Homework 1: Chapter 1

Mathematical Models

Author

Paul Beggs
BeggsPA@Hendrix.edu

Instructor

Dr. Christopher Camfield, Ph.D.

Due

January 27, 2026

1. An automobile manufacturer makes a profit of \$1,500 on the sale of a certain model. It is estimated that for every \$100 of rebate, sales increase by 15%. Python Code

(a) What amount of rebate will maximize profit? Use the five-step method, and model as one-variable optimization problem.

Solution. Five-step method:

(1) What amount of rebate will maximize profit?

(2) Single variable optimization.

(3) Label variables and parameters, and relate them with equations:

r	*rebate amount (\$)*
N	*cars sold*
$p_0 = 1500$	*initial profit per sale (\$)*
$s = .15$	*sales rate increase per r_0 rebate*
$r_0 = 100$	*initial rebate value (\$)*
$i(r) = 1 + (r/r_0) \cdot s$	*sales increase for r rebate*
$s(r) = (p_0 - r)N$	*profit after r rebate for N cars sold (\$)*
$n(r) = s(r)i(r)$	*net profit after r rebate (\$)*

(4) Plugging in the parameters into our equations, we get

$$i(r) = 1 + \frac{0.15r}{100}, \quad s(r) = 1500 - r, \quad n(r) = (1500 - r)N\left(1 + \frac{0.15r}{100}\right)$$

Then, we take the derivative of $n(r)$ and set it to 0 to get the optimal rebate value:

$$n'(r) = -0.003r + 1.25 \implies r = \frac{1250}{3}.$$

(5) To maximize the profit on the sale of the model, the automobile manufacturer should include a \$416.68 rebate on every purchase. This results in a \$1760.42 profit for each car sold.

(c) Suppose that rebates actually generate only a 10% increase in sales per \$100. What is the effect? What if the response is somewhere between 10 and 15% per \$100 of rebate?

Solution. If the rebates only generate a 10% increase, then the best rebate value would be \$250. This would drop our expected profit to \$1562.50 per car. For an increase in sales between 11 and 14%, we would require a rebate of \$295.45, \$333.33, \$365.38, and \$392.86 respectively. In turn, we would expect profit values of \$1596.02, \$1633.33, \$1673.56, and \$1716.07 per rebate.

(d) Under what circumstances would a rebate offer cause a random reduction in profit?

Solution. For a sales rate of 10% per \$100 of rebate, any amount of rebate over \$500 would start to return profits less than \$1500. If the sales rate were 15%, then any rebate above \$833.33 would also return less than \$1500.

5. It is estimated that the growth rate of the fin whale population (per year) is $rx(1 - x/K)$, where $r = 0.08$ is the intrinsic growth rate, $K = 400{,}000$ is the maximum sustainable population, and x is the current population, now around 70,000. It is further estimated that the number of whales harvested per year is about $0.00001\,Ex$, where E is the level of fishing effort in boat-days. Given a fixed level of effort, population will eventually stabilize at the level where growth rate equals harvest rate. Python Code

(a) What level of effort will maximize the sustained harvest rate? Model as a one-variable optimization problem using the five-step method.

Solution. The five-step method:

(1) What level of effort will maximize the sustained harvest rate?

(2) Single variable optimization.

(3) Label variables and parameters, and relate them with equations:

E	*level of effort (boat-days)*
x	*population (whales per year)*
$r = 0.08$	*intrinsic growth rate (whales per year)*
$K = 400000$	*max sustainable population (whales per year)*
$g(x) = rx(1 - x/K)$	*growth rate of fin whales (whales per year)*
$h(E) = 0.00001Ex$	*fin whales harvested (whales per year)*

(4) To determine the level of fishing effort to maximize the harvest rate, we first must find the optimal whale population for x. Then, we can plug this number in for $h(E)$ and solve for E. Treating x as a variable and setting $g'(x)$ to 0, we find that the optimal population is 200,000. This leaves the growth rate at 8,000 whales per year, so this is the most that can be harvested per year. With these numbers, we can solve for E:

$$8000 = 0.00001(200000)E \implies E = 4000.$$

Thus, the optimal level of effort is 4000 boat-days per year.

(5) To optimize a sustainable harvest rate, we should harvest at most 8,000 whales per year. To keep this population, we must keep a level of effort of 4,000 boat-days at the maximum. “My quote”

6. In Exercise 5, suppose that the cost of whaling is \$500 per boat-day, and the price of a fin whale carcass is \$6,000. Python Code

(a) Find the level of effort that will maximize profit over the long term. Model as a one-variable optimization problem using the five-step method.

Solution. The five-step method:

(1) What level of effort will maximize the profit over the long term?

(2) Single variable optimization.

(3) Label variables and parameters, and relate them with equations:

E	*level of effort (boat-days)*
x	*population (whales per year)*
$r = 0.08$	*intrinsic growth rate (whales per year)*
$w = 500$	*cost of whaling per boat-day (\$ per boat day)*
$f = 6000$	*price of a fin whale carcass (\$ per whale)*
$K = 400000$	*max sustainable population (whales per year)*
$g(x) = rx(1 - x/K)$	*growth rate of fin whales (whales per year)*
$R(x) = h(E)f$	*revenue (\$ per year)*
$c(x) = wE$	*cost (\$ per year)*
$p(x) = R(x) - c(x)$	*profit (\$ per year)*
$h(E) = 0.00001Ex$	*number of whales harvested (whales per year)*

(4) To relate variables, we can set $h(E) = g(x)$ and solve for E in terms of x:

$$\begin{aligned} h(E) &= g(x) \\ 0.00001Ex &= 0.08x(1 - x/400000) \\ E &= -0.02x + 8000. \end{aligned}$$

Now, we make the substitution of E, and update our equations:

$$R(x) = 0.06x\left(-0.02x + 8000\right), \quad c(x) = 500\left(-0.02x + 8000\right).$$

This gives us an updated profit equation:

$$p(x) = \left(-0.02x + 8000\right)\left(0.06x - 500\right).$$

Taking the derivative and setting it to 0, we find that the optimum amount of whales to be kept alive is 204,167 (x). With this number, we can find the optimum level of effort by solving

$$E = -0.02x + 8000 \implies E = 3917 \text{ boat-days per year.}$$

Then, we know the number of whales that must be harvested is

$$0.00001(3917)(204167) = 7997 \text{ whales per year.}$$

These harvests will gain the whaling company 46 million dollars per year.

(5) While maintaining a steady yearly whale population of 204,167 with 3917 boat-days per year, the whaling company should harvest 7997 whales per year. In turn, they will profit 46 million dollars per year.

```
# Problem 1.1
r = var('r')  # Rebate in dollars
N = var('N')  # Number of cars

# Parameters
p0 = 1500 # initial profit per sale ($)
s  = .15  # sales rate (%)
r0 = 100  # initial rebate value

# Functions
i(r) = 1 + (r/r0) * s  # sales increase for r rebate
b(r) = p0 - r          # profit after r rebate
p(r) = b(r) * i(r) * N # net profit for r rebate

# Solve the model
pprime = p.derivative(r)       # derivative of profit
pcrits = solve(pprime(r)==0,r) # critical points
optim_r = pcrits[0].rhs()      # best rebate value
maxprofit = p(optim_r)         # maximum profit given the best rebate value

print(f'The best rebate is ${round(optim_r, 2)}.')
print(f'The maximum profit is ${maxprofit}.')
```

Figure 1: Exercise 1 Code

```
# Problem 1.5
E = var('E')
x = var('x')

# Parameters
r = 0.08    # intrinsic growth rate
K = 400_000 # maximum sustainable population
# x = 70_000  # current population

# Functions
g(x) = r*x*(1 - x / K)
# h(E) = 0.00001 * E * x

# Find Maximum Sustainable Population
gprime = g.derivative(x)
gcrit = solve(gprime(x)==0,x)
optim_g = gcrit[0].rhs()
maxharvest = g(optim_g)

print(optim_g)
print(maxharvest)
```

Figure 2: Exercise 5 Code

```
# Problem 1.6
E = var('E') # level of effort (boat-days)
x = var('x') # population (whales per year)

# Parameters
r = 0.08 # intrinsic growth rate (whales per year)
w = 500 # cost of whaling per boat-day ($ per boat-day)
f = 6000 # price of a fin whale carcass ($ per whale)
K = 400_000 # max sustainable whale population (whales per year)

# Equations
g(x) = r*x * ( 1 - x/K) # growth rate of whales (whales per year)
h(E) = 0.00001 *E * x # number of whales harvested (whales per year)
R(x) = h(E) * f # revenue ($ per year)
c(x) = w *E # cost ($ per year)
p(x) = R(x) - c(x) # profit ($ per year)

# Solve for E in terms of x
solve(h(E)==g(x),E) # Gives [E == -1/50*x + 8000]

# Substitute E in equations
hn(x) = 0.00001 * (-1/50*x + 8000) * x
Rn(x) = hn(x) * f
cn(x) = w * (-1/50*x + 8000)
pn(x) = Rn(x) - cn(x)

# Solve the model
pprime = pn.derivative(x)
pcrits = solve(pprime(x)==0,x)
xoptim = pcrits[0].rhs()
maxprofit = pn(xoptim)

print(f"To maximize profits, the you must keep {xoptim} whales alive. This
↪  gives a profit of {maxprofit}")

# Find the optimum number of whales to be harvested and the effort required
lvl_o_effort = 8000 - .02 * xoptim # Change to .02 to fix pmdas problem
harvest = 0.00001 * lvl_o_effort * xoptim

print(lvl_o_effort, harvest)
```

Figure 3: Exercise 6 Code