Multivariable Calculus Exam I Corrections

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In Class Portion

1. Consider the parametric curve defined by $x(t) = 3t^2 - 8t + 1$, $y(t) = e^{-t^2}$, for $0 \leq t \leq 2$.

   (a) (4 points) Find the equation, in regular Cartesian coordinates, of the tangent line to this curve at $t = 1$. Please use exact values here!

   Solution. First, we compute the derivatives of $x(t)$ and $y(t)$ with respect to t:

   $$\tfrac{dx}{dt} = 6t - 8 \quad \text{and} \quad \tfrac{dy}{dt} = -2te^{-t^2}.$$

   Plugging this into the formula for slope, we see that:

   $$\frac{dx}{dy} = \frac{dy/dt}{dx/dt} = \frac{-2te^{-t^2}}{6t-8}.$$

   To get our points, we plug in $t = 1$:

   $$x(1) = 3(1)^2 - 8(1) + 1 = -4 \quad \text{and} \quad y(1) = e^{-1}.$$

   Thus, our point is $(-4, e^{-1})$. Plugging in $t = 1$ into the slope formula, we get:

   $$\left.\frac{dy}{dx}\right|_{t=1} = \frac{-2e^{-1}}{6-8} = e^{-1}.$$

   Thus, the equation of the tangent line is:

   $$\boxed{y = e^{-1}(x+4) + e^{-1}.}$$

   (b) (4 points) Is this curve concave up, down, or neither when $t = 1$? Justify this answer.

   Solution. To determine concavity, we must solve the following equation:

   $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt} = \frac{\frac{d}{dt}(-2te^{-t^2})}{6t-8} = \frac{-2e^{-t^2} + 4t^2e^{-t^2}}{6t-8} \Rightarrow t = 1 \Rightarrow -e^{-1}.$$

   Since $-e^{-1} < 0$, the curve is concave down at $t = 1$. (*Correction Explained*: My equation for concavity was incorrect on my test sheet.)

2. (4 points each) Let $\mathbf{u} = 5\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{v} = -\mathbf{j} + 2\mathbf{k}$.

   (c) Determine $\text{proj}_{\mathbf{v}}\mathbf{u}$. Leave all components as exact values.

Solution. We know that $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$. Thus, we compute:

$$\mathbf{u}\cdot\mathbf{v} = 5(0) + 2(-1) + (-3)(2) = -8 \quad \text{and} \quad \|\mathbf{v}\|^2 = \left(\sqrt{5}\right)^2 = 5.$$

Thus, $\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{-8}{5}(-\mathbf{j} + 2\mathbf{k}) = \boxed{\frac{8}{5}\mathbf{j} - \frac{16}{5}\mathbf{k}.}$

3. (4 points) Let $\mathbf{u} = \langle 5, -1, 2\rangle$ and $\mathbf{v} = \langle -2, y, z\rangle$. What is the relationship between y and z which makes $\mathbf{u}$ orthogonal to $\mathbf{v}$?

Solution. We know that two vectors are orthogonal if their dot product is zero. Thus:

$$\mathbf{u}\cdot\mathbf{v} = 5(-2) + (-1)y + 2z = -10 - y + 2z.$$

Therefore, the relationship between y and z which makes $\mathbf{u}$ orthogonal to $\mathbf{v}$ is:

$$\boxed{-10 - y + 2z = 0 \quad \Rightarrow \quad y = 2z - 10.}$$

5. (6 points) Find an equation in scalar form of the plane which passes through $(-2, 7, 1)$ and is perpendicular to the planes $3x + y - z = 0$ and $-2x - y + 5z + 1 = 0$ [Hint: Think about what the relationship among the various normal vectors must be.]

Solution. For the plane to be perpendicular to a given plane, its normal vector must lie in that given plane. Hence, our normal vector must be orthogonal to both $\langle 3, 1, -1\rangle$ and $\langle -2, -1, 5\rangle$. Thus, we can take the cross product of these two vectors to get our normal vector:

$$\langle 3, 1, -1\rangle \times \langle -2, -1, 5\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ -2 & -1 & 5 \end{vmatrix} = \langle 4, -13, -1\rangle .$$

With our normal vector found, we can plug in our point to get our scalar equation:

$$\boxed{4(x+2) - 13(y-7) - (z-1) = 0 \quad \Rightarrow \quad 4x - 13y - z + 100 = 0}$$

6. (6 points) Find the exact value of curvature κ for the curve defined by $\mathbf{r}(t) = (t^2 - t)\mathbf{i} + (t^3 - 7t + 1)\mathbf{j} + t^3\mathbf{k}$ at the point $t = 1$. [Hint: Since this is defined in $\mathbb{R}^3$, it is *significantly* easier to use the version of κ which uses a cross product!] Numerical approximations, rounded to 4 decimal places, are appropriate here.

Solution. For this problem, we will use the following formula for κ:

$$\frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}.$$

Thus, we need to find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$. We find:

$$\mathbf{r}'(t) = (2t-1)\mathbf{i} + (3t^2 - 7)\mathbf{j} + 3t^2\mathbf{k} \quad \text{and} \quad \mathbf{r}''(t) = 2\mathbf{i} + 6t\mathbf{j} + 6t\mathbf{k}.$$

Plugging in $t = 1$, we get:

$$\mathbf{r}'(1) = \mathbf{i} - 4\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{r}''(1) = 2\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}.$$

Next, we need to find $||\mathbf{r}'(t)||^3$:

$$||\mathbf{r}'(t)||^3 = \left(\sqrt{1+16+9}\right)^3 = \frac{\left(\sqrt{\sqrt{\sqrt{\left(\sum_{n=0}^{\infty} \frac{\left(\frac{3}{2}\ln 26\right)^n}{n!}\right)^5}}}\right)^{\frac{729}{8}}}{\int_0^{2\pi} \sqrt{(\frac{1}{8} - \frac{1}{8}\cos\beta)^2 + (\frac{1}{8}\sin\beta)^2}\, d\beta}.$$

Taking the cross product of $\mathbf{r}'(1)$ and $\mathbf{r}''(1)$, we get:

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 3 \\ 2 & 6 & 6 \end{vmatrix} = \big((-24) - 18\big)\mathbf{i} - (6-6)\mathbf{j} + \big(6 - (-8)\big)\mathbf{k} = -42\mathbf{i} + 14\mathbf{k}.$$

Then, we calculate the magnitude of this cross product:

$$||\mathbf{r}'(1) \times \mathbf{r}''(1)|| = \sqrt{42^2 + 0 + 14^2} = \sqrt{42^2 + 14^2} = \sqrt{1764 + 196} = 14\sqrt{10}.$$

Putting everything together:

$$\kappa = \frac{14\sqrt{10}}{26^{3/2}} = \boxed{0.3339.}$$