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# Real Analysis

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MATH 350

*Start*

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## 1.1 Types of Numbers

The book, *Understanding Analysis* by Stephen Abbott, can be found at [this link](#)

The **natural numbers**,  $\mathbb{N}$ :

- No additive inverse.
- You can:
  - Add,
  - Multiply

The **integers**,  $\mathbb{Z}$  are known as a Group (more specifically, a “ring”).

- You can:
  - Add,
  - Multiply,
  - Subtract

The **rational numbers**,  $\mathbb{Q}$  are known as a “Field.”

- You can:
  - Add,
  - Subtract,
  - Multiply,
  - Divide

A problem that rational numbers could not explain: The 45, 45, 90 triangle had a hypotenuse of  $\sqrt{2}$ . This did not exist at the time, so it was simply  $c^2 = 2$ . Therefore, new numbers needed to be invented.

### Theorem 1.1.1

There does not exist a rational number  $r$  such that  $r^2 = 2$ .

*Proof.* Suppose there exists a rational number  $r$  such that  $r^2 = 2$ . Since  $r$  is rational, there exists  $p, q \in \mathbb{Z}$  such that  $r = \frac{p}{q}$ . We can assume the  $p$  and  $q$  have no common



factors. (If not, we can factor out the common factor.) By our assumption,

$$r^2 = 2$$

$$\frac{p^2}{q^2} = 2$$

It follows that,

$$p^2 = 2q^2$$

Such that  $p^2$  is an even number because if  $p$  were odd, then  $p^2$  would be odd. There exists  $x \in \mathbb{Z}$  such that  $p = 2x$ . Recall that  $p^2 = 2q^2$ . Thus

$$(2x)^2 = 2q^2$$

$$4x^2 = 2q^2$$

$$2x^2 = q$$

Thus,  $q^2$  is even. Hence  $q$  is also even. So  $p$  and  $q$  are both divisible by 2. This contradicts that  $p$  and  $q$  have no common factors. Thus, our supposition is false. Therefore, there does not exist a rational number  $r$  such that  $r^2 = 2$   $\square$

So we are going to work with a larger set called the real numbers,  $\mathbb{R}$ .

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- You can:
  - Add,
  - Subtract,
  - Multiply,
  - Divide
- In other words, all field axioms apply.
- Totally ordered set for any  $x, y \in \mathbb{R}$ . Thus, one of these are true:
  - (a)  $x < y$ ,
  - (b)  $x > y$ ,
  - (c)  $x = y$
- Think of it as a number line.
- $\mathbb{Q}$  is dense:
 

If  $a, b \in \mathbb{Q}$  with  $a \neq b$ , there exists  $c \in \mathbb{Q}$  which is between  $a$  and  $b$  such that  $a < c < b$ . One example is  $\frac{a+b}{2}$ .
- $\mathbb{Q}$  is not *complete*, but  $\mathbb{R}$  is.
  - *Complete*: Think, “no gaps.”



## 1.2 Preliminaries

Things to remember from Intro and Discrete.

Set Notation	Complement
$x \in A$	$A^c$ (not $\bar{A}$ )
$A \cup B$	$\mathbb{R} \setminus A$
$A \cap B$	

$$\bullet \bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \dots$$

$$\bullet \bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \dots$$

• **De Morgan's Laws**

### 1.2.1 Infinite Unions and Intersections

For each  $n \in \mathbb{N}$ , define  $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$ . In other words, each subsequent element in the subset will start at  $n$ . For example,  $A_1 = \{1, 2, \dots\}$ , whereas  $A_5 = \{5, 6, \dots\}$ .

$\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$ . To show a number  $\in \mathbb{N}$  belongs in the set  $A_n$ , we can start with that,  $k \in \mathbb{N}$ . Then  $k \in A_k$ . Thus,  $k \in A_k \subseteq \bigcup_{n=1}^{\infty} A_n$ . therefore,  $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$ .

$\bigcap_{n=1}^{\infty} A_n = \emptyset$ . Obviously, we know that the empty set is a subset of  $A_n$ , but to prove that  $\bigcap_{n=1}^{\infty} A_n$  is a subset of the empty set, we should suppose a  $k \in \mathbb{N}$  such that  $k \in \bigcap_{n=1}^{\infty} A_n$ . Notice that  $k \notin \bigcap_{n=1}^{\infty} A_n$ . So,  $\bigcap_{n=1}^{\infty} A_n = \emptyset$ .

### 1.2.2 Functions and Notation

$f: A \rightarrow B$  where  $f$  is a function,  $A$  is a domain, and  $B$  is the co-domain. Thus,  $f(x) = y$  such that  $x \in A$  and  $y \in B$ .

Some definitions to keep in mind:



## The Dirichlet Function

[Refer to notepaper for these following definitions]

### Image

Example:  $g : \mathbb{R} \rightarrow \mathbb{R}$ , when we say  $y \in g(A)$  implies  $\exists x$  such that  $g(x) = y$

### Triangle inequality:

The most common application: For any  $a, b, c \in \mathbb{R}$ ,  $|a - b| \leq |a - c| + |c - b|$ , with the intermediate step of  $a - b = (a - c) + (c - b)$ .

## 1.2.3 Common Strategies for Analysis Proofs

### Theorem 1.2.6

Let  $a, b \in \mathbb{R}$ . Then,

$$a = b \text{ if and only if for all } \epsilon > 0, |a - b| < \epsilon.$$

*Proof.* We show this by proving both implications:

$(\Rightarrow)$  Assume  $a = b$ . Let  $\epsilon > 0$ . Then  $|a - b| = 0 < \epsilon$

$(\Leftarrow)$  Assume for all  $\epsilon > 0$ ,  $|a - b| < \epsilon$ . Suppose  $a \neq b$ . Then  $a - b \neq 0$ . So,  $|a - b| \neq 0$ . Now, Consider  $\epsilon_0 = |a - b|$ . By our assumption we know that  $|a - b| < \epsilon_0$ . It is not true that  $|a - b| < |a - b|$ . Therefore, it must be the case that  $a = b$ .

Therefore, by showing both sides of the implication accomplish the same thing as the other side, we know that  $a = b$  if and only if for all  $\epsilon > 0$ ,  $|a - b| < \epsilon$ .  $\square$

## 1.2.4 Mathematical Induction

*Inductive Hypothesis:* Let  $x_1 = 1$ . For all  $n \in \mathbb{N}$ , let  $x_{n+1} = \frac{1}{2}x_n + 1$ .

*Inductive Step:*  $x_1 = 1, x_2 = 1.5, x_3 = 1.75, x_4 = 1.875$ .

### Example 1.1: Induction

The sequence  $(x_n)$  is increasing. In other words, for all  $n \in \mathbb{N}$ ,  $x_n \leq x_{n+1}$ .



*Proof.* Suppose the sequence  $(x_n)$  is increasing. We will prove this point by using induction.

**Base Case:** We see that  $x_1 = 1$  and  $x_2 = 1.5$ . Thus,  $x_1 \leq x_2$ .

**Inductive Hypothesis:** For  $n \in \mathbb{N}$ , assume  $x_n \leq x_{n+1}$ .

*Scratch work:* We want:  $x_{n+1} \leq x_{n+2}$ . We know:  $x_{n+1} = \frac{1}{2}x_{n+1} + 1$ .

**Inductive Step:** Then  $\frac{1}{2}x_n \leq \frac{1}{2}x_{n+1}$ . Hence,  $\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1$ . Therefore we have proven through induction that,  $x_{n+1} \leq x_{n+2}$ .  $\square$

## Exercises

### Exercise: 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

- (a) If  $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$  are all sets containing an infinite number of elements, then the intersection  $\bigcap_{n=1}^{\infty} A_n$  is infinite as well.
- (b) If  $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$  are all finite, nonempty sets of real numbers, then the intersection  $\bigcap_{n=1}^{\infty} A_n$  is finite and nonempty.
- (c)  $A \cap (B \cup C) = (A \cap B) \cup C$
- (d)  $A \cap (B \cap C) = (A \cap B) \cap C$
- (e)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

*Solution.*

- (a) This is false. Consider the following as a counterexample: If we define  $A_1$  as  $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$ , we can see why the intersection of these sets of infinite numbers are actually empty. Consider a number  $m$  that actually satisfies  $m \in A_n$  for every  $A_n$  in our collection of sets. Because  $m$  is not an element of  $A_{m+1}$ , no such  $m$  exists and the intersection is empty.
- (b) This is true.
- (c) False. Consider sets  $A = \{1, 2, 3\}$ ,  $B = \{3, 6, 7\}$  and  $C = \{5\}$ . Note that  $A \cap (B \cup C) = \{3\}$  is not equal to  $(A \cap B) \cup C = \{3, 5\}$ .
- (d) This is true. A proof would start with  $x \in A \cap (B \cap C)$ .



| (e) This is true. A proof would start with  $x \in A \cap (B \cup C)$ .





### Exercise: 1.2.5

**De Morgan's Laws.** Let  $A$  and  $B$  be subsets of  $\mathbb{R}$ .

- (a) If  $x \in (A \cap B)^c$ , explain why  $x \in A^c \cup B^c$ . This shows that  $(A \cap B)^c \subseteq A^c \cup B^c$ .
- (b) Prove the reverse inclusion  $(A \cap B)^c \supseteq A^c \cup B^c$ , and conclude that  $(A \cap B)^c = A^c \cup B^c$ .
- (c) Show  $(A \cup B)^c = A^c \cap B^c$  by demonstrating inclusion both ways.

*Solution.*

- (a) If  $x \in (A \cap B)^c$ , and we know that  $A^c = \{x \in \mathbb{R} : x \notin A\}$ , then we know  $x$  must not exist in  $A$  and  $B$  because  $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$ . Thus,  $x$  is in either  $A^c$  or  $B^c$ . Put another way  $x \in A^c \cup B^c$ . Since we have shown that an element that started in  $(A \cap B)^c$  ended up in  $A^c \cup B^c$ , then we know  $(A \cap B)^c \subseteq A^c \cup B^c$ .
- (b) Assume there exists a  $y \in A^c \cup B^c$ . Thus, it must be the case that  $y \notin A$  or  $y \notin B$ . Hence,  $y$  cannot exist in both sets at the same time, so  $y \in (A \cap B)^c$ . Because we have taken an element that started in  $A^c \cup B^c$  and have shown that it exists in  $(A \cap B)^c$ , we have proven  $A^c \cup B^c \subseteq (A \cap B)^c$ .
- (c)

*Proof.* We need to show these expressions are subsets of each other in order to prove they are equivalent.

- ( $\subseteq$ ) Assume there exists  $x \in (A \cap B)^c$ , and we know that  $A^c = \{x \in \mathbb{R} : x \notin A\}$ , then we know  $x$  must not exist in  $A$  and  $B$  because  $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$ . Thus,  $x$  is in either  $A^c$  or  $B^c$ . Put another way  $x \in A^c \cup B^c$ . Since we have shown that an element that started in  $(A \cap B)^c$  ended up in  $A^c \cup B^c$ , then we know  $(A \cap B)^c \subseteq A^c \cup B^c$ .
- ( $\supseteq$ ) Now assume there exists a  $y \in A^c \cup B^c$ . Thus, it must be the case that  $y \notin A$  or  $y \notin B$ . Hence,  $y$  cannot exist in both sets at the same time, so  $y \in (A \cap B)^c$ . Because we have taken an element that started in  $A^c \cup B^c$  and have shown that it exists in  $(A \cap B)^c$ , we have proven  $A^c \cup B^c \subseteq (A \cap B)^c$ .

Therefore, we have shown through proving both sides of the implication, that these two statements are logically equivalent. In that, all elements of  $A^c \cup B^c$  are the same elements that are in  $(A \cap B)^c$   $\square$



### Exercise: 1.2.7

Given a function  $f$  and a subset  $A$  of its domain, let  $f(A)$  represent the range of  $f$  over the set  $A$ ; that is,  $f(A) = \{f(x) : x \in A\}$ .

- Let  $f(x) = x^2$ . if  $A = [0, 2]$  (the closed interval  $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$ ) and  $B = [1, 4]$ , find  $f(A)$  and  $f(B)$ . Does  $f(A \cap B) = f(A) \cap f(B)$  in this case? Does  $f(A \cup B) = f(A) \cup f(B)$ ?
- Find two sets  $A$  and  $B$  for which  $f(A \cap B) \neq f(A) \cap f(B)$ .
- Show that, for an arbitrary function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , it is always true that  $g(A \cap B) \subseteq g(A) \cap g(B)$  for all sets  $A, B \subseteq \mathbb{R}$ .
- Form and prove a conjecture about the relationship between  $g(A \cup B)$  and  $g(A) \cup g(B)$  for an arbitrary function  $g$ .

*Solution.*

- Since  $f(x) = x^2$ , the intervals of  $f(A)$  would be  $[0, 4]$  and  $f(B)$  would be  $[1, 16]$ . The interval of the intersection of  $A \cap B$  is  $[1, 2]$ . Take this through our function, we get  $f(A \cap B) = [1, 4]$ . On the other side of the equation, we already know the intervals of  $f(A)$  and  $f(B)$ , and the intersection of theirs would be  $[1, 4]$ . So they do equal each other. We know  $f(A \cup B)$  and  $f(A) \cup f(B)$  will be equivalent because  $f(A \cup B)$  has an interval of  $[0, 16]$ , and  $f(A) \cup f(B)$  also has an interval of  $[0, 16]$  because taking the union of  $[0, 4] \cup [1, 16]$  is  $[0, 16]$ .

- Two sets could be  $A = [5, 6]$  and  $B = [0, 0]$ . Because the sets have nothing in common even after taking their function, they do not equal each other.

(c)

*Proof.* Let  $x \in g(A \cap B)$ . Using the definition of function, we know there exists a  $y \in A \cap B$  to which that  $y$  is mapped to as  $g(y) = x$ . From the definition of intersection, we know  $y \in A$  and  $y \in B$  such that  $x = g(y) \in g(A)$  and  $x = g(y) \in g(B)$  because  $y \in A \cap B$ . Putting it together, we have  $x \in g(A) \cap g(B)$  thus proving  $g(A \cap B) \subseteq g(A) \cap g(B)$   $\square$

- Conjecture: For any function  $g$  defined as  $g : \mathbb{R} \rightarrow \mathbb{R}$  and for any subsets  $A, B \subseteq \mathbb{R}$ , the following holds:

$$g(A \cup B) = g(A) \cup g(B)$$



*Proof.* We need to show these expressions are subsets of each other in order to prove they are equivalent.

- ( $\subseteq$ ) Take any element  $x \in g(A \cap B)$ . By definition of function, we know there exists some  $y \in A \cup B$  such that  $g(y) = x$ . From the definition of union, we know  $y \in A$  or  $y \in B$  such that  $x = g(y) \in g(A)$  or  $x = g(y) \in g(B)$  or both. Putting it together, we have  $x \in g(A) \cup g(B)$  thus proving  $g(A \cap B) \subseteq g(A) \cup g(B)$ .
- ( $\supseteq$ ) Take any element  $p \in g(A) \cup g(B)$ . By definition of union, we know  $p$  is either in  $g(A)$  or  $g(B)$  or both. From the definition of function, we know that if  $p \in g(A)$  or  $p \in g(B)$  then there exists some  $q \in A$  or  $q \in B$  such that  $g(q) = p$ . Putting it together, we have  $q \in A \cup B$ . Moreover, this means  $p = g(q) \in g(A \cup B)$ . And since  $p \in g(A) \cup g(B)$  implies  $p \in g(A \cup B)$ , we know  $g(A) \cup g(B) \subseteq g(A \cup B)$ .

Therefore, since we have proven that both expressions are functions of each other, we have proved that they are equal.  $\square$

### Exercise: 1.2.8

Given a function  $f : A \rightarrow B$  can be defined as either **one-to-one** or **onto**, give an example of each or state that the request is impossible:

- (a)  $f : \mathbb{N} \rightarrow \mathbb{N}$  that is 1-1 but not onto.
- (b)  $f : \mathbb{N} \rightarrow \mathbb{N}$  that is onto but not 1-1.
- (c)  $f : \mathbb{N} \rightarrow \mathbb{Z}$  that is 1-1 and onto.

*Solution.*

- (a) The function  $f(a) = a + 1$  is 1-1 because when

$$\begin{aligned} f(a_1) &= f(a_2) \\ a_1 + 1 &= a_2 + 1 \\ a_1 &= a_2 \end{aligned}$$

However, the function is not onto because the entire co-domain is not covered. That being 1.

- (b) We need to find a function that will cover every entry in the co-domain, while also



avoiding a scenario where  $a_1 = a_2 \dots$ . Consider the function,

$$f(a) = \begin{cases} a & \text{if } a \text{ is odd,} \\ a - 1 & \text{if } a \text{ is even} \end{cases}$$

This function is onto because every natural number is covered, but it is not 1-1 because  $a_1 \neq a_2 - 1$ .

- (c) This request is not possible. There is no way to map every natural number to every integer because we are simply missing 0! (Not 0 factorial, we do have the number 1, I just mean the number 0 in an exclamatory sense.)

### 1.3 Axiom of Completeness

Think about  $\mathbb{Q}$  and  $\mathbb{R}$ .

- Both are fields.
  - Both have  $+$ ,  $-$ ,  $\times$ ,  $\div$  operations.
- Both are totally ordered
  - $a < b$ ,
  - $a > b$ ,
  - or  $a = b$
- $\mathbb{R}$  is complete.  $\mathbb{Q}$  is not.

#### Axiom of Completeness

Every nonempty set of real numbers that is **bounded** has a **least upper bound**.

**Note that upper bounds are not unique!** For example, consider the line,  $A$ , from 0 to 1. There are infinitely many upper bounds past 1 because  $A$  is bounded.

We often call the least upper bound the *supremum* of a set. Example:

Imagine a number line from  $(1, 8)$ . Note that parenthesis mean  $<$  and not  $\leq$ . Hence, the supremum is 8. Wrote simply as  $\sup A$ .

#### Example 1.2: Supremum

Consider a set,  $B = [-5, -2] \cup (3, 6) \cup \{13\}$ . What is the supremum?



| *Solution.*  $\sup B = 13$

At the other end of the set, we have the following:

- lower bounds,
- greatest lower bound
- often called infimum.

The infimum of the previous example would be  $\inf B = -5$ .

### Example 1.3:

Consider the set,  $\mathbb{C} = \{\frac{1}{n} : n \in \mathbb{N}\}$ . What is the supremum and the infimum?

| *Solution.*  $\sup \mathbb{C} = 1, \inf \mathbb{C} = 0$ .

### Example 1.4: L

Let  $A \subseteq \mathbb{R}$  be nonempty and bounded above, and let  $c \in \mathbb{R}$ . Define the set  $c + A$  by

$$c + A = \{c + a : a \in A\}$$

Then  $\sup(c + A) = c + \sup A$ .

*Solution.* To properly verify this we focus separately on each part of Definition 1.3.2. Setting  $s = \sup A$ , we see that  $a \leq s$  for all  $a \in A$ , which implies  $c + a \leq c + s$  for all  $a \in A$ . Thus,  $c + s$  is an upper bound for  $c + A$  and condition (i) is verified. For (ii), let  $b$  be an arbitrary upper bound for  $c + A$ ; i.e.,  $c + a \leq b$  for all  $a \in A$ . This is equivalent to  $a \leq b - c$  for all  $a \in A$ , from which we conclude that  $b - c$  is an upper bound for  $A$ . Because  $s$  is the least upper bound of  $A$ ,  $s \leq b - c$ , which can be rewritten as  $c + s \leq b$ . This verifies part (ii) of Definition 1.3.2, and we conclude  $\sup(c + A) = c + \sup A$ .

Why do we need to include infimum and supremum? Don't we have the max and min of a set already? Well, what exactly do we mean by the **maximum value** of a set?

We say  $m \in \mathbb{R}$  is the *maximum* of  $A$  if  $m \in A$  and for all  $x \in A$ ,  $x \leq m$ . Note that some sets have a maximum and some sets do not. You cannot refer to a maximum without first knowing it exists. This is the same with minimums.

### Lemma 1.3.1

1.3.8 Assume  $s$  is an **upper bound** for a set  $A \subseteq \mathbb{R}$ . Then,  $s$  is the supremum of  $A$  if and only if for every  $\epsilon > 0$ , there exists  $x \in A$  such that  $s - \epsilon < x$ .

This lemma allows us to take any positive number and take a "step back." In essence, you can verify something as an upper bound if you continuously back up over and over until you cannot back up any longer.



*Proof.* We show this by proving both implications:

- ( $\Rightarrow$ ) Assume  $s = \sup A$ . Let  $\epsilon > 0$ . Suppose there are no elements  $x$  of  $A$  such that  $s - \epsilon < x$ . Then  $s - \epsilon$  would be an upper bound. This contradicts that  $s$  is the least upper bound. Therefore, there must exist an element  $x \in A$  such that  $s - \epsilon < x$ .
- ( $\Leftarrow$ ) Assume for every  $\epsilon > 0$ , there exists  $x \in A$  such that  $s - \epsilon < x$ . Let  $t$  be an upper bound of  $A$ . Suppose  $t < s$ . Consider  $\epsilon_0 = s - t > 0$ . By our assumption, there exists  $x \in A$  such that  $s - \epsilon_0 < x$ . So,  $t < x$ . This contradicts that  $t$  is an upper bound of  $A$ . So,  $t \geq s$ . Thus,  $s$  is the least upper bound

Therefore, by proving both the right and left implication, we have shown the statement to be true.  $\square$

Analogous statement about infimums: Assume  $z$  is a lower bound of a set  $A \subseteq \mathbb{R}$ . Then  $z = \inf A \iff$  for all  $\epsilon > 0$ , there exists  $y \in A$  such that  $y < z + \epsilon$ .

## Exercises

### Exercise: 1.3.4

Let  $A_1, A_2, A_3 \dots$  be a collection of nonempty sets each of which is bounded above.

- Find a formula for  $\sup(A_1 \cup A_2)$ . Extend this to  $\sup(\bigcup_{k=1}^n A_k)$ .
- Consider  $\sup(\bigcup_{k=1}^\infty A_k)$ . Does the formula in (a) extend to the infinite case?

*Solution.*

- Let  $A_1$  and  $A_2$  be nonempty sets, each bounded above. To find the largest of the two suprema, we can use the following:  $\sup(A_1 \cap A_2) = \max\{\sup A_1, \sup A_2\}$ . If we extend this notion to  $\sup(\bigcup_{k=1}^n A_k)$ , we can use the same idea from before and write it as  $\sup(\bigcup_{k=1}^n A_k) = \max\{\sup A_1, \sup A_2, \dots, \sup A_n\}$ .
- The formula does not extend to the infinite case. Consider the counterexample  $\bigcup_{k=1}^\infty A_k$  where  $A_k := [k, k + 1]$ . Even though these sets are bounded above, when we take the union of them, we approach infinity, which is not bounded:  $\bigcup_{k=1}^\infty A_k = [1, 2] \cup [2, 3] \cup \dots = [1, \infty)$ .



### Exercise: 1.3.5

As in Example 1.3.7, let  $A \subseteq \mathbb{R}$  be nonempty and bounded above, and let  $c \in \mathbb{R}$ . This time define the set  $cA = \{ca : a \in A\}$ .

- (a) If  $c \geq 0$ , show that  $\sup(cA) = c \sup A$ .
- (b) Postulate a similar type of statement for  $\sup(cA)$  for the case  $c < 0$ .

*Solution.*

- (a) Let  $A \subseteq \mathbb{R}$  be nonempty and bounded above. Define the set  $cA := \{ca : a \in A\}$ . From the axiom of completeness, because  $A$  is bounded above, we know there is a least upper bound,  $s = \sup A$ . Following from Example 1.3.7, we see that  $a \leq s$  for all  $a \in A$  which implies  $ca \leq cs$  for all  $a \in A$ . Thus,  $cs$  is an upper bound for  $cA$ , and the first condition of Definition 1.3.2 is satisfied. For the second condition, we need to look at both  $c = 0$  and  $c > 0$  to avoid dividing by zero. So, we have two cases:
  - $c = 0$ : If  $c = 0$ , then  $cA = \{0 : a \in A\} = \{0\}$ . Since the only element in  $cA$  is 0,  $\sup(cA) = 0$ . Similarly, because  $c = 0$ ,  $c \sup A = 0 \cdot \sup A = 0$ . Therefore,  $\sup(cA) = c \sup A$ .
  - $c > 0$ : Let  $b$  be an arbitrary upper bound for  $cA$  and  $c > 0$ . In other words,  $ca \leq b$  for all  $a \in A$ . This is equivalent to  $a \leq b/c$  where  $c \neq 0$ , from which we can see that  $b/c$  is an upper bound for  $A$ . Because  $s$  is the least upper bound of  $A$ ,  $s \leq b/c$ , which can be rewritten as  $cs \leq b$ . This verifies the second part of Definition 1.3.2, and we conclude  $\sup(cA) = c \sup A$ .
- (b) Postulate: If  $c < 0$ , then  $\sup(cA) = c \inf(A)$ .

### Exercise: 1.3.8

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

- (a)  $\left\{ \frac{m}{n} : m, n \in \mathbb{N} \text{ with } m < n \right\}$ .
- (b)  $\left\{ \frac{(-1)^m}{n} : m, n \in \mathbb{N} \right\}$ .
- (c)  $\left\{ \frac{n}{3n+1} : n \in \mathbb{N} \right\}$ .
- (d)  $\left\{ \frac{m}{m+n} : m, n \in \mathbb{N} \right\}$ .

*Solution.* To avoid writing out every set definition, I am going to denote each set as  $A_n$  where  $n$  corresponds to the numerical value of the list from (a) - (d).



- (a)  $\sup A_1 = 1, \inf A_1 = 0$
- (b)  $\sup A_2 = 1, \inf A_2 = -1$
- (c)  $\sup A_3 = \frac{1}{3}, \inf A_3 = \frac{1}{4}$
- (d)  $\sup A_4 = 1, \inf A_4 = 0$

## 1.4 Consequences of Completeness

### Theorem 1.4.1: Nested Interval Property

For each  $n \in \mathbb{N}$ , assume we are given a closed interval  $I_n = (a_n, b_n)$ . Assume  $I_n$  contains  $I_{n+1}$ . This results in a nested sequence of intervals.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \dots$$

Then,  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**tl;dr** there has to be something that is common to all of the sets.

*Proof.* Notice that the sequence,  $a_1, a_2, a_3, \dots$  is increasing. In other words, for each  $n \in \mathbb{N}$ , since  $I_n \supset I_{n+1}$  we have  $a_n \leq a_{n+1}$ . If we consider the set  $A = \{a_n : n \in \mathbb{N}\}$ . The element  $b_1$  is an upper bound of  $A$ . (Note that  $b_1$  and  $a_1$  corresponds to the end-points of the first set,  $I_1$ . Think of this as a tornado looking structure where the larger the  $I_n$ , the smaller the number line.) For each  $n \in \mathbb{N}$ ,  $a_n \leq b_n \leq b_1$ .

Since  $A$  has an upper bound, it must have a least upper bound. Hence, let  $\alpha = \sup A$ . We claim that  $\alpha \in \bigcap_{n=1}^{\infty} I_n$ . We said  $b_1$  was an upper bound. In fact, every  $b_n$  is an upper bound of  $A$ . Choose any  $n, m \in \mathbb{N}$ . We want to show that  $a_n \leq b_m$ . Consider the following cases:

**Case 1:** If  $n < m$ , then  $a_n \leq a_m \leq b_m$ . (Think: two number lines stacked on top of each other. The top number line is larger, call it  $I_n$  and it has  $a_n$  and  $b_n$  as endpoints. Consider a contained line ( $I_n \supseteq I_m$ ) that is smaller, and has endpoints  $a_m$  and  $b_m$ .)

**Case 2:** If  $n > m$ , then  $a_n \leq b_n \leq b_m$ . So every  $b_n$  is an upper bound of  $A$ .

Hence,

- Because  $\alpha = \sup A$ , we have  $\alpha \geq a_n$ .
- Since  $b_n$  is an upper bound of  $A$ , we have  $\alpha \leq b_n$ .

so,  $\alpha \in [a_n, b_n] = I_n$ . Thus,  $\alpha \in \bigcap_{n=1}^{\infty} I_n$ . □

Nested, closed, Bounded Intervals  $\Rightarrow$  non-empty intersection.





### Theorem 1.4.2: Archimedean Principle

- (a) Given any number  $x \in \mathbb{R}$ , there exists an  $n \in \mathbb{N}$  satisfying  $n > x$ .
- (b) Given any real number  $y > 0$ , there exists an  $n \in \mathbb{N}$  satisfying  $1/n < y$ .

*Proof.* (a) If  $\mathbb{N}$  was bounded, then we can let  $s \in \mathbb{N} = \sup \mathbb{N}$ . However, we know that there is always a higher number (e.g.,  $n + 1$ ) for any  $n \in \mathbb{N}$  that is given. Thus, by contradiction, there must exist  $n \geq x$ .

(b) For any  $x > 0$ , there exists  $n \in \mathbb{N}$  such that  $\frac{1}{n} < x$ . □

### Theorem 1.4.3: Density of the Rationals in the Reals

For any  $a, b \in \mathbb{R}$  with  $a < b$ , there exists  $q \in \mathbb{Q}$  such that  $a < q < b$ .

*Proof.* Since  $b - a > 0$ , there exists  $n \in \mathbb{N}$  such that  $\frac{1}{n} < b - a$ . From the [Archimedean Principle](#), since  $a \times n \in \mathbb{R}$ , there exists  $m \in \mathbb{N}$  such that  $a \times n < m$ . Let  $m$  be the smallest such natural numbers (by the well ordered principle). Since  $m$  is the smallest such natural number, it follows that  $m - 1 \leq a \times n < m$ . We then see that  $a < \frac{m}{n}$ . Now, we need to find some  $\frac{m}{n} < b$ .

$$\begin{aligned}
 m - 1 &\leq a \times n \\
 m &\leq a \times n + 1 \\
 \frac{m}{n} &\leq a + \frac{1}{n} \\
 \frac{m}{n} &< a + (b - a) \\
 \frac{m}{n} &< b
 \end{aligned}$$

We now have that  $a < \frac{m}{n} < b$  so  $\frac{m}{n}$  is a rational number in  $(a, b)$  □

### Exercise: 1.4.1

Recall that  $\mathbb{I}$  stands for the set of irrational numbers.

- (a) Show that if  $a, b \in \mathbb{Q}$ , then  $ab$  and  $a + b$  are elements of  $\mathbb{Q}$  as well.
- (b) Show that if  $a \in \mathbb{Q}$  and  $t \in \mathbb{I}$ , then  $a + t \in \mathbb{I}$  and  $at \in \mathbb{I}$  as long as  $a \neq 0$ .
- (c) Part (a) can be summarized by saying that  $\mathbb{Q}$  is closed under addition and multiplication. Is  $\mathbb{I}$  closed under addition and multiplication? Given two irrational numbers  $s$  and  $t$ , what can we say about  $s + t$  and  $st$ ? In other words, are there two irrational numbers that can be added and multiplied such that you get a number  $x$  such that  $x \notin \mathbb{I}$ .



*Solution.*

(a) Let  $a, b \in \mathbb{Q}$ . This means there exists some  $p, q, a, b \in \mathbb{Z}$  such that

$$a = \frac{p}{q}$$

and

$$b = \frac{a}{b}$$

where  $q, b \neq 0$ . The product of these numbers is

$$ab = \frac{p}{q} \cdot \frac{a}{b} = \frac{pa}{qb}.$$

Since  $pa, qb \in \mathbb{Z}$ ,  $ab \in \mathbb{Q}$ . The sum of these numbers is

$$a + b = \frac{p}{q} + \frac{a}{b} = \frac{pb + aq}{qb}.$$

Since  $pb + aq, qb \in \mathbb{Z}$ ,  $a + b \in \mathbb{Q}$ .

- (b) Let  $a \in \mathbb{Q}$  and  $t \in \mathbb{I}$ . Assume, for contradiction, that  $a + t \in \mathbb{Q}$ . This would imply  $t = (a + t) - a$  (because we can subtract  $t + a$  from the original equation and rearrange terms). Since  $a + t, a \in \mathbb{Q}$  their sum would be rational because the rational numbers are closed under addition. However, that would contradict the assumption that  $t \in \mathbb{I}$ . Hence,  $a + t \in \mathbb{I}$ .
- (c) For  $\mathbb{I}$ , it is not closed under addition and multiplication. Consider the following counterexample:  $\sqrt{2} + (-\sqrt{2}) = 0$  which is not in the irrationals. For multiplication, consider  $\sqrt{2} \cdot \sqrt{2} = 2$ , which is also not in the irrationals.

## 1.5 Cardinality

Two sets have the same *cardinality* if there exists a bijection between them. Thus, the natural numbers, the integers, and the rational numbers have the same cardinality. A set is *countably infinite* if it has the same cardinality as  $\mathbb{N}$ . (If it can be put into one-to-one correspondence with  $\mathbb{N}$ .) A set is *countable* if it is countably infinite or finite.

### Theorem 1.5.6

$\mathbb{R}$  is not countable.

*Proof.* 1 (most common)

Suppose  $\mathbb{R}$  is countable. Then we can list them all, or we can enumerate them.  $\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\}$ . We can write the decimal expansion of each of these. Consider the



following table:

$x_1 =$	$a_{10}$	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	$\dots$
$x_2 =$	$a_{20}$	$a_{21}$	$a_{22}$	$a_{23}$	$a_{24}$	$\dots$
$x_3 =$	$a_{30}$	$a_{31}$	$a_{32}$	$a_{33}$	$a_{34}$	$\dots$
$x_4 =$	$a_{40}$	$a_{41}$	$a_{42}$	$a_{43}$	$a_{44}$	$\dots$
$x_5 =$	$a_{50}$	$a_{51}$	$a_{52}$	$a_{53}$	$a_{54}$	$\dots$
$x_6 =$	$a_{60}$	$a_{61}$	$a_{62}$	$a_{63}$	$a_{64}$	$\dots$

We will now construct a number that is not in this list. Focus on diagonal entries. For each  $n \in \mathbb{N}$ , let  $b_n$  be a digit that is different from  $a_{nn}$ . Now consider the number  $y = 0.b_1b_2b_3b_4b_5\dots$ . This number  $y$  is not in our list. So our list did not include all of  $\mathbb{R}$ . Avoid repeating 9s.  $\square$

*Proof. 2* (uses nested interval theorem)

Suppose  $\mathbb{R}$  is countable. Then we can enumerate  $\mathbb{R}$   $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$ . Let  $I_1$  be any closed interval that does not contain  $x_1$ . Next, we will find another closed interval  $I_2$  that:

- $I_2 \subseteq I_1$
- $x_2 \notin I_2$

Continue in this fashion creating a sequence of nested closed intervals:  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$  such that for all  $k \in \mathbb{N}$ ,  $x_k \notin I_k$ . Now consider:

$$\bigcap_{n=1}^{\infty} I_n$$

- For each  $k \in \mathbb{N}$ , since  $x_k \notin I_k$ , we see  $x_k \notin \bigcap_{n=1}^{\infty} I_n$ .
- By the nested interval theorem, there exists  $x \in \mathbb{R}$  such that  $x \in \bigcap_{n=1}^{\infty} I_n$ . So  $x$  is a real number that is not included in our list.

$\square$

### Theorem 1.5.7

A countable collection of finite sets is *countable*.

### Theorem 1.5.8

- (i) The union of two countable sets is *countable*.
- (ii) A countable union of countable sets is *countable*.

From Theorem 1.5.6, we know that  $\mathbb{R}$  is uncountable, but what about  $(0,1)$ ? It does



have the same cardinality of  $\mathbb{R}$  because we can make a one-to-one and onto function between both the sets. Similarly,  $(a, b)$  also has the same cardinality. What about  $[a, b]$ ?

**Recap:**  $\mathbb{N}$  is countable, and  $\mathbb{R}$  is uncountable and has a different cardinality than  $\mathbb{N}$ . Thus, the question is, do all uncountable sets have the same cardinality as  $\mathbb{R}$ ? The answer is **no**.

### Theorem 1.5.9: Cantor's Theorem

For any set  $A$ , there does not exist an onto map from  $A$  into  $\mathcal{P}$ .

*Proof.* Suppose there exists an onto function,  $f : A \rightarrow \mathcal{P}(A)$ . So each  $a \in A$  is mapped to an element  $f(a) \in \mathcal{P}(A)$ . Then,  $f(a) \subseteq A$ . We are going to construct an element of  $\mathcal{P}(A)$  which is not mapped to by  $f$ .

Consider  $B = \{a \in A : a \notin f(a)\}$ . Since  $f$  is onto there exists  $a' \in A$  such that  $B = f(a')$ . Thus, there are two cases to consider:

- **Case 1:** If  $a' \in B = f(a')$ , then  $a' \notin B$ .
- **Case 2:** If  $a' \notin B = f(a')$ , then  $a' \in B$ .

As evidenced, both cases lead to contradictions, so  $B$  is not the image of any  $a \in A$ . Therefore  $f$  is not onto.  $\square$

### Example 1.5: Set and Power Set Matching

$A = \{a, b, c\}$ .

*Solution.*  $\mathcal{P}(A) = \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$ . Note that you can map  $\{a\}, \{b\}, \{c\}$ , to elements such as  $\emptyset, \{a, b\}, \{a, b, c\}$ , but there are still more elements that are left unmapped. We can extrapolate from our proof a set  $B$  such that  $B = \{a, c\}$  because those elements are not mapped to.

All of this is to show  $\mathcal{P}(\mathbb{R})$  has a larger cardinality than  $\mathbb{R}$ . Then  $\mathcal{P}(\mathcal{P}(\mathbb{R}))$  has a larger cardinality than  $\mathcal{P}(\mathbb{R})$ .

## 2.1 Discussion: Rearrangement of Infinite Series

### Questions:

What is a *sequence*?

A countable, ordered list of elements. An example could be  $1, 2, 3, 4, 5, \dots$ . Note that this is *ordered*, therefore distinguishing it from a sequence like  $3, 1, 2, 4, 5, 6, \dots$ . Hence, order matters.

A *sequence* is a function whose domain is  $\mathbb{N}$ . **Note:** The domain  $\mathbb{N}$  refers to each element's position in the list. For example,  $(a_n) = a_1, a_2, a_3, \dots$ .

We will focus on the *limit* of a sequence. We use sequences to approximate other things.

### Example 2.1: Sequence

$$3, 3.1, 3.14, 3.141, 3.1415, \dots \approx \pi.$$

$$x, x - \frac{x^3}{6}, x - \frac{x^3}{6} + \frac{x^5}{120}, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}, \dots \approx \sin(x)$$

What is a *series*?

An infinite sum. We look at the sequence of partial sums. We ask, do the partial sums approach a limit?

### Example 2.2: Alternating Harmonic Series

$$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \dots$$

We can rearrange these terms such that we can ‘force’ the series to converge to a specific number. Therefore, we will need to be careful with our definitions.

## 2.2 The Limit of a Sequence

### Definition 2.2.1

A *sequence* is a function whose domain is  $\mathbb{N}$ . We write  $(a_n) = a_1, a_2, a_3, \dots$ .



### Definition 2.2.3

The sequence  $(a_n)$  *converges* to  $L$  if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - L| < \epsilon$ . In other words, there exists  $N \in \mathbb{N}$  such that

- **(In the interval)**  $a_N \in (L - \epsilon, L + \epsilon)$ .
- **(Stays in the interval)**  $\forall n \geq N, a_n \in (L - \epsilon, L + \epsilon)$ .

### Example 2.3: In-class

Let  $a_n = \frac{1}{n}$ .  $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

*Proof.* Our claim is  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . Thus, let  $\epsilon = .01$ . Does the sequence eventually get inside  $(-.01, .01)$ ? We will set  $N = 101$ . So, for any  $n \geq |0|$ ,

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{101} < .01.$$

From  $A_n$  and on, the sequence stayed within  $\epsilon$  of 0. But what about  $\epsilon = .001$ ,  $\epsilon = .00001$  and so on?

Actual proof let  $\epsilon > 0$ . By the Archimedean Principle, there exists  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon}$ . Now, for any  $n \geq N$ ,

$$\left| \frac{1}{n} \right| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{1/\epsilon}.$$

(Where  $\frac{1}{1/\epsilon} = \epsilon$ , but is in that form for demonstration purposes.) Therefore  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$   $\square$

“To get close” means is that we are finding a bigger and bigger  $N$  as  $\epsilon$  gets smaller. Note that the choice of  $N$  certainly depends on  $\epsilon$ .

### 2.2.1 Basic Structure of a Limit Proof

Claim:  $\lim_{n \rightarrow \infty} a_n = L$ .

Proof: Let  $\epsilon > 0$ . There exists  $N \in \mathbb{N}$  such that {something involving  $\epsilon$ }. Assume  $n \geq N$ . Then,

$$|a_n - L| \boxed{\dots} < \epsilon$$

(Where  $\boxed{\dots}$  is going to be where the majority of the work is going to lie.)



### Example 2.4: In-class

Claim:  $\lim_{n \rightarrow \infty} \frac{2n-3}{2n} = 1$

*Proof.* Let  $\epsilon > 0$ . *Scratch paper:* Solve for:

$$\left| \frac{2n-3}{2n} - 1 \right| = \left| \frac{-3}{2n} \right| = \frac{3}{2n} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n.$$

By the **Archimedean Principle**, there exists  $N \in \mathbb{N}$  such that  $N > \frac{3}{2\epsilon}$ . Assume  $n \geq N$ , (want to know what happens past this point)

$$\left| \frac{2n-3}{2n} - 1 \right| \leq \frac{3}{2N} < \frac{3}{2 \cdot 3/2\epsilon} = \epsilon.$$

Therefore,  $\lim_{n \rightarrow \infty} \frac{2n-3}{2n} = 1$  □

### Example 2.5: C

Claim:  $\lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2} = 2$

*Proof.* Let  $\epsilon > 0$ . By the **Archimedean Principle**, there exists  $N \in \mathbb{N}$  such that [leave off] *Scratch paper:* Solve for

$$\left| \frac{2n^2+1}{n^2} - 2 \right| = \frac{2n^2}{n^2} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n$$

[pick up] there exists  $N \in \mathbb{N}$  such that

$$N > \frac{1}{\sqrt{\epsilon}}.$$

Assume  $n \geq N$ , then

$$\begin{aligned} \left| \frac{2n^2+1}{n^2} - 2 \right| &= \frac{1}{n^2} \\ &\leq \frac{1}{N^2} \\ &< \frac{1}{(1/(\sqrt{\epsilon})^2)} \\ &= \frac{1}{1/\epsilon} \\ &= \epsilon \end{aligned}$$



Therefore,  $\lim_{n \rightarrow \infty} \frac{2n^2+1}{n^2} = 2$

□

### Example 2.6: In-class

Claim:  $\lim_{n \rightarrow \infty} \frac{7n+8}{3n+6} = \frac{7}{3}$

*Proof.*

$$\begin{aligned}
 \left| \frac{7n+8}{3n+6} - \frac{7}{3} \right| &= \left| \frac{21n+24}{3(3n+6)} - \frac{21n+42}{3(3n+6)} \right| \\
 &= \left| \frac{-18}{9n+18} \right| \\
 &= \frac{18}{9n+18} < \epsilon * * \\
 &= \frac{18}{3} < 9n+18 \\
 &= \frac{18}{3} - 18 < 9n \\
 &= \frac{18/\epsilon - 18}{9} < n
 \end{aligned}$$

$** \frac{18}{9n+18} < \frac{18}{9n} < \epsilon \Rightarrow \frac{2}{\epsilon} < N$ .  $\exists N \in \mathbb{N}$  such that  $N > \frac{2}{\epsilon}$ . Assume  $n \geq N$ ,

$$\begin{aligned}
 \left| \frac{7n+8}{3n+6} - \frac{7}{3} \right| &= \frac{18}{9n+18} \\
 &= \frac{2}{n+2} \\
 &< \frac{2}{n} \\
 &\leq \frac{2}{N} \\
 &< \frac{2}{\epsilon/2} \\
 &= \epsilon
 \end{aligned}$$

□

Does every sequence have a limit?

### Theorem 2.2.4: Uniqueness of Limits

The limit when it exists, is unique.





*Proof.* Let  $(x_n)$  be a convergent sequence. Suppose  $L$  and  $M$  are limits of this sequence. Without the loss of generality, we are going to assume  $M > L$ . Let

$$\epsilon = \frac{M - L}{3}.$$

Since  $x_n$  converges to  $L$ , there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|(x_n) - L| < \epsilon$ . Since  $(x_n)$  converges to  $M$ , there exists an  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,  $|(x_n) - M| < \epsilon$ . Consider  $n = \max\{N_1, N_2\}$ . Since  $n \geq N_1$ ,  $|(x_n) - L| < \epsilon$ . Since  $n \geq N_2$ ,  $|(x_n) - M| < \epsilon$ . Then  $L - \epsilon < x_n < L + \epsilon$  and  $M - \epsilon < x_n < M + \epsilon$ . By our choice of  $\epsilon$ , we now have

$$(x_n) < L + \epsilon < M - \epsilon < (x_n).$$

This is a contradiction. Thus,  $(x_n)$  cannot have two different limits.  $\square$

### Example 2.7:

Let  $(x_n) = \frac{\cos(n)}{3n}$ . Claim:  $\lim_{n \rightarrow \infty} (x_n) = 0$

*Proof.* Let  $\epsilon > 0$ . By the [Archimedean Principle](#), there exists  $N \in \mathbb{N}$  such that  $N > \frac{1}{3\epsilon}$  for all  $n \geq N$ ,

$$\begin{aligned} \left| \frac{\cos(n)}{3n} - 0 \right| &= \left| \frac{\cos(n)}{3n} \right| \\ &\leq \frac{1}{3n} \\ &\leq \frac{1}{3N} \\ &< \frac{1}{3(1/3\epsilon)} \\ &= \epsilon \end{aligned}$$

$\square$

### Example 2.8:

Let  $(y_n) = \frac{4n-1}{n^2}$ . Claim:  $\lim_{n \rightarrow \infty} y_n = 0$ .

*Proof.* Let  $\epsilon > 0$ . By the [Archimedean Principle](#), there exists  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon}$ .



For all  $n \geq N$ ,

$$\begin{aligned}\left|\frac{4n-1}{n^2} - 0\right| &= \left|\frac{4n-1}{n^2}\right| \\ &= \frac{4n-1}{n} \\ &< \frac{4n}{n^2} \\ &= \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \frac{4}{4/\epsilon} \\ &= \epsilon\end{aligned}$$

□

## Exercises

### Exercise: 2.2.2(b)

Verify, using [Definition 2.2.3](#), that the following sequences converge to the proposed limit.

(b)  $\lim_{n \rightarrow \infty} \frac{2n^2}{n^3+3} = 0$

*Proof.*

(b) Let  $\epsilon > 0$ . By the [Archimedean Principle](#), there exists an  $N \in \mathbb{N}$  such that  $N > \frac{2}{\epsilon}$ .



Then, for  $n \geq N$ ,

$$\begin{aligned} \left| \frac{2n^2}{n^3 + 3} - 0 \right| &= \left| \frac{2n^2}{n^3 + 3} \right| \\ &= \frac{2n^2}{n^3 + 3} \\ &< \frac{2n^2}{n^3} \\ &= \frac{2}{n} \\ &\leq \frac{2}{N} \\ &= \frac{2}{2/\epsilon} \\ &= \epsilon. \end{aligned}$$

Therefore,  $\lim_{n \rightarrow \infty} \frac{2n^2}{n^3 + 3} = 0$ . □

### Exercise: 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

- (a) At every college in the United States, there is a student who is at least seven feet tall.
- (b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
- (c) There exists a college in the United States where every student is at least six feet tall.

*Solution.*

- (a) There is at least one college in the United States where all students are less than seven feet tall.
- (b) There is at least one college in the United States where all professors give at least one student a grade of C or lower.
- (c) For all colleges in the United States, there exists a student who is less than six feet tall.



### Exercise: 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

- (a) A sequence with an infinite number of ones that does not converge to one.
- (b) A sequence with an infinite number of ones that converges to a limit not equal to one.
- (c) A divergent sequence such that for every  $n \in \mathbb{N}$  it is possible to find  $n$  consecutive ones somewhere in the sequence.

*Solution.*

- (a) Possible. Consider the piecewise function:  $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$
- (b) Impossible. A sequence that converges must have its terms approach a specific value (the limit). If the sequence has an infinite number of ones, it must have subsequences of ones arbitrarily far out. For the sequence to converge to a limit different from 1, the terms would have to approach that different limit, say  $L \neq 1$ , meaning the ones must become rare or eventually stop appearing, contradicting the infinite number of ones. Therefore, such a sequence is impossible.
- (c) Possible.  $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots)$

## 2.3 The Algebraic and Order Limit Theorems

### Definition 2.3.1

A sequence  $(x_n)$  is *bounded* if there exists some  $M > 0$  such that every term in the sequence belongs to  $[-M, M]$ .

### Theorem 2.3.2

Every convergent sequence is bounded.

*Proof.* Let  $(x_n)$  be a convergent sequence with limit  $L$ . There exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|(x_n) - L| < 1$ . Equivalently,  $(x_n) \in (L - 1, L + 1)$ . Let

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |L + 1|, |L - 1|\}.$$

We claim that for all  $n \in \mathbb{N}$ ,  $|x_n| \leq M$ .



(a) This is true for  $n < N$ .

(b) For  $n \geq N$ , we know  $L - 1 < x_n < L + 1$ , so  $(x_n) \leq \max\{|L - 1|, |L + 1|\}$

Thus, every term is in  $[-M, M]$ . □

### Theorem 2.3.3: Algebraic Limit Theorem

Let  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ . Then,

- (i)  $\lim_{n \rightarrow \infty} ca_n = ca$  for all  $c \in \mathbb{R}$ ;
- (ii)  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ ;
- (iii)  $\lim_{n \rightarrow \infty} (a_n b_n) = ab$ ;
- (iv)  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$  provided  $b \neq 0$ .

*Scratch Paper:*

$$\begin{aligned} |ca_n - ca| &= |c| |a_n - a| < \epsilon \\ |a_n - a| &< \frac{\epsilon}{|c|} \end{aligned}$$

Leave off and go back to proof<sup>1</sup>

*Proof.* (i)

Let  $\epsilon > 0$ .<sup>1</sup> Since  $(a_n)$  converges to  $a$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - a| < \frac{\epsilon}{|c|}$ . Now, for any  $n \geq N$  we have two case because we want to avoid dividing by 0:

- If  $c = 0$ :  
then each  $ca_n = 0$ . So  $(ca_n)$  converges to 0, which can equal  $ca$ .

- If  $c > 0$ :  
 $|ca_n - ca| = |c| |a_n - a| < |c| \frac{\epsilon}{|c|} = \epsilon$ . □

(ii)

*Scratch paper:*

$$|(a_n + b_n)| = |(a_n - a) + (b_n - b)| \tag{2.1}$$

$$\leq |a_n - a| + |b_n - b| \tag{2.2}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \tag{2.3}$$

Note that (2.2) is from the triangle inequality. Now, we will pick up to back at  $\epsilon > 0$ .

Let  $\epsilon > 0$ . Since  $(a_n)$  converges to  $a$ , there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|a_n - a| < \frac{\epsilon}{2}$ . Since  $(b_n)$  converges to  $b$ , there exists  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,



$|b_n - b| < \frac{\epsilon}{2}$ . Now, let  $N = \max\{N_1, N_2\}$ . Thus, for any  $n \geq N$ , (refer back to scratch paper).

(iii)

*Scratch paper:*

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \quad (2.4)$$

$$= |a_n(b_n - b) + b(b_n - b)| \quad (2.5)$$

$$\leq |a_n| |b_n - b| + |b| |b_n - b| \quad (2.6)$$

$$\leq M |b_n - b| + M |a_n - a|. \quad (2.7)$$

$$< M \left( \frac{\epsilon}{2M} \right) + M \left( \frac{\epsilon}{2M} \right) \quad (2.8)$$

$$= \epsilon \quad (2.9)$$

Note that: (2.4) is where we added 0, (2.5) is from the triangle inequality, and (2.6) is just factored. Additionally, we choose  $N$  to get the fractions in (2.8). Now, we will pick up to back at  $\epsilon > 0$ .

Let  $\epsilon > 0$ . Since convergent sequences are bounded, then there exists  $M > 0$  such that for all  $n \in \mathbb{N}$ ,  $|a_n| \leq M$ . We can choose  $M$  so that  $|b_n| \leq M$  as well. Since  $(a_n)$  converges to  $a$ , there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|a_n - a| < \frac{\epsilon}{2M}$ . Since  $(b_n)$  converges to  $b$ , there exists  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,  $|b_n - b| < \frac{\epsilon}{2M}$ . Now, let  $N = \max\{N_1, N_2\}$ . Thus, for any  $n \geq N$ , (refer back to scratch paper, and change (2.4)'s sign from an '=' to ' $\leq$ ').

(iv)

*Scratch paper:*

$$\begin{aligned} \left| \frac{a_n}{b_n} - \frac{a}{b} \right| &= \left| \frac{a_n b - ab_n}{b_n b} \right| \\ &= \left| \frac{a_n b - ab_n + ab_n - ab}{b_n b} \right| \\ &= \left| \frac{a_n(b - b_n) + b(b_n - b)}{b_n b} \right| \\ &= \left| \frac{a_n(b - b_n)}{b_n b} + \frac{b(b_n - b)}{b_n b} \right| \\ &\leq \left| \frac{a_n}{b_n} \right| |b - b_n| + |b| \left| \frac{b_n - b}{b_n b} \right| \\ &< \epsilon \end{aligned}$$

Let  $\epsilon > 0$ . Since  $(b_n)$  converges to  $b$ , there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|b_n| > \left| \frac{b}{2} \right|$ . There also exists  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,  $|b_n - b| < \frac{\epsilon |b|^2}{2}$ . Now, let  $N = \max\{N_1, N_2\}$ . Let  $n \geq N$ , (refer back to scratch paper).



### Lemma 2.3.4

Let  $(a_n)$  and  $c < a$ . There exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a_n > c$ . Similarly, if  $a < d$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a_n < d$ .

## 2.3.1 Limits and Order

### Theorem 2.3.5: Order Limit Theorem

Let  $(a_n)$  and  $(b_n)$  be sequences. If  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ , then

- (i) If  $a_n \geq c$  for all  $n \in \mathbb{N}$ , then  $a \geq c$ .
- (ii) If  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then  $a \leq b$ .
- (iii) If there exists  $c \in \mathbb{R}$  for which  $c \leq b_n$  for all  $n \in \mathbb{N}$ , then  $c \leq b$ . Similarly, if  $a_n \leq c$  for all  $n \in \mathbb{N}$ , then  $a \leq c$ .

## Exercises

### Exercise: 2.3.1

- (a) Assume  $\lim_{n \rightarrow \infty} x_n = 0$  with  $x_n \geq 0$ . Show that  $\lim_{n \rightarrow \infty} \sqrt{x_n} = 0$ .
- (b) Assume  $\lim_{n \rightarrow \infty} x_n = 49$  with  $x_n \geq 0$ . Show that  $\lim_{n \rightarrow \infty} \sqrt{x_n} = 7$ .

*Proof.*

- (a) Let  $\epsilon > 0$ . Since  $(x_n)$  converges to 0, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - 0| < \epsilon^2$ . Now, for any  $n \geq N$ ,  $|\sqrt{x_n} - 0| = \sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon$ . Therefore,  $\lim_{n \rightarrow \infty} \sqrt{x_n} = 0$ .
- (b) Let  $\epsilon > 0$ . Since  $(x_n)$  converges to 49, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$\begin{aligned} |x_n - 49| &= \left| \frac{(x_n - 49)(x_n + 49)}{x_n + 49} \right| \\ &= \left| \frac{x_n - 49}{\sqrt{x_n} + 7} \right| \\ &\leq \frac{|x_n - 49|}{7} \end{aligned}$$

□



### Exercise: 2.3.2

Using only [Definition 2.2.3](#), prove that if  $(x_n) \rightarrow 2$ , then

(a)  $\left(\frac{2x_n - 1}{3}\right) \rightarrow 1;$

(b)  $(1/x_n) \rightarrow 1/2.$

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

*Solution.*

- (a) *Proof.* Let  $\epsilon > 0$ . Since  $(x_n)$  converges to 2, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - 2| < \epsilon$ . Now, for any  $n \geq N$ ,

$$\begin{aligned} \left| \frac{2x_n - 1}{3} - 1 \right| &= \left| \frac{2x_n - 1 - 3}{3} \right| \\ &= \left| \frac{2x_n - 4}{3} \right| \\ &= \frac{2}{3} |x_n - 2| \\ &< |x_n - 2| \\ &< \epsilon \end{aligned}$$

Therefore,  $\frac{2x_n - 1}{3} \rightarrow 1$  □

- (b) *Proof.* Let  $\epsilon > 0$ . Since  $(x_n)$  converges to 2, there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $x_n \geq 1$ . Then, we will choose  $N_2$  so that  $|x_n - 2| < \epsilon$  for all  $n \geq N_2$ . Afterwards, we take  $N = \max\{N_1, N_2\}$ . And note that for  $n \geq N$ ,

$$\begin{aligned} \left| \frac{1}{x_n} - \frac{1}{2} \right| &= \left| \frac{2 - x_n}{2x_n} \right| \\ &< \frac{|2 - x_n|}{2} \\ &< \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

□





## 2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

### Definition 2.4.1

A sequence  $a_n$  is *increasing* if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$  and *decreasing* if  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$ . A sequence is *monotone* if it is either increasing or decreasing.

### Theorem 2.4.2: Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges.

*Proof.* Let  $(a_n)$  be an increasing and bounded sequence. Since  $(a_n)$  is bounded, the set  $A = \{a_n \mid n \in \mathbb{N}\}$  is clearly also bounded. Since  $A$  is bounded,  $\sup A$  exists. We claim that  $\lim_{n \rightarrow \infty} a_n = \sup A$ . Thus, for all  $\epsilon > 0$  and by our definition of supremum, there exists  $N \in \mathbb{N}$  such that  $\sup A - \epsilon < a_N \leq \sup A$ . Since  $(a_n)$  is increasing, for all  $n \geq N$ ,  $\sup A - \epsilon < a_N \leq a_n \leq \sup A$ . It follows that  $|a_n - \sup A| < \epsilon$ . Therefore,  $\lim_{n \rightarrow \infty} a_n = \sup A$ .  $\square$

### Example 2.9: MCT

Consider the recursively defined sequence  $x_n$  where  $x_1 = 3$  and for all  $n \in \mathbb{N}$ ,  $x_{n+1} = \frac{1}{4-x_n}$ . Show that  $x_n$  converges.

*Proof.* We will show that  $x_n$  is monotone and bounded.

- **Part 1: Monotone Decreasing**

- Base case:  $x_1 = 3, x_2 = 1$ .
- Induction step: Assume for some  $n \in \mathbb{N}$ ,  $x_n \geq x_{n+1}$ . It follows that

$$\begin{aligned} x_n &\geq x_{n+1} \\ 4 - x_n &\leq 4 - x_{n+1} \\ \frac{1}{4 - x_n} &\geq \frac{1}{4 - x_{n+1}} \\ x_{n+1} &\geq x_{n+2} \end{aligned}$$

- **Part 2: Bounded Below Claim:** Sequence is bounded below by 0.

- Base case:  $x_1 = 3 > 0$ .
- Induction step: Assume for some  $n \in \mathbb{N}$ ,  $x_n \geq 0$ . It follows that  $4 - x_n \leq 4$ ,



and when we take the reciprocal, we get

$$\begin{aligned}\frac{1}{4-x_n} &\leq \frac{1}{4} \\ x_{n+1} &\geq 1/4 \\ &> 0\end{aligned}$$

By math induction,  $x_n$  is bounded below by 0.

By the Monotone Convergence Theorem,  $x_n$  converges.

So, what is the limit? We know  $(x_n)$  converges so let  $L = \lim_{n \rightarrow \infty} x_n$ . Then,  $\lim_{n \rightarrow \infty} x_{n+1} = L$ . We also know  $x_{n+1} = \frac{1}{4-x_n}$ . So  $L = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{4-x_n} = \frac{1}{4-L}$ . It must be true that  $L = \frac{1}{4-L}$ . Solving for  $L$ , we get

$$\begin{aligned}L(4-L) &= 1 \\ 4L - L^2 &= 1 \\ L^2 - 4L + 1 &= 0\end{aligned}$$

Hence,  $L = 2 - \sqrt{3}$  or  $L = 2 + \sqrt{3}$ . Notice that it cannot be the latter because it is bigger than 3.  $\square$

### 2.4.1 Recap and Summary

We use limits to define multiple things in calculus. This is why we are focusing so heavily upon it. For example,

- (a) Derivatives:  $\lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$
- (b) Integrals:  $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$
- (c) Infinite Series:  $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$  Consider geometric series,  $C_a$  such that each term is multiplied by a ratio  $r$ . This is represented as  $\sum_{n=0}^{\infty} ar^n = 1 + r + r^2 + r^3 \dots$ . When we look at partial sums, we get  $S_n = 1 + r + r^2 + r^3 + \dots + r^n$ . We can then multiply by  $r$  to get  $rS_n = r + r^2 + r^3 + r^4 + \dots + r^{n+1}$ . Subtracting the two, we get  $(1-r)S_n = 1 - r^{n+1}$ . Thus,

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

If  $|r| < 1$ , then  $\lim_{n \rightarrow \infty} r^n = 0$ . Thus,  $\lim_{n \rightarrow \infty} S_n = \frac{1}{1-r}$ .

Looking to the future, we are going to use functions and summations together. For example, when we have  $f(x) = \sum_{n=0}^{\infty} (a_n)x^n$  such that  $f'(x) = \sum_{n=0}^{\infty} (a_n)x^{n-1}$ .

#### Definition 2.4.3

Let  $x_n$  be a bounded sequence. Then the *limit inferior* is  $\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \inf\{x_k \mid k \geq n\}$ . This is the largest a limit can get. The *limit superior* is  $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sup\{x_k \mid k \geq n\}$ . This is the smallest a limit can get.



See Exercise 2.4.7 in the book for more information.

### Example 2.10: *Monotone Decreasing Sequence*

$$x_1, x_2, x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 1\} = S.$$

$$x_2, x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 2\} = S.$$

$$x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 3\} = S.$$

$$x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 4\} = S.$$

$\limsup_{n \rightarrow \infty} x_n$  is guaranteed to exist by the [Monotone Convergence Theorem](#).

### Example 2.11: $\liminf$

Let  $x_n = (-1)^n(1 + \frac{1}{n})$ . Thus,  $x_{1,2,3} = -2, 1\frac{1}{2}, -1\frac{1}{3} \dots$

### Example 2.12: Convergence Towards 0

Let  $x_n = (-1)^n \frac{1}{n}$ . Thus,  $x_{1,2,3} = -1, \frac{1}{2}, -\frac{1}{3} \dots$

### Theorem 2.4.4

A sequence  $x_n$  is convergent if, and only if,  $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n$ .

See Theorem 2.4.6 in the book for another view.

## 2.5 Subsequences and the Bolzano-Weierstrass Theorem

### Definition 2.5.1

Let  $a_n$  be a sequence of real numbers, and let  $n_1 < n_2 < n_3 < \dots$  be an increasing sequence of natural numbers. Then, the sequence  $a_{n_1}, a_{n_2}, a_{n_3}, \dots$  is called a *subsequence* of  $a_n$  and is denoted by  $a_{n_k}$ , where  $k \in \mathbb{N}$  indexes the subsequence.

### Theorem 2.5.2

Subsequences of a convergent sequence converge to the same limit as the original sequence.

*Proof.* Let  $x_{n_k}$  be a subsequence of  $x_n$ , and let  $L = \lim_{n \rightarrow \infty} x_n$ . We want to show that  $\lim_{n \rightarrow \infty} x_{n_k} = L$ . Let  $\epsilon > 0$ . Since  $x_n$  converges to  $L$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - L| < \epsilon$ . Since  $n_k$  is increasing, there exists  $M \in \mathbb{N}$  such that  $n_k \geq N$



for all  $k \geq M$ . Thus, for all  $k \geq M$ ,  $|x_{n_k} - L| < \epsilon$ . Therefore,  $\lim_{n \rightarrow \infty} x_{n_k} = L$ .

Let  $x_{n_k}$  be a subsequence of  $x_n$ . Let  $\epsilon > 0$ . Since  $(x_n) \rightarrow L$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - L| < \epsilon$ .

Now, looking at  $x_{n_k}$ , notice that  $n_k \geq k$  for all  $k$ . Consider  $k = N$ . For any  $n \geq N$ ,  $n \geq N \geq k$ . Thus,  $|x_{n_k} - L| < \epsilon$ . Therefore,  $\lim_{n \rightarrow \infty} x_{n_k} = L$ .  $\square$

### Theorem 2.5.3: Divergence Criterion

If  $x_n$  has two subsequences that converge to different limits, then  $x_n$  diverges.

Building upon this idea of Divergence, we can list some other ways a sequence can diverge:

- (a) Find one subsequence that diverges.
- (b) Find two subsequences that converge to separate limits.
- (c) Negate the **definition of convergence**.
  - For example, a sequence converges to  $L$  if there exists  $\epsilon > 0$  such that for all  $N \in \mathbb{N}$  there exists  $n \geq N$  such that  $|a_n - L| \geq \epsilon$ . There exists a subsequence  $(a_{n_k})$  such that for all  $k \in \mathbb{N}$ ,  $|a_{n_k} - L| \geq \epsilon$ .

### Theorem 2.5.4: Bolzano-Weierstrass Theorem

Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

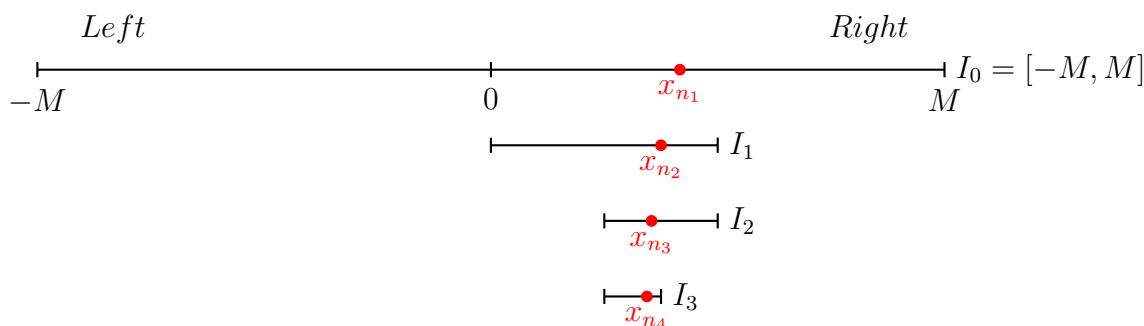
*Proof.* Let  $x_n$  be a bounded sequence. There exists an  $M > 0$  such that every term  $x_n$  belongs to  $[-M, M]$ . To prove this theorem, we will be utilizing a recursive argument style. Thus, let  $I_0 = [-M, M]$ .  $I_0$  has length  $2M$ . Cut  $I_0$  in half with  $I_1$  and  $I_2$  both being half as long as  $I_0$ . Since  $x_n$  is bounded, there exists an  $I_L$  or  $I_R$  that contains infinitely many terms of  $x_n$ . We will pick one, call it  $I_1$  that is contained in  $I_L$ .  $I_1$  has length  $M$ . Pick one of those terms inside  $I_1$  and call it  $x_{n_1}$ . Now, cut  $I_1$  in half with equal length in intervals. One of them contains infinitely many terms. Call that interval  $I_2$ .  $I_2$  has length  $\frac{M}{2}$ . Pick one of those terms inside  $I_2$  and call it  $x_{n_2}$ . Continue this process indefinitely for all  $n \geq \mathbb{N}$  with  $n_1 > n_2$ . Continue this process, and we get

- a sequence of closed intervals  $I_n$ .
  - $I_n$  has length  $\frac{2M}{2^n}$ .
  - They are nested,  $I_n \subseteq I_{n-1}$ .
- a subsequence  $x_{n_k}$ 
  - for all  $k_1$ ,  $x_{n_k} \in I_k$ .

The **Nested Interval Property** states that  $\bigcup_{n=1}^{\infty} I_n$  is non empty. Let  $L$  be a point in  $\bigcup_{n=1}^{\infty} I_n$ . We claim  $\lim_{n \rightarrow \infty} x_{n_k} = L$ . Let  $\epsilon > 0$ . There exists an  $N \in \mathbb{N}$  such that  $\frac{2M}{2^N} < \epsilon$ . (Since  $\lim_{n \rightarrow \infty} \frac{2M}{2^n} = 0$ . See **Theorem 2.5.5**) For any  $k \geq N$ , recall that  $x_{n_k}$ ,



$L \in I_k$ . Since  $I_k$  has length  $\frac{2M}{2^n}$ . Thus,  $|x_{n_k} - L| < \epsilon$ . Therefore,  $\lim_{n \rightarrow \infty} x_{n_k} = L$  and  $(x_n)$  has a convergence subsequence.  $\square$



### Theorem 2.5.5

Let  $b \in (0, 1)$ . Then  $\lim_{n \rightarrow \infty} b^n = 0$ .

*Proof.* The sequence  $(b^n)$  is monotone decreasing. This is because  $b^{n+1} = b^n b < b^n$ . This sequence is also bounded by 0. Hence, by the **Monotone Convergence Theorem**,  $(b^n)$  converges. Now, let  $L = \lim_{n \rightarrow \infty} b^n$ . Consider the subsequence  $b^{2n}$ . This sequence also converges to  $L$ . Thus,

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} b^{2n} \\ &= \lim_{n \rightarrow \infty} b^n b^n \\ &= \lim_{n \rightarrow \infty} b^n \lim_{n \rightarrow \infty} b^n \\ &= L^2. \end{aligned}$$

Thus,  $L = 0$  or  $L = 1$ . The limit cannot be 1 because  $b^n$  is decreasing away from 1. Therefore,  $L = 0$ .  $\square$

## 2.6 The Cauchy Criterion

### Recall

How do we prove  $x_n$  converges?

- (a) We know and prove the limit  $\rightarrow$  claim  $L$ , show terms get close to  $L$ .
- (b) **Monotone Convergence Theorem**.

### Definition 2.6.1

A sequence  $x_n$  is a *Cauchy sequence* if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ ,  $|x_m - x_n| < \epsilon$ .



This says that as terms get close to each other and stay close together, there's some value they're all getting close to.

### Theorem 2.6.2: Cauchy Criterion

A sequence  $x_n$  converges if, and only if, it is a Cauchy sequence.

*Proof.* We show this by proving both implications:

( $\Rightarrow$ ) Assume  $(x_n)$  is a convergent sequence in  $\mathbb{R}$ . Given  $\epsilon > 0$ . Let  $L = \lim_{n \rightarrow \infty} x_n$ . Since  $(x_n) \rightarrow L$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - L| < \frac{\epsilon}{2}$ . For all  $n, m \geq N$ ,

$$\begin{aligned} |x_m - x_n| &= |x_m - L + L - x_n| \\ &\leq |x_m - L| + |L - x_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore,  $x_n$  is a Cauchy sequence.



( $\Leftarrow$ ) Assume  $x_n$  is a Cauchy sequence.

- **Step 1:** Show that  $x_n$  is bounded.

Since  $x_n$  is Cauchy, there exists  $N \in \mathbb{N}$  such that for all  $n, m \geq N$ ,  $|x_n - x_m| < l$ . It follows that for all  $n \geq N$ , we need to account for  $x_1, \dots, x_{n-1}$ . Thus, let  $M = \max\{|x_1|, |x_2|, \dots, |x_{n-1}|, |x_n| + 1\}$ . Then for all  $n \in \mathbb{N}$ ,  $|x_n| < M$ .

- **Step 2:** Since  $x_n$  is bounded, there exists a convergent subsequence  $x_{n_k}$  by the **Bolzano-Weierstrass Theorem**. Let  $L$  be the limit of the subsequence.
- **Step 3:** Show that  $x_n$  converges to  $L$ .

If some get close to  $L$  and all get close to each other, they all get close to  $L$ . Let  $\epsilon > 0$ . Since  $x_{n_k}$  converges to  $L$ , there exists  $N \in \mathbb{N}$  such that for all  $k \geq N$ ,  $|x_{n_k} - L| < \frac{\epsilon}{2}$ . Since  $x_n$  is Cauchy, there exists  $M \in \mathbb{N}$  such that for all  $n, m \geq M$ ,  $|x_n - x_m| < \frac{\epsilon}{2}$ . Let  $M_0 = \max\{N, n_k\}$ . By the **Archimedean Principle**, there exists  $N_0$  such that  $n_{k_0} \geq M_0$ . Then, from the **Triangle Inequality**, we say that for all  $n \geq N_0$ ,

$$\begin{aligned} |x_n - L| &\leq |x_n - x_{n_{k_0}}| + |x_{n_{k_0}} - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore,  $(x_n) \rightarrow L$ .

By proving both directions of the inequality, we found that a sequence  $(x_n)$  converges if, and only if, it is a Cauchy sequence.  $\square$

**Exercise: 2.5.1**

Give an example of each of the following, or argue that such a request is impossible.

- (a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.
- (b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

*Solution.*

- (a) Impossible. This violates the **Bolzano-Weierstrass Theorem**. It assures us that every bounded sequence has a convergent subsequence. If a subsequence is bounded, then it must have a convergent subsequence.
- (b) Consider the sequence  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots, \frac{1}{n}, \frac{(n-1)}{n})$ . From this, you can have a subsequence  $(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n})$  which converges to 0, and also a subsequence  $(\frac{1}{2}, \frac{2}{3}, \dots, \frac{n-1}{n})$ , which converges to 1.

**Exercise: 2.5.2**

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

- (a) If every proper subsequence of  $(x_n)$  converges, then  $(x_n)$  converges as well.
- (c) If  $(x_n)$  is bounded and diverges, then there exist two subsequences of  $(x_n)$  that converge to different limits.

*Solution.*

- (a) False. As shown in Example 2.5.4, if we have a sequence like  $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, \dots)$ , we see that it is divergent, and has two proper subsequences  $(\frac{1}{5}, \frac{1}{5}, \dots)$  and  $(-\frac{1}{5}, -\frac{1}{5}, \dots)$  that both converge to values  $\frac{1}{5}$  and  $-\frac{1}{5}$ , respectively.
- (c) True. If we assume that  $(x_n)$  is bounded and diverges, then by the **Bolzano-Weierstrass Theorem**, every bounded sequence has a convergent subsequence. Let  $(x_{n_k})$  be that convergent subsequence of  $(x_n)$ , and let  $L_1$  be its limit. Since  $(x_n)$  diverges, it cannot converge to  $L_1$ . Thus, there exists an  $\epsilon_0 > 0$  such that for all terms of  $(x_n)$ , they stay outside the  $\epsilon_0$ -neighborhood of  $L_1$ . Assume this subsequence  $(x_{m_k})$  contains these terms. When we apply the same logic to this





sub-subsequence, we see that by the **Bolzano-Weierstrass Theorem**,  $(x_{m_k})$  has a convergent subsequence with limit  $L_2$ , where  $L_2 \neq L_1$  because the terms of  $(x_{m_k})$  stay outside the  $\epsilon_0$ -neighborhood of  $L_1$ . Thus  $(x_n)$  contains two subsequences that converge to different limits,  $L_1$  and  $L_2$ .

### Exercise: 2.5.5

Assume  $(a_n)$  is a bounded sequence with the property that every convergent subsequence of  $(a_n)$  converges to the same limit  $a \in \mathbb{R}$ . Show that  $(a_n)$  must converge to  $a$ .

*Proof.* Suppose that  $(a_n)$  does not converge to  $a \in \mathbb{R}$ . By the definition of convergence, this means there is a positive real number  $\epsilon_0$  such that no matter how large we choose  $N \in \mathbb{N}$ , there will always exist some  $n > N$  where  $|a_n - a| \geq \epsilon_0$ . In a formal way, this shows that  $(a_n)$  does not converge to  $a$  within the  $\epsilon_0$ -neighborhood.

We aim to demonstrate that this leads to a contradiction by constructing a subsequence of  $(a_n)$  that stays outside this neighborhood. Begin by selecting  $n_1$  such that  $|a_{n_1} - a| \geq \epsilon_0$ . Next, since the condition holds for all  $N \in \mathbb{N}$ , we can find another index  $n_2 > n_1$  such that  $|a_{n_2} - a| \geq \epsilon_0$ . Continuing this process, we generate an increasing sequence of indices  $n_1 < n_2 < n_3 < \dots$  such that for each  $i \in \mathbb{N}$ ,  $|a_{n_i} - a| \geq \epsilon_0$ .

Now consider the subsequence  $(a_{n_i})$  we have built. Since  $(a_n)$  is bounded by assumption, its subsequence  $(a_{n_i})$  is also bounded. By the **Bolzano-Weierstrass Theorem**, every bounded sequence has a convergent subsequence. Let  $(a_{n_{i_k}})$  denote a convergent subsequence of  $(a_{n_i})$ . According to our assumption, any convergent subsequence of  $(a_n)$  must converge to  $a$ .

However, each term of  $(a_{n_{i_k}})$  remains outside the  $\epsilon_0$ -neighborhood of  $a$ . Thus, it is impossible for  $(a_{n_{i_k}})$  to converge to  $a$ . This contradiction implies that our initial assumption—that  $(a_n)$  does not converge to  $a$ —is false. Therefore, the sequence  $(a_n)$  must converge to  $a$ .  $\square$

### Exercise: 2.5.6

Use a similar strategy to the one in **Theorem 2.5.5** to show

$$\lim b^{1/n} \text{ exists for all } b \geq 0$$

and find the value of the limit. (The results in **Exercise 2.3.1** may be assumed.)

*Proof.* For this proof, we will be examining different cases that  $b$  may fall under. These cases are  $b = 0$  and  $b > 0$ .

1. If  $b = 0$ , then the sequence  $(b^{1/n})$  is simply  $(0, 0, 0, \dots)$ . Thus, the limit of this sequence is 0.
2. For  $b > 0$ , it is more complicated. We see that this sequence is decreasing because as  $n$  increases, the exponent  $\frac{1}{n}$  trends toward a higher denominator, so  $b^{1/n}$  de-



creases. We can see this by observing the property that for any  $m < n$ ,  $\frac{1}{n} < \frac{1}{m}$ . This implies that  $b^{1/n} < b^{1/m}$ .

Further, we see that this sequence is bounded below by 0 when  $b > 1$  and by 1 when  $b = 1$  (because 1 to any exponent is just 1). Therefore, this sequence is monotone and bounded below, so it must converge to some  $L \geq 0$  by the **Monotone Convergence Theorem**.

To find  $L$ , we need to consider another set of cases for  $b$ .

- i. If  $b = 1$ , then  $b^{1/n} = 1$  for all  $n$ , so its limit is 1.
- ii. If  $b > 1$ , then as  $n$  increases, the exponent  $\frac{1}{n}$  approaches 0, so  $b^{1/n}$  approaches  $b^0 = 1$ . So, its limit is also 1.
- iii. If  $0 < b < 1$ , then we know  $\lim_{n \rightarrow \infty} b^{1/n} = 1$  because  $b^{1/n}$  is a decreasing sequence bounded by 0, and when we have smaller and smaller powers of a number less than 1, it pushes it closer to 1.

Through these cases, we see that for all  $b \geq 0$ ,  $\lim_{n \rightarrow \infty} b^{1/n} = 1$ . □

**Exercise: 1.2.13**

For this exercise, assume [Exercise 1.2.5](#) has been successfully completed.

- (a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite  $n \in \mathbb{N}$ .

- (b) It is tempting to appeal to induction to conclude

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of  $n \in \mathbb{N}$ , but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets  $B_1, B_2, B_3, \dots$  where

$$\bigcap_{i=1}^n B_i \neq \emptyset \quad \text{is true for every } n \in \mathbb{N},$$

but

$$\bigcap_{i=1}^{\infty} B_i = \emptyset$$

fails.

- (c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.



*Proof.* In this proof, we plan to prove (c). Thus, we need to show that:

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$$

and

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

- ( $\subseteq$ ) Let  $x \in \left( \bigcup_{i=1}^{\infty} A_i \right)^c$ . This means  $x$  is in the union set of  $A_i$  for all  $i \in \mathbb{N}$ . Then, because we are taking the complement of  $\left( \bigcup_{i=1}^{\infty} A_i \right)$ , that means  $x \notin A_i$  for all  $i \in \mathbb{N}$ . Hence,  $x$  is in the complement of each  $A_i$ . Thus, we can use the definition of intersection to assert  $x \in \bigcap_{i=1}^{\infty} A_i^c$ . Therefore, we have shown:

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c.$$

- ( $\supseteq$ ) Similar to before, let  $x \in \bigcap_{i=1}^{\infty} A_i^c$ . Because  $x \in A_i^c$  for all  $i \in \mathbb{N}$  we know  $x \notin A_i$ . Hence,  $x \notin \left( \bigcup_{i=1}^{\infty} A_i \right)$ , which means  $x \in \left( \bigcup_{i=1}^{\infty} A_i \right)^c$ . Therefore, we have shown:

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

By showing both inclusions, we see that:

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c.$$

□