Multivariable Calculus Take Home Exam I

Paul Beggs

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1. (6 points) Consider the parametric curve defined by $x(t) = t^2 + 3t$, $y(t) = \sin(\pi t)$, for $0 \leq t \leq 1$. Determine the area between this curve and the x-axis. Give your answer as exact, and it might be useful to use Integration by Parts for some of this!

Solution. To find the area between the curve and the x-axis, we can use:

$$\text{Area} = \int_0^1 y(t)\, x'(t)\, dt = \int_0^1 \sin(\pi t)\, \big(2t + 3\big)\, dt.$$

Split this into two integrals:

$$\int_0^1 \sin(\pi t)\,(2t)\, dt \quad + \quad \int_0^1 3\, \sin(\pi t)\, dt.$$

For $\int_0^1 3\, \sin(\pi t)\, dt$, we get:

$$3 \int_0^1 \sin(\pi t)\, dt = \frac{3}{\pi} \Big[-\cos(\pi t)\Big]_0^1 = \frac{3}{\pi}\big(-\cos(\pi) - (-\cos(0))\big) = \frac{3}{\pi}\big(-(-1) - (-1)\big) = \frac{3}{\pi}(1+1) = \frac{6}{\pi}.$$

For $\int_0^1 2t\, \sin(\pi t)\, dt$, use integration by parts with $u = 2t$ and $dv = \sin(\pi t) dt$. We get $du = 2\, dt$ and $v = -\frac{1}{\pi}\cos(\pi t)$. Then:

$$\int u\, dv = uv\Big|_0^1 - \int v\, du.$$

Evaluating,

$$2t \cdot \Big(-\tfrac{1}{\pi}\cos(\pi t)\Big)\Big|_0^1 = -\tfrac{2}{\pi}\big(1 \cdot \cos(\pi) - 0\big) = -\tfrac{2}{\pi}\big(-1\big) = \tfrac{2}{\pi},$$

and

$$-\int_0^1 \Big(-\tfrac{1}{\pi}\cos(\pi t)\Big) \cdot 2\, dt = \tfrac{2}{\pi}\int_0^1 \cos(\pi t)\, dt = \tfrac{2}{\pi} \times \frac{1}{\pi}\Big[\sin(\pi t)\Big]_0^1 = \tfrac{2}{\pi^2}\big(\sin(\pi) - \sin(0)\big) = 0.$$

Summing up, $\int_0^1 2t\, \sin(\pi t)\, dt = \frac{2}{\pi}$. Therefore, the total area is:

$$\frac{2}{\pi} + \frac{6}{\pi} = \frac{8}{\pi}.$$

2. (6 points) Determine an equation, in symmetric form, for the line of intersection of the two planes with equations $4x + 2y - z + 5 = 0$ and $3x - y + z + 1 = 0$.

Solution. The direction vector for the line is found by taking the cross product of the normals:

$$\langle 4, 2, -1\rangle \times \langle 3, -1, 1\rangle = \langle 1, -7, -10\rangle.$$

Next, we find a point on the line by solving the system. Setting $x = 0$ gives

$$2y - z + 5 = 0, \quad -y + z + 1 = 0.$$

From the second equation $z = y - 1$. Substitute into the first to get $2y - (y - 1) + 5 = 0$. This yields $y = -6$ and $z = -7$. So a point on the line is $(0, -6, -7)$.

Parametric form:

$$(x, y, z) = (0, -6, -7) + t\,\langle 1, -7, -10\rangle.$$

Symmetric form:

$$\frac{x - 0}{1} = \frac{y + 6}{-7} = \frac{z + 7}{-10}.$$

3. (4 points each) Let $\mathbf{r}(t) = \sin(t)\mathbf{i} + 7t\mathbf{j} - \cos(t)\mathbf{k}$ represent the position of a particle at time t.

(a) Find the velocity vector, $\mathbf{v}(t) = \mathbf{r}'(t)$.

Solution.

$$\mathbf{v}(t) = \frac{d}{dt}\big(\sin(t)\big)\,\mathbf{i} + \frac{d}{dt}\big(7t\big)\,\mathbf{j} - \frac{d}{dt}\big(\cos(t)\big)\,\mathbf{k} = \cos(t)\,\mathbf{i} + 7\,\mathbf{j} + \sin(t)\,\mathbf{k}.$$

(b) What total distance is travelled by the particle over the time interval $[0,\,4]$? Find this as an exact value – though, you are welcome to check the integral with your calculator.

Solution. The speed is $\|\mathbf{v}(t)\| = \sqrt{\cos^2(t) + 7^2 + \sin^2(t)} = \sqrt{49 + (\cos^2(t) + \sin^2(t))} = \sqrt{50} = 5\sqrt{2}$.

Hence, the distance travelled over $[0, 4]$ is

$$\int_0^4 \|\mathbf{v}(t)\|\,dt = \int_0^4 5\sqrt{2}\,dt = 5\sqrt{2} \times 4 = 20\sqrt{2}.$$

(c) Find the unit tangent vector, $\mathbf{T}(t)$.

Solution.

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\langle \cos(t),\, 7,\, \sin(t)\rangle}{5\sqrt{2}} = \Big\langle \frac{\cos(t)}{5\sqrt{2}}, \frac{7}{5\sqrt{2}}, \frac{\sin(t)}{5\sqrt{2}}\Big\rangle.$$

(d) Find the unit normal vector, $\mathbf{N}(t)$.

Solution. We can compute $\mathbf{N}(t)$ by taking $\mathbf{T}'(t)$ and dividing by its magnitude. In practice, you would differentiate each component of $\mathbf{T}(t)$, then normalize. The detailed expression for $\mathbf{T}'(t)$ has terms involving $\sin(t)$ and $\cos(t)$, and care must be taken in simplification:

$$\mathbf{T}'(t) = \Big\langle -\frac{\sin(t)}{5\sqrt{2}}, 0, \frac{\cos(t)}{5\sqrt{2}}\Big\rangle,$$

and $\|\mathbf{T}'(t)\| = \frac{1}{5\sqrt{2}}\sqrt{\sin^2(t) + \cos^2(t)} = \frac{1}{5\sqrt{2}}$.

Thus,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||} = \big(-\sin(t),\ 0,\ \cos(t)\big).$$

4. (4 points each) Consider the vector valued function $\mathbf{r}(t) = \langle f(t), g(t)\rangle$. Let α be a fixed number and define

$$\mathbf{r}_\alpha(t) = \langle f(t)\cos(\alpha) - g(t)\sin(\alpha),\ f(t)\sin(\alpha) + g(t)\cos(\alpha)\rangle\,.$$

   (a) Show that for any t, $||\mathbf{r}(t)|| = ||\mathbf{r}_\alpha(t)||$.

   Solution.

   (b) Show that for any t, $||\mathbf{r}'(t)|| = ||\mathbf{r}'_\alpha(t)||$. [Hint: Remember that α is a constant!]

   Solution.

   (c) Determine $\mathbf{T}(t)$ and $\mathbf{T}_\alpha(t)$, the unit tangent vectors for $\mathbf{r}$ and $\mathbf{r}_\alpha$, respectively.

   Solution.

   (d) Find the angle θ between the unit tangent vectors for $\mathbf{r}(t)$ and $\mathbf{r}_\alpha(t)$.

   Solution.

5. (3 points each) Suppose that each of $\mathbf{u}$ and $\mathbf{v}$ are unit vectors and are orthogonal. Let r be a fixed positive number and define the vector-valued function $\mathbf{r}(t) = r\cos(t)\mathbf{u} + r\sin(t)\mathbf{v}$.

   (a) Explain why there is a single plane that contains $\mathbf{r}(t)$ for each choice of t.

   Solution.

   (b) Determine $||\mathbf{r}(t)||$.

   Solution.

   (c) Determine each of $\text{proj}_{\mathbf{u}}\mathbf{r}(t)$ and $\text{proj}_{\mathbf{v}}\mathbf{r}(t)$.

   Solution.

   (d) What does this tell you about the curve generated by $\mathbf{r}(t)$?

   Solution.