



# HENDRIX

COLLEGE

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## Homework 6: Sections 13 & 14

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### Algebra

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## Section 13

In Exercises 4 and 5, determine whether the given map  $\varphi$  is a homomorphism. [*Hint:* The straightforward way to proceed is to check whether  $\varphi(ab) = \varphi(a)\varphi(b)$  for all  $a$  and  $b$  in the domain of  $\varphi$ . However, if we should happen to notice that  $\varphi^{-1}[\{e'\}]$  is not a subgroup whose left and right cosets coincide, or that  $\varphi$  does not satisfy the properties given in Exercise 44 or 45 for finite groups, then we can say at once that  $\varphi$  is not a homomorphism.]

4. Let  $\varphi : \mathbb{Z}_6 \rightarrow \mathbb{Z}_2$  be given by  $\varphi(x) =$  the remainder of  $x$  when divided by 2, as in the division algorithm.

*Solution.* We know  $\varphi(x) = x \bmod 2$ , and we have binary operators  $(a + b) \bmod 6$  for  $\mathbb{Z}_6$  and  $(a + b) \bmod 2$  for  $\mathbb{Z}_2$ . This leaves us with the following equation to check:

$$\varphi((a + b) \bmod 6) \stackrel{?}{=} (\varphi(a) + \varphi(b)) \bmod 2$$

For the left-hand side:

$$\varphi((a + b) \bmod 6) = ((a + b) \bmod 6) \bmod 2.$$

This simplifies down to  $(a + b) \bmod 2$  since  $6 \equiv 0 \pmod{2}$ . For the right-hand side:

$$(\varphi(a) + \varphi(b)) \bmod 2 = (a \bmod 2 + b \bmod 2) \bmod 2.$$

Since addition is commutative and associative in both groups, we can rewrite this as:

$$(a + b) \bmod 2.$$

Therefore, the equation holds, and the map is a homomorphism.

5. Let  $\varphi : \mathbb{Z}_9 \rightarrow \mathbb{Z}_2$  be given by  $\varphi(x) =$  the remainder of  $x$  when divided by 2, as in the division algorithm.

*Solution.* This is not a homomorphism because the two sides of the equation are not equal:

$$\varphi((3 + 7) \bmod 9) = \varphi(1) = 1,$$

but

$$(\varphi(3) + \varphi(7)) \bmod 2 = (1 + 1) \bmod 2 = 0.$$



In Exercises 19 and 23, compute the indicated quantities for the given homomorphism  $\varphi$ . (See Exercise 46.)

19.  $\ker(\varphi)$  and  $\varphi(20)$  for  $\varphi : \mathbb{Z} \rightarrow S_8$  such that  $\varphi(1) = (1, 4, 2, 6)(2, 5, 7)$ .

*Solution.* We know  $\varphi(1) = (1, 4, 2, 6)(2, 5, 7) = (1, 4, 2, 5, 7, 6)$  has order 6, so  $\ker(\varphi) = 6\mathbb{Z}$ . Then, we know from the homomorphism property:

$$\varphi(20) = (\varphi(1))^{20} = (\varphi(1))^{18}(\varphi(1))^2 = (\varphi(1))^2 = (1, 2, 7)(4, 5, 6).$$

23.  $\ker(\varphi)$  and  $\varphi(4, 6)$  for  $\varphi : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$  where  $\varphi(1, 0) = (2, -3)$  and  $\varphi(0, 1) = (-1, 5)$ .

*Solution.* We have:

$$\varphi(x, y) = \varphi(x, 0) + \varphi(0, y) = x\varphi(1, 0) + y\varphi(0, 1) = (2x - y, -3x + 5y).$$

To find the kernel, we set  $\varphi(x, y) = (0, 0)$ :

$$2x - y = 0 \quad \text{and} \quad -3x + 5y = 0$$

From the first equation, we have  $y = 2x$ . Substituting into the second equation gives:

$$-3x + 5(2x) = -3x + 10x = 7x = 0 \implies x = 0.$$

Thus,  $y = 0$  as well. Therefore,  $\ker(\varphi) = \{(0, 0)\}$ . Now, we find  $\varphi(4, 6)$ :

$$\varphi(4, 6) = (2(4) - 6, -3(4) + 5(6)) = (8 - 6, -12 + 30) = (2, 18).$$

44. Let  $\varphi : G \rightarrow G'$  be a group homomorphism. Show that if  $|G|$  is finite, then  $|\varphi[G]|$  is finite and is a divisor of  $|G|$ .

*Proof.* Let  $\varphi : G \rightarrow G'$  be a group homomorphism, and let  $H = \ker(\varphi)$ . Our goal is to show that the order of the preimage of  $G$ ,  $|\varphi[G]|$ , is finite, and a divisor of  $|G|$ , given  $G$  is finite. The theorem  $aH = \varphi^{-1}[\{\varphi(a)\}]$  maps a single coset to the same single element  $\varphi(a)$  in the image. This establishes a one-to-one correspondence between the set of all cosets  $(G/H)$  and the set of all images,  $\varphi[G]$ . Because there is a one-to-one correspondence, the two sets must have equal size:  $|G/H| = |\varphi[G]|$ . Then, by Lagrange's Theorem, since  $G$  is finite, the number of cosets  $(G/H)$  must be a finite number that divides the order of the group,  $|G|$ . Since  $|G/H| = |\varphi[G]|$ , it follows that  $|\varphi[G]|$  must also be a finite number that divides  $|G|$ .  $\square$



45. Let  $\varphi : G \rightarrow G'$  be a group homomorphism. Show that if  $|G'|$  is finite, then  $|\varphi[G]|$  is finite and is a divisor of  $|G'|$ .

*Proof.* From Theorem 13.12 (3) (the fundamental properties of homomorphisms), if  $H$  is a subgroup of  $G$ , then  $\varphi[H]$  is a subgroup of  $G'$ . Now, let  $H = G$ . It is certainly true that  $G$  is a subgroup of  $G$ , so  $\varphi[G]$  is a subgroup of  $G'$ . Then, by Lagrange's Theorem, since  $|G'|$  is finite, the order of its subgroup,  $|\varphi[G]|$  is also a finite number that divides  $|G'|$ .  $\square$

49. Show that if  $G$ ,  $G'$ , and  $G''$  are groups and if  $\varphi : G \rightarrow G'$  and  $\gamma : G' \rightarrow G''$  are homomorphisms, then the composite map  $\gamma\varphi : G \rightarrow G''$  is a homomorphism.

*Proof.* Let  $a, b \in G$ . Then,

$$(\gamma\varphi)(ab) = \gamma(\varphi(ab)) \quad (1)$$

$$= \gamma(\varphi(a)\varphi(b)) \quad (2)$$

$$= \gamma(\varphi(a))\gamma(\varphi(b)) \quad (3)$$

$$= (\gamma\varphi)(a)(\gamma\varphi)(b). \quad (4)$$

Equations (1) and (4) are from the definition of a composite map, and (2) and (3) are from the homomorphic properties of  $\varphi$  and  $\gamma$ , respectively. Therefore, we have shown that the composite map is a homomorphism.  $\square$

## Section 14

In Exercises 2 and 6, find the order of the given factor group.

2.  $(\mathbb{Z}_4 \times \mathbb{Z}_{12}) / (\langle 2 \rangle \times \langle 2 \rangle)$

*Solution.* In  $\mathbb{Z}_4$ , the subgroup generated by  $\langle 2 \rangle$  is  $\{0, 2\}$ . Then, in  $\mathbb{Z}_{12}$ , the subgroup generated by  $\langle 2 \rangle$  is  $\{0, 2, 4, 6, 8, 10\}$ . Thus, the subgroup  $\langle 2 \rangle \times \langle 2 \rangle$  has order  $2 \times 6 = 12$ , and  $\mathbb{Z}_4 \times \mathbb{Z}_{12}$  has order  $4 \times 12 = 48$ . By Lagrange's Theorem, the order of the factor group is  $48/12 = 4$ .

6.  $(\mathbb{Z}_{12} \times \mathbb{Z}_{18}) / \langle (4, 3) \rangle$

*Solution.* The order of  $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$  is  $12 \times 18 = 216$ . In  $\mathbb{Z}_{12} \times \mathbb{Z}_{18}$ , the subgroup generated by  $\langle (4, 3) \rangle$  has order 6. So, by Lagrange's Theorem, the order of the factor group is  $216/6 = 36$ .



In Exercises 11 and 15, give the order of the element in the factor group.

11.  $(2, 1) + \langle(1, 1)\rangle$  in  $(\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle(1, 1)\rangle$

*Solution.* In  $\mathbb{Z}_3 \times \mathbb{Z}_6$ ,  $\langle(1, 1)\rangle = \{(1, 1), (2, 2), (0, 3), (1, 4), (2, 5), (0, 0)\}$ . Then, we test the powers of  $(2, 1)$  until we find one that is in  $\langle(1, 1)\rangle$ :

$$(2, 1)(2, 1) = (1, 2), \quad (1, 2)(2, 1) = (0, 3).$$

So, the order of  $(2, 1) + \langle(1, 1)\rangle$  in  $(\mathbb{Z}_3 \times \mathbb{Z}_6)/\langle(1, 1)\rangle$  is 3.

15.  $(2, 0) + \langle(4, 4)\rangle$  in  $(\mathbb{Z}_6 \times \mathbb{Z}_8)/\langle(4, 4)\rangle$

*Solution.* Since  $(2, 0)$  is in  $\langle(4, 4)\rangle$ ,  $(2, 0) + \langle(4, 4)\rangle$  has order 1.

27. A subgroup  $H$  is **conjugate to a subgroup**  $K$  of a group  $G$  if there exists an inner automorphism  $i_g$  of  $G$  such that  $i_g[H] = K$ . Show that conjugacy is an equivalence relation on the collection of subgroups of  $G$ .

*Proof.* Let  $H$ ,  $K$ , and  $L$  be subgroups of  $G$ . To show that conjugacy is an equivalence relation on the collection of subgroups of  $G$ , we must show the following:

- **Reflexive:** We must show that  $H$  is conjugate to itself. That is, we must find an element  $g \in G$  such that  $H = i_g[H] = gHg^{-1}$ . The obvious choice is the identity element  $e \in G$ :

$$H = i_e[H] = eHe^{-1} = \{ehe^{-1} \mid h \in H\} = \{h \mid h \in H\} = H.$$

Thus, we have found an element  $g = e$  such that  $i_g[H] = H$ , so the subgroup is conjugate to itself. Therefore, the relation is reflexive.

- **Symmetric:** Assume  $H$  to be congruent to  $K$  (i.e.,  $K = i_g[H] = gHg^{-1}$  for some  $g \in G$ ). We must show  $K$  is congruent to  $H$ . That is, for some  $g' \in G$ ,  $H = i_{g'}[K] = g'Kg'^{-1}$ . If we multiply  $K = gHg^{-1}$  by  $g^{-1}$  and  $g$  on the left and right, respectively, we get  $g^{-1}Kg = H$ . Thus, we have found our elements  $g' = g^{-1}$ , and  $g'^{-1} = g$ . Therefore, the relation is symmetric.
- **Transitive:** Assume  $H$  to be congruent to  $K$  and  $K$  to be congruent to  $L$ . Then,  $K = gHg^{-1}$  and  $L = pKp^{-1}$  for some  $p \in G$ . We want to show  $L = g''Hg''^{-1}$ . We can do that by using substitution:

$$L = pKp^{-1} = pgHg^{-1}p^{-1}.$$

Notice  $pg = g''$  and  $g^{-1}p^{-1} = (pg)^{-1} = g''^{-1}$ . The product  $pg$  and  $(pg)^{-1}$  are both in  $G$  because  $G$  is a group. Thus, we have shown that  $H$  is conjugate to  $L$ . Therefore, the relation is transitive.  $\square$



30. Let  $H$  be a normal subgroup of a group  $G$ , and let  $m = (G : H)$ . Show that  $a^m \in H$  for every  $a \in G$ .

*Proof.* Let  $H$  be a normal subgroup of  $G$  and let  $m = (G : H)$ . Because  $H$  is normal, we can form the factor group  $G/H$ . The order of this group,  $|G/H|$ , is equal to the index of  $H$  in  $G$ , so  $|G/H| = m$ . Now, let  $a$  be any element in  $G$ . The corresponding element in the factor group  $G/H$  is the coset  $aH$ . Let  $k$  be the order of this element  $aH$  in the factor group. By the definition of order,  $(aH)^k = H$  (where  $H$  is the identity element of the factor group). By Lagrange's Theorem, the order of an element must divide the order of the group. Therefore,  $k$  divides  $m$ . This implies that  $m = kq$  for some integer  $q$ . Now, we can compute  $(aH)^m$ :

$$(aH)^m = (aH)^{kq} = ((aH)^k)^q.$$

Since  $(aH)^k = H$ , we can substitute this back in:

$$((aH)^k)^q = H^q = H.$$

Thus, we have shown that  $(aH)^m = a^m H = H$ . By the properties of cosets,  $gH = H$  implies  $g \in H$ . Consequently,  $a^m \in H$ .  $\square$