HENDRIX COLLEGE

Real Analysis

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CHAPTER 1

THE REAL NUMBERS

1.1 Types of Numbers

Definition 1.1.1

The *natural numbers* contain all positive, non-zero, and non-fractional numbers. Expressed as $\mathbb{N} = \{1, 2, 3, 4 \ldots\}$. They do not have an additive inverse, but you can add and multiply them.

Definition 1.1.2

The *integers* contains all non-fractional numbers. Expressed as: $\mathbb{Z} = \{\cdots -2, -1, 0, 1, 2 \ldots\}$—are known as a Group (more specifically, a "ring"). You can add, multiply, and subtract these numbers.

Definition 1.1.3

The *rational numbers* contain all numbers, except irrational numbers. Expressed as: $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$—are known as a "Field." You can add, subtract, multiply, and divide these numbers.

A problem that rational numbers could not explain: The $45, 45, 90$ triangle had a hypotenuse of $\sqrt{2}$. This did not exist at the time, so it was simply $c^2 = 2$. Therefore, new numbers needed to be invented.

Theorem 1.1.4

There does not exist a rational number r such that $r^2 = 2$.

Proof. Suppose there exists a rational number r such that $r^2 = 2$. Since r is rational, there exists $p, q \in \mathbb{Z}$ such that $r = \frac{p}{q}$. We can assume the p and q have no common factors. (If not, we can factor out the common factor.) By our assumption,

$$r^2 = 2$$
$$\frac{p^2}{q^2} = 2$$

It follows that,

$$p^2 = 2q^2$$

Such that p^2 is an even number because if p were odd, then p^2 would be odd. There exists $x \in \mathbb{Z}$ such that $p = 2x$. Recall that $p^2 = 2q^2$. Thus

$$\begin{aligned} (2x)^2 &= 2q^2 \\ 4x^2 &= 2q^2 \\ 2x^2 &= q \end{aligned}$$

Thus, q^2 is even. Hence q is also even. So p and q are both divisible by 2. This contradicts that p and q have no common factors. Thus, our supposition is false. Therefore, there does not exist a rational number r such that $r^2 = 2$ □

So we are going to work with a larger set called the real numbers, $\mathbb{R}$.

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- You can:
  - Add,
  - Subtract,
  - Multiply,
  - Divide
- In other words, all field axioms apply.
- Totally ordered set for any $x, y \in \mathbb{R}$. Thus, one of these are true:
  1. $x < y$,
  2. $x > y$,
  3. $x = y$
- Think of it as a number line.
- $\mathbb{Q}$ is dense:
  If $a, b \in \mathbb{Q}$ with $a \neq b$, there exists $c \in \mathbb{Q}$ which is between a and b such that $a < c < b$. One example is $\frac{a+b}{2}$.
- $\mathbb{Q}$ is not *complete*, but $\mathbb{R}$ is.
  - *Complete:* Think, "no gaps."

1.2 Preliminaries

Things to remember from Intro and Discrete.

- $\displaystyle\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \ldots$

Set Notation	Complement
$x \in A$	A^c (not $\overline{A}$)
$A \cup B$	$\mathbb{R} \setminus A$
$A \cap B$	

- $\displaystyle\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \ldots.$

Definition 1.2.1

De Morgan's Laws are defined as $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$.

1.2.1 Infinite Unions and Intersections

For each $n \in \mathbb{N}$, define $A_n = \{n, n+1, n+2, \ldots\} = \{k \in \mathbb{N} \mid k \geq n\}$. In other words, each subsequent element in the subset will start at n. For example, $A_1 = \{1, 2, \ldots\}$, whereas $A_5 = \{5, 6, \ldots\}$.

$\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$. To show a number $\in \mathbb{N}$ belongs in the set A_n, we can start with that, $k \in \mathbb{N}$. Then $k \in A_k$. Thus, $k \in A_k \subseteq \bigcup_{n=1}^{\infty} A_n$. Therefore, $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$.

$\bigcap_{n=1}^{\infty} A_n = \emptyset$. Obviously, we know that the empty set is a subset of A_n, but to prove that $\bigcap_{n=1}^{\infty} A_n$ is a subset of the empty set, we should suppose a $k \in \mathbb{N}$ such that $k \in \bigcap_{n=1}^{\infty} A_n$. Notice that $k \notin \bigcap_{n=1}^{\infty} A_n$. So, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

1.2.2 Functions and Notation

$f\colon A \to B$ where f is a function, A is a domain, and B is the co-domain. Thus, $f(x) = y$ such that $x \in A$ and $y \in B$.

Some definitions to keep in mind

Definition 1.2.2

The *Dirichlet Function* is defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Definition 1.2.3

Let $f : \mathbb{R} \to \mathbb{R}$. If $E \subseteq \mathbb{R}$, then $f(E) = \{f(x) \mid x \in E\}$.

Example: $g : \mathbb{R} \to \mathbb{R}$, when we say $y \in g(A)$ implies there exists an x such that $g(x) = y$

Definition 1.2.4

The *Triangle Inequality* is defined as: For any $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.

The most common application: For any $a, b, c \in \mathbb{R}$, $|a - b| \leq |a - c| + |c - b|$, with the intermediate step of $a - b = (a - c) + (c - b)$.

Definition 1.2.5

A function f is *injective* (or *one-to-one*) if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$ in B. Note the contrapositive of this definition: If $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Definition 1.2.6

A function f is *surjective* (or *onto*) if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. Note the contrapositive of this definition: If there exists a $b \in B$ such that there is no $a \in A$ such that $f(a) = b$, then the function is not surjective.

1.2.3 Common Strategies for Analysis Proofs

Theorem 1.2.6

Let $a, b \in \mathbb{R}$. Then,

$$a = b \text{ if and only if for all } \epsilon > 0, |a - b| < \epsilon.$$

Proof. We will show this by proving both implications:

($\Rightarrow$) Assume $a = b$. Let $\epsilon > 0$. Then $|a - b| = 0 < \epsilon$

($\Leftarrow$) Assume for all $\epsilon > 0$, $|a - b| < \epsilon$. Suppose $a \neq b$. Then $a - b \neq 0$. So, $|a - b| \neq 0$. Now, Consider $\epsilon_0 = |a - b|$. By our assumption we know that $|a - b| < \epsilon_0$. It is not true that $|a - b| < |a - b|$. Therefore, it must be the case that $a = b$.

Therefore, by showing both sides of the implication accomplish the same thing as the other side, we know that $a = b$ if and only if for all $\epsilon > 0, |a - b| < \epsilon$. $\square$

1.2.4 Mathematical Induction

Inductive Hypothesis: Let $x_1 = 1$. For all $n \in \mathbb{N}$, let $x_{n+1} = \frac{1}{2}x_n + 1$.

Inductive Step: $x_1 = 1, x_2 = 1.5, x_3 = 1.75, x_4 = 1.875$.

Example 1.1: Induction

The sequence (x_n) is increasing. In other words, for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$.

Proof. Suppose the sequence (x_n) is increasing. We will prove this point by using induction.

Base Case: We see that $x_1 = 1$ and $x_2 = 1.5$. Thus, $x_1 \leq x_2$.

Inductive Hypothesis: For $n \in \mathbb{N}$, assume $x_n \leq x_{n+1}$.

Scratch work: We want: $x_{n+1} \leq x_{n+2}$. We know: $x_{n+1} = \frac{1}{2}x_{n+1} + 1$.

Inductive Step: Then $\frac{1}{2}x_n \leq \frac{1}{2}x_{n+1}$. Hence, $\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1$. Therefore we have proven through induction that, $x_{n+1} \leq x_{n+2}$. □

1.2.5 Exercises

Exercise: 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$

(d) $A \cap (B \cap C) = (A \cap B) \cap C$

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution.

(a) This is false. Consider the following as a counterexample: If we define A_1 as $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$, we can see why the intersection of these sets of infinite numbers are actually empty. Consider a number m that actually satisfies $m \in A_n$ for every A_n in our collection of sets. Because m is not an element of A_{m+1}, no such m exists and the intersection is empty.

(b) This is true.

(c) False. Consider sets $A = \{1, 2, 3\}$, $B = \{3, 6, 7\}$ and $C = \{5\}$. Note that $A \cap (B \cup C) = \{3\}$ is not equal to $(A \cap B) \cup C = \{3, 5\}$.

(d) This is true. A proof would start with $x \in A \cap (B \cap C)$.

(e) This is true. A proof would start with $x \in A \cap (B \cup C)$.

Exercise: 1.2.5

De Morgan's Laws Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution.

(a) If $x \in (A \cap B)^c$, and we know that $A^c = \{x \in \mathbb{R} : x \notin A\}$, then we know x must cannot exist in A^c and B^c because $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$. Thus, x is in either A^c or B^c. Put another way $x \in A^c \cup B^c$. Since we have shown that an element that started in $(A \cap B)^c$ ended up in $A^c \cup B^c$, then we know $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Assume there exists a $y \in A^c \cup B^c$. Thus, it must be the case that $y \notin A$ or $y \notin B$. Hence, y cannot be exist in both sets at the same time, so $y \in (A \cap B)^c$. Because we have taken an element that started in $A^c \cup B^c$ and have shown that it exists in $(A \cap B)^c$, we have proven $A^c \cup B^c \subseteq (A \cap B)^c$.

(c) *Proof.* We need to show these expressions are subsets of each other in order to prove they are equivalent.

($\subseteq$) Assume there exists $x \in (A \cap B)^c$, and we know that $A^c = \{x \in \mathbb{R} : x \notin A\}$, then we know x must cannot exist in A^c and B^c because $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$. Thus, x is in either A^c or B^c. Put another way $x \in A^c \cup B^c$. Since we have shown that an element that started in $(A \cap B)^c$ ended up in $A^c \cup B^c$, then we know $(A \cap B)^c \subseteq A^c \cup B^c$.

($\supseteq$) Now assume there exists a $y \in A^c \cup B^c$. Thus, it must be the case that $y \notin A$ or $y \notin B$. Hence, y cannot be exist in both sets at the same time, so $y \in (A \cap B)^c$. Because we have taken an element that started in $A^c \cup B^c$ and have shown that it exists in $(A \cap B)^c$, we have proven $A^c \cup B^c \subseteq (A \cap B)^c$.

Therefore, we have shown through proving both sides of the implication, that these two statements are logically equivalent. In that, all elements of $A^c \cup B^c$ are the same elements that are in $(A \cap B)^c$ $\square$

Exercise: 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(a) = \{f(x) : x \in A\}$.

(a) Let $f(x) = x^2$. If $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution.

(a) Since $f(x) = x^2$, the intervals of $f(A)$ would be $[0, 4]$ and $f(B)$ would be $[1, 16]$. The interval of the intersection of $A \cap B$ is $[1, 2]$. Take this through our function, we get $f(A \cap B) = [1, 4]$. On the other side of the equation, we already know the intervals of $f(A)$ and $f(B)$, and the intersection of theirs would be $[1, 4]$. So they do equal each other. We know $f(A \cup B)$ and $f(A) \cup f(B)$ will be equivalent because $f(A \cup B)$ has an interval of $[0, 16]$, and $f(A) \cup f(B)$ also has an interval of $[0, 16]$ because taking the union of $[0, 4] \cup [1, 16]$ is $[0, 16]$.

(b) Two sets could be $A = [5, 6]$ and $B = [0, 0]$. Because the sets have nothing in common even after taking their function, they do not equal each other.

(c) *Proof.* Let $x \in g(A \cap B)$. Using the definition of function, we know there exists a $y \in A \cap B$ to which that y is mapped to as $g(y) = x$. From the definition of intersection, we know $y \in A$ and $y \in B$ such that $x = g(y) \in g(A)$ and $x = g(y) \in g(B)$ because $y \in A \cap B$. Putting it together, we have $x \in g(A) \cap g(B)$ thus proving $g(A \cap B) \subseteq g(A) \cap g(B)$ $\square$

(d) Conjecture: For any function g defined as $g : \mathbb{R} \to \mathbb{R}$ and for any subsets $A, B \subseteq \mathbb{R}$, the following holds:

$$g(A \cup B) = g(A) \cup g(B)$$

Proof. We need to show these expressions are subsets of each other in order to prove they are equivalent.

($\subseteq$) Take any element $x \in g(A \cap B)$. By definition of function, we know there exists some $y \in A \cup B$ such that $g(x) = y$. From the definition of union, we know $y \in A$ or $y \in B$ such that $x = g(y) \in g(A)$ or $x = g(y) \in g(B)$ or both. Putting it together, we have $x \in g(A) \cup g(B)$ thus proving $g(A \cup B) \subseteq g(A) \cup g(B)$.

($\supseteq$) Take any element $p \in g(A) \cap g(B)$. By definition of union, we know p is either in $g(A)$ or $g(B)$ or both. From the definition of function, we know that if $p \in g(A)$ or $p \in g(B)$ then there exists some $q \in A$ or $q \in B$ such that $g(q) = p$. Putting it together, we have $q \in A \cup B$. Moreover, this means $p = g(x) \in g(A \cup B)$. And since $p \in g(A) \cup g(B)$ implies $p \in g(A \cup B)$, we know $g(A) \cup g(B) \subseteq g(A \cup B)$.

Therefore, since we have proven that both expressions are functions of each other, we have proved that they are equal. □

Exercise: 1.2.8

Given a function $f : A \to B$ can be defined as either injective or surjective, give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

Solution.

(a) The function $f(a) + 1$ is 1-1 because when

$$f(a_1) = f(a_2)$$
$$a_1 + 1 = a_2 + 1$$
$$a_1 = a_2$$

However, the function is not onto because the entire co-domain is not covered. That being 1.

(b) *We need to find a function that will cover every entry in the co-domain, while also* *avoiding a scenario where* $a_1 = a_2 \ldots$ Consider the function,

$$f(a) = \begin{cases} a & \text{if } a \text{ is odd,} \\ a-1 & \text{if } a \text{ is even} \end{cases}$$

This function is onto because every natural number is covered, but it is not 1-1 because $a_1 \neq a_2 - 1$.

(c) This request is not possible. There is no way to map every natural number to every integer because we are simply missing 0! (Not 0 factorial, we do have the number 1, I just mean the number 0 in a exclamatory sense.)

Exercise: 1.2.13

For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where

$$\bigcap_{i=1}^{n} B_i \neq \emptyset \quad \text{is true for every } n \in \mathbb{N},$$

but

$$\bigcap_{i=1}^{\infty} B_i = \emptyset$$

fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Proof. In this proof, we plan to prove (c). Thus, we need to show that:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$$

and

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

($\subseteq$) Let $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c$. This means x is in the union set of A_i for all $i \in \mathbb{N}$. Then, because we are taking the complement of $\left(\bigcup_{i=1}^{\infty} A_i\right)$, that means $x \notin A_i$ for all $i \in \mathbb{N}$. Hence, x is in the complement of each A_i. Thus, we can use the definition of intersection to assert $x \in \bigcap_{i=1}^{\infty} A_i^c$.
Therefore, we have shown:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c.$$

($\supseteq$) Similar to before, let $x \in \bigcap_{i=1}^{\infty} A_i^c$. Because $x \in A_i^c$ for all $i \in \mathbb{N}$ we know $x \notin A_i$. Hence, $x \notin \left(\bigcup_{i=1}^{\infty} A_i\right)$, which means $x \in \left(\bigcup_{i=1}^{\infty} A_i\right)^c$.
Therefore, we have shown:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

By showing both inclusions, we see that:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c.$$

□

1.3 Axiom of Completeness

Axiom of Completeness

Every nonempty set of real numbers that is bounded has a least upper bound.

Think about $\mathbb{Q}$ and $\mathbb{R}$.

- Both are fields.
  - Both have $+, -, \times, \div$ operations.
- Both are totally ordered

  - $a < b$,
  - $a > b$,
  - or $a = b$

- $\mathbb{R}$ is complete. $\mathbb{Q}$ is not.

1.3.1 Least Upper Bounds and Greatest Lower Bounds

Definition 1.3.1

A set $A \subseteq \mathbb{R}$ is *bounded above* if there exists a number $b \in \mathbb{R}$ such that $a \leq b$ for all $a \in A$. The number b is called an *upper bound* of A.
Similarly, a set $A \subseteq \mathbb{R}$ is *bounded below* if there exists a *lower bound* $l \in \mathbb{R}$ satisfying $l \leq a$ for every $a \in A$.

Note that upper bounds are not unique! For example, consider the line, A, from 0 to 1. There are infinitely many upper bounds past 1 because A is bounded.

Definition 1.3.2

A number s is a *least upper bound* for a set $A \subseteq \mathbb{R}$ if it meets the following two criteria:

(i) s is an upper bound for A;

(ii) if b is any upper bound for A, then $s \leq b$.

We often call the least upper bound the *supremum* of a set.

Example 1.2: Supremum

Imagine a number line from $(1, 8)$. Note that parenthesis mean $<$ and not $\leq$. Hence, the supremum is 8. Wrote simply as $\sup A$.

Example 1.3: Supremum and Infimum 1

Consider a set, $B = [-5, -2] \cup (3, 6) \cup \{13\}$. What is the supremum and the infimum?

Solution. $\sup B = 13$; $\inf B = -5$ because -5 is the greatest lower bound.

Example 1.4: Supremum and Infimum 2

Consider the set, $\mathbb{C} = \{\frac{1}{n} : n \in \mathbb{N}\}$. What is the supremum and the infimum?

Solution. $\sup \mathbb{C} = 1$, $\inf \mathbb{C} = 0$.

Example 1.5: L

et $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}$$

Then $\sup(c + A) = c + \sup A$.

Solution. To properly verify this we focus separately on each part of Definition 1.3.2. Setting $s = \sup A$, we see that $a \leq s$ for all $a \in A$, which implies $c + a \leq c + s$ for all $a \in A$. Thus, $c + s$ is an upper bound for $c + A$ and condition (i) is verified. For (ii), let b be an arbitrary upper bound for $c + A$; i.e., $c + a \leq b$ for all $a \in A$. This is equivalent to $a \leq b - c$ for all $a \in A$, from which we conclude that $b - c$ is an upper bound for A. Because s is the least upper bound of A, $s \leq b - c$, which can be rewritten as $c + s \leq b$. This verifies part (ii) of Definition 1.3.2, and we conclude $\sup(c + A) = c + \sup A$.

Definition 1.3.4

A real number a_0 is a *maximum* of the set A if a_0 is an element of A and $a_a \geq a$ for all $a \in A$. Similarly, a number a_1 is a *minimum* of A if a_1 is an element of A and $a_1 \leq a$ for all $a \in A$.

Note that some sets have a maximum and some sets do not. You cannot refer to a maximum without first knowing it exists. This is the same with minimums.

Lemma 1.3.5

Assume s is an upper bound for a set $A \subseteq \mathbb{R}$. Then, s is the supremum of A if and only if for every $\epsilon > 0$, there exists $x \in A$ such that $s - \epsilon < x$.

This lemma allows us to take any positive number and take a "step back." In essence, you can verify something as an upper bound if you continuously back up over and over until you cannot back up any longer.

Proof. We show this by proving both implications:

($\Rightarrow$) Assume $s = \sup A$. Let $\epsilon > 0$. Suppose there are no elements x of A such that $s - \epsilon < x$. Then $s - \epsilon$ would be an upper bound. This contradicts that s is the least upper bound. Therefore, there must exist an element $x \in A$ such that $s - \epsilon < x$.

($\Leftarrow$) Assume for every $\epsilon > 0$, there exists $x \in A$ such that $s - \epsilon < x$. Let t be an upper bound of A. Suppose $t < s$. Consider $\epsilon_0 = s - t > 0$. By our assumption, there exists $x \in A$ such that $s - \epsilon_0 < x$. So, $t < x$. This contradicts that t is an upper bound of A. So, $t \geq s$. Thus, s is the least upper bound

Therefore, by proving both the right and left implication, we have shown the statement to be true. $\square$

Analogous statement about infimums: Assume z is a lower bound of a set $A \subseteq \mathbb{R}$. Then $z = \inf A \iff$ for all $\epsilon > 0$, there exists $y \in A$ such that $y < z + \epsilon$.

1.3.2 Exercises

Exercise: 1.3.4

Let $A_1, A_2, A_3 \ldots$ be a collection of nonempty sets each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

Solution.

(a) Let A_1 and A_2 be nonempty sets, each bounded above. To find the largest of the two suprema, we can use the following: $\sup(A_1 \cap A_2) = \max\{\sup A_1, \sup A_2\}$. If we extend this notion to $\sup(\bigcup_{k=1}^{n} A_k)$, we can use the same idea from before and write it as $\sup(\bigcup_{k=1}^{n} A_k) = \max\{\sup A_1, \sup A_2, \ldots, \sup A_n\}$.

(b) The formula does not extend to the infinite case. Consider the counterexample $\bigcup_{k=1}^{\infty} A_k$ where $A_k := [k, k+1]$. Even though these sets are bounded above, when we take the union of them, we approach infinity, which is not bounded: $\bigcup_{k=1}^{\infty} A_k = [1, 2] \cup [2, 3] \cup \cdots = [1, \infty)$.

Exercise: 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution.

(a) Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Define the set $cA := \{ca : a \in A\}$.

From the axiom of completeness, because A is bounded above, we know there is a least upper bound, $s = \sup A$. Following from Example 1.3.7, we see that $a \leq s$ for all $a \in A$ which implies $ca \leq cs$ for all $a \in A$. Thus, cs is an upper bound for cA, and the first condition of Definition 1.3.2 is satisfied. For the second condition, we need to look at both $c = 0$ and $c > 0$ to avoid dividing by zero. So, we have two cases:

- **$c = 0$:** If $c = 0$, then $cA = \{0 : \ a \in A\} = \{0\}$. Since the only element in cA is 0, $\sup(cA) = 0$. Similarly, because $c = 0$, $c \sup A = 0 \cdot \sup A = 0$. Therefore, $\sup(cA) = c \sup(A)$.
- **$c > 0$:** Let b be an arbitrary upper bound for cA and $c > 0$. In other words, $ca \leq b$ for all $a \in A$. This is equivalent to $a \leq b/c$ where $c \neq 0$, from which we can see that b/c is an upper bound for A. Because s is the least upper bound of A, $s \leq b/c$, which can be rewritten as $cs \leq b$. This verifies the second part of Definition 1.3.2, and we conclude $\sup(cA) = c \sup A$.

(b) Postulate: If $c < 0$, then $\sup(cA) = c \inf(A)$.

Exercise: 1.3.8

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\left\{\frac{m}{n} : m, n \in \mathbb{N} \text{ with } m < n\right\}$.

(b) $\left\{\frac{(-1)^m}{n} : m, n \in \mathbb{N}\right\}$.

(c) $\left\{\frac{n}{3n+1} : n \in \mathbb{N}\right\}$.

(d) $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$.

Solution. To avoid writing out every set definition, I am going to denote each set as A_n where n corresponds to the numerical value of the list from (a) - (d).

(a) $\sup A_1 = 1$, $\inf A_1 = 0$

(b) $\sup A_2 = 1$, $\inf A_2 = -1$

(c) $\sup A_3 = \frac{1}{3}$, $\inf A_3 = \frac{1}{4}$

(d) $\sup A_4 = 1$, $\inf A_3 = 0$

1.4 Consequences of Completeness

Theorem 1.4.1: Nested Interval Property

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n]$. Assume I_n contains I_{n+1}. This results in a nested sequence of intervals.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \ldots$$

Then, $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

tl;dr there has to be something that is common to all of the sets.

Proof. Notice that the sequence, $a_1, a_2, a_3, \ldots$ is increasing. In other words, for each $n \in \mathbb{N}$, since $I_n \supset I_{n+1}$ we have $a_n \leq a_{n+1}$. If we consider the set $A = \{a_n : n \in \mathbb{N}\}$. The element b_1 is an upper bound of A. (Note that b_1 and a_1 corresponds to the end-points of the first set, I_1. Think of this as a tornado looking structure where the larger the I_n, the smaller the number line.) For each $n \in \mathbb{N}$, $a_n \leq b_n \leq b_1$.

Since A has an upper bound, it must have a least upper bound. Hence, let $\alpha = \sup A$. We claim that $\alpha \in \bigcap_{n=1}^{\infty} I_n$. We said b_1 was an upper bound. In fact, every b_n is an upper bound of A. Choose any $n, m \in \mathbb{N}$. We want to show that $a_n \leq b_m$. Consider the following cases:

Case 1: If $n < m$, then $a_n \leq a_m \leq b_m$. (Think: two number lines stacked on top of each other. The top number line is larger, call it I_n and it has a_n and b_n as endpoints. Consider a contained line ($I_n \supseteq I_m$) that is smaller, and has endpoints a_m and b_m.)

Case 2: If $n > m$, then $a_n \leq b_n \leq b_m$. So every b_n is an upper bound of A.

Hence,

- Because $\alpha = \sup A$, we have $\alpha \geq a_n$.
- Since b_n is an upper bound of A, we have $\alpha \leq b_n$.

so, $\alpha \in [a_n, b_n] = I_n$. Thus, $\alpha \in \bigcap_{n=1}^{\infty} I_n$. □

Nested, closed, Bounded Intervals $\Rightarrow$ non-empty intersection.

Theorem 1.4.2: Archimedean Principle

1. Given any number $x \in R$, there exists an $n \in N$ satisfying $n > x$.
2. Given any real number $y > 0$, there exists an $n \in N$ satisfying $1/n < y$.

Proof.

1. If $\mathbb{N}$ was bounded, then we can let $s \in \mathbb{N} = \sup \mathbb{N}$. However, we know that there is always a higher number (e.g., $n+1$) for any $n \in \mathbb{N}$ that is given. Thus, by contradiction, there must exist $n \geq x$.

2. For any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

□

Theorem 1.4.3: Density of the Rationals in the Reals

For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Proof. Since $b - a > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. From the Archimedean Principle, since $a \times n \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that $a \times n < m$. Let m be there smallest such natural numbers (by the well ordered principle). Since m is the smallest such natural number, it follows that $m - 1 \leq a \times n < m$. We then see that $a < \frac{m}{n}$. Now, we need to find some $\frac{m}{n} < b$.

$$\begin{aligned} m - 1 &\leq a \times n \\ m &\leq a \times n + 1 \\ \frac{m}{n} &\leq a + \frac{1}{n} \\ \frac{m}{n} &< a + (b - a) \\ \frac{m}{n} &< b \end{aligned}$$

We now have that $a < \frac{m}{n} < b$ so $\frac{m}{n}$ is a rational number in (a, b) □

Exercise: 1.4.1

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

1. Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

2. Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

3. Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st? In other words, are there two irrational numbers that can be added and multiplied such that you get a number x such that $x \notin \mathbb{I}$.

Solution.

1. Let $a, b \in \mathbb{Q}$. This means there exists some $p, q, a, b \in \mathbb{Z}$ such that
$$a = \frac{p}{q}$$
and
$$b = \frac{a}{b}$$
where $q, b \neq 0$. The product of these numbers is
$$ab = \frac{p}{q} \cdot \frac{a}{b} = \frac{pa}{qb}.$$
Since $pa, qb \in \mathbb{Z}$, $ab \in \mathbb{Q}$. The sum of these numbers is
$$a + b = \frac{p}{q} + \frac{a}{b} = \frac{pb + aq}{qb}.$$
Since $pb + aq, qb \in \mathbb{Z}$, $a + b \in \mathbb{Q}$.

2. Let $a \in \mathbb{Q}$ and $t \in \mathbb{I}$. Assume, for contradiction, that $a + t \in \mathbb{Q}$. This would imply $t = (a + t) - a$ (because we can subtract $t + a$ from the original equation and rearrange terms). Since $a + t, a \in \mathbb{Q}$ their sum would be rational because the rational numbers are closed under addition. However, that would contradict the assumption that $t \in \mathbb{I}$. Hence, $a + t \in \mathbb{I}$.

3. For $\mathbb{I}$, it is not closed under addition and multiplication. Consider the following counterexample: $\sqrt{2} + (-\sqrt{2}) = 0$ which is not in the irrationals. For multiplication, consider $\sqrt{2} \cdot \sqrt{2} = 2$, which is also not in the irrationals.

1.5 Cardinality

Two sets have the same *cardinality* if there exists a bijection between them. Thus, the natural numbers, the integers, and the rational numbers have the same cardinality. A set is *countably infinite* if it has the same cardinality as $\mathbb{N}$. (If it can be put into one-to-one correspondence with $\mathbb{N}$.) A set is *countable* if it is countably infinite or finite.

Theorem 1.5.6

$\mathbb{R}$ is not countable.

Proof. 1 (most common)
Suppose $\mathbb{R}$ is countable. Then we can list them all, or we can enumerate them. $\mathbb{R} = \{x_1, x_2, x_3, x_4, \ldots\}$. We can write the decimal expansion of each of these. Consider the following table:

$x_1 =$	$\boxed{a_{10}}$	a_{11}	a_{12}	a_{13}	a_{14}	$\dots$
$x_2 =$	a_{20}	$\boxed{a_{21}}$	a_{22}	a_{23}	a_{24}	$\dots$
$x_3 =$	a_{30}	a_{31}	$\boxed{a_{32}}$	a_{33}	a_{34}	$\dots$
$x_4 =$	a_{40}	a_{41}	a_{42}	$\boxed{a_{43}}$	a_{44}	$\dots$
$x_5 =$	a_{50}	a_{51}	a_{52}	a_{53}	$\boxed{a_{54}}$	$\dots$
$x_6 =$	a_{60}	a_{61}	a_{62}	a_{63}	a_{64}	$\dots$

We will now construct a number that is not in this list. Focus on diagonal entries. For each $n \in \mathbb{N}$, let b_n be a digit that is different fron a_{nn}. Now consider the number $y = 0.b_1b_2b_3b_4b_5\dots$. This number y is not in our list. So our list did not include all of $\mathbb{R}$. Avoid repeating 9s. $\square$

Proof. 2 (uses nested interval theorem)
Suppose $\mathbb{R}$ is countable. Then we can enumerate $\mathbb{R}$ $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$. Let I_1 be any closed interval that does not contain x_1. Next, we will find another closed interval I_2 that:

- $I_2 \subseteq I_1$
- $x_2 \notin I_2$

Continue in this fashion creating a sequence of nested closed intervals: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ such that for all $k \in \mathbb{N}$, $x_k \notin I_k$. Now consider:

$$\bigcap_{n=1}^{\infty} I_n$$

- For each $k \in \mathbb{N}$, since $x_k \notin I_k$, we see $x_k \notin \bigcap_{n=1}^{\infty} I_n$.
- By the nested interval theorem, there exists $x \in \mathbb{R}$ such that $x \in \bigcap_{n=1}^{\infty} I_n$. So x is a real number that is not included in our list.

$\square$

Theorem 1.5.7

A countable collection of finite sets is *countable*.

Theorem 1.5.8

(i) The union of two countable sets is *countable*.

(ii) A countable union of countable sets is *countable*.

From Theorem 1.5.6, we know that $\mathbb{R}$ is uncountable, but what about $(0, 1)$? It does have the same cardinality of $\mathbb{R}$ because we can make a one-to-one and onto function between both the sets. Similarly, (a, b) also has the same cardinality. What about $[a, b]$?

Recap: $\mathbb{N}$ is countable, and $\mathbb{R}$ is uncountable and has a different cardinality than $\mathbb{N}$. Thus, the question is, do all uncountable sets have the same cardinality as $\mathbb{R}$? The answer is **no**.

Theorem 1.5.9: Canter's Theorem

For any set A, there does not exist an onto map from A into $\mathcal{P}$.

Proof. Suppose there exists an onto function, $f : A \to \mathcal{P}(A)$. So each $a \in A$ is mapped to an element $f(a) \in \mathcal{P}(A)$. Then, $f(a) \subseteq A$. We are going to construct an element of $\mathcal{P}(A)$ which is not mapped to by f.

Consider $B = \{a \in A : a \notin f(a)\}$. Since f is onto there exists $a' \in A$ such that $B = f(a')$. Thus, there are two cases to consider:

- **Case 1:** If $a' \in B = f(a')$, then $a' \notin B$.
- **Case 2:** If $a' \notin B = f(a')$, then $a' \in B$.

As evidenced, both cases lead to contradictions, so B is not the image of any $a \in A$. Therefore f is not onto. □

Example 1.6: Set and Power Set Matching

$A = \{a, b, c\}$.

Solution. $\mathcal{P}(A) = \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$. Note that you can map $\{a\}$, $\{b\}$, $\{c\}$, to elements such as $\emptyset$, $\{a, b\}$, $\{a, b, c\}$, but there are still more elements that are left unmapped. We can extrapolate from our proof a set B such that $B = \{a, c\}$ because those elements are not mapped to.

All of this is to show $\mathcal{P}(\mathbb{R})$ has a larger cardinality than $\mathbb{R}$. Then $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ has a larger cardinality than $\mathcal{P}(\mathbb{R})$.

CHAPTER 2

SEQUENCES AND SERIES

2.1 Discussion: Rearrangement of Infinite Series

Questions:

What is a *sequence*?

A countable, ordered list of elements. An example could be $1, 2, 3, 4, 5, \ldots$. Note that this is *ordered*, therefore distinguishing it from a sequence like $3, 1, 2, 4, 5, 6, \ldots$. Hence, order matters.

A *sequence* is a function whose domain is $\mathbb{N}$. **Note:** The domain $\mathbb{N}$ refers to each element's position in the list. For example, $(a_n) = a_1, a_2, a_3, \ldots$.

We will focus on the *limit* of a sequence. We use sequences to approximate other things.

Example 2.1: Sequence

$3, 3.1, 3.14, 3.141, 3.1415, \cdots \approx \pi$.
$x, x - \frac{x^3}{6}, x - \frac{x^3}{6} + \frac{x^5}{120}, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}, \cdots \approx \sin(x)$

What is a *series*?

An infinite sum. We look at the sequence of partial sums. We ask, do the partial sums approach a limit?

Example 2.2: Alternating Harmonic Series

$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots$

We can rearrange these terms such that we can 'force' the series to converge to a specific number. Therefore, we will need to be careful with our definitions.

2.2 The Limit of a Sequence

Definition 2.2.1

A *sequence* is a function whose domain is $\mathbb{N}$. We write $(a_n) = a_1, a_2, a_3, \ldots$.

Definition 2.2.3

The sequence (a_n) *converges* to L if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$, $|a_n - L| < \epsilon$. In other words, there exists $N \in \mathbb{N}$ such that

- **(In the interval)** $a_N \in (L - \epsilon, L + \epsilon)$.
- **(Stays in the interval)** $\forall n \geq N$, $a_n \in (L - \epsilon, L + \epsilon)$.

Example 2.3: Limit Proof 1

Let $a_n = \frac{1}{n}$. $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$

Proof. Our claim is $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus, let $\epsilon = .01$. Does the sequence eventually get inside $(-.01, .01)$? We will set $N = 101$. So, for any $n \geq |0|$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{101} < .01.$$

From A_n and on, the sequence stayed within ϵ of 0. But what about $\epsilon = .001$, $\epsilon = .00001$ and so on?

Actual proof let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Now, for any $n \geq N$,

$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{1/\epsilon}.$$

(Where $\frac{1}{1/\epsilon} = \epsilon$, but is in that form for demonstration purposes.) Therefore $\lim_{n\to\infty} \frac{1}{n} = 0$ □

"To get close" means is that we are finding a bigger and bigger N as ϵ gets smaller. Note that the choice of N certainly depends on ϵ. This idea of "getting close" can be seen in the following definition:

Definition 2.2.3B

A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).

2.2.1 Basic Structure of a Limit Proof

Claim: $\lim_{n\to\infty} a_n = L$.
Proof: Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that {something involving ϵ}. Assume $n \geq N$. Then,

$$|a - n - L| \boxed{\ldots} < \epsilon$$

(Where $\boxed{\ldots}$ is going to be where the majority of the work is going to lie.

Example 2.4: Limit Proof 2

Claim: $\lim\limits_{n\to\infty} \dfrac{2n-3}{2n} = 1$

Proof. Let $\epsilon > 0$. *Scratch paper:* Solve for:

$$\left|\frac{2n-3}{2n} - 1\right| = \left|\frac{-3}{2n}\right| = \frac{3}{2n} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n.$$

By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{3}{2\epsilon}$. Assume $n \geq N$, (want to know what happens past this point)

$$\left|\frac{2n-3}{2n} - 1\right| \leq \frac{3}{2N} < \frac{3}{2 \cdot 3/2\epsilon} = \epsilon.$$

Therefore, $\lim\limits_{n\to\infty} \dfrac{2n-3}{2n} = 1$ $\square$

Example 2.5: Limit Proof 3

Claim: $\lim_{n\to\infty} \frac{2n^2+1n^2}{=} 2$

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that [leave off] *Scratch paper:* Solve for

$$\left|\frac{2n^2+1}{n^2} - 2\right| = \frac{2n^2}{n^2} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n$$

[pick up] there exists $N \in \mathbb{N}$ such that

$$N > \frac{1}{\sqrt{\epsilon}}.$$

Assume $n \geq N$, then

$$\begin{aligned}
\left|\frac{2n^2+1}{n^2} - 2\right| &= \frac{1}{n^2} \\
&\leq \frac{1}{N^2} \\
&< \frac{1}{(1/(\sqrt{\epsilon})^2} \\
&= \frac{1}{1/\epsilon} \\
&= \epsilon
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{2n^2+1n^2}{=} 2$ □

Example 2.6: Limit Proof 4

Claim: $\lim_{n\to\infty} \frac{7n+83n+6}{=} \frac{7}{3}$

Proof.

$$\begin{aligned}
\left|\frac{7n+8}{3n+6} - \frac{7}{3}\right| &= \left|\frac{21n+24}{3(3n+6)} - \frac{21n+42}{3(3n+6)}\right| \\
&= \left|\frac{-18}{9n+18}\right| \\
&= \frac{18}{9n+18} < \epsilon ** \\
&= \frac{18}{3} < 9n+18 \\
&= \frac{18}{3} - 18 < 9n \\
&= \frac{18/\epsilon - 18}{9} < n
\end{aligned}$$

$**\ \frac{18}{9n+8} < \frac{18}{9n} < \epsilon \Rightarrow \frac{2}{\epsilon} < N$. $\exists N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Assume $n \geq N$,

$$\begin{aligned}\left|\frac{7n+8}{3n+6} - \frac{7}{3}\right| &= \frac{18}{9n+18} \\ &= \frac{2}{n+2} \\ &< \frac{2}{n} \\ &\leq \frac{2}{N} \\ &< \frac{2}{\epsilon/2} \\ &= \epsilon\end{aligned}$$

□

Does every sequence have a limit?

Theorem 2.2.4: Uniqueness of Limits

The limit when it exists, is unique.

Proof. Let (x_n) be a convergent sequence. Suppose L and M are limits of this sequence. Without the loss of generality, we are going to assume $M > L$ Let

$$\epsilon = \frac{M-L}{3}.$$

Since n_x converges to L, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N$, $|(x_n) - L| < \epsilon$. Since (x_n) converges to M, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|(x_n) - M| < \epsilon$. Consider $n = \max\{N_1, N_2\}$. Since $n \geq N_1$, $|(x_n) - L| < \epsilon$. Since $n \geq N_2$, $|(x_n) - M| < \epsilon$. Then $L - \epsilon < x_n < L + \epsilon$ and $M - \epsilon < x_n < M + \epsilon$. By our choice of ϵ, we now have

$$(x_n) < L + \epsilon < M - \epsilon < (x_n).$$

This is a contradiction. Thus, (x_n) cannot have two different limits. □

Example 2.7: Limit Proof 5

Let $(x_n) = \dfrac{\cos(n)}{3n}$. Claim: $\lim_{n\to\infty}(x_n) = 0$

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{3\epsilon}$ for all $n \geq N$,

$$\begin{aligned} \left|\frac{\cos(n)}{3n} - 0\right| &= \left|\frac{\cos(n)}{3n}\right| \\ &\leq \frac{1}{3n} \\ &\leq \frac{1}{3N} \\ &< \frac{1}{3(1/3\epsilon)} \\ &= \epsilon \end{aligned}$$

□

Example 2.8: Limit Proof 6

Let $(y_n) = \frac{4n-1}{n^2}$. Claim: $\lim_{n\to\infty} y_n = 0$.

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. For all $n \geq N$,

$$\begin{aligned} \left|\frac{4n-1}{n^2} - 0\right| &= \left|\frac{4n-1}{n^2}\right| \\ &= \frac{4n-1}{n} \\ &< \frac{4n}{n^2} \\ &= \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \frac{4}{4/\epsilon} \\ &= \epsilon \end{aligned}$$

□

2.2.2 Exercises

Exercise: 2.1.1

What happens if we reverse the order of the quantifiers in Definition 2.2.3?
Definition: A sequence x_n *verconges* to x if there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x - n - x| < \epsilon$.

(a) Give an example of a vercongent sequence.

(b) Is there an example of a vercongent sequence that is divergent?

(c) Can a sequence verconge to two different values?

(d) What exactly is being described in this strange definition?

Solution.

(a) Pick $\epsilon = 2$, $x_n = (-1)^n$ and $x = 0$. This sequence will stay within the bounds of $(-2, 2)$ for all $N \in \mathbb{N}$ and $n \geq N$.

(b) There cannot be a divergent vercongent sequence because vercongence wants us to be bounded, and divergence wants it to grow outside the bounds. These two ideas are mutually exclusive.

(c) Yes. For example, $x_n = 0$ and $x_n = 1$.

(d) This definition is describing a sequence that is bounded. It is a sequence that is not growing outside of a certain range.

Exercise: 2.2.2

Verify, using Definition 2.2.3, that the following sequences converge to the proposed limit.

(a) $\lim_{n\to\infty} \frac{2n+15n+4}{=} \frac{2}{5}$.

(b) $\lim_{n\to\infty} \frac{2n^2 n^3+3}{=} 0$

Proof.

(a) Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \dfrac{3}{25\epsilon}$. Then for all $n \geq N$,

$$\begin{aligned}
\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \left|\frac{-3}{5(5n+4)}\right| \\
&= \frac{3}{25n+20} \\
&\leq \frac{3}{25n} \\
&\leq \frac{3}{25N} \\
&< \epsilon
\end{aligned}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) Let $\epsilon > 0$. By the Archimedean Principle, there exists an $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Then, for $n \geq N$,

$$\begin{aligned}
\left|\frac{2n^2}{n^3+3} - 0\right| &= \left|\frac{2n^2}{n^3+3}\right| \\
&= \frac{2n^2}{n^3+3} \\
&< \frac{2n^2}{n^3} \\
&= \frac{2}{n} \\
&\leq \frac{2}{N} \\
&= \frac{2}{2/\epsilon} \\
&= \epsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{2n^2n^3+3}{=}0$. $\square$

Exercise: 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution.

(a) There is at least one college in the United States where all students are less than seven feet tall.

(b) There is at least one college in the United States where all professors give at least one student a grade of C or lower.

(c) For all colleges in the United States, there exists a student who is less than six feet tall.

Exercise: 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution.

(a) Possible. Consider the sequence $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$. This sequence has infinitely many ones but does not converge to one.

(b) Impossible. Suppose (a_n) is a sequence that converges to a limit $L \neq 1$ and has infinitely many ones. Since (a_n) converges to L, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$. Choose $\epsilon = \frac{|1-L|}{2} > 0$. Then, for $n \geq N$,

$|a_n - L| < \epsilon$, which implies $a_n \neq 1$ beyond this N. This contradicts the existence of infinitely many ones. Therefore, such a sequence is impossible.

(c) Possible. Define a sequence by concatenating increasing blocks of ones separated by zeros: $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \ldots)$. Specifically, the sequence consists of n ones followed by a zero for $n = 1, 2, 3, \ldots$. For every $n \in \mathbb{N}$, there is a block of n consecutive ones somewhere in the sequence. The sequence does not converge, so it is divergent.

Exercise: 2.2.5

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim_{n\to\infty} a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$

(b) $a_n = [[(12 + 4n)/3n]]$

Reflecting on these examples, comment on the statement following Definition 2.2.3B that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution.

(a) We will show that $\lim\limits_{n\to\infty} a_n = 0$.

Proof. For $n \geq 6$, we have $\frac{5}{n} \leq \frac{5}{6} < 1$, so $a_n = [[5/n]] = 0$.
Let $\epsilon > 0$. Choose $N = 6$. Then for all $n \geq N$,

$$|a_n - 0| = |0 - 0| = 0 < \epsilon.$$

Therefore, by the definition of convergence, $\lim\limits_{n\to\infty} a_n = 0$. □

(b) We will show that $\lim\limits_{n\to\infty} a_n = 1$.

Proof. Observe that:

$$a_n = \left[\frac{12+4n}{3n}\right] = \left[\frac{4n+12}{3n}\right] = \left[\frac{4}{3}+\frac{4}{n}\right].$$

As $n \to \infty$, $\frac{4}{n} \to 0$, so $\frac{4}{3}+\frac{4}{n} \to \frac{4}{3} \approx 1.333$.
For $n \geq 7$, we have:

$$\frac{4}{n} \leq \frac{4}{7} \approx 0.571, \quad \frac{4}{3}+\frac{4}{n} \leq 1.333+0.571 = 1.904.$$

Since $1 < \frac{4}{3}+\frac{4}{n} < 2$ for $n \geq 7$, we have:

$$a_n = \left[\frac{4}{3}+\frac{4}{n}\right] = 1.$$

Let $\epsilon > 0$. Choose $N = 7$. Then for all $n \geq N$,

$$|a_n - 1| = |1-1| = 0 < \epsilon.$$

Therefore, by the definition of convergence, $\lim_{n\to\infty} a_n = 1$. □

Reflection: In these examples, we see that once the sequence reaches a certain point (i.e., $n \geq N$), the terms remain constant. This means that for any $\epsilon > 0$, we can find a fixed N to satisfy the definition of convergence, regardless of how small ϵ is. However, in general, smaller ϵ-neighborhoods may require larger N because the sequence may not settle into its limit as neatly as it does in these cases.

Exercise: 2.2.6

Prove the Uniqueness of Limits theorem. To get started, assume $(a_n) \to a$ and $(a_n) \to b$. Now argue $a = b$.

Proof. Since $(a_n) \to a$, this means for all $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon/2$. Similarly, since $(a_n) \to b$, this means for all $\epsilon > 0$, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - b| < \epsilon/2$.

Now, let $N = \max\{N_1, N_2\}$ so that

$$\begin{aligned} |a-b| &= |a - a_n + a_n - b| \\ &\leq |(a_n - a) + (a_n - b)| \\ &< \epsilon/2 + \epsilon/2 \\ &< \epsilon \end{aligned}$$

Then, by Theorem 1.2.6, $a = b$. $\square$

Exercise: 2.2.7

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually, or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution.

(a) The sequence $(-1)^n$ is *frequently* in the set $\{1\}$ because for every $N \in \mathbb{N}$, we can find an $n \geq N$ such that $(-1)^n = 1$.

(b) The definition of *eventually* is stronger because *eventually* implies *frequently*, but *frequently* does not imply *eventually*.

(c) An alternate rephrasing of Definition 2.2.3B using *eventually* is: A sequence (a_n) converges to a if, given any ϵ-neighborhood—$V_\epsilon(a)$ of a—(a_n) is *eventually* in $V_\epsilon(a)$. The term we want is eventually.

(d) If an infinite number of terms of a sequence (x_n) are equal to 2, (x_n) is not *eventually* in $(1.9, 2.1)$ because we can have a sequence (a_n) that will not settle in $(1.9, 2.1)$. For example, $(a_n) = (0, 2, 0, 2, \cdots)$ does not settle in $(1.9, 2.1)$. Whereas, (x_n) is *frequently* in the interval $(1.9, 2.1)$ because for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $x_n \in (1.9, 2.1)$ for all $n \geq N$. We can see an instance of this being true by examining the previous example.

2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1

A sequence (x_n) is *bounded* if there exists some $M > 0$ such that every term in the sequence belongs to $[-M, M]$.

Theorem 2.3.2

Every convergent sequence is bounded.

Proof. Let (x_n) be a convergent sequence with limit L. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|(x_n) - L| < 1$. Equivalently, $(x_n) \in (L-1, L+1)$. Let

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |L+1|, |L-1|\}.$$

We claim that for all $n \in \mathbb{N}$, $|x_n| \leq M$.

1. This is true for $n < N$.
2. For $n \geq N$, we know $L - 1 < x_n < L + 1$, so $(x_n) \leq \max\{|L-1|, |L+1|\}$

Thus, every term is in $[-M, M]$. □

Theorem 2.3.3: Algebraic Limit Theorem

Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then,

(i) $\lim_{n\to\infty} ca_n = ca$ for all $c \in \mathbb{R}$;

(ii) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(iii) $\lim_{n\to\infty}(a_n b_n) = ab$;

(iv) $\lim_{n\to\infty} \frac{a_n b_n}{=} \frac{a}{b}$ provided $b \neq 0$.

Scratch Paper:

$$\begin{aligned} |ca_n - ca| = |c|\,|a_n - a| &< \epsilon \\ |a_n - a| &< \frac{\epsilon}{|c|} \end{aligned}$$

Leave off and go back to proof[1]

Proof. (i)
Let $\epsilon > 0$.[1] Since (a_n) converges to a, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \frac{\epsilon}{|c|}$. Now, for any $n \geq N$ we have two case because we want to avoid dividing

by 0:

- If $c = 0$:
  then each $ca_n = 0$. So (ca_n) converges to 0, which can equal ca.

- If $c > 0$:
  $|ca_n - ca| = |c|\,|a_n - a| < |c|\,\frac{\epsilon}{|c|} = \epsilon$.

(ii)
Scratch paper:

$$\begin{aligned} |(a_n + b_n)| &= |(a_n - a) + (b_n - b)| && (2.1)\\ &\le |a_n - a| + |b_n - b| && (2.2)\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} && (2.3) \end{aligned}$$

Note that (2.2) is from the triangle inequality. Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, $|a_n - a| < \frac{\epsilon}{2}$. Since (b_n) converges to b, there exists $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, $|b_n - b| < \frac{\epsilon}{2}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \ge N$, (refer back to scratch paper).

(iii)
Scratch paper:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| && (2.4)\\ &= |a_n(b_n - b) + b(b_n - b)| && (2.5)\\ &\le |a_n|\,|b_n - b| + |b|\,|b_n - b| && (2.6)\\ &\le M\,|b_n - b| + M\,|a_n - a|\,. && (2.7)\\ &< M\left(\frac{\epsilon}{2M}\right) + M\left(\frac{\epsilon}{2M}\right) && (2.8)\\ &= \epsilon && (2.9) \end{aligned}$$

Note that: (2.4) is where we added 0, (2.5) is from the triangle inequality, and (2.6) is just factored. Additionally, we choose N to get the fractions in (2.8) Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since convergent sequences are bounded, then there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \le M$. We can choose M so that $|b_n| \le M$ as well. Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, $|a_n - a| < \frac{\epsilon}{2M}$. Since (b_n) converges to b, there exists $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, $|b_n - b| < \frac{\epsilon}{2M}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \ge N$, (refer back to scratch paper, and change (2.4)'s sign from an '$=$' to '$\le$').

(iv)
Scratch paper:

$$\begin{aligned}\left|\frac{a_n}{b_n}-\frac{a}{b}\right| &= \left|\frac{a_nb-ab_n}{b_nb}\right| \\ &= \left|\frac{a_nb-ab_n+ab_n-ab}{b_nb}\right| \\ &= \left|\frac{a_n(b-b_n)+b(b_n-b)}{b_nb}\right| \\ &= \left|\frac{a_n(b-b_n)}{b_nb}+\frac{b(b_n-b)}{b_nb}\right| \\ &\leq \left|\frac{a_n}{b_n}\right||b-b_n|+|b|\left|\frac{b_n-b}{b_nb}\right| \\ &< \epsilon\end{aligned}$$

Let $\epsilon > 0$. Since (b_n) converges to b, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|b_n| > \left|\frac{b}{2}\right|$. There also exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon|b|^2}{2}$. Now, let $N = \max\{N_1, N_2\}$. Let $n \geq N$, (refer back to scratch paper). □

Lemma 2.3.4

Let (a_n) and $c < a$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > c$. Similarly, if $a < d$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n < d$.

2.3.1 Limits and Order

Theorem 2.3.5: Order Limit Theorem

Let (a_n) and (b_n) be sequences. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

(i) If $a_n \geq c$ for all $n \in \mathbb{N}$, then $a \geq c$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(iii) If there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

2.3.2 Exercises

Exercise: 2.3.1

(a) If $\lim_{n\to\infty} x_n = 0$, show that $\lim_{n\to\infty} \sqrt{x_n} = 0$.

(b) If $\lim_{n\to\infty} x_n = x$, show that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

Proof.

(a) *Solution.* Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$|x_n| < \epsilon^2.$$
Then, for all $n \geq N$,
$$|\sqrt{x_n} - 0| = \sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon.$$
Therefore, $\lim_{n\to\infty} \sqrt{x_n} = 0$.

(b) *Solution.* Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$|x_n - x| < \delta.$$
We consider two cases:

Case 1: $x > 0$.

Since $x > 0$, choose $\delta = \min\left\{\epsilon\left(2\sqrt{x}\right), \frac{x}{2}\right\}$. Then for all $n \geq N$, we have $x_n > x - \frac{x}{2} = \frac{x}{2} > 0$. Thus,
$$\sqrt{x_n} + \sqrt{x} \geq \sqrt{\frac{x}{2}} + \sqrt{x} > 0.$$
Now,
$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{\delta}{\sqrt{\frac{x}{2}}} \leq \epsilon.$$

Case 2: $x = 0$.

From part (1), we have $\lim_{n\to\infty} \sqrt{x_n} = 0 = \sqrt{0}$.

Therefore, $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

□

Exercise: 2.3.2

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\dfrac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution.

(a) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - 2| < \epsilon$. Now, for any $n \geq N$,

$$\begin{aligned} \left|\frac{2x_n - 1}{3} - 1\right| &= \left|\frac{2x_n - 1 - 3}{3}\right| \\ &= \left|\frac{2x_n - 4}{3}\right| \\ &= \frac{2}{3}|x_n - 2| \\ &< |x_n - 2| \\ &< \epsilon \end{aligned}$$

Therefore, $\frac{2x_n - 1}{3} \to 1$ □

(b) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $x_n \geq 1$. Then, we will choose N_2 so that $|x_n - 2| < \epsilon$ for all $n \geq N_2$. Afterwards, we take $N = \max\{N_1, N_2\}$. And note that for $n \geq N$,

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{2}\right| &= \left|\frac{2 - x_n}{2x_n}\right| \\ &< \frac{|2 - x_n|}{2} \\ &< \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

□

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Definition 2.4.1

A sequence a_n is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.2: Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges.

Proof. Let (a_n) be an increasing and bounded sequence. Since (a_n) is bounded, the set $A = \{a_n \mid n \in \mathbb{N}\}$ is clearly also bounded. Since A is bounded, $\sup A$ exists. We claim that $\lim_{n\to\infty} a_n = \sup A$. Thus, for all $\epsilon > 0$ and by our definition of supremum, there exists $N \in \mathbb{N}$ such that $\sup A - \epsilon < a_N \leq \sup A$. Since (a_n) is increasing, for all $n \geq N$, $\sup A - \epsilon < a_N \leq a_n \leq \sup A$. It follows that $|a_n - \sup A| < \epsilon$. Therefore, $\lim_{n\to\infty} a_n = \sup A$. □

Example 2.9: MCT

Consider the recursively defined sequence x_n where $x_1 = 3$ and for all $n \in \mathbb{N}$, $x_{n+1} = \frac{1}{4-x_n}$. Show that x_n converges.

Proof. We will show that x_n is monotone and bounded.

- **Part 1: Monotone Decreasing**
  - Base case: $x_1 = 3$, $x_2 = 1$.
  - Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq x_{n+1}$. It follows that

$$\begin{aligned} x_n &\geq x_{n+1} \\ 4 - x_n &\leq 4 - x_{n+1} \\ \frac{1}{4-x_n} &\geq \frac{1}{4-x_{n+1}} \\ x_{n+1} &\geq x_{n+2} \end{aligned}$$

- **Part 2: Bounded Below** Claim: Sequence is bounded below by 0.
  - Base case: $x_1 = 3 > 0$.
  - Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq 0$. It follows that $4 - x_n \leq 4$, and when we take the reciprocal, we get

$$\begin{aligned} \frac{1}{4-x_n} &\leq \frac{1}{4} \\ x_{n+1} &\geq 1/4 \\ &> 0 \end{aligned}$$

  By math induction, x_n is bounded below by 0.

By the Monotone Convergence Theorem, x_n converges.

So, what is the limit? We know (x_n) converges so let $L = \lim_{n\to\infty} x_n$. Then, $\lim_{n\to\infty} x_{n+1} = L$. We also know $x_{n+1} = \frac{1}{4-x_n}$. So $L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{4-x_n} =$

$\frac{1}{4-L}$. It must be true that $L = \frac{1}{4-L}$. Solving for L, we get

$$L(4-L) = 1$$
$$4L - L^2 = 1$$
$$L^2 - 4L + 1 = 0$$

Hence, $L = 2 - \sqrt{3}$ or $L = 2 + \sqrt{3}$. Notice that it cannot be the latter because it is bigger than 3. □

2.4.1 Recap and Summary

We use limits to define multiple things in calculus. This is why we are focusing so heavily upon it. For example,

1. Derivatives: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
2. Integrals: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$
3. Infinite Series: $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ Consider geometric series, C_a such that each term is multiplied by a ratio r. This is represented as $\sum_{n=0}^{\infty} ar^n = 1 + r + r^2 + r^3 \ldots$. When we look at partial sums, we get $S_n = 1 + r + r^2 + r^3 + \cdots + r^n$. We can then multiply by r to get $rS_n = r + r^2 + r^3 + r^4 + \cdots + r^{n+1}$. Subtracting the two, we get $(1-r)S_n = 1 - r^{n+1}$. Thus,
$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$
If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. Thus, $\lim_{n\to\infty} S_n = \frac{1}{1-r}$.

Looking to the future, we are going to use functions and summations together. For example, when we have $f(x) = \sum_{n=0}^{\infty}(a_n)x^n$ such that $f'(x) = \sum_{n=0}^{\infty}(a_n)x^{n-1}$.

Definition 2.4.3

Let (x_n) be a bounded sequence. Then the *limit inferior* is $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k \mid k \geq n\}$. This is the largest a limit can get. The *limit superior* is $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k \mid k \geq n\}$. This is the smallest a limit can get.

See Exercise 2.4.7 in the book for more information.

Example 2.10: Monotone Decreasing Sequence

The following sequence is an example of a monotone decreasing sequence.

$$x_1, x_2, x_3, x_4, x_5, x_6, \ldots \sup\{x_k \mid k \geq 1\} = S.$$
$$x_2, x_3, x_4, x_5, x_6, \ldots \sup\{x_k \mid k \geq 2\} = S.$$
$$x_3, x_4, x_5, x_6, \ldots \sup\{x_k \mid k \geq 3\} = S.$$
$$x_4, x_5, x_6, \ldots \sup\{x_k \mid k \geq 4\} = S.$$

$\limsup_{n\to\infty} x_n$ is guaranteed to exist by the Monotone Convergence Theorem.

Example 2.11: liminf

Let $x_n = (-1)^n(1 + \frac{1}{n})$. Thus, $x_{1,2,3} = -2, 1\frac{1}{2}, -1\frac{1}{3} \ldots$

Example 2.12: Convergence Towards 0

Let $x_n = (-1)^n \frac{1}{n}$. Thus, $x_{1,2,3} = -1, \frac{1}{2}, -\frac{1}{3} \ldots$.

Theorem 2.4.4

A sequence x_n is convergent if, and only if, $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

See Theorem 2.4.6 in the book for another view.

2.4.2 Exercises

Exercise: 2.4.7 (Limit Superior)

Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k : k \geq n\}$ converges.

(b) The *limit superior* of (a_n) or $\limsup_{n\to\infty} a_n$, is defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf_{n\to\infty} a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution.

(a) We will show that (y_n) converges.

Proof. Since (a_n) is bounded, there exists $M > 0$ such that
all n.
For each n, define $y_n = \sup\{a_k : k \geq n\}$. As n increases,
$k \geq n\}$ becomes smaller, so the supremum cannot increase.
sequence (y_n) is non-increasing:

$$y_{n+1} \leq y_n \quad \text{for all } n.$$

Additionally, since (a_n) is bounded below, so is (y_n). There
bounded, non-increasing sequence.
By the Monotone Convergence Theorem, every bounded,
quence converges. Thus, (y_n) converges.

(b) A reasonable definition for $\liminf_{n\to\infty} a_n$ is to define $z_n = \inf\{a_k : k \geq n\}$ for each n. Then, the *limit inferior* of (a_n) is defined by:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} z_n.$$

Since (a_n) is bounded, each z_n exists and the sequence (z_n) is non-decreasing. As n increases, the set $\{a_k : k \geq n\}$ becomes smaller, so the infimum cannot decrease. Therefore, (z_n) is a bounded, non-decreasing sequence, which converges by the Monotone Convergence Theorem. Hence, $\liminf_{n\to\infty} a_n$ always exists for any bounded sequence.

(c) We will show that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence.

Proof. For each n, we have $z_n = \inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\} = y_n$. This implies:

$$z_n \leq y_n \quad \text{for all } n.$$

Taking limits as $n \to \infty$, we get:

$$\lim_{n\to\infty} z_n \leq \lim_{n\to\infty} y_n,$$

which means:

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

For an example where the inequality is strict, consider the sequence $a_n = (-1)^n$. Then:

$$y_n = \sup\{(-1)^k : k \geq n\} = 1, \quad z_n = \inf\{(-1)^k : k \geq n\} = -1.$$

Therefore:

$$\limsup_{n\to\infty} a_n = 1, \quad \liminf_{n\to\infty} a_n = -1, \quad \liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n.$$

□

(d) We will show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. We will show this by proving both implications:

(⇒) Suppose $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$. We wi
$\lim a_n$ exists and equals L.
Let $\epsilon > 0$. Since $\limsup_{n\to\infty} a_n = L$, there exists N
for all $n \geq N_1$:

$$y_n = \sup\{a_k : k \geq n\} < L + \epsilon.$$

Similarly, since $\liminf_{n\to\infty} a_n = L$, there exists N_2 s
all $n \geq N_2$:

$$z_n = \inf\{a_k : k \geq n\} > L - \epsilon.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$:

$$L - \epsilon < z_n \leq a_n \leq y_n < L + \epsilon,$$

which implies:

$$|a_n - L| < \epsilon.$$

Therefore, $\lim a_n = L$.

(⇐) Conversely, suppose $\lim a_n = L$. Then, for every ϵ
exists N such that for all $n \geq N$:

$$|a_n - L| < \epsilon.$$

This implies that for all $n \geq N$, the set $\{a_k : k \geq n\}$
in $(L - \epsilon, L + \epsilon)$. Therefore:

$$y_n = \sup\{a_k : k \geq n\} \leq L + \epsilon, \quad z_n = \inf\{a_k : k \geq$$

Taking limits, we get:

$$\limsup_{n\to\infty} a_n \leq L + \epsilon, \quad \liminf_{n\to\infty} a_n \geq L - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\lim su$
$\liminf_{n\to\infty} a_n = L$.

Exercise: 2.4.10 (Infinite Products)

A close relative of infinite series is the infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots ,$$

which is understood in terms of its sequence of partial products

$$p_m = \prod_{n=1}^{m} b_n = b_1 b_2 b_3 \cdots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty}(1+a_n) = (1+a_1)(1+a_2)(1+a_3)\cdots , \quad \text{where} \quad a_n \geq 0.$$

(a) Find an explicit formula for the sequence of partial products in the case where $a_n = \frac{1}{n}$ and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where $a_n = \frac{1}{n^2}$ and make a conjecture about the convergence of this sequence.

(b) Show, in general, the sequence of partial products converges if and only if $\sum_{n=1}^{\infty} a_n$ converges. (The inequality $1 + x \leq 3^x$ for positive x will be useful in one direction.)

Solution.

(a) For $a_n = \dfrac{1}{n}$:

The sequence of partial products is:

$$p_m = \prod_{n=1}^{m}\left(1+\frac{1}{n}\right) = \prod_{n=1}^{m}\frac{n+1}{n}$$

This telescopes:

$$p_m = \frac{2}{1}\times\frac{3}{2}\times\frac{4}{3}\times\cdots\times\frac{m+1}{m} = \frac{m+1}{1} = m+1$$

Therefore, the sequence diverges as $m \to \infty$.

For $a_n = \dfrac{1}{n^2}$:

Compute the first few terms:

$$\begin{aligned}
p_1 &= 1 + \frac{1}{1^2} = 2 \\
p_2 &= \left(1 + \frac{1}{1^2}\right)\left(1 + \frac{1}{2^2}\right) = 2 \times \frac{5}{4} = \frac{5}{2} \\
p_3 &= p_2 \times \left(1 + \frac{1}{3^2}\right) = \frac{5}{2} \times \frac{10}{9} = \frac{25}{9} \\
p_4 &= p_3 \times \left(1 + \frac{1}{4^2}\right) = \frac{25}{9} \times \frac{17}{16} = \frac{425}{144}
\end{aligned}$$

The sequence increases slowly, suggesting that the infinite product is monotone increasing, and thus it converges.

(b) We will provide an if and only if proof below.

Proof. We will show that the infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n$ converges.

($\Rightarrow$) If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$. For $a_n \geq 0$, we have $\ln(1 + a_n) \leq a_n$. Thus,

$$\sum_{n=1}^{\infty} \ln(1 + a_n) \leq \sum_{n=1}^{\infty} a_n < \infty$$

So the series $\sum_{n=1}^{\infty} \ln(1 + a_n)$ converges, which implies that the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges.

($\Leftarrow$) Conversely, if $\prod_{n=1}^{\infty}(1+a_n)$ converges, then the partial products are bounded. For $a_n \geq 0$ and $1 + x \geq e^{x/2}$ for small x, we have

$$\ln(1 + a_n) \geq \frac{a_n}{2}$$

For sufficiently large n, this gives

$$\sum_{n=1}^{\infty} a_n \leq 2 \sum_{n=1}^{\infty} \ln(1 + a_n)$$

Since $\sum_{n=1}^{\infty} \ln(1 + a_n)$ converges, so does $\sum_{n=1}^{\infty} a_n$. $\square$

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Definition 2.5.1

Let a_n be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \ldots$ be an increasing sequence of natural numbers. Then, the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is called a *subsequence* of a_n and is denoted by a_{n_k}, where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 2.5.2

Subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let x_{n_k} be a subsequence of x_n, and let $L = \lim_{n\to\infty} x_n$. We want to show that $\lim_{n\to\infty} x_{n_k} = L$. Let $\epsilon > 0$. Since x_n converges to L, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$. Since n_k is increasing, there exists $M \in \mathbb{N}$ such that $n_k \geq N$ for all $k \geq M$. Thus, for all $k \geq M$, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n\to\infty} x_{n_k} = L$.

Let x_{n_k} be a subsequence of x_n. Let $\epsilon > 0$. Since $(x_n) \to L$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$.

Now, looking at x_{n_k}, notice that $n_k \geq k$ for all k. Consider $k = N$. For any $n \geq N$, $n \geq N \geq k$. Thus, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n\to\infty} x_{n_k} = L$. ☐

Theorem 2.5.3: Divergence Criterion

If x_n has two subsequences that converge to different limits, then x_n diverges.

Building upon this idea of Divergence, we can list some other ways a sequence can diverge:

1. Find one subsequence that diverges.
2. Find tow subsequences that converge to separate limits.
3. Negate the definition of convergence.
   - For example, a sequence converges to L if there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - a| \geq \epsilon$. There exists a subsequence (a_{n_k}) such that for all $k \in \mathbb{N}$, $|a_{n_k} - L| \geq \epsilon$.

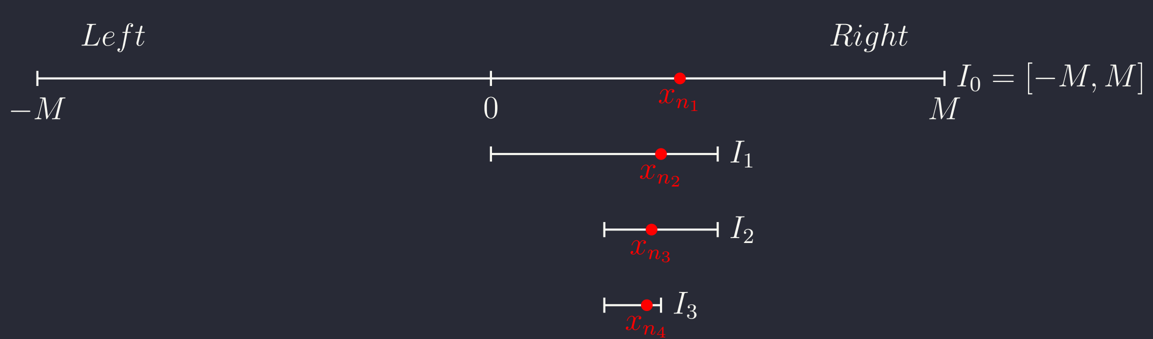
Theorem 2.5.4: Bolzano-Weierstrass Theorem

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. Let x_n be a bounded sequence. There exists an $M > 0$ such that every term x_n belongs to $[-M, M]$. To prove this theorem, we will be utilizing a recursive argument style. Thus, let $I_0 = [-M, M]$. I_0 has length $2M$. Cut I_0 in half with I_1 and I_2 both being half as long as I_0. Since x_n is bounded, there exists an I_L or I_R that contains infinitely many terms of x_n. We will pick one, call it I_1 that is contained in I_L. I_1 has length M. Pick one of those terms inside I_1 and call it x_{n_1}. Now, cut I_1 in half with equal length in intervals. One of them contains infinitely many terms. Call that interval I_2. I_2 has length $\frac{M}{2}$. Pick one of those terms inside I_2 and call it x_{n_2}. Continue this process indefinitely for all $n \geq \mathbb{N}$ with $n_1 > n_2$. Continue this process, and we get

- a sequence of closed intervals I_n.
  - I_n has length $\frac{2M}{2^n}$.
  - They are nested, $I_n \subseteq I_{n-1}$.
- a subsequence x_{n_k}
  - for all k_1, $x_{n_k} \in I_k$.

The Nested Interval Property states that $\bigcup_{n=1}^{\infty} I_n$ is non empty. Let L be a point in $\bigcup_{n=1}^{\infty} I_n$. We claim $\lim_{n\to\infty} x_{n_k} = L$. Let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that $\frac{2M}{2^n} < \epsilon$. (Since $\lim_{n\to\infty} \frac{2M2^n}{=} 0$. See Theorem 2.5.5) For any $k \geq N$, recall that x_{n_k}, $L \in I_k$. Since I_k has length $\frac{2M}{2^n}$. Thus, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n\to\infty} x_{n_k} = L$ and (x_n) has a convergence subsequence. □



Theorem 2.5.5

Let $b \in (0, 1)$. Then $\lim_{n\to\infty} b^n = 0$.

Proof. The sequence (b^n) is monotone decreasing. This is because $b^{n+1} = b^n b < b^n$. This sequence is also bounded by 0. Hence, by the Monotone Convergence Theorem, (b^n) converges. Now, let $L = \lim_{n\to\infty} b^n$. Consider the subsequence b^{2n}. This sequence also converges to L. Thus,

$$\begin{aligned} L &= \lim_{n\to\infty} b^{2n} \\ &= \lim_{n\to\infty} b^n b^n \\ &= \lim_{n\to\infty} b^n \lim_{n\to\infty} b^n \\ &= L^2. \end{aligned}$$

Thus, $L = 0$ or $L = 1$. The limit cannot be 1 because b^n is decreasing away from 1. Therefore, $L = 0$. □

2.5.1 Exercises

Exercise: 2.5.1

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

Solution.

(a) **Impossible.** This violates the Bolzano-Weierstrass Theorem. It assures us that every bounded sequence has a convergent subsequence. If a subsequence is bounded, then it must have a convergent subsequence.

(b) Consider the sequence $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \cdots \frac{1}{n}, \frac{(n-1)}{n})$. From this, you can have a subsequence $(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n})$ which converges to 0, and also a subsequence $(\frac{1}{2}, \frac{2}{3}, \ldots \frac{n-1}{n})$, which converges to 1.

Exercise: 2.5.2

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

Solution.

(a) **True.** If every proper subsequence of (x_n) converges, then (x_n) must converge to the same limit. If (x_n) did not converge, there would exist at least one divergent subsequence or two subsequences converging to different limits, contradicting the assumption.

(b) **True.** If (x_n) contained a divergent subsequence, then (x_n) cannot converge. A convergent sequence has all its subsequences converging to the same limit, so the existence of a divergent subsequence implies that (x_n) diverges (contrapositive).

(c) **True.** Since (x_n) is bounded and diverges, the Bolzano-Weierstrass Theorem guarantees the existence of at least one convergent subsequence. Let this subsequence converge to L_1. Because (x_n) does not converge to L_1, there is an $\epsilon > 0$ and infinitely many terms of (x_n) such that $|x_n - L_1| \geq \epsilon$. Extracting a subsequence from these terms, the Bolzano-Weierstrass Theorem ensures a further subsequence converging to a limit $L_2 \neq L_1$. Thus, (x_n) has two subsequences converging to different limits.

Exercise: 2.5.5

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Proof. Suppose that (a_n) does not converge to $a \in \mathbb{R}$. By the definition of convergence, this means there is a positive real number ϵ_0 such that no matter how large we choose $N \in \mathbb{N}$, there will always exist some $n > N$ where $|a_n - a| \geq \epsilon_0$. In a formal way, this shows that (a_n) does not converge to a within the ϵ_0-neighborhood.

We aim to demonstrate that this leads to a contradiction by constructing a subsequence of (a_n) that stays outside this neighborhood. Begin by selecting n_1 such that $|a_{n_1} - a| \geq \epsilon_0$. Next, since the condition holds for all $N \in \mathbb{N}$, we can find another index $n_2 > n_1$ such that $|a_{n_2} - a| \geq \epsilon_0$. Continuing this process, we generate an increasing sequence of indices $n_1 < n_2 < n_3 < \dots$ such that for each $i \in \mathbb{N}$, $|a_{n_i} - a| \geq \epsilon_0$.

Now consider the subsequence (a_{n_i}) we have built. Since (a_n) is bounded by assumption, its subsequence (a_{n_i}) is also bounded. By the Bolzano-Weierstrass Theorem, every bounded sequence has a convergent subsequence. Let $(a_{n_{i_k}})$ denote a convergent subsequence of (a_{n_i}). According to our assumption, any convergent subsequence of (a_n) must converge to a.

However, each term of $(a_{n_{i_k}})$ remains outside the ϵ_0-neighborhood of a. Thus, it is impossible for $(a_{n_{i_k}})$ to converge to a. This contradiction implies that our initial assumption—that (a_n) does not converge to a—is false. Therefore, the sequence (a_n) must converge to a. $\square$

Exercise: 2.5.6

Use a similar strategy to the one in Theorem 2.5.5 to show

$$\lim b^{1/n} \text{ exists for all } b \geq 0$$

and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Proof. We will show that $\lim_{n\to\infty} b^{1/n}$ exists for all $b \geq 0$ and find its value.

- **Case 1:** $b = 0$.

  When $b = 0$, the sequence becomes $a_n = 0^{1/n} = 0$ for all n. Thus, $\lim_{n\to\infty} b^{1/n} = 0$.

- **Case 2:** $b > 0$.

  Suppose, for contradiction, that $\lim_{n\to\infty} b^{1/n} \neq 1$. Then there exists $\epsilon > 0$ and infinitely many n such that $|b^{1/n} - 1| \geq \epsilon$. Extract a subsequence (b^{1/n_k}) where this inequality holds for all k.

  Since $b^{1/n} > 0$ and bounded, by the Bolzano-Weierstrass Theorem, the subsequence (b^{1/n_k}) has a further subsequence that converges to a limit L. According to Exercise 2.3.1, any convergent subsequence of $(b^{1/n})$ must have its limit equal to $\lim_{n\to\infty} b^{1/n}$.

  Consider $\ln b^{1/n} = \frac{\ln b}{n}$. As $n \to \infty$, $\frac{\ln b}{n} \to 0$, so $\ln b^{1/n} \to 0$, which implies $b^{1/n} \to e^0 = 1$.

  This contradicts the assumption that $|b^{1/n_k} - 1| \geq \epsilon$, so $\lim_{n\to\infty} b^{1/n} = 1$.

Conclusion:

$$\lim_{n\to\infty} b^{1/n} = \begin{cases} 0, & \text{if } b = 0, \\ 1, & \text{if } b > 0. \end{cases} \quad \square$$

2.6 The Cauchy Criterion

Recall

How do we prove x_n converges?

1. We know and prove the limit $\to$ claim L, show terms get close to L.
2. Monotone Convergence Theorem.

Definition 2.6.1

A sequence (x_n) is a *Cauchy sequence* if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|x_m - x_n| < \epsilon$.

This says that as terms get close to each other and stay close together, there's some value they're all getting close to.

Geometric Series Review

Remember that geometric series consist of terms that are multiplied by a ratio r. For example, that could look like $1 + r + r^2 + r^3 + \cdots$.

We are most interested in **partial sums**. That is,

$$1 + r + r^2 + \cdots r^{n-1} + r^n = S_n.$$

From here, we we would multiply both sides by r. This gives

$$r + r^2 + \cdots r^n + r^{n+1} = rS_n.$$

When we subtract these two from each other, we get

$$1 - r^{n+1} = S_n - rS_n.$$

This yields the identity

$$1 + r + r^2 + \cdots r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Example 2.13: Cauchy Sequence

Consider the sequence $a_1 = 1, a_2 = 2$, where

$$a_n = \frac{a_{n-1} + a_{n-2}}{2} \text{ for all } n \geq 2.$$

Show this sequence is Cauchy.

Proof. Look at the differences of consequtive terms, $|a_1 - a_2| = 1$, $|a_2 - a_3| = 1/2$, we can see a formula $a_n - a_{n+1} = 1/2^{n-1}$. Assume $|a_n - a_m| = |a_n - a_{n+1} - a_{n+2}| - \cdots -$

$a_{m-1} - a_m$ with $n < m$. From the Triangle Inequality,

$$|a_n - a_m| \leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \tag{2.10}$$
$$= \frac{1}{2^{n-1}} + \frac{1}{2}^n + \cdots + \frac{1}{2}^{m-2} \tag{2.11}$$
$$= \frac{1}{2^{n-1}} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2}^{m-n-1}\right) \tag{2.12}$$
$$= \frac{1}{2^{n-1}} \left(\frac{1 - \frac{1}{2}^{m-n}}{1 - \frac{1}{2}}\right) \tag{2.13}$$
$$= \frac{1}{2^n} \left(1 - \frac{1}{2^{m-n}}\right) \tag{2.14}$$
$$< \frac{1}{2^n}. \tag{2.15}$$

Notice that we were able to pull out the 1/2 and use the geometric series formula at step 2.12. From here we know that $|a_n - a_m| < \frac{1}{2^n}$.

Now, conclude the proof by letting $\epsilon > 0$. We know $(1/2^n) \to 0$. Thus, there exists $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$. For all $n, m \geq N$, (without loss of generality $n < m$) $|a_n - a_m| < \frac{1}{2^n} \leq \frac{1}{2^N} < \epsilon$. Therefore, a_n is Cauchy and it converges. □

Note: To find the limit of this series, a proof strategy is finding subsequences that are odd and even, and show the converge to the same limit.

Theorem 2.6.2: Cauchy Criterion

A sequence x_n converges if, and only if, it is a Cauchy sequence.

Proof. We will show this by proving both implications:

($\Rightarrow$) Assume (x_n) is a convergent seqeuence in $\mathbb{R}$. Given $\epsilon > 0$. Let $L = \lim_{n\to\infty} x_n$. Since $(x_n) \to L$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \frac{\epsilon}{2}$. For all $n, m \geq N$,

$$\begin{aligned} |x_m - x_n| &= |x_m - L + L - x_n| \\ &\leq |x_m - L| + |L - x_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, x_n is a Cauchy sequence.

($\Leftarrow$) Assume x_n is a Cauchy sequence.

- **Step 1:** Show that x_n is bounded.

  Since x_n is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < l$. It follows that for all $n \geq N$, we need to account for $x_1 \ldots, x_{n-1}$. Thus, let $M = \max\{|x_1|, |x_2|, \ldots, |x_{n-1}|, |x_n| + 1\}$. Then for all $n \in \mathbb{N}$, $|x_m| < M$.
- **Step 2:** Since x_n is bounded, there exits a convergent subsequence x_{n_k} by the Bolzano-Weierstrass Theorem. Let L be the limit of the subsequence.
- **Step 3:** Show that x_n converges to L.

  If some get close to L and all get close to each other, they all get close to L. Let $\epsilon > 0$. Since x_{n_k} converges to L, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $|x_{n_k} - L| < \frac{\epsilon}{2}$. Since x_n is Cauchy, there exists $M \in \mathbb{N}$ such that for all $n, m \geq M$, $|x_n - x_m| < \frac{\epsilon}{2}$. Let $M_0 = \max\{N_1, n_k\}$. By the Archimedean Principle, there exists N_0 such that $n_{k_0} \geq M_0$. Then, from the Triangle Inequality, we say that for all $n \geq N_0$,

  $$\begin{aligned} |x_n - L| &\leq \left|x_n - x_{n_{k_0}}\right| + \left|x_{n_{k_0}} - L\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

  Therefore, $(x_n) \to L$.

By proving both directions of the inequality, we found that a sequence (x_n) converges if, and only if, it is a Cauchy sequence. □

Definition 2.6.3

A sequence is called *contracting* is there exists $0 < C < 1$ such that for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq C\,|x_n - x_{n-1}|$.

How this works: we take a sequence $a_1, a_2, \ldots$ and subtract $a_1 - a_2$. Then, we have the inequality:

$$\begin{aligned} |a_2 - a_1| &\leq C\,|a_1 - a_0| \\ |a_3 - a_2| &\leq C\,|a_2 - a_1| \leq C^2\,|a_1 - a_2| \\ |a_4 - a_3| &\leq C\,|a_3 - a_2| \leq C^3\,|a_1 - a_2| \\ &\vdots \end{aligned}$$

From this, a theorem emerges:

Theorem 2.6.4

If a sequence is contracting, then it is Cauchy, and thus converges.

Proof. Let (a_n) be a contracting sequence; that is, there exists a constant $0 < C < 1$ such that for all $n \in \mathbb{N}$,

$$|a_{n+1} - a_n| \leq C|a_n - a_{n-1}|.$$

We will show that (a_n) is a Cauchy sequence.

First, we observe by induction that for all $k \geq 1$,

$$|a_{n+k} - a_{n+k-1}| \leq C^k|a_n - a_{n-1}|.$$

Proof by induction:

Base case $(k = 1)$**:**

$$|a_{n+1} - a_n| \leq C|a_n - a_{n-1}|.$$

Inductive step: Assume that for some $k \geq 1$,

$$|a_{n+k} - a_{n+k-1}| \leq C^k|a_n - a_{n-1}|.$$

Then,

$$\begin{aligned} |a_{n+k+1} - a_{n+k}| &\leq C|a_{n+k} - a_{n+k-1}| \\ &\leq C\left(C^k|a_n - a_{n-1}|\right) \\ &= C^{k+1}|a_n - a_{n-1}|. \end{aligned}$$

Thus, the inequality holds for $k + 1$, completing the induction.

Next, for any integers $m > n$, we have:

$$|a_m - a_n| = \left|\sum_{j=n}^{m-1} (a_{j+1} - a_j)\right| \leq \sum_{j=n}^{m-1} |a_{j+1} - a_j|.$$

Applying the inequality obtained from the induction,

$$|a_{j+1} - a_j| \leq C^{j-n+1}|a_n - a_{n-1}|.$$

Therefore, Since $C^{m-n} \geq 0$, we have:

$$|a_m - a_n| \leq |a_n - a_{n-1}| \left(\frac{C}{1-C}\right).$$

As $n \to \infty$, the term $|a_n - a_{n-1}|$ tends to zero because:

$$|a_n - a_{n-1}| \leq C^{n-1}|a_1 - a_0| \to 0 \quad \text{as } n \to \infty.$$

Therefore, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - a_{n-1}| < \epsilon \left(\frac{1-C}{C}\right).$$

Then, for all $m, n \geq N$ (with $m > n$),

$$|a_m - a_n| \leq |a_n - a_{n-1}| \left(\frac{C}{1-C}\right) < \epsilon.$$

This shows that (a_n) is a Cauchy sequence. Since every Cauchy sequence in $\mathbb{R}$ converges, the sequence (a_n) converges. $\square$

CHAPTER 4

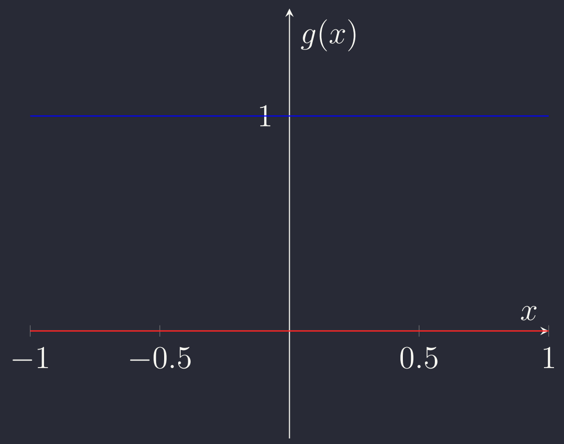
FUNCTIONAL LIMITS AND CONTINUITY

4.1 Discussion: Examples of Dirichlet and Thomae

Definition 4.1.1

The *Dirichlet function* $\lim_{x\to c} g(x)$ does not exist for any $c \in \mathbb{R}$.

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$



Definition 4.1.2

The *Thomae function* is defined as

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is in lowest terms}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Thomae's function, $t(x)$ is continuous at all $x \notin \mathbb{Q}$. It is not continuous at any $x \in \mathbb{Q}$.

4.2 Functional Limits

Recall from calculus I, that a function $f(x)$ is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$.

Definition 4.2.1

Let $f\colon A \to \mathbb{R}$ be a function and let c be a limit point of A. We say $\lim_{x\to c} f(x) = L$, if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Example 4.1: Functional Limit (From book) 1

Let $f(x) = 3x + 1$. Claim: $\lim_{x\to 2} f(x) = 7$.

Proof. Let $\epsilon > 0$. After we have done our scratch work, we can choose $\delta = \epsilon/3$, then $0 < |x - 2| < \delta$ implies $|f(x) - 7| < 3(\epsilon/3) = 3$. □

Scratch Paper. Definition 4.2.1 requires that we produce a $\delta > 0$ so that $0 < |x - 2| < \delta$ leads to the conclusion that $|f(x) - 7| < \epsilon$. Notice that

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3\,|x - 2|\,.$$

Example 4.2: Functional Limit (From book) 2

Let $g(x) = x^2$. Claim: $\lim_{x\to 2} g(x) = 4$.

Proof. Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/5\}$. If $0 < |x - 2| < \delta$, then

$$\begin{aligned} |g(x) - 4| &= \left|x^2 - 4\right| \\ &= |x - 2|\,|x + 2| \\ &< 5\delta \\ &= (5)\frac{\epsilon}{5} \\ &= \epsilon. \end{aligned}$$

□

Scratch Paper. Our goal this time is to make $|g(x) - 4| < \epsilon$ by restricting $|x - 2|$ to be smaller than some carefully chosen δ. As in the previous example, a little algebra reveals

$$|g(x) - 4| = \left|x^2 - 4\right| = |x - 2|\,|x + 2|\,.$$

We can make $|x + 2|$ as small as we like, but we need an upper bound on $|x + 2|$ in order to know how small to choose δ. The presence of the variable x causes some initial confusion, but keep in mind that we are discussing the limt as x approaches 2. If we agree that our $\delta-$neighborhood around $c = 2$ must have radius no bigger than $\delta = 1$, then we get the upper bound $|x + 2| < |3 + 2| = 5$ for all $x \in V_\delta(c)$.

Example 4.3: Functional Limit 1

Let $f(x) = 3x + 1$. Show that $\lim_{x\to 2} f(x) = 7$.

Proof. Let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{3}$. Assume $0 < |x-2| < \delta$. Since $\delta > 0$, $2 - \delta < x < 2 + \delta$. Then,

$$\begin{aligned} |x-2| &< \delta, \\ |f(x) - 7| &= |3x + 1 - 7| \\ &= |3x - 6| \\ &= 3\,|x-2| \\ &< 3\delta \\ &= \epsilon. \end{aligned}$$

Therefore, $\lim_{x\to 2} f(x) = 7$. $\square$

Example 4.4: Functional Limit 3

Let $f(x) = x^2$. Claim: $\lim_{x\to 7} f(x) = 49$

Proof. Let $\epsilon > 0$. Set $\delta = \min\{\frac{\epsilon}{8}, 1\}$. If $0 < |x-7| < \delta$, then

$$\begin{aligned} |f(x) - 49| &= \left|x^2 - 49\right| \\ &= |x-7|\,|x+7| \\ &< 8\delta \\ &< 8\left(\frac{\epsilon}{8}\right) \\ &= \epsilon. \end{aligned}$$

$\square$

Scratch Paper. Always start with the goal statement: $|f(x) - 49| = |x^2 - 49|$. This factors into $|x-7|\,|x+7|$. Then, if $\delta < 1$, $|x-7| < \delta$ and $|x+7| < 8$. All together, we have $8\delta < \epsilon < \frac{\epsilon}{8}$.

Example 4.5: Functional Limit 4

Claim: $\lim_{x\to 3} \frac{1}{x+1} = \frac{1}{4}$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{12\epsilon, 1\}$. If $0 < |x-3| < \delta$, then

$$\begin{aligned}\left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{4-(x+1)}{4(x+1)}\right| \\ &= \left|\frac{3-x}{4(x+1)}\right| \\ &< \frac{\delta}{4(3)} \\ &= \frac{12\epsilon}{12} \\ &= \epsilon.\end{aligned}$$

Therefore, $\lim_{x\to 3} \frac{1}{x+1} = \frac{1}{4}$ □

Scratch Paper. Goal: $\left|\frac{1}{x+1} - \frac{1}{4}\right|$. Hence,

$$\begin{aligned}\left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{4-(x+1)}{4(x+1)}\right| \\ &= \left|\frac{3-x}{4(x+1)}\right| \\ &< \frac{\delta}{4\,|x+1|} \\ &< \frac{\delta}{4(3)} \\ &= \frac{\delta}{12} \\ &< \epsilon.\end{aligned}$$

Thus, we need a $\delta < 1$, and we can choose $\delta = \min\{12\epsilon, 1\}$. Note: When we are determining the value for $|x+2|$, we solve for $\delta = 3 \pm 1 \Rightarrow x \in (2,4)$. Then, we find $x+1 = (3,5)$. We choose 3 rather than 5 because of division. We want to be as close as possible.

Example 4.6: Functional Limit 5

Claim: $\lim_{x\to 3}(x^2 + 7x) = 30$.

Proof. Let $\epsilon > 0$ and set $\delta = \min\{\frac{\epsilon}{14}, 1\}$. If $0 < |x-3| < \delta$, then

$$\begin{aligned}\left|x^2 + 7x - 30\right| &= |x-3|\,|x+10| \\ &< 14\delta \\ &= 14\left(\frac{\epsilon}{14}\right) \\ &= \epsilon.\end{aligned}$$

□

Example 4.7: Functional Limit 6

Claim: $\lim_{x\to 3} \frac{2x+3}{4x-9} = 3$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{\frac{\epsilon}{10}, \frac{1}{2}\}$. (Note: We are choosing $\frac{1}{2}$ because we want to avoid having 0 anywhere in the interval.) Assume $0 < |x-3| < \delta$. Since $\delta < \frac{1}{2}$, $\frac{5}{2} < x < \frac{7}{2}$, then $1 < |4x-9| < 5$. (Thus, 0 can not possibly be in the denominator.) ☐

Scratch Paper.

$$\begin{aligned}\left|\frac{2x+3}{4x+9} - 3\right| &= \left|\frac{2x+3-3(4x+9)}{4x+9}\right| \\ &= \left|\frac{2x+3-12x-27}{4x+9}\right| \\ &= 10\left|\frac{x-3}{4x-4}\right| \\ &< 10\frac{\epsilon/10}{1} \\ &= \epsilon.\end{aligned}$$

Example 4.8: Functional Limit 7

Claim: $\lim_{x\to 4}\sqrt{x} = 2$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{1, 3\epsilon\}$. Assume $0 < |x-4| < \delta$. Then (refer to scratch work). ☐

Scratch Paper.

$$\begin{aligned}\left|\sqrt{x} - 2\right| &= \left|\sqrt{x} - 2\right| \\ &= \left|\frac{(\sqrt{x}-2)\cdot(\sqrt{x}+2)}{\sqrt{x}+2}\right| \\ &= \left|\frac{x-4}{\sqrt{x}+2}\right| \\ &< \frac{\delta}{3} \\ &< \frac{3\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Notice that we picked $\delta < 1$ such that $3 < x < 4$ so $1 < \sqrt{x} < 2$ and $3 < \sqrt{x}+2 < 4$.

Theorem 4.2.2: Sequential Criterion for Functional Limits

The following statements are equivalent:

1. $\lim_{x\to c} f(x) = L$.
2. For all sequences (x_n) where $x_n \neq c$ and $(x_n) \to c$, we have $\lim_{n\to\infty} f(x_n) = L$.

Proof. (1) $\to$ (2)

Assume $\lim_{x\to c} f(x) = L$.
Let $(x_n) \to c$ with $x_n \neq c$
Let $\epsilon > 0$.

- Since $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.
- Since $x_n \to c$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - c| < \delta$.
- Now, for all $n \geq N$, it follows that $x_n - c < \delta$ and thus $|f(x) - L| < \epsilon$.

Thus, $\lim_{n\to\infty} f(x_n) = L$.

(2) $\to$ (1)

Proof by contrapositive.
Assume (1) is not true. Thus,

$$\lim_{x\to c} f(x) \neq L.$$

There exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists an x with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon_0$.
For each $n \in \mathbb{N}$, consider $\delta = \frac{1}{n}$. There exists $x_n \in (c - \frac{1}{n}, c + \frac{1}{n})$ with $x_n \neq c$ such that $|f(x) - L| \geq \epsilon_0$.

- Since $|x_n - c| < \frac{1}{n}$, we see that $(x_n) \to c$.
- Since for all $n \in \mathbb{N}$, $|f(x) - L| \geq \epsilon_0$. Then, $\lim_{n\to\infty} f(x) \neq L$.

Thus, $\neg(1) \to \neg(2)$. So (2) $\to$ (1) and (1) $\to$ (2). $\square$

If functional limits and sequential limits are the same thing, then everything we know about sequential limits is also true about functional limits.
Recall Algebraic Limit Theorem. From this, we can write the functional equivalent:
Assume $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$. Then,

- $\lim_{x\to c}(f(x) + g(x)) = L + M$
- $\lim_{x\to c}(f(x) - g(x)) = L - M$
- $\lim_{x\to c}(f(x)g(x)) = LM$
- $\lim_{x\to c}(\frac{f(x)}{g(x)}) = \frac{L}{M}$ unless $M = 0$.

Theorem 4.2.3: Divergence Criterion

Let $f\colon A \to \mathbb{R}$ with c as a limit point of A. If there exists two sequences (x_n) and (y_n) in $A \setminus \{c\}$ (that both converge to c) such that $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, then $\lim_{x\to c} f(x)$ does not exist.

Example 4.9: Divergence Criterion 1

$f(x) = \frac{x}{|x|} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$ Our goal is to show that $\lim_{x\to 0} f(x)$ does not exist.

Proof. Let $(x_n) = (\frac{1}{n})$ and let $(y_n) = (\frac{-1}{n})$. We will see that as $n \to \infty$, $\lim_{n\to\infty} f(x_n) = 1$ and $\lim_{n\to\infty} f(y_n) = -1$. Thus, $\lim_{x\to 0} f(x)$ does not exist. □

Example 4.10: Divergence Criterion 2

$g(x) = \sin(\frac{1}{x})$. Show that $\lim_{x\to 0} g(x)$ does not exist.

Proof. Let $(x_n) = (\frac{1}{2\pi n})$ and let $(y_n) = (\frac{1}{2\pi n + \frac{\pi}{2}})$. We will see that as $n \to \infty$, $\lim_{n\to\infty} g(x_n) = 1$ and $\lim_{n\to\infty} g(y_n) = -1$. Thus, $\lim_{x\to 0} g(x)$ does not exist. □

We say $\lim_{n\to\infty} x_n = \infty$ if for all $M > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n > M$.

We say $\lim_{x\to c} f(x) = \infty$ if for all $M > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $f(x) > M$. Think of vertical asymptotes.

Theorem 4.2.4: Infinite Limits Cauchy Criterion

If $(x_n) \to \infty$, (x_n) will not be Cauchy. It is possible to have $x_{n+1} - x_n$ approach 0, but (x_n) converges to ∞.

4.3 Continuous Functions

Definition 4.3.1

We say a function f is *continuous* at c if

$$\lim_{x \to c} f(x) = f(c).$$

Equivalent definition:
For all $\epsilon > 0$, there exists $\delta > 0$ such that if $|x - c| < \delta$, then

$$|f(x) - f(c)| < \epsilon.$$

Idea: When x is close to c, $f(x)$ is close to $f(c)$. Then, for the topological definition, we can say if $x \in V_\delta(c)$ then $f(x) \in V_\epsilon(f(c))$.

Definition 4.3.2

We say function f is *continuous* on a set D if f is continuous at every point in D.

The following are equivalent (TFAE):

1. $\lim_{x \to c} f(x) = L$
2. For all sequences (x_n) such that $(x_n) \to c$, we have $\lim_{n \to \infty} f(x_n) = L$.

Continuous Functions (THM 4.3.2 in book)

Claim: Let $a \in \mathbb{R}$. Then $f(x) = a$ is continuous.

Proof. Let $c \in \mathbb{R}$. Let $\epsilon > 0$. Set $\delta = 8$. Now, if $x - c < 8$, then $|f(x) - f(c)| = a - a = 0 < \epsilon$. Thus, constant functions are continuous. □

Claim: $f(x) = x$ is continuous.

Proof. Let $c \in \mathbb{R}$. Let $\epsilon > 0$. Set $\delta = \epsilon$. If $|x - c| < \delta$, then $|f(x) - f(c)| = |x - c| < \delta = \epsilon$. Thus, the identity function is continuous. □

Claim: $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Proof.

- **Case 1:** $c \neq 0$

  Let $c \in [0, \infty)$. Let $\epsilon > 0$. Set $\delta < \epsilon$. If $|x - c| < \delta$, then $|g(x) - g(c)| = |\sqrt{x} - \sqrt{c}| = \frac{|x-c|}{\sqrt{x}+\sqrt{c}} < \frac{\delta}{\sqrt{c}} < \epsilon$. Thus, $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

- **Case 2:** $c = 0$

  Let $c \in [0, \infty)$. Let $\epsilon > 0$ Set $\delta = \epsilon^2$. If $|x - 0| < \delta$, then $|g(x) - g(0)| = |\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \epsilon$. Thus, $g(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

Therefore, $g(x)$ is continuous for all $c \in [0, \infty)$. $\square$

Theorem 4.3.3: Compositions of Continuous Functions

Let f be continuous at c. Let g be continuous at $f(x)$. Then,

$$g \circ f(x) = g(f(x)) \text{ is continuous.}$$

Proof. Let $\epsilon > 0$. Since g is continuous at $f(c)$, there exists $\delta_1 > 0$ such that if $|x - f(c)| < \delta_1$, then $|g(x) - g(f(c))| < \epsilon$. Since f is continuous at c, there exists $\delta_2 > 0$ such that if $|x - c| < \delta_2$, then $|f(x) - f(c)| < \delta_1$. Thus, if $|x - c| < \delta_2$, then $|g(f(x)) - g(f(c))| < \epsilon$. Therefore, $g \circ f(x)$ is continuous. $\square$

$$\frac{\text{Most Common Applications of Continuity}}{\text{Is with limits.}}$$

If f is continuous at c and $(x_n) \to c$, then $\lim_{n\to\infty} f(x_n) = f(c)$. Hence.

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n).$$

4.4 Continuous Functions on Compact Sets

Theorem 4.4.1: Extreme Value Theorem

If K is compact and $f : K \to \mathbb{R}$ is continuous, then f attains a maximum and minimum value on K.
In other words, there exists $a \in K$ such that $f(a) = \sup\{f(x) \mid x \in K\}$. Also, there exists $b \in K$ such that $f(b) = \inf\{f(x) \mid x \in K\}$.

Proof. We know f is bounded on K from Lemma 4.4.2 below. Hence,

$$S = \sup\{f(x) \mid x \in K\} \text{ exists.}$$

For every natural number, there exists an $x_n \in K$ such that $S - \frac{1}{n} < f(x_n) \leq S$. It follows that $\lim_{n\to\infty} f(x_n) = S$. So, now we have a sequence, (x_n) in the compact set K. Since K is compact, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence,

$$(x_{n_j}) \text{ with } a = \lim_{j\to\infty} x_{n_j} \in K.$$

Since f is continuous, we have,

$$f(a) = \lim_{j\to\infty} f(x_{n_j}) = S.$$

By a similar method, there exists $b \in K$ such that

$$f(b) = \inf\{f(x) \mid x \in K\}.$$

Note: This proof hinges on the fact that f is bounded! We need to show that f is bounded on K with a proof with subcovers. $\square$

Lemma 4.4.2

How do we know f is bounded on K? That is,

$$f(K) = \{f(x) \mid x \in K\}.$$

Show that $f(K)$ is bounded.

Proof. Let $c \in K$. Since f is continuous at c, there exists $\delta_c > 0$ such that if $|x - c| < \delta_c$, then

$$|f(x) - f(c)| < 1.$$

Do this over every $c \in K$. We get an open cover of K.

$$\mathcal{O} = \{V_{\delta_c}(c) \mid c \in K\}.$$

Since K is compact, the Heine-Borel Theorem says there exists a finite subcover. We get $c_1, c_2 \ldots, c_n \in K$ such that $K \subseteq \bigcup_{i=1}^{n} V_{\delta_{c_i}}(c_i)$. Thus,

$$\begin{aligned} f(K) &\subseteq f\left(\bigcup_{i=1}^{n} V_{\delta_{c_i}}(c_i)\right) \\ &\subseteq \bigcup_{i=1}^{n} f(V_{\delta_{c_i}}(c_i)) \\ &\subseteq \bigcup_{i=1}^{n} (f(c_i) - 1, f(c_i) + 1) \\ &\subseteq [\min f(c_i) - 1, \max f(c_i) + 1]. \end{aligned}$$

Therefore, $f(K)$ is bounded. $\square$

Theorem 4.4.3: Preservation of Compact Sets

If K is compact and $f \colon K \to \mathbb{R}$ is continuous, then $f(K)$ is compact.

Proof. Let (y_n) be a sequence in $f(K)$. We will show (y_n) has a convergent subsequence with its limit in $f(K)$.

For each n there exists $x_n \in K$ such that $f(x_n) = y_n$. So (x_n) is a sequence in a compact set K. There exists a convergent subsequence (x_{n_j}) with

$$a = \lim_{j\to\infty} x_{n_j} \in K.$$

Now consider the corresponding subsequence (y_{n_j}) in $f(K)$. Since f is continuous, we have

$$\begin{aligned}\lim_{j\to\infty} y_{n_j} &= \lim_{j\to\infty} f(x_{n_j}) \\ &= f(a) \in f(K).\end{aligned}$$

So, x_{n_j} is a convergent subsequence with limit in $f(K)$. Therefore, $f(K)$ is compact. □

Definition 4.4.4

A function $f\colon A \to \mathbb{R}$ is *uniformly continuous* on A if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, c \in A$, if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

Compare this definition with Definition 4.3.1. The difference is that the δ is independent of x. That is, we have to find one δ that needs to work for every point x.

Definition 4.4.5

A function $f\colon A \to \mathbb{R}$ is *not uniformly continuous* on A if there exists $\epsilon_0 > 0$ such that for all $\delta > 0$, there exists $x, c \in A$ such that $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon_0$.

Theorem 4.4.6

If $K \subseteq \mathbb{R}$ is compact and $f\colon K \to \mathbb{R}$ is continuous, then f is uniformly continuous on K.

Proof. Suppose f is not uniformly continuous on K. Then, there exists $\epsilon_0 > 0$ such that for all $n \in \mathbb{N}$, there exists $x_n, y_n \in K$ such that $\left|x_n - y_n < \frac{1}{n}\right|$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$.

We now have two sequences (x_n) and (y_n) in K. Since K is compact, by the heine-borel theorem, there exists a convergent subsequence (x_{n_i}) which converges to a point $x_0 \in K$.

Since K is compact, (y_{n_i}) has a convergent subsequence $\left(y_{n_{i_j}}\right)$ which converges to a point $y_0 \in K$. Notice that since $\left(x_{n_{i_j}}\right)$ is a subsequence of (x_{n_i}), it converges to x_0. Since f is continuous:

$$\lim_{j\to\infty} f\left(x_{n_{i_j}}\right) = f(x_0) \quad \text{and} \quad \lim_{j\to\infty} f\left(y_{n_{i_j}}\right) = f(y_0).$$

Because
$$\left|f\left(x_{n_{i_j}}\right) - f\left(y_{n_{i_j}}\right)\right| < \frac{1}{n_{i_j}},$$
we can see that
$$\lim_{j\to\infty}\left|x_{n_{i_j}} - y_{n_{i_j}}\right| = 0.$$
It follows that $x_0 = y_0$. But this is a contradiction because $|f(x_0) - f(y_0)| \geq \epsilon_0$. Therefore, f is uniformly continuous on K. □

4.5 The Intermediate Value Theorem

Theorem 4.5.1: Intermediate Value Theorem

Let $f\colon [a,b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(b) < L < f(a)$, then there exists $c \in (a,b)$ such that $f(c) = L$.

Note: IVT does not guarantee where, or how many c's are in the interval. It only guarantees that at least one c exists.

Proof. (Using Nested Interval Property) Without the loss of generality, assume $f(a) < f(b)$ and let $y \in (f(a), f(b))$. Let $I_1 = [a_1, b_1]$. Bisect I_1 into two intervals $[a_1, d]$ and d, b_1 where $d = \dfrac{a_1 + b_1}{2}$.

- If $f(d) < y$, set $a_2 = d$, $b_2 = b_1$, and $I_2 = [a_2, b_2]$. Notice that $f(a_2) < y < f(b_2)$.
- If $f(d) > y$, then set $a_2 = a_1$, $b_2 = d$, and $I_2 = [a_2, b_2]$. Notice that $f(a_2) < y < f(b_2)$.

Repeat this process indefinitely. We end up with a sequence of nested intervals $I_n = [a_n, b_n]$, where

- $I_n \subseteq I_{n-1}$
- $f(a_n) < y < f(b_n)$
- $|a_n - b_n| = \dfrac{a_n - b_n}{2^{n-1}}$

By the Nested Interval Property, there exists a point c such that $c \in \bigcap_{n=1}^{\infty} I_n$. In fact, there is a unique point c in the intersection. It follows that
$$c = \lim_{n\to\infty} a_n \quad \text{and} \quad c = \lim_{n\to\infty} b_n.$$
Since f is continuous, we have
$$f(c) = \lim_{n\to\infty} f(a_n) \leq y$$

$$f(c) = \lim_{n\to\infty} f(b_n) \geq y.$$

Therefore, $f(c) = y$. $\square$

What Is Important About Continuous Functions?

If $\lim_{n\to\infty}(x_n = x)$, then $\lim_{n\to\infty} f(x_{n)} = f(x)$.
Think about $f(x) = 2^x$. Thus, fx makes sense if $x \in \mathbb{Q}$:

$$2^{\frac{p}{q}} = \sqrt[q]{2^p}.$$

But how do we make sense of something like 2^π?
We can find $f\colon \mathbb{Q} \to \mathbb{R}$ is continuous.
We can define $f\colon \mathbb{R} \to \mathbb{R}$ to be continuous
If (q_n) is in $\mathbb{Q}$ and $(a_n \to \pi)$, then we define

$$f(\pi) = \lim_{n\to\infty} f(q_{n)}.$$

Exercise: 4.3.11 (Contraction Mapping Theorem)

Let f be a function defined on all of $\mathbb{R}$, and assume there is a constant c such that $0 < c < 1$ and

$$|f(x) - f(y)| \leq c|x - y|$$

for all $x, y \in \mathbb{R}$.

(a) Show that f is continuous on $\mathbb{R}$.

(b) Pick some point $y_1 \in \mathbb{R}$ and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \ldots).$$

In general, if $y_{n+1} = f(y_n)$, show that the resulting sequence (y_n) is a Cauchy sequence (Definition 2.6.1). Hence, we may let $y = \lim y_n$.

(c) Prove that y is a fixed point of f (i.e., $f(y) = y$) and that it is unique in this regard.

(d) Finally, prove that if x is *any* arbitrary point in $\mathbb{R}$, then the sequence $(x, f(x), f(f(x)), \ldots)$ converges to y defined in (b).

Solution.

(a) Let $\epsilon > 0$. Choose $\delta = \dfrac{\epsilon}{c}$. Then, for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have

$$|f(x) - f(y)| \leq c|x - y| < c\left(\frac{\epsilon}{c}\right) = \epsilon.$$

Thus, f is continuous at every point in $\mathbb{R}$.

(b) *Proof.* Since $y_{n+1} = f(y_n)$, we have $y_2 = f(y_1)$, $y_3 = f(y_2) = f(f(y_1))$, and so on. Thus, the difference between consequent terms in the sequence is:

$$|y_{n+1} - y_n| = |f(y_n) - f(y_{n-1})| \leq c|y_n - y_{n-1}|.$$

Substituting, we see that:

$$|y_{n+1} - y_n| \leq c|y_n - y_{n-1}|.$$

This inequality shows that our sequence is contracting (Definition 2.6.3) with $0 < c < 1$. Thus, by Theorem 2.6.4, the sequence is Cauchy.

□

(c) Taking the limit as $n \to \infty$, and using the continuity of f, we have:

$$f(y) = f\left(\lim_{n\to\infty} y_n\right) = \lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} y_{n+1} = y.$$

Suppose there is another fixed point z such that $f(z) = z$. Then

$$|y - z| = |f(y) - f(z)| \leq c|y - z|.$$

Since $0 < c < 1$, this implies $y - z = 0$, so $y = z$. Thus, the fixed point is unique.

(d) From the work we did in (b), we know the fixed point must be unique. Therefore, for any $x \in \mathbb{R}$, it must be the case that the sequence $(x, f(x), f(f(x)), \ldots)$ converges to the fixed point y.

Exercise: 4.3.13

Let f be a function defined on all of $\mathbb{R}$ that satisfies the additive condition $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Show that $f(0) = 0$ and that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(b) Let $k = f(1)$. Show that $f(n) = kn$ for all $n \in \mathbb{N}$, and then prove that $f(z) = kz$ for all $z \in \mathbb{Z}$. Now, prove that $f(r) = kr$ for any rational number r.

(c) Show that if f is continuous at $x = 0$, then f is continuous at every point in $\mathbb{R}$ and conclude that $f(x) = kx$ for all $x \in \mathbb{R}$. Thus, any additive function that is continuous at $x = 0$ must necessarily be a linear function through the origin.

Solution.

(a) For $f(0)$, we have:

$$\begin{aligned} f(0) &= f(0+0) \\ f(0) &= f(0)+f(0) \\ f(0) &= 2f(0) \\ f(0)-f(0) &= 2f(0)-f(0) \\ 0 &= f(0). \end{aligned}$$

For all $x \in \mathbb{R}$, we have:

$$\begin{aligned} f(0) &= f(x+(-x)) \\ 0 &= f(x)+f(-x) \\ -f(x) &= f(-x). \end{aligned}$$

(b) Let $k = f(1)$.

First, we will show that $f(n) = kn$ for all $n \in \mathbb{N}$ by induction.

Base case: For $n = 1$,

$$f(1) = k = k \cdot 1.$$

Inductive step: Assume $f(n) = kn$ for some arbitrary $n \in \mathbb{N}$. Then,

$$\begin{aligned} f(n+1) &= f(n)+f(1) \\ &= kn+k \\ &= k(n+1). \end{aligned}$$

Thus, by induction, $f(n) = kn$ for all $n \in \mathbb{N}$.

Next, from (a), we have $f(-x) = -f(x)$. Hence, for $z \in \mathbb{Z}$, if $z = -n$ where $n \in \mathbb{N}$,

$$f(z) = f(-n) = -f(n) = -kn = k(-n) = kz. \tag{1}$$

Therefore, $f(z) = kz$ for all integers z.

Now, observe that for $p, q \in \mathbb{Z}$ with $q \neq 0$, we have:

$$f\left(\underbrace{\frac{p}{q}+\frac{p}{q}+\cdots+\frac{p}{q}}_{q \text{ times}}\right) = f(p).$$

Similarly, by the additive condition (q times on each side):

$$f\left(\frac{p}{q}+\frac{p}{q}+\cdots+\frac{p}{q}\right) = f\left(\frac{p}{q}\right)+f\left(\frac{p}{q}\right)+\cdots+f\left(\frac{p}{q}\right).$$

This is equivalent to:

$$q \cdot f\left(\frac{p}{q}\right) = f(p).$$

Therefore,

$$f\left(\frac{p}{q}\right) = \frac{1}{q} f(p)$$

Putting everything together, we let $r = \frac{p}{q}$. Then,

$$\begin{aligned} f\left(\frac{p}{q}\right) &= \frac{1}{q} f(p) \\ &= \frac{1}{q}(kp) \quad \text{(from (1))} \\ &= k\left(\frac{p}{q}\right) \\ &= kr. \end{aligned}$$

Thus, $f(r) = kr$ for any rational number r.

(c) Since f is continuous at $x = 0$, we will show f is continuous everywhere.

Let $c \in \mathbb{R}$ and let $\epsilon > 0$. Since f is continuous at 0, there exists $\delta > 0$ such that if $|h| < \delta$, then $|f(h)| < \epsilon$.

For $x = c + h$, we have:

$$\begin{aligned} |f(x) - f(c)| &= |f(c+h) - f(c)| \\ &= |f(c) + f(h) - f(c)| \\ &= |f(h)| \\ &< \epsilon. \end{aligned}$$

Thus, f is continuous at c.

Since f agrees with the continuous function kx on all rational numbers and f is continuous on $\mathbb{R}$, it follows that $f(x) = kx$ for all $x \in \mathbb{R}$.

Therefore, any additive function that is continuous at $x = 0$ must be linear, $f(x) = kx$.

Exercise: 4.4.3

Show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on the set $[1, \infty)$ but not on the set $(0, 1]$.

Solution.

(a) **Uniform Continuity on $[1, \infty)$:**

Observe that for all $x, y \in [1, \infty)$,

$$|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{y^2 - x^2}{x^2 y^2} \right| = |y - x| \left(\frac{y + x}{x^2 y^2} \right).$$

Since $x, y \geq 1$, we have $x^2 y^2 \geq 1$. Therefore,

$$\frac{y + x}{x^2 y^2} = \frac{y}{x^2 y^2} + \frac{x}{x^2 y^2} \leq \frac{1}{x^2 y} + \frac{1}{x y^2} \leq 1 + 1 = 2.$$

Now, let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Then, when $|x - y| < \delta$ for $x, y \in [1, \infty)$, we have:

$$|f(x) - f(y)| = |y - x| \left(\frac{y + x}{x^2 y^2} \right) < \delta \cdot 2 = \epsilon.$$

(b) **Not Uniformly Continuous on $(0, 1]$:**

We will show that f is not uniformly continuous on $(0, 1]$ by demonstrating that no matter how small we choose $\delta > 0$, there exist $x, y \in (0, 1]$ such that $|x - y| < \delta$ but $|f(x) - f(y)|$ is arbitrarily large.

Let $\epsilon = 1$. For the sake of contradiction, assume that f is uniformly continuous on $(0, 1]$; then, there exists $\delta > 0$ such that for all $x, y \in (0, 1]$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.

Choose $x = \frac{\delta}{2}$ and $y = \frac{\delta}{2} + h$ for some h with $0 < h < \frac{\delta}{2}$. Then $|x - y| = h < \delta$. Compute $|f(x) - f(y)|$:

$$|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{y^2 - x^2}{x^2 y^2} \right| = \frac{|y^2 - x^2|}{x^2 y^2}.$$

As $x \to 0^+$, $x^2 y^2 \to 0$, making the denominator very small and the entire expression very large.

Specifically, take $x_n = \dfrac{1}{n}$ and $y_n = \dfrac{1}{n+1}$ for $n \in \mathbb{N}$. Then,

$$|x_n - y_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \delta \quad \text{for large } n.$$

However,

$$|f(x_n) - f(y_n)| = \left|n^2 - (n+1)^2\right| = \left|n^2 - n^2 - 2n - 1\right| = 2n+1.$$

Thus, as $|x_n - y_n| \to 0$, $|f(x_n) - f(y_n) \to \infty$. Therefore, by the Sequential Criterion for Absence of Uniform Continuity theorem, f is not uniformly continuous on $(0,1]$.

Exercise: 4.4.8

Give an example of each of the following, or provide a short argument for why the request is impossible.

(a) A continuous function defined on $[0,1]$ with range $(0,1)$.

(b) A continuous function defined on $(0,1)$ with range $[0,1]$.

(c) A continuous function defined on $(0,1]$ with range $(0,1)$.

Solution.

(a) **Impossible**. There does not exist a continuous function defined on $[0,1]$ with range $(0,1)$. This is because $[0,1]$ is a compact set, and the continuous image of a compact set is also compact by the Preservation of Compact Sets theorem. However, $(0,1)$ is not a compact set since it is not closed in $\mathbb{R}$. Therefore, a continuous function cannot map $[0,1]$ onto $(0,1)$.

(b) **Possible**. An example of a continuous function defined on $(0,1)$ with range $[0,1]$ is:

$$f(x) = \frac{1}{2}\sin(4\pi x) + \frac{1}{2}.$$

This is because $\sin(4\pi x)$ oscillates between -1 and 1. Therefore, $\frac{1}{2}\sin(4\pi x)$ oscillates between $-\frac{1}{2}$ and $\frac{1}{2}$. Adding $\frac{1}{2}$ shifts the range to $[0,1]$.

(c) **Impossible**. There does not exist a continuous function defined on $(0,1]$ with range $(0,1)$. Since $x=1$ is in the domain $(0,1]$ and f is continuous at $x=1$, the value $f(1)$ exists and must be in the range $(0,1)$. However, because f approaches $f(1)$ as $x \to 1^-$, the function attains its supremum at $x=1$, meaning $f(1)$ should be included in the range. But the range $(0,1)$ excludes its endpoints, leading to a contradiction. Therefore, such a function cannot exist.