Probability and Statistics: Practice Set 4

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1. (2 points) A device that continuously measures and records seismic activity is placed in a remote region. The time, T, to failure of this device is exponentially distributed with mean 3 years. Since the device will not be monitored during its first two years of service, the time to discovery of its failure is $X = \max(T, 2)$. Find $E(X)$. You may approximate any integrals using your calculator.

   Solution.
   $$E(X) = \int_0^\infty \max(t,2) \cdot \frac{1}{3}e^{-\frac{t}{3}}dt = \int_0^2 2 \cdot \frac{1}{3}e^{-\frac{t}{3}}\,dt + \int_2^\infty t \cdot \frac{1}{3}e^{-\frac{t}{3}}\,dt \approx 3.540.$$

2. (1 point each) An actuary determines that the annual number of tornadoes in counties P and Q are jointly distributed as follows:

	County Q			
County P	0	1	2	3
0	0.12	0.06	0.05	0.02
1	0.13	0.15	0.12	0.03
2	0.05	0.15	0.10	0.02

   (a) Determine the conditional expected number of tornados in county Q, given that there are no tornados in county P.

   Solution.
   $$E(Q|P=0) = 0 \cdot \frac{0.12}{0.25} + 1 \cdot \frac{0.06}{0.25} + 2 \cdot \frac{0.05}{0.25} + 3 \cdot \frac{0.02}{0.25} = 0.88.$$

   (b) Calculate the conditional variance of the annual number of tornadoes in county Q, given that there are no tornadoes in county P.

   Solution.
   $$\operatorname{Var}(Q|P=0) = E(Q^2|P=0) - (E(Q|P=0))^2 = 1.76 - 0.88^2 = 0.9856.$$

3. Let X and Y be random variables with joint pmf $f(x,y) = c(xy^2 + x)$, where $x = 1, 2, 3$ and $|y-3| + x = 0, 1, 2, 3$.

   (a) (2 points) Explicitly list the 9 points which are in the support (i.e. the sample space):

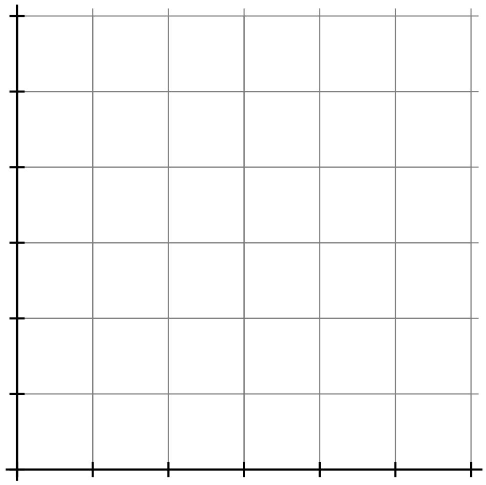
   Solution. $S = S_X \cdot S_Y$. $S_Y = \{1,2,3\}$ for $x = 1$, $S_Y = \{0,1,2,3\}$ for $x = 2$, and $S_Y = \{0,1,2\}$ for $x = 3$. Thus, $S = \{(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3),(3,0),(3,1),(3,2)\}$

   (b) (1 point) Determine a value of c that makes this a pmf.

Solution. Thus, $c = \frac{1}{28}$.

(c) (1 point) On the axes given, show the sample space, the individual probabilities, and the marginal pmfs.

Solution.



(d) (2 points) Find each of μ_X and μ_Y.

Solution.

(e) (2 points) find each of σ_X^2 and σ_Y^2.

Solution.

(f) (2 points) Find each of $\mathrm{Cov}(X, Y)$ and ρ.

Solution.

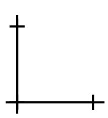
(g) (1 point) Find the equation of the least squares regression line.

Solution.

4. Let X and Y be continuous random variables with joint pdf $f(x, y) = kxy^2$, for $0 \le x \le 1$ and $0 \le y \le x^2$, and some constant k.

(a) (1 point) On the axes given, show the sample space

Solution.



(b) (1 point) Find the value of k which makes this a pdf.

Solution.

(c) (2 points) Find each of $f_X(x)$ and $f_Y(y)$.

Solution.

(d) (2 points) Find each of μ_X and μ_Y.

Solution.

(e) (2 points) Find each of σ_X^2 and σ_Y^2.

Solution.

(f) (2 points) Find each of $\mathrm{Cov}(X, Y)$ and ρ.

Solution.

(g) (1 point) Are X and Y independent? Justify this answer.

Solution.

5. (2 points each) Let X be the weight of robin eggs, in grams, and Y be the daily high temperature, in degrees Celsius. Assume that X and Y have a bivariate normal distribution with $\mu_X = 145.2$, $\sigma_X^2 = 109.2$, $\mu_Y = 23.5$, $\sigma_Y^2 = 21.8$, and $\rho = -0.34$.

(a) Find the probability that a robin egg weights between 142 and 152 grams.

Solution.

(b) Given that it has been a warm year, so that $Y = 25.1$, find the probability that a robin egg weights between 142 and 152 grams.

Solution.