Problems

1. Show that G with the φ-product is a group.

Proof. For any $a, b, c \in A$ and $x, y, z \in X$, we will show that G with the φ-product is a group.

(a) **Associativity:** We will show that

$$[(a,x)(b,y)](c,z) = (a,x)[(b,y)(c,z)],$$

which proves that G is associative. Thus, consider the left side of the equation:

$$\begin{aligned}[(a,x)(b,y)](c,z) &= (a\varphi_x(b), xy)(c,z) \\ &= (a\varphi_x(b)\varphi_{xy}(c), xyz).\end{aligned}$$

Now for the right side:

$$\begin{aligned}(a,x)[(b,y)(c,z)] &= (a,x)(b\varphi_y(c), yz) \\ &= (a\varphi_x(b\varphi_y(c)), xyz) \\ &= (a\varphi_x(b)\varphi_x(\varphi_y(c)), xyz) \\ &= (a\varphi_x(b)\varphi_{xy}(c), xyz).\end{aligned}$$

Therefore, we have shown

$$[(a,x)(b,y)](c,z) = (a,x)[(b,y)(c,z)],$$

and G is associative.

(b) **Identity:** We conjecture that G's identity element is (e, e'). Hence, we will show

$$(e,e')(a,x) = (a,x)(e,e') = (a,x)$$

for $e \in A$ and $e' \in X$ to prove this conjecture. Thus, consider the following:

$$(e,e')(a,x) = (e\varphi_{e'}(a), e'x) = (ea, x) = (a,x).$$

Now for the other direction:

$$(a,x)(e,e') = (a\varphi_x(e), xe') = (ae, x) = (a,x).$$

Therefore, (e, e') is the identity of G.

(c) **Inverses:** We conjecture that the inverses of (a,x) is $(\varphi_{x^{-1}}(a^{-1}), x^{-1})$. To prove this, we must show

$$(a,x)(\varphi_{x^{-1}}(a^{-1}), x^{-1}) = (\varphi_{x^{-1}}(a^{-1}), x^{-1})(a,x) = (e,e').$$

Thus, consider the following:

$$\begin{aligned}(a,x)(\varphi_{x^{-1}}(a^{-1}),x^{-1}) &= \left(a\varphi_x\left(\varphi_{x^{-1}}(a^{-1})\right), xx^{-1}\right)\\ &= (a\varphi_{xx^{-1}}(a^{-1}), e')\\ &= (a\varphi_{e'}(a^{-1}), e')\\ &= (aa^{-1}, e')\\ &= (e, e').\end{aligned}$$

Now, for the other direction:

$$\begin{aligned}(\varphi_{x^{-1}}(a^{-1}),x^{-1})(a,x) &= (\varphi_{x^{-1}}(a^{-1})\varphi_{x^{-1}}(a), x^{-1}x)\\ &= (\varphi_{x^{-1}}(a^{-1}a), e')\\ &= (\varphi_{x^{-1}}(e), e')\\ &= (e, e').\end{aligned}$$

Therefore, we have shown the identity element is $(\varphi_{x^{-1}}(a^{-1}),x^{-1}) \in G$. □

2. Let $\tilde{A} = \{(a,e') \mid a \in A\}$ and $\tilde{X} = \{(e,x) \mid x \in X\}$.

(a) Prove that $A \simeq \tilde{A}$ by showing $(a,e')(b,e') = (ab,e')$.

Proof. Consider the following:

$$(a,e')(b,e') = (a\varphi_{e'}(b), e'e') = (ab, e').$$

This shows that A is isomorphic to $\tilde{A}$. □

(b) Prove that $X \simeq \tilde{X}$ by showing $(e,x)(e,y) = (e,xy)$.

Proof. Consider the following:

$$(e,x)(e,y) = (e\varphi_x(e), xy) = (e, xy)$$

This shows that X is isomorphic to $\tilde{X}$. □

As a result, we can think of A and X as subgroups of G.

3. Show that in general, G is not abelian.

(a) Do this by comparing $(a,e')(e,x)$ and $(e,x)(a,e')$.

Solution. Consider the following:

$$(a,e')(e,x) = (a\varphi_{e'}(e), e'x) = (a,x).$$

However,

$$(e,x)(a,e') = (e\varphi_x(a), xe') = (\varphi_x(a), x).$$

(b) If φ is not trivial, then there exists $x \in X$ and $a \in A$ with $\varphi_x(a) \neq a$. This and part (a) show G is not abelian (even if A and X are both abelian).

4. Show that A (actually $\tilde{A}$) is a normal subgroup of G.

(a) Let $(a, x) \in G$ and $b \in A$. Show that $(a, x)(b, e')(a, x)^{-1} \in \tilde{A}$.

Proof. Let $(a, x) \in G$ and $b \in A$. By showing $(a, x)(b, e')(a, x)^{-1} \in \tilde{A}$, we will prove that A is a normal subgroup of G. Thus, consider the following:

$$\begin{aligned}(a, x)(b, e')(a, x)^{-1} &= (a\varphi_x(b), xe')(\varphi_{x^{-1}}(a^{-1}), x^{-1}) \\ &= (a\varphi_x(b)\varphi_{e'}(a^{-1}), xx^{-1}e') \\ &= (a\varphi_x(b)a^{-1}, e').\end{aligned}$$

Since a, $\varphi_x(b)$, a^{-1} are all in A, their product is also in A. Thus, $(a\varphi_x(b)a^{-1}, e')$ belongs to $\tilde{A}$, and $\tilde{A}$ is a normal subgroup of G. □

Application

5. Let $\varphi : \mathbb{Z}_2 \to Aut(\mathbb{Z}_4)$ be defined as follows:

$$\begin{aligned}\varphi_0 &: \mathbb{Z}_4 \to \mathbb{Z}_4 \text{ is the identity map: } \varphi_0(n) = n \\ \varphi_1 &: \mathbb{Z}_4 \to \mathbb{Z}_4 \text{ maps to inverses: } \varphi_1(n) = n^{-1}\end{aligned}$$

Show that D_4 is isomorphic to $\mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_2$.

Proof. To relate the structures, it will be easiest to use the 2-generator form of D_4 where $D_4 = \{r, s \mid r^4 = e, s^2 = e, rs = sr^{-1}\}$ from our notes. This is because the order of $\mathbb{Z}_4 = 4$ and $\mathbb{Z}_2 = 2$, implying $\mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_2$'s elements must be of the form (n, m) where $n = \{0, 1, 2, 3\}$ and $m = \{0, 1\}$, matching the order of D_4's generators. The operation for the group is addition, making the product rule $(a, x)(b, y) = (a + \varphi_x(b), x + y)$. Now, we must check that $rs = sr^{-1}$ is true for the corresponding elements in $\mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_2$. By using orders, we can match r to $(1, 0)$, and s to $(0, 1)$. First, we check the left side:

$$(1, 0)(0, 1) = (1 + \varphi_0(0), 1 + 0) = (1, 1).$$

For the right side:

$$(0, 1)(1, 0)^{-1} = (0, 1)(3, 0) = (0 + \varphi_1(3), 0 + 1) = (1, 1).$$

Hence, the two sides match. Therefore, we have shown that both groups have the same structural relationships between generators, so $D_4 \simeq \mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_2$. □