

Real Analysis

MATH 350

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Exercise: 4.5.4

Let g be continuous on an interval A and let F be the set of points where g fails to be one-to-one; that is,

$$F = \{x_1 \in A : f(x_1) = f(x_2) \text{ for some } x_1 \neq x_2 \text{ and } x_2 \in A\}.$$

Show F is either empty or uncountable.

Proof. Assume F is not empty; that is, there exists distinct points $x_1, x_2 \in A$ such that $g(x_1) = g(x_2)$. (Note that if F were empty, then we would be done.)

Without loss of generality, assume $x_1 < x_2$.

From here, we consider two cases. Either g is constant, or g is not constant on $[x_1, x_2]$. For the first case:

Case 1: g is constant on $[x_1, x_2]$.

Since g(x) = y for all $x \in [x_1, x_2]$, every point in this interval belongs to F. Then, because $[x_1, x_2]$ is uncountable (intervals in \mathbb{R} contain uncountably many points), it follows that F is uncountable.

Case 2:
$$g$$
 is not constant on $[x_1, x_2]$.

Because g is not constant, there exists some $c \in (x_1, x_2)$ such that $g(c) \neq y$. By the continuity of g, we can apply the Intermediate Value Theorem (IVT) to analyze g on the intervals $[x_1, c]$ and $[c, x_2]$:

- (i) On $[x_1, c]$: $g(x_1) = y$ and $g(c) \neq y$. By the IVT, g(x) attains every value between y and g(c) on $[x_1, c]$.
- (ii) On $[c, x_2]$: $g(c) \neq y$ and $g(x_2) = y$. By the IVT, g(x) attains every value between g(c) and y on $[c, x_2]$.

For any value k between y and g(c), there exist at least two points $x'_1 \in [x_1, c]$ and $x'_2 \in [c, x_2]$ such that $g(x'_1) = g(x'_2) = k$. These points x'_1 and x'_2 are distinct because they belong to different subintervals. Thus, every such k corresponds to multiple $k \in [x_1, x_2]$ where g(k) = g(k), and these k belong to k.

Since there are uncountably many such k (as the range of g over $[x_1, x_2]$ is uncountable), it follows that there are uncountably many $x \in [x_1, x_2]$ in F.

Therefore, because there are uncountably many such x, it follows that F is uncountable.

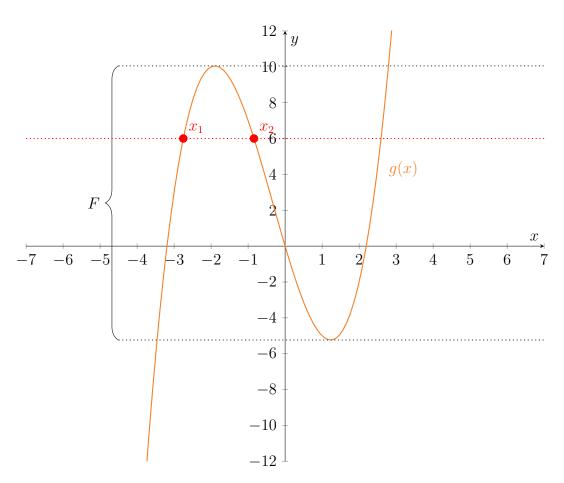


Figure 1: Graph of $g(x) = x^3 + x^2 - 7x$ (Non-empty)