Homework 7: Section 15

Algebra

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Section 15

In Exercises 1, 3, 4, and 7, classify the given group according to the fundamental theorem of finitely generated abelian groups.

1. $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,1)\rangle$

Solution. $\langle(0,1)\rangle$ has order 4, so $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,1)\rangle$ has order 2. Therefore, this leaves only one choice: $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0,1)\rangle \simeq \mathbb{Z}_2$.

3. $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle$

Solution. $\langle(1,2)\rangle$ has order 2, so $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle$ has order 4. This leaves us with two choices: $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. Since $(1,1) + \langle(1,2)\rangle$ has order 4 in the factor group, we must have $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1,2)\rangle \simeq \mathbb{Z}_4$.

4. $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle$

Solution. $|\langle(1,2)\rangle| = 8 \implies |(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle| = 8$. Therefore, either $\mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$. Since $|(0,1) + \langle(1,2)\rangle| = 8$, $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2)\rangle \simeq \mathbb{Z}_8$.

7. $(\mathbb{Z} \times \mathbb{Z})/\langle(1,2)\rangle$

Solution. In the factor group, everything in the subgroup becomes the identity, so $(1,2) = (0,0)$. This implies $x + 2y = 0$, or $x = -2y$. Therefore, every element in the factor group can be written as $(-2y, y) = y(-2,1)$. Since x depends on y, the factor group is isomorphic to $\mathbb{Z}$.

In Exercises 20 and 21, let F be the additive group of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, and let F^* be the multiplicative group of all elements of F that do not assume the value 0 at any point of $\mathbb{R}$.

20. Let K be the subgroup of F consisting of the constant functions. Find a subgroup of F to which F/K is isomorphic.

Solution. Define a homomorphism $\varphi : F \to F$ by $\varphi(f) = f(x) - f(0)$. The kernel of this map consists of all functions where $f(x) - f(0) = 0$, which implies $f(x)$ is constant, so $\ker(\varphi) = K$. The image of the map, $\varphi[F]$, is the set of all functions that evaluate to 0 at $x = 0$. By the fundamental theorem of homomorphisms, F/K is isomorphic to the subgroup $\{f \in F \mid f(0) = 0\}$.

21. Let K^* be the subgroup of F^* consisting of the nonzero constant functions. Find a subgroup of F^* to which F^*/K^* is isomorphic.

Solution. This problem is almost exactly like the last one: Define a homomorphism $\varphi : F^* \to F^*$ by $\varphi(f) = f(x)/f(0)$. The kernel of this map consists of all functions where $f(x)/f(0) = 1$, which implies $f(x)$ is constant, so $\ker(\varphi) = K^*$. The image of the map, $\varphi[F^*]$, is the set of all functions that evaluate to 1 at $x = 0$. By the fundamental theorem of homomorphisms, F^*/K^* is isomorphic to the subgroup $\{f \in F^* \mid f(0) = 1\}$.

28. Give an example of a group G having no elements of finite order > 1 but having a factor group G/H, all of whose elements are of finite order.

Proof. The group $G = \mathbb{Z}$ contains only the identity element, 0, with finite order. Every other integer has infinite order. Thus, G has no elements of finite order > 1. Now, consider G/H where $H = 2\mathbb{Z}$. Since this factor group has order 2, it is isomorphic to $\mathbb{Z}_2$. Then, since $\mathbb{Z}_2$ is itself finite, all of its elements must also have finite order. $\square$

29. Let H and K be normal subgroups of a group G. Give an example showing that we may have $H \simeq K$ while G/H is not isomorphic to G/K.

Proof. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$. Consider the subgroups $H = \langle (2, 0) \rangle$ and $K = \langle (0, 1) \rangle$. Both H and K are subgroups of order 2. Since there is only one group of order 2 up to isomorphism, $H \simeq K$. Notice, however, that $G/H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$, while $G/K \simeq \mathbb{Z}_4$. Therefore, $G/H \not\simeq G/K$. $\square$