

# Multivariable Calculus Practice Set II

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1. (2 points) Write, in general equation form, an equation of the plane which contains the three points  $P = (2, 7, 3)$ ,  $Q = (-5, 0, 1)$ , and  $R = (-3, 1, 2)$ .

*Solution.* First, we find  $\mathbf{PQ}$  and  $\mathbf{PR}$ :

$$\mathbf{PQ} = \langle -7, -7, -2 \rangle \quad \text{and} \quad \mathbf{PR} = \langle -5, -6, -1 \rangle.$$

With  $\mathbf{PQ}$  and  $\mathbf{PR}$ , we can find  $\mathbf{n}$  by solving for the cross product:

$$\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & -7 & -2 \\ -5 & -6 & -1 \end{vmatrix} = (7 - 12)\mathbf{i} - (7 - 10)\mathbf{j} + (42 - 35)\mathbf{k} = -5\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}.$$

With  $\mathbf{n}$ , we get the general formula:

$$\boxed{-5(x - 2) + 3(y - 7) + 7(z - 3) = 0.}$$

2. (2 points) Write, in scalar form, an equation of the plane which contains the point  $(5, 2, 1)$  and the line given by  $x + 2 = \frac{y}{4} = \frac{z - 5}{2}$ .

*Solution.* We start by parametrizing the line with common parameter  $t$ :

- $x + 2 = t \Rightarrow x = t - 2,$
- $\frac{y}{4} = t \Rightarrow y = 4t,$  and
- $\frac{z - 5}{2} = t \Rightarrow z = 2t + 5.$

This gives us the parametric form:

$$(x, y, z) = (-2, 0, 5) + t(1, 4, 2)$$

Thus, the line passes through the point  $(-2, 0, 5)$  and has the direction vector

$$\mathbf{v}_1 = \langle 1, 4, 2 \rangle.$$

Since the plane is two-dimensional, we need 2 independent directions within it. We got the first through our line, but we need another because there are infinitely many planes that contain the same line. Thus, we can form a second vector  $\mathbf{v}_2$  by taking the difference between the given point and a point on the line:

$$\mathbf{v}_2 = (5, 2, 1) - (-2, 0, 5) = \langle 7, 2, -4 \rangle.$$



3. (3 points) Determine the arc length parametrization for the curve  $\mathbf{r}(t) = 3e^t \sin(t)\mathbf{i} + 3e^t \cos(t)\mathbf{j}$ , where you start from  $t = 0$ .

*Solution.* From equation 3.11 from Theorem 3.4 in the book, (and [this website](#)) we know that we can rewrite the arc length parametrization as:

$$s = \int_0^t \|\mathbf{r}'(\tau)\| d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2} d\tau,$$

where  $f(\tau) = 3e^\tau \sin(\tau)$  and  $g(\tau) = 3e^\tau \cos(\tau)$ . Thus, we find:

$$\begin{aligned} f'(\tau) &= 3e^\tau (\sin(\tau) + \cos(\tau)) \\ g'(\tau) &= 3e^\tau (\cos(\tau) - \sin(\tau)). \end{aligned}$$

Thus, we have:

$$\begin{aligned} s &= \int_0^t \sqrt{[3e^\tau (\sin(\tau) + \cos(\tau))]^2 + [3e^\tau (\cos(\tau) - \sin(\tau))]^2} d\tau \\ &= \int_0^t \sqrt{9e^{2\tau} (\sin^2(\tau) + 2\sin(\tau)\cos(\tau) + \cos^2(\tau)) + 9e^{2\tau} (\cos^2(\tau) - 2\sin(\tau)\cos(\tau) + \sin^2(\tau))} d\tau \\ &= \int_0^t \sqrt{9e^{2\tau} [2(\sin^2(\tau) + \cos^2(\tau)) + (2\sin(\tau)\cos(\tau) - 2\sin(\tau)\cos(\tau))]} d\tau \\ &= \int_0^t \sqrt{9e^{2\tau} \cdot [2(1 + 0)]} d\tau \\ &= \int_0^t 3e^\tau \sqrt{2} d\tau \\ &= 3\sqrt{2} \int_0^t e^\tau d\tau \\ &= 3\sqrt{2}(e^t - 1). \end{aligned}$$

With  $s$ , we know that  $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$ , so we need to find  $t$  in terms of  $s$ :

$$\begin{aligned} s &= 3\sqrt{2}(e^t - 1) \\ e^t &= \frac{s}{3\sqrt{2}} + 1 \\ t &= \ln\left(\frac{s}{3\sqrt{2}} + 1\right). \end{aligned}$$

Finally, by replacing  $t$  with  $t(s)$  in the original equation, we can get the arc length parametrization:

$$\boxed{\mathbf{r}(s) = \left(\frac{s}{\sqrt{2}} + 3\right) \sin\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right) \mathbf{i} + \left(\frac{s}{\sqrt{2}} + 3\right) \cos\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right) \mathbf{j}.}$$

4. (3 points) Use curvature to find the equation of the osculating circle at the planar curve  $y = x^3 - 4x + 1$  at  $x = 1$ . Then, check your answer by graphing both the curve and its circle on the same axes. [you do not need to include the graph in your work turned in – but you should be able to tell if your work is correct.]

*Solution.* First, we need to find the curvature of the curve at  $x = 1$ . We start by finding the first and second derivatives of the function:

$$\begin{aligned}y(x) &= x^3 - 4x + 1 \\y'(x) &= 3x^2 - 4 \\y''(x) &= 6x.\end{aligned}$$

Then, we evaluate the point and the first and second derivatives at  $x = 1$ :

$$\begin{aligned}y(1) &= 1 - 4 + 1 = -2 \\y'(1) &= 3(1)^2 - 4 = -1 \\y''(1) &= 6(1) = 6.\end{aligned}$$

With these values, we can find the curvature:

$$\kappa = \frac{|y''(x)|}{(1 + y'(x)^2)^{3/2}} = \frac{6}{(1 + (-1)^2)^{3/2}} = \frac{6}{2^{3/2}} = \frac{3\sqrt{2}}{2}.$$

With the curvature, we can find the radius of the osculating circle:

$$R = \frac{1}{\kappa} = \frac{1}{\frac{3\sqrt{2}}{2}} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}.$$

To find the center, we need the unit normal vector at  $x = 1$ :

$$\mathbf{N} = \frac{(-y', 1)}{\sqrt{1 + (y')^2}} = \frac{(-(-1), 1)}{\sqrt{1 + (-1)^2}} = \frac{(1, 1)}{\sqrt{2}}$$

The center  $C$  can be found by moving our point  $P(1, -2)$  the distance  $R$  along the unit normal vector:

$$\begin{aligned}C &= P + R\mathbf{N} \\&= (1, -2) + \frac{\sqrt{2}}{3} \frac{(1, 1)}{\sqrt{2}} \\&= (1, -2) + \left(\frac{1}{3}, \frac{1}{3}\right) \\&= \left(\frac{4}{3}, -\frac{5}{3}\right).\end{aligned}$$

This gives the equation for the osculating circle:

$$\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{5}{3}\right)^2 = \frac{2}{9}.$$

5. (3 points each) Suppose the position of some particle is given by  $\mathbf{r}(t) = \sin(t)\mathbf{i} + t\mathbf{j} + 3t\mathbf{k}$ .

(a) Find the velocity vector,  $\mathbf{v}(t)$ .

*Solution.*

$$\mathbf{v}(t) = \mathbf{r}'(t) = \cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

(b) What total distance is travelled by the particle over the time period  $[0, 3\pi]$ ? (You can set up the necessary integral, and calculate it using your calculator up to 3 decimal places.)

*Solution.*

$$\int_0^{3\pi} \|\mathbf{r}'(t)\| dt = \int_0^{3\pi} \sqrt{\cos^2(t) + 1 + 9} dt = 9.709$$

(c) Find the unit tangent vector  $\mathbf{T}(t)$ .

*Solution.*

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{\cos^2(t) + 10}}$$

(d) Find unit normal vector  $\mathbf{N}(t)$ .

*Solution.* To find the unit normal vector, we need to find the derivative of the unit tangent vector. To avoid making mistakes (and making differentiating easier), let's break  $\mathbf{T}(t)$  into separate functions  $u(t)$  and  $v(t)$ :

$$\mathbf{T}(t) = \underbrace{(\cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k})}_{u(t)} \cdot \underbrace{(\cos^2(t) + 10)^{-1/2}}_{v(t)}$$

Differentiating  $u(t)$ :

$$u'(t) = -\sin(t)\mathbf{i},$$

and differentiating  $v(t)$  with the chain rule:

$$v'(t) = -\frac{1}{2}(\cos^2 + 10)^{-3/2} \cdot 2\cos(t)(-\sin(t)) = \cos(t)\sin(t)(\cos^2(t) + 10)^{-3/2}.$$

Thus, we apply the product rule for  $\mathbf{T}'(t)$ :

$$\mathbf{T}'(t) = \left[-\sin(t)\mathbf{i}\right](\cos^2(t) + 10)^{-1/2} + \left[\cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k}\right]\left[\cos(t)\sin(t)(\cos^2(t) + 10)^{-3/2}\right].$$

Notice that both terms contain a factor of  $\sin(t)$  and a power of  $\cos^2(t) + 10$ , so we can factor them out:

$$\mathbf{T}'(t) = \sin(t)(\cos^2(t) + 10)^{-3/2} \left\{ -\left[\cos^2(t) + 10\right]\mathbf{i} + \cos(t)\left[\cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k}\right] \right\}.$$

Inside the braces, multiply and combine terms:

$$\begin{aligned}\{\dots\} &= -\cos^2(t) \mathbf{i} - 10 \mathbf{i} + \cos^2(t) \mathbf{i} + \cos(t) \mathbf{j} + 3 \cos(t) \mathbf{k} \\ &= -10 \mathbf{i} + \cos(t) \mathbf{j} + 3 \cos(t) \mathbf{k}.\end{aligned}$$

This gives us:

$$\mathbf{T}'(t) = \frac{\sin(t)}{(\cos^2(t) + 10)^{3/2}} \left[ -10 \mathbf{i} + \cos(t) \mathbf{j} + 3 \cos(t) \mathbf{k} \right].$$

Now we need to find the magnitude of  $\mathbf{T}'(t)$ :

$$\|\mathbf{T}'(t)\| = \frac{|\sin(t)|}{(\cos^2(t) + 10)^{3/2}} \sqrt{(-10)^2 + (\cos(t))^2 + (3 \cos(t))^2}.$$

Simplify inside the square root and factor:

$$\|\mathbf{T}'(t)\| = \frac{|\sin(t)|}{(\cos^2(t) + 10)^{3/2}} \cdot \sqrt{10} (\cos^2(t) + 10)^{1/2}.$$

This simplifies to:

$$\|\mathbf{T}'(t)\| = \frac{|\sin(t)| \sqrt{10}}{\cos^2(t) + 10}.$$

Finally, we can find the unit normal vector:

$$\mathbf{N}(t) = \frac{\frac{\sin(t)}{(\cos^2(t)+10)^{3/2}} \left[ -10 \mathbf{i} + \cos(t) \mathbf{j} + 3 \cos(t) \mathbf{k} \right]}{\frac{|\sin(t)| \sqrt{10}}{\cos^2(t)+10}}.$$

After further simplification, we see:

$$\mathbf{N}(t) = \frac{-10 \mathbf{i} + \cos(t) \mathbf{j} + 3 \cos(t) \mathbf{k}}{\sqrt{10} \sqrt{\cos^2(t) + 10}}.$$

(e) Find binormal vector,  $\mathbf{B}(t)$ .

*Solution.* The binomial vector is the cross product of the unit tangent and unit normal vectors:

$$\mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\cos(t)}{\sqrt{\cos^2(t)+10}} & \frac{1}{\sqrt{\cos^2(t)+10}} & \frac{3}{\sqrt{\cos^2(t)+10}} \\ \frac{-10}{\sqrt{10}\sqrt{\cos^2(t)+10}} & \frac{\cos(t)}{\sqrt{10}\sqrt{\cos^2(t)+10}} & \frac{3\cos(t)}{\sqrt{10}\sqrt{\cos^2(t)+10}} \end{vmatrix}.$$

Thankfully, we can factor out the common term  $\frac{1}{\sqrt{\cos^2(t)+10}}$  from each vector in the cross product:

$$\mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{\cos^2(t)+10}} \cdot \frac{1}{\sqrt{\cos^2(t)+10}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & 1 & 3 \\ \frac{-10}{\sqrt{10}} & \frac{\cos(t)}{\sqrt{10}} & \frac{3\cos(t)}{\sqrt{10}} \end{vmatrix}.$$

Now we can find the cross product:

$$\frac{1}{\cos^2(t)+10} \left[ \left( 1 \cdot \frac{3\cos(t)}{\sqrt{10}} \right) - \left( \frac{\cos(t)}{\sqrt{10}} \cdot 3 \right), \right. \\ \left. \left( 3 \cdot \frac{-10}{\sqrt{10}} \right) - \left( \cos(t) \cdot \frac{3\cos(t)}{\sqrt{10}} \right), \left( \cos(t) \cdot \frac{\cos(t)}{\sqrt{10}} \right) - \left( 1 \cdot \frac{-10}{\sqrt{10}} \right) \right].$$

Multiplying, we see that:

$$\frac{1}{\cos^2(t)+10} \left\langle 0, \frac{-3(\cos^2(t)+10)}{\sqrt{10}}, \frac{\cos^2(t)+10}{\sqrt{10}} \right\rangle.$$

Notice  $\frac{1}{\cos^2(t)+10}$  cancels with the  $\mathbf{j}^{\text{th}}$  and  $\mathbf{k}^{\text{th}}$  terms. Thus, we can further simplify this expression to:

$$\left\langle 0, \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.$$