Multivariable Calculus Notes

MATH 230

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CHAPTER 1

PARAMETRIC EQS AND POLAR COORDS

1.1 Parametric Equations

1.1.1 Introduction

Most of your calculus experience has been single variable, so that the functions under consideration were typically $f : \mathbb{R} \to \mathbb{R}$. Our course is divided into roughly 3 sections:

- Parametric Equations/Functions: Functions of the form $f : \mathbb{R} \to \mathbb{R}^n$ (Chapters 1 - 3)
- Scalar Functions: Functions of the form $f : \mathbb{R}^n \to \mathbb{R}$ (Chapters 4 - 5)
- Vector Fields: Functions of the form $f : \mathbb{R}^n \to \mathbb{R}^n$ (Chapter 6)

1.1.2 Parametric Equations

A *parametric equation* (or, *sometimes parametric function* or *vector-valued function*) is a function of the form $f\colon \mathbb{R} \to \mathbb{R}^n$. We will typically consider $n = 2$ or $n = 3$ and call the input variable the parameter, usually denoted by t. We write them as

$$f(t) = \begin{cases} x(t) \\ y(t) \end{cases} \qquad \text{or} \qquad f(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases} .$$

A *parametric curve* is the set of points $(x(t), y(t))$ in $\mathbb{R}^2$ or $(x(t), y(t), z(t))$ in $\mathbb{R}^3$ traced out. Note that in general, the curve may not be a function for y in terms of x, but is a function of the parameter t.

1.1.3 Graphing Parametric Curves in the Second Dimension

Elimination of the Parameter

In some cases, we can explicitly solve for t in terms of one of x or y. When this is possible, you can write $y(x)$ or $x(y)$ and use your "regular" algebraic knowledge. We call this process *eliminating the parameter*.

Using Technology

- Your TI-84 can graph this if you switch to `par` mode.
- Likewise, GeoGebra can do this, using the `curve` function.
  - In general, the syntax is: `curve(x(t), y(t), t, min, max)`

1.1.4 The Cycloid

A wheel of radius a is rolling along a flat road at a constant velocity. The curve generated by a point along the edge of the wheel traces out a shape called a *cycloid*. Let t represent the angle - in radians!!!! - rotated through, and that the point of interest starts at the origin. Before we find the equations for the point, let's find the location of the center of the circle:

$$f_{\text{center}}(t) = \begin{cases} x(t) = at \\ y(t) = a \end{cases}$$

Then, relative to the center, our point along the edge has equations

$$f(t) = \begin{cases} x(t) = -a\sin t \\ y(t) = -a\cos t \end{cases}$$

Thus, our point has parametric equations

$$f(t) = \begin{cases} x(t) = a(t - \sin t) \\ y(t) = a(1 - \cos t) \end{cases}$$

1.1.5 Final Notes

Next time, we'll start asking Calculus-y questions: What are the velocities in the x, y, and total directions? What total distance does it travel? What is the area of the region under one period of the cycloid?

- The syllabus has a number of practice problems to work on. These are not required, and not to be turned in, but are for you to work before class next time.
- We will talk about them at the start of the next class. You should try them beforehand.
- The most common reason for a lack of success in this class is not spending time working problems on your own.

1.2 Calculus of Parametric Curves

For this section, we will have a parametric curve in $\mathbb{R}^2$, defined by $f(t) = \begin{cases} x(t) \\ y(t) \end{cases}$. In many cases, the curve does not describe y as a function of x. However, we can still carry over many ideas from single variable calculus.

1.2.1 Slope for a Parametric Curve

Given a point t_0, the *slope of the curve* in the xy-plane is given by

$$\left.\frac{dy}{dx}\right|_{t=t_0} = \left.\frac{dy/dt}{dx/dt}\right|_{t=t_0}.$$

Note that this is undefined when $x'(t_0) = 0$.

The *tangent line* at t_0 is given by

$$y = \left(\left.\frac{dy}{dx}\right|_{t=t_0}\right)(x - x(t_0)) + y(t_0).$$

1.2.2 Second Derivative

The value of the second derivative for the curve at t_0 is given by

$$\left.\frac{d^2y}{dx^2}\right|_{t=t_0} = \left.\left(\frac{\frac{d}{dt}(dy/dx)}{dx/dt}\right)\right|_{t=t_0}.$$

Note the benefit of Leibnitz notation for each of these two derivatives!

1.2.3 Area Under a Curve

Suppose that a parametric curve is non-self intersecting. Then, the signed area of the region between the curve and the x-axis on the t interval $[t_a, t_b]$ is given by

$$A = \int_{t_a}^{t_b} y(t)\frac{dx}{dt}\,dt.$$

1.2.4 Arc Length

The *arc length* of a parametric curve over the t interval $[t_a, t_b]$ is given by

$$s = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

1.2.5 Surface Area

The *surface area* of the region obtained by rotating a non-self intersecting parametric curve is given by

$$S = \int_{t_a}^{t_b} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

1.2.6 The Cycloid

We can apply each of the above to the cycloid:

- *Derivative*: $\frac{dy}{dx} = \frac{dy}{dx} = \frac{\sin t}{1-\cos t}$. Note that the slope is then independent of the radius of the wheel and that the slope is undefined at each of $t = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$.

- *Cartesian Equation*: With radius of 3 and when $t = \frac{\pi}{3}$, the point is found by solving for $x(\frac{\pi}{3})$ and $y(\frac{\pi}{3})$:

$$\begin{aligned} x\left(\frac{\pi}{3}\right) &= 3\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right) = \pi - \frac{3\sqrt{3}}{2} \\ y\left(\frac{\pi}{3}\right) &= 3\left(1 - \cos\left(\frac{\pi}{3}\right)\right) = \frac{3}{2} \\ (x, y) &= \left(\pi - \frac{3\sqrt{3}}{2}, \frac{3}{2}\right) \end{aligned}$$

Plugging in our t value into our derivative, we get a slope of

$$\frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Now, we can write the equation of the tangent line as

$$y = \sqrt{3}\left(x - \pi + \frac{3\sqrt{3}}{2}\right) + \frac{3}{2}.$$

- *Concavity*: $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)$.

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d/dt(dy/dx)}{dx/dt} \\
&= \frac{\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)}{a - a\cos t} \\
&= \frac{\frac{\cos t(1-\cos t) - \sin t \sin t}{(1-\cos t)^2}}{a - a\cos t} \\
&= \frac{\cos t - \cos^2 - \sin^2(t)}{(1-\cos t)^2 a(1-\cos t)} \\
&= \frac{\cos t - 1}{a(1-\cos t)^2} \\
&= -\frac{1}{a(1-\cos t)^2} \\
&= -\frac{a}{a^2(1-\cos t)^2} \\
&= -\frac{a}{y^2}
\end{aligned}$$

  After some work, we find that $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$, which shows that the cycloid is always concave down.

- *Area*: The area of one period of the cycloid $A = 3\pi a^2$, after some work:

$$\begin{aligned}
A &= \int_{t_a}^{t_b} y(t)x'(t)dt \\
&= \int_0^{2\pi} (a - a\cos t)(a - a\cos t)dt \\
&= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)dt \\
&= a^2 \left(t + \frac{t}{2} + \frac{1}{4}\sin(2t)\right)\Bigg|_0^{2\pi} \\
&= a^2 \left[\left(2\pi + \frac{2\pi}{2} + \frac{1}{4}\sin(2\pi)\right) - \left(0 + \frac{0}{2} + \frac{1}{4}\sin(0)\right)\right] \\
&= a^2[2\pi + \pi] \\
&= 3\pi a^2.
\end{aligned}$$

- *Arc Length*: The arc length of one period of the cycloid is $s = 8a$, again after some work:

$$\begin{aligned}
s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt \\
&= \int_0^{2\pi} \sqrt{(a - a\cos t)^2 + (a\sin t)^2}\ dt \\
&= a\int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\ dt \\
&= a\int_0^{2\pi} \sqrt{2 - 2\cos t}\ dt \\
&= \sqrt{2}a\int_0^{2\pi} \sqrt{1 - \cos t}\ dt \\
&= \sqrt{2}a\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{t}{2}\right)}\ dt \\
&= \sqrt{2}a \cdot \sqrt{2}\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\ dt \\
&= 2a\left(-2\cos\left(\frac{t}{2}\right)\right)\Bigg|_0^{2\pi} \\
&= 8a.
\end{aligned}$$

- *Surface Area*: The surface area of the solid obtained by rotating one period of the cycloid around the x-axis is $S = \frac{64\pi a^2}{3}$, after a lot of tedious work.

$$S = \int_0^{2\pi} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

CHAPTER 2

VECTORS IN SPACE

2.1 Vectors in the Plane

2.1.1 Notation

In print, we write vectors in bold like: $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$, $\dots$. In handwriting, we often write vectors with an arrow over the top: $\vec{v}$, $\vec{w}$, $\vec{u}$, $\dots$.

2.1.2 Vectors

A *vector* is a quantity with both *magnitude* (size, length, strength, $\dots$) and *direction*. Given two points in the plane $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, the vector from P to Q, denoted $\overrightarrow{PQ} = \mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

We can also simply state components (known as *component form*): $\mathbf{v} = \langle x, y \rangle$.

In $\mathbb{R}^2$ the *standard unit vectors* are $\hat{\imath} = \mathbf{i} = \langle 1, 0 \rangle$ and $\hat{\jmath} = \mathbf{j} = \langle 0, 1 \rangle$. This allows us to write $\mathbf{v} = \langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j}$, for example.

In $\mathbb{R}^3$, we have three stand unit vectors, $\hat{\imath} = \mathbf{i} = \langle 1, 0, 0 \rangle$, $\hat{\jmath} = \mathbf{j} = \langle 0, 1, 0 \rangle$, and $\hat{k} = \mathbf{k} = \langle 0, 0, 1 \rangle$.

It is a picky detail, but $\mathbf{i} \in \mathbb{R}^2 \neq \mathbf{i} \in \mathbb{R}^3$.

The *zero vector*, denoted $\mathbf{0}$, is $\mathbf{0} = \langle 0, 0 \rangle$. Note that $\mathbf{0} \neq 0$.

A *scalar* is a real number (or a magnitude), without direction.

If c is a scalar and $\mathbf{v} = \langle x, y \rangle$, then

$$c\mathbf{v} = c\langle x, y \rangle = \langle cx, cy \rangle.$$

This operation is called *scalar multiplication*. Scalar multiplication changes the magnitude of a vector, but not its direction.

Note that the individual components of a vector are themselves *scalars*. You need to keep track of which is which.

If $\mathbf{v} = \langle x_1, y_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2 \rangle$, then the *vector sum*

$$\mathbf{v} + \mathbf{w} = \langle x_1 + x_2, y_1 + y_2 \rangle.$$

That is, we add component wise.

If $\mathbf{v} = \langle x_1, y_1 \rangle$, then the *magnitude* of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2}.$$

This is really just the Pythagorean theorem.

2.2 Vectors in Space

In $\mathbb{R}^3$, we have three axes, x, y, and z, which follow the *right-hand rule*: point the fingers of the right hand in the direction of the positive x-axis, curl them towards the positive y-axis, and the thumb points in the direction of the positive z-axis.

Since the distance formula in $\mathbb{R}^3$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$, then $\mathbf{u} = \langle x, y, z \rangle$ we have $\|\mathbf{u}\| = \sqrt{x^2 + y^2 + z^2}$.

To *normalize* a vector, we divide by its magnitude: $\mathbf{v} = \langle x, y, z \rangle$, then $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left\langle \frac{x}{\|\mathbf{v}\|}, \frac{y}{\|\mathbf{v}\|}, \frac{z}{\|\mathbf{v}\|} \right\rangle$. This gives us a *unit vector* in the direction of $\mathbf{v}$.

Thus, a *unit vector* is a vector $\mathbf{u}$ such that $\|\mathbf{u}\| = 1$.

2.2.1 Vector Properties

Suppose that each of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and r and s are scalars. Then the following properties hold:

- *Additive Commutativity*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
- *Additive Associativity*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- *Additive Identity*: $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- *Additive Inverse*: $-\mathbf{v} = (-1)\mathbf{v}$ and $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- *Scalar Associativity*: $r(s\mathbf{u}) = (rs)\mathbf{u}$.
- *Scalars Distributive over Vectors*: $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.
- *Vectors Distributive over Scalars*: $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$.
- *Multiplicative Identity*: $1\mathbf{u} = \mathbf{u}$.
- *Zero Scalar*: $0\mathbf{u} = \mathbf{0}$.

2.3 The Dot Product

Suppose $\mathbf{u} = \langle u_1, u_2, \dots u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \dots v_n \rangle$ are vectors in $\mathbb{R}^n$. Then the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

That is, we multiply the corresponding components and sum the results.

It should be clear that $\mathbf{u} \cdot \mathbf{v}$ results in a scalar. The dot product is a special type of inner product.

Think of the dot product as a way to measure how much of one vector points in the same direction as another.

2.3.1 Properties of the Dot Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and c be a scalar. Then the following properties hold:

- *Commutativity*: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- *Distributive Property*: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
- *Scalar Associativity*: $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.
- *Self-Product*: $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.
- *Magnitude*: $\|\mathbf{v}\| = \sqrt{v} \cdot v$
- *Angle*: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos(\theta)$, where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. (Law of Cosines.)
- *Orthogonality*: $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

2.3.2 Projections

The *projection* of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\mathrm{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

This is a vector parallel to $\mathbf{v}$, which has length equal to the amount of $\mathbf{u}$ which points in the same direction as $\mathbf{v}$.

Think of a projection as a measure of how much of one vector points in the same direction as another.

2.3.3 Work

If a constant force $\mathbf{F}$ moved an object from P to Q, the *work* done is given by

$$W = \mathbf{F} \cdot \overrightarrow{PQ}.$$

Thus, if that force acts at an angle θ to the line of motion, the work is:

$$W = (\|\mathbf{F}\|)\, \|\mathbf{PQ}\| \cos(\theta).$$

Later this semester, we will learn how to compensate for a non-constant force, and over a non-linear path.

2.4 The Cross Product

Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Then, the *cross product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is the unique right-hand rule vector orthogonal to each of $\mathbf{u}$ and v whose magnitude is equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then,

$$\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 \rangle .$$

NOTE: You will never multiply an v_1-coordinate by an u_1-coordinate. This is true for all v_n and u_n coordinates.

You can show by working the algebra that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

With determinants, you can do this in one step:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Oddly, we can only define a cross-product in $\mathbb{R}$, $\mathbb{R}^3$, and $\mathbb{R}^7$, while the dot product is *always* defined.

Example

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle 2, 1, 4 \rangle \cdot \langle 1, -3, 1 \rangle \\ &= \langle (1)(1) - 4(-3), 4(1) - 2(1), 2(-3) - 1(1) \rangle \\ &= \langle 13, 2, -7 \rangle . \end{aligned}$$

2.4.1 Properties of the Cross Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and c be a scalar. Then the following properties hold:

- *Anticommutativity*: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- *Distributive Property*: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
- *Scalar Associativity*: $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v})$.
- *Zero*: $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
- *Nilpotence*: $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- *Scalar Triple Product*: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.
- *Angle*: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$, where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

2.4.2 Standard Unit Vectors and the Cross Product

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
- $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
- $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
- $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
- $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

2.4.3 Torque

Torque, denoted by τ, measures the tendency to produce a rotation about an axis.

If $\mathbf{r}$ is a radial vector from an axis to a force and $\mathbf{F}$ is the force, then the torque induced on the axis by the force is given by:

$$\tau = \mathbf{r} \times \mathbf{F} \qquad \text{or} \qquad \|\tau\| = \|\mathbf{r}\| \, \|\mathbf{F}\| \sin(\theta).$$

2.5 Equations of Lines and Planes

For the vector equation, parametric equation, and the symmetric equation, use these points for the examples: $(3, 5, 1) + (9, 1, 2)$.

2.5.1 Lines

Lines in Two Dimensions

A line in $\mathbb{R}^2$ which contains the point (x_0, y_0) and is parallel to the vector $\mathbf{v} = \langle a, b\rangle$ has parametric form

$$f(t) = \begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \end{cases}.$$

Lines in Three Dimensions

In $\mathbb{R}^3$, we have more options for the form of a line. Suppose that our line contains the point $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ and is parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$. Then the following properties hold:

Vector Equation

The *vector equation* of a line is given by $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

Example: Find all 3 equations of lines

From our example, $\mathbf{v} = \langle 6, -4, 1\rangle$ and $\mathbf{r}_0 = \langle 3, 5, 1\rangle$.

Vector equation: $\mathbf{r}(t) = \langle 3, 5, 1\rangle + t\,\langle 6, -4, 1\rangle$.

Parametric Equation

The *parametric equation* of a line is given by

$$f(t) = \begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \\ z(t) = z_0 + tc \end{cases}.$$

From our example, we would get $x(t) = 3 + 6t$, $y(t) = 5 - 4t$, and $z(t) = 1 + t$.

Symmetric Equation

For the following formula, we get a, b and c from subtracting the x, y, and z components of the direction vector from the point vector.

As long as each of a, b, $c \neq 0$, the symmetric equation is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

(Notice that in two dimensions, this is just the equation of the line: $\left(\frac{b}{a}\right)(x - x_0) + y_0 = y$, when solved for y.)
From our example, we would get

$$\frac{x - 3}{9 - 3} = \frac{y - 5}{1 - 5} = \frac{z - 1}{1 - 2} \implies \frac{x - 3}{6} = \frac{y - 5}{-4} = -z + 1.$$

Line Segment

Suppose that $P = (x_0, y_0, z_0)$ and $Q = (x_1, y_1, z_1)$. The line segment from P to Q is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{p} + t\mathbf{q},$$

where $\mathbf{p} = \langle x_0, y_0, z_0 \rangle$, $\mathbf{q} = \langle x_1, y_1, z_1 \rangle$, and $0 \leq t \leq 1$.

The parametric equations for this segment are

$$f(t) = \begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \\ z(t) = z_0 + t(z_1 - z_0) \end{cases}.$$

Distance Between Point and Line

The distance from a point M to a line which contains the point P and has direction vector $\mathbf{v}$ is given by

$$d = \frac{\|\mathbf{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}.$$

Notice that you are free to choose any point on the line you'd like!

Relationships Between Lines

- *Equal*: Same direction vector, share a point.
- *Parallel*: Same direction vector, do not share a point.
- *Intersecting*: Different direction vectors, share a point.
- *Skew*: Different direction vectors, do not share a point.

2.5.2 Planes

A plane can be defined by:

- any three non-colinear points,
- any two intersection points,
- a line and a point not on the line, or
- given two orthogonal vectors with a common starting point: "spin" one vector in place; notice the other sweeps out a circle, which can be extended to a plane. * In notes *

Of particular importance for a plane is a *normal vector*. A vector **n** is a normal vector provided it is orthogonal to $\overrightarrow{PQ}$ for any two points P and Q which are in the plane.

2.5.3 Equations of a Plane

Like lines, we have three equations of a plane. Let P and Q be points in the plane and $n = \langle a, b, c \rangle$.

Vector Equation

The *vector equation* of a plane is $n \cdot \overrightarrow{PQ} = 0$. Note that this is an implicit definition (i.e. it is not useful for directly writing down an equation, but is the fundamental idea of why this all works)!

Scalar Equation

If (x_0, y_0, z_0) is any point in the plane, the *scalar equation* of the plane is given by

$$\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

General Form

The *general form* of the equation of a plane is given by $ax + by + cz + d = 0$, where $d = -ax_0 - by_0 - cz_0$.

Distance Between Point and Plane

- Equal: Share a common point, have parallel normal vectors
- Parallel: Do not share a common point, do have parallel normal vectors
- Intersecting: If their normal vectors are not parallel, the two planes intersect in a line.
  - You can use algebra to find a point in common – i.e. solve both equations for the planes
  - Find the line's direction vector by taking the cross product of the planes' normal vectors.

2.5.4 Examples

Use the points $P(3,5,1)$, $Q(9,1,2)$, and $R(0,2,5)$.

Example 1

Find the scalar equation of the plane containing P, Q, and R.

We know $\mathbf{PQ} = \langle 6, -4, 1\rangle$ and $\mathbf{PR} = \langle -3, -3, 4\rangle$. Then, $\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \langle 13, -27, -30\rangle$.

Thus, the equation of the plane is $13(x-3) - 27(y-5) - 30(z-1) = 0$.

To check, plug in the points: $13(3) - 27(5) - 30(1) = 0$, $13(9) - 27(1) - 30(2) = 0$, and $13(0) - 27(2) - 30(5) = 0$.

Distance from point to line:

Suppose M is a point and P is any point on some line l. Refer to notes for graph. Let $d = \|\mathbf{PM}\| \sin\theta$. Let $\mathbf{v}$ be a directional vector of l.

$\|\mathbf{PM} \times \mathbf{v}\| = \|\mathbf{PM}\| \, \|\mathbf{v}\| \sin\theta$. This gives us $d = \frac{\|\mathbf{PM}\times\mathbf{v}\|}{\|\mathbf{v}\|}$.

Distance from point to plane:

Let P be the point with $\mathbf{n}$ as the norm vector. We start with $d = \|\text{proj}_{\mathbf{n}}\mathbf{PQ}\|$. Then, $d = \frac{\|\mathbf{PQ}\times\mathbf{n}\|}{\|\mathbf{n}\|\cdot\|\mathbf{n}\|} \, \|\mathbf{n}\|$. Look in the book for the rest of this.

2.6 Quadratic Surfaces

2.6.1 Spheres

A *sphere*, centered at (x_0, y_0, z_0) with radius r, is given by the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

2.6.2 Cylinder

A *cylinder* is a surface in $\mathbb{R}^3$ which consists of all lines that are parallel to a given line and pass through a given plane curve. The lines that make up a cylinder are called *rulings*. The *trace* of a cylinder is the cross section generated by intersecting the cylinder with a coordinate plane.

2.6.3 Quadratic Surfaces

A *quadratic surface* is a surface in $\mathbb{R}^3$ whose equation can be written as

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

By a change of axes (rotations) and origin (translations), we can rewrite these always as one of

$$Ax^2 + By^2 + Cz^2 = 1, \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

While cylinders have rulings made of parallel lines, quadratic surfaces do not (at least, not in general). However, their traces are always conic sections: lines, parabolas, circles, ellipses, or hyperbolas.

CHAPTER 3

VECTOR-VALUED FUNCTIONS

3.1 Vector Valued Functions and Space Curves

3.1.1 Vector-Valued Functions

Recall that a *function* f from a domain D to codomain E is a rule which assigns a single element of E to each element of D.

If each of $f_1, f_2, \ldots, f_n \colon \mathbb{R} \to \mathbb{R}$ is a function we can then define the *vector-valued function* $\mathbf{r} \colon \mathbb{R} \to \mathbb{R}^n$ by

$$\mathbf{r}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t) \rangle$$

- When $n = 2$, we might write $\mathbf{r} = \langle f(t), g(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath}$,
- and when $n = 3$, we might write $\mathbf{r} = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$.

3.1.2 Curves

A *plane curve* is the set of points satisfying the parameterized curve $\mathbf{r} \colon \mathbb{R} \to \mathbb{R}^2$ over a given domain and for a given function $\mathbf{r}$.

A *space curve* is the set of points satisfying the parameterized curve $\mathbf{r} \colon \mathbb{R} \to \mathbb{R}^3$ over a given domain and for a given function $\mathbf{r}$.

We will call these *vector parameterizations* of the curve.

3.1.3 Limits

Formal Definition of a Limit

Suppose that $\mathbf{r} \colon \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function and that $a \in \mathbb{R}$, though perhaps into in the domain of $\mathbf{r}$.

If there exists a vector $\mathbf{L} \in \mathbb{R}^n$ such that for each choice of $\epsilon > 0$ there exists $\delta > 0$ such that whenever $t \in \mathbb{R}$, $t \neq a$ and $|t - a| < \delta$, then

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon,$$

then we say that $\mathbf{r}$ has *limit* $\mathbf{L}$ as t approaches a, and we write

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{L}.$$

That is, if we want the output from $\mathbf{r}$ to be close to $\mathbf{L}$, we can always choose inputs close to a to make that occur.

Limit Properties

Suppose that $f, g, h\colon \mathbb{R} \to \mathbb{R}$ are functions and that each of $\lim_{t\to a} f(t)$, $\lim_{t\to a} g(t)$, and $\lim_{t\to a} h(t)$ exist. Then, if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then $\lim_{t\to a} \mathbf{r}(t)$ exists as well, and has value

$$\lim_{t\to a} \mathbf{r}(t) = \left(\lim_{t\to a} f(t)\right)\mathbf{i} + \left(\lim_{t\to a} g(t)\right)\mathbf{j} + \left(\lim_{t\to a} h(t)\right)\mathbf{k}.$$

That is, we can essentially do limits component-wise.
Also, there is nothing special here about $\mathbb{R}^3$. This works equally well for $\mathbb{R}^n$ for any n.

3.1.4 Continuity

Suppose that $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function and that $a \in \mathbb{R}$. The statement that $\mathbf{r}$ is *continuous* at $t = a$ means that

- $\lim_{t\to a} \mathbf{r}(t)$ exists,
- $\mathbf{r}(a)$ is defined, and
- $\mathbf{r}(a) = \lim_{t\to a} \mathbf{r}(t)$.

This is of course essentially the same as our Calculus I definition of continuity, but now in $\mathbb{R}^n$.

Like that definition, informally, it means that the curve generated by the function is in "one piece."

3.2 Calculus of Vector Valued Functions

Last time, we saw that limits and continuity for vector valued functions work component-wise. This is mostly the case for derivatives and integrals as well.

3.2.1 Derivatives

Suppose that $\mathbf{r}\colon \mathbb{R} \to \mathbb{R}^n$ is a vector valued function. We define the *derivative* of $\mathbf{r}$ as

$$\mathbf{r}'(t) = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h},$$

when this limit exists. We can of course also define the left- or right- hand derivatives as needed.

3.2.2 Properties of Derivatives

Suppose that $f, \mathbf{r}, \mathbf{u}$ are differentiable and c is a scalar. Then, we have the following properties:

- $\frac{d}{dt}(c\mathbf{r}) = c\frac{d}{dt}\mathbf{r}$
- $\frac{d}{dt}(\mathbf{r} \pm \mathbf{u}) = \frac{d}{dt}\mathbf{r} \pm \frac{d}{dt}\mathbf{u}$
- $\frac{d}{dt}(f(t)\mathbf{r}(t)) = f'(t)\mathbf{r}(t) + f(t)\mathbf{r}'(t)$
- $\frac{d}{dt}(\mathbf{r} \cdot \mathbf{u}) = \mathbf{r}' \cdot \mathbf{u} + \mathbf{r} \cdot \mathbf{u}'$
- $\frac{d}{dt}(\mathbf{r} \times \mathbf{u}) = \mathbf{r}' \times \mathbf{u} + \mathbf{r} \times \mathbf{u}'$
- $\frac{d}{dt}(\mathbf{r}(f(t))) = \mathbf{r}'(f(t))f'(t)$

From the fourth above, note that if $\mathbf{r} \cdot \mathbf{r}$ is constant, then $\mathbf{r}$ is orthogonal to $\mathbf{r}'$.

3.2.3 Tangent Vectors

Suppose that $\mathbf{r}(t)$ is a differentiable vector-valued function and t_0 is the domain. This means that $\mathbf{r}(t_0)$ is tangenet to the curve generated by $\mathbf{r}(t)$ at $t = t_0$. In particular, it is a tangent vector in the sense that if we interpret $\mathbf{r}(t)$ to refer to the position of some particle as a function of time, then $\mathbf{r}'(t_0)$ is the velocity vector, and $\|\mathbf{r}'(t_0)\|$ describes the speed of the particle.

Unit Tangent Vector

Of importance is the *principle unit tangent vector*, $\mathbf{T}$, which is defined by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}.$$

Note that this is only defined when $\mathbf{r}'(t) \neq \mathbf{0}$.

The unit tangent vector is simply a vector, of length 1, which points in the tangent direction of the curve. It is useful for describing the direction of motion of a particle along a curve.

3.2.4 Integrals

Like derivatives, we do integrals component-wise:

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= \int \langle f_1(t), f_2(t), \dots, f_n(t)\rangle\,dt \\ &= \left\langle \int f_1(t)\,dt, \int f_2(t)\,dt, \dots, \int f_n(t)\,dt \right\rangle,\end{aligned}$$

where since these are indefinite, each produce a constant $C = \langle C_1, C_2, \dots, C_n\rangle$. Definite intergrals can be defined in the same way.

3.3 Arc Length and Curvature

3.3.1 The Unit Normal Vector

For a curve C defined by $\mathbf{r}$ in $\mathbb{R}^3$, we have the *unit normal vector*, $\mathbf{N}(t)$, which is defined by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

The normal vector points orthogonally to the tangent vector; it points in the direction the curve is turning.

We show that $\mathbf{T}$ and $\mathbf{N}$ are orthogonal:

$$\begin{aligned}
\mathbf{T} \cdot \mathbf{N} &= \mathbf{T}(t) \cdot \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} \\
&= \frac{1}{\|\mathbf{T}'(t)\|}\mathbf{T}(t) \cdot \mathbf{T}'(t) \\
&= \frac{1}{\|\mathbf{T}'(t)\|} \cdot 0 \\
&= 0.
\end{aligned}$$

The last step is true since $\mathbf{T}$ is a *unit* vector, and so that $\|\mathbf{T}\|$, and from our rules for differentiation, if a vector $\mathbf{T}$ has a constant magnitude, then $\mathbf{T} \cdot \mathbf{T}' = 0$.

3.3.2 The Binormal Vector

The *binormal vector*, denoted by $\mathbf{B}$, is defined by

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t).$$

Since $\mathbf{T}$ and $\mathbf{N}$ are each unit vectors which are orthogonal, by its definition, $\mathbf{B}$ is also a unit vector which is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$.

You can think of $\mathbf{T}$ and $\mathbf{N}$ as defining the instantaneous plane of motion of a particle. Then, $\mathbf{B}$ is the normal to that plane. If $\mathbf{B}$ is constant, then the particle stays in a single plane, otherwise, $\frac{d\mathbf{B}}{dt}$ measures your "twisting" or torsion of motion.

3.3.3 Arc Length

If $\mathbf{r}(t)$ defines a smooth curve in $\mathbb{R}^n$, then the *arc length* of the curve from $t = a$ to $t = b$ is given by

$$\int_a^b \|\mathbf{r}'(t)\|\, dt.$$

That is, we are just adding up the "speed" of the curve to find its length (i.e., its distance).

Arc Length Parameterization

We can define the *arc length parameterization* of a curve C by:

- Define the arc length $s(t) = \int_a^t \|\mathbf{r}'(t)\|\, dt$.
- Solving, if possible, the resulting expression for t as a function of s.
- Rewriting $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so that the curve is written as a function of its length, from a given starting point.
- This is useful since there are many possible parameterizations of a given curve, but only a single arc length parameterization.

As you may recall from Calculus II, arc length can only be explicitly worked out for carefully selected problems. The same applies here.

3.3.4 Curvature

We define the *curvature*, denoted by κ, for a smooth curve given by $\mathbf{r}(s)$ as

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

that is, how fast is the unit tangent vector changing, relative to the length of the curve itself. We use s because we want our answer to be independent of the parameterization.

However, though this formula is the easiest to reason with, it is in practice not useful in most cases. You would have to find the arc-length parameterization, and then find the unit tangent vector from that.

3.3.5 Calculating Curvature

- For all $\mathbf{r}$: $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$.
- For $\mathbb{R}^3$: $\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$.
- If $y = f(x)$: $\kappa = \frac{|y''(x)|}{[1+(y'(x)^2)]^{3/2}}$

3.3.6 Curvature of a Circle

We use each of the three equations to determine the curvature of a circle of radius a.

Parameterized Circle

A circle of radius a is parameterized by $\mathbf{r}(t) = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j}$. Thus, $\mathbf{r}'(5) = -a\sin(t)\mathbf{i} + a\sin(t)\mathbf{j}$, so that $\|\mathbf{r}'(t)\| = a$. So, using the definition above, we get:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{-a\sin(t)\mathbf{i} + a\sin(t)\mathbf{j}}{a} = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$$

and therefore

$$\mathbf{T}'(t) = -\cos(t)\mathbf{i} - \sin(t)\mathbf{j}$$

so that $\|\mathbf{T}(t)\| = 1$. Finally, we see that

$$\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{a}.$$

Therefore, a circle has constant curvature, and in fact the curvature is just the reciprocal of the radius. This should make some intuitive sense, as the radius gets smaller, the curvature will increase and vice-versa. Notice also that the curvature of a line is 0 – in some sense, a line is a circle, but with infinite radius!

Curvature of a Circle in Three Dimensions

We can also use the second formula, again with a circle, noting that a circle of radius a is parameterized by $\mathbf{r} = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j} + 0\mathbf{k}$ so that $\mathbf{r}'(t) = -a\sin(t)\mathbf{i} + a\cos(t)\mathbf{j} + 0\mathbf{k}$ so that $\|\mathbf{r}'(t)\| = a$. Thus, $\mathbf{r}'' = -a\cos(t)\mathbf{i} - a\sin(t)\mathbf{j}$, so that

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin(t) & a\cos(t) & 0 \\ -a\cos(t) & -a\sin(t) & 0 \end{vmatrix} = a^2\mathbf{k}.$$

Then,

$$\kappa = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3} = \frac{a^2}{a^3} = \frac{1}{a}.$$

Curvature as a Function

We can also write (the top half) of a circle as $y(x) = \sqrt{a_2 - x_2}$. Then, $y' = \dfrac{-x}{\sqrt{a^2 - x^2}}$ and

$$\begin{aligned} y'' &= \frac{-\sqrt{a^2 - x^2} - \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)(-x)}{a^2 - x^2} \\ &= \frac{-\sqrt{a^2 - x^2} - x^2/\sqrt{a^2 - x^2}}{a^2 - x^2} \\ &= \frac{-(a^2 - x^2) - x^2}{(a^2 - x^2)^{3/2}} \\ &= \frac{-a^2}{(a^2 - x^2)^{3/2}}. \end{aligned}$$

Thus,

$$\kappa = \frac{|y''(x)|}{[1 + (y'(x)^2)]^{3/2}} = \frac{1}{a},$$

after some algebra.

Osculating Circle

In general, of course, curvature is not constant. At a particular point of interest, the curvature finds the reciprocal of the radius of the *osculating circle*. That is, the “tangent circle”–the circle that best matches the curve at the point of interest.

3.4 Motion in Space

3.4.1 Motion

Let $\mathbf{r}(t)$ represent the position of some object as a function of time in $\mathbb{R}^3$. Then, as in Calculus I:

- Velocity vector: $\mathbf{v}(t) = \mathbf{r}'(t)$
- Speed: $\|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\|$
- Acceleration vector: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$

Note that $\mathbf{v}(t) \neq \mathbf{T}(t)$, though they do point in the same direction. More importantly, the relationship between $\mathbf{a}(t)$ and $\mathbf{N}(t)$ is complicated. $\mathbf{N}(t)$ is not acceleration, or even the direction of acceleration, but is the direction of "turning."

Acceleration

In fact, $\mathbf{a}(t) = v'(t)\mathbf{T}(t) + (v(t))^2\kappa\mathbf{N}(t)$.

Proof. Notice that $\mathbf{v}(t) = v(t)\mathbf{T}(t)$. Then,

$$\begin{aligned}
\mathbf{a}(t) &= v'(t)\mathbf{T}(t) + v(t)\mathbf{T}'(t) \\
&= v'(t)\mathbf{T}(t) + v(t)\|\mathbf{T}'(t)\|\mathbf{N}(t) \\
&= v'(t)\mathbf{T}(t) + v(t)\kappa\|\mathbf{r}'(t)\|\mathbf{N}(t) \\
&= v'(t)\mathbf{T}(t) + (v(t))^2\kappa\mathbf{N}(t).
\end{aligned}$$

Since $\mathbf{T}$ and $\mathbf{T}'$ are orthogonal, this decomposes $\mathbf{a}$ into two pieces – the first is the change in speed, and the second the change in direction. □

3.4.2 Projectile Motion

The motion of an object – in 2-dimensions, typically, acted on only by gravity $\mathbf{F}_g = -mg\mathbf{j}$, where $g \approx 9.8$ m/s^2 and m is the mass of the object. By Newton's second law, $\mathbf{F} = m\mathbf{a}$, so we have

$$\mathbf{a}(t) = -g\mathbf{j}.$$

Thus, $\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$, where $\mathbf{v}_0$ is the initial velocity vector, and

$$\mathbf{s}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{s}_0,$$

where $\mathbf{s}$ is the position, and $\mathbf{s}_0$ is the initial position vector.

Often, we have an object starting at the origin (so $\mathbf{s}_0 = \mathbf{0}$) and fired at a velocity of v_0 at an angle θ above the horizon. Then,

$$\mathbf{s}(t) = v_0 t\cos(\theta)\mathbf{i} + \left(v_0 t\sin(\theta) - \frac{1}{2}gt^2\right)\mathbf{j}.$$

What is the horizontal distance travelled by the object before it hits the ground again?

Solving $\left(v_0 t\sin(\theta) - \frac{1}{2}gt^2\right) = 0$ gives $t_1 = \emptyset,\ \frac{2v_0\sin(\theta)}{g}$. We plug this time into the horizontal component and find

$$s(t_1) = \frac{v_0^2}{g}\sin(2\theta).$$

We note the unsurprising result that this is maximized when $\theta = \frac{\pi}{4}$.

3.4.3 Planetary Motion

There is a handout (on teams) that works through the mathematics of planetary motion –specifically:

- Starting from Newton's Law of Gravity and Newton's Second Law, we determine the position of a planet circling a (fixed) star.
- Describe the conditions under which the planet is bound to the star, or separately, flung off into space!

These are not exam questions, but are the sort of things that this material is used for in the "real world."

CHAPTER 4

DIFF. OF FUN. OF SEVERAL VARIABLES

4.1-4.3 Functions of Several Variables

A *scalar function* is a function $f\colon \mathbb{R}^n \to \mathbb{R}$. This is in some sense a reverse of a parametric function. Parametric functions take in a single real variable t and output an answer of multiple variable (which we typically interpreted as a vector). A scalar function takes in multiple independent variables (typically interpreted as a point in $\mathbb{R}^n$) and outputs a single real variable.

Notation and Graphs

When $f\colon \mathbb{R}^2 \to \mathbb{R}$, we can write $z = f(x, y)$. For each point (x, y) in the domain of f, we can interpret z as a height, so that the set of points (x, y, z) which satisfy $z = f(x, y)$ form a surface in $\mathbb{R}^3$. You can graph this in GeoGebra using `geogebra.com/3d`.

When $f\colon \mathbb{R}^3 \to \mathbb{R}$, we often write $w = f(x, y, z)$. Then, the point (x, y, z) is in the domain and w is still a height, though since the graph lives we are in $\mathbb{R}^4$, we cannot draw it in a simple way.

Contour Plots

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a function. The graph is a surface in $\mathbb{R}^3$, but often it can be difficult to show this on paper. For a given $z_0 \in \mathbb{R}$ a *level curve* consists of all points $(x, y) \in \mathbb{R}^2$ for which $f(x, y) = z_0$. If we think of the function f as describing a height at a function of two-dimension position, a level curve is the set of all points with a particular given height.

A *contour plot* consists of a collection of level curves – typically, we select some number of equally spaced $\{z_1, z_2, \ldots, z_n\}$ and build the level curves for each. This gives a "topographical map" of the function f.

4.1.1 Limits

Formal Definition of $\lim_{(x,y)\to(a,b)} f(x, y)$

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$, $(a, b) \in \mathbb{R}^2$, and $L \in \mathbb{R}$. The statement that f has limit L as (x, y) approaches (a, b) means that for each choice of $\epsilon > 0$, there exists $\delta > 0$ such that for all $(x, y) \neq (a, b)$ within a disk of radius δ, centered at (a, b), then

$$|f(x, y) - L| < \epsilon.$$

Essentially, this says that the limit exists whenever we can make the output from f as close to L as we'd like by choosing (x, y) sufficiently close to (a, b).

In Calculus I, you showed that a limit $\lim_{x\to a} f(x)$ exists if and only if the left and right hand limits both exist and match. Basically, you showed that no matter if you approach a from the left or from the right, you always get the same answer. The same idea applies here, but there is an important complication: We need to check every possible path into the point (a, b) – and rather than just two, there are infinitely many paths!

In general, there is not a simple recipe that will work. You can try various combinations of paths until something goes wrong. For practice set/exam purposes, I will always tell you that the limit does not exist, and ask you to prove it. Showing that a limit does exist is a completely different procedure – just because two paths happen to produce the same answer does not prove that all do.

Problem #4 on Practice Set #3 will ask you to show a limit exists. It provides step-by-step instructions and this is the only time I will ask you to do this.

Limit Laws

The basic rules are all the same. Suppose that $a, b, c \in \mathbb{R}$ and $f, g\colon \mathbb{R}^2 \to \mathbb{R}$ and that $\lim_{(x,y)\to(a,b)} f(x, y) = L$ and $\lim_{(x,y)\to(a,b)} g(x, y) = M$.

- $\lim_{(x,y)\to(a,b)} c = c$
- $\lim_{(x,y)\to(a,b)} x = a$
- $\lim_{(x,y)\to(a,b)} y = b$
- $\lim_{(x,y)\to(a,b)}[f(x, y) + g(x, y)] = L + M$
- $\lim_{(x,y)\to(a,b)}[cf(x, y)] = cL$
- if f has a single, simple algebraic definition and (a, b) is in the domain, then $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.

4.1.2 Continuity

A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(a, b) \in \mathbb{R}^2$ provided that $\lim_{(x,y)\to(a,b)} f(x, y)$ exists and is equal in value to $f(a, b)$. This is exactly our usual definition of continuity from Calculus I again – and there is nothing special here about being in two dimensions.

All of the usual algebraic functions (polynomials, root functions, trig functions, exponentials, and logarithms) and their combinations by addition, subtraction, multiplication, division, exponentiation, and composition are continuous everywhere in their domains.

4.1.3 Partial Derivatives

Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ is a scalar function. Then, the *partial derivative* of f with respect to x_i is

$$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, x_2, \ldots, x_i + h, \ldots, x_n) - f(x_1, x_2, \ldots, x_i, \ldots, x_n)}{h},$$

if this limit exists.

That is, we hold all other variables constant and take the derivative as though x_i is the only variable. Just like in Calculus I, the partial derivative is a rate of change – it measures the ratio of how much the output of f changes relative to a small change in x_i.

Notation

Suppose that f is a function and x is one of its variables. Then, we denote the partial derivative of f with respect to x by any of:

- Leibnitz/Condorcet Notation: $\dfrac{\partial f}{\partial x}$
- Jacobi Notation: f_x
- Euler Operation Notation: $D_x f$

Higher order derivatives are notated as:

- Leibnitz/Condorcet Notation: $\dfrac{\partial^n f}{\partial x^n}$
- Jacobi Notation: $f_{xxx\ldots x}$
- Euler Operation Notation: $D_n f$

4.1.4 Mixed Partials

Given a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ we can determine the *mixed partial derivative* of f with respect to x and then y. In Leibnitz notation, this would be

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \quad \text{or in Jacobi notation} \quad f_{xy}.$$

Clairaut's Theorem

Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is defined on an open disk D containing the point (a, b). If each of f_{xy} and f_{yx} are continuous on D, then $f_{xy}(a, b) = f_{yx}(a, b)$.

That is, for pretty much any function you'll even encounter, order does not matter for the mixed partial derivatives! This is especially nice, since notationally, $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$ and $\dfrac{\partial^2 f}{\partial y \partial x} = f_{xy}$, and who wants to worry about keeping up with that!

4.4 Tangent Planes

4.4.1 Tangent Lines

For a single-variable differentiable function $f\colon \mathbb{R} \to \mathbb{R}$, in Calculus I we defined the *tangent line* at $x = x_0$ as the unique line

$$y = f'(x_0)(x - x_0) + f(x_0).$$

This is the line which most closely matches the graph of $y = f(x)$ at the desired point. Functions which have tangent lines are said to be "smooth" or "locally linear."

4.4.2 Tangent Planes

For functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ we have a similar idea. If the surface generated by such a function has no sharp corners or edges, you might see that as you zoom in, the surface becomes flatter and flatter – and will eventually resemble a plane. In fact, we define the *tangent plane* as the unique plane at $(x, y) = (x_0, y_0)$ which satisfies

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

But notice that $f(x_0, y_0)$ is just z_0, the height of the surface at (x_0, y_0). So, the tangent plane is the unique plane which most closely matches the graph of $z = f(x, y)$ at the desired point.

4.4.3 Linearizations

Recall from Calculus I that if f is differentiable at x_0 and x is close to x_0 then

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

This is the *linear approximation* of f at x_0. In the same way, if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0) and (x, y) is near (x_0, y_0), then

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

This is the *linearization* of f at (x_0, y_0).

4.4.4 Examples

$f(x, y) = x\cos(\pi x)\sin(\pi y)$, $(x_0, y_0) = \left(\frac{1}{3}, \frac{1}{2}\right)$. Our point of interest is the pair $\left(\frac{1}{3}, \frac{1}{2}\right)$ and $\frac{1}{6}$, because we can just plug in the values of x_0 and y_0 into the function to get the height.

To find the equation of the tangent plane, we need to find the partial derivatives to fill out the following equation:

$$z = \_\_\_\_ + \_\_\_\_ \left(x - \frac{1}{3}\right) + \_\_\_\_ \left(y - \frac{1}{2}\right).$$

We found z_0 to be $\frac{1}{6}$, and the partial derivatives are

$$f_x(x, y) = \cos(\pi x)\sin(\pi y) - \pi x \sin(\pi x)\sin(\pi y),$$

and

$$f_x\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{2} - \frac{\pi}{3}\left(\frac{\sqrt{3}}{2}\right)(1) = \frac{1}{2} - \frac{\pi\sqrt{3}}{3}.$$

Similarly, we have

$$f_y(x, y) = \pi x \cos(\pi x)\cos(\pi y),$$

and

$$f_y\left(\frac{1}{3}, \frac{1}{2}\right) = 0.$$

Now, we can fill out the rest of our equation:

$$z = \frac{1}{6} + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{3}\right)\left(x - \frac{1}{3}\right) + 0\left(y - \frac{1}{2}\right).$$

4.5 The Chain Rule

4.5.1 One Dimensional Chain Rule

Suppose that $f(x)$ and $g(x)$ are both single variable, differentiable functions. Then, if $h(x) = f(g(x))$, h is also differentiable and

$$h'(x) = f'\big(g(x)\big)g'(x).$$

Alternatively, if we write $u = g(x)$, then

$$h'(x) = f'(u)g'(x) \quad \text{or} \quad \frac{dh}{dx} = \frac{df}{du}\frac{du}{dx}.$$

4.5.2 Higher Dimensional Chain Rules

The Chain Rule – One Independent Variable

Suppose that $x = f(t)$, $y = h(t)$, and $z = f(x, y)$ are all differentiable functions. Notice that z is "really" a function of the single independent variable t. Then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

The Chain Rule – Two Independent Variables

Suppose that $x = g(u, v)$, $y = h(u, v)$, and $z = f(x, y)$. Then,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u},$$

and

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

The Generalized Chain Rule

Let $w = f(x_1, x_2, \ldots, x_m)$ be a function of m independent variables and each $x_i = x_i(t_1, t_2, \ldots, t_n)$ has n independent variables. Then,

$$\frac{\partial w}{\partial t_j} = \frac{\partial w}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \frac{\partial w}{\partial x_2} \cdot \frac{\partial x_2}{\partial t_j} + \cdots + \frac{\partial w}{\partial x_m} \cdot \frac{\partial x_m}{\partial t_j} = \sum_{i=1}^{m} \frac{\partial w}{\partial x_i} \cdot \frac{\partial x_i}{\partial t_j}.$$

4.5.3 Implicit Differentiation

Suppose we have an equation involving x and y. We can rewrite this as an equation of the form $z = f(x, y)$, where we let $z = 0$. Then,

$$z = 0, \text{ defines a curve for } x, y$$

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \cdot \frac{\partial f}{\partial y}.$$

This gives the same answer as the Calculus I *implicit differentiation* procedure.

4.6 Directional Derivatives and the Gradient

The function $z = f(x, y)$ defines a 2-dimensional surface in $\mathbb{R}^3$. The values of $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ tell us about the rate of change of z relative to the two independent (and orthogonal) directions x and y. What if we want to know about the rate of change in some other direction?

Let v be a vector in the direction we are interested in. Then, the *directional derivative* at the point $(x_1, x_2, \ldots, x_n)$ of the function f in the direction of a vector v is given by

$$D_v = \frac{\partial f_{x_1}(x_1, x_2, \ldots, x_n)}{\|v\|}v_1 + \frac{\partial f_{x_2}(x_1, x_2, \ldots, x_n)}{\|v\|}v_2 + \cdots + \frac{\partial f_{x_n}(x_1, x_2, \ldots, x_n)}{\|v\|}v_n.$$

4.6.1 The Gradient, ∇f

Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$. We define the *gradient* of f, denoted by $\nabla f(x, y)$, as

$$\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\mathbf{i} + \frac{\partial f}{\partial y}(x, y)\mathbf{j}.$$

We can extend this: If $g\colon \mathbb{R}^3 \to \mathbb{R}$, then

$$\nabla g(x, y, z) = \frac{\partial g}{\partial x}(x, y, z)\mathbf{i} + \frac{\partial g}{\partial y}(x, y, z)\mathbf{j} + \frac{\partial g}{\partial z}(x, y, z)\mathbf{k},$$

and so on to higher dimensions.

The symbol ∇ is called "nabla" or "del." Sometimes the gradient is written as grad f. The gradient is a vector which points in the direction of largest increase in the value of the function – if interpreted as height, it points in the direction that is the steepest uphill.

Note that the ∇ lives in the domain of the function. That is, if $f\colon \mathbb{R}^2 \to \mathbb{R}$, then $\nabla f\colon \mathbb{R}^2 \to \mathbb{R}^2$. The gradient is a vector field – it assigns a vector to each point in the domain of the function.

The gradient on a contour plot is the vector which is orthogonal to the level curve at that point. This is because the gradient points in the direction of the greatest increase, and the level curve is the set of points where the function does not change.

Properties of the Gradient

- If $\nabla f(x_0, y_0) = 0$, then $D_u f(x_0, y_0) = 0$ for all u.
- If $\nabla f(x_0, y_0) \neq 0$, then $D_u f(x_0, y_0)$ is maximized when u and $\nabla f(x_0, y_0)$ point in the same direction.
- This maximum value is $\|\nabla f(x_0, y_0)\|$.

- The gradient vector is always orthogonal to any level curve.

Del – Vector Definition

We can think of ∇ as the vector

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle.$$

From chapter 2, we know that when we multiply a vector (like ∇) by a scalar (like f), we simply multiply each coordinate of the vector by the scalar. Thus, we can think of ∇f as a "simple" vector operation. Later, when we have functions which have vector-values themselves, we will talk about the meaning of $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$.

As a final note, we will start to blur a bit the distinction between $\mathbb{R}^2$ and $\mathbb{R}^3$ here. We can think of always working within $\mathbb{R}^3$, and that $\mathbb{R}^2$ is the special case when the k component is 0.

4.6.2 Example

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$. Find the direction derivative of f at $(4, 1)$, in the $\langle 1, 3\rangle$ direction.

Solution. We first find the partial derivative of f with respect to x and y:

$$f_x(x, y) = 2x - 2 \quad \text{and} \quad f_y(x, y) = 2y - 6.$$

Then, at the point $(4, 1)$:

$$f_x(4, 1) = 6 \quad \text{and} \quad f_y(4, 1) = -4.$$

Then, we find the unit vector in the direction of $\langle 1, 3\rangle$:

$$\frac{\langle 6, -4\rangle \cdot \langle 1, 3\rangle}{\sqrt{10}} = \frac{-6}{\sqrt{10}}.$$

4.7 Maxima and Minima

For this section, we will be considering the specific case $f\colon \mathbb{R}^2 \to \mathbb{R}$.

4.7.1 Critical Points

The statement that (x_0, y_0) is a *critical point* of f means that either:

- both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or
- one or both partials does not exist.

4.7.2 Maxima and Minima

Local Extrema

The function f has a *local maximum* at (x_0, y_0) provided that $f(x_0, y_0) \geq f(x, y)$ for all choices of (x, y) in some disk centered at (x_0, y_0) – that is, in some neighborhood of (x_0, y_0).

In a similar manner, we can define a local minimum.

We will use the term *extremum* to refer to something that is either a maximum or minimum; the plural is *extrema*.

Note that if there is a local extrema, $\nabla f = \mathbf{0}$. This is because the gradient points in the direction of greatest increase, and if we are at a maximum or minimum, the function does not change.

Fermat's Theorem for Extrema: Suppose that $z = f(x, y)$ and that (x_0, y_0) is a local extremum. Then, (x_0, y_0) is also a critical point.

Second Derivative Test

Calculus I Version: In Calculus I, the sign of the second derivative tells you whether a critical point is a local max/min, or inconclusive: Suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is a function of one variable, and x_0 is a critical point.

- if $g''(x_0) > 0$, then x_0 is a local minimum
- if $g''(x_0) < 0$, then x_0 is a local maximum
- if $g''(x_0) = 0$, this test is inconclusive – it could be a max, min, or neither.

Multivariable Calculus Version: We have a similar test for $f\colon \mathbb{R}^2 \to \mathbb{R}$, where (x_0, y_0) is a critical point. Define

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2.$$

- if $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum
- if $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum
- if $D < 0$, then f has a saddle point
- if $D = 0$, then the test is inconclusive

Absolute Extrema

The function f has an *absolute maximum* at (x_0, y_0) provided that $f(x_0, y_0) \geq f(x, y)$ for all choices of (x, y) in the domain of f. Likewise, we can define an *absolute minimum*.

Extreme Value Theorem: Suppose that f is continuous on a closed, bounded domain D. Then f has both an absolute maximum and absolute minimum.

In Calculus I, we saw that a continuous function, defined on a closed interval, has its absolute extrema either at interior critical points or at the endpoints of the domain interval. This idea transfers over here – but is somewhat more complicated.

If $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuous on a closed and bounded domain, then by the Extreme Value Theorem, it has absolute extrema. These occur either at interior critical points of the domain or somewhere along the edge of the domain. The interior critical points are fairly straightforward. However, the edge of the domain is more complicated.

Extrema Finding Algorithm

- Find all critical points inside the region of interest by setting the two first partials equal to zero
- For each boundary, use the relationship between x and y to convert $f(x, y)$ to a new function, say $g(x)$, of a single variable
- This function will have an interval $[a, b]$ as its domain
- Work the Calculus I problem of finding absolute extrema for g on $[a, b]$. This will generate additional points of interests.
  - Absolute extrema occur at:
    1. Interior critical points.
    2. Along the boundary.
- Find the value of $f(x, y)$ for each point of interest – the largest value is the absolute maximum, the smallest is the absolute minimum.

4.7.3 Examples

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Solution. We know that the critical points are $(1, 3)$. We can use the second derivative test to determine that this is a local minimum.

$$f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = 0 \quad D = 4 > 0.$$

Let $f(x, y) = x^4 + y^4 - 4xy + 1$.

Solution. Find critical points:

$$f_x = 4x^3 - 4y \quad f_y = 4y^3 - 4x,$$

and set to 0:

$$x^3 = y \quad y^3 = x.$$

Substituting, we see:

$$\begin{aligned} x^9 - x &= 0 \\ x(x^8 - 1) &= 0 \\ x(x^4 + 1)(x^2 + 1)(x + 1)(x - 1) &= 0. \\ x &= 0, \pm 1 \\ \therefore y &= 0, \pm 1. \end{aligned}$$

Now we have our critical points $(0, 0), (1, 1), (-1, -1)$. We can use the second derivative test to determine the maximum or minimums. But, to use this formula, we need the second partials for the formula:

$$f_{xx} = 12x^2 \quad f_{yy} = 12y^2 \quad f_{xy} = -4.$$

For $(0, 0)$:

$$0 \cdot 0 - (4)^2 = -16 < 0.$$

Therefore, $(0, 0)$ is a saddle point. For $(1, 1)$:

$$12 \cdot 12 - 16 = 128 > 0.$$

Therefore, $(1, 1)$ is a local minimum. For the same reasons, $(-1, -1)$ is also a local minimum.

From Calculus I, remember we found the absolute extrema. Thus, let $g(x) = x^2 - 6x + 1$, on $[0, 4]$.

Solution. To find the absolute extrema of this function, get the critical points from the function, and also evaluate the function at the endpoints of the interval.

$$g'(x) = 2x - 6 = 0 \implies x = 3.$$

Then, $g(0) = 1$, $g(3) = -8$, and $g(4) = -7$. Thus, the absolute minimum is (-8) and the absolute maximum is 1.

With that refresher, lets return to our $f(x, y)$ function and find the absolute extrema. Specifically, find absolute extreme on the triangle with vertices $(0, 0), (0, 1), (\frac{1}{2}, 1)$.

Solution.

CHAPTER 5

MULTIPLE INTEGRATION

5.1 Double Integrals

5.1.1 Integration in $\mathbb{R}$

Recall that for a continuous function $f : \mathbb{R} \to \mathbb{R}$, the *signed area* between the graph of $y = f(x)$ and the x-axis on the interval $[a, b]$ is given by

$$\int_a^b f(x)\,dx.$$

Recall also that

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$ and x_i^* is any convenient point in the ith subinterval. The sum above is called a *Riemann Sum*.

5.1.2 Integration in $\mathbb{R}^2$

Let R be a rectangle defined by

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\}.$$

Suppose that $f : R \to \mathbb{R}$ be continuous. If we interpret $f(x, y)$ as a height above or below the xy-plane, we can find the *signed volume* of the solid

$$S = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in R, z \text{ lives between } 0 \text{ and } f(x, y)\}.$$

We can divide R into $m \cdot n$ rectangles by

$$\Delta x = \frac{b-a}{m} \quad \text{and} \quad \Delta y = \frac{d-c}{n}$$

and then write

$$\Delta A = \Delta x \Delta y.$$

Then,

$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A,$$

where (x_{ij}^*, y_{ij}^*) is a point selected from the ijth rectangle.

If f is continuous, then

$$\lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

exists, and we define the *double integral* of f over the rectangle R as

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Double Integral – Properties

Basically, the properties are exactly what you'd expect – we'll assume R is a rectangle, c is constant, and f, g are continuous:

- $\iint_R (f+g)\,dA = \iint_R f\,dA + \iint_R g\,dA$
- $\iint_R cf\,dA = c\iint_R f\,dA$
- $\iint_R f\,dA = \iint_S f\,dA + \iint_T f\,dA$, if S and T are rectangles, $R = S\cup T$ and $S\cap T$ is one-dimensional
- if $f \geq g$ on R, then $\iint_R f\,dA \geq \iint_R g\,dA$
- if $m \leq f(x,y) \leq M$ on R then $m\cdot \text{area}(R) \leq \iint_R f\,dA \leq M\cdot\text{area}(R)$

Factorization Property

Suppose that $f(x,y) = g(x)\cdot h(y)$ – that is, suppose we can factor f into a product of one function involving only x and another involving only y. Then,

$$\iint_R f(x,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

Iterated Integrals

For a fixed x, define

$$A(x) = \int_c^d f(x,y)\,dy.$$

This can be called a *partial integral*, and is a function of x. It is the area of one slice of the solid. Then,

$$\int_a^b A(x)\,dx = \int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_a^b\int_c^d f(x,y)\,dydx,$$

where we call this an *iterated integral*. (There is, of course, nothing special about doing y first and then x here....)

Fubini's Theorem

If f is continuous on $R = [a, b] \times [c, d]$ then

$$\iint_R f(x, y)\, dA = \int_b^a \int_d^c f(x, y)\, dydx = \int_d^c \int_b^a f(x, y)\, dxdy.$$

This is just like Clariaut's Theorem, but for integrals, rather than partial derivatives. You can work the integral in whatever order is most convenient.

5.2 Double Integrals Over General Regions

5.2.1 General Regions

Suppose that D is a general region of finite size in $\mathbb{R}^2$. Then, there exists some rectangle R which contains D. We can take a function $f(x, y)$ and define

$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D, \\ 0, & \text{if otherwise.} \end{cases}$$

This does likely introduce discontinuities into our function, but is not a problem – trust me. Then,

$$\iint_R F(x,y)\, dA = \iint_D f(x,y)\, dA.$$

So, we can always actually integrate over a rectangle!

Two Special Cases

Suppose we have a general region D. Then,

- **Type I Region** – we say that D is a Type I region provided there exists constants a, b and continuous functions g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ so that

$$D = \{(x,y) : a \leq x \leq b, \text{ and } g_1(x) \leq y \leq g_2(x)\}.$$

- **Type II Region** – we say that D is a Type II region provided there exists constants c, d and continuous functions h_1, $h_2 : \mathbb{R} \to \mathbb{R}$ so that

$$D = \{(x,y) : h_1(y) \leq x \leq h_2(y), \text{ and } c \leq y \leq d\}.$$

Type I Regions

Suppose that D is a type I region:

$$\begin{aligned} \iint_D f(x,y)\, dA &= \iint_R F(x,y)\, dA \\ &= \int_b^a \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy dx. \end{aligned}$$

The "see below" line is true since $F(x, y) = 0$ if $y > g_2(x)$ or $y < g_1(x)$.

Type II Regions

In the same way, if D is type II, we have

$$\iint_D f(x,y)\, dA = \int_d^c \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx dy.$$

How do you tell? DRAW A PICTURE! (In practice, you don't typically explicitly note what type an integral is.)

Area

Suppose that D is a region. Then, the *area* of D is given by

$$\text{area}(D) = \iint_D 1\, dA.$$

Average Value

The *average value* of f over D is given by

$$\text{ave}(f) = \frac{1}{\text{area}(D)} \iint_D f(x, y)\, dA.$$

5.3 Double Integrals and Polar Coordinates

5.3.1 Polar Coordinates

Let $(x, y) \in \mathbb{R}^2$. Then, in polar coordinates

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ r &= \sqrt{x^2+y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right), \end{aligned}$$

and recall that θ is not uniquely determined by the last identity.

5.3.2 Polar Rectangle

A region $R \subseteq \mathbb{R}^2$ is called a polar rectangle provided it can be written as

$$R = \{(r, \theta) : a \leq r \leq b, \alpha \leq \theta \leq \beta\},$$

for choices of a, b, α, β.

We can divide each polar rectangle into smaller ones by

$$\Delta r = \frac{b-a}{m} \quad \text{and} \quad \Delta\theta = \frac{\beta - \alpha}{n}.$$

(This should look very familiar!!) Then, a little polar rectangle $R_{ij} = \{(r, \theta) : r_{i-1} \leq r \leq r_i, \theta_{j-1} \leq \theta \leq \theta_j\}$ has its center at

$$\begin{aligned} r_i^* &= \tfrac{1}{2}(r_{i-1} + r_i) \\ \theta_i^* &= \tfrac{1}{2}(\theta_{j-1} + \theta_j) \end{aligned}$$

The area of this polar rectangle is

$$\begin{aligned} \Delta A &= \tfrac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta \\ &= \tfrac{1}{2}\left(r_i^2 - r_{i-1}^2\right)\Delta\theta \\ &= \tfrac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta \\ &= r_i^*\Delta r\Delta\theta \end{aligned}$$

5.3.3 Polar Integrals

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \\
&= \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(r_i^*\cos\left(\theta_j^*\right), r_i^*\sin\left(\theta_j^*\right)\right) r_i^*\Delta r\Delta\theta \\
&= \int_\alpha^\beta \int_a^b f\big(r\cos(\theta), r\sin(\theta)\big) r\,dr\,d\theta
\end{aligned}$$

Thus, we can convert each x to $r\cos(\theta)$, each y to $r\sin(\theta)$ and replace $dA = dxdy$ with $dA = r\,drd\theta$. You have made a change in coordinates and are now integrating over a polar region, so the results of sections 5.1 & 5.2 now apply.

5.3.4 Change of Coordinates

Suppose that we wish to change from one two-dimensional coordinate system to another.

- Initial: (x, y)
- New: (α, β): That is, we have $x(\alpha, \beta)$ and $y(\alpha, \beta)$.
- Then,
$$dA = dxdy = \left(\frac{\partial x}{\partial \alpha}\cdot\frac{\partial y}{\partial \beta} - \frac{\partial y}{\partial \alpha}\cdot\frac{\partial y}{\partial \beta}\right) d\alpha d\beta$$

This idea is called the *Jacobian*. We can think of this as being the determinant of the Jacobian Matrix:

$$J = \begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \beta} \\ \frac{\partial y}{\partial \alpha} & \frac{\partial y}{\partial \beta} \end{bmatrix}$$

5.4 Triple Integrals

5.4.1 Triple Integrals

Triple Integrals over a Rectangular Box

Suppose that $B \subseteq \mathbb{R}^3$ is a solid box, so that

$$B = \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq b,\, c \leq y \leq d,\, e \leq z \leq f\}.$$

Suppose $f : B \to \mathbb{R}$ is continuous. We define the *triple integral* of f over B as

$$\iiint_B f(x, y, z)\, dV = \lim_{\ell, m, n \to \infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta x \Delta y \Delta z,$$

if this limit exists.

Fubini's Theorem

Given a box B as defined above and a continuous function f, we each of the following gives the same answer:

$$\begin{aligned}
\iiint_B f(x, y, z)\, dV &= \int_a^b \int_c^d \int_e^f f(x, y, z)\, dz dy dx \\
&= \int_a^b \int_e^f \int_c^d f(x, y, z)\, dy dz dx \\
&= \int_c^d \int_a^b \int_e^f f(x, y, z)\, dz dx dy \\
&= \int_c^d \int_e^f \int_a^b f(x, y, z)\, dx dz dy \\
&= \int_e^f \int_a^b \int_c^d f(x, y, z)\, dy dx dz \\
&= \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx dy dz
\end{aligned}$$

In other words, we can integrate over a *box* in whatever order is most convenient.

General Regions

Suppose that $D \subseteq \mathbb{R}^2$ is a region in the plane and $u_1,\ u_2 : \mathbb{R}^2 \to \mathbb{R}$ are defined so that $u_1(x, y) \le u_2(x, y)$. Then, if the general region

$$E = \{(x, y, z) \in \mathbb{R}^3 : (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\},$$

we have

$$\iiint_E f(x, y, z)\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right) dA.$$

There is nothing special about the z-axis, of course.

Now, if D is a Type I region with $g_1(x) \le y \le g_2(x)$, we have

$$\iiint_E f(x, y, z)\, dV = \int_b^a \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dzdydx.$$

Likewise, if D happens to be Type II, we have

$$\iiint_E f(x, y, z)\, dV = \int_d^c \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dzdxdy.$$

Volume

The *volume* of the solid region $E \subseteq \mathbb{R}^3$ is given by

$$\text{vol}(E) = \iiint_E 1\, dV.$$

Average Value

The *average value* of the continuous function f over the solid region $E \subseteq \mathbb{R}^3$ is given by

$$\text{ave}(f) = \frac{1}{\text{vol}(E)} \iiint_E f(x, y, z)\, dV.$$

5.5 Triple Integrals in Cylindrical and Spherical Coordinates

5.5.1 Cylindrical Coordinates

Cylindrical Coordinate System

This is a straightforward extension of polar coordinates in $\mathbb{R}^2$ to $\mathbb{R}^3$. In cylindrical coordinates,

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= z \end{aligned}$$

Cylindrical Triple Integrals

If you have a solid general region $E \subseteq \mathbb{R}^3$ such that you can describe E as

$$E = \{(r, \theta, z) : (r, \theta) \in D, u_1(r, \theta) \leq z \leq u_2(r, \theta)\},$$

for some $D \subseteq \mathbb{R}^2$ such that

$$D = \{(r, \theta) : \alpha \leq \theta \leq \beta, g_1(\theta) \leq r \leq g_2(\theta)\}$$

then

$$\iiint_E f(x, y, z)\, dV = \int_\beta^\alpha \int_{g_1(\theta)}^{g_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos(\theta), r\sin(\theta), z) r\, dz dr d\theta.$$

5.5.2 Spherical Coordinates

We can also describe a point in $\mathbb{R}^3$ by use of spherical coordinates. We let ρ be the distance from the origin to the point, θ remains the angle the point makes in the xy plane, and ϕ is the polar angle, the angle made between the point and the positive z axis.

Important. In physics, typically, the notation is reversed – θ is commonly the polar angle and ϕ is the planar angle. This is admittedly stupid that we cannot agree on this – it is just arbitrary notation.

Spherical Coordinate System

Using ρ as the radius (in 3 dimensions, from the origin), θ as the angle in the xy plane and ϕ as the polar angle (letting $\phi = 0$ mean the positive z axis), we have:

$$\begin{aligned} x &= \rho\sin(\phi)\cos(\theta) \\ y &= \rho\sin(\phi)\sin(\theta) \\ z &= \rho\cos(\phi) \end{aligned}$$

$$\begin{aligned}\rho &= \sqrt{x^2+y^2+z^2}\\ \theta &= \arctan(\tfrac{y}{x})\\ \phi &= \arccos(\tfrac{z}{\rho})\end{aligned}$$

Spherical Triple Integrals

In spherical coordinates, a *spherical wedge* is a region

$$E = \{(p, \theta, \phi) : a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \phi \le \psi\}.$$

Using an analysis similar to what we did for a polar rectangle, you can show that

$$\Delta V = \rho^2 \sin(\phi) \Delta\rho\Delta\theta\Delta\phi.$$

Thus,

$$\iiint_E f(x, y, z)\, dV = \int_\psi^\gamma \int_\beta^\alpha \int_b^a f(\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi))\rho^2 \sin(\phi)\, d\rho d\theta d\phi.$$

We can also describe a point in $\mathbb{R}^3$ by use of spherical coordinates. We let ρ be the distance from the origin to the point, θ remains the angle the point makes in the xy plane, and ϕ is the polar angle, the angle made between the point and the positive z axis.

Conversion from Spherical to Cylindrical

We have spherical coordinates (ρ, θ, ϕ) and cylindrical coordinates (r, θ, z). Converting spherical to cylindrical, we have

$$\begin{aligned}r &= \rho \sin(\phi)\\ \theta &= \theta\\ z &= \rho \cos(\phi).\end{aligned}$$

5.5.3 Jacobian

Cylindrical

We have

$$\begin{aligned} x(r,\theta,z) &= r\cos(\theta) \\ y(r,\theta,z) &= r\sin(\theta) \\ z(r,\theta,z) &= z \end{aligned}$$

so that

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} \\ &= \begin{bmatrix} \cos(\theta) & -r\sin(\theta) & 0 \\ \sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ |J| &= r \end{aligned}$$

Spherical

We have

$$\begin{aligned} x(\rho,\theta,\phi) &= \rho\sin(\phi)\cos(\theta) \\ y(\rho,\theta,\phi) &= \rho\sin(\phi)\sin(\theta) \\ z(\rho,\theta,\phi) &= \rho\cos(\phi) \end{aligned}$$

so that

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{bmatrix} \\ &= \begin{bmatrix} \sin(\phi)\cos(\theta) & -\rho\sin(\phi)\sin(\theta) & \rho\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & \rho\sin(\phi)\cos(\theta) & \rho\cos(\phi)\sin(\theta) \\ \cos(\phi) & 0 & -\rho\sin(\phi) \end{bmatrix} \\ |J| &= \rho^2\sin(\phi) \end{aligned}$$

CHAPTER 6

VECTOR CALCULUS

6.1 Vector Fields

6.1.1 Introduction

So far, we have talked about:

- Parametric Functions: $f: \mathbb{R} \to \mathbb{R}^n$ (also called vector-valued functions)
- Scalar Functions: $f: \mathbb{R}^n \to \mathbb{R}$

We conclude the course by considering:

- $f: \mathbb{R}^n \to \mathbb{R}^n$, specifically:
  - $f: \mathbb{R}^2 \to \mathbb{R}^2$
  - $f: \mathbb{R}^3 \to \mathbb{R}^3$

6.1.2 Vector Field

A vector field is an assignment to each point in $\mathbb{R}^n$ a vector in $\mathbb{R}^n$. For example:

- At each point on the Earth's surface, assign the current wind speed and direction.
- A force field – gravity, electromagnetism, etc.
- Fluid flow – in a pipe, the rate and direction of flow of water.
- A gradient field – given a surface, its gradient at each point.

Formally, a vector field is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ with some domain $D \subseteq \mathbb{R}^n$. We will only be concerned with $n = 2$ and $n = 3$.

Common notation:

$$\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

and

$$\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), \mathbf{R}(x, y, z)\rangle = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + \mathbf{R}(x, y, z)\mathbf{k}$$

6.1.3 Graph

To truly graph a vector field, you need 4 or 6 dimensions, which is somewhat inconvenient. Instead, we plot representative vectors in either $\mathbb{R}^2$ or $\mathbb{R}^3$.
The applet at https://www.geogebra.org/m/QPE4PaDZ is one way to graph a 2-D vector field.

6.1.4 Special Vector Fields

There are a few commonly seen special types of vector fields:

- **Radial** – each vector points either directly toward or away from the origin. For example, $\langle x, y \rangle$.
- **Rotational** – each vector points tangent to a circle centered at the origin. For example, $\langle -y, x \rangle$.
  - **Think:** At the point $(2, 1)$, we have the vector $\langle -1, 2 \rangle$. This vector is tangent to the circle of radius $\sqrt{5}$ centered at the origin. The vector is perpendicular to the radius of the circle at that point.
- **Unit** – each output vector has magnitude 1. For example, $\langle \sin(x + y), \cos(x + y) \rangle$.
- **Velocity** – the vectors represent the velocity of some particle.
- **Gradient** – the vectors are the gradient of some function $\mathbf{F} \colon \mathbb{R}^n \to \mathbb{R}$.

6.1.5 Gradient Field

A vector field $\mathbf{F}$ is called a gradient vector field if there exists some scalar function $\mathbf{F} \colon \mathbb{R}^n \to \mathbb{R}$ such that $\nabla f = \mathbf{F}$. That is, if $\mathbf{F}$ is the gradient of some scalar function f.

6.1.6 Conservative Vector Fields

A vector field $\mathbf{F}$ is a conservative vector field when it is the gradient field for some scalar function f. We will call f the **potential function** of $\mathbf{F}$.

Theorem. Potential functions are unique, except for a constant. That is, if f and g are each potential functions for $\mathbf{F}$, then there exists some constant c such that $f - g = c$.

6.1.7 Cross Partial Property

Theorem. Suppose that $\mathbf{F}$ is a conservative vector field with continuous mixed second partials. Then:

- In two dimensions, if $\mathbf{F} = \langle P, Q \rangle$, then $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
- In three dimensions, if $\mathbf{F} = \langle P, Q, R \rangle$, then:

- $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
- $\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$
- $\frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$

6.1.8 Examples

Example 1: Gradient Field

$f(x, y) = x^2y+y^3$ is a potential function for the vector field $\nabla f = \mathbf{F}(x, y) = \langle 2xy, x^2+3y^2\rangle = \langle P, Q\rangle$.

If we find the second derivative of these functions, we see that $\frac{\partial P}{\partial y} = 2x$; $\frac{\partial Q}{\partial x} = 2x$. Thus, the cross partial property holds.

6.2 Line Integrals

6.2.1 Scalar Line Integral

Suppose that $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}$ is a continuous scalar function and that C is a continuous curve in $\mathbb{R}^n$ included in the domain of f. We want to find the area of the sheet which extends down from f to the domain along the curve C. We parameterize C as $\mathbf{r}(t)$, $a \leq t \leq b$ and break C into small subintervals of length $\Delta t = \frac{b-a}{n}$. Then, we have:

$$\begin{aligned}
\Delta s_1 &= [\mathbf{r}(a), \mathbf{r}(a+\Delta t)] \\
\Delta s_2 &= [\mathbf{r}(a+\Delta t), \mathbf{r}(a+2\Delta t)] \\
&\vdots \\
\Delta s_n &= [\mathbf{r}(a+(n-1)\Delta t), \mathbf{r}(a+n\Delta t)]
\end{aligned}$$

In each Δs_i, choose t_i^* and find $f(\mathbf{r}(t_i^*))$. Then, we have a Riemann sum:

$$\sum_{i=1}^{n} f(\mathbf{r}(t_i^*))\Delta s_i,$$

which, if the limit as $n \to \infty$ exists, we will write as one of:

$$\int_C f(x,y)\,ds \quad \text{or} \quad \int_C f(x,y,z)\,ds.$$

The term ds represents the infinitesimal step taken along the path C, analogous to dx in single-variable integration. Since working with ds directly is cumbersome, we use:

$$\int_C f(x,y)\,ds = \int_C f(x(t),y(t))\sqrt{(x'(t))^2+(y'(t))^2}\,dt = \int_a^b f(\mathbf{r}(t))\,\underbrace{\|\mathbf{r}'(t)\|}_{*}\,dt, \|\mathbf{r}\|$$

(*: Area of sheet under surface above curve.)
which simplifies computations and ensures that parametrization does not affect the integral's value, though orientation does. If C and D are the same curve with opposite orientations:

$$\int_C f\,ds = -\int_D f\,ds,$$

analogous to:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

6.2.2 Line Integrals over Vector Fields

Suppose that $\mathbf{F}\colon \mathbb{R}^n \to \mathbb{R}^n$ is a vector field, and C is a continuous curve in $\mathbb{R}^n$ within the domain of $\mathbf{F}$. We wish to evaluate:

$$\int_C \mathbf{F} \cdot d\mathbf{s},$$

which essentially "multiplies" $\mathbf{F}$ by a small infinitesimal along the curve C (denoted by $d\mathbf{s}$).

We choose the points $P_0, P_1, P_2, \ldots, P_i, \ldots, P_n$ to divide up our curve C and then approximate our line integral as a Riemann sum. We can write the line integral as:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{F}(P_i^*) \cdot \Delta\mathbf{s}_i.$$

Since $d\mathbf{s} = \mathbf{r}'(t)dt$, the integral simplifies to:

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,$$

Where $\mathbf{r}(t)$, $a \le t \le b$ is any parametrization of C.

Equivalent notations include:

$$\int_C \mathbf{F} \cdot d\mathbf{r}, \quad \int P\,dx + Q\,dy, \quad \text{or} \quad \int P\,dx + Q\,dy + R\,dz.$$

These alternate notations come from expanding the dot product: $\int \langle P, Q\rangle \cdot \langle x', y'\rangle\, dt$. Solving, we get the notation above.

6.2.3 Properties of Line Integrals over Vector Fields

- $\int_C (\mathbf{F} + G) \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r} + \int_C G \cdot d\mathbf{r}$
- $\int_C k\mathbf{F} \cdot d\mathbf{r} = k \int_C \mathbf{F} \cdot d\mathbf{r}$, for constant k
- $\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_{-C} \mathbf{F} \cdot d\mathbf{r}$, where $-C$ denotes C with reversed orientation
- If $C = C_1 + C_2$ with intersection only at endpoints, then:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

6.2.4 Circulation Along a Curve

The circulation of $\mathbf{F}$ along C is defined as:

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

If **F** represents a force field, this integral gives the total work done by **F** in moving an object along C.

6.2.5 Flux Across a Curve

Given a vector field **F** and curve C in R^2, the *flux* of **F** across C is the total amount of **F** that crosses orthogonally across C. We note that there is an orientation issue here. We define positive flux as moving from left to right across the curve where forward is the direction of the curve's orientation. In two dimensions, this is actually opposite of N from chapter 2, but we'll fix that easily by defining n to be the unit vector which points left-to-right across the curve. We can see that $n(t) = \langle y'(t), -x'(t) \rangle$. Thus, the flux integral is given by:

$$\int_C \mathbf{F} \cdot \mathbf{N} \, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \langle y'(t), -x'(t) \rangle \, dt,$$

or equivalently:

$$\int_C -Q \, dx + P \, dy.$$

6.2.6 Examples

Example 1: Scalar Line Integral

Evaluate $\int_C (x^2 + y^2) ds$ where C is the quarter-circle $\mathbf{r}(t) = (\cos t, \sin t)$, $0 \leq t \leq \frac{\pi}{2}$.
Solution:

$$ds = \|\mathbf{r}'(t)\| dt = \sqrt{(-\sin t)^2 + (\cos t)^2} dt = dt.$$
$$f(\mathbf{r}(t)) = \cos^2 t + \sin^2 t = 1.$$
$$\int_C (x^2 + y^2) ds = \int_0^{\pi/2} 1 \, dt = \frac{\pi}{2}.$$

Example 2: Scalar Line Integral

Let $f(x, y) = x^2 y$ and C by the curve from $(0, 0)$ to $(2, 4)$ to $(0, 4)$. Call lines C_1 and C_2. Find the value of $\int_C f(x, y) ds$.
Solution:

First, we find C_1: $\mathbf{r}(t) = \langle t, 2t \rangle$, $0 \le t \le 2$. This turns into the first integral:

$$\begin{aligned}
\int_{C_1} f(x,y)\,ds &= \int_0^2 t^2(2t)\sqrt{(1)^2+(2)^2}\,dt \\
&= 2\sqrt{5}\int_0^2 t^3\,dt \\
&= 2\sqrt{5}\left[\frac{t^4}{4}\right]_0^2 \\
&= 2\sqrt{5}\left[\frac{16}{4}-0\right] \\
&= 8\sqrt{5}.
\end{aligned}$$

Now, for C_2: $\mathbf{r}(t) = \langle 2-t, 4 \rangle$, $0 \le t \le 2$. This turns into the second integral:

$$\begin{aligned}
\int_{C_2} f(x,y)\,ds &= \int_0^2 (2-t)^2(4)\sqrt{(-1)^2+(0)^2}\,dt \\
&= 4\int_0^2 (2-t)^2\,dt \\
&= \frac{-4}{3}\left[(2-t)^3\right]_0^2 \\
&= \frac{-4}{3}\left[0-\frac{8}{3}\right] \\
&= \frac{32}{3}.
\end{aligned}$$

Example 3: Vector Line Integral

Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = \langle y, x \rangle$ and C is the line from $(0,0)$ to $(1,1)$.
Solution:

$$\begin{aligned}
\mathbf{r}(t) &= (t,t), \quad 0 \le t \le 1. \\
\mathbf{r}'(t) &= (1,1). \\
F(\mathbf{r}(t)) &= \langle t, t \rangle. \\
\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \langle t,t\rangle\cdot\langle 1,1\rangle\,dt = \int_0^1 (t+t)\,dt = \int_0^1 2t\,dt = 2.
\end{aligned}$$

6.3 Conservative Vector Fields

6.3.1 Conservative Vector Fields

Previously, we saw that a conservative vector field, $\mathbf{F}$, is one for which there exists a scalar potential function f such that $\nabla f = \mathbf{F}$. That is, $\mathbf{F}$ is the gradient of some potential function f.

6.3.2 Fundamental Theorem of Line Integrals

Suppose that C is piecewise smooth, parameterized by $\mathbf{r}$ on $a \le t \le b$, and f is differentiable in two or three variables. If $\mathbf{F} = \nabla f$ (i.e., f is conservative) is continuous on C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof. This is an immediate result of the chain rule and the "regular" Fundamental Theorem of Calculus:

$$\begin{aligned}\int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt \\ &= f(\mathbf{r}(b)) - f(\mathbf{r}(a)).\end{aligned}$$ □

Thus, when $\mathbf{F}$ is the gradient of some potential function – i.e., when $\mathbf{F}$ is conservative – we can evaluate a line integral by:

- determining a potential function for $\mathbf{F}$
- evaluating the potential function at each of $t = a$ and $t = b$ and finding the difference.

The line integral of a conservative vector field $\mathbf{F}$ is just the change in potential between the ending and starting points on the curve.

Therefore, for a conservative vector field, the circulation along a curve from the point P to Q is independent of the path to get from P to Q. In fact, we call this the *path independence property* of $\mathbf{F}$. Moreover, if you have 2 curves C_1 and C_2, and $\mathbf{F}$ is conservative, then

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

This is very much like the Calculus I version of the Fundamental Theorem of Calculus – in that case, to determine $\int_a^b f(x)\, dx$, we find an antiderivative, $F(x)$, for f and evaluate that antiderivative at each of $x = a$ and $x = b$ and subtract. We did not notice "path independence" since there is (essentially) only one path in $\mathbb{R}$ to go from $x = a$ to $x = b$.

Line Integrals on a Closed Curve

If C is a closed curve, we often emphasize this for a line integral by writing

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

Notice that the Fundamental Theorem of Line Integrals implies that whenever $\mathbf{F}$ is conservative, then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Path Independence

The Fundamental Theorem of Line Integral implies that any $\int_C \mathbf{F} \cdot d\mathbf{r}$ is path independent whenever $\mathbf{F}$ is conservative. The reverse is almost true:

Theorem. If $\mathbf{F}$ is a continuous vector field that is path independent with open, connected domain, then $\mathbf{F}$ is conservative.

Cross Partial Property

Theorem. Suppose that $\mathbf{F}$ is a continuous vector field on an open, simply connected region D in $\mathbb{R}^2$ or $\mathbb{R}^3$. If the cross partial property holds in D, then $\mathbf{F}$ is conservative.

6.3.3 Finding a Potential Function

To find a potential function for a conservative vector field, we can use the following algorithm:

Given a conservative vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$, we can find a potential function f such that $\nabla f = \mathbf{F}$. The algorithm is as follows:

- We can find $\int P(x, y)\, dx$ (i.e., hold y constant) $= g(x, y) + h(y)$, for some currently unknown function h – that is, g is "just" the antiderivative of P, only with respect to x.
- Then $\frac{\partial}{\partial y}(g(x, y) + h(y)) = g_y(x, y) + h'(y)$.
- This should be $Q(x, y) = g_y(x, y) + h'(y)$, so we know $h'(y)$.
- Then, $h(y) = \int h'(y)\, dy$.
- Finally, $f(x, y) = g(x, y) + h(y) + K$, where K is a constant of integration.

6.3.4 Examples

Example 1

Let $\mathbf{F} = \langle y^2, 2xy + 2\rangle$ and C be the curve parameterized by $\mathbf{r}(t) = \langle t\cos(t), t^2 + e^t - \sin(t)\rangle$, $0 \le t \le 2$. Find $\int_C \mathbf{F} \cdot d\mathbf{r}$.

Solution:
We first check if $\mathbf{F}$ is conservative. We find the cross partials:

$$\frac{\partial P}{\partial y} = 2y, \quad \text{and} \quad \frac{\partial Q}{\partial x} = 2y.$$

Since these are equal, we know that $\mathbf{F}$ is conservative.

We can find a potential function for $\mathbf{F}$ by using the algorithm above. We have:

$$\begin{aligned}
\int P(x,y)\,dx = \int y^2\,dx &= xy^2 + h(y), \\
\frac{\partial}{\partial y}(xy^2 + h(y)) = 2xy + h'(y) &= Q(x,y) = 2xy + 2. \\
h'(y) &= 2, \\
h(y) &= 2y + K. \\
f(x,y) &= xy^2 + 2y + K.
\end{aligned}$$

Now, we can evaluate the line integral:

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= f(\mathbf{r}(2)) - f(\mathbf{r}(0)) \\
&= f(2\cos(2), 4 + e^2 - \sin(2)) - f(0, e^2) \\
&= (2\cos(2))(4 + e^2 - \sin(2)) + 2(4 + e^2 - \sin(2)) - (0) - (e^2) \\
&= 8\cos(2) + 3e^2 - 2\sin(2).
\end{aligned}$$

6.4 Green's Theorem

6.4.1 Circulation over Closed Curves

Let C be a closed curve and $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field. Then, the circulation of $\mathbf{F}$ along C is given by:

$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \oint_C \mathbf{F} \cdot d\mathbf{r} = \oint P\, dx + Q\, dy.$$

You can think of this as measuring the total amount that $\mathbf{F}$ pushes along the curve C.

We will define the closed curve C to have positive orientation if the interior of the region enclosed by the curve is always to the left as you traverse the curve – we often call this counter-clockwise orientation.

Notice that if $\mathbf{F}$ happens to be conservative, then $P_y = Q_x$ by the cross partial property. Thus, $Q_x - P_y = 0$. Then, if D is the region defined by the interior of the curve C, we have:

$$\iint_D (Q_x - P_y)\, dA = 0,$$

which happens to match:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

But what if $\mathbf{F}$ is not conservative?

6.4.2 Green's Theorem – Circulation Form

Suppose that C is a positively oriented closed curve and that D is the region bounded by C. If $\mathbf{F} = \langle P, Q \rangle$ is any vector field with continuous partial derivatives, then:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

We can write each of these as:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D (Q_x - P_y) dA.$$

Thus, the total circulation of $\mathbf{F}$ along the closed curve C is equal to the sum (i.e. integral) of the difference between the cross partials over the entire region D. The difference of the partials measures how non-conservative the vector field is – and measures the total microscopic circulation at each point in D. But, in the interior all of the microscopic circulations cancel out, leaving only the circulation along the boundary.

6.4.3 Interpretation as a Fundamental Theorem

We can interpret the quantity $Q_x - P_y$ as a type of derivative of $\mathbf{F}$. The total double sum of the quantity $Q_x - P_y$ over some region matches the sum of the original function $\mathbf{F}$ over the boundary of that region.

6.4.4 Flux over Closed Curves

The flux of $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, where we write $\mathbf{F} = \langle P, Q \rangle$ over a closed curve C is given by:

$$\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \oint_C \mathbf{F} \cdot \mathbf{n}\, dt = \oint_C \mathbf{F} \cdot \langle y'(t), -x'(t) \rangle\, dt = \oint_C -Q\, dx + P\, dy.$$

Green's Theorem – Flux Form

Suppose that C is a positively oriented closed curve and that D is the region bounded by C. If $\mathbf{F} = \langle P, Q \rangle$ is any vector field with continuous partial derivatives, then:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, dt = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA.$$

We can write each of these as:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, dt = \oint_C -Q\, dx + P\, dy = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA = \iint_D (P_x + Q_y) dA.$$

Thus, the total flux of $\mathbf{F}$ across the closed curve C is equal to the sum (i.e. integral) of the sum of the regular partials over the entire region D. The sum of the partials measures how much the vector field is running away (expanding away) from a point. But, as the animations in the slides show, in the interior all of the expansions cancel out, leaving only the expansion (i.e. flux) across the boundary.

6.4.5 Interpretation as a Fundamental Theorem

We can interpret the quantity $P_x + Q_y$ as a type of derivative of $\mathbf{F}$. The total double sum of the quantity $P_x + Q_y$ over some region matches the sum of the original function $\mathbf{F}$ across the boundary of that region.

6.5 Divergence and Curl

6.5.1 Divergence

If $\mathbf{F} = \langle P, Q, R\rangle$ is a vector field in $\mathbb{R}^3$, then we can find the divergence of $\mathbf{F}$ as follows:

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The divergence of a vector field is a scalar quantity that measures the rate at which the vector field diverges from a point. A large positive divergence at a point indicates that the vector field is "running away" from that point, on average more than it is coming into that point. See the slides from today for an animated example.

Let C be a closed curve in $\mathbb{R}^2$ and D the region it contains and $\mathbf{F}$ a vector field. Then, by the flux form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, dt = \iint_D \nabla \cdot \mathbf{F}\, dA.$$

We will see a version of this in $\mathbb{R}^3$ in section 6.8.

Interpretation of Divergence

The divergence at a point find the microscopic net flux through that point.

Suppose that D is a region and that the sum of the divergence over that region is positive. Then, D contains at least one *source*. If that sum is negative, we say that D contains a *sink*.

A vector field for which $\nabla \cdot \mathbf{F} = 0$ is called *source free* or *solenoidal*. A rotational field is the stereotypical source free vector field.

6.5.2 Curl

Here, it does matter that we are in $\mathbb{R}^3$. Define the *curl* of the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ as:

$$\begin{aligned}\operatorname{curl}\mathbf{F} &= \nabla \times \mathbf{F}\\ &= (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}\\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.\end{aligned}$$

Notice a few things: The divergence is a scalar quantity, while the curl is a vector quantity. Also, the curl is exactly the difference of the various cross partials – so it is a vector quantity which measures how non-conservative a vector field is.

Interpretation of Curl

For a point P in the domain, the curl of $\mathbf{F}$ at P measures the microscopic circulation at that point – since it is a vector quantity, it follows the right-hand rule, with the vector's magnitude measuring the strength of the circulation and direction the right-hand rule of that circulation.

We see immediately that any conservative vector field has $\nabla \times \mathbf{F} = 0$.

Curl in Two Dimensions

We cannot define the curl for $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, since the cross product is not defined in two dimensions. However, we can extend $\mathbf{F} = \langle P, Q \rangle$ in this case to the three dimensional vector field $\mathbf{F} = \langle P, Q, 0 \rangle$. In this case:

$$\operatorname{curl} \mathbf{F} = (Q_x - P_y)\mathbf{k}.$$

This should look familiar!

The circulation form of Green's Theorem says that – with the extension noted above:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dA.$$

6.5.3 Curl and Divergence

Since curl itself is a vector field, we can find $\nabla \cdot (\nabla \times \mathbf{F})$:

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(P_z - R_x) + \frac{\partial}{\partial z}(Q_x - P_y) \\
&= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \\
&= 0, \text{ by Clairaut's Theorem.}
\end{aligned}$$

6.6 Parameterized Surfaces

A region $D \subseteq \mathbb{R}^2$ can be defined by two parameters, x and y, or r and θ, or other suitable pairs. These regions are "flat." We can relax that constraint.

If each of $x(u,v)$, $y(u,v)$, and $z(u,v)$ is a continuous function with domains $u \in [a,b]$ and $v \in [c,d]$, then the vector-valued function

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

maps points in the uv-plane to a set of points in $\mathbb{R}^3$. This image, being two-dimensional, can be regarded as a surface.

6.6.1 Surface Infinitesimals

We divide the region in the uv-plane into rectangles of area $\Delta u \, \Delta v$. Under the parametrization, we approximate the image of these rectangles under $\mathbf{r}$ by considering tangent vectors:

$$\mathbf{t}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle, \quad \mathbf{t}_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

That is, each of $\mathbf{t}_u$ and $\mathbf{t}_v$ measures how a small change in u and v affects their image in $\mathbb{R}^3$.

The rectangles generated by $\Delta u \, \Delta v$ become parallelograms in $\mathbb{R}^3$, and the area of such a parallelogram is given by the magnitude of the cross product:

$$\|\mathbf{t}_u \times \mathbf{t}_v\|$$

6.6.2 Scalar Surface Integrals

Surface Area

Let S be a surface parameterized by $\mathbf{r}(u,v)$, with the domain of $\mathbf{r}$ denoted by D – that is, D is the domain for the parameters u and v. Then, the surface area of S is given by

$$\text{Area}(S) = \iint_D \|\mathbf{t}_u \times \mathbf{t}_v\| \, dA$$

6.6.3 General Scalar Surface Integrals

Suppose that S is a surface parameterized by $\mathbf{r}(u,v)$ and $f(x,y,z)$ is a scalar function. Then the surface integral of f over S is given by:

$$\iint_S f(x,y,z) \, dS = \iint_D f(\mathbf{r}(u,v)) \, \|\mathbf{t}_u \times \mathbf{t}_v\| \, dA.$$

6.6.4 Oreintable Surfaces

Let S be a smooth surface and P be a point on the surface. Then, there are two unit vectors, $\mathbf{N}$ and $-\mathbf{N}$ which point orthogonally to the surface. That is, the normal vectors to the tangent plane at P. We say that S is orientable provided that as we move P, then $\mathbf{N}$ and $-\mathbf{N}$ move continuously – so that they do not jump around.

We select our orientation as follows:

- If the surface is closed – that is, it encloses a volume – we select $\mathbf{N}$ so that it points outward.
  - For a sphere, we can choose $\mathbf{N}$ to be either of $t_\theta \times t_\phi$ or $t_\phi \times t_\theta$ – the latter is the positive orientation.
  - For a cylinder, we choose $t_\theta \times t_z$.
- For a surface defined by $z = f(x, y)$, we want $\mathbf{N}$ to point upward. Then, $t_x = \langle 1, 0, f_x \rangle$ and $t_y = \langle 0, 1, f_y \rangle$ and choose $t_x \times t_y$.

$$\begin{aligned} t_x \times t_y &= \langle -f_x, -f_y, 1 \rangle \\ \mathbf{N} &= \frac{1}{\sqrt{1 + (f_x)^2 + (f_y)^2}} \langle -f_x, -f_y, 1 \rangle \end{aligned}$$

Flux

Let S be an oriented surface with unit normal vector $\mathbf{N}$ and $\mathbf{F}$ be continuous on S. Then, the surface integral of $\mathbf{F}$ over S is given by:

$$\iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{N}\, dS.$$

The value of this integral is called the flux of $\mathbf{F}$ across S.

Like line integrals we have a bit of a shortcut for computational purposes: Given a parametrization $\mathbf{r}(u, v)$ of S, we have

$$\begin{aligned} \iint_S \mathbf{F} \cdot dS &= \iint_S \mathbf{F} \cdot \mathbf{N}\, dS \\ &= \iint_S \mathbf{F} \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{||\mathbf{t}_u \times \mathbf{t}_v||}\, dS \\ &= \iint_D \mathbf{F} \cdot \frac{\mathbf{t}_u \times \mathbf{t}_v}{||\mathbf{t}_u \times \mathbf{t}_v||}\, ||\mathbf{t}_u \times \mathbf{t}_v||\, dudv \\ &= \iint_D \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v)\, dudv, \end{aligned}$$

where D is the region in the uv-plane which maps to S.

This is (somewhat) easier to calculate than its scalar equivalent, typically. For all surface integrals, you are strongly advised to draw a picture and think a bit before just blindly trying to guess about your situation.

6.6.5 Examples

Example 1: Surface Area

Solve $\iint_S (x^2z + y^z)\, dS$, where S is the upperhemisphere of $x^2 + y^2 + z^2 = 4$.

Solution:
First, it's clear that we are going to need to use spherical coordinates. From page 5 in the handout, we know that:

$$\mathbf{r}(\theta, \phi) = \langle 2\sin(\theta)\cos(\phi), 2\sin(\theta)\sin(\phi), 2\cos(\phi)\rangle, \quad 0 \le \theta \le \tfrac{\pi}{2}, \quad 0 \le \phi < 2\pi.$$

Thus, we know this is also equal to $\|\mathbf{t}_\theta \times \mathbf{t}_\phi\| = 4\sin(\phi)$.

We can now solve our integral:

$$\begin{aligned}
\iint_S (x^2z + y^z)\, dS &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \big((2\cos\theta\sin\phi)^2 2\cos\phi + (2\sin\theta\sin\phi)^2 2\cos\phi\big)\cdot 4\sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \big((8\cos^2\theta\sin^2\phi)\cos\phi + (8\sin^2\theta\sin^2\phi)\cos\phi\big)\cdot 4\sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} 32\sin^3\phi\cos\phi\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}\, d\phi\, d\theta
\end{aligned}$$

With the u-substitution $u = \sin\phi$, we have $du = \cos\phi\, d\phi$.

$$\begin{aligned}
&= 32\cdot 2\pi\cdot\left[\frac{1}{4}u^4\right]_0^1 \\
&= 16\pi.
\end{aligned}$$