2. (2 points) Show that $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ does not exist.

Solution.

- $x = 0$ path: $\lim_{(x,y)\to(0,0)} \dfrac{0\cdot y^2}{0+y^4} = \dfrac{0}{y^2} = 0.$
- $y = 0$ path: $\lim_{(x,y)\to(0,0)} \dfrac{x\cdot 0}{x^2+0} = \dfrac{0}{x^2} = 0.$
- $x = y^2$ path: $\lim_{(x,y)\to(0,0)} \dfrac{y^2\cdot y^2}{y^4+y^4} = \dfrac{y^4}{2y^4} = \dfrac{1}{2}.$

Since the limit is not the same along all paths, the limit does not exist.

3. (2 points each) Find each indicated partial derivative:

(a) $\dfrac{\partial}{\partial x}\left(xy^2\cos(x+y^3) - e^{xy}\right)$

Solution.

$$\begin{aligned}
\tfrac{\partial}{\partial x}\left(xy^2\cos(x+y^3) - e^{xy}\right) &= y^2\tfrac{\partial}{\partial x}\left[x\cos(x+y^3)\right] - \tfrac{\partial}{\partial x}\left[e^{xy}\right] \\
&= y^2\Big(\cos(x+y^3) + x\big(-\sin(x+y^3)\big)\Big) - ye^{xy} \\
&= y^2\big(\cos(x+y^3) - x\sin(x+y^3)\big) - ye^{xy}.
\end{aligned}$$

(b) $\frac{\partial}{\partial y}\left(\ln(x+y+z) - y^2z^3 + x\right)$

Solution.

$$\begin{aligned}
\tfrac{\partial}{\partial y}\left(\ln(x+y+z) - y^2z^3 + x\right) &= \tfrac{\partial}{\partial y}\left[\ln(x+y+z)\right] - z^3\tfrac{\partial}{\partial y}[y^2] + \tfrac{\partial}{\partial y}[x] \\
&= \tfrac{1}{x+y+z} - 2yz^3.
\end{aligned}$$

(c) $\dfrac{\partial^2}{\partial x\partial y}\left(x^3y - y^3\tan(xy)\right)$

Solution.

$$\begin{aligned}
\tfrac{\partial^2}{\partial x\partial y}\left(x^3y - y^3\tan(xy)\right) &= \tfrac{\partial}{\partial x}\left[\tfrac{\partial}{\partial y}[x^3y] - \tfrac{\partial}{\partial y}\left[y^3\tan(xy)\right]\right] \\
&= \tfrac{\partial}{\partial x}\left[x^3 - (3y^2\tan(xy) + xy^3\sec^2(xy))\right] \\
&= \tfrac{\partial}{\partial x}[x^3] - \tfrac{\partial}{\partial x}\left[3y^2\tan(xy)\right] - \tfrac{\partial}{\partial x}\left[xy^3\sec^2(xy)\right].
\end{aligned}$$

Splitting this into 3 partial derivatives:

$$\tfrac{\partial}{\partial x}[x^3] = 3x^2, \quad -\tfrac{\partial}{\partial x}\left[3y^2\tan(xy)\right] = -3y^3\sec^2(xy),$$

with the final derivative worked out:

$$\begin{aligned}
-\tfrac{\partial}{\partial x}\left[xy^3\sec^2(xy)\right] &= y^3\sec^2(xy) + \left[(xy^3)\cdot 2y\sec^2(xy)\tan(xy)\right] \\
&= -y^3\sec^2(xy) - 2xy^4\sec^2(xy)\tan(xy).
\end{aligned}$$

Combining these results, we have:

$$3x^2 - 3y^3\sec^2(xy) - y^3\sec^2(xy) - 2xy^4\sec^2(xy)\tan(xy).$$

Since three terms contain a factor of $y^3\sec^2(xy)$, we can factor this out to get:

$$3x^2 - y^3\sec^2(xy)\big(3 + 1 + 2xy\tan(xy)\big).$$

Adding and simplifying further, we get:

$$\boxed{3x^2 - 2y^3\sec^2(xy)\big(2 + xy\tan(xy)\big).}$$

4. (3 points) Complete each of the following steps to prove that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Let $\epsilon > 0$. Choose $\delta = \epsilon/3$. Suppose that (x, y) is chosen so that $||(x, y) - (0, 0)|| < \delta$ and $(x, y) \neq (0, 0)$.

(a) Explain why $\sqrt{x^2 + y^2} < \delta$.

Solution. Since we have that $||(x, y) - (0, 0)|| < \delta$, when we find the magnitude of this difference, we get:

$$\boxed{\sqrt{x^2 + y^2} < \delta.}$$

(b) Explain why $x^2 \leq x^2 + y^2$, and thus $\frac{x^2}{(x^2+y^2)} \leq 1$.

Solution. Because y^2 will always be positive and $x^2 = x^2$, it must be the case that $x^2 \leq x^2 + y^2$. Hence, when we divide both sides by $x^2 + y^2$, we get:

$$\boxed{\frac{x^2}{(x^2+y^2)} \leq 1.}$$

(c) Explain why $\frac{3x^2}{(x^2+y^2)} \leq 3$.

Solution. Since $\frac{x^2}{(x^2+y^2)} \leq 1$, multiplying both sides by 3 gives us:

$$\boxed{\frac{3x^2}{(x^2+y^2)} \leq 3.}$$

(d) Explain why $\frac{3x^2|y|}{(x^2+y^2)} \leq 3|y|$.

Solution. Similarly to the previous step, we know that since $|y| \geq 0$, so when we multiply both sides of the inequality by $|y|$, the inequality is unchanged.

(e) Now, show that $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right| \le 3\sqrt{x^2+y^2}$.

Solution. First, note that $|y| \le \sqrt{x^2+y^2}$ (since $y^2 \le x^2+y^2$ by the same logic in (b)). Then, recall that $\sqrt{x^2+y^2} < \delta$. We can multiply both sides by 3 to get $3\sqrt{x^2+y^2} < 3\delta$. Since $\delta = \epsilon/3$, when we substitute this δ for ϵ in our equation, we get:

$$\boxed{\left|\frac{3x^2y}{x^2+y^2} - 0\right| \le 3|y| \le 3\sqrt{x^2+y^2} < 3\delta = 3\cdot\tfrac{\epsilon}{3} = \epsilon.}$$

(f) Conclude that whenever (x,y) is in the δ-disk centered at $(0,0)$, then $\left|\dfrac{3x^2y}{x^2+y^2} - 0\right| < \epsilon$.

Solution. Combining the previous steps, we have shown that when (x,y) is in the δ-disk centered at $(0,0)$ (i.e., $||(x,y)-(0,0)||$), we have:

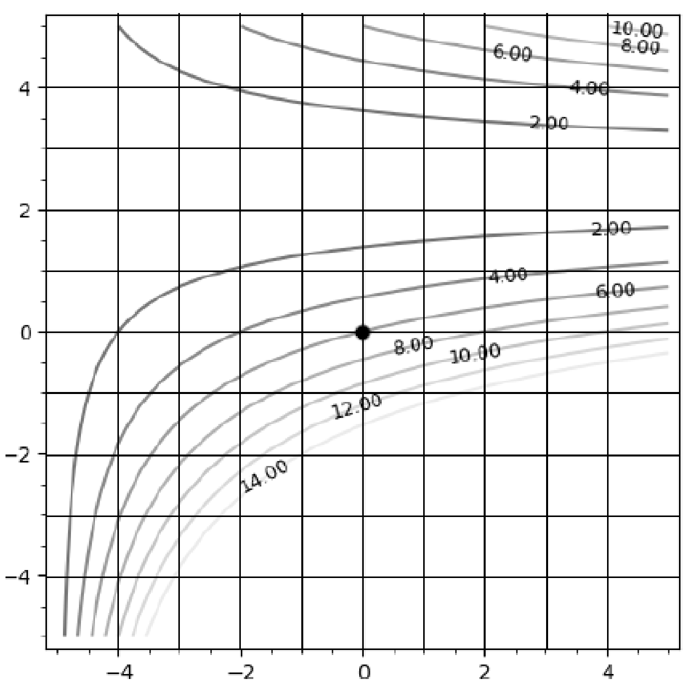
$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \epsilon.$$

This shows that for every $\epsilon > 0$, we can choose a $\delta = \frac{\epsilon}{3}$ so that whenever $||(x,y)-(0,0)|| < \delta$, the inequality holds. Hence, this proves that:

$$\boxed{\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.}$$

$\square$

5. (1 point each) Consider the contour plot of the function $f(x,y)$ shown below.



Determine the sign (+, −, or 0) of each of the following partial derivatives, including a *brief* justification.

(a) $f_x(0,0)$

Solution. We see that $f(0,0) = 6$, and when we move in the x-direction, the function increases slightly. Thus, $f_x(0,0)$ is positive.

(b) $f_y(0,0)$

Solution. If we move in the y-direction, the function decreases slightly towards 4. Thus, $f_y(0,0)$ is negative.

(c) $f_{xx}(0,0)$

Solution. We can see in the plot that as we continue in the x-direction, the lines get closer together, indicating that the function is increasing. Thus, $f_{xx}(0,0)$ is positive.

(d) $f_{yy}(0,0)$

Solution. As we move in the y-direction, the lines get further apart, indicating that the function is decreasing. Hence, $f_{yy}(0,0)$ is negative.

(e) $f_{xy}(0,0)$

Solution. The function increasing in the x-direction, but when we start to move up, we can see that the lines begin to get further apart from each other. Therefore, $f_{xy}(0,0)$ is negative.

6. (2 points) Find an equation of the tangent plane to $f(x,y) = x^2y - \sqrt{x} + y$ at the point $(3,1)$.

Solution. To find the equation of the tangent plane, we need to fill out the following equation:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

First, we find z_0 by plugging in the point $(3,1)$ into the function f:

$$z_0 = f(3,1) = 3^2 \cdot 1 - \sqrt{3} + 1 = 9 - \sqrt{3} + 1 = 10 - \sqrt{3}.$$

Thus, we must find the partial derivatives of f:

$$f_x(x,y) = 2xy - \tfrac{1}{2}x^{-1/2}, \quad f_y(x,y) = x^2 + 1.$$

Plugging in the point $(3,1)$ into these partial derivatives, we get:

$$f_x(3,1) = \tfrac{12\sqrt{3}-1}{2\sqrt{3}}, \quad f_y(3,1) = 10.$$

Finally, we can plug in these values into the equation of the tangent plane to get:

$$\boxed{z = 10 - \sqrt{3} + \tfrac{12\sqrt{3}-1}{2\sqrt{3}}(x-3) + 10(y-1).}$$

7. (2 points) For the function $f(x,y,z) = \dfrac{x+\sin(xy)}{x^2+y^2+z^2+1}$, find $\nabla f(x,y,z)$.

Solution. From the handout, we know that:

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z)\mathbf{i} + \frac{\partial f}{\partial y}(x,y,z)\mathbf{j} + \frac{\partial f}{\partial z}(x,y,z)\mathbf{k}.$$

Thus, we must find the partial derivatives of f:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{x+\sin(xy)}{x^2+y^2+z^2+1}\right] \\
&= \frac{(x^2+y^2+z^2+1)\big(1+y\cos(xy)\big) - 2x\big(x+\sin(xy)\big)}{(x^2+y^2+z^2+1)^2} \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left[\frac{x+\sin(xy)}{x^2+y^2+z^2+1}\right] \\
&= \frac{(x^2+y^2+z^2+1)\big(x\cos(xy)\big) - 2y\big(x+\sin(xy)\big)}{(x^2+y^2+z^2+1)^2} \\
\frac{\partial f}{\partial z} &= \frac{\partial}{\partial z}\left[\frac{x+\sin(xy)}{x^2+y^2+z^2+1}\right] \\
&= -\frac{2z\big(x+\sin(xy)\big)}{(x^2+y^2+z^2+1)^2}
\end{aligned}$$

Therefore, the gradient of f is:

$$\begin{aligned}
\nabla f(x,y,z) = \Bigg\langle & \frac{(x^2+y^2+z^2+1)\big(1+y\cos(xy)\big) - 2x\big(x+\sin(xy)\big)}{(x^2+y^2+z^2+1)^2}, \\
& \frac{(x^2+y^2+z^2+1)\big(x\cos(xy)\big) - 2y\big(x+\sin(xy)\big)}{(x^2+y^2+z^2+1)^2}, \frac{-2z\big(x+\sin(xy)\big)}{(x^2+y^2+z^2+1)^2}\Bigg\rangle.
\end{aligned}$$

8. (2 points) Consider the function $f(x,y) = x^2y - y^3$. Find the directional derivative for f, at $(3,4)$, in the direction of $\mathbf{u} = 5\mathbf{i} - 2\mathbf{j}$.

Solution. We first find the partial derivative of f with respect to x and y:

$$f_x(x,y) = 2xy, \quad f_y(x,y) = x^2 - 3y^2.$$

Then, at the point $(3,4)$:

$$f_x(3,4) = 2\cdot 3\cdot 4 = 24, \quad f_y(3,4) = 3^2 - 3\cdot 4^2 = -39.$$

Then, we find the unit vector in the direction of $\langle 5,-2\rangle$:

$$\boxed{\frac{\langle 24,-39\rangle \cdot \langle 5,-2\rangle}{\sqrt{29}} = \frac{198}{\sqrt{29}}.}$$