Multivariable Calculus Exam 2

Practice Set # 3

1. Show that $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.
   - $x = 0$ path: $\lim_{(x,y)\to(0,0)} \frac{0\cdot y^2}{0+y^4} = \frac{0}{y^2} = 0$.
   - $y = 0$ path: $\lim_{(x,y)\to(0,0)} \frac{x\cdot 0}{x^2+0} = \frac{0}{x^2} = 0$.
   - $x = y^2$ path: $\lim_{(x,y)\to(0,0)} \frac{y^2\cdot y^2}{y^4+y^4} = \frac{y^4}{2y^4} = \frac{1}{2}$.

   Since the limit is not the same along all paths, the limit does not exist.

2. $\frac{\partial^2}{\partial x \partial y}\left(x^3y - y^3\tan(xy)\right)$

$$\begin{aligned}\frac{\partial^2}{\partial x\partial y}\left(x^3y - y^3\tan(xy)\right) &= \frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}[x^3y] - \frac{\partial}{\partial y}\left[y^3\tan(xy)\right]\right] \\ &= \frac{\partial}{\partial x}\left[x^3 - (3y^2\tan(xy) + xy^3\sec^2(xy))\right] \\ &= \frac{\partial}{\partial x}[x^3] - \frac{\partial}{\partial x}\left[3y^2\tan(xy)\right] - \frac{\partial}{\partial x}\left[xy^3\sec^2(xy)\right].\end{aligned}$$

   Splitting this into 3 partial derivatives:

$$\frac{\partial}{\partial x}[x^3] = 3x^2, \quad -\frac{\partial}{\partial x}\left[3y^2\tan(xy)\right] = -3y^3\sec^2(xy),$$

   with the final derivative worked out:

$$\begin{aligned}-\frac{\partial}{\partial x}\left[xy^3\sec^2(xy)\right] &= y^3\sec^2(xy) + \left[(xy^3)\cdot 2y\sec^2(xy)\tan(xy)\right] \\ &= -y^3\sec^2(xy) - 2xy^4\sec^2(xy)\tan(xy).\end{aligned}$$

   Combining these results, we have:

$$3x^2 - 3y^3\sec^2(xy) - y^3\sec^2(xy) - 2xy^4\sec^2(xy)\tan(xy).$$

   Since three terms contain a factor of $y^3\sec^2(xy)$, we can factor this out to get:

$$3x^2 - y^3\sec^2(xy)\left(3 + 1 + 2xy\tan(xy)\right).$$

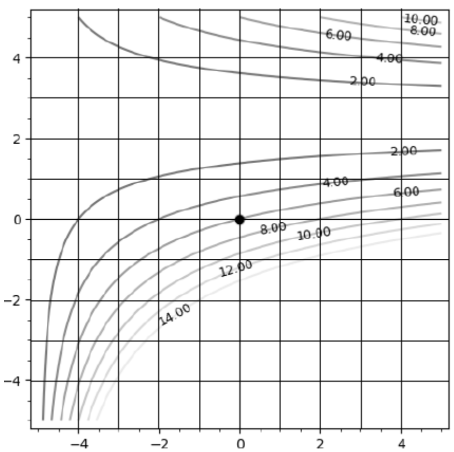
   Adding and simplifying further, we get:

$$\boxed{3x^2 - 2y^3\sec^2(xy)\left(2 + xy\tan(xy)\right).}$$

3. For the function $f(x,y,z) = \dfrac{x+\sin(xy)}{x^2+y^2+z^2+1}$, find $\nabla f(x,y,z)$.
   From the handout, we know that:

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)\mathbf{i} + \frac{\partial f}{\partial y}(x,y,z)\mathbf{j} + \frac{\partial f}{\partial z}(x,y,z)\mathbf{k}.$$

Practice Set # 3 (cont.)



Determine the sign $(+, -, 0)$ for each of the following partial derivatives.

4. $f_x(0,0)$
   We see $f(0,0) \approx 6$. As we move right (positive x), f increases, toward value 8. Thus, $+$.

5. $f_y(0,0)$
   As we move up f decreases toward 4. Thus, $-$.

6. $f_{xx}(0,0)$
   The contours are evenly spread in the x-direction through $(0,0)$. We are increasing at a <u>constant rate</u>. Hence, 0.

7. $f_{yy}(0,0)$
   As we move in positive y-direction, we decrease, but less rapidly. The amount by which we are changing is *increasing* (becoming less negative). Thus, $+$.

8. $f_{xy}(0,0)$
   If we move in positive x, the slope in the y-direction becomes more negative (i.e., decreases). Thus, $-$.

9. Find an equation of the tangent plane to $f(x,y) = x^2y - \sqrt{x} + y$ at the point $(3,1)$.
   Solve for $f_x(x,y)$, then $f_x(3,1)$, and $f_y(x,y)$, then $f_y(3,1)$ to get the values of the partial derivatives at the point $(3,1)$:

$$\begin{aligned}z &= f(3,1) + f_x(3,1)(x-3) + f_y(3,1)(y-1) \\ &= 7 + \tfrac{23}{4}(x-3) + \tfrac{35}{4}(y-1)\end{aligned}$$

10. Consider the function $f(x,y) = x^2y - y^3$. Find the directional derivative for f, at $(3,4)$, in the direction of $\mathbf{u} = 5\mathbf{i} - 2\mathbf{j}$.
    Find

$$f_x(x,y) = 2xy, \quad f_y(x,y) = x^2 - 3y^2.$$

    Then, at the point $(3,4)$:

$$f_x(3,4) = 2\cdot 3\cdot 4 = 24, \quad f_y(3,4) = 3^2 - 3\cdot 4^2 = -39.$$

    Then, we find the unit vector in the direction of $\langle 5,-2\rangle$:

$$\frac{\langle 24,-39\rangle\cdot\langle 5,-2\rangle}{\sqrt{29}} = \frac{198}{\sqrt{29}}$$

Practice Set # 4

1. (3 points) Determine the absolute extrema for the function $f(x,y) = x^2+3y^2-2x-y-xy$ on the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$.
We first find the critical points of the function:

$$\begin{aligned}\nabla f(x,y) &= \langle 2x-2-y,\, 6y-1-x\rangle = \mathbf{0}\\ &\implies y = 2x-2 \quad \text{and} \quad x = 6(2x-2)-1-x\\ &\implies y = \tfrac{4}{11} \quad \text{and} \quad x = \tfrac{13}{11}\end{aligned}$$

This gives the critical point $\boxed{\left(\frac{13}{11}, \frac{4}{11}\right)}$. We also need to check the boundary of the region. Thus:

(ℓ_1): $y = 0,\ 0 \le x \le 2 \implies f(x,y) = g(x) = x^2 + 3(0)^2 - 2x - (0) - x(0) = x^2 - 2x \implies g'(x) = 2x - 2$. This gives $(1,0)$.

(ℓ_2): $x = 0,\ 0 \le y \le 1 \implies f(x,y) = h(y) = (0)^2+3y^2-(0)-y-0 = 3y^2-y \implies h'(y) = 6y-1$. This gives $\left(0, \frac{1}{6}\right)$

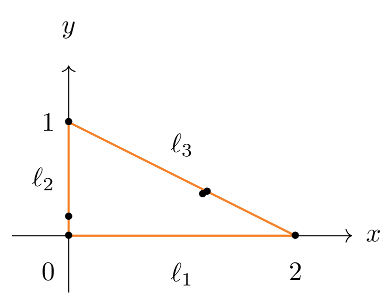
(ℓ_3): $y = 1 - \frac{1}{2}x,\ 0 \le x \le 2 \implies f(x,y) = k(x) = x^2 + 3\left(1-\frac{1}{2}x\right)^2 - 2x - \left(1-\frac{1}{2}x\right) - x\left(1-\frac{1}{2}x\right)$.

$$\begin{aligned}k(x) &= x^2 + 3\left(1 - \tfrac{1}{2}x - \tfrac{1}{2}x + \tfrac{1}{4}x^2\right) - 2x - 1 + \tfrac{1}{2}x - x + \tfrac{1}{2}x^2\\ &= \tfrac{1}{4}(9x^2 - 22x + 8)\\ \implies k'(x) &= \tfrac{1}{4}\cdot \tfrac{d}{dx}[9x^2 - 22x + 8]\\ x &= \tfrac{11}{9}\end{aligned}$$

Using this x-value, we plug it back into our equation for y to get the critical point $\left(\frac{11}{9}, \frac{7}{18}\right)$.

The vertices of the triangle give $f(0,0) = 0$, $f(2,0) = -2$, and $f(0,1) = 2$. We can do the same for the other points and add them to our table.

Point	$f(x,y)$	**Type**
$\left(\frac{13}{11}, \frac{4}{11}\right)$	-1.364	Interior CP
$(1,0)$	-1	ℓ_1
$\left(0, \frac{1}{6}\right)$	-0.083	ℓ_2
$\left(\frac{11}{9}, \frac{7}{18}\right)$	$\boxed{-1.361}$	ℓ_3
$(0,0)$	0	Vertex 1
$(2,0)$	-2	Vertex 2
$(0,1)$	$\boxed{2}$	Vertex 3



Practice Set # 4 (cont.)

1. Convert the rectangular point $(-5,1)$ to polar coordinates.

$$\begin{aligned}r &= \sqrt{(-5)^2+1^2} = \sqrt{26}\\ \theta &= \arctan\left(\tfrac{1}{-5}\right) = \arctan\left(-\tfrac{1}{5}\right) = \tfrac{7\pi}{6} + \pi \text{ (2nd quadrant)}\end{aligned}$$

The polar coordinates are $\boxed{\left(\sqrt{26}, \frac{7\pi}{6}+\pi\right)}$.

2. Convert the cylindrical point $(5, \frac{7\pi}{6}, 2)$ to rectangular.

$$\begin{aligned}x &= 5\cos\left(\tfrac{7\pi}{6}\right) = 5\left(-\tfrac{\sqrt{3}}{2}\right) = -\tfrac{5\sqrt{3}}{2}\\ y &= 5\sin\left(\tfrac{7\pi}{6}\right) = 5\left(-\tfrac{1}{2}\right) = -\tfrac{5}{2}\\ z &= 2\end{aligned}$$

The rectangular coordinates are $\boxed{\left(-\frac{5\sqrt{3}}{2}, -\frac{5}{2}, 2\right)}$.

3. Convert the rectangular point $(-2,4,-1)$ to spherical.

$$\begin{aligned}\rho &= \sqrt{(-2)^2+4^2+(-1)^2} = \sqrt{21}\\ \theta &= \arctan\left(\tfrac{4}{-2}\right) = \arctan(-2)\\ \phi &= \arccos\left(\tfrac{-1}{\sqrt{21}}\right) = \arccos\left(-\tfrac{1}{\sqrt{21}}\right)\end{aligned}$$

Since the point $(-2,4)$ is in the second quadrant, we add π to the arctan value. Hence, the spherical coordinates are $\boxed{\left(\sqrt{21}, \pi+\arctan(-2), \arccos\left(-\frac{1}{\sqrt{21}}\right)\right)}$.

4. Convert the spherical point $(4, \frac{11\pi}{6}, \frac{3\pi}{4})$ to cylindrical.
The conversion from spherical to cylindrical follows the following equations:

$$r = \rho\sin\phi, \quad \theta = \theta, \quad \text{and} \quad z = \rho\cos\phi.$$

Thus, we have:

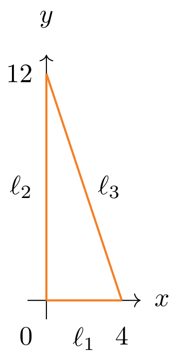
$$\begin{aligned}r &= 4\sin\left(\tfrac{3\pi}{4}\right) = 4\left(\tfrac{\sqrt{2}}{2}\right) = 2\sqrt{2}\\ \theta &= \tfrac{11\pi}{6}\\ z &= 4\cos\left(\tfrac{3\pi}{4}\right) = 4\left(-\tfrac{\sqrt{2}}{2}\right) = -2\sqrt{2}\end{aligned}$$

Therefore, we get the cylindrical coordinates $\boxed{\left(2\sqrt{2}, \frac{11\pi}{6}, -2\sqrt{2}\right)}$.

Practice Set # 4 (cont.)

1. $\iint_D (x^2 + 6xy)\, dA$ where D is the triangle with vertices $(0,0)$, $(4,0)$, and $(0,12)$

Solution. We can see that this triangle is bounded by three lines:



$$\ell_1 : y = 0$$
$$\ell_2 : x = 0$$
$$\ell_3 : y = -3x + 12$$

This gives us the limits of integration as follows:

$$\{(x, y) \colon 0 \le x \le 4, \quad 0 \le y \le -3x + 12\}.$$

Thus, we can write the double integral as:

$$\begin{aligned}
\iint_D (x^2 + 6xy)\, dA &= \int_0^4 \int_0^{-3x+12} (x^2 + 6xy)\, dy\, dx \\
&= \int_0^4 \left[x^2 y + 3xy^2\right]_0^{-3x+12} \, dx \\
&= \int_0^4 \left[x^2(-3x+12) + 3x(-3x+12)^2\right] \, dx \\
&= 6\left[x^4 - \tfrac{34}{3}x^3 + 36x^2\right]_0^4 \\
&= 48\left[32 - \tfrac{34}{3}(8) + 36(2)\right] \\
&= \boxed{640}
\end{aligned}$$

General Regions

Suppose we have a general region D. Then,

- **Type I Region** – we say that D is a Type I region provided there exists constants a, b and continuous functions g_1, $g_2 : \mathbb{R} \to \mathbb{R}$ so that

$$D = \{(x, y) : a \le x \le b, \text{ and } g_1(x) \le y \le g_2(x)\}.$$

- **Type II Region** – we say that D is a Type II region provided there exists constants c, d and continuous functions h_1, $h_2 : \mathbb{R} \to \mathbb{R}$ so that

$$D = \{(x, y) : h_1(y) \le x \le h_2(y), \text{ and } c \le y \le d\}.$$

0.0.1 Type I Regions

Suppose that D is a type I region:

$$\begin{aligned}
\iint_D f(x, y)\, dA &= \iint_R F(x, y)\, dA \\
&= \int_b^a \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy dx.
\end{aligned}$$

The "see below" line is true since $F(x, y) = 0$ if $y > g_2(x)$ or $y < g_1(x)$.

0.0.2 Type II Regions

In the same way, if D is type II, we have

$$\iint_D f(x, y)\, dA = \int_d^c \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx dy.$$

How do you tell? DRAW A PICTURE! (In practice, you don't typically explicitly note what type an integral is.)

0.0.3 Area

Suppose that D is a region. Then, the ***area*** of D is given by

$$\text{area}(D) = \iint_D 1\, dA.$$

0.0.4 Average Value

The ***average value*** of f over D is given by

$$\text{ave}(f) = \frac{1}{\text{area}(D)} \iint_D f(x, y)\, dA.$$

Max and Min

The statement that (x_0, y_0) is a ***critical point*** of f means that either:

- both $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or
- one or both partials does not exist.

0.1 Maxima and Minima

0.1.1 Local Extrema

The function f has a ***local maximum*** at (x_0, y_0) provided that $f(x_0, y_0) \geq f(x, y)$ for all choices of (x, y) in some disk centered at (x_0, y_0) – that is, in some neighborhood of (x_0, y_0).

Note that if there is a local extrema, $\nabla f = \mathbf{0}$. This is because the gradient points in the direction of greatest increase, and if we are at a maximum or minimum, the function does not change.

0.1.2 Second Derivative Test

Calculus I Version: In Calculus I, the sign of the second derivative tells you whether a critical point is a local max/min, or inconclusive: Suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is a function of one variable, and x_0 is a critical point.

- if $g''(x_0) > 0$, then x_0 is a local minimum
- if $g''(x_0) < 0$, then x_0 is a local maximum
- if $g''(x_0) = 0$, this test is inconclusive – it could be a max, min, or neither.

Multivariable Calculus Version: We have a similar test for $f\colon \mathbb{R}^2 \to \mathbb{R}$, where (x_0, y_0) is a critical point. Define

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - \big(f_{xy}(x_0, y_0)\big)^2.$$

- if $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then f has a local minimum
- if $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then f has a local maximum
- if $D < 0$, then f has a saddle point
- if $D = 0$, then the test is inconclusive

Tangent Planes

For functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ we have a similar idea. If the surface generated by such a function has no sharp corners or edges, you might see that as you zoom in, the surface becomes flatter and flatter – and will eventually resemble a plane. In fact, we define the ***tangent plane*** as the unique plane at $(x, y) = (x_0, y_0)$ which satisfies

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Recall from Calculus I that if f is differentiable at x_0 and x is close to x_0 then

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

This is the ***linear approximation*** of f at x_0. In the same way, if $f\colon \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (x_0, y_0) and (x, y) is near (x_0, y_0), then

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

This is the ***linearization*** of f at (x_0, y_0).

0.2 Examples

$f(x, y) = x\cos(\pi x)\sin(\pi y)$, $(x_0, y_0) = (\frac{1}{3}, \frac{1}{2})$. Our point of interest is $(\frac{1}{3}, \frac{1}{2}), \frac{1}{6}$, because we can just plug in the values of x_0 and y_0 into the function to get the height.

To find the equation of the tangent plane, we need to find the partial derivatives to fill out the following equation:

$$z = \underline{\qquad} + \underline{\qquad}\left(x - \frac{1}{3}\right) + \underline{\qquad}\left(y - \frac{1}{2}\right).$$

We found z_0 to be $\frac{1}{6}$, and the partial derivatives are

$$f_x(x, y) = \cos(\pi x)\sin(\pi y) - \pi x \sin(\pi x)\sin(\pi y),$$

and

$$f_x\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{2} - \frac{\pi}{3}\left(\frac{\sqrt{3}}{2}\right)(1) = \frac{1}{2} - \frac{\pi\sqrt{3}}{3}.$$

Similarly, we have

$$f_y(x, y) = \pi x \cos(\pi x)\cos(\pi y),$$

and

$$f_y\left(\frac{1}{3}, \frac{1}{2}\right) = 0.$$

Now, we can fill out the rest of our equation:

$$z = \frac{1}{6} + \left(\frac{1}{2} - \frac{\pi\sqrt{3}}{3}\right)\left(x - \frac{1}{3}\right) + 0\left(y - \frac{1}{2}\right).$$