



# HENDRIX

COLLEGE

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## Multivariable Calculus Notes

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### MATH 230

*Start*

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# TABLE OF CONTENTS

<b>1</b>	<b>Parametric Eqs and Polar Coords</b>	<b>2</b>
1.1	Parametric Equations . . . . .	2
1.1.1	Introduction . . . . .	2
1.1.2	Parametric Equations . . . . .	2
1.1.3	Graphing Parametric Curves in the Second Dimension . . . . .	2
1.1.4	The Cycloid . . . . .	3
1.1.5	Final Notes . . . . .	3
1.2	Calculus of Parametric Curves . . . . .	3
1.2.1	Slope for a Parametric Curve . . . . .	4
1.2.2	Second Derivative . . . . .	4
1.2.3	Area Under a Curve . . . . .	4
1.2.4	Arc Length . . . . .	4
1.2.5	Surface Area . . . . .	4
1.2.6	The Cycloid . . . . .	5
<b>2</b>	<b>Vectors in Space</b>	<b>8</b>
2.1	Vectors in the Plane . . . . .	8
2.1.1	Notation . . . . .	8
2.1.2	Vectors . . . . .	8
2.2	Vectors in Space . . . . .	9
2.2.1	Vector Properties . . . . .	9
2.2.2	Special Vectors . . . . .	9
2.3	The Dot Product . . . . .	10
2.3.1	Properties of the Dot Product . . . . .	10
2.3.2	Projections . . . . .	10
2.3.3	Work . . . . .	11
2.4	The Cross Product . . . . .	11
2.4.1	Properties of the Cross Product . . . . .	12
2.4.2	Standard Unit Vectors and the Cross Product . . . . .	12
2.4.3	Torque . . . . .	12
2.5	Equations of Lines and Planes . . . . .	12
2.5.1	Lines . . . . .	12
2.5.2	Planes . . . . .	14
2.5.3	Equations of a Plane . . . . .	15
2.5.4	Examples . . . . .	15
2.6	Quadratic Surfaces . . . . .	16
2.6.1	Spheres . . . . .	16
2.6.2	Cylinder . . . . .	16
2.6.3	Quadratic Surfaces . . . . .	16

## 1.1 Parametric Equations

### 1.1.1 Introduction

Most of your calculus experience has been single variable, so that the functions under consideration were typically  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Our course is divided into roughly 3 sections:

- Parametric Equations/Functions: Functions of the form  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  (Chapters 1 - 3)
- Scalar Functions: Functions of the form  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  (Chapters 4 - 5)
- Vector Fields: Functions of the form  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (Chapter 6)

### 1.1.2 Parametric Equations

A *parametric equation* (or, *sometimes parametric function* or *vector-valued function*) is a function of the form  $f : \mathbb{R} \rightarrow \mathbb{R}^n$ . We will typically consider  $n = 2$  or  $n = 3$  and call the input variable the parameter, usually denoted by  $t$ . We write them as

$$f(t) = \begin{cases} x(t) \\ y(t) \end{cases} \quad \text{or} \quad f(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}.$$

A *parametric curve* is the set of points  $(x(t), y(t))$  in  $\mathbb{R}^2$  or  $(x(t), y(t), z(t))$  in  $\mathbb{R}^3$  traced out. Note that in general, the curve may not be a function for  $y$  in terms of  $x$ , but is a function of the parameter  $t$ .

### 1.1.3 Graphing Parametric Curves in the Second Dimension

#### Elimination of the Parameter

In some cases, we can explicitly solve for  $t$  in terms of one of  $x$  or  $y$ . When this is possible, you can write  $y(x)$  or  $x(y)$  and use your “regular” algebraic knowledge. We call this process *eliminating the parameter*.

#### Using Technology

- Your TI-84 can graph this if you switch to **par** mode.
- Likewise, GeoGebra can do this, using the **curve** function.
  - In general, the syntax is: `curve(x(t), y(t), t, min, max)`



### 1.1.4 The Cycloid

A wheel of radius  $a$  is rolling along a flat road at a constant velocity. The curve generated by a point along the edge of the wheel traces out a shape called a *cycloid*. Let  $t$  represent the angle - in radians!!!! - rotated through, and that the point of interest starts at the origin. Before we find the equations for the point, let's find the location of the center of the circle:

$$f_{\text{center}}(t) = \begin{cases} x(t) = at \\ y(t) = a \end{cases}$$

Then, relative to the center, our point along the edge has equations

$$f(t) = \begin{cases} x(t) = -a \sin t \\ y(t) = -a \cos t \end{cases}$$

Thus, our point has parametric equations

$$f(t) = \begin{cases} x(t) = a(t - \sin t) \\ y(t) = a(1 - \cos t) \end{cases}$$

### 1.1.5 Final Notes

Next time, we'll start asking Calculus-y questions: What are the velocities in the  $x$ ,  $y$ , and total directions? What total distance does it travel? What is the area of the region under one period of the cycloid?

- The syllabus has a number of practice problems to work on. These are not required, and not to be turned in, but are for you to work before class next time.
- We will talk about them at the start of the next class. You should try them beforehand.
- The most common reason for a lack of success in this class is not spending time working problems on your own.

## 1.2 Calculus of Parametric Curves

For this section, we will have a parametric curve in  $\mathbb{R}^2$ , defined by  $f(t) = \begin{cases} x(t) \\ y(t) \end{cases}$ . In many cases, the curve does not describe  $y$  as a function of  $x$ . However, we can still carry over many ideas from single variable calculus.



### 1.2.1 Slope for a Parametric Curve

Given a point  $t_0$ , the *slope of the curve* in the  $xy$ -plane is given by

$$\left. \frac{dy}{dx} \right|_{t=t_0} = \left. \frac{dy/dt}{dx/dt} \right|_{t=t_0}.$$

Note that this is undefined when  $x'(t_0) = 0$ .

The *tangent line* at  $t_0$  is given by

$$y = \left( \left. \frac{dy}{dx} \right|_{t=t_0} \right) (x - x(t_0)) + y(t_0).$$

### 1.2.2 Second Derivative

The value of the second derivative for the curve at  $t_0$  is given by

$$\left. \frac{d^2y}{dx^2} \right|_{t=t_0} = \left. \frac{d}{dt} \left( \frac{dy}{dx} \right) \right|_{t=t_0} = \left. \frac{d}{dt} \left( \frac{dy/dt}{dx/dt} \right) \right|_{t=t_0}.$$

Note the benefit of Leibnitz notation for each of these two derivatives!

### 1.2.3 Area Under a Curve

Suppose that a parametric curve is non-self intersecting. Then, the signed area of the region between the curve and the  $x$ -axis on the  $t$  interval  $[t_a, t_b]$  is given by

$$A = \int_{t_a}^{t_b} y(t) \frac{dx}{dt} dt.$$

### 1.2.4 Arc Length

The *arc length* of a parametric curve over the  $t$  interval  $[t_a, t_b]$  is given by

$$s = \int_{t_a}^{t_b} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt.$$

### 1.2.5 Surface Area

The *surface area* of the region obtained by rotating a non-self intersecting parametric curve is given by

$$S = \int_{t_a}^{t_b} 2\pi y(t) \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt.$$



### 1.2.6 The Cycloid

We can apply each of the above to the cycloid:

- *Derivative:*  $\frac{dy}{dx} = \frac{dy}{dt} = \frac{\sin t}{1 - \cos t}$ . Note that the slope is then independent of the radius of the wheel and that the slope is undefined at each of  $t = \dots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \dots$
- *Cartesian Equation:* With radius of 3 and when  $t = \frac{\pi}{3}$ , the point is found by solving for  $x(\frac{\pi}{3})$  and  $y(\frac{\pi}{3})$ :

$$\begin{aligned}x\left(\frac{\pi}{3}\right) &= 3\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right) = \pi - \frac{3\sqrt{3}}{2} \\y\left(\frac{\pi}{3}\right) &= 3\left(1 - \cos\left(\frac{\pi}{3}\right)\right) = \frac{3}{2} \\(x, y) &= \left(\pi - \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)\end{aligned}$$

Plugging in our  $t$  value into our derivative, we get a slope of

$$\frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Now, we can write the equation of the tangent line as

$$y = \sqrt{3}\left(x - \pi + \frac{3\sqrt{3}}{2}\right) + \frac{3}{2}.$$



- *Concavity*:  $\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{\sin t}{1 - \cos t} \right)$ .

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= \frac{d/dt(dy/dx)}{dx/dt} \\
 &= \frac{\frac{d}{dt} \left( \frac{\sin t}{1 - \cos t} \right)}{a - a \cos t} \\
 &= \frac{\frac{\cos t(1 - \cos t) - \sin t \sin t}{(1 - \cos t)^2}}{a - a \cos t} \\
 &= \frac{\cos t - \cos^2 t - \sin^2 t}{(1 - \cos t)^2 a (1 - \cos t)} \\
 &= \frac{\cos t - 1}{a(1 - \cos t)^2} \\
 &= -\frac{1}{a(1 - \cos t)^2} \\
 &= -\frac{a}{a^2(1 - \cos t)^2} \\
 &= -\frac{a}{y^2}
 \end{aligned}$$

After some work, we find that  $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$ , which shows that the cycloid is always concave down.

- *Area*: The area of one period of the cycloid  $A = 3\pi a^2$ , after some work:

$$\begin{aligned}
 A &= \int_{t_a}^{t_b} y(t)x'(t)dt \\
 &= \int_0^{2\pi} (a - a \cos t)(a - a \cos t)dt \\
 &= a^2 \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t)dt \\
 &= a^2 \left( t + \frac{t}{2} + \frac{1}{4} \sin(2t) \right) \Big|_0^{2\pi} \\
 &= a^2 \left[ \left( 2\pi + \frac{2\pi}{2} + \frac{1}{4} \sin(2\pi) \right) - \left( 0 + \frac{0}{2} + \frac{1}{4} \sin(0) \right) \right] \\
 &= a^2 [2\pi + \pi] \\
 &= 3\pi a^2.
 \end{aligned}$$



- *Arc Length*: The arc length of one period of the cycloid is  $s = 8a$ , again after some work:

$$\begin{aligned}
 s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_0^{2\pi} \sqrt{(a - a \cos t)^2 + (a \sin t)^2} dt \\
 &= a \int_0^{2\pi} \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} dt \\
 &= a \int_0^{2\pi} \sqrt{2 - 2 \cos t} dt \\
 &= \sqrt{2}a \int_0^{2\pi} \sqrt{1 - \cos t} dt \\
 &= \sqrt{2}a \int_0^{2\pi} \sqrt{2 \sin^2 \left(\frac{t}{2}\right)} dt \\
 &= \sqrt{2}a \cdot \sqrt{2} \int_0^{2\pi} \sin \left(\frac{t}{2}\right) dt \\
 &= 2a \left( -2 \cos \left(\frac{t}{2}\right) \right) \Big|_0^{2\pi} \\
 &= 8a.
 \end{aligned}$$

- *Surface Area*: The surface area of the solid obtained by rotating one period of the cycloid around the  $x$ -axis is  $S = \frac{64\pi a^2}{3}$ , after a lot of tedious work.

$$S = \int_0^{2\pi} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$



## 2.1 Vectors in the Plane

### 2.1.1 Notation

In print, we write vectors in bold like:  $\mathbf{v}$ ,  $\mathbf{w}$ ,  $\mathbf{u}$ ,  $\dots$ . In handwriting, we often write vectors with an arrow over the top:  $\vec{v}$ ,  $\vec{w}$ ,  $\vec{u}$ ,  $\dots$ .

### 2.1.2 Vectors

A *vector* is a quantity with both *magnitude* (size, length, strength,  $\dots$ ) and *direction*. Given two points in the plane  $P = (x_1, y_1)$  and  $Q = (x_2, y_2)$ , the vector from  $P$  to  $Q$ , denoted  $\overrightarrow{PQ} = \mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$ .

We can also simply state components (known as *component form*):  $\mathbf{v} = \langle x, y \rangle$ .

The *zero vector*, denoted  $\mathbf{0}$ , is  $\mathbf{0} = \langle 0, 0 \rangle$ . Note that  $\mathbf{0} \neq 0$ .

A *scalar* is a real number (or a magnitude), without direction.

If  $c$  is a scalar and  $\mathbf{v} = \langle x, y \rangle$ , then

$$c\mathbf{v} = c\langle x, y \rangle = \langle cx, cy \rangle.$$

This operation is called *scalar multiplication*. Scalar multiplication changes the magnitude of a vector, but not its direction.

Note that the individual components of a vector are themselves *scalars*. You need to keep track of which is which.

If  $\mathbf{v} = \langle x_1, y_1 \rangle$  and  $\mathbf{w} = \langle x_2, y_2 \rangle$ , then the *vector sum*

$$\mathbf{v} + \mathbf{w} = \langle x_1 + x_2, y_1 + y_2 \rangle.$$

That is, we add component wise.

If  $\mathbf{v} = \langle x_1, y_1 \rangle$ , then the *magnitude* of  $\mathbf{v}$  is given by

$$\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2}.$$

This is really just the Pythagorean theorem.



## 2.2 Vectors in Space

In  $\mathbb{R}^3$ , we have three axes,  $x$ ,  $y$ , and  $z$ , which follow the *right-hand rule*: point the fingers of the right hand in the direction of the positive  $x$ -axis, curl them towards the positive  $y$ -axis, and the thumb points in the direction of the positive  $z$ -axis.

Since the distance formula in  $\mathbb{R}^3$  is  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ , then  $\mathbf{u} = \langle x, y, z \rangle$  we have  $\|\mathbf{u}\| = \sqrt{x^2 + y^2 + z^2}$ .

To *normalize* a vector, we divide by its magnitude:  $\mathbf{v} = \langle x, y, z \rangle$ , then  $\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \left\langle \frac{x}{\|\mathbf{v}\|}, \frac{y}{\|\mathbf{v}\|}, \frac{z}{\|\mathbf{v}\|} \right\rangle$ . This gives us a *unit vector* in the direction of  $\mathbf{v}$ .

Everything else is basically the same.

### 2.2.1 Vector Properties

Suppose that each of  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  are vectors and  $r$  and  $s$  are scalars. Then the following properties hold:

- *Additive Commutativity*:  $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$ .
- *Additive Associativity*:  $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ .
- *Additive Identity*:  $\mathbf{v} + \mathbf{0} = \mathbf{v}$ .
- *Additive Inverse*:  $-\mathbf{v} = (-1)\mathbf{v}$  and  $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ .
- *Scalar Associativity*:  $r(s\mathbf{u}) = (rs)\mathbf{u}$ .
- *Scalars Distributive over Vectors*:  $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$ .
- *Vectors Distributive over Scalars*:  $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$ .
- *Multiplicative Identity*:  $1\mathbf{u} = \mathbf{u}$ .
- *Zero Scalar*:  $0\mathbf{u} = \mathbf{0}$ .

### 2.2.2 Special Vectors

A *unit vector* is a vector  $\mathbf{u}$  such that  $\|\mathbf{u}\| = 1$ .

In  $\mathbb{R}^2$  the *standard unit vectors* are  $\hat{i} = \mathbf{i} = \langle 1, 0 \rangle$  and  $\hat{j} = \mathbf{j} = \langle 0, 1 \rangle$ . This allows us to write  $\mathbf{v} = \langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j}$ , for example.



In  $\mathbb{R}^3$ , we have three stand unit vectors,  $\hat{i} = \mathbf{i} = \langle 1, 0, 0 \rangle$ ,  $\hat{j} = \mathbf{j} = \langle 0, 1, 0 \rangle$ , and  $\hat{k} = \mathbf{k} = \langle 0, 0, 1 \rangle$ .

It is a picky detail, but  $\mathbf{i} \in \mathbb{R}^2 \neq \mathbf{i} \in \mathbb{R}^3$ .

## 2.3 The Dot Product

Suppose  $\mathbf{u} = \langle u_1, u_2, \dots, u_n \rangle$  and  $\mathbf{v} = \langle v_1, v_2, \dots, v_n \rangle$  are vectors in  $\mathbb{R}^n$ . Then the *dot product* of  $\mathbf{u}$  and  $\mathbf{v}$  is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + \dots + u_nv_n.$$

That is, we multiply the corresponding components and sum the results.

It should be clear that  $\mathbf{u} \cdot \mathbf{v}$  results in a scalar. The dot product is a special type of inner product.

Think of the dot product as a way to measure how much of one vector points in the same direction as another.

### 2.3.1 Properties of the Dot Product

Let  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$  and  $c$  be a scalar. Then the following properties hold:

- *Commutativity*:  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ .
- *Distributive Property*:  $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ .
- *Scalar Associativity*:  $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$ .
- *Self-Product*:  $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$ .
- *Magnitude*:  $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$ .
- *Angle*:  $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)$ , where  $0 \leq \theta \leq \pi$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ . (Law of Cosines.)
- *Orthogonality*:  $\mathbf{u} \cdot \mathbf{v} = 0$  if and only if  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal.

### 2.3.2 Projections

The *projection* of  $\mathbf{u}$  onto  $\mathbf{v}$  is given by

$$\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

This is a vector parallel to  $\mathbf{v}$ , which has length equal to the amount of  $\mathbf{u}$  which points in the same direction as  $\mathbf{v}$ .



Think of a projection as a measure of how much of one vector points in the same direction as another.

### 2.3.3 Work

If a constant force  $\mathbf{F}$  moved an object from  $P$  to  $Q$ , the *work* done is given by

$$W = \mathbf{F} \cdot \overrightarrow{PQ}.$$

Thus, if that force acts at an angle  $\theta$  to the line of motion, the work is:

$$W = (\|\mathbf{F}\|) \|\mathbf{PQ}\| \cos(\theta).$$

Later this semester, we will learn how to compensate for a non-constant force, and over a non-linear path.

## 2.4 The Cross Product

Suppose that  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ . Then, the *cross product* of  $\mathbf{u}$  and  $\mathbf{v}$ , denoted by  $\mathbf{u} \times \mathbf{v}$ , is the unique right-hand rule vector orthogonal to each of  $\mathbf{u}$  and  $\mathbf{v}$  whose magnitude is equal to the area of the parallelogram spanned by  $\mathbf{u}$  and  $\mathbf{v}$ .

Let  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ . Then,

$$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 \rangle.$$

NOTE: You will never multiply an  $v_1$ -coordinate by an  $u_1$ -coordinate. This is true for all  $v_n$  and  $u_n$  coordinates.

You can show by working the algebra that  $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$  and  $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ .

With determinants, you can do this in one step:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Oddly, we can only define a cross-product in  $\mathbb{R}$ ,  $\mathbb{R}^3$ , and  $\mathbb{R}^7$ , while the dot product is *always* defined.

### Example

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle 2, 1, 4 \rangle \cdot \langle 1, -3, 1 \rangle \\ &= \langle (1)(1) - 4(-3), 4(1) - 2(1), 2(-3) - 1(1) \rangle \\ &= \langle 13, 2, -7 \rangle. \end{aligned}$$



### 2.4.1 Properties of the Cross Product

Let  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$  and  $c$  be a scalar. Then the following properties hold:

- *Anticommutativity:*  $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$ .
- *Distributive Property:*  $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ .
- *Scalar Associativity:*  $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v})$ .
- *Zero:*  $\mathbf{u} \times \mathbf{u} = \mathbf{0}$ .
- *Nilpotence:*  $\mathbf{u} \times \mathbf{v} = \mathbf{0}$  if and only if  $\mathbf{u}$  and  $\mathbf{v}$  are parallel.
- *Scalar Triple Product:*  $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ .
- *Angle:*  $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$ , where  $0 \leq \theta \leq \pi$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .

### 2.4.2 Standard Unit Vectors and the Cross Product

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

- |  |  |
|--|--|
| • $\mathbf{i} \times \mathbf{j} = \mathbf{k}$  | • $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$ |
| • $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$ | • $\mathbf{k} \times \mathbf{i} = \mathbf{j}$  |
| • $\mathbf{j} \times \mathbf{k} = \mathbf{i}$  | • $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ |

### 2.4.3 Torque

*Torque*, denoted by  $\tau$ , measures the tendency to produce a rotation about an axis.

If  $\mathbf{r}$  is a radial vector from an axis to a force and  $\mathbf{F}$  is the force, then the torque induced on the axis by the force is given by:

$$\tau = \mathbf{r} \times \mathbf{F} \quad \text{or} \quad \|\tau\| = \|\mathbf{r}\| \|\mathbf{F}\| \sin(\theta),$$

## 2.5 Equations of Lines and Planes

For the vector equation, parametric equation, and the symmetric equation, use these points for the examples:  $(3, 5, 1)$  and  $(9, 1, 2)$ .

### 2.5.1 Lines

#### Lines in Two Dimensions

A line in  $\mathbb{R}^2$  which contains the point  $(x_0, y_0)$  and is parallel to the vector  $\mathbf{v} = \langle a, b \rangle$  has parametric form



$$f(t) = \begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \end{cases}.$$

### Lines in Three Dimensions

In  $\mathbb{R}^3$ , we have more options for the form of a line. Suppose that our line contains the point  $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$  and is parallel to the vector  $\mathbf{v} = \langle a, b, c \rangle$ . Then:

### Vector Equation

The *vector equation* of a line is given by  $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$ .

**Example:** Find all 3 equations of lines

From our example,  $\mathbf{v} = \langle 6, -4, 1 \rangle$  and  $\mathbf{r}_0 = \langle 3, 5, 1 \rangle$ .

Vector equation:  $\mathbf{r}(t) = \langle 3, 5, 1 \rangle + t \langle 6, -4, 1 \rangle$ .

### Parametric Equation

The *parametric equation* of a line is given by

$$f(t) = \begin{cases} x(t) = x_0 + ta \\ y(t) = y_0 + tb \\ z(t) = z_0 + tc \end{cases}.$$

From our example, we would get  $x(t) = 3 + 6t$ ,  $y(t) = 5 - 4t$ , and  $z(t) = 1 + t$ .

### Symmetric Equation

For the following formula, we get  $a$ ,  $b$  and  $c$  from subtracting the  $x$ ,  $y$ , and  $z$  components of the direction vector from the point vector.

As long as each of  $a$ ,  $b$ ,  $c \neq 0$ , the symmetric equation is

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

(Notice that in two dimensions, this is just the equation of the line:  $(\frac{b}{a})(x - x_0) + y_0 = y$ , when solved for  $y$ .)

From our example, we would get

$$\frac{x - 3}{9 - 3} = \frac{y - 5}{1 - 5} = \frac{z - 1}{1 - 2} \implies \frac{x - 3}{6} = \frac{y - 5}{-4} = -z + 1.$$



### Line Segment

Suppose that  $P = (x_0, y_0, z_0)$  and  $Q = (x_1, y_1, z_1)$ . The line segment from  $P$  to  $Q$  is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{p} + t\mathbf{q},$$

where  $\mathbf{p} = \langle x_0, y_0, z_0 \rangle$ ,  $\mathbf{q} = \langle x_1, y_1, z_1 \rangle$ , and  $0 \leq t \leq 1$ .

The parametric equations for this segment are

$$f(t) = \begin{cases} x(t) = x_0 + t(x_1 - x_0) \\ y(t) = y_0 + t(y_1 - y_0) \\ z(t) = z_0 + t(z_1 - z_0) \end{cases}.$$

### Distance Between Point and Line

The distance from a point  $M$  to a line which contains the point  $P$  and has direction vector  $\mathbf{v}$  is given by

$$d = \left| \frac{\overrightarrow{PM} \times \mathbf{v}}{\|\mathbf{v}\|} \right|.$$

Notice that you are free to choose any point on the line you'd like!

### Relationships Between Lines

- *Equal*: Same direction vector, share a point.
- *Parallel*: Same direction vector, do not share a point.
- *Intersecting*: Different direction vectors, share a point.
- *Skew*: Different direction vectors, do not share a point.

### 2.5.2 Planes

A plane can be defined by:

- any three non-colinear points,
- any two intersection points,
- a line and a point not on the line, or
- given two orthogonal vectors with a common starting point: “spin” one vector in place; notice the other sweeps out a circle, which can be extended to a plane. \* In notes \*

Of particular importance for a plane is a *normal vector*. A vector  $\mathbf{n}$  is a normal vector provided it is orthogonal to  $\overrightarrow{PQ}$  for any two points  $P$  and  $Q$  which are in the plane.



### 2.5.3 Equations of a Plane

Like lines, we have three equations of a plane. Let  $P$  and  $Q$  be points in the plane and  $n = \langle a, b, c \rangle$ .

#### Vector Equation

The *vector equation* of a plane is  $n \cdot \overrightarrow{PQ} = 0$ . Note that this is an implicit definition (i.e. it is not useful for directly writing down an equation, but is the fundamental idea of why this all works)!

#### Scalar Equation

If  $(x_0, y_0, z_0)$  is any point in the plane, the *scalar equation* of the plane is given by

$$\begin{aligned}\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0\end{aligned}$$

#### General Form

The *general form* of the equation of a plane is given by  $ax + by + cz + d = 0$ , where  $d = -ax_0 - by_0 - cz_0$ .

#### Distance Between Point and Plane

- Equal: Share a common point, have parallel normal vectors
- Parallel: Do not share a common point, do have parallel normal vectors
- Intersecting: If their normal vectors are not parallel, the two planes intersect in a line.
  - You can use algebra to find a point in common – i.e. solve both equations for the planes
  - Find the line's direction vector by taking the cross product of the planes' normal vectors.

### 2.5.4 Examples

Use the points  $P(3, 5, 1)$ ,  $Q(9, 1, 2)$ , and  $R(0, 2, 5)$ .

#### Example 1

Find the scalar equation of the plane containing  $P$ ,  $Q$ , and  $R$ .

We know  $\mathbf{PQ} = \langle 6, -4, 1 \rangle$  and  $\mathbf{PR} = \langle -3, -3, 4 \rangle$ . Then,  $\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \langle 13, -27, -30 \rangle$ .





Thus, the equation of the plane is  $13(x - 3) - 27(y - 5) - 30(z - 1) = 0$ .

To check, plug in the points:  $13(3) - 27(5) - 30(1) = 0$ ,  $13(9) - 27(1) - 30(2) = 0$ , and  $13(0) - 27(2) - 30(5) = 0$ .

## 2.6 Quadratic Surfaces

### 2.6.1 Spheres

A *sphere*, centered at  $(x_0, y_0, z_0)$  with radius  $r$ , is given by the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

### 2.6.2 Cylinder

A *cylinder* is a surface in  $\mathbb{R}^3$  which consists of all lines that are parallel to a given line and pass through a given plane curve. The lines that make up a cylinder are called *rulings*. The *trace* of a cylinder is the cross section generated by intersecting the cylinder with a coordinate plane.

### 2.6.3 Quadratic Surfaces

A *quadratic surface* is a surface in  $\mathbb{R}^3$  whose equation can be written as

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

By a change of axes (rotations) and origin (translations), we can rewrite these always as one of

$$Ax^2 + By^2 + Cz^2 = 1, \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

While cylinders have rulings made of parallel lines, quadratic surfaces do not (at least, not in general). However, their traces are always conic sections: lines, parabolas, circles, ellipses, or hyperbolas.

