Real Analysis: Exam 2

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"All work on this take-home exam is my own." **Signature:** Paul Beggs

Part 1. These two problems will give results that are useful in the next part.
Throughout this test, $f^{(j)}(x)$ denotes the j^{th} derivative of f at x.

(1) Let $c_0, c_1, c_2, \ldots, c_k$ be real numbers. Prove there exists a unique polynomial $p(x)$ of order at most k such that for each integer j between 0 and k, $p^{(j)}(0) = c_j$. In other words,

$$p(0) = c_0, \qquad p'(0) = c_1, \qquad p''(0) = c_2, \qquad \ldots, \qquad p^{(k)}(0) = c_k.$$

If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$, give formulas for $a_0, \ldots, a_k$ in terms of $c_0, \ldots, c_k$.

(2) Let φ be a function that is differentiable $k+1$ times on an interval $[a, b]$. This means $\varphi', \varphi'', \ldots, \varphi^{k+1}$ all exist on $[a, b]$. Assume that

$$\begin{aligned} \varphi(a) &= 0 \qquad\qquad \text{and} \qquad\qquad \varphi(b) = 0. \\ \varphi'(a) &= 0 \\ &\vdots \\ \varphi^{(k)}(a) &= 0 \end{aligned}$$

Prove there exists a point $c \in (a, b)$ such that $\varphi^{k+1}(c) = 0$.

Part 2. These problems will walk you through an important concept and result in Calculus.

Let I be an interval with zero in its interior and $f(x)$ be a function that is $k+1$ times differentiable on I.

(3) Construct the unique polynomial $P_k(x)$ of order at most k which satisfies that for all integers j between 0 and k, $P_k^{(j)}(0) = f^{(j)}(0)$. *This should be a direct application of Problem (1).*

(4) Let x be a fixed nonzero point in I. Define a new function g on I as follows:

$$g(t) = f(t) - P_k(t) - \left(\frac{f(x) - P_k(x)}{x^{k+1}}\right) t^{k+1}.$$

Show that

$$\begin{aligned} g(0) &= 0 \qquad\qquad \text{and} \qquad\qquad g(x) = 0. \\ g'(0) &= 0 \\ &\vdots \\ g^{(k)}(0) &= 0 \end{aligned}$$

Conclude there exists a point c between 0 and x such that $g^{(k+1)}(c) = 0$.

(5) Use the above problem to prove the existence of a point c between 0 and x for which

$$f(x) = P_k(x) + \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1}.$$

(6) This polynomial P_k is used as an approximation of f. If it is known that $|f^{(k+1)}|$ is bounded by some number M on the interval I, prove the error bound formula

$$|f(x) - P_k(x) \leq \frac{M|x|^{k+1}}{(k+1)!}.$$

Part 3. Now you get to enjoy using your result!

(7) Consider the function $f(x) = e^x$. Give the expression of the polynomial approximation P_k for an arbitrary $k \in \mathbb{N}$. Use what you know about f and its derivatives on the interval $[0, 1]$ to determine an integer k for which you can guarantee that $|f(1) - P_k(1)| < 10^{-12}$. Use this (and a calculator) to generate an approximation of e to 12 decimal places.

Solutions

(1) To ensure that we have a polynomial with at order of at most k, we need to observe some behaviors of derivatives. For example, for the polynomial x^j,

$$\begin{aligned}(x^j)' &= jx^{j-1} \\ (x^j)'' &= j(j-1)x^{j-2} \\ &\vdots \\ (x^j)^{(j)} &= j! \cdot x^0 = j!\end{aligned}$$

Notice the factorial arises from the recursive application of the power rule. Thus, when we combine this with the coefficients $a_0, a_1, \ldots, a_k$, we get,

$$\begin{aligned}p(x) &= a_0 + a_1x + a_2x^2 + \cdots + a_kx^k \\ p'(x) &= a_1 + (a_2 \cdot 2)x + (a_3 \cdot 3)x^2 \cdots (a_k \cdot k)x^{k-1} \\ &\vdots \\ p^{(j)}(x) &= a_j \cdot j! +\end{aligned}$$

To solve this problem, we'll determine the coefficients $a_0, a_1, \ldots, a_k$ in terms of $c_0, c_1, \ldots, c_k$.
Step 1: Calculate the derivatives of $p(x)$ at $x = 0$.
Given:

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k$$

Compute the j-th derivative of $p(x)$:
- The j-th derivative $p^{(j)}(x)$ is:

$$p^{(j)}(x) = a_j \cdot j! + \text{terms involving higher powers of } x$$

- Evaluated at $x = 0$:

$$p^{(j)}(0) = a_j \cdot j!$$

Step 2: Solve for a_j using $p^{(j)}(0) = c_j$.
Since $p^{(j)}(0) = c_j$, we have:

$$c_j = a_j \cdot j!$$

Therefore:

$$a_j = \frac{c_j}{j!}$$

for each $j = 0, 1, 2, \ldots, k$.
Final Result:
The coefficients are: - $a_0 = c_0$ - $a_1 = \frac{c_1}{1!}$ - $a_2 = \frac{c_2}{2!}$ - $\vdots$ - $a_k = \frac{c_k}{k!}$
Conclusion:
- **Existence:** The polynomial $p(x)$ exists with coefficients defined by $a_j = \frac{c_j}{j!}$. - **Uniqueness:** The polynomial is unique because each a_j is uniquely determined by c_j.

This shows that there exists a unique polynomial $p(x)$ of degree at most k satisfying the given conditions, with coefficients explicitly given in terms of c_j.