Real Analysis Exam 1

1.2 Some Preliminaries

Infinite Unions and Intersections:

Note: For the following, we define $A_n = \{n, n+1, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$. In other words, each subsequent element in the subset will start at n. For example, $A_1 = \{1, 2, \dots\}$, whereas $A_5 = \{5, 6, \dots\}$.

Union: $\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$. To show a number $\in \mathbb{N}$ belongs in the set A_n, we can start with that, $k \in \mathbb{N}$. Then $k \in A_k$. Thus, $k \in A_k \subseteq \bigcup_{n=1}^{\infty} A_n$. therefore, $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$.

Inters.: $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Obviously, we know that the empty set is a subset of A_n, but to prove that $\bigcap_{n=1}^{\infty} A_n$ is a subset of the empty set, we should suppose a $k \in \mathbb{N}$ such that $k \in \bigcap_{n=1}^{\infty} A_n$. Notice that $k \notin \bigcap_{n=1}^{\infty} A_n$. So, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Proof Definitions and Structure:

Tri. Ineq.: For any $a, b, c \in \mathbb{R}$, $|a-b| \leq |a-c| + |c-b|$, with the intermediate step of $a - b = (a-c) + (c-b)$.

Induction Proof and Set Theory Exercises

EX 1: Let $x_1 = 1$. Show that the sequence $x_{n+1} = \frac{1}{2}x_n + 1$ is increasing. Or, in other words, for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$.

1. **Base Case:** We see that $x_1 = 1$ and $x_2 = 1.5$. Thus, $x_1 \leq x_2$
2. **Inductive Hypothesis:** For some $n \in \mathbb{N}$, assume $x_n \leq x_{n+1}$
3. **Inductive Step:** $\frac{1}{2}x_n \leq \frac{1}{2}x_{n+1}$. Hence, $\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1$. Therefore we have proven through induction that, $x_{n+1} \leq x_{n+2}$. □

EX 2: Evaluate as either true or false. If false, provide a counterexample.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.
This is false. Consider the following as a counterexample: If we define A_1 as $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$, we can see why the intersection of these sets of infinite numbers are actually empty. Consider a number m that actually satisfies $m \in A_n$ for every A_n in our collection of sets. Because m is not an element of A_{m+1}, no such m exists and the intersection is empty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$
This is true.

1.2 Some Preliminaries (cont.)

EX 2: Cont.

(d) $A \cap (B \cap C) = (A \cap B) \cap C$
This is false. Let the following sets be defined as $A = \{1\}$, $B = \{2\}$, $C = \{3\}$. If we start on the left side of the equation: $A \cap (B \cup C)$ implies $\{1\} \cap (\{2\} \cup \{3\})$ implies $\{1\} \cap \{2, 3\}$ implies $\emptyset$. From the right: $(A \cap B) \cup C$ implies $(\{1\} \cap \{2\}) \cup \{3\}$ implies $\emptyset \cup \{3\}$ implies $\{3\}$.

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
This is true.

1.3 Axiom of Completeness

AXIOM: *Every non-empty set of real numbers that is bounded above has a least upper bound (suprema).*

Definitions:

Functions: A *function* is the association $x \in X$, with a unique $y \in Y$. Denoted as $f(x) = y$.

One-to-one: f is *one-to-one* if for each $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, and $x_1 = x_2$.

Onto: f is onto if there exists an $x \in E$ so that $f(x) = y$.

2.3 Functions (cont.)

Onto and One-to-One Examples:

1. Let $f\colon \mathbb{N} \to \mathbb{N}$ by $f(x) = 3n + 1$. Then f is one-to-one.

   Let $x_1, x_2 \in \mathbb{N}$ so that $f(x_1) = f(x_2)$. We hope to show $x_1 = x_2$. $f(x_1) = 3x_1+1$ and $f(x_2) = 3x_2 + 1$. Then,

$$\begin{aligned} 3x_1 + 1 &= 3x_2 + 1 \\ 3x_1 &= 3x_2 \\ x_1 &= x_2 \end{aligned}$$

   Therefore, f is one-to-one.

2. Suppose that each of $f\colon B \to C$ and $A \to B$ are one-to-one and onto functions. Let $h\colon A \to C$ be defined as $h = f \circ g$.

   (a) Show that h is one-to-one.
   Let $x_1, x_2 \in A$ and suppose $h(x_1) = h(x_2)$. By definition of h, $h(x_1) = f(g(x_1))$ and $h(x_2) = f(g(x_2))$. $\therefore$ $f(g(x_1)) = f(g(x_2))$. Now, since f is one-to-one, $f(g(x_1)) = f(g(x_2)) \to g(x_1) = g(x_2)$. And since g is one-to-one as well, $g(x_1) = g(x_2)$ and $x_1 = x_2$.

   (b) Show that h is onto.
   Let $c \in C$. Since f is onto, there exists $b \in B\colon f(b) = c$. And by the same logic, since g is onto, for this $b \in B$, $\exists a \in A\colon g(a) = b$. Now, consider $h(a) = f(g(a)) = f(b) = c$, by substitution. Thus, $\forall c \in C, \exists a \in A\colon h(a) = c$. $\therefore$ h is onto.

3. Find functions for all $f, g, h\colon \mathbb{Z} \to \mathbb{Z}$ so that (a) f is one-to-one and not onto, (b) g is not one-to-one and is onto, and...

   (a) Let f(x) = 2x. Assume $f(x_1) = f(x_2)$. Then $2x_1 = 2x_2$. Then, $x_1 = x_2$. Therefore, f is one-to-one. Now, let $y, k \in \mathbb{Z}\colon y = 2k + 1$ (y is odd). There is no integer x such that $2x = 2k + 1$ because the left side is always even, and the right side is always odd. $\therefore$ f is not onto.

   (b) Consider the piece-wise function,

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even,} \\ \frac{-(x+1)}{2} & \text{if } x \text{ is odd.} \end{cases}$$

   If $y \geq 0$, let $y = 2y$. Then $g(x) = g(2y) = y$ because x is even by definition. If $y < 0$, let $x = -2y - 1$. Then $g(x) = g(-2y-1) = \frac{-(2y-1)}{2} = y$ because x is odd by definition $\therefore$ for any $y \in \mathbb{Z}, \exists x\colon g(x) = y, \therefore g$ is onto. Now, consider $x_1 = 2$ and $x_2 = -3$. Then $g(2) = \frac{2}{2} = 1$ and $g(-3) = \frac{-((-3)+1}{2} = 1$. Since $2 \neq 3$, $x_1 \neq x_2 \therefore g$ is not one-to-one.

2.3 Functions (cont.)

3. ... h is neither.

   (c) Consider $h(x) = x^2$. Let $x_1 = -1$, and $x_2 = 1$. $h(-1) = 1$ and $h(1) = 1$. Hence, $x_1 \neq x_2$ and is not one-to-one. Because $h(x)$ only produces positive numbers, it cannot cover every possible integer, and hence, $h(x) \neq y$ for $y \in \mathbb{Z} \therefore h$ is not onto.

2.4 Relations

Definitions:

Relation:	Defined as $r\colon X \to Y$ if $R \subseteq X \times Y$.
Reflexive:	If for each $x \in X$, $x \ R \ x$.
Symmetric:	If when $a \ R \ b$, then $a \ R \ b$.
Transitive:	If when $a \ R \ b$, and $b \ R \ c$, then $a \ R \ c$.
Equivalence relation:	Has all three above traits.
Equivalence class:	Because equivalence relations allow us to partition the domain, each partition is then labeled as $[a]$.
Modulo (%):	For two integers, a and b, a is said to be equivalent to b modulo n if the difference between a and b (that is, $a-b$) is divisible by n. This means there exists an integer k such that $a - b = kn$.

Note: All functions are relations, but not all relations are functions. Abiding by the definition of a function, if x can be mapped to more than one y-value, the statement is not a function. Also note for the relation traits, those items are true on the basis that $R\colon X \to X$ is a relation.

Prove Modulo Is an Equivalence Class

Let $n \in \mathbb{N}$, and let $a, b, c \in \mathbb{Z}$.

- Is $a \equiv a \mod n$ reflexive? $a - a = 0$, which is divisible by n. So, Yes.
- Is $a \equiv a \mod n$ symmetric? Well, we must find out if $a \equiv b \mod n$ is equivalent to $b \equiv a \mod n$. We know that $(a - b)$ is divisible by n, and there are no restrictions to also include that $b - a$ is also divisible by n. Yes.
- Is $a \equiv a \mod n$ transitive? Let $a \equiv b \mod n$ and $b \equiv c \mod n$. This means that $a = b + kn$ and $b = c + ln$ for some integers k and l. Substituting the second equation into the first, we find that $a = (c+ln)+kn = c+(l+k)n$. So, $a \equiv c \mod n$, as required for transitivity.

Integers Modulo n

The set of equivalence classes form by this equivalence relation is called the *integers modulo n*, and is denoted as $\mathbb{Z}/n$. We use $[k]$ to denote the equivalence class of $[k]$. For example, elements of $\mathbb{Z}$ 3 are $[0] = \{\ldots, -9, -8, -6, -3, 0, 3, 6, 9, \ldots\}$, and $[1] = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$.

2.6 Graph Theory

__Definitions__:

Graphs

Graph: $G = (V, E)$, is a pair of sets V, the *vertex set*, and E the *edge set*, so that each element of E has the form $\{v_i, v_j\}, v_1, v_j \in V$.

Degree: The number of edges which include v. Granted that $v \in V$.

Adjacent: Vertices u, v belong to $\{u, v\} \in E$.

Path: From vertex v_0 to vertex v_n is a sequence $v_0, v_1, v_2 \ldots, v_n$, where each $v_i \in V$ and $\{v_i, v_{i+1}\} \in E$.

Simple: No edge occurs twice in a path.

Connected: If each pair of vertices are adjoined by an edge.

Circuits

Circuit: A path with the same starting and ending vertex.

Complete: On n vertices, K_n, is the connected graph where each vertex is adjacent to each other.

Bipartite Graphs

Bipartite: The vertex set $V = v_1 \cup v_2, v_1 \cap v_2 = \emptyset$, and no vertex in V_1 is adjacent to any other in V_1, and no vertex in V_2 is adjacent to any other in V_2.

Trees

Tree: A connected graph that has no circuit.

BiSTree: For every node, all elements in the left subtree are less than the node's value, and all elements in the right subtree are greater.

***Lemma*:** If G is a tree, G has at least one vertex of degree 1.

Proof. For the sake of contradiction, suppose each vertex has degree ≥ 2. Pick a vertex, v_0. Since, $\deg(v_0) \geq 2$, it is adjacent to some v_j. Because $\deg(v_1) \geq 2$, it has an edge distinct from $\{v_0, v_1\}$, follow it to v_2. Then, v_2 has edge distinct from $\{v_1, v_2\}$, follow it to $v - 3$. If $v_3 = v_0(v_1, v_2)$ we have a circuit. v_3 has edge distinct from $\{v_2, v_3\}$. Go to v_4. Continue .... Either some v_j is visited again, or $v_0, v_1, v_2, \ldots v_n$. Therefore, it must be a circuit.

Hence, G must have a vertex with degree of at least 1 such that $1 \leq$. □

***Theorem 1*:** A tree with n vertices always has $n - 1$ edges.

Proof. By the Lemma, there exists a vertex of degree 1. Remove it and its edge. We still have a tree. This new tree has vertex of degree 1. Remove it and its edge. Continue until you get to k_2, then k_1. We stop with 1 vertex, 0 edges, we have removed $n - 1$ vertices. Each edge was removed. Thus, we threw out *all* $n - 1$ edges. □

2.6 Graph Theory (cont.)

Euler Circuits

Euler circuit: A circuit graph which uses each edge only once.

Euler path: A path which uses each edge – start and end vertices are distinct.

Note: G has an Euler circuit if, and only if, it is connected and each vertex has an even degree. Intuitively, if each vertex has an even degree, then if you come into the vertex through the entrance (first edge), and you leave through the exit (second edge) you have used up both openings.

G has an Euler path if, and only if, it is connected and has exactly two vertices of odd degree.

Isomorphisms

Two graphs, $G = (V, E)$ and $H = (W, f)$ are *isomorphic* if there is an $f: V \to W$ which is one-to-one, onto, and $\{v_i, v_j\} \in E \iff \{f(v_i), f(v_j)\} \in F$.

Vertex Colorings

If G contains a triangle (i.e., if it has a copy of k_3), we need at least 3 colors. If G contains a copy of K_n, we need at least n.

If a graph has no overlapping paths, the graph requires no more than 4 colors.

Hamilton Graphs

A graph has a *Hamilton Circuit* if there is a circuit that uses each vertex once.

Note: This is different from Euler, as Euler uses edges. This is specifically for vertices. If it has a vertex of degree 1, it cannot have a Hamilton circuit.

Estimating Big Θ

Sets

1. Suppose that the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the particular sets $A = \{2, 3, 5\}$ and $B = \{1, 3, 5, 7, 9\}$. Find each of the following:

   (a) $A \cup B = \{1, 2, 3, 5, 7, 9\}$

   (b) $A \cap B = \{3, 5\}$

   (c) $\{x \in U : x^2 < 10 \land x \in A\} = \{2, 3\}$

   (d) $B' \cap A = \{2\}$

   (e) $\{x \in A : x \geq 7\} = \emptyset$

   (f) $\mathcal{P}(A) = \{\emptyset, \{2\}, \{3\}, \{5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{2, 3, 5\}\}$