# Some Common Parameterizations and Integrals

## Shapes in the $xy$-plane, $\mathbb{R}^2$

### Curves

Each curve is one-dimensional, which means it can be parameterized by a single variable, $t$. In general, we have:

- the parameterization, $\mathbf{r}(t) = \langle x(t), y(t) \rangle$,
- the tangent vector, $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$,
- the magnitude of the tangent vector, $\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$, and
- the normal vector, $\mathbf{n}(t) = \langle y'(t), -x'(t) \rangle$, where we adopt the convention that this vector points left to right as we move along the curve.

For scalar integrals, $ds = \|\mathbf{r}'(t)\| dt$ and for vector integrals, $\mathbf{T}\, ds = d\mathbf{r} = \mathbf{r}'(t)\, dt$ and $\mathbf{N}\, ds = \mathbf{n}(t)\, dt$.

#### Line Segments

The line segment which goes from $\mathbf{a} = \langle a_x, a_y \rangle$ to $\mathbf{b} = \langle b_x, b_y \rangle$ is parameterized by:

- $\mathbf{r}(t) = \langle a_x + t(b_x - a_x), a_y + t(b_y - a_y) \rangle$, $0 \leq t \leq 1$,
- $\mathbf{r}'(t) = \langle b_x - a_x, b_y - a_y \rangle$,
- $\|\mathbf{r}'(t)\| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2}$ (which we note is always constant for any given segment!), and
- $\mathbf{n}(t) = \langle b_y - a_y, a_x - b_x \rangle$.

#### Circle

The circle of constant radius $r$ and whose center is at $(h, k)$ is parameterized by:

- $\mathbf{r}(t) = \langle h + r\cos(t), k + r\sin(t) \rangle$, $0 \leq t \leq 2\pi$,
- $\mathbf{r}'(t) = \langle -r\sin(t), r\cos(t) \rangle$,
- $\|\mathbf{r}'(t)\| = r$, and
- $\mathbf{n}(t) = \langle r\cos(t), r\sin(t) \rangle$.

#### Ellipse

The ellipse with center at $(h, k)$ and "radius" $a$ along the $x$-axis and $b$ along the $y$-axis is parameterized by:

- $\mathbf{r}(t) = \langle h + a\cos(t), k + b\sin(t) \rangle$, $0 \leq t \leq 2\pi$,
- $\mathbf{r}'(t) = \langle -a\sin(t), b\cos(t) \rangle$,
- $\|\mathbf{r}'(t)\| = \sqrt{a^2\sin^2(t) + b^2\cos^2(t)}$, and
- $\mathbf{n}(t) = \langle b\cos(t), a\sin(t) \rangle$.

## Regions

### Disk

The disk centered at $(h, k)$ with radius $a$ is parameterized by:

- $\mathbf{r}(t, \theta) = \langle h + r\cos(\theta), k + r\sin(\theta)\rangle$, $0 \leq r \leq b$, $0 \leq \theta \leq 2\pi$.

We then have

- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta)\rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta)\rangle$,

and thus

- $\|\mathbf{t}_r \times \mathbf{t}_\theta\| = r$,

where it is understood that we are doing the two-dimensional analog to the cross product. Note that this is not defined without its magnitude, since it would need to be a vector in the $k$ direction. Further, we see that $dS = dA = dx\,dy = r\,dr\,d\theta$, not too much of a surprise.

# Shapes in the $xyz$-plane, $\mathbb{R}^3$

## Curves

Much of the earlier material can be easily raised to three dimensions. However, note that our normal vector is missing. We can still define $\mathbf{N}(t)$ in many cases, but its main use in the previous section was flux. Flux is not well-defined over a single curve in $\mathbb{R}^3$, and so we will not consider it here. In general, we have:

- the parameterization, $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$,
- the tangent vector, $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$,
- the magnitude of the tangent vector, $\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$,
- for scalar integrals, $ds = \|\mathbf{r}'(t)\|\,dt$, and
- for vector integrals, $\mathbf{T}\,ds = d\mathbf{r} = \mathbf{r}'(t)\,dt$.

### Line Segments

The line segment which goes from $\mathbf{a} = \langle a_x, a_y, a_z\rangle$ to $\mathbf{b} = \langle b_x, b_y, b_z\rangle$ is parameterized by:

- $\mathbf{r}(t) = \langle a_x + t(b_x - a_x), a_y + t(b_y - a_y), a_z + t(b_z - a_z)\rangle$, for $0 \leq t \leq 1$,
- $\mathbf{r}'(t) = \langle (b_x - a_x), (b_y - a_y), (b_z - a_z)\rangle$, and
- $\|\mathbf{r}'(t)\| = \sqrt{(b_x - a_x)^2 + (b_y - a_y)^2 + (b_z - a_z)^2}$, which we note is always constant for any given segment!

### Circle

The circle with center $(h, k, \ell)$, constant radius $r$, and whose plane is parallel to the orthogonal unit vectors $\mathbf{a} = \langle a_x, a_y, a_z\rangle$ and $\mathbf{b} = \langle b_x, b_y, b_z\rangle$ is parameterized by:

- $\mathbf{r}(t) = \langle h + r(a_x\cos(t) + b_x\sin(t)), k + r(a_y\cos(t) + b_y\sin(t)), \ell + r(a_z\cos(t) + b_z\sin(t))\rangle$, $0 \leq t \leq 2\pi$,
- $\mathbf{r}'(t) = r\,\langle -a_x\sin(t) + b_x\cos(t), -a_y\sin(t) + b_y\cos(t), -a_z\sin(t) + b_z\cos(t)\rangle$, and
- $\|\mathbf{r}'(t)\| = r$, after some work!

### Helix

A circular helix with constant radius $r$ and $z$-coordinate at for constant $a$ is parameterized by:

- $\mathbf{r}(t) = \langle r\cos(t), r\sin(t), at\rangle$,
- $\mathbf{r}'(t) = \langle -r\sin(t), r\cos(t), a\rangle$, and
- $\|\mathbf{r}'(t)\| = \sqrt{r^2 + a^2}$, which is constant for any given helix.

## Surfaces

We finally turn to the truly new idea, that of a surface $S \subseteq \mathbb{R}^3$. As these are two dimensional, we will parameterize with two independent variables, $u$ and $v$:

- $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$.

We then have

- $\mathbf{t}_u = \left\langle \dfrac{\partial x}{\partial u}, \dfrac{\partial y}{\partial u}, \dfrac{\partial z}{\partial u} \right\rangle$,
- $\mathbf{t}_v = \left\langle \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial v} \right\rangle$,

and thus,

- $dS = \|\mathbf{t}_u \times \mathbf{t}_v\|\, du\, dv$, and
- $\mathbf{N}\, dS = dS = \mathbf{t}_u \times \mathbf{t}_v\, du\, dv$.

where we have an orientation of the surface. We can choose to use $\mathbf{t}_v \times \mathbf{t}_u$ to flip the orientation. When the surface of interest is closed (i.e., is the boundary of a solid) we will always want the normal to point "outward." When the surface is $z = f(x, y)$, we typically want the normal to point "up." In other cases, the orientation will be given.

### Scalar Surface

Consider the scalar surface defined by $z = f(x, y)$. We can use $x$ and $y$ themselves as the parameters to find

- $\mathbf{r}(x, y) = \langle x, y, f(x, y)\rangle$,
- $\mathbf{t}_x = \langle 1, 0, f_x(x, y)\rangle$,
- $\mathbf{t}_y = \langle 0, 1, f_y(x, y)\rangle$,
- $\mathbf{t}_x \times \mathbf{t}_y = \langle -f_x(x, y), -f_y(x, y), 1\rangle$, which points upward, since the $z$-component is $+1$, and
- $\|\mathbf{t}_x \times \mathbf{t}_y\| = \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2}$.

### Scalar Surface – Polar

We can also write the scalar surface $z = f(x, y)$ where we think of $x = r\cos(\theta)$ and $y = r\sin(\theta)$. This is useful if the projection of the surface into the $xy$-plane is easier to think of in polar coordinates:

- $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), f(r, \theta)\rangle$,
- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta), f_r(r, \theta)(r, \theta)\rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta), f_\theta(r, \theta)(r, \theta)\rangle$,
- $\mathbf{t}_r \times \mathbf{t}_\theta = \langle \sin(\theta) f_\theta(r, \theta) - r\cos(\theta) f_r(r, \theta), -\cos(\theta) f_\theta(r, \theta) - r\sin(\theta) f_r(r, \theta), r\rangle$, which points upward, since the $z$-component is $+r$, and
- $\|\mathbf{t}'(r)\| \times \mathbf{t}_\theta = \sqrt{(f_\theta(r, \theta))^2 + r^2 + r^2(f_r(r, \theta))^2}$.

**Plane – Cartesian**

Consider the plane $z = ax + by + d$, which is parameterized with variables $x$ and $y$:

- $\mathbf{r}(x, y) = \langle x, y, ax + by + d \rangle$,
- $\mathbf{t}_x = \langle 1, 0, a \rangle$,
- $\mathbf{t}_y = \langle 0, 1, b \rangle$,
- $\mathbf{t}_x \times \mathbf{t}_y = \langle -a, -b, 1 \rangle$, which points upward, since the $z$-component is $+1$, and
- $\|\mathbf{t}_x \times \mathbf{t}_y\| = \sqrt{1 + a^2 + b^2}$.

**Plane – Polar**

Consider the plane $z = ax + by + d$ where the projection of the region of interest is more easily expressed in polar coordinates:

- $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), ar\cos(\theta) + br\sin(\theta) + d \rangle$,
- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta), a\cos(\theta) + b\sin(\theta) \rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta), -ar\sin(\theta) + br\cos(\theta) \rangle$,
- $\mathbf{t}_r \times \mathbf{t}_\theta = \langle -ar, -br, r \rangle$, which points upward, since the $z$-component is $+1$, and
- $\|\mathbf{t}_r \times \mathbf{t}_\theta\| = r\sqrt{1 + a^2 + b^2}$.

**Cylinder**

A cylinder, with constant radius $r$ and axis matching the $z$-axis, is parameterized by $\theta$ and $z$ as:

- $\mathbf{r}(\theta, z) = \langle r\cos(\theta), r\sin(\theta), z \rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle$,
- $\mathbf{t}_z = \langle 0, 0, 1 \rangle$,
- $\mathbf{t}_\theta \times \mathbf{t}_z = \langle r\cos(\theta), r\sin(\theta), 0 \rangle$, which points outward, and
- $\|\mathbf{t}_\theta \times \mathbf{t}_z\| = r$, also hopefully not a surprise.

**Cone – Polar**

The cone, with axis aligned with the $z$-axis and "slope" a constant $a$ is parameterized by $r$ and $\theta$ so that:

- $\mathbf{r}(r, \theta) = \langle r\cos(\theta), r\sin(\theta), ar \rangle$,
- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta), a \rangle$,
- $\mathbf{t}_\theta = \langle -r\sin(\theta), r\cos(\theta), 0 \rangle$,
- $\mathbf{t}_\theta \times \mathbf{t}_r = \langle ar\cos(\theta), ar\sin(\theta), -r \rangle$, where we need $\mathbf{t}_\theta \times \mathbf{t}_r$, so our normal points out of the cone, and
- $\|\mathbf{t}_\theta \times \mathbf{t}_r\| = r\sqrt{a^2 + 1}$.

### Cone – Spherical

The cone, with axis aligned with the $z$-axis and "slope" given by the constant polar angle $\phi$ is parameterized by $\rho$ and $\theta$ so that:

- $\mathbf{r}(\rho, \theta) = \langle \rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi) \rangle$,
- $\mathbf{t}_\rho = \langle \sin(\phi) \cos(\theta), \sin(\phi) \sin(\theta), \cos(\phi) \rangle$,
- $\mathbf{t}_\theta = \langle -\rho \sin(\phi) \sin(\theta), \rho \sin(\phi) \cos(\theta), 0 \rangle$, and
- $\mathbf{t}_\theta \times \mathbf{t}_\rho = \left\langle \rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), -\rho \sin^2(\phi) \right\rangle$, where we need to flip around the order so our normal points out of the cone.
- $\|\mathbf{t}_\theta \times t_\rho\| = \rho \sin(\phi)$

### Paraboloid – Cartesian

This is a special case of the general surface where $z = a(x^2 + y^2)$; this is useful if we have simple Cartesian bounds for $x$ and $y$:

- $\mathbf{r}(x, y) = \left\langle x, y, a(x^2 + y^2) \right\rangle$,
- $\mathbf{t}_x = \langle 1, 0, 2ax \rangle$,
- $\mathbf{t}_y = \langle 0, 1, 2ay \rangle$,
- $\mathbf{t}_x \times \mathbf{t}_y = \langle -2ax, -2ay, 1 \rangle$, and
- $\|\mathbf{t}_x \times t_y\| = \sqrt{1 + 4a^2x^2 + 4a^2y^2}$.

### Paraboloid – Polar

This is the special case of the general polar surface where $z = ar^2$; this version is more useful if our projection down into the $xy$-plane is circular:

- $\mathbf{r}(r, \theta) = \left\langle r \cos(\theta), r \sin(\theta), ar^2 \right\rangle$,
- $\mathbf{t}_r = \langle \cos(\theta), \sin(\theta), 2ar \rangle$,
- $\mathbf{t}_\theta = \langle -r \sin(\theta), r \cos(\theta), 0 \rangle$,
- $\mathbf{t}_\theta \times t_r = \left\langle 2ar^2 \cos(\theta), 2ar^2 \sin(\theta), -r \right\rangle$, where we again flip around the order so the normal points outward, and
- $\|\mathbf{t}_\theta \times t_r\| = r\sqrt{1 + 4a^2r^2}$.

### Sphere

The sphere of constant radius $\rho$ centered at the origin is parameterized by $\theta$ and $\phi$, where $\theta$ is the planar angle and $\phi$ is the polar angle:

- $\mathbf{r}(\theta, \phi) = \langle \rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi) \rangle$,
- $\mathbf{t}_\phi = \langle \rho \cos(\phi) \cos(\theta), \rho \cos(\phi) \sin(\theta), -\rho \sin(\phi) \rangle$,
- $\mathbf{t}_\theta = \langle -\rho \sin(\phi) \sin(\theta), \rho \sin(\phi) \cos(\theta), 0 \rangle$,
- $\mathbf{t}_\phi \times t_\theta = \left\langle \rho^2 \sin^2(\phi) \cos(\theta), \rho^2 \sin^2(\phi) \sin(\theta), \rho^2 \sin(\phi) \cos(\phi) \right\rangle$, where we again flip around the order so the normal points outward, and
- $\|\mathbf{t}_\phi \times t_\theta\| = \rho^2 \sin(\phi)$, as expected.

# Integrals

## Curves and Regions in 2-D

Let $C$ be a curve in $\mathbb{R}^2$ which is not necessarily closed, parameterized by $\mathbf{r}(t)$, $a \leq t \leq b$.

- The arc length of $C$ is given by:
$$s = \int_C ds = \int_a^b \|\mathbf{r}'(t)\| \, dt.$$
- If $f(x, y)$ is a continuous scalar function, then the signed area of the curtain from $z = f(x, y)$ back down to the $xy$-axis which lies above $C$ is given by:
$$\int_C f(x(t), y(t)) \, ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt.$$
- If $\mathbf{F}$ is a vector field, the circulation along $C$ is given by:
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \, dt.$$
  - In the special case where $C$ is closed, by Green's Theorem
$$\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA,$$
    where $D$ is the region enclosed by $C$.
- If $\mathbf{F}$ is a vector field, the flux across $C$ is given by:
$$\int_C \mathbf{F} \cdot \mathbf{N} \, ds = \int_C \mathbf{F} \cdot \mathbf{n}(t) \, dt.$$
  - In the special case where $C$ is closed, by Green's Theorem
$$\oint_C \mathbf{F} \cdot \mathbf{N} \, ds = \iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dA,$$
    where $D$ is the region enclosed by $C$.

## Curves in 3-D

Now, let $C$ be a curve in $\mathbb{R}^3$, again not necessarily closed, parameterized by $\mathbf{r}(t)$, $a \leq t \leq b$.

- The arc length of $C$ is given by:
$$s = \int_C ds = \int_a^b \|\mathbf{r}'(t)\| \, dt.$$
- If $f(x, y, z)$ is a continuous scalar function, then the following integral calculates the average value of $f$ along $C$, multiplied by the arc length of $C$:
$$\int_C f(x(t), y(t), z(t)) \, ds = \int_a^b f(\mathbf{r}(t)) \|\mathbf{r}'(t)\| \, dt.$$
- If $\mathbf{F}$ is a vector field, the circulation along $C$ is given by:
$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T} \, ds = \int_a^b \mathbf{F} \cdot \mathbf{r}'(t) \, dt.$$
  - In the special case where $C$ is closed, by Stokes' Theorem:
$$\oint_C \mathbf{F} \cdot \mathbf{T} \, ds = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S},$$
    where $S$ is any positively oriented surface with $C$ as its boundary.

(We will not worry about flux over a curve in $\mathbb{R}^3$.)

## Surfaces in 3-D

Let $S$ be a surface in $\mathbb{R}^3$, not necessarily closed, parameterized by $\mathbf{r}(u, v)$.

- If $f(x, y, z)$ is a scalar function, we can find the average value of $f$ over $S$ multiplied by the area of $S$ by finding:

$$\iint_S f(x, y, z)\|\mathbf{t}'(u)\| \times \mathbf{t}_v \, dS.$$

  - For example, if $f$ is the density of some object, this integral finds the mass, that is the average density times the area.

- If $\mathbf{F}$ is a vector field, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{N} \, dS = \iint_S \mathbf{F} \cdot (\mathbf{t}_u \times \mathbf{t}_v) \, du \, dv$$

  measures the flux of $\mathbf{F}$ through $S$ – that is the amount of $\mathbf{F}$ which passes orthogonally through $S$, where positive flux matches the direction of the normal.

  - If $\mathbf{G} = \nabla \times \mathbf{F}$, then

$$\iint_S \mathbf{G} \cdot d\mathbf{S} = \iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

    measures the flux of the curl of $\mathbf{F}$ across $S$ – that is the total amount of circulation (or torque) that $\mathbf{F}$ induces in $S$.

  - by Stokes' Theorem, if $C$ is the positively oriented boundary curve of $S$:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_C \mathbf{F} \cdot \mathbf{T} \, ds.$$

- In the special case when $S$ is closed with positive orientation – that is, $S$ encloses a solid region $E \subseteq \mathbb{R}^3$ with outward pointing normal – then $\iint_S \mathbf{F} \cdot d\mathbf{S}$ measures the total flux of $\mathbf{F}$ out of $E$. by the Divergence Theorem,

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\nabla \cdot \mathbf{F}) \, dV,$$

  since the second integral adds up all the "outflow" of $\mathbf{F}$ at each point inside the solid region $E$.

  - In this special case when $S$ is closed, it does not have a boundary curve, and therefore the total circulation, or total flux of the curl of any vector field $\mathbf{F}$ across $S$ is therefore 0. We can also see this since we know that for any vector field, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.