Probability and Statistics: Practice Set 3

Paul Beggs

September 26, 2025

1. (2 points each) Consider the function $f(x) = cx^5$ on $[0, 1]$.

   (a) Find the value of c for which this is a probability density function.

   Solution.

   $$\int_0^1 cx^5\,dx = 1 \implies c\int_0^1 x^5\,dx = 1 \implies c\left[\frac{x^6}{6}\right]_0^1 = 1 \implies c\cdot\frac{1}{6} = 1 \implies c = 6.$$

   (b) Using this value of c, determine $E(X)$.

   Solution.

   $$E(X) = \int_0^1 x\cdot 6x^5\,dx = 6\left[\frac{x^7}{7}\right]_0^1 = \frac{6}{7}.$$

   (c) Again, using this c, determine $\mathrm{Var}(X)$.

   Solution.

   $$\begin{aligned}\mathrm{Var}(X) &= \int_0^1 \left(x - \frac{6}{7}\right)^2 6x^5\,dx\\ &= \int_0^1 \left(x^2 - \frac{12}{7}x + \frac{36}{49}\right) 6x^5\,dx\\ &= \int_0^1 6x^7 - \frac{72}{7}x^6 + \frac{216}{49}x^5\,dx\\ &= \frac{6}{8} - \frac{72}{49} + \frac{36}{49}\\ &= \frac{3}{196}.\end{aligned}$$

   (d) Find $\pi_{0.25}$.

   Solution.

   $$\int_0^b 6x^5\,dx = 0.25 \implies 6\left[\frac{x^6}{6}\right]_0^b = 0.25 \implies b^6 = 0.25 \implies b = 0.25^{\frac{1}{6}} \implies b \approx 0.7937.$$

   Therefore, $\pi_{0.25} \approx 0.7937$.

2. (2 points) Consider the function $f(x) = \dfrac{\ln(x)}{x^2}$ on the interval $[1, \infty)$.

(a) Show that this is a probability distribution. [Hint: You will likely need integration by parts here.]

Solution. We choose $u = \ln(x)$ so that $du = \frac{1}{x}$, and $dv = \frac{1}{x^2}$ so that $v = -\frac{1}{x}$. This leaves us with the following:

$$\begin{aligned}
[uv]_1^\infty - \int_1^\infty v du &= \left[-\frac{\ln(x)}{x}\right]_1^\infty + \int_1^\infty \frac{1}{x^2}\,dx \\
&= \left(\frac{\ln(1)}{1} - \lim_{x\to\infty}\frac{\ln(x)}{x}\right) - \left(\lim_{x\to\infty}\frac{1}{x} - \frac{1}{1}\right) \\
&= (0-0) - (0-1) \\
&= 1
\end{aligned}$$

Since $f(x) \geq 0$ for all $x \in [1, \infty)$ and the integral over this interval is 1, this is a valid probability density function.

(b) Show that $E(X)$ is undefined.

Solution.

$$E(X) = \int_1^\infty x \cdot \frac{\ln(x)}{x^2}\,dx = \int_1^\infty \frac{\ln(x)}{x}\,dx.$$

For this problem, we can use u-substitution. Let $u = \ln(x)$, then $du = \frac{1}{x}\,dx$, which transforms the limits from 1 to ∞ into 0 to ∞:

$$E(X) = \int_0^\infty u\,du = \left[\frac{u^2}{2}\right]_0^\infty = \infty.$$

Therefore, $E(X)$ is undefined.

3. (2 points) A pdf is given by $f(x) = \dfrac{1}{2}$ for $x \in [0,1] \cup [2,3]$, and 0 otherwise. [i.e., this is like a uniform distribution, but with a "gap" from 1 to 2.] Determine $E(X)$ and $\mathrm{Var}(X)$.

Solution.

4. (2 points each) We have shown in class that the gamma function Γ has the two properties:

- $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$.
- $\Gamma(x) = (x-1)\Gamma(x-1)$.

Use these to find the exact values of:

(a) $\Gamma\left(\dfrac{5}{2}\right)$.

(b) $\Gamma\left(-\dfrac{7}{2}\right)$.

5. (2 points each) Customers arrive at a certain bank according to an approximate Poisson process at a mean rate of 15 per hour.

   (a) What is the probability that between 10 and 13 customers come in a particular hour?

   Solution.

   (b) What is the probability that the first customer comes between 5 and 10 minutes into the bank's opening?

   Solution.

   (c) What is the probability that the *third* customer arrives between 5 and 10 minutes into the bank's opening?

   Solution.

6. (1 point each) The mean airspeed of an unladen swallow is 30 ft/s, with a standard deviation of 4.3 ft/s. Assuming the distribution of speed is normal:

   (a) What is the probability that a randomly selected swallow will have a speed between 28 and 38 ft/s?

   Solution.

   (b) Suppose that 12 swallows are selected at random. What is the probability that exactly 8 have their speed between 28 and 38 ft/s?

   Solution.

   (c) Determine $\pi_{0.90}$ for the swallow speed.

   Solution.