# Writing Assignment 4

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## 1 Introduction

In this paper, our goal is to analyze the second derivative test for functions of two variables. We will start with a quadratic function and show that the second derivative test works in this case. Then, we will generalize this to any twice-differentiable scalar function using Taylor polynomials. Finally, we will summarize the results and their implications for classifying critical points.

## 2 Investigating the Quadratic Function

Let's start by looking at a quadratic function of the form $q(x, y) = ax^2 + bxy + cy^2$, where we assume that $a, c \neq 0$ and $b^2 - 4ac \neq 0$. We want to look into how $q$ behaves at its critical point. Remember that a critical point for a function of two variables can be found by finding the points where the gradient is zero. Thus, to find the critical points, we need to find the partial derivatives of $q$ with respect to $x$ and $y$ and set them equal to zero:

$$\frac{\partial q}{\partial x} = 2ax + by = 0, \quad \frac{\partial q}{\partial y} = bx + 2cy = 0.$$

This leaves us with the system of equations:

$$\begin{aligned} 2ax + by &= 0 \\ bx + 2cy &= 0 \end{aligned}$$

We can rewrite the first equation as $y = -\frac{2a}{b}x$ and substitute this into the second equation to get:

$$bx + 2c\left(-\frac{2a}{b}x\right) = 0 \implies \left(b - \frac{4ac}{b}\right)x = 0.$$

This gives us two cases: either $x = 0$ or $b - \frac{4ac}{b} = 0$. If $x = 0$, then substituting back into the first equation gives us $y = 0$. Thus, we have one critical point at $(0, 0)$. If $b - \frac{4ac}{b} = 0$, then we have $b^2 - 4ac = 0$, which is not allowed by our assumption. Therefore, the only critical point is at $(0, 0)$.

Now that we've identified the critical point, our next step is to understand the nature of this point – specifically, whether it corresponds to a local minimum, local maximum, or a saddle point. To analyze the function's behavior, particularly its sign, around this critical point, we will rewrite the expression for $q(x, y)$ by completing the square with respect to the $x$ terms and treating $y$ as a constant:

$$q(x, y) = ax^2 + bxy + cy^2$$

Factor out $a$ from the terms involving $x$:

$$= a\left(x^2 + \frac{b}{a}xy\right) + cy^2$$

Now, we need to complete the square for the expression in parentheses. Remember that we need to calculate $(\frac{b}{2})^2$ for our function of the form $ax^2 + bx + c = 0$.

$$= a$$

$$\begin{aligned}
q(x, y) &= ax^2 + bxy + cy^2 \\
&= a\left(x^2 + \frac{b}{a}xy + \frac{c}{a}y^2\right) \\
&= a\left(x^2 + \frac{b}{a}xy + \frac{b^2}{4a^2}y^2 - \frac{b^2}{4a^2}y^2 + \frac{c}{a}y^2\right) \\
&= a\left(\left(x + \frac{b}{2a}y\right)^2 - \frac{b^2 - 4ac}{4a^2}y^2\right).
\end{aligned}$$

This shows that we can express $q(x, y)$ in the form:

$$\begin{aligned}
q(x, y) &= a\left(\left(x + \frac{b}{2a}y\right)^2 + \frac{4ac - b^2}{4a^2}y^2\right) \\
&= a\left(\left(x + \frac{b}{2a}y\right)^2 + \frac{D}{4a^2}y^2\right),
\end{aligned}$$

where $D = 4ac - b^2$. This expression allows us to analyze the behavior of $q(x, y)$ at the critical point $(0, 0)$.