Definitions

Recall that an **automorphism** of a group G is an isomorphism $\varphi : G \to G$. The **automorphism group**, $Aut(G)$, is the group of all automorphisms on G with function composition as its operation.

For two groups A and X with respective identity elements e and e', let G be the set of ordered pairs,

$$G = \{(a, x) \mid a \in A, x \in X\}.$$

Let φ be a homomorphism from X to the automorphism group on A,

$$\varphi : X \to Aut(A).$$

This means that for each $x \in X$, we have an automorphism $\varphi_x : A \to A$. Since φ is a homomorphism, then for any $x, y \in X$ and $a \in A$, $\varphi_{xy}(a) = \varphi_x(\varphi_y(a))$. Also, $\varphi e' : A \to A$ is the identity map on A.

For your exam, we will define a binary operation on G that we are going to call the φ-product. Multiplication in G is defined as

$$(a, x)(b, y) = (a\varphi_x(b), xy).$$

Since $\varphi_x(b) \in A$, the φ-product does indeed belong to G. Notice that this multiplication is very similar to a direct product. We just use the automorphism φ_x to do something to b before multiplying the terms. In fact, if φ is trivial (meaning every φ_x is the identity map), then G is the direct product $A \times X$.

Problems

1. Show that G with the φ-product is a group.

(a) **Associativity:** For any $a, b, c \in A$ and $x, y, z \in X$, show that

$$[(a, x)(b, y)](c, z) = (a, x)[(b, y)(c, z)]$$

(b) **Identity:** Show that (e, e') is the identity element of G.

(c) **Inverses:** the inverses of (a, x) is $(\varphi_{x^{-1}}(a^{-1}), x^{-1})$.

Solution.

2. Let $\tilde{A} = \{(a, e') \mid a \in A\}$ and $\tilde{X} = \{(e, x) \mid x \in X\}$.

(a) Prove that $A \simeq \tilde{A}$ by showing $(a, e')(b, e') = (ab, e')$.

Solution.

(b) Prove that $X \simeq \tilde{X}$ by showing $(e, x)(e, y) = (e, xy)$.

Solution.

As a result, we can think of A and X as subgroups of G.

3. Show that in general, G is not abelian.

(a) Do this by comparing $(a, e')(e, x)$ and $(e, x)(a, e')$.

(b) If φ is not trivial, then there exists $x \in X$ and $a \in A$ with $\varphi_x(a) \neq a$. This and part (a) show G is not abelian (even if A and X are both abelian).

Solution.

4. Show that A (actually $\tilde{A}$) is a normal subgroup of G.

(a) Let $(a, x) \in G$ and $b \in A$. Show that $(a, x)(b, e')(a, x)^{-1} \in \tilde{A}$.

Solution.

Application

5. Let $\varphi : \mathbb{Z}_2 \to Aut(\mathbb{Z}_4)$ be defined as follows:

$$\varphi_0 : \mathbb{Z}_4 \to \mathbb{Z}_4 \text{ is the identity map: } \varphi_0(n) = n$$
$$\varphi_1 : \mathbb{Z}_4 \to \mathbb{Z}_4 \text{ maps to inverses: } \varphi_1(n) = n^{-1}$$

Show that D_4 is isomorphic to $\mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_2$.