Real Analysis: Exam 2

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"All work on this take-home exam is my own." **Signature:** Paul Beggs

(1) Let $0 < a_1 < b_1$. Then, for each $n \in \mathbb{N}$, define

$$a_{n+1} = \sqrt{a_n b_n}, \qquad b_{n+1} = \frac{a_n + b_n}{2}.$$

Prove the following (in order):

(a) For any numbers $x, y \in \mathbb{R}^+$ with $x \neq y$, $\sqrt{xy} < \dfrac{x+y}{2}$.
Conclude that for all $n \in \mathbb{N}$, $0 < a_n < b_n$.

(b) (a_n) is increasing and (b_n) is increasing.

(c) Both sequences (a_n) and (b_n) must be convergent.

(d) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

(2) Recall the definitions of the limit superior and limit inferior of a sequence:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k \mid k \geq n\} \quad \text{and} \quad \liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k \mid k \geq n\}.$$

Let (x_n) and (y_n) be bounded sequences.

(a) Show that

$$\begin{aligned} \liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\leq \liminf_{n\to\infty}(x_n + y_n) \\ &\leq \limsup_{n\to\infty}(x_n + y_n) \\ &\leq \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n. \end{aligned}$$

(b) Give an example of a pair of sequences (x_n) and (y_n) for which all three of the above inequalities are strict. *That means* $<$ *instead of* $\leq$.

Solutions

(1) (a) For $x, y > 0$ with $x \neq y$, note the following property:

$$\sqrt{xy} < \frac{x+y}{2}.$$

Since $a_n, b_n > 0$ and $a_n < b_n$, it follows that:

$$a_{n+1} = \sqrt{a_n b_n} < \frac{a_n + b_n}{2} = b_{n+1}.$$

Therefore, $0 < a_{n+1} < b_{n+1}$, and by induction, $0 < a_n < b_n$ for all $n \in \mathbb{N}$.

(b) Since $a_n < b_n$, we know $a_n^2 < a_n b_n$. Taking the square root of both sides, we get:

$$\sqrt{a_n^2} = a_n < a_{n+1} = \sqrt{a_n b_n}.$$

Thus, (a_n) is increasing.
Similarly, since $a_n < b_n$, we have:

$$a_n + b_n < b_n + b_n = 2b_n.$$

Dividing by 2, we get:

$$\frac{a_n + b_n}{2} < b_n,$$

which means

$$b_{n+1} < b_n.$$

So, (b_n) is decreasing.

(c) The sequence (a_n) is increasing and bounded above by b_1, so it converges.
The sequence (b_n) is decreasing and bounded below by a_1, so it converges.

(d) Since (a_n) is increasing and is bounded above by b_1, it converges to some limit L:

$$L = \lim_{n \to \infty} a_n.$$

Conversely, since (b_n) is decreasing and is bounded below by a_1, it converges to some limit M:

$$M = \lim_{n \to \infty} b_n.$$

Thus, when we take the limit of both a_{n+1} and b_{n+1}, we get:

$$a_{n+1} = \sqrt{a_n b_n} \quad \text{and} \quad b_{n+1} = \frac{a_n + b_2}{2}$$

$$L = \sqrt{LM} \qquad \text{and} \qquad M = \frac{L+M}{2}.$$

When we square the left equation, and then multiply both sides by 2 for the right equation, we have:

$$L^2 = LM \quad \text{and} \quad 2M = L + M.$$

Because we know $0 < a_1 < b_1$ and a_n is increasing, then we know the limit $L \neq 0$ as $n \to \infty$. Additionally, because b_1 is decreasing and is bounded below by a_1, we know its limit $M \neq 0$. Thus, for the left equation, we can divide both sides by L, and then for the right equation, we can subtract M from both sides. This gives us the following:

$$L = M \qquad \text{and} \qquad M = L.$$

(2) (a) Let (x_n) and (y_n) be bounded sequences.
For each $n \in \mathbb{N}$, we have:

$$\inf_{k \geq n} x_k \leq x_k, \quad \inf_{k \geq n} y_k \leq y_k.$$

Adding these inequalities:

$$\inf_{k \geq n} x_k + \inf_{k \geq n} y_k \leq x_k + y_k.$$

Since this holds for all $k \geq n$, it follows that:

$$\inf_{k \geq n} x_k + \inf_{k \geq n} y_k \leq \inf_{k \geq n} (x_k + y_k).$$

Taking the limit inferior as $n \to \infty$:

$$\liminf_{n \to \infty} x_n + \liminf_{n \to \infty} y_n \leq \liminf_{n \to \infty} (x_n + y_n).$$

Similarly, for the limit superior, note that for each $n \in \mathbb{N}$:

$$x_k + y_k \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k.$$

Taking the supremum over $k \geq n$:

$$\sup_{k \geq n} (x_k + y_k) \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k.$$

Taking the limit superior as $n \to \infty$:

$$\limsup_{n \to \infty} (x_n + y_n) \leq \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n.$$

Combining these results:

$$\begin{aligned}\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n &\le \liminf_{n\to\infty}(x_n + y_n)\\ &\le \limsup_{n\to\infty}(x_n + y_n)\\ &\le \limsup_{n\to\infty} x_n + \limsup_{n\to\infty} y_n.\end{aligned}$$

Note: we know $\liminf\limits_{n\to\infty} x_n, \liminf\limits_{n\to\infty} y_n$ and $\limsup\limits_{n\to\infty} x_n, \limsup\limits_{n\to\infty} y_n$ exist because these sequences are bounded.

(b) Example:
Let $x_n = (-1)^n$, so $\liminf\limits_{n\to\infty} x_n = -1$, $\limsup\limits_{n\to\infty} x_n = 1$.
Let $y_n = (-1)^{n+1}$, so $\liminf\limits_{n\to\infty} y_n = -1$, $\limsup\limits_{n\to\infty} y_n = 1$.
Then $x_n + y_n = 0$ for all n.
Therefore:

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n = (-1) + (-1) = -2 < 0 = \liminf_{n\to\infty}(x_n + y_n),$$

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n = -2 < 0 = \limsup_{n\to\infty}(x_n + y_n),$$

and

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n = -2 < 2 = 1 + 1 = \limsup_{n\to\infty} x_n \limsup_{n\to\infty} y_n.$$

Hence, all three inequalities are strict.