

Multivariable Calculus Exam 1

Derivation

Basic Derivatives

$$\begin{aligned}\frac{d}{dx} e^{f(x)} &= f'(x) e^{f(x)} \\ \frac{d}{dx} \sin f(x) &= \cos f(x) \cdot f'(x) \\ \frac{d}{dx} \cos f(x) &= -\sin f(x) \cdot f'(x) \\ \frac{d}{dx} \tan f(x) &= \sec^2 f(x) \cdot f'(x) \\ \frac{d}{dx} \cot f(x) &= -\csc^2 f(x) \cdot f'(x) \\ \frac{d}{dx} \sec f(x) &= \sec f(x) \tan f(x) \cdot f'(x) \\ \frac{d}{dx} \csc f(x) &= -\csc f(x) \cot f(x) \cdot f'(x) \\ \frac{d}{dx} \ln f(x) &= \frac{f'(x)}{f(x)} \\ \frac{d}{dx} \log_a f(x) &= \frac{f'(x)}{f(x) \ln a} \\ \frac{d}{dx} (f(x))^n &= n(f(x))^{n-1} f'(x) \\ \frac{d}{dx} \sqrt{f(x)} &= \frac{f'(x)}{2\sqrt{f(x)}} \\ \frac{d}{dx} a^x &= a^x \ln a \\ \frac{d}{dx} b^{g(x)} &= b^{g(x)} \ln b \cdot g'(x)\end{aligned}$$

Product and Quotient

$$\begin{aligned}\frac{d}{dx} [u \cdot v] &= u' \cdot v + u \cdot v' \\ \frac{d}{dx} \left(\frac{u}{v} \right) &= \frac{u' \cdot v - u \cdot v'}{v^2}\end{aligned}$$

Inverse Trigonometric

$$\begin{aligned}\frac{d}{dx} \arcsin f(x) &= \frac{f'(x)}{\sqrt{1 - (f(x))^2}} \\ \frac{d}{dx} \arccos f(x) &= -\frac{f'(x)}{\sqrt{1 - (f(x))^2}} \\ \frac{d}{dx} \arctan f(x) &= \frac{f'(x)}{1 + (f(x))^2} \\ \frac{d}{dx} \operatorname{arccot} f(x) &= -\frac{f'(x)}{1 + (f(x))^2} \\ \frac{d}{dx} \operatorname{arcsec} f(x) &= \frac{f'(x)}{|f(x)| \sqrt{(f(x))^2 - 1}} \\ \frac{d}{dx} \operatorname{arccsc} f(x) &= -\frac{f'(x)}{|f(x)| \sqrt{(f(x))^2 - 1}}\end{aligned}$$

Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Higher-Order Derivatives

$$\begin{aligned}\frac{d^2}{dx^2} e^x &= e^x \\ \frac{d^3}{dx^3} \sin x &= -\cos x \\ \frac{d^4}{dx^4} \cos x &= \cos x\end{aligned}$$

Integration

Trigonometric Integrals

$$\begin{aligned}\int \sin x \, dx &= -\cos x + C \\ \int \cos x \, dx &= \sin x + C \\ \int \sin^2 x &= \frac{1}{2}(x - \sin x \cos x) + C \\ \int \cos^2 x &= \frac{1}{2}(x + \sin x \cos x) + C \\ \int \tan x \, dx &= -\ln |\cos x| + C \\ \int \cot x \, dx &= \ln |\sin x| + C \\ \int \sec x \, dx &= \ln |\sec x + \tan x| + C \\ \int \csc x \, dx &= -\ln |\csc x + \cot x| + C \\ \int \sec^2 x \, dx &= \tan x + C \\ \int \csc^2 x \, dx &= -\cot x + C \\ \int \sec x \tan x \, dx &= \sec x + C \\ \int \csc x \cot x \, dx &= -\csc x + C\end{aligned}$$

Inverse Trigonometric Integrals

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 - x^2}} \, dx &= \arcsin \left(\frac{x}{a} \right) + C \\ \int \frac{1}{a^2 + x^2} \, dx &= \frac{1}{a} \arctan \left(\frac{x}{a} \right) + C \\ \int \frac{1}{x\sqrt{x^2 - a^2}} \, dx &= \frac{1}{a} \operatorname{arcsec} \left(\frac{x}{a} \right) + C\end{aligned}$$

Regular Integrals and e

$$\begin{aligned}\int x^n \, dx &= \frac{1}{n+1} x^{n+1} + C \\ \int \frac{1}{x} \, dx &= \ln |x| + C \\ \int e^x \, dx &= e^x + C \\ \int e^{ax} \, dx &= \frac{1}{a} e^{ax} + C \\ \int e^{f(x)} f'(x) \, dx &= e^{f(x)} + C\end{aligned}$$

Exponential

Reduction Formulas for Sine and Cosine

$$\begin{aligned}\int \sin^n x \, dx &= -\frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \\ \int \cos^n x \, dx &= \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx\end{aligned}$$

$$\begin{aligned}\int_0^b e^x \, dx &= e^b - 1 \\ \int_0^b e^{-x} \, dx &= 1 - e^{-b} \\ \int_0^\infty e^{-x} \, dx &= 1 \\ \int_0^\infty x^n e^{-x} \, dx &= n!\end{aligned}$$

n	$\int_0^{\pi/2} \sin^n x \, dx$	$\int_0^{\pi/2} \cos^n x \, dx$	$\int_0^\pi \sin^n x \, dx$	$\int_0^\pi \cos^n x \, dx$	$\int_0^{2\pi} \sin^n x \, dx$	$\int_0^{2\pi} \cos^n x \, dx$
1	1	1	2	0	0	0
2	$\pi/4$	$\pi/4$	$\pi/2$	$\pi/2$	π	π
3	$2/3$	$2/3$	$4/3$	0	0	0
4	$3\pi/16$	$3\pi/16$	$3\pi/8$	$3\pi/8$	$3\pi/4$	$3\pi/4$
5	$8/15$	$8/15$	$16/15$	0	0	0
6	$5\pi/32$	$5\pi/32$	$5\pi/16$	$5\pi/16$	$5\pi/8$	$5\pi/8$

Radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	—
π	0	−1	0
$3\pi/2$	−1	0	—

Trigonometric Identities

Pythagorean

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

Half Angle

$$\sin^2 \left(\frac{x}{2} \right) = \frac{1 - \cos x}{2}$$

$$\cos^2 \left(\frac{x}{2} \right) = \frac{1 + \cos x}{2}$$

$$\tan^2 \left(\frac{x}{2} \right) = \frac{1 - \cos x}{1 + \cos x}$$

Double Angle

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2 \cos^2 x - 1$$

$$\cos 2x = 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Product to Sum

$$\sin x \sin y = \frac{1}{2} [\cos(x - y) - \cos(x + y)]$$

$$\cos x \cos y = \frac{1}{2} [\cos(x - y) + \cos(x + y)]$$

$$\sin x \cos y = \frac{1}{2} [\sin(x + y) + \sin(x - y)]$$

$$\cos x \sin y = \frac{1}{2} [\sin(x + y) - \sin(x - y)]$$

Sum to Product

$$\sin x + \sin y = 2 \sin \left(\frac{x + y}{2} \right) \cos \left(\frac{x - y}{2} \right)$$

$$\sin x - \sin y = 2 \cos \left(\frac{x + y}{2} \right) \sin \left(\frac{x - y}{2} \right)$$

$$\cos x + \cos y = 2 \cos \left(\frac{x + y}{2} \right) \cos \left(\frac{x - y}{2} \right)$$

$$\cos x - \cos y = -2 \sin \left(\frac{x + y}{2} \right) \sin \left(\frac{x - y}{2} \right)$$

Chapter 1: Parametric Equations and Polar Coordinates

- **Slope:** $\left. \frac{dy}{dx} \right|_{t=t_0} = \left. \frac{dy/dt}{dx/dt} \right|_{t=t_0}$.

The **tangent line** at t_0 is given by

$$y = \left(\left. \frac{dy}{dx} \right|_{t=t_0} \right) (x - x(t_0)) + y(t_0).$$

- **Concavity:** $\left. \frac{d^2y}{dx^2} \right|_{t=t_0} = \left. \frac{\frac{d}{dt}(dy/dx)}{dx/dt} \right|_{t=t_0}$

- **Area Under a Curve:** $\int_{t_a}^{t_b} y(t) \frac{dx}{dt} dt.$

- **Arc Length:** $\int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt.$

- **Surface Area:** $\int_{t_a}^{t_b} 2\pi y(t) \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt.$

Chapter 2: Vectors in Space

- **Direction:** $P = (x_1, y_1)$ and $Q = (x_2, y_2)$: $\mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

- **Vector Sum:** $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.

- **Magnitude:** $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2} = \sqrt{u} \cdot u.$

- **Dot Product:** $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$.

– **Angle:** $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)$, where $0 \leq \theta \leq \pi$ is between \mathbf{u} & \mathbf{v} .

– **Self-Product:** $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.

– **Work:** $W = \mathbf{F} \cdot \mathbf{PQ} = (\|\mathbf{F}\|) \|\mathbf{PQ}\| \cos(\theta).$

- To **Normalize** a vector, divide it by its magnitude $\mathbf{v} = \langle x, y, z \rangle$, then $\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \left\langle \frac{x}{\|\mathbf{v}\|}, \frac{y}{\|\mathbf{v}\|}, \frac{z}{\|\mathbf{v}\|} \right\rangle$. $\therefore \mathbf{u} :=$ **Unit Vector** in direction of \mathbf{v} .

- **Projection:** $\text{proj}_{\mathbf{b}} \mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \right) \mathbf{b}.$

- **Cross product:** $\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 \rangle$.

– **Angle:** $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$, where $0 \leq \theta \leq \pi$ is between \mathbf{u} & \mathbf{v} .

– **Torque:** $\tau = \mathbf{r} \times \mathbf{F}$ or $\|\tau\| = \|\mathbf{r}\| \|\mathbf{F}\| \sin(\theta)$

Parametric Equations Revisted

- To **Parameterize** an equation such as $y = x^3 - 4x + 1$ we can let $x = t$ and $y = t^3 - 4t + 1$. This allows us to write the equation as $\mathbf{r}(t) = \langle t, t^3 - 4t + 1 \rangle$.

- **Vector Equation:** $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}.$

- **Parametric Equation:** $x = x_0 + at, y = y_0 + bt, z = z_0 + ct.$

- **Symmetric Equation:** $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}.$

- The **Line Segment** from P to Q : $\mathbf{r}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$ (where \mathbf{p}, \mathbf{q} are the vector forms of P, Q and $0 \leq t \leq 1$).

- **Shortest Distance:** $d = \frac{\|\mathbf{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$

– **Equal:** Same direction vector, share a point.

– **Parallel:** Same direction vector, do not share a point.

– **Intersecting:** Different direction vectors, share a point.

– **Skew:** Different direction vectors, do not share a point.

- If (x_0, y_0, z_0) is a point on a plane, the **Scalar Equation** would be: $\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0 \implies a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$

Chapter 3: Vector-Valued Functions

If each of $f_1, f_2, \dots, f_n: \mathbb{R} \rightarrow \mathbb{R}$ is a function we can then define the **vector-valued function** $\mathbf{r}: \mathbb{R} \rightarrow \mathbb{R}^n$ by $\mathbf{r}(t) = \langle f_1(t), f_2(t), \dots, f_n(t) \rangle$

- When $n = 2$, we might write $\mathbf{r} = \langle f(t), g(t) \rangle = f(t)\hat{i} + g(t)\hat{j}$,
- and when $n = 3$, we might write $\mathbf{r} = \langle f(t), g(t), h(t) \rangle = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$.

Note: Deriving and integrating vector-valued functions follow the same rules as regular derivatives.

- **Principle unit tangent vector** $\mathbf{T}(t)$: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.
 - This vector, of length 1, points in the tangent direction of the curve.
- **Unit Normal Vector** $\mathbf{N}(t)$: $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$.
 - This vector points in the direction the curve is turning.
- **Binormal Vector** $\mathbf{B}(t)$: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

Arc Length Parameterization

We can define the **arc length parameterization** of a curve C by:

- Define the arc length $s(t) = \int_0^t \|\mathbf{r}'(\tau)\| d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2} d\tau$.
(Where $f(\tau), g(\tau)$ correspond to the x, y components of $\mathbf{r}(t)$).
- Solving, if possible, the resulting expression for t as a function of s .
- Rewriting $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so that the curve is written as a function of its length, from a given starting point.

Curvature

For all \mathbf{r} : $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$; for \mathbb{R}^3 : $\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$; if $y = f(x)$: $\kappa = \frac{|y''(x)|}{[1+(y'(x)^2)]^{3/2}}$

Motion

- **Velocity**: $\mathbf{v}(t) = \mathbf{r}'(t)$.
- **Speed**: $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\|$.
- **Acceleration**: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

The motion of an object – in 2-dimensions, typically, acted on only by gravity $\mathbf{F}_g = -mg\mathbf{j}$, where $g \approx 9.8 \text{ m/s}^2$ and m is the mass of the object. By Newton's second law, $\mathbf{F} = m\mathbf{a}$, so we have $\mathbf{a}(t) = -g\mathbf{j}$. Thus, $\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$, where \mathbf{v}_0 is the initial velocity vector, and $\mathbf{s}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0t + \mathbf{s}_0$, where \mathbf{s} is the position, and \mathbf{s}_0 is the initial position vector. Often, we have an object starting at the origin (so $\mathbf{s}_0 = \mathbf{0}$) and fired at a velocity of v_0 at an angle θ above the horizon. Then, $\mathbf{s}(t) = v_0t \cos(\theta)\mathbf{i} + (v_0t \sin(\theta) - \frac{1}{2}gt^2)\mathbf{j}$.

Chapter 1 Examples

1. Consider the curve defined by the parametric equations $x(t) = \sin(2t)$, $y(t) = \cos(t)$, for $0 \leq t \leq 2\pi$.

(a) Find the equation of the tangent line to the curve at the point where $t = \pi/3$.

Solution. Solve for $\frac{dy}{dt}$ and $\frac{dx}{dt}$:

$$y(t) = \cos t \Rightarrow \frac{dy}{dt} = -\sin t.$$

$$x(t) = \sin 2t \Rightarrow \frac{dx}{dt} = 2 \cos 2t.$$

Now, we have our slope, which we evaluate at $t = \pi/3$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{2 \cos 2t} \Rightarrow \frac{-\sin t}{2 \cos 2t} = \frac{-\sqrt{3}/2}{-1} = \frac{\sqrt{3}}{2}.$$

With our slope, we need the points:

$$x(\pi/3) = \sin(2(\pi/3)) = \sqrt{3}/2 \quad \text{and} \quad y(\pi/3) = \cos(\pi/3) = 1/2.$$

Putting it all together, we have $y = \frac{\sqrt{3}}{2}(x - \frac{\sqrt{3}}{2}) + \frac{1}{2}$. (Simplify.)

(b) Determine geometric area enclosed by the curve.

Solution. Our equation has 4. We find 1 quadrant and multiply it by 4, we can get the total geometric area for the whole shape.

$$\begin{aligned} 4 \int_0^{\pi/2} y(t) \frac{dx}{dt} dt &= 4 \int_0^{\pi/2} \cos t (2 \cos 2t) dt \\ &= 8 \int_0^{\pi/2} \cos t (1 - 2 \sin^2 t) dt \\ &= 8 \int_0^{\pi/2} \cos t - 2 \cos t \sin^2 t dt \\ &= 8 \left[\int_0^{\pi/2} \cos t dt - 2 \int_0^{\pi/2} \cos t \sin^2 t dt \right]. \end{aligned}$$

Thus, let $u = \sin t$ such that $\frac{du}{\cos t} = dt$. Hence,

$$\begin{aligned} 8 \left[\int_0^{\pi/2} \cos t dt - 2 \int_0^{\pi/2} \cos t \sin^2 t dt \right] &= 8 \left[\int_0^{\pi/2} \cos t dt - 2 \int_0^{\pi/2} u^2 du \right] \\ &= 8 \left[\sin t - \frac{2}{3} \sin^3 t \right]_0^{\pi/2} \end{aligned}$$

2. Find angle between $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + \mathbf{j} - 7\mathbf{k}$.

Solution. Find the magnitudes of \mathbf{u} and \mathbf{v} . Then solve:

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|} = \frac{13}{\sqrt{65}\sqrt{66}} \approx 0.198.$$

Thus, $\theta \approx \cos^{-1}(0.198) \approx \boxed{1.371}$ radians.

Chapter 1 & 2 Examples (cont.)

3. Determine a parametric equation for the line *segment* that goes from the point $P = (6, 1, -2)$ to $Q = (-2, 0, 5)$.

Solution. $x(t) = 6 - 8t$; $y(t) = 1 - t$; $z(t) = -2 + 7t$, for $0 \leq t \leq 1$.

4. Find a symmetric equation for the line which contains the points $R = (4, -6, 1)$ and $S = (1, 2, 3)$.

Solution. $\frac{x-4}{-3} = \frac{y+6}{8} = \frac{z-1}{2}$.

5. Find the general form of an equation of the plane which contain the three points $P = (3, 1, -4)$, $Q = (-2, 0, 5)$ and $R = (4, -6, 1)$.

Solution. Let $\mathbf{PQ} = \langle -5, -1, 9 \rangle$ and $\mathbf{PR} = \langle 1, -7, 5 \rangle$. Thus, $\mathbf{n} = \mathbf{PQ} \times \mathbf{QR} = \langle 58, 34, 36 \rangle$. General equation:

$$58(x - 3) + 34(y - 1) + 36(z + 4) = 0 \Rightarrow 58x + 34y + 36z - 64 = 0.$$

6. Find an equation, in symmetric form, of the line of intersection between the planes $2x + y - z + 4 = 0$ and $x - y + 3z = 1$.

Solution. Add the plane equations to eliminate y so that $3x + 2z = -3$. Thus, $x = -1 - \frac{2}{3}z$. Substitute this equation into the first equation to express y in terms of z , giving $y = -2 + \frac{7}{3}z$. Define z in terms of t . Choose parameter t as $t = -\frac{1}{3}z$. This gives $z = -3t$. When we substitute our value t back into the previous two equations, we see that the parametric equations for the line of intersection are $x = -1 + 2t$, $y = -2 - 7t$, and $z = -3t$. Therefore, the

$$\text{symmetric equations for the line are } \frac{x+1}{2} = \frac{y+2}{-7} = \frac{z}{-3}.$$

Chapter 2 Examples

1. Write, in scalar form, an equation of the plane which contains the point $(5, 2, 1)$ and the line given by $x + 2 = \frac{y}{4} = \frac{z - 5}{2}$.

Solution. We start by parametrizing the line with common parameter t : $x + 2 = t \Rightarrow x = t - 2$, $\frac{y}{4} = t \Rightarrow y = 4t$, and $\frac{z - 5}{2} = t \Rightarrow z = 2t + 5$. This gives us the parametric form: $(x, y, z) = (-2, 0, 5) + t(1, 4, 2)$. Thus, the line passes through the point $(-2, 0, 5)$ and has the direction vector $\mathbf{v}_1 = \langle 1, 4, 2 \rangle$. Form a second vector \mathbf{v}_2 by taking the difference between the given point and a point on the line: $\mathbf{v}_2 = (5, 2, 1) - (-2, 0, 5) = \langle 7, 2, -4 \rangle$. With \mathbf{v}_1 and \mathbf{v}_2 , we find the normal vector:

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 2 \\ 7 & 2 & -4 \end{vmatrix} = (-16 - 4)\mathbf{i} - (-4 - 14)\mathbf{j} + (2 - 28)\mathbf{k} = -20\mathbf{i} + 18\mathbf{j} - 26\mathbf{k}$$

Therefore, we find the scalar form to be:

$$-20(x + 2) + 18y - 26(z - 5) = 0.$$

Chapter 3 Examples

2. Determine the arc length parametrization for the curve $\mathbf{r}(t) = 3e^t \sin(t)\mathbf{i} + 3e^t \cos(t)\mathbf{j}$, where you start from $t = 0$.

Solution. Rewrite the arc length parametrization as: $s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2} d\tau$. Thus, $f'(\tau) = 3e^\tau(\sin(\tau) + \cos(\tau))$ and $g'(\tau) = 3e^\tau(\cos(\tau) - \sin(\tau))$. Thus, we have:

$$\begin{aligned} s &= \int_0^t \sqrt{[3e^\tau(\sin(\tau) + \cos(\tau))]^2 + [3e^\tau(\cos(\tau) - \sin(\tau))]^2} d\tau \\ &= \int_0^t \sqrt{9e^{2\tau} [2(\sin^2(\tau) + \cos^2(\tau)) + (2\sin(\tau)\cos(\tau) - 2\sin(\tau)\cos(\tau))]} d\tau \\ &= \int_0^t \sqrt{9e^{2\tau} \cdot [2(1 + 0)]} d\tau = \int_0^t 3e^\tau \sqrt{2} d\tau = 3\sqrt{2} \int_0^t e^\tau d\tau = 3\sqrt{2}(e^t - 1). \end{aligned}$$

With s , we know that $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so we need to find t in terms of s : $s = 3\sqrt{2}(e^t - 1) \Rightarrow e^t = \frac{s}{3\sqrt{2}} + 1 \Rightarrow t = \ln\left(\frac{s}{3\sqrt{2}} + 1\right)$. Finally, by replacing t with $t(s)$ in the original equation, we can get the arc length parametrization:

$$\mathbf{r}(s) = \left(\frac{s}{\sqrt{2}} + 3\right) \sin\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right) \mathbf{i} + \left(\frac{s}{\sqrt{2}} + 3\right) \cos\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right) \mathbf{j}.$$

3. Use curvature to find the equation of the osculating circle at the planar curve $y = x^3 - 4x + 1$ at $x = 1$.

Solution. First, we need to find the curvature of the curve at $x = 1$. We start by finding the first and second derivatives of the function: $y(x) = x^3 - 4x + 1 \Rightarrow y'(x) = 3x^2 - 4 \Rightarrow y''(x) = 6x$.

Then, we evaluate the point and the first and second derivatives at $x = 1$: $y(1) = -2$; $y'(1) = -1$; $y''(1) = 6$.

$$\text{Find the curvature: } \kappa = \frac{|y''(x)|}{(1 + y'(x)^2)^{3/2}} = \frac{6}{(1 + (-1)^2)^{3/2}} = \frac{3\sqrt{2}}{2}.$$

$$\text{Find radius: } R = \frac{1}{\kappa} = \frac{1}{\frac{3\sqrt{2}}{2}} = \frac{2\sqrt{2}}{3} = \frac{\sqrt{2}}{3}.$$

$$\text{For the center, find } \mathbf{N} \text{ at } x = 1: \mathbf{N} = \frac{(-y', 1)}{\sqrt{1 + (y')^2}} = \frac{(-(-1), 1)}{\sqrt{1 + (-1)^2}} = \frac{(1, 1)}{\sqrt{2}}.$$

The center C can be found by moving our point $P(1, -2)$ the distance R along the unit normal vector: $C = P + R\mathbf{N} = (1, -2) + \frac{\sqrt{2}}{3} \frac{(1, 1)}{\sqrt{2}} = (1, -2) + \frac{1}{3}(1, 1) = (\frac{4}{3}, -\frac{5}{3})$. This gives the equation for the osculating circle:

$$\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{5}{3}\right)^2 = \frac{2}{9}.$$