Homework 1: Chapter 1

Mathematical Models

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Due

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1. Ecologists use the following model to represent the growth process of two competing species, x and y:

$$\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{K_1} \right) - \alpha_1 xy, \quad \frac{dy}{dt} = r_2 y \left(1 - \frac{y}{K_2} \right) - \alpha_2 xy.$$

The variables x and y represent the number in each population; the parameters r_i represent the intrinsic growth rates of each species; K_i represents the maximum sustainable population in the absence of competition; and α_i represents the effects of competition. Studies of the blue whale and fin whale populations have determined the following parameter values (t in years):

	Blue	Fin
r	0.05	0.08
K	150,000	400,000
α	10^{-8}	10^{-8}

(a) Determine the population levels x and y that maximize the number of new whales born each year. Use the five-step method, and model as an unconstrained optimization problem. Python Code

> *Solution.* Our goal is to find the optimal population for blue and fin whales that maximize the number of new whales born each year. We will model this as an unconstrained optimization problem. After some work (see python code), we find that the optimal number of blue and fin whales is 69,104 and 196,545, respectively. Maintaining these populations result in an annual growth rate of 9,589 new whales per year.

2. Reconsider the whale problem of Exercise 1, but now consider the economic ramifications of harvesting.

(a) A blue whale carcass is worth \$12,000, and a fin whale carcass is worth about half as much. Assuming that controlled harvesting can be used to maintain x and y at any desired level, what population levels will produce the maximum revenue? (Once population reaches the desired levels, the population levels will be kept constant by harvesting at a rate equal to the growth rate.) Use the five-step method. Model as an unconstrained optimization problem. Python Code

> *Solution.* Our goal is to maximize revenue for the whaling operation. We model this as an unconstrained optimization problem. We find that the optimal number of blue and fin whales is 70,619 and 194,704, respectively. Maintaining these populations yields an annual revenue of approximately \$67,914,605.90.

```
 1  # Variables
 2  x = var('x') # Blue whale population
 3  y = var('y') # Fin whale population
 4
 5  # Parameters
 6  r_x = 0.05 # Growth rate for blue whales (whale per year)
 7  r_y = 0.08 # '' fin whales (whale per year)
 8  K_x = 150_000 # Max sustainable population for blue whales
 9  K_y = 400_000 # '' for fin whales
10  alpha_x = 10 ** -8 # Effects of competition for blue whales
11  alpha_y = 10 ** -8 # '' for fin whales
12
13  # Equations
14  dx_dt(x,y) = r_x * x * (1 - x / K_x) - alpha_x * x * y # Growth rate for
    ↪  blue whales
15  dy_dt(x,y) = r_y * y * (1 - y / K_y) - alpha_y * x * y # '' for fin whales
16  G(x,y) = dx_dt(x,y) + dy_dt(x,y) # Growth rate for whales
17
18  # Solve the model
19  growth_x = G.derivative(x) # Partial derivative with respect to x
20  growth_y = G.derivative(y) # '' with respect to y
21  solutions = solve([growth_x(x,y) == 0, growth_y(x,y) == 0], x, y) # get
    ↪  critical points
22  # print(solutions)
23  optimal_x = round(solutions[0][0].rhs(), 0)
24  optimal_y = round(solutions[0][1].rhs(), 0)
25  print(f"The optimal number of blue whales is {optimal_x}. For fin whales,
    ↪  it is {optimal_y}.")
26
27  most_whales = G(optimal_x, optimal_y)
28  print(most_whales)
```

Figure 1: Exercise 1 Code

```
1  # Variables
2  x = var('x') # Blue whale population
3  y = var('y') # Fin whale population
4
5  # Parameters
6  r_x = 0.05 # Growth rate for blue whales (whale per year)
7  r_y = 0.08 # '' fin whales (whale per year)
8  K_x = 150_000 # Max sustainable population for blue whales
9  K_y = 400_000 # '' for fin whales
10 alpha_x = 10 ** -8 # Effects of competition for blue whales
11 alpha_y = 10 ** -8 # '' for fin whales
12 price_x = 12_000 # Blue whale carcass ($ / whale)
13 price_y = 6_000 # Fin whale carcass ($ / whale)
14
15 # Equations
16 dx_dt(x,y) = r_x * x * (1 - x / K_x) - alpha_x * x * y # Growth (& harvest)
   ↪    rate for blue whales
17 dy_dt(x,y) = r_y * y * (1 - y / K_y) - alpha_y * x * y # '' for fin whales
18 revenue(x,y) = (price_x * dx_dt) + (price_y * dy_dt)
19 G(x,y) = dx_dt(x,y) + dy_dt(x,y) # Growth rate for whales
20
21 # Solve the model
22 revenue_x = revenue.derivative(x) # Partial derivative with respect to x
23 revenue_y = revenue.derivative(y) # Partial derivative with respect to x
24 solutions = solve([revenue_x(x,y) == 0, revenue_y(x,y) == 0], x, y) # get
   ↪    critical points
25 optimal_x = round(solutions[0][0].rhs(), 0)
26 optimal_y = round(solutions[0][1].rhs(), 0)
27 print(f"The optimal number of blue whales is {optimal_x}. For fin whales,
   ↪    it is {optimal_y}.")
28
29 most_revenue = revenue(optimal_x, optimal_y)
30 print(most_revenue)
```

Figure 2: Exercise 3 Code