Writing Assignment 3

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1 Introduction

In statistics and probability, one of the most important mathematical equations is known as the normal curve (or the Gaussian), defined as $g(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$. Because this curve is used notably for probability, an important characteristic of it is that the area under it must be equal to 1 over the entire real line. In other words, we want to show that:

$$\int_{-\infty}^{+\infty} g(x)\,dx = 1.$$

However, we cannot expect to work this integral directly. The reason for this is that the function $g(x)$ is not easily integrable using standard techniques such as u-substitution or integration by parts. The exponential function in the integrand does not lend itself to these methods, and thus we need to pivot to a different approach.

2 Integral in Polar Coordinates

To overcome this difficulty, we consider the square of the one-dimensional integral by looking at a related two-dimensional integral. Let D_a be a disk centered at the origin with radius a. We can use polar coordinates to evaluate the integral:

$$\iint_{D_a} e^{-(x^2+y^2)/2}\,dA.$$

In polar coordinates, we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$, where $dA = r\,dr\,d\theta$. The limits of integration for r will be from 0 to a, and for θ from 0 to 2π because we are integrating over the entire disk D_a. This ensures that every point on the disk is accounted for exactly once in the integration. Thus, we can rewrite the integral as:

$$\begin{aligned}
\iint_{D_a} e^{-(x^2+y^2)/2}\,dA &= \int_0^{2\pi}\int_0^a e^{-r^2/2} r\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^a re^{-r^2/2}\,dr.
\end{aligned}$$

When we go to solve the inner integral, we see that it can be solved using the substitution $u = -\frac{r^2}{2}$, which gives us $du = -r\,dr$. Thus, the limits changed from $r = 0$ to $r = a$ (which gives $u = 0$ and $u = -\frac{a^2}{2}$ as our upper and lower bounds, respectively). Hence, the inner integral becomes:

$$\begin{aligned}
\int_0^a re^{-r^2/2}\,dr &= -\int_{-\frac{a^2}{2}}^0 e^u\,du \\
&= -\left[e^u\right]_{-\frac{a^2}{2}}^0 \\
&= 1 - e^{-a^2/2}.
\end{aligned}$$

Then, solving for the outer integral, we have:

$$\begin{aligned}\iint_{D_a} e^{-(x^2+y^2)/2}\, dA &= \int_0^{2\pi} d\theta \left(1 - e^{-a^2/2}\right) \\ &= 2\pi \left(1 - e^{-a^2/2}\right).\end{aligned}$$

This result shows that the area under the curve of the normal distribution over the disk D_a is equal to $2\pi \left(1 - e^{-a^2/2}\right)$.

3 Limit as $a \to +\infty$

With this expression, we must take the limit as a approaches infinity because we want to find the area under the curve of the normal distribution over the entire real line. Taking the limit, we can evaluate the integral as a approaches infinity, which gives us the total area under the curve. Thus, we have:

$$\lim_{a\to+\infty} \iint_{D_a} e^{-(x^2+y^2)/2}\, dA = \lim_{a\to+\infty} 2\pi \left(1 - e^{-a^2/2}\right) = 2\pi\,(1-0) = 2\pi.$$

This result above shows that integrating over the whole plane in polar coordinates is equivalent to the iterated integral:

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\, dx\, dy.$$

Then, because the integrand separates in Cartesian coordinates as $(e^{-x^2/2})(e^{-y^2/2})$, we can write the double integral as a product of two single integrals:

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\, dx\, dy = \left(\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx\right)\left(\int_{-\infty}^{+\infty} e^{-y^2/2}\, dy\right).$$

This equality follows from the fact that both x and y are independent variables ranging over the same domain (the entire real line), and both integrals have the same form. The variables x and y are essentially interchangeable in this context, since they both represent coordinates in our two-dimensional plane, and the exponential function treats them symmetrically.

Furthermore, since the form of both integrals is identical, with each being an integral of $e^{-t^2/2}$ for a variable t over the entire real line from $-\infty$ to $+\infty$, they must evaluate to the same value. Therefore, we can simplify by writing:

$$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\, dx\, dy = \left(\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx\right)^2.$$

This allows us to relate our two-dimensional integral, which we calculated to be equal to 2π, to the square of the one-dimensional integral in which we are interested. Thus, we can write:

$$\left(\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx\right)^2 = 2\pi.$$

Taking the square root of both sides, we find that:

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx = \sqrt{2\pi}.$$

Finally, when we substitute back into the definition of the normal curve, we see:

$$\int_{-\infty}^{+\infty} g(x)\, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-x^2/2}\, dx = \frac{1}{\sqrt{2\pi}} \cdot \sqrt{2\pi} = 1.$$

This shows that the area under the normal curve is equal to 1, as we hoped.

4 Conclusion

Through this process, we have shown that the area under the normal curve is equal to 1 by using polar coordinates and the properties of the Gaussian function. By evaluating the integral over a disk and taking the limit as the radius approaches infinity, we were able to relate the two-dimensional integral to the one-dimensional integral, ultimately leading us to the conclusion that $\int_{-\infty}^{+\infty} g(x)\,dx = 1$.

This paper demonstrates the power of changing variables and using polar coordinates to simplify complex integrals when integration techniques, such as u-substitution or integration by parts, are not applicable. By transforming the problem into a more manageable form, we can derive important results in mathematics and statistics.