

1. Consider the vector field  $\mathbf{F} = \langle 2x - 2y, 2x + 2y, 0 \rangle$

(a) (2 points) Show that  $\mathbf{F}$  is not conservative.

*Solution.* For a vector field to be conservative, its curl must be zero. Computing the curl of  $\mathbf{F}$ :

$$\begin{aligned}\nabla \times \mathbf{F} &= \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k} \\ &= (0 - 0)\mathbf{i} + (0 - 0)\mathbf{j} + (2 - (-2))\mathbf{k} \\ &= 4\mathbf{k}.\end{aligned}$$

Since  $\nabla \times \mathbf{F} = \langle 0, 0, 4 \rangle \neq \langle 0, 0, 0 \rangle$ ,  $\mathbf{F}$  is not conservative.

(b) (2 points) Show that  $\mathbf{F}$  is not solenoidal.

*Solution.* A vector field is solenoidal if its divergence is zero. Thus:

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(2x - 2y) + \frac{\partial}{\partial y}(2x + 2y) + \frac{\partial}{\partial z}(0) \\ &= 2 + 2 + 0 \\ &= 4.\end{aligned}$$

Since  $\nabla \cdot \mathbf{F} = 4 \neq 0$ ,  $\mathbf{F}$  is not solenoidal.

(c) (2 points each) Let  $\mathbf{G} = \langle 2x, 2y, 0 \rangle$  and  $\mathbf{H} = \langle -2y, 2x, 0 \rangle$ . Notice that  $\mathbf{F} = \mathbf{G} + \mathbf{H}$ .

i. Show that  $\mathbf{G}$  is conservative, and find a potential function  $g$ .

*Solution.*

$$\nabla \times \mathbf{G} = (0 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + \left( \frac{\partial}{\partial x}(2y) - \frac{\partial}{\partial y}(2x) \right) \mathbf{k}.$$

Since  $\nabla \times \mathbf{G} = \langle 0, 0, 0 \rangle$ ,  $\mathbf{G}$  is conservative.

A potential function is one where  $\nabla g = \mathbf{G}$ . Thus, since we have a (relatively) straightforward vector field, the following will work for a potential function:

$$g(x, y, z) = x^2 + y^2 + K. \quad (\text{Setting } K = 0 \text{ for simplicity.})$$

We can check this by finding  $\nabla g$ :

$$\nabla g(x, y, z) = \left\langle \frac{\partial}{\partial x}(x^2 + y^2), \frac{\partial}{\partial y}(x^2 + y^2), \frac{\partial}{\partial z}(x^2 + y^2) \right\rangle = \langle 2x, 2y, 0 \rangle = \mathbf{G}.$$

- ii. Show that  $\mathbf{H}$  is solenoidal – so that it is the curl of some other vector field  $\mathbf{C}$ . Find such a  $\mathbf{C}$ . [Hint: You might want to choose the  $z$ -component to be 0.]

*Solution.*

$$\begin{aligned}\nabla \cdot \mathbf{H} &= \frac{\partial}{\partial x}(-2y) + \frac{\partial}{\partial y}(2x) + \frac{\partial}{\partial z}(0) \\ &= 0 + 0 + 0 \\ &= 0.\end{aligned}$$

Since  $\nabla \cdot \mathbf{H} = 0$ , the vector field  $\mathbf{H}$  is solenoidal.

For finding the vector field  $\mathbf{C}$ , we can look at the formula for the curl of a vector field from (a). We see that we need to have our  $x$ -component and  $y$ -component to be equal to  $\mathbf{H}$ . Thus, for the  $x$ -component, we need to get  $-2y$ . The only part of the formula that allows us that is the  $\frac{\partial R}{\partial y}$  part. The same argument can be said for the  $y$ -component. We need to get a positive  $2x$ , and we can get that through  $(-\frac{\partial R}{\partial x})$ . Thus, this leads me to believe that we need to keep our function in  $R$  of  $\mathbf{C}$ , and ensure  $P$  and  $Q$  are both 0. This leaves us with:

$$\mathbf{C} = \langle 0, 0, -x^2 - y^2 \rangle.$$

Taking the curl of this vector field, we can confirm our choice:

$$\begin{aligned}\nabla \times \mathbf{C} &= \left\langle \frac{\partial}{\partial y}(-x^2 - y^2) - 0, 0 - \frac{\partial}{\partial x}(-x^2 - y^2), 0 - 0 \right\rangle \\ &= \langle -2y, -(-2x), 0 \rangle \\ &= \langle -2y, 2x, 0 \rangle.\end{aligned}$$

- (d) (2 points) Conclude that we have decomposed  $\mathbf{F}$  into a purely conservative (i.e., irrotational) part and a purely solenoidal (i.e., divergence-free) part, so that  $\mathbf{F} = \nabla g + \nabla \times \mathbf{C}$ . [This can always be done, so long as everything is continuous enough.]

*Solution.* We have shown that:

- $\mathbf{G} = \langle 2x, 2y, 0 \rangle$  is conservative with potential function  $g(x, y, z) = x^2 + y^2$ , so  $\mathbf{G} = \nabla g$ .
- $\mathbf{H} = \langle -2y, 2x, 0 \rangle$  is solenoidal and can be expressed as  $\mathbf{H} = \nabla \times \mathbf{C}$  where  $\mathbf{C} = \langle 0, 0, -x^2 - y^2 \rangle$ .
- $\mathbf{F} = \mathbf{G} + \mathbf{H}$ .

Therefore, we have successfully decomposed  $\mathbf{F}$  as follows:

$$\begin{aligned}\mathbf{F} &= \mathbf{G} + \mathbf{H} \\ &= \nabla g + \nabla \times \mathbf{C} \\ &= \nabla(x^2 + y^2) + \nabla \times \langle 0, 0, -x^2 - y^2 \rangle \\ &= \langle 2x, 2y, 0 \rangle + \langle -2y, 2x, 0 \rangle \\ &= \langle 2x - 2y, 2x + 2y, 0 \rangle.\end{aligned}$$

2. (8 points) Let  $\mathbf{F} = \langle 6x^2y, -6x - 4y \rangle$ . Let  $C$  be the rectangle with endpoints  $(0, 0)$ ,  $(4, 0)$ ,  $(4, 1)$ , and  $(0, 1)$ , with positive orientation. Determine the exact value of the flux of  $\mathbf{F}$  over  $C$ :  $\oint \mathbf{F} \cdot \mathbf{N} \, ds$ .

*Solution.* Since the rectangle is closed, we can use Green's Theorem for flux:

$$\oint \mathbf{F} \cdot \mathbf{N} \, ds = \iint_D (\nabla \cdot \mathbf{F}) \, dA = \iint_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA.$$

Finding the integrand:

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial x}(6x^2y) = 12xy, \quad \frac{\partial Q}{\partial y} = \frac{\partial}{\partial y}(-6x - 4y) = -4 \quad \Rightarrow \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 12xy - 4.$$

Leaving us with:

$$\int_0^4 \int_0^1 (12xy - 4) \, dy \, dx.$$

Integrating with respect to  $y$ :

$$\int_0^1 (12xy - 4) \, dy = [6xy^2 - 4y]_0^1 = 6x - 4.$$

Then, integrating with respect to  $x$ :

$$\int_0^4 (6x - 4) \, dx = [3x^2 - 4x]_0^4 = (3 \cdot 16 - 4 \cdot 4) = \boxed{32}.$$

3. (8 points) Determine  $\iint_S z \, dS$  where  $S$  is the surface  $y = 3x + z^2$ , where  $0 \leq x \leq 1$  and  $0 \leq z \leq 2$ .

*Solution.* Parameterize  $S$  with:

$$r(x, z) = (x, 3x + z^2, z), \quad 0 \leq x \leq 1, \quad 0 \leq z \leq 2.$$

Then, we need to find  $dS$ :

$$dS = \|r_x \times r_z\| \, dx \, dz,$$

where

$$r_x = (1, 3, 0) \quad \text{and} \quad r_z = (0, 2z, 1)$$

gives

$$r_x \times r_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 0 \\ 0 & 2z & 1 \end{vmatrix} = \langle 3, -1, 2z \rangle \quad \Rightarrow \quad \|r_x \times r_z\| = \sqrt{10 + 4z^2}.$$

Setting up the integral:

$$\iint_S z \, dS = \int_0^1 \int_0^2 z \sqrt{10 + 4z^2} \, dz \, dx.$$

Since the integrand does not depend on  $x$ , we simply find:

$$\int_0^1 dx = 1.$$

This leaves us with the  $z$ -integral:

$$\int_0^2 z \sqrt{10 + 4z^2} \, dz.$$

Using  $u$ -substitution:

$$u = 10 + 4z^2 \quad \Rightarrow \quad du = 8z \, dz,$$

with new bounds 10 and 26:

$$\int_{10}^{26} \frac{1}{8} \sqrt{u} \, du = \frac{1}{8} \left[ \frac{2}{3} u^{3/2} \right]_{10}^{26} = \boxed{\frac{1}{12} (26^{3/2} - 10^{3/2})}.$$

4. (8 points) Let  $\mathbf{F}(x, y, z) = \langle y, x, xz \rangle$  and the surface  $S$  be the part of the paraboloid  $z = 4 - x^2 - y^2$  that lies above  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ , where positive orientation is directed upward. Determine

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

[Hint: The surface, projected into the  $xy$ -plane, is *not* a quarter of the unit circle, and therefore, it is likely easiest to parameterize  $S$  in Cartesian coordinates.]

*Solution.* Parameterize  $S$  with:

$$\mathbf{r}(x, y) = (x, y, 4 - x^2 - y^2), \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1.$$

Then, finding  $d\mathbf{S}$ :

$$d\mathbf{S} = r_x \times r_y \, dx \, dy,$$

where

$$r_x = (1, 0, -2x) \quad \text{and} \quad r_y = (0, 1, -2y)$$

gives

$$r_x \times r_y = (2x, 2y, 1).$$

Therefore:

$$d\mathbf{S} = (2x, 2y, 1) \, dx \, dy.$$

This leaves us with the following:

$$\begin{aligned} \iint_S \mathbf{F}(\mathbf{r}(x, y)) \cdot (r_x \times r_y) \, dx \, dy &= \int_0^1 \int_0^1 (y, x, x(4 - x^2 - y^2)) \cdot (2x, 2y, 1) \, dx \, dy \\ &= \int_0^1 \int_0^1 (4xy + 4x - x^3 - xy^2) \, dx \, dy \\ &= \int_0^1 \left[ 2xy^2 + 4xy - x^3y - \frac{1}{3}xy^3 \right]_0^1 \, dy \\ &= \int_0^1 \left( 2x + 4x - \frac{1}{3}x - x^3 \right) \, dx \\ &= \left[ x^2 + 2x^2 - \frac{1}{6}x^2 - \frac{1}{4}x^4 \right]_0^1 \\ &= \boxed{\frac{31}{12}}. \end{aligned}$$

5. (8 points) Calculate the flux of  $\mathbf{F}(x, y, z) = \langle x^3 + y, y^3 + z^2, z^3 + x^3 \rangle$  across the surface of the sphere centered at the origin with radius 2, with positive orientation. [Since the surface is closed, there are two distinct ways to do this – though both are things you can work out, one is *significantly* easier than the other.]

*Solution.* The equation of a sphere with radius 2 gives:

$$x^2 + y^2 + z^2 = 4.$$

Then, by the divergence theorem:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_V (\nabla \cdot \mathbf{F}) dV, \quad \text{where } V = \{x^2 + y^2 + z^2 \leq 4\}.$$

Solving for  $\nabla \cdot \mathbf{F}$ :

$$\frac{\partial}{\partial x}(x^3 + y) + \frac{\partial}{\partial y}(y^3 + z^2) + \frac{\partial}{\partial z}(z^3 + x^3) = 3x^2 + 3y^2 + 3z^2 = 3(x^2 + y^2 + z^2).$$

Switching to spherical coordinates:

$$3(x^2 + y^2 + z^2) = 3r^2, \quad \text{with } dV = r^2 \sin(\phi) dr d\phi d\theta.$$

This leaves us with the integral:

$$\iiint_V 3r^2 dV = 3 \int_0^2 \int_0^\pi \int_0^{2\pi} r^2(r^2 \sin(\phi)) d\theta d\phi dr = 3(2\pi)(2) \int_0^2 r^4 dr = 12\pi \left[ \frac{1}{5} r^5 \right]_0^2 = \boxed{\frac{384\pi}{5}}.$$