Homework 2: Sections 3 & 4

Algebra

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Due

September 11, 2025

Section 3

6. $\langle \mathbb{Q}, \cdot \rangle$ with $\langle \mathbb{Q}, \cdot \rangle$ where $\phi(x) = x^2$ for $x \in \mathbb{Q}$.

Solution. The binary operation ϕ is not an isomorphism because it is not onto. A counterexample is that ϕ is not one-to-one because $\phi(1) = 1$ and $\phi(-1) = 1$.

7. $\langle \mathbb{R}, \cdot \rangle$ with $\langle \mathbb{R}, \cdot \rangle$ where $\phi(x) = x^3$ for $x \in \mathbb{R}$.

Solution. The binary operation ϕ is an isomorphism because it has the following properties:

- **One-to-one:** If $\phi(x) = \phi(y)$, then $x^3 = y^3$. Taking the cube root of both sides, we have $x = y$. Thus, ϕ is one-to-one.
- **Onto:** For any $z \in \mathbb{R}$, we can find an $x \in \mathbb{R}$ such that $\phi(x) = z$. Specifically, we can choose $x = \sqrt[3]{z}$. Therefore, ϕ is onto.
- **Homomorphism:** For any $x, y \in \mathbb{R}$, we have

$$\phi(x \cdot y) = (x \cdot y)^3 = x^3 \cdot y^3 = \phi(x) \cdot \phi(y).$$

  Therefore, ϕ preserves the binary operation.

11. $\langle F, + \rangle$ with $\langle F, + \rangle$ where $\phi(f) = f'$, the derivative of f.

Solution. The binary operation ϕ is not an isomorphism because it is not one-to-one. A counterexample is that $\phi(f) = \phi(g)$ for $f(x) = x^2$ and $g(x) = x^2 + 1$, but $f \neq g$.

12. $\langle F, + \rangle$ with $\langle \mathbb{R}, + \rangle$ where $\phi(f) = f'(0)$.

Solution. Similarly to the previous problem, the binary operation ϕ is not an isomorphism because it is not one-to-one. A counterexample is that $\phi(f) = \phi(g)$ for $f(x) = \frac{x}{2}$ and $g(x) = x^2 + \frac{x}{2}$, but $f \neq g$.

Section 4

3. Let $*$ be defined on $\mathbb{R}^+$ by letting $a * b = \sqrt{ab}$.

Solution. The binary operation $*$ does not give a group structure on $\mathbb{R}^+$ because it fails $\mathcal{G}_1$. A counterexample is that $(1 * 2) * 3 = \sqrt{2} * 3 = \sqrt{3\sqrt{2}}$, but $1 * (2 * 3) = 1 * \sqrt{6} = \sqrt{\sqrt{6}}$. Since $\sqrt{3\sqrt{2}} \neq \sqrt{\sqrt{6}}$, the operation is not associative.

12. All $n \times n$ diagonal matrices under matrix multiplication.

Solution. The set of all $n \times n$ diagonal matrices under matrix multiplication does not form a group because it fails $\mathcal{G}_3$. A counterexample is the diagonal matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which is not invertible since $\det(A) = 0$.

19. Let S be the set of all real numbers except -1. Define $*$ on S by $a * b = a + b + ab$.

a. Show that $*$ gives a binary operation on S.

Solution. Let $a, b \in S$. Assume, for the sake of contradiction, that $a * b = -1$. Then,

$$\begin{aligned} a * b &= -1 \\ a + b + ab &= -1 \\ ab + a + b + 1 &= 0 \\ (a+1)(b+1) &= 0. \end{aligned}$$

Notice this equation is only true if $a = -1$ and $b = -1$, which contradicts our assumption that $a, b \in S$. Thus, $a * b \neq -1$ and so $a * b \in S$. Therefore, $*$ is a binary operation on S.

b. Show that $(S, *)$ is a group.

Solution. We will verify that $(S, *)$ satisfies the group axioms:

- $\mathcal{G}_1$ (Associativity): Let $a, b, c \in S$. Then,

$$\begin{aligned} (a * b) * c &= (a + b + ab) * c \\ &= (a + b + ab) + c + (a + b + ab)c \\ &= a + b + ab + c + ac + bc + abc. \end{aligned}$$

Similarly,

$$\begin{aligned} a * (b * c) &= a * (b + c + bc) \\ &= a + (b + c + bc) + a(b + c + bc) \\ &= a + b + c + bc + ab + ac + abc. \end{aligned}$$

Since both expressions are equal, we have $(a*b)*c = a*(b*c)$, confirming associativity.

- $\mathcal{G}_2$ (Identity Element): We need to find an element $e \in S$ such that for all $a \in S$, $a * e = e * a = a$. Let $e = 0$. Then,

$$a * 0 = a + 0 + a(0) = a,$$

and

$$0 * a = 0 + a + 0(a) = a.$$

Since $0 \in S$, it serves as the identity element.

- $\mathcal{G}_3$ (Inverse Element): For each $a \in S$, we need to find an element $a' \in S$ such that $a * a' = a' * a = e$, where e is the identity element found above. We want to solve for a' in the equation:

$$a * a' = 0$$

This gives us:

$$a + a' + aa' = 0.$$

Rearranging, we have:

$$aa' + a' = -a,$$

or

$$a'(a + 1) = -a.$$

Since $a \in S$, we have $a + 1 \neq 0$, and thus we can divide by $a + 1$ to find:

$$a' = \frac{-a}{a+1}.$$

With candidate a' identified, we can plug in to verify:

$$a * a' = a + \frac{-a}{a+1} + a\left(\frac{-a}{a+1}\right) = 0,$$

and
$$a' * a = \frac{-a}{a+1} + a + \left(\frac{-a}{a+1}\right) a = 0.$$
Since $a' \neq -1$, we have $a' \in S$. Therefore, every element in S has an inverse in S.

c. Find the solution of the equation $2 * x * 3 = 7$ in S.

Solution. Since we are working with an expression in S, we have to be mindful of the order of operations. Since we don't have a nice way to "decouple" x from its coefficient, we'll have to work from the outside in. First, we must use the inverses (and Theorem 4.15) to get rid of the 2 and 3 from the left side:
$$2 * (x * 3) = 7 \quad \Rightarrow \quad 2' * (2 * x * 3) * 3' = 2' * 7 * 3'.$$
Using the associative property, we can rewrite the left side as:
$$(2' * 2) * x * (3 * 3') = 2' * 7 * 3'.$$
Since $2' * 2 = 0$ and $3 * 3' = 0$:
$$0 * x * 0 = 2' * 7 * 3'.$$
The identity property tells us that $0 * x = x$ and $x * 0 = x$, so:
$$x = 2' * 7 * 3'.$$
Using the formula for inverses found in part (b), we have:
$$2' = \frac{-2}{2+1} = -\frac{2}{3}, \quad 3' = \frac{-3}{3+1} = -\frac{3}{4}.$$
Substituting these values back into our equation for x, we have:
$$x = \left(-\frac{2}{3}\right) * 7 * \left(-\frac{3}{4}\right).$$
Thus, we can expand the right side:
$$\begin{aligned} x &= \left[\left(-\frac{2}{3}\right) + 7 + \left(-\frac{2}{3}\right)(7)\right] * \left(-\frac{3}{4}\right) \\ &= \frac{5}{3} * \left(-\frac{3}{4}\right) \\ &= \frac{5}{3} + \left(-\frac{3}{4}\right) + \left(\frac{5}{3}\right)\left(-\frac{3}{4}\right) \\ &= -\frac{1}{3}. \end{aligned}$$

41. Let G be a group and let g be one fixed element of G. Show that the map i_g, such that $i_g(x) = gxg^{-1}$ for $x \in G$, is an isomorphism of G with itself.

Solution. We will verify that i_g satisfies the properties of an isomorphism:

- **One-to-one:** Assume $i_g(x) = i_g(y)$ for some $x, y \in G$. Then,

$$gxg^{-1} = gyg^{-1}.$$

Utilizing Theorem 4.15, we can multiply both sides on the left by g^{-1} and on the right by g, revealing:

$$g^{-1}(gxg^{-1})g = g^{-1}(gyg^{-1})g,$$

which we can rearrange using the associative property to get:

$$(g^{-1}g)x(g^{-1}g) = (g^{-1}g)y(g^{-1}g).$$

Since $g^{-1}g = e$, the identity element of G, we have:

$$x = y.$$

Thus, i_g is one-to-one.

- **Onto:** Let $y \in G$. We need to find an $x \in G$ such that $i_g(x) = y$. Consider the element $x = g^{-1}yg$. Since $g, y \in G$, it follows that $g^{-1} \in G$, and by the closure property of the group, $x = g^{-1}yg \in G$. Now we can verify that this choice of x maps to y:

$$i_g(x) = i_g(g^{-1}yg) = g(g^{-1}yg)g^{-1} = (gg^{-1})y(gg^{-1}) = y.$$

Therefore, i_g is onto.

- **Homomorphism:** Let $x, y \in G$. Then,

$$i_g(x)i_g(y) = gxg^{-1}gyg^{-1} = gx(g^{-1}g)yg^{-1} = gxyg^{-1} = i_g(xy).$$

Therefore, i_g preserves the group operation.

Since i_g is a bijection and a homomorphism, it is an isomorphism of G with itself.