Multivariable Calculus Practice Set II

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1. (2 points) Write, in general equation form, an equation of the plane which contains the three points $P = (2, 7, 3)$, $Q = (-5, 0, 1)$, and $R = (-3, 1, 2)$.

Solution. First, we find $\mathbf{PQ}$ and $\mathbf{PR}$:

$$\mathbf{PQ} = \langle -7, -7, -2 \rangle \qquad \text{and} \qquad \mathbf{PR} = \langle -5, -6, -1 \rangle .$$

With $\mathbf{PQ}$ and $\mathbf{PR}$, we can find $\mathbf{n}$ by solving for the cross product:

$$\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & -7 & -2 \\ -5 & -6, & -1 \end{vmatrix} = (7 - 12)\mathbf{i} - (7 - 10)\mathbf{j} + (42 - 35)\mathbf{k} = -5\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}.$$

With $\mathbf{n}$, we get the general formula:

$$\boxed{-5(x-2) + 3(y-7) + 7(z-3) = 0.}$$

2. (2 points) Write, in scalar form, an equation of the plane which contains the point $(5, 2, 1)$ and the line given by $x + 2 = \dfrac{y}{4} = \dfrac{z-5}{2}$.

Solution. We start by parametrizing the line with common parameter t:

- $x + 2 = t \quad \Rightarrow \quad x = t - 2$,
- $\frac{y}{4} = t \quad \Rightarrow \quad y = 4t$, and
- $\frac{z-5}{2} = t \quad \Rightarrow \quad z = 2t + 5$.

This gives us the parametric form:

$$(x, y, z) = (-2, 0, 5) + t(1, 4, 2)$$

Thus, the line passes through the point $(-2, 0, 5)$ and has the direction vector

$$\mathbf{v}_1 = \langle 1, 4, 2 \rangle .$$

Since the plane is two-dimensional, we need 2 independent directions within it. We got the first through our line, but we need another because there are infinitely many planes that contain the same line. Thus, we can form a second vector $\mathbf{v}_2$ by taking the difference between the given point and a point on the line:

$$\mathbf{v}_2 = (5, 2, 1) - (-2, 0, 5) = \langle 7, 2, -4 \rangle .$$

With $\mathbf{v}_1$ and $\mathbf{v}_2$, we find the normal vector:

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 2 \\ 7 & 2 & -4 \end{vmatrix} = (-16-4)\mathbf{i} - (-4-14)\mathbf{j} + (2-28)\mathbf{k} = -20\mathbf{i} + 20\mathbf{j} - 26\mathbf{k}$$

Therefore, we find the scalar form to be:

$$\boxed{-20(x+2) + 18y - 26(z-5) = 0.}$$

3. (3 points) Determine the arc length parametrization for the curve $\mathbf{r}(t) = 3e^t \sin(t)\mathbf{i} + 3e^t \cos(t)\mathbf{j}$, where you start from $t = 0$.

Solution. From equation 3.11 from Theorem 3.4 in the book, (and this website) we know that we can rewrite the arc length parametrization as:

$$s = \int_0^t ||\mathbf{r}'(\tau)||\, d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2}\, d\tau,$$

where $f(\tau) = 3e^\tau \sin(\tau)$ and $g(\tau) = 3e^\tau \cos(\tau)$. Thus, we find:

$$\begin{aligned} f'(\tau) &= 3e^\tau(\sin(\tau) + \cos(\tau)) \\ g'(\tau) &= 3e^\tau(\cos(\tau) - \sin(\tau)). \end{aligned}$$

Thus, we have:

$$\begin{aligned} s &= \int_0^t \sqrt{\left[3e^\tau\left(\sin(\tau) + \cos(\tau)\right)\right]^2 + \left[3e^\tau\left(\cos(\tau) - \sin(\tau)\right)\right]^2}\, d\tau \\ &= \int_0^t \sqrt{9e^{2\tau}\left(\sin^2(\tau) + 2\sin(\tau)\cos(\tau) + \cos^2(\tau)\right) + 9e^{2\tau}\left(\cos^2(\tau) - 2\sin(\tau)\cos(\tau) + \sin^2(\tau)\right)}\, d\tau \\ &= \int_0^t \sqrt{9e^{2\tau}\left[2\left(\sin^2(\tau) + \cos^2(\tau)\right) + \left(2\sin(\tau)\cos(\tau) - 2\sin(\tau)\cos(\tau)\right)\right]} \\ &= \int_0^t \sqrt{9e^{2\tau} \cdot [2(1 + 0)]} \\ &= \int_0^t 3e^\tau\sqrt{2}\, d\tau \\ &= 3\sqrt{2}\int_0^t e^\tau\, d\tau \\ &= 3\sqrt{2}(e^t - 1). \end{aligned}$$

With s, we know that $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so we need to find t in terms of s:

$$\begin{aligned} s &= 3\sqrt{2}(e^t - 1) \\ e^t &= \frac{s}{3\sqrt{2}} + 1 \\ t &= \ln\left(\frac{s}{3\sqrt{2}} + 1\right). \end{aligned}$$

Finally, by replacing t with $t(s)$ in the original equation, we can get the arc length parametrization:

$$\boxed{\mathbf{r}(s) = \left(\frac{s}{\sqrt{2}} + 3\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right)\mathbf{i} + \left(\frac{s}{\sqrt{2}} + 3\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right)\mathbf{j}.}$$

4. (3 points) Use curvature to find the equation of the osculating circle at the planar curve $y = x^3 - 4x + 1$ at $x = 1$. Then, check your answer by graphing both the curve and its circle on the same axes. [you do not need to include the graph in your work turned in – but you should be able to tell if your work is correct.]

Solution. First, we need to find the curvature of the curve at $x = 1$. We start by finding the first and second derivatives of the function:

$$\begin{aligned} y(x) &= x^3 - 4x + 1 \\ y'(x) &= 3x^2 - 4 \\ y''(x) &= 6x. \end{aligned}$$

Then, we evaluate the point and the first and second derivatives at $x = 1$:

$$\begin{aligned} y(1) &= 1 - 4 + 1 = -2 \\ y'(1) &= 3(1)^2 - 4 = -1 \\ y''(1) &= 6(1) = 6. \end{aligned}$$

With these values, we can find the curvature:

$$\kappa = \frac{|y''(x)|}{\left(1 + y'(x)^2\right)^{3/2}} = \frac{6}{\left(1 + (-1)^2\right)^{3/2}} = \frac{6}{2^{3/2}} = \frac{3\sqrt{2}}{2}.$$

With the curvature, we can find the radius of the osculating circle:

$$R = \frac{1}{\kappa} = \frac{1}{\frac{3\sqrt{2}}{2}} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}.$$

To find the center, we need the unit normal vector at $x = 1$:

$$\mathbf{N} = \frac{(-y, 1)}{\sqrt{1 + (y')^2}} = \frac{\left(-(-1), 1\right)}{\sqrt{1 + (-1)^2}} = \frac{(1, 1)}{\sqrt{2}}$$

The center C can be found by moving our point $P(1, -2)$ the distance R along the unit normal vector:

$$\begin{aligned} C &= P + R\mathbf{N} \\ &= (1, -2) + \frac{\sqrt{2}}{3}\frac{(1, 1)}{\sqrt{2}} \\ &= (1, -2) + \left(\frac{1}{3}, \frac{1}{3}\right) \\ &= \left(\frac{4}{3}, -\frac{5}{3}\right). \end{aligned}$$

This gives the equation for the osculating circle:

$$\boxed{\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{5}{3}\right)^2 = \frac{2}{9}.}$$

5. (3 points each) Suppose the position of some particle is given by $\mathbf{r}(t) = \sin(t)\mathbf{i} + t\mathbf{j} + 3t\mathbf{k}$.

(a) Find the velocity vector, $\mathbf{v}(t)$.

Solution.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

(b) What total distance is travelled by the particle over the time period $[0, 3\pi]$? (You can set up the necessary internal, and calculate it using your calculator up to 3 decimal places.)

Solution.

$$\int_0^{3\pi} ||\mathbf{r}'(t)||\, dt = \int_0^{3\pi} \sqrt{\cos^2(t) + 1 + 9}\, dt = 9.709$$

(c) Find the unit tangent vector $\mathbf{T}(t)$.

Solution.

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{||\mathbf{v}(t)||} = \frac{\cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{\cos^2(t) + 10}}$$

(d) Find unit normal vector $\mathbf{N}(t)$.

Solution. To find the unit normal vector, we need to find the derivative of the unit tangent vector. To avoid making mistakes (and making differentiating easier), lets break $\mathbf{T}(t)$ into separate functions $u(t)$ and $v(t)$:

$$\mathbf{T}(t) = \underbrace{\left(\cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k}\right)}_{u(t)} \cdot \underbrace{\left(\cos^2(t) + 10\right)^{-1/2}}_{v(t)}$$

Differentiating $u(t)$:

$$u'(t) = -\sin(t)\mathbf{i},$$

and differentiating $v(t)$ with the chain rule:

$$v'(t) = -\frac{1}{2}(\cos^2 + 10)^{-3/2} \cdot 2\cos(t)(-\sin(t)) = \cos(t)\sin(t)(\cos^2(t) + 10)^{-3/2}.$$

Thus, we apply the product rule for $\mathbf{T}'(t)$:

$$\mathbf{T}'(t) = \Big[-\sin(t)\,\mathbf{i}\Big]\left(\cos^2(t) + 10\right)^{-1/2} + \Big[\cos(t)\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}\Big]\Big[\cos(t)\sin(t)\left(\cos^2(t) + 10\right)^{-3/2}\Big].$$

Notice that both terms contain a factor of $\sin(t)$ and a power of $\cos^2(t) + 10$, so we can factor them out:

$$\mathbf{T}'(t) = \sin(t)\left(\cos^2(t) + 10\right)^{-3/2}\left\{-\Big[\cos^2(t) + 10\Big]\mathbf{i} + \cos(t)\Big[\cos(t)\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}\Big]\right\}.$$

Inside the braces, multiply and combine terms:

$$\begin{aligned}\{\ldots\} &= -\cos^2(t)\,\mathbf{i} - 10\,\mathbf{i} + \cos^2(t)\,\mathbf{i} + \cos(t)\,\mathbf{j} + 3\cos(t)\,\mathbf{k} \\ &= -10\mathbf{i} + \cos(t)\mathbf{j} + 3\cos(t)\mathbf{k}.\end{aligned}$$

This gives us:

$$\mathbf{T}'(t) = \frac{\sin(t)}{\left(\cos^2(t)+10\right)^{3/2}}\left[-10\,\mathbf{i} + \cos(t)\,\mathbf{j} + 3\cos(t)\,\mathbf{k}\right].$$

Now we need to find the magnitude of $\mathbf{T}'(t)$:

$$||\mathbf{T}'(t)|| = \frac{|\sin(t)|}{\left(\cos^2(t)+10\right)^{3/2}}\sqrt{(-10)^2 + \left(\cos(t)\right)^2 + \left(3\cos(t)\right)^2}.$$

Simplify inside the square root and factor:

$$||\mathbf{T}'(t)|| = \frac{|\sin(t)|}{\left(\cos^2(t)+10\right)^{3/2}} \cdot \sqrt{10}\,(\cos^2(t)+10)^{1/2}.$$

This simplifies to:

$$||\mathbf{T}'(t)|| = \frac{|\sin(t)|\,\sqrt{10}}{\cos^2(t)+10}.$$

Finally, we can find the unit normal vector:

$$\mathbf{N}(t) = \frac{\frac{\sin(t)}{\left(\cos^2(t)+10\right)^{3/2}}\left[-10\,\mathbf{i} + \cos(t)\,\mathbf{j} + 3\cos(t)\,\mathbf{k}\right]}{\frac{|\sin(t)|\,\sqrt{10}}{\cos^2(t)+10}}.$$

After further simplification, we see:

$$\boxed{\mathbf{N}(t) = \frac{-10\mathbf{i} + \cos(t)\mathbf{j} + 3\cos(t)\mathbf{k}}{\sqrt{10}\sqrt{\cos^2(t)+10}}}.$$

(e) Find binormal vector, $\mathbf{B}(t)$.

Solution. The binomial vector is the cross product of the unit tangent and unit normal vectors:

$$\mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\cos(t)}{\sqrt{\cos^2(t)+10}} & \dfrac{1}{\sqrt{\cos^2(t)+10}} & \dfrac{3}{\sqrt{\cos^2(t)+10}} \\ \frac{-10}{\sqrt{10}\sqrt{\cos^2(t)+10}} & \frac{\cos(t)}{\sqrt{10}\sqrt{\cos^2(t)+10}} & \frac{3\cos(t)}{\sqrt{10}\sqrt{\cos^2(t)+10}} \end{vmatrix}.$$

Thankfully, we can factor out the common term $\frac{1}{\sqrt{\cos^2(t)+10}}$ from each vector in the cross product:

$$\mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{\cos^2(t)+10}} \cdot \frac{1}{\sqrt{\cos^2(t)+10}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & 1 & 3 \\ \frac{-10}{\sqrt{10}} & \frac{\cos(t)}{\sqrt{10}} & \frac{3\cos(t)}{\sqrt{10}} \end{vmatrix}.$$

Now we can find the cross product:

$$\frac{1}{\cos^2(t)+10}\Bigg[\left(1 \cdot \frac{3\cos(t)}{\sqrt{10}}\right) - \left(\frac{\cos(t)}{\sqrt{10}} \cdot 3\right), \\ \left(3 \cdot \frac{-10}{\sqrt{10}}\right) - \left(\cos(t) \cdot \frac{3\cos(t)}{\sqrt{10}}\right), \left(\cos(t) \cdot \frac{\cos(t)}{\sqrt{10}}\right) - \left(1 \cdot \frac{-10}{\sqrt{10}}\right)\Bigg].$$

Multiplying, we see that:

$$\frac{1}{\cos^2(t)+10}\left\langle 0, \frac{-3\left(\cos^2(t)+10\right)}{\sqrt{10}}, \frac{\cos^2(t)+10}{\sqrt{10}} \right\rangle.$$

Notice $\frac{1}{\cos^2(t)+10}$ cancels with the $\mathbf{j}^{\text{th}}$ and $\mathbf{k}^{\text{th}}$ terms. Thus, we can further simply this expression to:

$$\boxed{\left\langle 0, \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}} \right\rangle.}$$