2. (2 points) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^4}$ does not exist.

Solution.

- $x = 0$ path: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{0 \cdot y^2}{0 + y^4} = \frac{0}{y^2} = 0.$ • $y = 0$ path: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x \cdot 0}{x^2 + 0} = \frac{0}{x^2} = 0.$

- $x = y^2$ path: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2 \cdot y^2}{y^4 + y^4} = \frac{y^4}{2y^4} = \frac{1}{2}.$

Since the limit is not the same along all paths, the limit does not exist.

3. (2 points each) Find each indicated partial derivative:

(a) $\frac{\partial}{\partial x}\left(xy^2 \cos(x + y^3) - e^{xy}\right)$

Solution.

$$\frac{\partial}{\partial x}\left(xy^2 \cos(x + y^3) - e^{xy}\right) = y^2 \frac{\partial}{\partial x}\left[x \cos(x + y^3)\right] - \frac{\partial}{\partial x}\left[e^{xy}\right]$$
$$= y^2\left(\cos(x + y^3) + x\left(-\sin(x + y^3)\right)\right) - ye^{xy}$$
$$= y^2\left(\cos(x + y^3) - x\sin(x + y^3)\right) - ye^{xy}.$$

(b) $\frac{\partial}{\partial y}\left(\ln(x + y + z) - y^2 z^3 + x\right)$

Solution.

$$\frac{\partial}{\partial y}\left(\ln(x + y + z) - y^2 z^3 + x\right) = \frac{\partial}{\partial y}\left[\ln(x + y + z)\right] - z^3 \frac{\partial}{\partial y}\left[y^2\right] + \frac{\partial}{\partial y}\left[x\right]$$
$$= \frac{1}{x+y+z} - 2yz^3.$$

(c) $\frac{\partial^2}{\partial x \partial y}\left(x^3 y - y^3 \tan(xy)\right)$

Solution.

$$\frac{\partial^2}{\partial x \partial y}\left(x^3 y - y^3 \tan(xy)\right) = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}[x^3 y] - \frac{\partial}{\partial y}[y^3 \tan(xy)]\right]$$
$$= \frac{\partial}{\partial x}\left[x^3 - (3y^2 \tan(xy) + xy^3 \sec^2(xy))\right]$$
$$= \frac{\partial}{\partial x}[x^3] - \frac{\partial}{\partial x}[3y^2 \tan(xy)] - \frac{\partial}{\partial x}[xy^3 \sec^2(xy)].$$

Splitting this into 3 partial derivatives:

$$\frac{\partial}{\partial x}[x^3] = 3x^2, \quad -\frac{\partial}{\partial x}[3y^2 \tan(xy)] = -3y^3 \sec^2(xy),$$

with the final derivative worked out:

$$-\frac{\partial}{\partial x}[xy^3 \sec^2(xy)] = y^3 \sec^2(xy) + \left[(xy^3) \cdot 2y \sec^2(xy) \tan(xy)\right]$$
$$= -y^3 \sec^2(xy) - 2xy^4 \sec^2(xy) \tan(xy).$$

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Combining these results, we have:

$$3x^2 - 3y^3 \sec^2(xy) - y^3 \sec^2(xy) - 2xy^4 \sec^2(xy)\tan(xy).$$

Since three terms contain a factor of $y^3 \sec^2(xy)$, we can factor this out to get:

$$3x^2 - y^3 \sec^2(xy)\big(3 + 1 + 2xy\tan(xy)\big).$$

Adding and simplifying further, we get:

$$\boxed{3x^2 - 2y^3 \sec^2(xy)\big(2 + xy\tan(xy)\big).}$$

4. (3 points) Complete each of the following steps to prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2 y}{x^2 + y^2} = 0$.

Let $\epsilon > 0$. Choose $\delta = \epsilon/3$. Suppose that (x, y) is chosen so that $\|(x, y) - (0, 0)\| < \delta$ and $(x, y) \neq (0, 0)$.

(a) Explain why $\sqrt{x^2 + y^2} < \delta$.

Solution. Since we have that $\|(x, y) - (0, 0)\| < \delta$, when we find the magnitude of this difference, we get:

$$\boxed{\sqrt{x^2 + y^2} < \delta.}$$

(b) Explain why $x^2 \leq x^2 + y^2$, and thus $\dfrac{x^2}{(x^2 + y^2)} \leq 1$.

Solution. Because y^2 will always be positive and $x^2 = x^2$, it must be the case that $x^2 \leq x^2 + y^2$. Hence, when we divide both sides by $x^2 + y^2$, we get:

$$\boxed{\frac{x^2}{(x^2 + y^2)} \leq 1.}$$

(c) Explain why $\dfrac{3x^2}{(x^2 + y^2)} \leq 3$.

Solution. Since $\dfrac{x^2}{(x^2 + y^2)} \leq 1$, multiplying both sides by 3 gives us:

$$\boxed{\frac{3x^2}{(x^2 + y^2)} \leq 3.}$$

(d) Explain why $\dfrac{3x^2 |y|}{(x^2 + y^2)} \leq 3|y|$.

Solution. Similarly to the previous step, we know that since $|y| \geq 0$, so when we multiply both sides of the inequality by $|y|$, the inequality is unchanged.

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(e) Now, show that $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right| \le 3\sqrt{x^2+y^2}$.

Solution. First, note that $|y| \le \sqrt{x^2+y^2}$ (since $y^2 \le x^2+y^2$ by the same logic in (b)). Then, recall that $\sqrt{x^2+y^2} < \delta$. We can multiply both sides by 3 to get $3\sqrt{x^2+y^2} < 3\delta$. Since $\delta = \epsilon/3$, when we substitute this δ for ϵ in our equation, we get:

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| \le 3|y| \le 3\sqrt{x^2+y^2} < 3\delta = 3 \cdot \tfrac{\epsilon}{3} = \epsilon.$$

(f) Conclude that whenever (x,y) is in the δ-disk centered at $(0,0)$, then $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right| < \epsilon$.

Solution. Combining the previous steps, we have shown that when (x,y) is in the δ-disk centered at $(0,0)$ (i.e., $\|(x,y)-(0,0)\|$), we have:
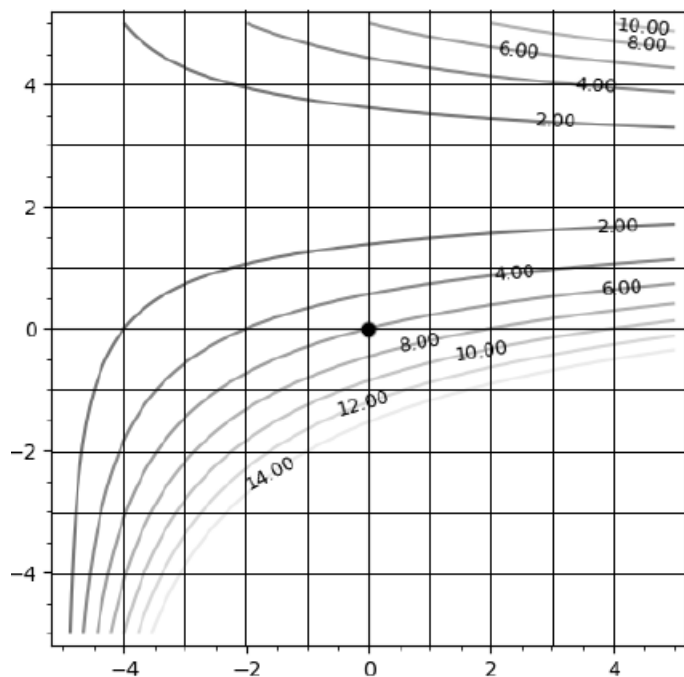
$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \epsilon.$$

This shows that for every $\epsilon > 0$, we can choose a $\delta = \tfrac{\epsilon}{3}$ so that whenever $\|(x,y)-(0,0)\| < \delta$, the inequality holds. Hence, this proves that:

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$$

□

5. (1 point each) Consider the contour plot of the function $f(x,y)$ shown below.

Determine the sign $(+, -,$ or $0)$ of each of the following partial derivatives, including a *brief* justification.

(a) $f_x(0,0)$

> *Solution.* We see that $f(0,0) = 6$, and when we move in the x-direction, the function increases slightly. Thus, $f_x(0,0)$ is positive.

(b) $f_y(0,0)$

> *Solution.* If we move in the y-direction, the function decreases slightly towards 4. Thus, $f_y(0,0)$ is negative.

(c) $f_{xx}(0,0)$

> *Solution.* We can see in the plot that as we continue in the x-direction, the lines get closer together, indicating that the function is increasing. Thus, $f_{xx}(0,0)$ is positive.

(d) $f_{yy}(0,0)$
> *Solution.* As we move in the y-direction, the lines get further apart, indicating that the function is decreasing. Hence, $f_{yy}(0,0)$ is negative.

(e) $f_{xy}(0,0)$
> *Solution.* The function increasing in the x-direction, but when we start to move up, we can see that the lines begin to get further apart from each other. Therefore, $f_{xy}(0,0)$ is negative.

6. (2 points) Find an equation of the tangent plane to $f(x,y) = x^2y - \sqrt{x} + y$ at the point $(3,1)$.

> *Solution.* To find the equation of the tangent plane, we need to fill out the following equation:
>
> $$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$
>
> First, we find z_0 by plugging in the point $(3,1)$ into the function f:
>
> $$z_0 = f(3,1) = 3^2 \cdot 1 - \sqrt{3} + 1 = 9 - \sqrt{3} + 1 = 10 - \sqrt{3}.$$
>
> Thus, we must find the partial derivatives of f:
>
> $$f_x(x,y) = 2xy - \tfrac{1}{2}x^{-1/2}, \quad f_y(x,y) = x^2 + 1.$$
>
> Plugging in the point $(3,1)$ into these partial derivatives, we get:
>
> $$f_x(3,1) = \tfrac{12\sqrt{3}-1}{2\sqrt{3}}, \quad f_y(3,1) = 10.$$
>
> Finally, we can plug in these values into the equation of the tangent plane to get:
>
> $$\boxed{z = 10 - \sqrt{3} + \tfrac{12\sqrt{3}-1}{2\sqrt{3}}(x-3) + 10(y-1).}$$

7. (2 points) For the function $f(x, y, z) = \dfrac{x + \sin(xy)}{x^2 + y^2 + z^2 + 1}$, find $\nabla f(x, y, z)$.

Solution. From the handout, we know that:

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}(x, y, z)\mathbf{i} + \frac{\partial f}{\partial y}(x, y, z)\mathbf{j} + \frac{\partial f}{\partial z}(x, y, z)\mathbf{k}.$$

Thus, we must find the partial derivatives of f:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\frac{x + \sin(xy)}{x^2 + y^2 + z^2 + 1}\right]$$
$$= \frac{(x^2 + y^2 + z^2 + 1)(1 + y\cos(xy)) - 2x(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\frac{x + \sin(xy)}{x^2 + y^2 + z^2 + 1}\right]$$
$$= \frac{(x^2 + y^2 + z^2 + 1)(x\cos(xy)) - 2y(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2}$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[\frac{x + \sin(xy)}{x^2 + y^2 + z^2 + 1}\right]$$
$$= -\frac{2z(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2}$$

Therefore, the gradient of f is:

$$\nabla f(x, y, z) = \left\langle \frac{(x^2 + y^2 + z^2 + 1)(1 + y\cos(xy)) - 2x(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2}, \right.$$
$$\left. \frac{(x^2 + y^2 + z^2 + 1)(x\cos(xy)) - 2y(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2}, \frac{-2z(x + \sin(xy))}{(x^2 + y^2 + z^2 + 1)^2} \right\rangle.$$

8. (2 points) Consider the function $f(x, y) = x^2 y - y^3$. Find the directional derivative for f, at $(3, 4)$, in the direction of $\mathbf{u} = 5\mathbf{i} - 2\mathbf{j}$.

Solution. We first find the partial derivative of f with respect to x and y:

$$f_x(x, y) = 2xy, \quad f_y(x, y) = x^2 - 3y^2.$$

Then, at the point $(3, 4)$:

$$f_x(3, 4) = 2 \cdot 3 \cdot 4 = 24, \quad f_y(3, 4) = 3^2 - 3 \cdot 4^2 = -39.$$

Then, we find the unit vector in the direction of $\langle 5, -2 \rangle$:

$$\boxed{\frac{\langle 24, -39 \rangle \cdot \langle 5, -2 \rangle}{\sqrt{29}} = \frac{198}{\sqrt{29}}.}$$