Homework 5: Sections 13 & 14

Algebra

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Due

November 18, 2025

Section 13

In Exercises 4 and 5, determine whether the given map φ is a homomorphism. [*Hint:* The straightforward way to proceed is to check whether $\varphi(ab) = \varphi(a)\varphi(b)$ for all a and b in the domain of φ. However, if we should happen to notice that $\varphi^{-1}[\{e'\}]$ is not a subgroup whose left and right cosets coincide, or that φ does not satisfy the properties given in Exercise 44 or 45 for finite groups, then we can say at once that φ is not a homomorphism.]

4. Let $\varphi : \mathbb{Z}_6 \to \mathbb{Z}_2$ be given by $\varphi(x) =$ the remainder of x when divided by 2, as in the division algorithm.

Solution.

5. Let $\varphi : \mathbb{Z}_9 \to \mathbb{Z}_2$ be given by $\varphi(x) =$ the remainder of x when divided by 2, as in the division algorithm.

Solution.

In Exercises 19 and 23, compute the indicated quantities for the given homomorphism φ. (See Exercise 46.)

19. $\ker(\varphi)$ and $\varphi(20)$ for $\varphi : \mathbb{Z} \to S_8$ such that $\varphi(1) = (1, 4, 2, 6)(2, 5, 7)$.[1]

Solution.

23. $\ker(\varphi)$ and $\varphi(4, 6)$ for $\varphi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ where $\varphi(1, 0) = (2, -3)$ and $\varphi(0, 1) = (-1, 5)$.

Solution.

44. Let $\varphi : G \to G'$ be a group homomorphism. Show that if $|G|$ is finite, then $|\varphi[G]|$ is finite and is a divisor of $|G|$.

Solution.

45. Let $\varphi : G \to G'$ be a group homomorphism. Show that if $|G'|$ is finite, then, $|\varphi[G]|$ is finite and is a divisor of $|G'|$.

Solution.

49. Show that if G, G', and G'' are groups and if $\varphi : G \to G'$ and $\gamma : G' \to G''$ are homomorphisms, then the composite map $\gamma\varphi : G \to G''$ is a homomorphism.

[1] I decided not to capitalize the 'k' in "ker" to keep with `amsmath`'s `\ker` command.

Solution.

Section 14

In Exercises 3 and 6, find the order of the given factor group.

2. $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/(\langle 2 \rangle \times \langle 2 \rangle)$

Solution.

6. $(\mathbb{Z}_{12} \times \mathbb{Z}_8)/\langle (4,3) \rangle$

Solution.

In Exercises 10 and 15, give the order of the element in the factor group.

11. $(2,1) + \langle (1,1) \rangle$ in $(\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle (1,1) \rangle$

Solution.

15. $(2,0) + \langle (4,4) \rangle$ in $(\mathbb{Z}_6 \times \mathbb{Z}_5)/\langle (4,4) \rangle$

Solution.

27. A subgroup H is **conjugate to a subgroup** K of a group G if there exists an inner automorphism i_g of G such that $i_g[H] = K$. Show that conjugacy is an equivalence relation on the collection of subgroups of G.

Solution.

30. Let H be a normal subgroup of a group G, and let $m = (G : H)$. Show that $a^m \in H$ for every $a \in G$.

Solution.