In Exercises 10 and 15, determine whether the given map φ is a homomorphism.

10. Let F be the additive group of all continuous functions mapping $\mathbb{R}$ to $\mathbb{R}$. Let $\mathbb{R}$ be the additive group of real numbers, and let $\varphi : F \to \mathbb{R}$ be given by

$$\varphi(f) = \int_0^4 f(x)\,dx.$$

Solution. This problem is exactly like Example 13.9, but we use $[0, 4]$ for the domain instead of $[0, 1]$. Thus, the following computation shows that φ is a homomorphism:

$$\begin{aligned}\varphi(f+g) &= \int_0^4 (f+g)(x)\,dx \\ &= \int_0^4 [f(x)+g(x)]\,dx \\ &= \int_0^4 f(x)\,dx + \int_0^4 g(x)\,dx \\ &= \varphi(f) + \varphi(g).\end{aligned}$$

15. Let F be the multiplicative group of all continuous functions mapping $\mathbb{R}$ into $\mathbb{R}$ that are nonzero at every $x \in \mathbb{R}$. Let $\mathbb{R}^*$ be the multiplicative group of nonzero real numbers. Let $\varphi : F \to \mathbb{R}^*$ be given by

$$\varphi(f) = \int_0^1 f(x)\,dx.$$

Solution. For this problem, φ is not a homomorphism. A counterexample can be shown by using $f(x) = x^2 + 1$ and $g(x) = x^4 + 1$. Consider the following:

$$\begin{aligned}\varphi(f \cdot g) &= \int_0^1 \left[x^6 + x^4 + x^2 + 1\right]\,dx \\ &= \frac{1}{7} + \frac{1}{5} + \frac{1}{3} + 1 = \frac{176}{105}.\end{aligned}$$

However,

$$\begin{aligned}\varphi(f) \cdot \varphi(g) &= \int_0^1 \left(x^2+1\right)\,dx \cdot \int_0^1 \left(x^4+1\right)\,dx \\ &= \left(1 + \frac{1}{3}\right)\left(1 + \frac{1}{5}\right) = \frac{8}{5}.\end{aligned}$$

Since $\varphi(f \cdot g) \neq \varphi(f) \cdot \varphi(g)$, φ is not a homomorphism.