Real Analysis: Take-Home Final Exam

Paul Beggs

December 17, 2024

"All work on this take-home exam is my own."[1] **Signature:** Paul Beggs

For this exam, you are going to use results from this semester to set up a big idea for next semester.

Part 1. These two problems will give results that are useful in the next part. Throughout this test, $f^{(j)}(x)$ denotes the j^{th} derivative of f at x.

(1) Let $c_0, c_1, c_2, \ldots, c_k$ be real numbers. Prove there exists a unique polynomial $p(x)$ of order at most k such that for each integer j between 0 and k, $p^{(j)}(0) = c_j$. In other words,

$$p(0) = c_0, \qquad p'(0) = c_1, \qquad p''(0) = c_2, \qquad \ldots, \qquad p^{(k)}(0) = c_k.$$

If $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$, give formulas for $a_0, \ldots, a_k$ in terms of $c_0, \ldots, c_k$.

(2) Let φ be a function that is differentiable $k+1$ times on an interval $[a, b]$. This means $\varphi', \varphi'', \ldots, \varphi^{k+1}$ all exist on $[a, b]$. Assume that

$$\begin{aligned} \varphi(a) &= 0 \qquad \text{and} \qquad \varphi(b) = 0. \\ \varphi'(a) &= 0 \\ &\vdots \\ \varphi^{(k)}(a) &= 0 \end{aligned}$$

Prove there exists a point $c \in (a, b)$ such that $\varphi^{k+1}(c) = 0$.

[1] Except for the calculator section where I needed to use Wolfram Alpha.

Part 2. These problems will walk you through an important concept and result in Calculus.

Let I be an interval with zero in its interior and $f(x)$ be a function that is $k+1$ times differentiable on I.

(3) Construct the unique polynomial $P_k(x)$ of order at most k which satisfies that for all integers j between 0 and k, $P_k^{(j)}(0) = f^{(j)}(0)$. *This should be a direct application of Problem (1).*

(4) Let x be a fixed nonzero point in I. Define a new function g on I as follows:

$$g(t) = f(t) - P_k(t) - \left(\frac{f(x) - P_k(x)}{x^{k+1}}\right) t^{k+1}.$$

Show that

$$\begin{aligned} g(0) &= 0 \qquad\qquad \text{and} \qquad\qquad g(x) = 0. \\ g'(0) &= 0 \\ &\vdots \\ g^{(k)}(0) &= 0 \end{aligned}$$

Conclude there exists a point c between 0 and x such that $g^{(k+1)}(c) = 0$.

(5) Use the above problem to prove the existence of a point c between 0 and x for which

$$f(x) = P_k(x) + \frac{f^{(k+1)}(c)}{(k+1)!} x^{k+1}.$$

(6) This polynomial P_k is used as an approximation of f. If it is known that $|f^{(k+1)}|$ is bounded by some number M on the interval I, prove the error bound formula

$$|f(x) - P_k(x)| \leq \frac{M|x|^{k+1}}{(k+1)!}.$$

Part 3. Now you get to enjoy using your result!

(7) Consider the function $f(x) = e^x$. Give the expression of the polynomial approximation P_k for an arbitrary $k \in \mathbb{N}$. Use what you know about f and its derivatives on the interval $[0, 1]$ to determine an integer k for which you can guarantee that $|f(1) - P_k(1)| < 10^{-12}$. Use this (and a calculator) to generate an approximation of e to 12 decimal places.

Solutions

(1) To ensure that we have a polynomial with at order of at most k, we first need to observe some behaviors of derivatives. For example, for the polynomial x^j:

$$(x^j)' = jx^{j-1}$$
$$(x^j)'' = j(j-1)x^{j-2}$$
$$\vdots$$
$$(x^j)^{(j)} = j! \cdot x^0 = j!$$

Notice the factorial arises from the recursive application of the power rule. Thus, find the j^{th} derivative, we combine this with the coefficients $a_0, a_1, \ldots, a_k$ to get:

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$$
$$p'(x) = a_1 + (a_2 \cdot 2)x + (a_3 \cdot 3)x^2 + \cdots + (a_k \cdot k)x^{k-1}$$
$$\vdots$$
$$p^{(j)}(x) = a_j \cdot j! + (\text{terms involving higher values of } x)$$

To derive formulas for $a_1, a_2, \ldots, a_k$, we need to set $x = 0$. Thus,

$$p^{(j)}(0) = a_j \cdot j!$$

Substitute c_j in for $p^{(j)}$ and solve for a_j:

$$c_j = a_j \cdot j!$$

Therefore:

$$a_j = \frac{c_j}{j!}$$

for each $j = 0, 1, 2, \ldots, k$. This gives the coefficients of

$$a_0 = c_0$$
$$a_1 = \frac{c_1}{1!}$$
$$a_2 = \frac{c_2}{2!}$$
$$\vdots$$
$$a_k = \frac{c_k}{k!}.$$

Thus, the polynomial $p(x)$ exists with unique coefficients defined by $a_j = \dfrac{c_j}{j!}$ because each a_j is uniquely determined by c_j.

(2) Since $\varphi(a) = 0$ and $\varphi(b) = 0$, then

$$\varphi(a) = \varphi(b).$$

Thus, by Rolle's Theorem, there exists point $c_1 \in (a, b)$ such that

$$\varphi'(c_1) = 0.$$

Using this c_1 as a point in a new closed interval, we have on $[a, c_1]$:

$$\varphi'(a) = \varphi'(c) = 0.$$

Again, by Rolle's Theorem, we have a point $c_2 \in (a, c_1)$ such that

$$\varphi''(c_2) = 0.$$

Repeat this up to $\varphi^{(k)}$ times to create a sequence of points $c_1, c_2, \ldots, c_k$ such that

$$\varphi^{(j)}(c_j) = 0 \text{ for } j = 1, 2, \ldots, k.$$

After k applications of Rolle's Theorem, we have

$$\varphi^{(k)}(c_k) = 0.$$

Now, consider $\varphi^{(k)}(x)$ on (a, b) such that

$$\varphi^{(k)}(a) = \varphi^{(k)}(c_k) = 0.$$

By Rolle's Theorem, there exists a point $c \in (c_k, b)$ such that

$$\varphi^{(k+1)}(c) = 0.$$

(3) From Problem (1), we know that the polynomial $P_k(x)$ can be written as:

$$P_k(x) = \sum_{j=0}^{k} \frac{f^{(j)}(0)}{j!} x^j$$

This polynomial satisfies $P_k^{(j)}(0) = f^{(j)}(0)$ for all integers j between 0 and k.

(4) First, we show that $g(0) = 0$:

$$g(0) = f(0) - P_k(0) - \left(\frac{f(x) - P_k(x)}{x^{k+1}}\right) 0^{k+1} = f(0) - P_k(0) = 0$$

since $P_k(0) = f(0)$.

Next, we show that $g'(0) = 0$:

$$g'(t) = f'(t) - P_k'(t) - \left(\frac{f(x) - P_k(x)}{x^{k+1}}\right)(k+1)t^k$$

$$g'(0) = f'(0) - P_k'(0) = 0$$

since $P_k'(0) = f'(0)$.

Similarly, we can show that $g^{(j)}(0) = 0$ for $j = 0, 1, \ldots, k$.

Finally, we show that $g(x) = 0$:

$$g(x) = f(x) - P_k(x) - \left(\frac{f(x) - P_k(x)}{x^{k+1}}\right)x^{k+1} = f(x) - P_k(x) - (f(x) - P_k(x)) = 0$$

Then, from the hard work we did in (2), we conclude from Rolle's Theorem that there exists a point $c \in (0, x)$ such that $g^{(k+1)}(c) = 0$.

(5) From the previous problem, we have $g^{(k+1)}(c) = 0$ for some $c \in (0, x)$. Therefore,

$$g^{(k+1)}(t) = f^{(k+1)}(t) - \left(\frac{f(x) - P_k(x)}{x^{k+1}}\right)(k+1)!$$

Evaluating at $t = c$, we get:

$$f^{(k+1)}(c) - \left(\frac{f(x) - P_k(x)}{x^{k+1}}\right)(k+1)! = 0$$

Solving for $f(x)$, we obtain:

$$f(x) = P_k(x) + \frac{f^{(k+1)}(c)}{(k+1)!}x^{k+1}$$

(6) From the previous problem, we have:

$$f(x) = P_k(x) + \frac{f^{(k+1)}(c)}{(k+1)!}x^{k+1}$$

Therefore, the error term is:

$$f(x) - P_k(x) = \frac{f^{(k+1)}(c)}{(k+1)!}x^{k+1}$$

Taking the absolute value and using the bound $|f^{(k+1)}(c)| \leq M$, we get:

$$|f(x) - P_k(x)| \leq \frac{M|x|^{k+1}}{(k+1)!}$$

(7) For the function $f(x) = e^x$, all derivatives are $f^{(j)}(x) = e^x$. At $x = 0$, we have $f^{(j)}(0) = 1$ for all j. Therefore, the polynomial approximation $P_k(x)$ is:

$$P_k(x) = \sum_{j=0}^{k} \frac{x^j}{j!}$$

To find k such that $|f(1) - P_k(1)| < 10^{-12}$, we use the error bound formula:

$$|e - P_k(1)| \leq \frac{e}{(k+1)!}$$

We need to find the smallest k such that:

$$\frac{e}{(k+1)!} < 10^{-12}$$

Using a calculator (Wolfram Alpha), we find that $k = 20$ satisfies this inequality. Therefore, the polynomial approximation $P_{20}(x)$ is:

$$P_{20}(x) = \sum_{j=0}^{20} \frac{x^j}{j!}$$

Evaluating at $x = 1$, we get:

$$P_{20}(1) = \sum_{j=0}^{20} \frac{1}{j!} \approx 2.718281828459$$

Thus, an approximation of e to 12 decimal places is 2.718281828459.