Real Analysis

MATH 350

Start

August 26, 2024

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End

December 2, 2024

CHAPTER 5

THE DERIVATIVE

5.2 Derivates and the Intermediate Value Property

Definition 5.2.1

Let $g\colon A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, we define the *derivative* of g at c to be

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c}$$

provided this limit exists. In this case, we say g is *differentiable* at c. If g' exists for all points $c \in A$, then we say g is *differentiable on* A.

Example 5.1: Differentiation 1

Let $g(x) = x^2$. Use Definition 5.2.1 to find $g'(c)$.

Solution.

$$\begin{aligned} g'(c) &= \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\ &= \lim_{x \to c} \frac{x^2 - c^2}{x - c} \\ &= \lim_{x \to c} \frac{(x-c)(x+c)}{x - c} \\ &= \lim_{x \to c} x + c \\ &= 2c \end{aligned}$$

Example 5.2: Differentiation 2

Let $g(x) = x^3$. Use Definition 5.2.1 to find $g'(c)$.

Solution.

$$\begin{aligned} g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= \lim_{x\to c} \frac{x^3-c^3}{x-c} \\ &= \lim_{x\to c} \frac{(x-c)(x^2+xc+c^2)}{x-c} \\ &= \lim_{x\to c} x^2+xc+c^2 \\ &= 3c^2 \end{aligned}$$

Example 5.3: Differentiation 3

Let $g(x) = x^4$. Use Definition 5.2.1 to find $g'(c)$.

Solution.

$$\begin{aligned} g'(c) &= \lim_{x\to c} \frac{g(x)-g(c)}{x-c} \\ &= \lim_{x\to c} \frac{x^4-c^4}{x-c} \\ &= \lim_{x\to c} \frac{(x-c)(x^3+x^2c+xc^2+c^3)}{x-c} \\ &= \lim_{x\to c} x^3+x^2c+xc^2+c^3 \\ &= 4c^3 \end{aligned}$$

Theorem 5.2.2: Power Rule

For any $n \in \mathbb{N}$, if $f(x) = x^n$, then $f'(c) = nc^{n-1}$.

Example 5.4: Differentiation 4

Let $f(x) = |x|$. Use Definition 5.2.1 to find $f'(c)$.

Solution. Is this differentiable at 0?

$$\lim_{x\to 0} \frac{|x|-|0|}{x-0} = \lim_{x\to 0} \frac{|x|}{x}$$

When we view this from the left and right definitions of the limit, we see that the limit does not exist:

$$\lim_{x\to 0^+}\frac{|x|}{x}=1 \quad\text{and}\quad \lim_{x\to 0^-}\frac{|x|}{x}=-1.$$

Hence, because these limits are not equal, the limit does not exist, and $f(x)=|x|$ is not differentiable at 0.

Example 5.5: Differentiation 5

Let $f(x)=\begin{cases}1 & \text{if } x>0\\ 0 & \text{if } x\le 0\end{cases}$

Solution. Is this differentiable at 0?

$$\lim_{x\to 0^+}\frac{f(x)}{x}=\infty \quad\text{and}\quad \lim_{x\to 0^-}\frac{f(x)}{x}=0.$$

Hence, because these limits are not equal, the limit does not exist, and $f(x)$ is not differentiable at 0.

Theorem 5.2.3: Algebraic Differentiation Rules

Let f and g be differentiable at c with $k\in\mathbb{R}$. Then the following functions are differentiable at c:

(a) $(f+g)'(c)=f'(c)+g'(c)$

$\Rightarrow \lim_{x\to c}\frac{f(x)+g(x)-f(c)-g(c)}{x-c}$

(b) $(kf)'(c)=kf'(c)$

(c) $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$

(d) $(f/g)'(c)=\frac{f'(c)g(c)-f(c)g'(c)}{g^2(c)}$

Proof. For (3), we have

$$\begin{aligned}\lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c}&=\lim_{x\to c}\frac{f(x)g(x)-f(x)g(c)+f(x)g(c)-f(c)g(c)}{x-c}\\&=\lim_{x\to c}\frac{f(x)(g(x)-g(c))+g(c)(f(x)-f(c))}{x-c}\\&=\lim_{x\to c}f(x)\lim_{x\to c}\frac{g(x)-g(c)}{x-c}+g(c)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\\&=f(c)g'(c)+g(c)f'(c).\end{aligned}$$

For (4), we have

$$\begin{aligned}
\lim_{x\to c}\frac{\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}}{x-c} &= \lim_{x\to c}\frac{f(x)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}\\
&= \lim_{x\to c}\frac{f(x)g(c)-f(c)g(c)+f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}\\
&= \lim_{x\to c}\frac{f(x)-f(c)}{g(x)g(c)(x-c)}+\lim_{x\to c}\frac{f(c)g(c)-f(c)g(x)}{g(x)g(c)(x-c)}\\
&= \frac{f'(c)g(c)-f(c)g'(c)}{g^2(c)}.
\end{aligned}$$

□

Theorem 5.2.4: Chain Rule

Let f and g be differentiable at c. Then the composition $g\circ f$ is differentiable at c and $(g\circ f)'(c) = g'(f(c))f'(c)$.

Proof.

$$\begin{aligned}
\lim_{x\to c}\frac{g(f(x))-g(f(c))}{x-c} &= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\cdot\frac{f(x)-f(c)}{x-c}\\
&= \lim_{x\to c}\frac{g(f(x))-g(f(c))}{f(x)-f(c)}\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\\
&= g'(f(c))f'(c).
\end{aligned}$$

□

Theorem 5.2.5: Interior Extremum Theorem

If f is differentiable at c and c is a local maximum of f, then $f'(c)=0$.

Proof. We know that $f'(c)=\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$. Then,

$$f'(x) = \lim_{x\to c^-}\frac{f(x)-f(c)}{x-c}\ge 0,$$

and

$$f'(x) = \lim_{x\to c^+}\frac{f(x)-f(c)}{x-c}\le 0,$$

So, $f'(c)=0$. □

Big Results From This Semester

(a) Extreme Value Theorem

- If f is continuous on a compact set, then f achieves a maximum and minimum on that set.

(b) Intermediate Value Theorem

- If f is continuous on an interval $[a, b]$ and y is between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$ such that $f(c) = y$.

Theorems for Today

(a) Rolle's Theorem

(b) Mean Value Theorem

(c) Maybe:

- Generalized MVT.
- L'Hopital's Rule.

(d) Darbaux's Theorem

Theorem 5.2.6: Rolle's Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Since f is continuous on $[a, b]$, the Extreme Value Theorem says f achieves a maximum and minimum in $[a, b]$.

- If either maximum or minimum is in the interior, then the thm:Interior Extremum Theorem says that $f'(c) = 0$.
- If the maximum or minimum is at the endpoints, then f is constant and $f'(c) = 0$ for all $[a, b]$.

□

Theorem 5.2.7: Mean Value Theorem

Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists a $c \in (a,b)$ such that
$$f'(c) = \frac{f(b) - f(a)}{b-a}.$$
This is saying that there exists a point where the slope of the tangent line is equal to the slope of the secant line. In other words, there is a point where $f'(c)$ happens.

Proof. Let $d(x) = f(x) + \frac{f(a)}{}$ □

EXAM 3 REVIEW

Able to have 3 pages front and back for notes. Focus on definitions, key theorems, and basic proof techniques.

Need to know vocabulary. Like, bounded, continuous, supremum, infinum, etc.

Know if a sequence is Cauchy, convergent, or divergent. Remember, if we can prove a sequence is convergent, then it is Cauchy.

Any theorem that has a name, know it. Like, Bolzano-Weierstrass, Heine-Borel, etc.

Know the hypotheses of the theorems. Like, if a function is continuous on a closed interval, then it is bounded and attains its maximum and minimum.

Proof rule of thumb for sequences: Know if and when to pick a sequence (there exists) vs. an arbitrary sequence (for all).

Know when a convergent subsequence exists (Bolzano-Weierstrass).

Proof techniques:

(a) Be able to use the $N - \epsilon$ definitions of a convergent sequence.

(b) Be able to use the $\delta - \epsilon$ definition for functional limits.

    Remember the specific rules for the range of δ. We do not want to have a fraction that has a 0 in the denominator.

(c) Prove a sequence is both monotone and bounded $\Rightarrow$ convergent (Monotone Convergence Theorem).

Will include true-false statements. Do you know this is false? Provide a counterexample. Is it true? Prove it.

On take home: part (4) will use part 2.

On part (7), you can assume that $f(x) = e^x$ is differentiable to e^x