Real Analysis: Exam 2 Corrections

Paul Beggs

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(1) Let $x_1 = 6$, and for all $n \in \mathbb{N}$, $x_{n+1} = 2 + \sqrt{x_n - 2}$. Show that (x_n) converges and find its limit.

Solution. **Step 1: Show that (x_n) is bounded and decreasing.**

Notice that if $x_n = 3$, then,

$$x_{n+1} = 2 + \sqrt{3 - 2} = 2 + 1 = 3.$$

This indicates that if x_n ever reaches 3, then $x_{n+1} = 3$. Thus, making it a fixed point. We conjecture that this is the lower bound for the sequence.

- **Base Case:** $x_1 = 6 > 3$.
- **Inductive Step:** Assume $x_n > 3$ for some $n \in \mathbb{N}$. Since $x_n > 3$, we have $x_n - 2 > 1$, which means $\sqrt{x_n - 2} > 1$. Thus,

  $$x_{n+1} = 2 + \sqrt{x_n - 2} > 2 + 1 = 3.$$

  This shows that $x_n > 3$ for all n.

Now we will show that $x_{n+1} < x_n$ when $x_n > 3$, which will prove that the sequence is decreasing.

Our goal is to prove

$$2 + \sqrt{x_n - 2} < x_n.$$

Rearranging this inequality, we get

$$\sqrt{x_n - 2} < x_n - 2.$$

Since $x_n > 3$, we know $x_n - 2 > 1$, and for numbers greater than 1, it holds that $\sqrt{x_n - 2} < x_n - 2$. Therefore, $x_{n+1} < x_n$, and the sequence is decreasing when $x_n > 3$. Thus, the sequence is bounded and decreasing.

Step 2: Conclude that (x_n) converges.

A monotonic sequence that is bounded converges by the Monotone Convergence Theorem. Therefore, (x_n) converges to some limit $L \geq 3$.

Step 3: Find the limit L.

Taking the limit on both sides of the recursion and solving for L:

$$\begin{aligned}
\lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \left(2 + \sqrt{x_n - 2}\right) \\
L &= 2 + \sqrt{L-2} \\
L - 2 &= \sqrt{L-2} \\
(L-2)^2 &= L - 2 \\
L^2 - 4L + 4 &= L - 2 \\
L^2 - 5L + 6 &= 0 \\
(L-2)(L-3) &= 0.
\end{aligned}$$

Therefore, $L = 2$ or $L = 3$. Since $L \geq 3$, the limit is $L = 3$.

The sequence (x_n) converges to 3.

(3) Let $K \subset \mathbb{R}$ be a nonempty compact set, and let $p \in K^c$. Define

$$d = \inf\{|x - p| \mid x \in K\}.$$

(a) Show that there exists a sequence (x_n) in K such that $\lim\limits_{n\to\infty} |x_n - p| = d$.

(b) Show there exists a point x_0 in K such that $|x_0 - p| = d$.
We think of x_0 as the closest point in K to p.

Solution.

(a) By the definition of the infimum, we know for all $\epsilon > 0$, there exists an $x \in K$ such that

$$d \leq |x - p| < d + \epsilon.$$

Use $\epsilon = \frac{1}{n}$. Then, for each $n \in \mathbb{N}$ there exists $x_n \in K$ such that

$$d \leq |x_n - p| < d + \frac{1}{n}.$$

This implies

$$||x_n - p| - d| < \frac{1}{n}.$$

Therefore,

$$\lim_{n\to\infty} |x_n - p| = d.$$

Hence, there exists a sequence (x_n) in K such that $\lim\limits_{n\to\infty} |x_n - p| = d$.

(b) Since K is compact and (x_n) is in K, by the Heine–Borel theorem, there exists a subsequence (x_{n_k}) that converges to some point $x_0 \in K$.
We have

$$\lim_{k\to\infty} |x_{n_k} - p| = d.$$

Using the triangle inequality,

$$||x_0 - p| - |x_{n_k} - p|| \leq |x_{n_k} - x_0|.$$

Taking the limit as $k \to \infty$,

$$||x_0 - p| - d| \leq 0.$$

Thus,

$$|x_0 - p| = d.$$

Therefore, there exists $x_0 \in K$ such that $|x_0 - p| = d$.