Multivariable Calculus Exam 1

Basic Derivatives

$$\frac{d}{dx}e^{f(x)} = \qquad f'(x)e^{f(x)}$$

$$\frac{d}{dx}\sin f(x) = \qquad \cos f(x) \cdot f'(x)$$

$$\frac{d}{dx}\cos f(x) = \qquad -\sin f(x) \cdot f'(x)$$

$$\frac{d}{dx}\tan f(x) = \qquad \sec^2 f(x) \cdot f'(x)$$

$$\frac{d}{dx}\cot f(x) = \qquad -\csc^2 f(x) \cdot f'(x)$$

$$\frac{d}{dx}\sec f(x) = \qquad \sec f(x)\tan f(x) \cdot f'(x)$$

$$\frac{d}{dx}\csc f(x) = \qquad -\csc f(x)\cot f(x) \cdot f'(x)$$

$$\frac{d}{dx}\ln f(x) = \qquad \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\log_a f(x) = \qquad \frac{f'(x)}{f(x)\ln a}$$

$$\frac{d}{dx}(f(x))^n = \qquad n\,(f(x))^{n-1} f'(x)$$

$$\frac{d}{dx}\sqrt{f(x)} = \qquad \frac{f'(x)}{2\sqrt{f(x)}}$$

$$\frac{d}{dx}a^x = \qquad a^x \ln a$$

$$\frac{d}{dx}b^{g(x)} = \qquad b^{g(x)} \ln b \cdot g'(x)$$

Product and Quotient

$$\frac{d}{dx}[u \cdot v] = \qquad u' \cdot v + u \cdot v'$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \qquad \frac{u' \cdot v - u \cdot v'}{v^2}$$

Inverse Trigonometric

$$\frac{d}{dx}\arcsin f(x) = \qquad \frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

$$\frac{d}{dx}\arccos f(x) = \qquad -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

$$\frac{d}{dx}\arctan f(x) = \qquad \frac{f'(x)}{1+(f(x))^2}$$

$$\frac{d}{dx}\text{arccot}\, f(x) = \qquad -\frac{f'(x)}{1+(f(x))^2}$$

$$\frac{d}{dx}\text{arcsec}\, f(x) = \qquad \frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$$

$$\frac{d}{dx}\text{arccsc}\, f(x) = \qquad -\frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$$

Chain Rule

$$\frac{d}{dx}f(g(x)) = \qquad f'(g(x)) \cdot g'(x)$$

Higher-Order Derivatives

$$\frac{d^2}{dx^2}e^x = \qquad e^x$$

$$\frac{d^3}{dx^3}\sin x = \qquad -\cos x$$

$$\frac{d^4}{dx^4}\cos x = \qquad \cos x$$

Trigonometric Integrals

$$\int \sin x\,dx = \qquad -\cos x + C$$

$$\int \cos x\,dx = \qquad \sin x + C$$

$$\int \sin^2 x = \qquad \frac{1}{2}(x - \sin x \cos x) + C$$

$$\int \cos^2 x = \qquad \frac{1}{2}(x + \sin x \cos x) + C$$

$$\int \tan x\,dx = \qquad -\ln|\cos x| + C$$

$$\int \cot x\,dx = \qquad \ln|\sin x| + C$$

$$\int \sec x\,dx = \qquad \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \qquad -\ln|\csc x + \cot x| + C$$

$$\int \sec^2 x\,dx = \qquad \tan x + C$$

$$\int \csc^2 x\,dx = \qquad -\cot x + C$$

$$\int \sec x \tan x\,dx = \qquad \sec x + C$$

$$\int \csc x \cot x\,dx = \qquad -\csc x + C$$

Reduction Formulas for Sine and Cosine

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

Inverse Trigonometric Integrals

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \qquad \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \qquad \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \qquad \frac{1}{a}\text{arcsec}\left(\frac{x}{a}\right) + C$$

Regular Integrals and e

$$\int x^n\,dx = \qquad \frac{1}{n+1}x^{n+1} + C$$

$$\int \frac{1}{x}\,dx = \qquad \ln|x| + C$$

$$\int e^x\,dx = \qquad e^x + C$$

$$\int e^{ax}\,dx = \qquad \frac{1}{a}e^{ax} + C$$

$$\int e^{f(x)} f'(x)\,dx = \qquad e^{f(x)} + C$$

Exponetials

$$\int_0^b e^x\,dx = e^b - 1$$

$$\int_0^b e^{-x}\,dx = 1 - e^{-b}$$

$$\int_0^\infty e^{-x}\,dx = 1$$

$$\int_0^\infty x^n e^{-x}\,dx = n!$$

n	$\int_0^{\pi/2} \sin^n x\,dx$	$\int_0^{\pi/2} \cos^n x\,dx$	$\int_0^{\pi} \sin^n x\,dx$	$\int_0^{\pi} \cos^n x\,dx$	$\int_0^{2\pi} \sin^n x\,dx$	$\int_0^{2\pi} \cos^n x\,dx$
1	1	1	2	0	0	0
2	$\pi/4$	$\pi/4$	$\pi/2$	$\pi/2$	π	π
3	2/3	2/3	4/3	0	0	0
4	$3\pi/16$	$3\pi/16$	$3\pi/8$	$3\pi/8$	$3\pi/4$	$3\pi/4$
5	8/15	8/15	16/15	0	0	0
6	$5\pi/32$	$5\pi/32$	$5\pi/16$	$5\pi/16$	$5\pi/8$	$5\pi/8$

Radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	$-$
π	0	-1	0
$3\pi/2$	-1	0	$-$

Trigonometric Identities

Pythagorean

$$\sin^2 \theta + \cos^2 \theta = 1$$
$$\tan^2 \theta + 1 = \sec^2 \theta$$
$$1 + \cot^2 \theta = \csc^2 \theta$$

Half Angle

$$\sin^2 \left(\frac{x}{2}\right) = \frac{1 - \cos x}{2}$$
$$\cos^2 \left(\frac{x}{2}\right) = \frac{1 + \cos x}{2}$$
$$\tan^2 \left(\frac{x}{2}\right) = \frac{1 - \cos x}{1 + \cos x}$$

Double Angle

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = 2 \cos^2 x - 1$$
$$\cos 2x = 1 - 2 \sin^2 x$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Product to Sum

$$\sin x \sin y = \frac{1}{2} \left[\cos(x - y) - \cos(x + y)\right]$$
$$\cos x \cos y = \frac{1}{2} \left[\cos(x - y) + \cos(x + y)\right]$$
$$\sin x \cos y = \frac{1}{2} \left[\sin(x + y) + \sin(x - y)\right]$$
$$\cos x \sin y = \frac{1}{2} \left[\sin(x + y) - \sin(x - y)\right]$$

Sum to Product

$$\sin x + \sin y = 2 \sin \left(\frac{x + y}{2}\right) \cos \left(\frac{x - y}{2}\right)$$
$$\sin x - \sin y = 2 \cos \left(\frac{x + y}{2}\right) \sin \left(\frac{x - y}{2}\right)$$
$$\cos x + \cos y = 2 \cos \left(\frac{x + y}{2}\right) \cos \left(\frac{x - y}{2}\right)$$
$$\cos x - \cos y = -2 \sin \left(\frac{x + y}{2}\right) \sin \left(\frac{x - y}{2}\right)$$

Chapter 1: Parametric Equations and Polar Coordinates

- **Slope:** $\left.\frac{dy}{dx}\right|_{t=t_0} = \left.\frac{dy/dt}{dx/dt}\right|_{t=t_0}$.

  The **tangent line** at t_0 is given by

  $$y = \left(\left.\frac{dy}{dx}\right|_{t=t_0}\right) (x - x(t_0)) + y(t_0).$$

- **Concavity:** $\left.\frac{d^2 y}{dx^2}\right|_{t=t_0} = \left.\frac{\frac{d}{dt}(dy/dx)}{dx/dt}\right|_{t=t_0}$

- **Area Under a Curve:** $\int_{t_a}^{t_b} y(t) \frac{dx}{dt} \, dt$.

- **Arc Length:** $\int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$.

- **Surface Area:** $\int_{t_a}^{t_b} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$.

Chapter 2: Vectors in Space

- ***Direction:*** $P = (x_1, y_1)$ and $Q = (x_2, y_2)$: $\mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

- ***Vector Sum:*** $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.

- ***Magnitude:*** $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2} = \sqrt{u \cdot u}$.

- ***Dot Product:*** $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$.

  - ***Angle:*** $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)$, where $0 \leq \theta \leq \pi$ is between $\mathbf{u}$ & $\mathbf{v}$.
  - ***Self-Product:*** $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.
  - ***Work:*** $W = \mathbf{F} \cdot \mathbf{PQ} = (\|\mathbf{F}\|) \|\mathbf{PQ}\| \cos(\theta)$.

- To ***Normalize*** a vector, divide it by its magnitude $\mathbf{v} = \langle x, y, z \rangle$, then $\mathbf{u} = \frac{1}{\|\mathbf{v}\|} \mathbf{v} = \left\langle \frac{x}{\|\mathbf{v}\|}, \frac{y}{\|\mathbf{v}\|}, \frac{z}{\|\mathbf{v}\|} \right\rangle$. $\therefore \mathbf{u} :=$ ***Unit Vector*** in direction of $\mathbf{v}$.

- ***Projection:*** $\text{proj}_{\mathbf{b}} \mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\right) \mathbf{b}$.

- ***Cross product:*** $\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2, \; u_3 v_1 - u_1 v_3, \; u_1 v_2 - u_2 v_1 \rangle$.

  - ***Angle:*** $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$, where $0 \leq \theta \leq \pi$ is between $\mathbf{u}$ & $\mathbf{v}$.
  - ***Torque:*** $\tau = \mathbf{r} \times \mathbf{F}$ or $\|\tau\| = \|\mathbf{r}\| \|\mathbf{F}\| \sin(\theta)$

Parametric Equations Revisted

- To ***Parameterize*** an equation such as $y = x^3 - 4x + 1$ we can let $x = t$ and $y = t^3 - 4t + 1$. This allows us to write the equation as $\mathbf{r}(t) = \langle t, t^3 - 4t + 1 \rangle$.

- ***Vector Equation:*** $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$.

- ***Parametric Equation:*** $x = x_0 + at, \; y = y_0 + bt, \; z = z_0 + ct$.

- ***Symmetric Equation:*** $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$.

- The ***Line Segment*** from P to Q: $\mathbf{r}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$ (where $\mathbf{p}, \mathbf{q}$ are the vector forms of P, Q and $0 \leq t \leq 1$).

- ***Shortest Distance:*** $d = \frac{\|\mathbf{PM} \times \mathbf{v}\|}{\|\mathbf{v}\|}$.

  - ***Equal***: Same direction vector, share a point.
  - ***Parallel***: Same direction vector, do not share a point.
  - ***Intersecting***: Different direction vectors, share a point.
  - ***Skew***: Different direction vectors, do not share a point.

- If (x_0, y_0, z_0) is a point on a plane, the ***Scalar Equation*** would be: $\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0 \implies a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Chapter 3: Vector-Valued Functions

If each of $f_1, f_2, \ldots, f_n \colon \mathbb{R} \to \mathbb{R}$ is a function we can then define the **vector-valued function** $\mathbf{r} \colon \mathbb{R} \to \mathbb{R}^n$ by $\mathbf{r}(t) = \langle f_1(t), f_2(t), \ldots, f_n(t) \rangle$

- When $n = 2$, we might write $\mathbf{r} = \langle f(t), g(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath}$,

- and when $n = 3$, we might write $\mathbf{r} = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$.

Note: Deriving and integrating vector-valued functions follow the same rules as regular derivates.

- **Principle unit tangent vector** $\mathbf{T}(t)$: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$.
    - This vector, of length 1, points in the tangent direction of the curve.

- **Unit Normal Vector N**: $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$.
    - This vector points in the direction the curve is turning.

- **Binormal Vector B**: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

Arc Length Parameterization

We can define the **arc length parameterization** of a curve C by:

- Define the arc length $s(t) = \int_0^t ||\mathbf{r}'(\tau)|| \, d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2} \, d\tau$. (Where $f(\tau), g(\tau)$ correspond to the x, y components of $\mathbf{r}(t)$).

- Solving, if possible, the resulting expression for t as a function of s.

- Rewriting $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so that the curve is written as a function of its length, from a given starting point.

Curvature

For all $\mathbf{r}$: $\kappa = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$; for $\mathbb{R}^3$: $\kappa = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$; if $y = f(x)$: $\kappa = \frac{|y''(x)|}{[1+(y'(x)^2)]^{3/2}}$

Motion

- **Velocity:** $\mathbf{v}(t) = \mathbf{r}'(t)$.

- **Speed:** $||\mathbf{v}(t)|| = ||\mathbf{r}'(t)||$.

- **Acceleration:** $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

The motion of an object – in 2-dimensions, typically, acted on only by gravity $\mathbf{F}_g = -mg\mathbf{j}$, where $g \approx 9.8$ m/s^2 and m is the mass of the object. By Newton's second law, $\mathbf{F} = m\mathbf{a}$, so we have $\mathbf{a}(t) = -g\mathbf{j}$. Thus, $\mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}_0$, where $\mathbf{v}_0$ is the initial velocity vector, and $\mathbf{s}(t) = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{v}_0 t + \mathbf{s}_0$, where $\mathbf{s}$ is the position, and $\mathbf{s}_0$ is the initial position vector. Often, we have an object starting at the origin (so $\mathbf{s}_0 = \mathbf{0}$) and fired at a velocity of v_0 at an angle θ above the horizon. Then, $\mathbf{s}(t) = v_0 t \cos(\theta)\mathbf{i} + \left(v_0 t \sin(\theta) - \frac{1}{2}gt^2\right)\mathbf{j}$.

Chapter 1 Examples

1. Consider the curve defined by the parametric equations $x(t) = \sin(2t)$, $y(t) = \cos(t)$, for $0 \le t \le 2\pi$.

(a) Find the equation of the tangent line to the curve at the point where $t = \pi/3$.
   Solution. Solve for $\frac{dy}{dt}$ and $\frac{dx}{dt}$:
   $y(t) = \cos t \Rightarrow \frac{dy}{dt} = -\sin t$.
   $x(t) = \sin 2t \Rightarrow \frac{dx}{dt} = 2\cos 2t$.
   Now, we have our slope, which we evaluate at $t = \pi/3$:

   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{2\cos 2t} \quad \Rightarrow \quad \frac{-\sin t}{2\cos 2t} = \frac{-\sqrt{3}/2}{-1} = \frac{\sqrt{3}}{2}.$$

   With our slope, we need the points:

   $$x(\pi/3) = \sin(2(\pi/3)) = \sqrt{3}/2 \quad \text{and} \quad y(\pi/3) = \cos(\pi/3) = 1/2.$$

   Putting it all together, we have $y = \frac{\sqrt{3}}{2}\left(x - \frac{\sqrt{3}}{2}\right) + \frac{1}{2}$. (Simplify.)

(b) Determine geometric area enclosed by the curve.
   Solution. Our equation has 4. We find 1 quadrant and multiply it by 4, we can get the total geometric area for the whole shape.

   $$4\int_0^{\pi/2} y(t)\frac{dx}{dt} \, dt = 4\int_0^{\pi/2} \cos t(2\cos 2t) \, dt$$
   $$= 8\int_0^{\pi/2} \cos t(1 - 2\sin^2 t) \, dt$$
   $$= 8\int_0^{\pi/2} \cos t - 2\cos t \sin^2 t \, dt$$
   $$= 8\left[\int_0^{\pi/2} \cos t \, dt - 2\int_0^{\pi/2} \cos t \sin^2 t \, dt\right].$$

   Thus, let $u = \sin t$ such that $\frac{du}{\cos t} = dt$. Hence,

   $$8\left[\int_0^{\pi/2} \cos t \, dt - 2\int_0^{\pi/2} \cos t \sin^2 t \, dt\right] = 8\left[\int_0^{\pi/2} \cos t \, dt - 2\int_0^{\pi/2} u^2 \, du\right]$$
   $$= 8\left[\sin t - \frac{2}{3}\sin^3 t\right]_0^{\pi/2}$$

2. Find angle between $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + \mathbf{j} - 7\mathbf{k}$.
   Solution. Find the magnitudes of $\mathbf{u}$ and $\mathbf{v}$. Then solve:

   $$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \, ||\mathbf{v}||} = \frac{13}{\sqrt{65}\sqrt{66}} \approx 0.198.$$

   Thus, $\theta \approx \cos^{-1}(0.198) \approx \boxed{1.371}$ radians.

3. Determine a parametric equation for the line *segment* that goes from the point $P = (6, 1, -2)$ to $Q = (-2, 0, 5)$.

   Solution. $x(t) = 6 - 8t$; $y(t) = 1 - t$ $z(t) = -2 + 7t$, for $0 \leq t \leq 1$.

4. Find a symmetric equation for the line which contains the points $R = (4, -6, 1)$ and $S = (1, 2, 3)$.

   Solution. $\frac{x-4}{-3} = \frac{y+6}{8} = \frac{z-1}{-2}$.

5. Find the general form of an equation of the plane which contain the three points $P = (3, 1, -4)$, $Q = (-2, 0, 5)$ and $R = (4, -6, 1)$.

   Solution. Let $\mathbf{PQ} = \langle -5, -1, 9 \rangle$ and $\mathbf{PR} = \langle 1, -7, 5 \rangle$. Thus, $\mathbf{n} = \mathbf{PQ} \times \mathbf{QR} = \langle 58, 34, 36 \rangle$. General equation:

$$58(x - 3) + 34(y - 1) + 36(z + 4) = 0 \quad \Rightarrow \quad 58x + 34y + 36z - 64 = 0.$$

6. Find an equation, in symmetric form, of the line of intersection between the planes $2x + y - z + 4 = 0$ and $x - y + 3z = 1$.

   Solution. Add the plane equations to eliminate y so that $3x + 2z = -3$. Thus, $x = -1 - \frac{2}{3}z$. Substitute this equation into the first equation to express y in terms of z, giving $y = -2 + \frac{7}{3}z$. Define z in terms of t. Choose parameter t as $t = -\frac{1}{3}z$. This gives $z = -3t$. When we substitute our value t back into the previous two equations, we see that the parametric equations for the line of intersection are $x = -1 + 2t$, $y = -2 - 7t$, and $z = -3t$. Therefore, the symmetric equations for the line are $\boxed{\dfrac{x+1}{2} = \dfrac{y+2}{-7} = \dfrac{z}{-3}}$.

1. Write, in scalar form, an equation of the plane which contains the point $(5, 2, 1)$ and the line given by $x + 2 = \dfrac{y}{4} = \dfrac{z-5}{2}$.

   Solution. We start by parametrizing the line with common parameter t: $x + 2 = t \Rightarrow x = t - 2$, $\frac{y}{4} = t \Rightarrow y = 4t$, and $\frac{z-5}{2} = t \Rightarrow z = 2t + 5$. This gives us the parametric form: $(x, y, z) = (-2, 0, 5) + t(1, 4, 2)$ Thus, the line passes through the point $(-2, 0, 5)$ and has the direction vector $\mathbf{v}_1 = \langle 1, 4, 2 \rangle$. Form a second vector $\mathbf{v}_2$ by taking the difference between the given point and a point on the line: $\mathbf{v}_2 = (5, 2, 1) - (-2, 0, 5) = \langle 7, 2, -4 \rangle$. With $\mathbf{v}_1$ and $\mathbf{v}_2$, we find the normal vector:

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 2 \\ 7 & 2 & -4 \end{vmatrix} = (-16-4)\mathbf{i} - (-4-14)\mathbf{j} + (2-28)\mathbf{k} = -20\mathbf{i} + 20\mathbf{j} - 26\mathbf{k}$$

Therefore, we find the scalar form to be:

$$\boxed{-20(x + 2) + 18y - 26(z - 5) = 0.}$$

2. Determine the arc length parametrization for the curve $\mathbf{r}(t) = 3e^t \sin(t)\mathbf{i} + 3e^t \cos(t)\mathbf{j}$, where you start from $t = 0$.

   Solution. Rewrite the arc length parametrization as: $s = \int_0^t \|\mathbf{r}'(\tau)\| \, d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2} \, d\tau$. Thus, $f'(\tau) = 3e^\tau (\sin(\tau) + \cos(\tau))$ and $g'(\tau) = 3e^\tau (\cos(\tau) - \sin(\tau))$. Thus, we have:

$$s = \int_0^t \sqrt{\left[3e^\tau (\sin(\tau) + \cos(\tau))\right]^2 + \left[3e^\tau (\cos(\tau) - \sin(\tau))\right]^2} \, d\tau$$

$$= \int_0^t \sqrt{9e^{2\tau} \left[2(\sin^2(\tau) + \cos^2(\tau)) + (2\sin(\tau)\cos(\tau) - 2\sin(\tau)\cos(\tau))\right]} \, d\tau$$

$$= \int_0^t \sqrt{9e^{2\tau} \cdot [2(1 + 0)]} = \int_0^t 3e^\tau \sqrt{2} \, d\tau = 3\sqrt{2} \int_0^t e^\tau \, d\tau = 3\sqrt{2}(e^t - 1).$$

With s, we know that $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so we need to find t in terms of s: $s = 3\sqrt{2}(e^t - 1) \Rightarrow e^t = \dfrac{s}{3\sqrt{2}} + 1 \Rightarrow t = \ln\left(\dfrac{s}{3\sqrt{2}} + 1\right)$. Finally, by replacing t with $t(s)$ in the original equation, we can get the arc length parametrization:

$$\boxed{\mathbf{r}(s) = \left(\frac{s}{\sqrt{2}} + 3\right) \sin\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right) \mathbf{i} + \left(\frac{s}{\sqrt{2}} + 3\right) \cos\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right) \mathbf{j}.}$$

3. Use curvature to find the equation of the osculating circle at the planar curve $y = x^3 - 4x + 1$ at $x = 1$.

   Solution. First, we need to find the curvature of the curve at $x = 1$. We start by finding the first and second derivatives of the function: $y(x) = x^3 - 4x + 1 \Rightarrow y'(x) = 3x^2 - 4 \Rightarrow y''(x) = 6x$.

   Then, we evaluate the point and the first and second derivatives at $x = 1$: $y(1) = -2$; $y'(1) = -1$; $y''(1) = 6$.

   Find the curvature: $\kappa = \dfrac{|y''(x)|}{\left(1 + y'(x)^2\right)^{3/2}} = \dfrac{6}{\left(1 + (-1)^2\right)^{3/2}} = \dfrac{3\sqrt{2}}{2}$.

   Find radius: $R = \dfrac{1}{\kappa} = \dfrac{1}{\frac{3\sqrt{2}}{2}} = \dfrac{2\sqrt{2}}{6} = \dfrac{\sqrt{2}}{3}$.

   For the center, find $\mathbf{N}$ at $x = 1$: $\mathbf{N} = \dfrac{(-y', 1)}{\sqrt{1 + (y')^2}} = \dfrac{(-(-1), 1)}{\sqrt{1 + (-1)^2}} = \dfrac{(1, 1)}{\sqrt{2}}$.

   The center C can be found by moving our point $P(1, -2)$ the distance R along the unit normal vector: $C = P + R\mathbf{N} = \left(\frac{4}{3}, -\frac{5}{3}\right)$. This gives the equation for the osculating circle:

$$\boxed{\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{5}{3}\right)^2 = \frac{2}{9}.}$$