# Exam 3 Corrections

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## In-Class Exam

2. Consider the recurrence relation, $B(n) = \begin{cases} 2, & \text{if } n = 0 \\ 3, & \text{if } n = 1 \\ 2B(n-1) - B(n-2), & \text{if } n > 1 \end{cases}$

   (b) What is the closed form solution to this recurrence relation?

   *Solution.* $B(n) = 2 + n$

   (c) Use induction to prove your answer in part (b).

   *Solution.*

   - **Base Cases**:
     For $n = 0$: $B(0) = 2 + 0 = 2$
     For $n = 1$: $B(1) = 2 + 1 = 3$

   - **Inductive Hypothesis**:
     Assume $B(n) = 2 + n$ is true for an $n \geq 1$.

   - **Inductive Step**:
     We must show that $B(n+1) = 2 + (n+1)$.
     Given the recurrence relationship that is defined, $B(n+1) = 2B(n) - B(n-1)$.
     We can substitute using the inductive hypothesis and solve:

$$\begin{aligned} B(n+1) &= 2(2+n) - (2+n-1) \\ &= 4 + 2n - 2 - n + 1 \\ &= 3 + n \\ &= 2 + (n+1) \end{aligned}$$

3. Show that $n^n \geq n!$ for all integers $n \geq 1$ using induction.

   *Solution.*

- **Base Case**:
  For $n = 1$: $1^1 \geq 1! \Rightarrow 1 \geq 1$.

- **Inductive Hypothesis**:
  Assume the inequality $n^n \geq n!$ is true for some integer $n \geq 1$.

- **Inductive Step**:
  We need to show that $(n+1)^{n+1} \geq (n+1)!$.

  To begin, we will thoroughly show that $(n+1)^n \geq n^n$.[1] Thus, for the left-hand side, we know that $(n+1)^n = (n+1) \times (n+1) \times \cdots \times (n+1)$ (Consisting of $n$ terms). Then, for the right-hand side, we can show $n^n = n \times n \times \cdots \times n$ (Consisting of $n$ terms). We know that mathematically, it must be the case that $n+1$ is greater than $n$, and we see this is true for each corresponding entry in both expanded expressions.

  So, because we know it is the case that $n+1$ is greater than or equal $n$ for $n$ terms, we can justify the statement $(n+1)^n \geq n^n$. Then, we can apply that to our inductive hypothesis to get $(n+1)^n \geq n^n \geq n!$, and by substitution, $(n+1)^n \geq n!$.

  In wrapping up, we can multiply both sides of the expression above by $n+1$ to get

$$(n+1) \times (n+1)^n \geq (n+1) \times n!$$
$$(n+1)^{n+1} \geq (n+1)!$$

4. Prof. Seme has 9 books that he's bought recently. He is planning on taking a trip and wants to select 4 to read. How many different combinations are there?

   *Solution.* $\binom{n}{k} = \dfrac{n!}{k!(n-k)!} \Rightarrow \binom{9}{4} = \dfrac{9!}{4!5!} \Rightarrow 126.$

5. What is the coefficient of $x^8$ in the expansion of $(5x-7)^{11}$?

   *Solution.* $\binom{11}{3}(5x)^8(-7)^3 = -22235625x^8$

**** SKIP 6 & 7 ****

8. give a big-$\Theta$ estimate for the number of additions, as a function of $n$ in the code shown:

```
y = 7
for i in [1,2,...,9]:
    for j in [1,2,...,n]:
        for k in [i,i+1,...,n]:
            y = y + i + j + k
```

   *Solution.* Outer loop: constant time.
   Middle loop: $n$ time.
   Inner loop: $n$ time.
   $\therefore \Theta(n^2)$

[1] This explanation is very pedantic: Of course we know that $(n+1)^n \geq n^n$, but for the sake of covering all bases, I must show for each term in both expressions that they are, indeed, greater than or equal.

# Take-Home Exam

1. Suppose you have an unlimited number of 3 and 7 cent stamps. Show, using induction, that you can make any value of 12 cents or more.[2]

   *Solution*.

   - **Base Cases**:
     $n = 12$: $3 \times 4 = 12$
     $n = 13$: $(3 \times 2) + 7 = 13$
     $n = 14$: $7 \times 2 = 14$
     $n = 15$: $3 \times 5 = 15$
     $n = 16$: $(3 \times 3) + 7 = 16$
     $n = 17$: $(7 \times 2) + 3 = 17$

   - **Inductive Hypothesis**:
     Suppose it is possible to create any postage value of $k$ cents, for all values of $k$ such that $12 \leq k \leq n$, where $n \geq 17$.

   - **Inductive Step**:
     *This is more for me as a way of laying out what the proof form is supposed to be like*:

     The goal is to show that for any stamp value $n$ that is 17 cents or more, we can subtract some amount of either 3 or 7 cents to fall back into the range set by the inductive hypothesis. This way, we can show that every larger amount can be 'built up' from smaller amounts that we have shown to be possible.

     Given that $n \geq 17$, we need to show that $n+1$ can be created from smaller combinations laid out by the inductive hypothesis. Thus, from the inductive hypothesis, we can make $n - 2$ cents because $n - 2$ is at least 15. Thus, adding one 3 cent stamp to $n - 2$ gives us $n + 1$.

**** SKIP 2-4 ****

5. M&M Candies come in 6 colors. How many do you need to select to be certain that you have at least 7 of the same color?

   *Solution*. There are 6 possible candies, with 6 possible colors. If we were to pick out 36 candies, it would be possible that for each candy, we got exactly 6 colors. Hence, we would need to pick one more candy to be certain that we have at least 7 of the same color. Thus, if you are *extremely unlucky*, you need to pick 37 M&M's to be sure that of the M&M's you have, at least 7 are of the same color.

[2] See: Fundamental Theorem of Arithmetic. (And here I thought you were trying to teach us about what to do if we ever found ourselves in a situation with an infinite amount of 3-cent and 7-cent stamps. Definitely fooled me!)