Multivariable Calculus Exam 1

Derivation

Basic Derivatives

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$
$$\frac{d}{dx}\sin f(x) = \cos f(x)\cdot f'(x)$$
$$\frac{d}{dx}\cos f(x) = -\sin f(x)\cdot f'(x)$$
$$\frac{d}{dx}\tan f(x) = \sec^2 f(x)\cdot f'(x)$$
$$\frac{d}{dx}\cot f(x) = -\csc^2 f(x)\cdot f'(x)$$
$$\frac{d}{dx}\sec f(x) = \sec f(x)\tan f(x)\cdot f'(x)$$
$$\frac{d}{dx}\csc f(x) = -\csc f(x)\cot f(x)\cdot f'(x)$$
$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$$
$$\frac{d}{dx}\log_a f(x) = \frac{f'(x)}{f(x)\ln a}$$
$$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1}f'(x)$$
$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$$
$$\frac{d}{dx}a^x = a^x\ln a$$
$$\frac{d}{dx}b^{g(x)} = b^{g(x)}\ln b\cdot g'(x)$$

Product and Quotient

$$\frac{d}{dx}[u\cdot v] = u'\cdot v + u\cdot v'$$
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'\cdot v - u\cdot v'}{v^2}$$

Inverse Trigonometric

$$\frac{d}{dx}\arcsin f(x) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$$
$$\frac{d}{dx}\arccos f(x) = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$$
$$\frac{d}{dx}\arctan f(x) = \frac{f'(x)}{1+(f(x))^2}$$
$$\frac{d}{dx}\operatorname{arccot} f(x) = -\frac{f'(x)}{1+(f(x))^2}$$
$$\frac{d}{dx}\operatorname{arcsec} f(x) = \frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$$
$$\frac{d}{dx}\operatorname{arccsc} f(x) = -\frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$$

Chain Rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x)$$

Higher-Order Derivatives

$$\frac{d^2}{dx^2}e^x = e^x$$
$$\frac{d^3}{dx^3}\sin x = -\cos x$$
$$\frac{d^4}{dx^4}\cos x = \cos x$$

Integration

Trigonometric Integrals

$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sin^2 x = \frac{1}{2}(x - \sin x\cos x) + C$$
$$\int \cos^2 x = \frac{1}{2}(x + \sin x\cos x) + C$$
$$\int \tan x\,dx = -\ln|\cos x| + C$$
$$\int \cot x\,dx = \ln|\sin x| + C$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$
$$\int \csc x\,dx = -\ln|\csc x + \cot x| + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x\tan x\,dx = \sec x + C$$
$$\int \csc x\cot x\,dx = -\csc x + C$$

Reduction Formulas for Sine and Cosine

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

Inverse Trigonometric Integrals

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$
$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$$

Regular Integrals and e

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$$
$$\int \frac{1}{x}\,dx = \ln|x| + C$$
$$\int e^x\,dx = e^x + C$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$$
$$\int e^{f(x)}f'(x)\,dx = e^{f(x)} + C$$

Exponetials

$$\int_0^b e^x\,dx = e^b - 1$$
$$\int_0^b e^{-x}\,dx = 1 - e^{-b}$$
$$\int_0^\infty e^{-x}\,dx = 1$$
$$\int_0^\infty x^n e^{-x}\,dx = n!$$

n	$\int_0^{\pi/2}\sin^n x\,dx$	$\int_0^{\pi/2}\cos^n x\,dx$	$\int_0^{\pi}\sin^n x\,dx$	$\int_0^{\pi}\cos^n x\,dx$	$\int_0^{2\pi}\sin^n x\,dx$	$\int_0^{2\pi}\cos^n x\,dx$
1	1	1	2	0	0	0
2	$\pi/4$	$\pi/4$	$\pi/2$	$\pi/2$	π	π
3	$2/3$	$2/3$	$4/3$	0	0	0
4	$3\pi/16$	$3\pi/16$	$3\pi/8$	$3\pi/8$	$3\pi/4$	$3\pi/4$
5	$8/15$	$8/15$	$16/15$	0	0	0
6	$5\pi/32$	$5\pi/32$	$5\pi/16$	$5\pi/16$	$5\pi/8$	$5\pi/8$

Radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	–
π	0	-1	0
$3\pi/2$	-1	0	–

Trigonometric Identities

Pythagorean

$$\sin^2\theta+\cos^2\theta=1$$
$$\tan^2\theta+1=\sec^2\theta$$
$$1+\cot^2\theta=\csc^2\theta$$

Half Angle

$$\sin^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{2}$$
$$\cos^2\left(\frac{x}{2}\right)=\frac{1+\cos x}{2}$$
$$\tan^2\left(\frac{x}{2}\right)=\frac{1-\cos x}{1+\cos x}$$

Double Angle

$$\sin 2x=2\sin x\cos x$$
$$\cos 2x=\cos^2 x-\sin^2 x$$
$$\cos 2x=2\cos^2 x-1$$
$$\cos 2x=1-2\sin^2 x$$
$$\tan 2x=\frac{2\tan x}{1-\tan^2 x}$$

Product to Sum

$$\sin x\sin y=\frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$$
$$\cos x\cos y=\frac{1}{2}\left[\cos(x-y)+\cos(x+y)\right]$$
$$\sin x\cos y=\frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$$
$$\cos x\sin y=\frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$$

Sum to Product

$$\sin x+\sin y=2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x-\sin y=2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x+\cos y=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x-\cos y=-2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Chapter 1: Parametric Equations and Polar Coordinates

- **Slope:** $\left.\frac{dy}{dx}\right|_{t=t_0}=\left.\frac{dy/dt}{dx/dt}\right|_{t=t_0}$.

  The ***tangent line*** at t_0 is given by

  $$y=\left(\left.\frac{dy}{dx}\right|_{t=t_0}\right)(x-x(t_0))+y(t_0).$$

- **Concavity:** $\left.\frac{d^2y}{dx^2}\right|_{t=t_0}=\left.\frac{d}{dt}\left(\frac{dy}{dx}\right)\right|_{t=t_0}=\left.\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\right|_{t=t_0}$.
- **Area Under a Curve:** $\int_{t_a}^{t_b} y(t)\frac{dx}{dt}\,dt$.
- **Arc Length:** $\int_{t_a}^{t_b}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$.
- **Surface Area:** $\int_{t_a}^{t_b} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$.

Chapter 2: Vectors in Space

- ***Direction:*** $P=(x_1,y_1)$ and $Q=(x_2,y_2)$: $\mathbf{PQ}=\langle x_2-x_1,y_2-y_1\rangle$.
- ***Vector Sum:*** $\mathbf{u}+\mathbf{v}=\langle u_1+v_1,u_2+v_2\rangle$.
- ***Magnitude:*** $\|\mathbf{u}\|=\sqrt{u_1^2+u_2^2}=\sqrt{u}\cdot u$.
- ***Dot Product:*** $\mathbf{u}\cdot\mathbf{v}=u_1v_1+u_2v_2$.
  - ***Angle:*** $\mathbf{u}\cdot\mathbf{v}=\|\mathbf{u}\|\,\|\mathbf{v}\|\cos(\theta)$, where $0\le\theta\le\pi$ is between $\mathbf{u}$ & $\mathbf{v}$.
  - ***Self-Product:*** $\mathbf{u}\cdot\mathbf{u}=\|\mathbf{u}\|^2$.
  - ***Work:*** $W=\mathbf{F}\cdot\mathbf{PQ}=(\|\mathbf{F}\|)\,\|\mathbf{PQ}\|\cos(\theta)$.
- To ***Normalize*** a vector, divide it by its magnitude $\mathbf{v}=\langle x,y,z\rangle$, then $\mathbf{u}=\frac{1}{\|\mathbf{v}\|}\mathbf{v}=\left\langle\frac{x}{\|\mathbf{v}\|},\frac{y}{\|\mathbf{v}\|},\frac{z}{\|\mathbf{v}\|}\right\rangle$. $\therefore\ \mathbf{u}:=$ ***Unit Vector*** in direction of $\mathbf{v}$.
- ***Projection:*** $\mathrm{proj}_{\mathbf{b}}\mathbf{a}=\left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}$.
- ***Cross product:*** $\mathbf{u}\times\mathbf{v}=\langle u_2v_3-u_3v_2,\ u_3v_1-u_1v_3,\ u_1v_2-u_2v_1\rangle$.
  - ***Angle:*** $\|\mathbf{u}\times\mathbf{v}\|=\|\mathbf{u}\|\,\|\mathbf{v}\|\sin(\theta)$, where $0\le\theta\le\pi$ is between $\mathbf{u}$ & $\mathbf{v}$.
  - ***Torque:*** $\tau=\mathbf{r}\times\mathbf{F}$ or $\|\tau\|=\|\mathbf{r}\|\,\|\mathbf{F}\|\sin(\theta)$

Parametric Equations Revisted

- To ***Parameterize*** an equation such as $y=x^3-4x+1$ we can let $x=t$ and $y=t^3-4t+1$. This allows us to write the equation as $\mathbf{r}(t)=\langle t,t^3-4t+1\rangle$.
- ***Vector Equation:*** $\mathbf{r}(t)=\mathbf{r}_0+t\mathbf{v}$.
- ***Parametric Equation:*** $x=x_0+at,\ y=y_0+bt,\ z=z_0+ct$.
- ***Symmetric Equation:*** $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$.
- The ***Line Segment*** from P to Q: $\mathbf{r}(t)=(1-t)\mathbf{p}+t\mathbf{q}$ (where $\mathbf{p},\mathbf{q}$ are the vector forms of P,Q and $0\le t\le 1$).
- ***Shortest Distance:*** $d=\frac{||\mathbf{PM}\times\mathbf{v}||}{\|\mathbf{v}\|}$.
  - ***Equal***: Same direction vector, share a point.
  - ***Parallel***: Same direction vector, do not share a point.
  - ***Intersecting***: Different direction vectors, share a point.
  - ***Skew***: Different direction vectors, do not share a point.
- If (x_0,y_0,z_0) is a point on a plane, the ***Scalar Equation*** would be: $\langle x-x_0,y-y_0,z-z_0\rangle\cdot\langle a,b,c\rangle=0\Longrightarrow a(x-x_0)+b(y-y_0)+c(z-z_0)=0$.

Chapter 3: Vector-Valued Functions

If each of $f_1, f_2, \ldots, f_n : \mathbb{R} \to \mathbb{R}$ is a function we can then define the ***vector-valued function*** $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^n$ by $\mathbf{r}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t)\rangle$

- When $n = 2$, we might write $\mathbf{r} = \langle f(t), g(t)\rangle = f(t)\hat{\imath} + g(t)\hat{\jmath}$,
- and when $n = 3$, we might write $\mathbf{r} = \langle f(t), g(t), h(t)\rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$.

Note: Deriving and integrating vector-valued functions follow the same rules as regular derivates.

- ***Principle unit tangent vector*** $\mathbf{T}(t)$: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$.
  - This vector, of length 1, points in the tangent direction of the curve.
- ***Unit Normal Vector*** $\mathbf{N}$: $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$.
  - This vector points in the direction the curve is turning.
- ***Binormal Vector*** $\mathbf{B}$: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

Arc Length Parameterization

We can define the ***arc length parameterization*** of a curve C by:

- Define the arc length $s(t) = \int_0^t ||\mathbf{r}'(\tau)||\, d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2}\, d\tau$. (Where $f(\tau), g(\tau)$ correspond to the x, y components of $\mathbf{r}(t)$).
- Solving, if possible, the resulting expression for t as a function of s.
- Rewriting $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so that the curve is written as a function of its length, from a given starting point.

Curvature

For all $\mathbf{r}$: $\kappa = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$; for $\mathbb{R}^3$: $\kappa = \frac{||\mathbf{r}'(t)\times\mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$; if $y = f(x)$: $\kappa = \frac{|y''(x)|}{[1+(y'(x)^2)]^{3/2}}$

Motion

- ***Velocity:*** $\mathbf{v}(t) = \mathbf{r}'(t)$.
- ***Speed:*** $||\mathbf{v}(t)|| = ||\mathbf{r}'(t)||$.
- ***Acceleration:*** $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

The motion of an object – in 2-dimensions, typically, acted on only by gravity $\mathbf{F}_g = -mg\mathbf{j}$, where $g \approx 9.8$ m/s^2 and m is the mass of the object. By Newton's second law, $\mathbf{F} = m\mathbf{a}$, so we have $\mathbf{a}(t) = -g\mathbf{j}$. Thus, $\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$, where $\mathbf{v}_0$ is the initial velocity vector, and $\mathbf{s}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{s}_0$, where $\mathbf{s}$ is the position, and $\mathbf{s}_0$ is the initial position vector. Often, we have an object starting at the origin (so $\mathbf{s}_0 = \mathbf{0}$) and fired at a velocity of v_0 at an angle θ above the horizon. Then, $\mathbf{s}(t) = v_0 t\cos(\theta)\mathbf{i} + \left(v_0 t\sin(\theta) - \frac{1}{2}gt^2\right)\mathbf{j}$.

Chapter 1 Examples

1. Consider the curve defined by the parametric equations $x(t) = \sin(2t)$, $y(t) = \cos(t)$, for $0 \le t \le 2\pi$.

(a) Find the equation of the tangent line to the curve at the point where $t = \pi/3$.
Solution. Solve for $\frac{dy}{dt}$ and $\frac{dx}{dt}$:
$y(t) = \cos t \Rightarrow \frac{dy}{dt} = -\sin t$.
$x(t) = \sin 2t \Rightarrow \frac{dx}{dt} = 2\cos 2t$.
Now, we have our slope, which we evaluate at $t = \pi/3$:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{2\cos 2t} \quad \Rightarrow \quad \frac{-\sin t}{2\cos 2t} = \frac{-\sqrt{3}/2}{-1} = \frac{\sqrt{3}}{2}.$$

With our slope, we need the points:

$$x(\pi/3) = \sin(2(\pi/3)) = \sqrt{3}/2 \quad \text{and} \quad y(\pi/3) = \cos(\pi/3) = 1/2.$$

Putting it all together, we have $y = \frac{\sqrt{3}}{2}(x - \frac{\sqrt{3}}{2}) + \frac{1}{2}$. (Simplify.)

(b) Determine geometric area enclosed by the curve.
Solution. Our equation has 4. We find 1 quadrant and multiply it by 4, we can get the total geometric area for the whole shape.

$$\begin{aligned} 4\int_0^{\pi/2} y(t)\frac{dx}{dt}\ dt &= 4\int_0^{\pi/2} \cos t(2\cos 2t)\ dt \\ &= 8\int_0^{\pi/2} \cos t(1 - 2\sin^2 t)\ dt \\ &= 8\int_0^{\pi/2} \cos t - 2\cos t\sin^2 t\ dt \\ &= 8\left[\int_0^{\pi/2} \cos t\ dt - 2\int_0^{\pi/2} \cos t\sin^2 t\ dt\right]. \end{aligned}$$

Thus, let $u = \sin t$ such that $\dfrac{du}{\cos t} = dt$. Hence,

$$\begin{aligned} 8\left[\int_0^{\pi/2} \cos t\ dt - 2\int_0^{\pi/2} \cos t\sin^2 t\ dt\right] &= 8\left[\int_0^{\pi/2} \cos t\ dt - 2\int_0^{\pi/2} u^2\ du\right] \\ &= 8\left[\sin t - \frac{2}{3}\sin^3 t\right]_0^{\pi/2} \end{aligned}$$

2. Find angle between $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + \mathbf{j} - 7\mathbf{k}$.
Solution. Find the magnitudes of $\mathbf{u}$ and $\mathbf{v}$. Then solve:

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{13}{\sqrt{65}\sqrt{66}} \approx 0.198.$$

Thus, $\theta \approx \cos^{-1}(0.198) \approx \boxed{1.371}$ radians.

Chapter 1 & 2 Examples (cont.)

3. Determine a parametric equation for the line *segment* that goes from the point $P = (6, 1, -2)$ to $Q = (-2, 0, 5)$.

   Solution. $x(t) = 6 - 8t$; $y(t) = 1 - t\; z(t) = -2 + 7t$, for $0 \le t \le 1$.

4. Find a symmetric equation for the line which contains the points $R = (4, -6, 1)$ and $S = (1, 2, 3)$.

   Solution. $\frac{x-4}{-3} = \frac{y+6}{8} = \frac{z-1}{-2}$.

5. Find the general form of an equation of the plane which contain the three points $P = (3, 1, -4)$, $Q = (-2, 0, 5)$ and $R = (4, -6, 1)$.

   Solution. Let $\mathbf{PQ} = \langle -5, -1, 9\rangle$ and $\mathbf{PR} = \langle 1, -7, 5\rangle$. Thus, $\mathbf{n} = \mathbf{PQ} \times \mathbf{QR} = \langle 58, 34, 36\rangle$. General equation:

   $$58(x-3) + 34(y-1) + 36(z+4) = 0 \quad \Rightarrow \quad 58x + 34y + 36z - 64 = 0.$$

6. Find an equation, in symmetric form, of the line of intersection between the planes $2x + y - z + 4 = 0$ and $x - y + 3z = 1$.

   Solution. Add the plane equations to eliminate y so that $3x + 2z = -3$. Thus, $x = -1 - \frac{2}{3}z$. Substitute this equation into the first equation to express y in terms of z, giving $y = -2 + \frac{7}{3}z$. Define z in terms of t. Choose parameter t as $t = -\frac{1}{3}z$. This gives $z = -3t$. When we substitute our value t back into the previous two equations, we see that the parametric equations for the line of intersection are $x = -1 + 2t$, $y = -2 - 7t$, and $z = -3t$. Therefore, the symmetric equations for the line are $\boxed{\frac{x+1}{2} = \frac{y+2}{-7} = \frac{z}{-3}}$.

Chapter 2 Examples

1. Write, in scalar form, an equation of the plane which contains the point $(5, 2, 1)$ and the line given by $x + 2 = \frac{y}{4} = \frac{z-5}{2}$.

   Solution. We start by parametrizing the line with common parameter t: $x+2 = t \;\Rightarrow\; x = t-2$, $\frac{y}{4} = t \;\Rightarrow\; y = 4t$, and $\frac{z-5}{2} = t \;\Rightarrow\; z = 2t+5$. This gives us the parametric form: $(x, y, z) = (-2, 0, 5) + t(1, 4, 2)$ Thus, the line passes through the point $(-2, 0, 5)$ and has the direction vector $\mathbf{v}_1 = \langle 1, 4, 2\rangle$. Form a second vector $\mathbf{v}_2$ by taking the difference between the given point and a point on the line: $\mathbf{v}_2 = (5, 2, 1) - (-2, 0, 5) = \langle 7, 2, -4\rangle$. With $\mathbf{v}_1$ and $\mathbf{v}_2$, we find the normal vector:

   $$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 2 \\ 7 & 2 & -4 \end{vmatrix} = (-16-4)\mathbf{i} - (-4-14)\mathbf{j} + (2-28)\mathbf{k} = -20\mathbf{i} + 20\mathbf{j} - 26\mathbf{k}$$

   Therefore, we find the scalar form to be:

   $$\boxed{-20(x+2) + 18y - 26(z-5) = 0.}$$

Chapter 3 Examples

2. Determine the arc length parametrization for the curve $\mathbf{r}(t) = 3e^t \sin(t)\mathbf{i} + 3e^t \cos(t)\mathbf{j}$, where you start from $t = 0$.

   Solution. Rewrite the arc length parametrization as: $s = \int_0^t \|\mathbf{r}'(\tau)\|\, d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2}\, d\tau$. Thus, $f'(\tau) = 3e^\tau(\sin(\tau) + \cos(\tau))$ and $g'(\tau) = 3e^\tau(\cos(\tau) - \sin(\tau))$. Thus, we have:

   $$\begin{aligned} s &= \int_0^t \sqrt{\left[3e^\tau\left(\sin(\tau) + \cos(\tau)\right)\right]^2 + \left[3e^\tau\left(\cos(\tau) - \sin(\tau)\right)\right]^2}\, d\tau \\ &= \int_0^t \sqrt{9e^{2\tau}\left[2\left(\sin^2(\tau) + \cos^2(\tau)\right) + \left(2\sin(\tau)\cos(\tau) - 2\sin(\tau)\cos(\tau)\right)\right]} \\ &= \int_0^t \sqrt{9e^{2\tau}\cdot[2(1+0)]} = \int_0^t 3e^\tau\sqrt{2}\, d\tau = 3\sqrt{2}\int_0^t e^\tau\, d\tau = 3\sqrt{2}(e^t - 1). \end{aligned}$$

   With s, we know that $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so we need to find t in terms of s: $s = 3\sqrt{2}(e^t - 1) \Rightarrow e^t = \frac{s}{3\sqrt{2}} + 1 \Rightarrow t = \ln\left(\frac{s}{3\sqrt{2}} + 1\right)$. Finally, by replacing t with $t(s)$ in the original equation, we can get the arc length parametrization:

   $$\boxed{\mathbf{r}(s) = \left(\frac{s}{\sqrt{2}} + 3\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right)\mathbf{i} + \left(\frac{s}{\sqrt{2}} + 3\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right)\mathbf{j}.}$$

3. Use curvature to find the equation of the osculating circle at the planar curve $y = x^3 - 4x + 1$ at $x = 1$.

   Solution. First, we need to find the curvature of the curve at $x = 1$. We start by finding the first and second derivatives of the function: $y(x) = x^3 - 4x + 1 \Rightarrow y'(x) = 3x^2 - 4 \Rightarrow y''(x) = 6x$.

   Then, we evaluate the point and the first and second derivatives at $x = 1$: $y(1) = -2$; $y'(1) = -1$; $y''(1) = 6$.

   Find the curvature: $\kappa = \frac{|y''(x)|}{\left(1+y'(x)^2\right)^{3/2}} = \frac{6}{\left(1+(-1)^2\right)^{3/2}} = \frac{3\sqrt{2}}{2}$.

   Find radius: $R = \frac{1}{\kappa} = \frac{1}{\frac{3\sqrt{2}}{2}} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}$.

   For the center, find $\mathbf{N}$ at $x = 1$: $\mathbf{N} = \frac{(-y',1)}{\sqrt{1+(y')^2}} = \frac{\left(-(-1),1\right)}{\sqrt{1+(-1)^2}} = \frac{(1,1)}{\sqrt{2}}$.

   The center C can be found by moving our point $P(1, -2)$ the distance R along the unit normal vector: $C = P + R\mathbf{N} = \left(\frac{4}{3}, -\frac{5}{3}\right)$. This gives the equation for the osculating circle:

   $$\boxed{\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{5}{3}\right)^2 = \frac{2}{9}.}$$