Mathematical Cryptography

MATH 490

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CHAPTER 1

THE REAL NUMBERS

1.1 Introduction

Before starting the course, it is important to understand that this document is for notetaking, and will therefore be a bit more informal than the actual textbook. The textbook is *An Introduction to Mathematical Cryptography* by Hoffstein, Pipher, and Silverman. The textbook can be located at this link

Definition Caesar Shift Cipher:

An encrypted text (by **shifting**), and you match it up with the alphabet. To encrypt, you write out a sentence, match it with a random assortment of letters by shifting the letters by a predetermined amount.

Definition Code:

Replace words / concepts. Example: Eagle has landed.

Definition Cipher:

Replacing characters or letters. Simply, replacing one letter for another.

Definition Scytale Cipher:

Used by the Spartans in the 5^{th} century B.C.. This is also known as a **Transposition Cipher**.

Definition Transposition Cipher:

Changed order, but the stayed the same.

Definition Plain Text:

Original message that is readable to humans. Abbreviated as [pt].

Definition Cipher Text:

Encrypted message that is unreadable to humans. Abbreviated as [ct].

Definition Encrypting:

From plain text to cipher text. The inverse of encrypting is decrypting; which is going from the Plain Text [pt] to the Cipher Text [ct].

Definition Key:

A secret number or word used in encoding and decoding using a certain algorithm. Example: Caesar shift: $A \to R$ rotated clockwise by 17

Definition Key Space:

The set of all keys, notated $\mathcal{K}$. The cardinality (amount of different keys) is notated with absolute value symbols. (E.g., for the Caesar shift, $|\mathcal{K}| = 26$ because there are 26 letters in the alphabet. Similarly, Scytale $|\mathcal{K}| =$ pt.)

Definition Brute Force Attack:

[During decryption] Trying all possible keys.

1.1.1 Goals of Cryptography

1. Provide confidentiality – You can't read the message.
2. Provide integrity – You can't change the message.
3. Provide authenticity – You can't forge the message.

Generally, these are the people and setting that will be used in examples: Alice and Bob are trying to communicate. Eve is trying to eavesdrop on the conversation.

1.1.2 Simple Substitution Ciphers (Mono-alphabetic Cipher)

Each letter can be replaced with any other letter. For example, you may have a key: $\{a, b, c, \ldots, z\} \to \{q, m, w, \ldots, t\}$. Its cardinality is $|\mathcal{K}| = 26!$.

Definition Cryptanalysis:

Process of decrypting without a key.

Definition Bigrams:

Two letters that are commonly placed together in language. For example, "Th", "is", or "He".

Definition Frequency Analysis:

English language patterns

Note that 13% of letters that are used in the alphabet are (in order from least to greatest): E, T, A, O, N. In brief, the longer the text, the more likely these letters will pop up.

1.2 Divisibility and Greatest Common Denominators

Can assume all the properties of $\mathbb{R}, \mathbb{Z},$ and $\mathbb{N}$. Note that $\mathbb{N}$ does not include 0.

Definition Divides:

Let a and b be integers with $b \neq 0$. We say that b *divides* a or that a is divisible by b, denoted by $b \mid a$, if there exists an integer n such that $a = nb$.

Example 1.1: Divisibility

Let $a = 100$ and $b = 4$. Is $b \mid a$?

Solution. Yes, because $100 = 4 \times 25$.

Example 1.2: Divisibility

Let $a = 100$ and $b = 8$. Is $b \mid a$?

Solution. No, because $100 = 8 \times 12 + 4$.

Proposition 1.4: Let $a, b, c \in \mathbb{Z}$:

1. If $a \mid b$ and $b \mid c$, then $a \mid c$.
2. If $a \mid b$ and $b \mid a$, then $a = \pm 2$.
3. If $a \mid b$ and $a \mid c$, then $a \mid (b + c) and a \mid (b - c)$.

Definition Greatest Common Divisor:

A *common divisor* of two integers a and b is a positive integer d that divides both of them. The *greatest common divisor* of a and b, denoted by $\gcd(a, b)$, is the largest positive integer such that $d \mid a$ and $d \mid b$.

This is less complicated than it sounds: we are simply factoring the integers and finding the largest common divisor between the two numbers.

Definition Divison with Remainder:

Let a, b be positive integers. Then we say that a *divided by* b gives a *quotient* q and a *remainder* r if $a = bq + r$ and $0 \leq r < b$.

Example 1.3: Division with Remainder

Let $a = 24$ and $b = 16$. Find the quotient and remainder.

Solution. $24 = 16(1) + 8$. Therefore, the quotient is 1 and the remainder is 8.

Example 1.4: Euclidean Algorithm

Compute $\gcd(2024, 748)$ using the Euclidean Algorithm.

Solution. Notice how the b and r values on each line become the new a and b values on the subsequent line:

$$\begin{aligned} 2024 &= 2(748) + 528 \\ 748 &= 1(528) + 220 \\ 528 &= 2(220) + 88 \\ 220 &= 2(88) + \boxed{44} \\ 88 &= 2(44) + 0 \end{aligned}$$

Therefore, the greatest common divisor is 44.

Theorem: Euclidean Algorithm

Let $a, b \in \mathbb{Z}^+$ with $a \geq b$. The following algorithm computes $\gcd(a, b)$ in a finite number of steps.

1. Let $r_0 = a, r_1 = b$;
2. Set $i = 1$;
3. Divide r_{i-1} by r_i to get quotient q_i and remainder r_{i+1};
4. If $r_{i+1} = 0$, stop, and $\gcd(a, b) = r_i$;
5. Otherwise, $r_{i+1} > 0$. Set $i = i + 1$, and go back to step 3.
6. Step 3 is executed at most $2\log_2(b) + 2$ times.

(Extended Version): There exists $u, v \in \mathbb{Z}$ such that $\gcd(a, b) = ua + cb$. Thus, $r_0 = a$, $r_1 = b$, $r_{i+1} = r_{i-1} - g_i r_i$, $s_0 = 1$, $s_1 = 0$, $s_{i+1} = s_{i-1} - q_i s_i$ (up 2) $-$ (q left) $\times$ (up 1), $t_0 = 0$, $t_1 = 1$, $t_{i+1} = t_{i-1} - q_i t_i$. Stop when $r_i = 0$, $u = s_{i-1}$, $v = t_{i-1}$

Example 1.5: Linear Combinations

Use Example 1.2 in determining the linear combination of 2024 and 748 that equals 44.

Solution. We let $a = 2024$ and $b = 748$. From the equation in Example 1.2, we read the first line:

$$528 = a - 2b.$$

We substitue this into the second line to get

$$b = (a - 2b) \cdot 1 + 220, \quad \text{so} \quad 220 = 2b - a.$$

We next substitute the expressions $528 = a - 2b$ and $220 = 2b - a$ into the third line to get

$$a - 2b = (-a + 3b) \cdot 2 + 88, \quad \text{so} \quad 88 = 3a - 8b.$$

Finally, we substitute the expressions $220 = -a + 3b$ and $88 = 3a - 8b$ into the fourth line to get

$$-a + 3b = (3a - 8b) \cdot 2 + 44, \quad \text{so} \quad 44 = 7a - 19b.$$

In other words,

$$-7 \cdot 2024 + 19 \cdot 748 = 44 = \gcd(2024, 748),$$

so we have found a way to write $\gcd(a, b)$ as a linear combination of a and b using integer coefficients.

Definition Relatively Prime:

Let a and b be *relatively prime* if $\gcd(a,b) = 1$. More generally, any equation $Au + Bv = \gcd(A,B)$ can be reduced to the case of relatively prime numbers by dividing both sides by $\gcd(A,B)$. Thus, $\frac{A}{\gcd(A,B)}u + \frac{B}{\gcd(A,B)} = 1$ where $a = A/\gcd(A,B)$ and $b = B/\gcd(A,B)$ are relatively prime and satisfy $au + bv = 1$.

Theorem: Extended Euclidean Algorithm

Let $a, b \in \mathbb{Z}^+$ with $a \geq b$. Then the equation $\gcd(a,b) = ua + vb$ always has a solution in integers u and v. If u_0, v_0 is any one solution, then every solution has the form

$$u = u_0 + \frac{b \cdot t}{\gcd(a,b)} \text{ and } v = v_0 - \frac{a \cdot t}{\gcd(a,b)} \text{ for some integer } t \in \mathbb{Z}.$$

See Exercise 1.10 for a detailed example.

Example 1.6: Relative Prime

What are the relative prime numbers of 2024 and 748?

Solution. In Example 1.2, we found that the gcd of 2024 and 748 have greatest common divisor of 44 and satisfy the equation $-7 \cdot 2024 + 19 \cdot 748 = 44$. We can divide both sides by 44 to get $46u + 17v = 1$. Therefore, 46 and 17 are relatively prime and $u = -7$ and $v = 19$ are the coefficents of a linear combination of 46 and 17 that equals 1.

1.3 Modular Arithmetic

Definition Modular Arithmetic:

Let $m \geq 1$ be an integer. We say that the integers a and b are congruent modulo m if their difference is divisible by m: $m \mid (b-a)$ or $m \mid (a-b)$. Notated as $a = b \mod m$

By the definition of division, we can write this as $b - a = mk$ for some $k \in \mathbb{Z}$ and $b = mk + a$ for some $k \in \mathbb{Z}$. For example, $17 \mod 4 \equiv 1$. Or for another example, $-17 \mod 4 \equiv -1 \equiv 3$. For addition, you can go in two separate directions. For the first, sequence of operations, we could add the numbers inside of the parentheses and then take the mod of the number as demonstrated $(26 + 14) \mod 5 \equiv 40 \mod 5 \equiv 0$, or we could take the mod of both numbers inside the parentheses, demonstrated as $26 \mod +14 \mod 5 = 1 + 4 = 0$.

Proposition 1.13: Let $m \in \mathbb{Z}^+$

1. If $a_1 \equiv a_2 \pmod{m}$ and $b_1 \equiv b_2 \pmod{m}$ then $a_1 \pm b_1 \equiv a_2 \pm b_2 \pmod{m} \equiv a_2 \pmod{m} + b_2 \pmod{m}$. Also, $a_1 b_1 = a_2 b_2 \pmod{m} = a_2 \pmod{m} b_2 \pmod{m}$
2. Let $a \in \mathbb{Z}$. Then $ab \equiv 1 \pmod{m}$ for some $b \equiv \mathbb{Z} \iff \gcd(a,m) = 1$

Definition Ring:

We write $\mathbb{Z}/m\mathbb{Z} = \{0, 1, 2, \ldots, m-1\}$ and call $\mathbb{Z}/m\mathbb{Z}$ the *ring of integers modulo* m. We add and multiplying them as integers and then dividing the result by m and taking the remainder in order to obtain an element in $\mathbb{Z}/m\mathbb{Z}$.

Note that we will be finding the more traditional rings that are brought up in Algebra. Thus, for a ring to be a ring, it must have the following properties:

1. **Additive Closure:** For any $a, b \in R$, the sum $a + b$ is also in R.
2. **Associativity of Addition:** For any $a, b, c \in R$, $(a + b) + c = a + (b + c)$.
3. **Commutativity of Addition:** For any $a, b \in R$, $a + b = b + a$.
4. **Additive Identity:** There exists an element $0 \in R$ such that for any $a \in R$, $a+0 = a$.
5. **Additive Inverses:** For each $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0$.
6. **Multiplicative Closure:** For any $a, b \in R$, the product $a \cdot b$ is also in R.
7. **Associativity of Multiplication:** For any $a, b, c \in R$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
8. **Distributive Property:** For any $a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.

Example 1.7: $(\mathbb{Z}_6, \cdot)$

Find the ring of integers modulo 6 bound under multiplication.

Solution. Note that from the following table, 6 only has 2 inverses, 1 and 5. This is because in their respective column, there is a 1 in each row.

a	0	1	2	3	4	5
$a \cdot 0$	0	0	0	0	0	0
$a \cdot 1$	0	1	2	3	4	5
$a \cdot 2$	0	2	4	0	2	4
$a \cdot 3$	0	3	0	3	0	3
$a \cdot 4$	0	4	2	0	4	2
$a \cdot 5$	0	5	4	3	2	1

Definition Unit:

Recall from Proposition 1.13 that a has an inverse modulo m if and only if $\gcd(a, m) = 1$. Numbers that have inverses are called *units*. We denote the set of all units by

$$\begin{aligned}(\mathbb{Z}/m\mathbb{Z})^* &= \{a \in \mathbb{Z}/m\mathbb{Z} \mid \gcd(a, m) = 1\} \\ &= \{a \in \mathbb{Z}/m\mathbb{Z} \mid \text{a has an inverse modulo } m\}\end{aligned}$$

Fast forward to section 2.5:

Definition Group:

A set G along with a binary operation (closure) such that for all $a, b \in G$, $a \times b \in G$ (closure), and there exists an $e \in G$ such that $a \times e = a$ and $e \times a = a$ (identity), for all $a \in G$, there exists $a^{-1} \in G$ such that $a \times a^{-1} = a^{-1} \times a = e$ (inverse), and for all $a, b, c \in G$, $(a \times b) \times c = a \times (b \times c)$ (associativity)

For commutativity, for all $a, b \in G$, $a \times b = b \times a$. Some groups have this, some do not.

Example 1.8: Integer Addition as a Group

Lets check to see addition among the integers are a group: $(\mathbb{Z}, +)$

Solution.

1. True. Let $a, b \in \mathbb{Z}$ $a + b \in \mathbb{Z}$.
2. True. $e = 0 \in \mathbb{Z}$, $a + 0 = a$ and $0 + a = a$
3. True. For all $a \in \mathbb{Z}$, $a^{-1} = -a$ because $a + (-a) = 0 = -a + a$
4. True. For all $a, b, c \in \mathbb{Z}$, $(a + b) + c = a + (b + c)$

Therefore, the additive property of the integers are a group. In fact, because $a+b = b+a$ $\mathbb{Z}$ are a commutative group (abelian group).

Example 1.9: Integer Multiplication

Lets check to see multiplication among the integers are a group: $(\mathbb{Z}, +)$

Solution.

1. True. Let $a, b \in \mathbb{Z}$ $ab \in \mathbb{Z}$.
2. True. $e = 1 \in \mathbb{Z}$, $a * 1 = a$ and $1 * a = a$
3. False. Counterexample: consider $2^{-1} = \frac{1}{2}$ because $2(\frac{1}{2}) = 1$ but $\frac{1}{2} \notin \mathbb{Z}$

Definition Euler's Phi Function:

The function $\phi(m)$ defined by the rule

$$\phi(m) = |\{0 \leq a < m \mid \gcd(a, m) = 1\}|$$

1.3.1 Modular Arithmetic and Shift Ciphers

Example 1.10: Shift Cipher

Let's say we have a shift cipher with a key of 3. We want to encrypt the message "HELLO". What is the encrypted message?

Solution. By shifting the letters by 3, we get "KHOOR". For the decryption, we would shift the letters by -3 to get the original message.

1.3.2 Fast Powering Algorithm

We can write the algorithm as follows:

1. Write the exponent in binary.
2. Compute the powers of the base in binary. For example,

$$\begin{aligned} k_0 &= a \pmod m \\ k_1 &= k_0^2 \pmod m = a^2 \pmod m \\ k_2 &= k_1^2 \pmod m = (a^2)^2 \pmod m \\ &\vdots \\ k_r &= k_{r-1}^2 \pmod m = (a^{2^{n-1}})^2 \pmod m \end{aligned}$$

3. Multiply the powers of the base that correspond to the 1s in the binary representation of the exponent. Compute $a_b \pmod m$ by using

$$a^b = a^{b_0 \cdot b_1 2 + \cdots + b_r 2^r}$$

Example 1.11: Fast Powering Algorithm

Compute $3^{218} \pmod{1000}$.

Solution. The first step is to write 218 in binary: $218 = 11011010$. Then, we can write

the powers of 3 in binary:

$$\begin{aligned}
3^1 &= 3 \pmod{1000} = 3 \\
3^2 &= 3^2 \pmod{1000} = 9 \\
3^4 &= 9^2 \pmod{1000} = 81 \\
3^8 &= 81^2 \pmod{1000} = 561 \\
3^{16} &= 561^2 \pmod{1000} = 721 \\
3^{32} &= 721^2 \pmod{1000} = 841 \\
3^{64} &= 841^2 \pmod{1000} = 281 \\
3^{128} &= 281^2 \pmod{1000} = 961
\end{aligned}$$

Now, we can calculate the value of 3^{218} by multiplying the values of 3^{128}, 3^{64}, 3^{16}, 3^8, and 3^2 together:

$$\begin{aligned}
3^{218} &= 3^{128} \times 3^{64} \times 3^{16} \times 3^8 \times 3^2 \\
&= 961 \times 281 \times 721 \times 561 \times 9 \\
&= 489 \pmod{1000}
\end{aligned}$$

1.4 Prime Numbers, Unique Factorization, and Finite Fields

Definition Prime Number:

A prime number is a positive integer greater than 1 whose only divisors are 1 and itself.

Proposition 1.19 Let p be a prime number with $a, b \in \mathbb{Z}$ such that $p \mid ab$. Then $p \mid a$ or $p \mid b$.

Theorem: Fundamental Theorem of Arithmetic

Let $a \geq 2$ be an integer. Then a can be factored as a product of prime numbers

$$a = p_1^{e_1} \cdot p_2^{e_2} \cdots p_r^{e_r}.$$

Further, other than rearranging the order of the factors, this factorization is unique.

Proposition 1.21: Let p be prime. Then every non-zero element of $\mathbb{Z}/p\mathbb{Z}$ has a multiplicative inverse.

Put another way, $x \in \mathbb{Z}_p$ has a multiplicative inverse if and only if $\gcd(x, p) = 1$ because of Proposition 1.13.

Definition Field:

If p is prime, then $\mathbb{Z}/p\mathbb{Z}$ of integers modulo p with its addition, subtraction, multiplication,

and division rules is a *field*. See the definition of Ring for more information on the properties of a field. (Note that a field is a type of ring, called a communicative ring.) We often notate fields as $\mathbb{F}_p$ and $\mathbb{F}_p^*$ as the group of units

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1.5 Powers and Primitive Roots in Finite Fields

Theorem: Fermat's Little Theorem

Let p be a prime number and a be an integer. Then,

$$a^{p-1} \equiv \begin{cases} 1 \pmod{p} & \text{if } p \nmid a \\ 0 \pmod{p} & \text{if } p \mid a \end{cases}$$

Definition Order:

The *order* of an element $a \pmod{p}$ is the smallest exponent $k \geq 1$ such that $a^k \equiv 1 \pmod{p}$.

Example 1.12: Order

Find the order of 2 modulo 7.

Solution. $2^3 = 8 \pmod{7} = 1$. Thus, the order of 2 modulo 7 is 3.

Example 1.13: Order

Find the order of 5 modulo 7.

Solution. $5^6 = 15625 \pmod{7} = 1$. Thus, the order of 5 modulo 7 is 6.

Theorem: Primitive Root Theorem

Let p be a prime number. Then there exists an integer g such that there exists an $x \in \mathbb{F}_p^*$ whose powers give every element of $\mathbb{F}_p^*$, i.e.,

$$\mathbb{F}_p^* = \{1, g, g^2, g^3 \ldots, g^{p-2}\}.$$

Elements with this property are called the *primitive roots of* $\mathbb{F}_p$ or *generators of* $\mathbb{F}_p^*$. They are elements of order $p-1$.

Example 1.14: Primitive Root

Find a primitive root modulo 7.

Solution. We can look to the Primitive Root Theorem to see how many primitive roots are 7 has. Because 7 is prime, we know that $\phi = 6$. Thus, we know that 7 will have 6 primitive roots. Because there are 6 primitive roots in total, and $\mathbb{F}_7^*$ has 7 elements (including 0), we know that 1-6 will be the primitive roots.

1.6 Symmetric and Antisymmetric Ciphers

1.6.1 Symmetric Ciphers

Definition Symmetric Cipher:

A cipher that uses the same key for both encryption and decryption.

Review of Notation

- k implies key;
- $\mathcal{K}$ implies key space;
- m implies plaintext plain text;
- c implies cipher text;
- $\mathcal{M}$ implies all possible messages (message space);
- $\mathcal{C}$ implies all possible cipher texts (cipher space);
- Encryption is a function that is defined as:

$$e : \mathcal{K} \times \mathcal{M} \to \mathcal{C} \text{ such that } d(k(e(k, m))) = m$$

- Decryption is a function that is defined as:

$$d : \mathcal{K} \times \mathcal{C} \to \mathcal{M} \text{ such that } e(k, d(k, c)) = c$$

This brings us to what it means to be a *successful* cipher. Thus, we look to *Kerckhoff's Principle* which states that the security of a cipher should not depend on the secrecy of the algorithm, but rather on the secrecy of the key. Therefore, we have the following properties that are required for each cipher:

1. For all k and m, it is easy to computer $e_k(m)$. (Note that *easy* is relative to the computational power of the adversary. For this course, *easy* denotes a decryption time of less than a second.)
2. For all k and c, it is easy to compute $e_k(c)$.
3. Given one or more ciphertext, $c_1, \ldots c_n$, all encrypted with the same key, it is hard to compute any plaintext, m, such that $d_k(m)$ without knowing the key.

The next traits are desired, but not required:

4. Given one or more PT and CT pair, $(m_1, c_1), \ldots, (m_n, c_n)$ it is decrypt another CT not on this list without knowing the key. An example would be the *enigma cipher* from WW-II.
5. For any PT chosen by the adversary and their CT's, $(c_1, \ldots c_n)$, it should be hard to decrypt any CT not in this list.

Types of Attacks

- *Brute Force Attack*: Trying all possible keys.
- *Known PT Attack*: The adversary knows the plaintext and the corresponding ciphertext.
- *Chosen PT Attack*: The adversary chooses the plaintext and receives the corresponding ciphertext.

Types of Ciphers

1. Multiplication modulo m: $c = m \times k \pmod{m}$.
   - $\mathcal{K} = \mathcal{M} = \mathcal{C} = \mathbb{F}_p^*$
   - $k \in \mathbb{F}_p^*$
   - $e_k(m) = k \cdot m \pmod{p}$
   - $d_k(m) = k^{-1} \cdot c \pmod{p}$
   - Example: $\mathbb{F}_{307}, k = 258, m = 444, e_{258}(444) = 258.444 \pmod{1307} = 843$. To find decrypt this message, Eve needs to iterate through $1307 - 1$ different keys. (Easy.)
2. Add (mod m): $c = m + k \pmod{m}$. (Caesar Cipher.)
3. Affine Cipher: Key $= (k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}$.
4. Hill Cipher.
5. Vernam's One-time Pad.

1.6.2 Encoding Schemes

Definition Encoding Scheme:

An *encoding scheme* is a method of converting plaintext into a form that can be transmitted over a channel. An encoding scheme is assumed to be entirely public knowledge and used by everyone for the same purposes. An encryption scheme is designed to hide information from anyone who dopes not know the key. Thus, an encoding scheme, like an encryption scheme, consists of an encoding functions are public knowledge and should be fast and easy to compute.

This section will cover ASCII: We can take strings of 8 bits and convert them into a single character. From 0 to 255, and use them to represent the letters of the alphabet via $\texttt{a} = 00000000, \texttt{b} = 00000001, \texttt{c} = 00000010, \ldots, \texttt{z} = 00011001$. To distinguish between upper and lower case letters, we can use the first bit to represent the case.

1.6.3 Asymmetric (public key) Cryptography

This section will lead us to the following chapter.

CHAPTER 2

DISCRETE LOGARITHMS AND DIFFIE-HELLMAN

2.1 The Birth of Public Key Cryptography

Definition One-way Function:

A *one-way function* that is easy to computer, but whose inverse is difficult.

Definition Trap-door Function:

A *trap-door function* is a one-way function with an extra piece of information that makes f^{-1} easy.

2.2 Discrete Logarithm Problem (DLP)

Definition Discrete Logarithm Problem:

Let g be a primitive root for $\mathbb{F}_p$ and let h be a nonzero element of $\mathbb{F}_p$. The *Discrete Logarithm Problem* is the problem of finding an exponent x such that

$$g^x \equiv h \pmod{p}.$$

The number x is called the *discrete logarithm* of h to the base g and is denoted by $\log_g(h)$.

Remember the rules of logarithms:

$$\begin{aligned} \log_b(a \cdot c) &= \log_b(a) + \log_b(c) \\ \log_b(a^c) &= c \cdot \log_b(a) \\ \log_b(a/c) &= \log_b(a) - \log_b(c) \end{aligned}$$

What is the value of x such that $20^x = 21 \pmod{23} \overset{\text{Brute-f}}{\Longrightarrow} \log_{20} 21 = \boxed{7} \pmod{23}$. (where 7 is from wolfram.)

Example 2.1: DLP

Find $\log_2(10) \pmod{11}$. In other words, find the value of x such that $2^x = 10 \pmod{11}$.

Solution.

$$\begin{aligned} 2^1 &= 2 \pmod{11} \\ 2^2 &= 4 \pmod{11} \\ 2^3 &= 8 \pmod{11} \\ 2^4 &= 5 \pmod{11} \\ 2^5 &= 10 \pmod{11} \\ \log_2(10) &= \boxed{5} \pmod{11}. \end{aligned}$$

2.2.1 Diffie-Hellman Key Exchange

D-H gives a way for Alice and Bob to get a secret shared key in an unsecure environment (i.e., when Eve is listening). Now, we will follow the steps of D-H below:

1. Alice and Bob choose large prime p and primitive root g and make public $k_{\text{pub}} = (p, g)$.
2. Alice and Bob each pick their own secret integers, a, b such that $k_{\text{priv } A} = a$ and $k_{\text{priv } B} = b$. Compute $g^a \pmod{p} = A$ and $g^b \pmod{p} = B$.
3. Exchange a and b over an insecure channel.
   i. Note that Eve would have to solve DLP if she obtained a and b where $a = \log_A(A)$ and $b = \log_g(B)$.
   ii. Guidelines $\approx 2^{1000} g \approx p/2$.
4. Alice computes $B^a \pmod{p} = A'$ and Bob computes $A^b \pmod{p} = B'$.

Example 2.2: D-H

Let $p = 23$ and $g = 5$. Alice chooses $a = 6$ and Bob chooses $b = 15$. Compute the shared secret key.

Solution.

$$\begin{aligned} A &= 5^6 \pmod{23} = 8 \\ B &= 5^{15} \pmod{23} = 19 \\ A' &= 19^6 \pmod{23} = 2 \\ B' &= 8^{15} \pmod{23} = 2. \end{aligned}$$

Definition Diffie-Hellman Problem:

Let p be a prime number and g an integer. The *Diffie-Hellman Problem* is the problem of computing the value of $g^{ab} \pmod{p}$ from the known values of $g^a \pmod{p}$ and $g^b \pmod{p}$.

2.3 Elgamal Public Key Cryptosystem

Public parameter creation	
A trusted party chooses and publishes a large prime p and an element g modulo p of large (prime) order.	
Key creation	
Alice	**Bob**
Choose private key $1 \leq a \leq p-1$. Compute $A = g^a \pmod{p}$. Publish the public key A.	
Encryption	
	Choose plaintext m. Choose random element k. Use Alice's public key A to compute $c_1 = g^k \pmod{p}$ and $c_2 = mA^k \pmod{p}$. Send ciphertext c_1, c_2 to Alice.
Decryption	
Compute $(a_1^a)^{-1} \cdot c_2 \pmod{p}$. This quantity is equal to m.	

Table 2.1: Elgamal Key Creation, Encryption, and Decryption

Example 2.3: Elgamal

Let $p = 29$ and $g = 2$. Alice chooses $a = 12$ and Bob chooses $k = 5$ and wants to send secret message $m = 26$. Compute the shared secret key.

Solution. First, we need to calculate Alice's A and Bob's B. Then, we can calculate the ciphertexts c_1 and c_2:

$$\begin{aligned} A &= g^a \pmod{p} && = 2^{12} \pmod{29} = 7 \\ B &= g^k \pmod{p} && = 2^5 \pmod{29} = 3 \\ c_1 &= g^k \pmod{p} && = 2^5 \pmod{29} = 3 \\ c_2 &= m(A^k) \pmod{p} && = 26(7^5 \pmod{29}) = 10 \end{aligned}$$

Now, for Alice to decrypt the message, she must compute $(c_1^a)^{-1} \cdot c_2 \pmod{p}$:

$$(c_1^a)^{-1} \cdot c_2 \pmod{p} = (3^{12})^{-1} \cdot 10 \pmod{29}$$

The order of operations to compute this is as follows:

1. **Compute** $3^{12} \pmod{29} = 16$;
2. **Compute** $16^{-1} \pmod{29} = 20$;
3. **Finish by multiplying** $20 \cdot 10 \pmod{29} = 26$.

Be aware: You should only use this encryption scheme once. If you use it more than once, it is possible for an attacker to decrypt the message. For example, Eve knows $m_1(c_1, c_2) \rightarrow$ Eve finds A by keeping record of the first message, then by solving for d_2 such that c_1, d_2 (where c_1 is the *same* as the first message) and d_2 is the second message. Then, Eve can solve for m_2 by computing $(c_1^d)^{-1} \cdot d_2 \pmod{p}$.

Notes (ver batim for the most part)
DLP $a^x \equiv b \pmod{p}$.
Fermat's Little theorem: $a^{p-1} \equiv \pmod{p} \forall a \neq 0$. Is FLT true if modulus is not prime?
No, $2^5 \pmod{6} = 32$

Example 2.4: Alterative to FLT for non-prime moduli

Find $a^4 \pmod{15}$ for all a.

$\equiv 1$, $a = 1, 2, 4, 7, 8, 11, 13, 14$ $(\gcd(a, 15) = 1)$ $\not\equiv$, $a = 3, 5, 6, 9, 10, 12$ $(\gcd(a, 15) \neq 1)$

EXERCISES PT. 1

Exercise 1.1

Build a cipher wheel as illustrated in Figure 1.1, but with an inner wheel that rotates, and use it to complete the following tasks.

(a) Encrypt the following plaintext using a rotation of 11 clockwise.

"A page of history is worth a volume of logic."

(b) Decrypt the following message, which was encrypted with a rotation of 7 clockwise.

`AOLYLHYLUVZLJYLAZILAALYAOHUAOLZLJYLALZAOHALCLYFIVKFNBLZZLZ`

Solution.

(a) `L ALRP ZQ STDEZCJ TD HZCES L GZWFXP ZQ WZRTN`

(b) THERE ARE NO SECRETS BETTER THAN THE SECRETES [sic] THAT EVERY BODY GUESSES
In the encrypted text, "Secrets" is `ZLJYLAZ`. Then, they use an incorrect spelling of the word, `ZLJYLALZ`, of which has an extra 'e' in it. That is what the "[sic]" is for.

Exercise 1.2

Decrypt each of the following Caesar encryptions by trying the various possible shifts until you obtain readable text.

(a) `LWKLQNWKDWLVKDOOQHYHUVHHDELOOERDUGORYHOBDVDWUHH`

(b) `UXENRBWXCUXENFQRLQJUCNABFQNWRCJUCNAJCRXWORWMB`

Solution.

(a) I THINK THAT I SHALL NEVER SEE A BILLBOARD LOVELY AS A TREE

(b) LOVE IS NOT LOVE WHICH ALTERS WHEN IT ALTERATION FINDS

Exercise 1.3

For this exercise, use the simple substitution table given in Table 1.11.

(a) Encrypt the plaintext message:

The gold is hidden in the garden

Solution.

(a) `IBX FEPA QL BQAAXW QW IBX FSVAXW`

Exercise 1.4

Each of the following messages has been encrypted using a simple substitution cipher. Decrypt them. For your convenience, we have given you a frequency table and a list of the most common bigrams that appear in the ciphertext. (If you do not want to recopy the ciphertexts by hand, they can be downloaded or printed from the web site listed in the preface.)

(a) "A Piratical Treasure"

```
JNRZR BNIGI BJRGZ IZLQR OTDNJ GRIHT USDKR ZZWLG OIBTM NRGJN
IJTZJ LZISJ NRSBL QVRSI ORIQT QDEKJ JNRQW GLOFN IJTZX QLFQL
WBIMJ ITQXT HHTBL KUHQL JZKMM LZRNT OBIMI EURLW BLQZJ GKBJT
QDIQS LWJNR OLGRI EZJGK ZRBGS MJLDG IMNZT OIHRK MOSOT QHIJL
QBRJN IJJNT ZFIZL WIZTO MURZM RBTRZ ZKBNN LFRVR GIZFL KUHIM
MRIGJ LJNRB GKHRT QJRUU RBJLW JNRZI TULGI EZLUK JRUST QZLUK
EURFT JNLKJ JNRXR S
```

Solution.

(a) THESE CHARACTERS AS ONE MIGHT READILY GUESS FORM A CIPHER THAT IS TO SAY THEY CONVEY A MEANING BUT THEN FROM WHAT IS KNOWN OF CAPTAIN KIDD I COULD NOT SUPPOSE HIM CAPABLE OF CONSTRUCTING ANY OF THE MORE ABSTRUSE CRYPTOGRAPHS I MADE UP MY MIND AT ONCE THAT THIS WAS OF A SIMPLE SPECIES SUCH HOW EVER AS WOULD APPEAR TO THE CRUDE INTELLECT OF THE SAILOR ABSOLUTELY INSOLUBLE WITHOUT THE KEY

Solver.

Exercise 1.5

Suppose that you have an alphabet of 26 letters.

(a) How many possible simple substitution ciphers are there?

(b) A letter in the alphabet is said to be fixed if the encryption of the letter is the letter itself. *Show an example of how the pieces work together*

Solution.

(a) 26!

(b) This is the formula used for solving for derangements (where n is the number of elements in the set, and $!n$ is the number of derangements [Definition: A permutation with no fixed points]): $!n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$. From Wikipedia. For $n = 2$, we can run through the following:

(1) $i = 0$: $\frac{(-1)^0}{0!} = 1$

(2) $i = 1$: $\frac{(-1)^1}{1!} = -1$

(3) $i = 2$: $\frac{(-1)^2}{2!} = 0.5$

Sum them together: $1 - 1 + 0.5 = 0.5$. Now we can get !2:

$$!2 = 2! \times (1 - 1 + 0.5) = 2 \times 0.5 = 1$$

Exercise 1.6

Let $a, b, c \in Z$. Use the definition of divisibility to directly prove the following properties of divisibility. (This is Proposition 1.4.)

(a) If $a \mid b$ and $b \mid c$, then $a \mid c$.

(b) If $a \mid b$ and $b \mid a$, then $a = \pm b$.

(c) If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$ and $a \mid (b - c)$.

Solution.

(a) Let $a, b, c \in \mathbb{Z}$ such that $a \mid b$ and $b \mid c$. We know there exists an $n \in \mathbb{Z}$ such that $a \times n = b$. Similarly, $b \mid c$ means there exists an $k \in \mathbb{Z}$ such that $b \times k = c$. We

can use the commutative property to show that:

$$\begin{aligned} k(an) &= (b)k \\ ank &= bk \\ bk &= c \\ a(nk) &= c \\ a &\mid c \end{aligned}$$

(b) From the problem statement, we can intuitively ascertain that because both a, b divide each other, then it must be the case that they are the same number. Moreover, because the criteria for dividing is not pertinent to whether the quotient is negative or positive, this number can be positive or negative. Now that we have an idea of what we are trying to accomplish, we can begin the proof: Let $a, b \in \mathbb{Z}$ such that $a \mid b$ and $b \mid a$. We know there exists an $n \in \mathbb{Z}$ such that $a \times n = b$. Similarly, $b \mid a$ means there exists an $k \in \mathbb{Z}$ such that $b \times k = a$. Then, we can utilize substitution to get the following:

$$\begin{aligned} bk &= a \\ (an)k &= a \end{aligned}$$

From here, we know that because a is present on both sides of the equation, we should divide by a to simplify. Thus, consider the following two cases.

- **Case 1:** $a \neq 0$
  Since a is not zero, we can divide both sides by a to get $nk = 1$. Since, $n, k \in \mathbb{Z}$, we do not need to worry about fractional reciprocals. Instead, we know from the identity property of multiplication, that n, k must both be ± 1.
  - n, k **are both** $+1$**:** Then, $a = b \times 1$ and $b = a \times 1$. Which simplifies to $a = b$ in both cases.
  - n, k **are both** -1**:** Then, $a = b \times (-1)$ and $b = a \times (-1)$ Which simplifies to $a = -b$ in both cases.
- **Case 2:** $a = 0$
  If $a = 0$, then $b = 0 \times n = 0$ and $0 = b \times k = 0$. Therefore, $a = b = 0$, and $a = \pm b$ is still true.

We have shown that in either case if $a \mid b$ and $b \mid a$, then $a = \pm b$.

(c) Because a needs to be divisible by both b and c, we know that there must exist an $n, k \in \mathbb{Z}$ such that $b = an$ and $c = ak$. Our goal is to get to the form, $a \times$ some integer $= b + c$ and $a \times$ some integer $= b - c$ so we can use the definition of divides to help us out here. Therefore, let us consider both $b + c$ and $b - c$ in two separate cases:

- **Case 1:** $b + c$

$$\begin{aligned} b + c &= (an) + (ak) \\ b + c &= a(n + k) \\ a &\mid (b + c) \end{aligned}$$

- **Case 2:** $b - c$

$$\begin{aligned} b - c &= (an) - (ak) \\ b - c &= a(n - k) \\ a &\mid (b - c) \end{aligned}$$

We have shown that if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$ and $a \mid (b - c)$.

Exercise 1.7

Use a calculator and the method described in Remark 1.9 to compute the following quotients and remainders.

(a) 34787 divided by 353.

(b) 238792 divided by 7843.

Solution.

(a) $a = 34787$ and $b = 353$. Then $a/b \approx 98.54674220$, so $q = 98$ and $r = a - b \cdot q = 34787 - 353 \cdot 98 = 193$.

(b) $a = 238792$ and $b = 7843$. Then $a/b \approx 30.446512$, so $q = 30$ and $r = a - b \cdot q = 238792 - 7843 \cdot 30 = 3502$.

Exercise 1.9

Use the Euclidean algorithm to compute the following greatest common divisors.

(a) $\gcd(291, 252)$.

(b) $\gcd(16261, 85652)$.

Solution.

(a) $\gcd(291, 252)$

(1) $r_0 = 291$, $r_1 = 252$.

(2) $i = 1$.

(3) Divide r_0 by r_1 to get a quotient, q_1 and a remainder, r_2:

$$291/252 = 1 = q_1$$
$$291 - (252 \times 1) = 39 = r_2$$

(4) $r_2 \neq 0$. So, we continue.

(5) $i = 2 + 1 = 3$.

---

(3) Divide r_1 by r_2 to get quotient, q_2 and a remainder, r_3:

$$252/39 = 6 = q_2$$
$$252 - (39 \times 6) = 18 = r_3$$

(4) $r_3 \neq 0$. So, we continue.

(5) $i = 3 + 1 = 4$.

---

(3) Divide r_2 by r_3 to get quotient q_3 and a remainder, r_4:

$$39/18 = 2 = q_3$$
$$39 - (18 \times 2) = 3 = r_4$$

(4) $r_4 \neq 0$. So, we continue.

(5) $i = 3 + 1 = 5$.

---

(3) Divide r_3 by r_4 to get quotient q_4 and a remainder, r_5:

$$18/3 = 6 = q_4$$
$$18 - (3 \times 6) = 0 = r_5$$

(4) $r_4 = 0$. So, we stop.

We have found that the greatest common divisor is 3.

(b) *To cut back on paper, I am going to avoid reiterating the steps of 4 and 5. If there is a continuation in the enumeration process, then $r_i \neq 0$, and the process needs to continue*: $\gcd(16261, 85652) \Rightarrow \gcd(85652, 16261)$.

(1)

$$85652/16261 = 5 = q_1$$
$$85652 - (16261 \times 5) = 4347 = r_2$$

(2)

$$16261/4347 = 3 = q_2$$
$$16261 - (4347 \times 3) = 3220 = r_3$$

(3)

$$4347/3220 = 1 = q_3$$
$$4347 - (3220 \times 1) = 1127 = r_4$$

(4)

$$3220/1127 = 2 = q_4$$
$$3220 - (1127 \times 2) = 966 = r_5$$

(5)

$$1127/966 = 1 = q_5$$
$$1127 - (966 \times 1) = 161 = r_6$$

(6)

$$966/161 = 6 = q_6$$
$$966 - (161 \times 6) = 0 = r_7$$

We have found that $\gcd(85652, 16261) = 161$.

Exercise 1.10

For each of the $\gcd(a, b)$ values in Exercise 1.9, use the extended Euclidean algorithm (Theorem 1.11) to find integers u and v such that $au + bv = \gcd(a, b)$.

Solution.

(a) We need to solve for the various u_i and v_i. We will start at $i = 2$.

i	u_i	*Formula*	*Evaluation*	v_i	*Formula*	*Evaluation*
2	1	$u_0 - q_1 \times u_1$	$1 - 1 \times 0$	-1	$v_0 - q_1 \times v_1$	$0 - 1 \times 1$
3	-6	$u_1 - q_2 \times u_2$	$0 - 6 \times 1$	7	$v_1 - q_2 \times v_2$	$1 - 6 \times (-1)$
4	13	$u_2 - q_3 \times u_3$	$1 - 2 \times (-6)$	-15	$v_2 - q_3 \times v_3$	$-1 - 2 \times 7$

Thus, we can now fill out the table in full:

i	r_i	q_i	r_{i+1}	u_i	v_i
0	291	–	–	1	0
1	252	1	39	0	1
2	39	6	18	1	−1
3	18	2	3	−6	7
4	3	6	0	13	−15

Now, we need to solve: $au + bv = \gcd(a, b) \Rightarrow 291(13) + 252(-15) = 3$. 3 matches the gcd that we found in Exercise 1.9, so this is the correct solution.

(b)

i	r_i	q_i	r_{i+1}	u_i	v_i
0	85652	–	–	1	0
1	16261	5	4347	0	1
2	4347	3	3220	1	−5
3	3220	1	1127	−3	16
4	1127	2	966	4	−21
5	966	1	161	−11	58
6	161	6	0	15	−79

$85652(15) + 16261(-79) = 161$.

Exercise 1.11

Let a and b be positive integers.

(a) Suppose that there are integers u and v satisfying $au + bv = 1$. Prove that $\gcd(a, b) = 1$.

Proof. (a) Suppose there are integers a, b, u, v such that $av + bv = 1$. Assume $d \in \mathbb{Z}$ such that $d = \gcd(a, b)$. Since d divides both a and b by definition of common divisor, it must also divide av and bv by definition of divisibility. Moreover, because $au + bv = 1$ and d is a common divisor of both av and bv, it must also divide 1 by Proposition 1.4 (c). Then, the only positive integer that divides 1 is 1 itself, so it must be the case that $d = 1$. Therefore, since $d = 1$ and $\gcd(a, b) = d$, it follows that $\gcd(a, b) = 1$. □

Exercise 1.14

Let $m \geq 1$ be an integer and suppose that

$$a_1 \equiv a_2 \pmod{m} \text{ and } b_1 \equiv b_2 \pmod{m}.$$

Prove that

$$a_1 \pm b_1 \equiv a_2 \pm b_2 \pmod{m} \text{ and } a_1 \cdot b_1 \equiv a_2 \cdot b_2 \pmod{m}.$$

(This is Proposition 1.13(a).)

Proof. Let $m \geq 1$ be an integer and suppose that $a_1 \equiv a_2 \pmod{m}$ and $b_1 \equiv b_2 \pmod{m}$. From the definition of modulo, we know the difference of $a_1 - a_2$ and $b_1 - b_2$ is divisible by m.

- **Addition:** We want to show that $a_1 + b_1 \equiv a_2 + b_2$. So, our goal is to achieve $m \mid ((a_1+b_1)-(a_2+b_2))$. Thus, consider $(a_1+b_1)-(a_2+b_2)$. We can distribute the minus sign to get $(a_1 - a_2) + (b_1 - b_2)$. From Proposition 1.4 (c), because we know that $m \mid (a_1 - a_2)$ and $m \mid (b_1 - b_2)$, we can write this as $m \mid ((a_1 - a_2) + (b_1 - b_2))$ which implies $m \mid ((a_1 + b_1) - (a_2 + b_2))$. This shows $a_1 + b_1 \equiv a_2 + b_2 \pmod{m}$.
- **Subtraction:** Similarly to addition, we want to show $a_1 - b_1 \equiv a_2 - b_2 \pmod{m}$. Thus, consider $(a_1 - b_1) - (a_2 - b_2)$ which implies $(a_1 - a_2) - (b_1 - b_2)$. From Proposition 1.4 (c), because we know that $m \mid (a_1 - a_2)$ and $m \mid (b_1 - b_2)$, we can write this as $m \mid ((a_1 - a_2) - (b_1 - b_2))$ which implies $m \mid ((a_1 - b_1) - (a_2 - b_2))$, so $a_1 - b_1 \equiv a_2 - b_2 \pmod{m}$.

Therefore we have shown $a_1 \pm b_1 \equiv a_2 \pm b_2 \pmod{m}$

- **Product** We want to show that $a_1 \cdot a_2 \equiv a_2 \cdot b_2 \pmod{m}$. Thus, consider $a_1 \cdot b_1 - a_2 \cdot b_2$:

$$\begin{aligned} a_1 \cdot b_1 - a_2 \cdot b_2 &= a_1 \cdot b_1 - a_1 \cdot b_2 + a_1 \cdot b_2 - a_2 \cdot b_2 \\ &= a_1 \cdot (b_1 - b_2) + b_2 \cdot (a_1 - a_2) \end{aligned}$$

  Because $m \mid (b_1 - b_2)$ and $m \mid (a_1 - a_2)$, we know from the definition of division that when we multiply those numbers by an integer like a_1 and b_2, m still divides the expression. Hence, $m \mid (a_1 \cdot (b_1 - b_2))$ and $m \mid (b_2 \cdot (a_1 - a_2))$. Therefore, $m \mid (a_1 \cdot b_1 - a_2 \cdot b_2)$ and $a_1 \cdot b_1 \equiv a_2 \cdot b_2 \pmod{m}$. □

Exercise 1.16

Do the following modular computations. In each case, fill in the box with an integer between 0 and $m-1$, where m is the modulus.

(a) $347 + 513 \equiv \boxed{} \pmod{763}$.

(c) $153 \cdot 287 \equiv \boxed{} \pmod{353}$

(e) $5327 \cdot 6135 \cdot 7139 \cdot 2187 \cdot 5219 \cdot 1873 \equiv \boxed{} \pmod{8157}$ (*Hint:* After each multiplication, reduce modulo 8157 before doing the next multiplication.)

(g) $373^6 \equiv \boxed{} \pmod{581}$.

Solution.

(a) $347 + 513 \pmod{763} = 97$

(c) $153 \cdot 287 \pmod{353} = 139$

(e)

$$\begin{aligned} 5327 \cdot 6135 \pmod{8157} &= 4203 \\ 4203 \cdot 7139 \pmod{8157} &= 3771 \\ 3771 \cdot 2187 \pmod{8157} &= 450 \\ 450 \cdot 5219 \pmod{8157} &= 7491 \\ 7491 \cdot 1873 \pmod{8157} &= \boxed{603} \end{aligned}$$

(g) $373^6 \pmod{581} = 463$

Exercise 1.17

Find all values of x between 0 and $m-1$ that are solutions of the following congruences. (*Hint:* If you can't figure out a clever way to find the solution(s), you can just substitute each value $x = 1$, $x = 2, \ldots$, $x = m - 1$ and see which ones work.)

(a) $x + 17 \equiv 23 \pmod{37}$.

(c) $x^2 \equiv 3 \pmod{11}$

(g) $x \equiv 1 \pmod{5}$ and also, $x \equiv 2 \pmod{7}$. (Find all solutions modulo 35, that is, find the solutions satisfying $0 \leq x \leq 34$.

Solution.

(a) $x = 6$

(c) We know that x cannot be any number who's square does not exceed 11 because we

cannot square a number to get 3 (other than $\sqrt{3}$, but these are only the integers, so we cannot use that). Hence, $x \neq 1, 2, 3$ because we know that the results of these, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$ are not equivalent to 3. Let's try some more values: $x = 4$: $4^2 = 16 \pmod{11} = 5 \not\equiv 3$; $x = 5$: $5^2 = 25 \pmod{11} = 3 \equiv 3$; $x = 6$: $6^2 = 36 \pmod{11} = 3 \equiv 3$; $x = 7$: $7^2 = 49 \pmod{11} = 5 \not\equiv 3$; $x = 8$: $8^2 = 64 \pmod{11} = 9 \not\equiv 3$; $x = 9$: $9^2 = 81 \pmod{11} = 4 \not\equiv 3$; $x = 10$: $10^2 = 100 \pmod{11} = 1 \not\equiv 3$.

Thus, we have it that $x = 5, 6$.

(g) Because we have $x \equiv 1 \pmod{5}$, verifying correct x's are straightforward. All we need to check for is if a multiple of $5 + 1$ satisfies $x \equiv 2 \pmod{7}$. Thus, the only solution is $x = 16$

Exercise 1.19

Prove that if a_1 and a_2 are units modulo m, then $a_1 a_2$ is a unit modulo m.

Proof. Suppose a_1 and a_2 are units modulo m. This means $a \in \mathbb{Z}/m\mathbb{Z}$: $gcd(a, m) = 1$. In other words, $a_1 b_1 \equiv 1 \pmod{m}$ and $a_2 b_2 \equiv 1 \pmod{m}$ for some $b_1, b_2 \in \mathbb{Z}$. When we multiply the equations together, we get $(a_1 b_1)(a_2 b_2) \equiv 1 \pmod{m}$ which can be rewritten as $(a_1 a_2)(b_1 b_2) \equiv 1 \pmod{m}$. We can multiply b_1 and b_2 to get an integer b_3. Thus, when we multiply $a_1 a_2$ by b_3 and get 1, we have shown that b_3 is a multiplicative inverse, and $a_1 a_2$ is a unit modulo m. □

Exercise (Additional)

Decide whether each of the following is a group:

(a) `All 2x2 matrices with real number entries` with operation matrix addition

(b) `All 2x2 matrices with real number entries` with operation matrix multiplication

Solution.

(a) **Matrix Addition: ✓**

(1) **Closure:** For addition to work between matrices, they must be of dimension $2 \times 2 + 2 \times 2$. Therefore, the dimensions do not change, and it is closed.

(2) **Associativity:** 2×2 matrix addition is associative, as it inherits this property from the properties of matrices.

(3) **Identity Element:** We can add a matrix Z that consists of only 0s to a matrix A, and matrix A will remain unchanged.

(4) **Inverse Element:** True. Consider the matrices, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$.

When we add these two together we get $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This shows that 2×2 matrices have additive inverses.

(b) **Matrix Multiplication: ✗**

(1) **Closure:** The dimensions will stay the same during multiplication because it is an $n \times n$ matrix.

(2) **Associativity:** 2×2 matrix multiplication is associative, as it inherits this property from the properties of matrices.

(3) **Identity Element:** True. Consider the identity matrix, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. When we multiply a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by I, we get

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

(4) **Inverse Element:** False. Matrices with a non-zero determinant fail this criteria. Consider the matrix $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. The determinant would be $\det((1)(4) - (2)(3) = -2$. Therefore, this matrix would not have an inverse.

Exercise (Additional)

Is `All 2x2 matrices with real number entries` a ring with operations matrix addition and matrix multiplication? Justify your answer.

Solution. ✓

(1) **Additive Closure:** True. (See the previous exercise (a), (1)).

(2) **Additive Associativity:** True. Inherited from the properties of matrices.

(3) **Additive Identity:** True. (See the previous exercise (a), (3)).

(4) **Additive Inverse:** True. You can take the difference between a matrix and its inverted duplicate (e.g., $-[A]$) and get 0.

(5) **Multiplicative Closure:** True. (See the previous exercise (b), (4)).

(6) **Distributive Property:** True. While this will pose an error on a calculator, you can do the equivalent: $[A]([B] + [C]) = [A][B] + [A][C]$. This is true because you are still taking the summation of each a_{ij}, b_{ij} and c_{ij}.