Some Common Parameterizations and Integrals

Shapes in the xy-plane, $\mathbb{R}^2$

Curves

Each curve is one-dimensional, which means it can be parameterized by a single variable, t. In general, we have:

- the parameterization, $\mathbf{r}(t) = \langle x(t), y(t) \rangle$,
- the tangent vector, $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$,
- the magnitude of the tangent vector, $\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$, and
- the normal vector, $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$, where we adopt the convention that this vector points left to right as we move along the curve.

For scalar integrals, $ds = \|\mathbf{r}'(t)\| dt$ and for vector integrals, $\mathbf{T}\, ds = d\mathbf{r} = \mathbf{r}'(t)\, dt$ and $\mathbf{N}\, ds = \mathbf{n}(t)\, dt$.

Line Segments

The line segment which goes from $\mathbf{a} = \langle a_x, a_y \rangle$ to $\mathbf{b} = \langle b_x, b_y \rangle$ is parameterized by:

- $\mathbf{r}(t) = \langle a_x + t(b_x - a_x), a_y + t(b_y - a_y) \rangle$, $0 \leq t \leq 1$,
- $\mathbf{r}'(t) = \langle b_x - a_x, b_y - a_y \rangle$,
- $\|\mathbf{r}'(t)\| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2}$ (which we note is always constant for any given segment!), and
- $\mathbf{n}(t) = \langle b_y - a_y, a_x - b_x \rangle$.

Circle

The circle of constant radius r and whose center is at (h, k) is parameterized by:

- $\mathbf{r}(t) = \langle h + r\cos(t), k + r\sin(t) \rangle$, $0 \leq t \leq 2\pi$,
- $\mathbf{r}'(t) = \langle -r\sin(t), r\cos(t) \rangle$,
- $\|\mathbf{r}'(t)\| = r$, and
- $\mathbf{n}(t) = \langle r\cos(t), r\sin(t) \rangle$.

Ellipse

The ellipse with center at (h, k) and “radius” a along the x-axis and b along the y-axis is parameterized by:

- $\mathbf{r}(t) = \langle h + a\cos(t), k + b\sin(t) \rangle$, $0 \leq t \leq 2\pi$,
- $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t) \rangle$,
- $\|\mathbf{r}'(t)\| = \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}$, and
- $\mathbf{n}(t) = \langle b\cos(t), a\sin(t) \rangle$.

Regions

Disk

The disk centered at (h, k) with radius a is parameterized by:

- $\mathbf{r}(t, \theta) = \langle h + r\cos(\theta), k + r\sin(\theta)\rangle$, $0 \le r \le b$, $0 \le \theta \le 2\pi$.

We then have

- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta)\rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta)\rangle$,

and thus

- $\|\mathbf{t}_r \times \mathbf{t}_\theta\| = r$,

where it is understood that we are doing the two-dimensional analog to the cross product. Note that this is not defined without its magnitude, since it would need to be a vector in the k direction. Further, we see that $dS = dA = dx\,dy = r\,dr\,d\theta$, not too much of a surprise.

Shapes in the xyz-plane, $\mathbb{R}^3$

Curves

Much of the earlier material can be easily raised to three dimensions. However, note that our normal vector is missing. We can still define $\mathbf{N}(t)$ in many cases, but its main use in the previous section was flux. Flux is not well-defined over a single curve in $\mathbb{R}^3$, and so we will not consider it here. In general, we have:

- the parameterization, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$,
- the tangent vector, $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$,
- the magnitude of the tangent vector, $\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$,
- for scalar integrals, $ds = \|\mathbf{r}'(t)\|\,dt$, and
- for vector integrals, $\mathbf{T}\,ds = d\mathbf{r} = \mathbf{r}'(t)\,dt$.

Line Segments

The line segment which goes from $\mathbf{a} = \langle a_x, a_y, a_z\rangle$ to $\mathbf{b} = \langle b_x, b_y, b_z\rangle$ is parameterized by:

- $\mathbf{r}(t) = \langle a_x + t(b_x - a_x), a_y + t(b_y - a_y), a_z + t(b_z - a_z)\rangle$, for $0 \le t \le 1$,
- $\mathbf{r}'(t) = \langle (b_x - a_x), (b_y - a_y), (b_z - a_z)\rangle$, and
- $\|\mathbf{r}'(t)\| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2 + (b_z - a_z)^2}$, which we note is always constant for any given segment!

Circle

The circle with center (h, k, ℓ), constant radius r, and whose plane is parallel to the orthogonal unit vectors $\mathbf{a} = \langle a_x, a_y, a_z\rangle$ and $\mathbf{b} = \langle b_x, b_y, b_z\rangle$ is parameterized by:

- $\mathbf{r}(t) = \langle h + r(a_x\cos(t) + b_x\sin(t)), k + r(a_y\cos(t) + b_y\sin(t)), \ell + r(a_z\cos(t) + b_z\sin(t))\rangle$, $0 \le t \le 2\pi$,
- $\mathbf{r}'(t) = r\,\langle -a_x\sin(t) + b_x\cos(t), -a_y\sin(t) + b_y\cos(t), -a_z\sin(t) + b_z\cos(t)\rangle$, and
- $\|\mathbf{r}'(t)\| = r$, after some work!

Helix

A circular helix with constant radius r and z-coordinate at for constant a is parameterized by:

- $\mathbf{r}(t) = \langle r\cos(t), r\sin(t), at\rangle$,
- $\mathbf{r}'(t) = \langle -r\sin(t), r\cos(t), a\rangle$, and
- $\|\mathbf{r}'(t)\| = \sqrt{r^2 + a^2}$, which is constant for any given helix.

Surfaces

We finally turn to the truly new idea, that of a surface $S \subseteq \mathbb{R}^3$. As these are two dimensional, we will parameterize with two independent variables, u and v:

- $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$.

We then have

- $\mathbf{t}_u = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u} \right\rangle$,
- $\mathbf{t}_v = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v} \right\rangle$,

and thus,

- $dS = \|\mathbf{t}_u \times \mathbf{t}_v\|\, du\, dv$, and
- $\mathbf{N}\, dS = dS = \mathbf{t}_u \times \mathbf{t}_v\, du\, dv$.

where we have an orientation of the surface. We can choose to use $\mathbf{t}_v \times \mathbf{t}_u$ to flip the orientation. When the surface of interest is closed (i.e., is the boundary of a solid) we will always want the normal to point "outward." When the surface is $z = f(x, y)$, we typically want the normal to point "up." In other cases, the orientation will be given.

Scalar Surface

Consider the scalar surface defined by $z = f(x, y)$. We can use x and y themselves as the parameters to find

- $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$,
- $\mathbf{t}_x = \langle 1, 0, f_x(x, y)\rangle$,
- $\mathbf{t}_y = \langle 0, 1, f_y(x, y)\rangle$,
- $\mathbf{t}_x \times \mathbf{t}_y = \langle -f_x(x, y), -f_y(x, y), 1\rangle$, which points upward, since the z-component is $+1$, and
- $\|\mathbf{t}_x \times \mathbf{t}_y\| = \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2}$.

Scalar Surface – Polar

We can also write the scalar surface $z = f(x, y)$ where we think of $x = r\cos(\theta)$ and $y = r\sin(\theta)$. This is useful if the projection of the surface into the xy-plane is easier to think of in polar coordinates:

- $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), f(r, \theta)\rangle$,
- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta), f_r(r, \theta)(r, \theta)\rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta), f_\theta(r, \theta)(r, \theta)\rangle$,
- $\mathbf{t}_r \times \mathbf{t}_\theta = \langle \sin(\theta) f_\theta(r, \theta) - r\cos(\theta) f_r(r, \theta), -\cos(\theta) f_\theta(r, \theta) - r\sin(\theta) f_r(r, \theta), r\rangle$, which points upward, since the z-component is $+r$, and
- $\|\mathbf{t}'(r)\| \times \mathbf{t}_\theta = \sqrt{(f_\theta(r, \theta))^2 + r^2 + r^2(f_r(r, \theta))^2}$.

Plane – Cartesian

Consider the plane $z = ax + by + d$, which is parameterized with variables x and y:

- $\mathbf{r}(x, y) = \langle x, y, ax + by + d\rangle$,
- $\mathbf{t}_x = \langle 1, 0, a\rangle$,
- $\mathbf{t}_y = \langle 0, 1, b\rangle$,
- $\mathbf{t}_x \times \mathbf{t}_y = \langle -a, -b, 1\rangle$, which points upward, since the z-component is $+1$, and
- $\|\mathbf{t}_x \times \mathbf{t}_y\| = \sqrt{1 + a^2 + b^2}$.

Plane – Polar

Consider the plane $z = ax+by+d$ where the projection of the region of interest is more easily expressed in polar coordinates:

- $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), ar\cos(\theta) + br\sin(\theta) + d\rangle$,
- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta), a\cos(\theta) + b\sin(\theta)\rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta), -ar\sin(\theta) + br\cos(\theta)\rangle$,
- $\mathbf{t}_r \times \mathbf{t}_\theta = \langle -ar, -br, r\rangle$, which points upward, since the z-component is $+1$, and
- $\|\mathbf{t}_r \times \mathbf{t}_\theta\| = r\sqrt{1 + a^2 + b^2}$.

Cylinder

A cylinder, with constant radius r and axis matching the z-axis, is parameterized by θ and z as:

- $\mathbf{r}(\theta, z) = \langle r\cos(\theta), r\sin(\theta), z\rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$,
- $\mathbf{t}_z = \langle 0, 0, 1\rangle$,
- $\mathbf{t}_\theta \times \mathbf{t}_z = \langle r\cos(\theta), r\sin(\theta), 0\rangle$, which points outward, and
- $\|\mathbf{t}_\theta \times \mathbf{t}_z\| = r$, also hopefully not a surprise.

Cone – Polar

The cone, with axis aligned with the z-axis and "slope" a constant a is parameterized by r and θ so that:

- $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), ar\rangle$,
- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta), a\rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0\rangle$,
- $\mathbf{t}_\theta \times \mathbf{t}_r = \langle ar\cos(\theta), ar\sin(\theta), -r\rangle$, where we need $\mathbf{t}_\theta \times \mathbf{t}_r$, so our normal points out of the cone, and
- $\|\mathbf{t}_\theta \times \mathbf{t}_r\| = r\sqrt{a^2 + 1}$.

Cone – Spherical

The cone, with axis aligned with the z-axis and "slope" given by the constant polar angle ϕ is parameterized by ρ and θ so that:

- $\mathbf{r}(\rho, \theta) = \langle \rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi) \rangle$,
- $\mathbf{t}_\rho = \langle \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi) \rangle$,
- $\mathbf{t}_\theta = \langle -\rho \sin(\phi) \sin(\theta), \rho \sin(\phi) \cos(\theta), 0 \rangle$, and
- $\mathbf{t}_\theta \times \mathbf{t}_\rho = \left\langle \rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), -\rho \sin^2(\phi) \right\rangle$, where we need to flip around the order so our normal points out of the cone.
- $\|\mathbf{t}_\theta \times t_\rho\| = \rho \sin(\phi)$

Paraboloid – Cartesian

This is a special case of the general surface where $z = a(x^2 + y^2)$; this is useful if we have simple Cartesian bounds for x and y:

- $\mathbf{r}(x, y) = \left\langle x, y, a(x^2 + y^2) \right\rangle$,
- $\mathbf{t}_x = \langle 1, 0, 2ax \rangle$,
- $\mathbf{t}_y = \langle 0, 1, 2ay \rangle$,
- $\mathbf{t}_x \times \mathbf{t}_y = \langle -2ax, -2ay, 1 \rangle$, and
- $\|\mathbf{t}_x \times t_y\| = \sqrt{1 + 4a^2x^2 + 4a^2y^2}$.

Paraboloid – Polar

This is the special case of the general polar surface where $z = ar^2$; this version is more useful if our projection down into the xy-plane is circular:

- $\mathbf{r}(r, \theta) = \left\langle r \cos(\theta), r \sin(\theta), ar^2 \right\rangle$,
- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta), 2ar \rangle$,
- $\mathbf{t}_\theta = \langle -r \sin(\theta), r \cos(\theta), 0 \rangle$,
- $\mathbf{t}_\theta \times t_r = \left\langle 2ar^2 \cos(\theta), 2ar^2 \sin(\theta), -r \right\rangle$, where we again flip around the order so the normal points outward, and
- $\|\mathbf{t}_\theta \times t_r\| = r\sqrt{1 + 4a^2r^2}$.

Sphere

The sphere of constant radius ρ centered at the origin is parameterized by θ and ϕ, where θ is the planar angle and ϕ is the polar angle:

- $\mathbf{r}(\theta, \phi) = \langle \rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi) \rangle$,
- $\mathbf{t}_\phi = \langle \rho \cos(\phi) \cos(\theta), \rho \cos(\phi) \sin(\theta), -\rho \sin(\phi) \rangle$,
- $\mathbf{t}_\theta = \langle -\rho \sin(\phi) \sin(\theta), \rho \sin(\phi) \cos(\theta), 0 \rangle$,
- $\mathbf{t}_\phi \times t_\theta = \left\langle \rho^2 \sin^2(\phi) \cos(\theta), \rho^2 \sin^2(\phi) \sin(\theta), \rho^2 \sin(\phi) \cos(\phi) \right\rangle$, where we again flip around the order so the normal points outward, and
- $\|\mathbf{t}_\phi \times t_\theta\| = \rho^2 \sin(\phi)$, as expected.

Integrals

Curves and Regions in 2-D

Let C be a curve in $\mathbb{R}^2$ which is not necessarily closed, parameterized by $\mathbf{r}(t)$, $a \le t \le b$.

- The arc length of C is given by:
$$s = \int_C ds = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$
- If $f(x, y)$ is a continuous scalar function, then the signed area of the curtain from $z = f(x, y)$ back down to the xy-axis which lies above C is given by:
$$\int_C f(x(t), y(t))\, ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, dt.$$
- If $\mathbf{F}$ is a vector field, the circulation along C is given by:
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$
  - In the special case where C is closed, by Green's Theorem
$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$
    where D is the region enclosed by C.
- If $\mathbf{F}$ is a vector field, the flux across C is given by:
$$\int_C \mathbf{F} \cdot \mathbf{N}\, ds = \int_C \mathbf{F} \cdot \mathbf{n}(t)\, dt.$$
  - In the special case where C is closed, by Green's Theorem
$$\oint_C \mathbf{F} \cdot \mathbf{N}\, ds = \iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA,$$
    where D is the region enclosed by C.

Curves in 3-D

Now, let C be a curve in $\mathbb{R}^3$, again not necessarily closed, parameterized by $\mathbf{r}(t)$, $a \le t \le b$.

- The arc length of C is given by:
$$s = \int_C ds = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$
- If $f(x, y, z)$ is a continuous scalar function, then the following integral calculates the average value of f along C, multiplied by the arc length of C:
$$\int_C f(x(t), y(t), z(t))\, ds = \int_a^b f(\mathbf{r}(t))\|\mathbf{r}'(t)\|\, dt.$$
- If $\mathbf{F}$ is a vector field, the circulation along C is given by:
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t)\, dt.$$
  - In the special case where C is closed, by Stokes' Theorem:
$$\oint_C \mathbf{F} \cdot \mathbf{T}\, ds = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$
    where S is any positively oriented surface with C as its boundary.

(We will not worry about flux over a curve in $\mathbb{R}^3$.)

Surfaces in 3-D

Let S be a surface in $\mathbb{R}^3$, not necessarily closed, parameterized by $\mathbf{r}(u, v)$.

- If $f(x, y, z)$ is a scalar function, we can find the average value of f over S multiplied by the area of S by finding:

$$\iint_S f(x, y, z)\|\mathbf{t}'(u)\| \times \mathbf{t}_v \, dS.$$

  - For example, if f is the density of some object, this integral finds the mass, that is the average density times the area.

- If $\mathbf{F}$ is a vector field, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iint_S \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, du \, dv$$

  measures the flux of $\mathbf{F}$ through S – that is the amount of $\mathbf{F}$ which passes orthogonally through S, where positive flux matches the direction of the normal.

  - If $\mathbf{G} = \nabla \times \mathbf{F}$, then

$$\iint_S \mathbf{G} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

    measures the flux of the curl of $\mathbf{F}$ across S – that is the total amount of circulation (or torque) that $\mathbf{F}$ induces in S.
  - by Stokes' Theorem, if C is the positively oriented boundary curve of S:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

- In the special case when S is closed with positive orientation – that is, S encloses a solid region $E \subseteq \mathbb{R}^3$ with outward pointing normal – then $\iint_S \mathbf{F} \cdot d\mathbf{S}$ measures the total flux of $\mathbf{F}$ out of E. by the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\nabla \cdot \mathbf{F}) \, dV,$$

  since the second integral adds up all the "outflow" of $\mathbf{F}$ at each point inside the solid region E.

  - In this special case when S is closed, it does not have a boundary curve, and therefore the total circulation, or total flux of the curl of any vector field $\mathbf{F}$ across S is therefore 0. We can also see this since we know that for any vector field, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.