent vector*, $\mathbf{T}$, which is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}.$$

Note that this is only defined when $\mathbf{r}'(t) \neq \mathbf{0}$.

The unit tangent vector is simply a vector, of length 1, which points in the tangent direction of the curve. It is useful for describing the direction of motion of a particle along a curve.

3.2.4 Integrals

Like derivatives, we do intergrals component-wise:

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= \int \langle f_1(t), f_2(t), \ldots, f_n(t)\rangle\,dt \\ &= \left\langle \int f_1(t)\,dt, \int f_2(t)\,dt, \ldots, \int f_n(t)\,dt \right\rangle,\end{aligned}$$

where since these are indefinite, each produce a constant $C = \langle C_1, C_2, \ldots, C_n\rangle$. Definite intergrals can be defined in the same way.

3.3 Arc Length and Curvature

3.3.1 The Unit Normal Vector

For a curve C defined by $\mathbf{r}$ in $\mathbb{R}^3$, we have the *unit normal vector*, $\mathbf{N}(t)$, which is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}.$$

The normal vector points orthogonally to the tangent vector; it points in the direction the curve is turning.

We show that $\mathbf{T}$ and $\mathbf{N}$ are orthogonal:

$$\begin{aligned}
\mathbf{T} \cdot \mathbf{N} &= \mathbf{T}(t) \cdot \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} \\
&= \frac{1}{||\mathbf{T}'(t)||} \mathbf{T}(t) \cdot \mathbf{T}'(t) \\
&= \frac{1}{||\mathbf{T}'(t)||} \cdot 0 \\
&= 0.
\end{aligned}$$

The last step is true since $\mathbf{T}$ is a *unit* vector, and so that $||\mathbf{T}||$, and from our rules for differentiation, if a vector $\mathbf{T}$ has a constant magnitude, then $\mathbf{T} \cdot \mathbf{T}' = 0$.

3.3.2 The Binormal Vector

The *binormal vector*, denoted by $\mathbf{B}$, is defined by

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

Since $\mathbf{T}$ and $\mathbf{N}$ are each unit vectors which are orthogonal, by its definition, $\mathbf{B}$ is also a unit vector which is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

You can think of $\mathbf{T}$ and $\mathbf{N}$ as defining the instantaneous plane of motion of a particle. Then, $\mathbf{B}$ is the normal to that plane. If $\mathbf{B}$ is constant, then the particle stays in a single plane, otherwise, $\frac{d\mathbf{B}}{dt}$ measures your "twisting" or torsion of motion.

3.3.3 Arc Length

If $\mathbf{r}(t)$ defines a smooth curve in $\mathbb{R}^n$, then the *arc length* of the curve from $t = a$ to $t = b$ is given by

$$\int_a^b |\mathbf{r}'(t)| \, dt.$$

That is, we are just adding up the "speed" of the curve to find its length (i.e., its distance).

Arc Length Parameterization

We can define the *arc length parameterization* of a curve C by:

- Define the arc length $s(t) = \int_a^t ||\mathbf{r}'(t)||\ dt$.
- Solving, if possible, the resulting expression for t as a function of s.
- Rewriting $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so that the curve is written as a function of its length, from a given starting point.
- This is useful since there are many possible parameterizations of a given curve, but only a single arc length parameterization.

As you may recall from Calculus II, arc length can only be explicitly worked out for carefully selected problems. The same applies here.

3.3.4 Curvature

We define the *curvature*, denoted by κ, for a smooth curve given by $\mathbf{r}(s)$ as

$$\kappa = \left|\left|\frac{d\mathbf{T}}{ds}\right|\right|.$$

that is, how fast is the unit tangent vector changing, relative to the length of the curve itself. We use s because we want our answer to be independent of the parameterization.

However, though this formula is the easiest to reason with, it is in practice not useful in most cases. You would have to find the arc-length parameterization, and then find the unit tangent vector from that.

3.3.5 Calculating Curvature

- For all $\mathbf{r}$: $\kappa = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$.
- For $\mathbb{R}^3$: $\kappa = \frac{||\mathbf{r}'(t)\times\mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$.
- If $y = f(x)$: $\kappa = \frac{|y''(x)|}{[1+(y'(x)^2)]^{3/2}}$

3.3.6 Curvature of a Circle

We use each of the three equations to determine the curvature of a circle of radius a.

Parameterized Circle

A circle of radius a is parameterized by $\mathbf{r}(t) = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j}$. Thus, $\mathbf{r}'(5) = -a\sin(t)\mathbf{i} + a\sin(t)\mathbf{j}$, so that $||\mathbf{r}'(t)|| = a$. So, using the definition above, we get:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||} = \frac{-a\sin(t)\mathbf{i} + a\sin(t)\mathbf{j}}{a} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$$

and therefore

$$\mathbf{T}'(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j}$$

so that $||\mathbf{T}(t)|| = 1$. Finally, we see that

$$\kappa = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||} = \frac{1}{a}.$$

Therefore, a circle has constant curvature, and in fact the curvature is just the reciprocal of the radius. This should make some intuitive sense, as the radius gets smaller, the curvature will increase and vice-versa. Notice also that the curvature of a line is 0 – in some sense, a line is a circle, but with infinite radius!

Curvature of a Circle in Three Dimensions

We can also use the second formula, again with a circle, noting that a circle of radius a is parameterized by $\mathbf{r} = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j} + 0\mathbf{k}$ so that $\mathbf{r}'(t) = -a\sin(t)\mathbf{i} + a\cos(t)\mathbf{j} + 0\mathbf{k}$ so that $||\mathbf{r}'(t)|| = a$. Thus, $\mathbf{r}'' = -a\cos(t)\mathbf{i} - a\sin(t)\mathbf{j}$, so that

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin(t) & a\cos(t) & 0 \\ -a\cos(t) & -a\sin(t) & 0 \end{vmatrix} = a^2\mathbf{k}.$$

Then,

$$\kappa = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3} = \frac{a^2}{a^3} = \frac{1}{a}.$$

Curvature as a Function

We can also write (the top half) of a circle as $y(x) = \sqrt{a_2 - x_2}$. Then, $y' = \dfrac{-x}{\sqrt{a^2 - x^2}}$ and

$$y'' = \frac{-1}{\sqrt{a_2 - x_2}} + \frac{x^2}{(a^2 - x^2)^{3/2}} = \frac{-a^2}{(a^2 - x^2)^{3/2}}.$$

Thus,

$$\kappa = \frac{|y''(x)|}{[1 + (y'(x)^2)]^{3/2}} = \frac{1}{a},$$

after some algebra.

Osculating Circle

In general, of course, curvature is not constant. At a particular point of interest, the curvature finds the reciprocal of the radius of the *osculating circle*,. That is, the "tangent circle"–the circle that best matches the curve at the point of interest.