# Real Analysis

## MATH 350

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*Author*

Paul Beggs
BeggsPA@Hendrix.edu

*Instructor*

Prof. Christopher Camfield, Ph.D.

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## Exercise: 4.5.4

Let $g$ be continuous on an interval $A$ and let $F$ be the set of points where $g$ fails to be one-to-one; that is,

$$F = \{x_1 \in A : f(x_1) = f(x_2) \text{ for some } x_1 \neq x_2 \text{ and } x_2 \in A\}.$$

Show $F$ is either empty or uncountable.

*Proof.* Assume $F$ is not empty; that is, there exists distinct points $x_1, x_2 \in A$ such that $g(x_1) = g(x_2)$. (Note that if $F$ were empty, then we would be done.)

Without loss of generality, assume $x_1 < x_2$.

From here, we consider two cases. Either $g$ is constant, or $g$ is not constant on $[x_1, x_2]$. For the first case:

**Case 1: $g$ is constant on $[x_1, x_2]$.**

Since $g(x) = y$ for all $x \in [x_1, x_2]$, every point in this interval belongs to $F$. Then, because $[x_1, x_2]$ is uncountable (intervals in $\mathbb{R}$ contain uncountably many points), it follows that $F$ is uncountable.

**Case 2: $g$ is not constant on $[x_1, x_2]$.**

Because $g$ is not constant, there exists some $c \in (x_1, x_2)$ such that $g(c) \neq y$. By the continuity of $g$, we can apply the Intermediate Value Theorem (IVT) to analyze $g$ on the intervals $[x_1, c]$ and $[c, x_2]$:

(i) On $[x_1, c]$: $g(x_1) = y$ and $g(c) \neq y$. By the IVT, $g(x)$ attains every value between $y$ and $g(c)$ on $[x_1, c]$.

(ii) On $[c, x_2]$: $g(c) \neq y$ and $g(x_2) = y$. By the IVT, $g(x)$ attains every value between $g(c)$ and $y$ on $[c, x_2]$.

For any value $k$ between $y$ and $g(c)$, there exist at least two points $x_1' \in [x_1, c]$ and $x_2' \in [c, x_2]$ such that $g(x_1') = g(x_2') = k$. These points $x_1'$ and $x_2'$ are distinct because they belong to different subintervals. Thus, every such $k$ corresponds to multiple $x \in [x_1, x_2]$ where $g(x) = g(x')$, and these $x$ belong to $F$.

Since there are uncountably many such $k$ (as the range of $g$ over $[x_1, x_2]$ is uncountable), it follows that there are uncountably many $x \in [x_1, x_2]$ in $F$.

Therefore, because there are uncountably many such $x$, it follows that $F$ is uncountable. $\square$

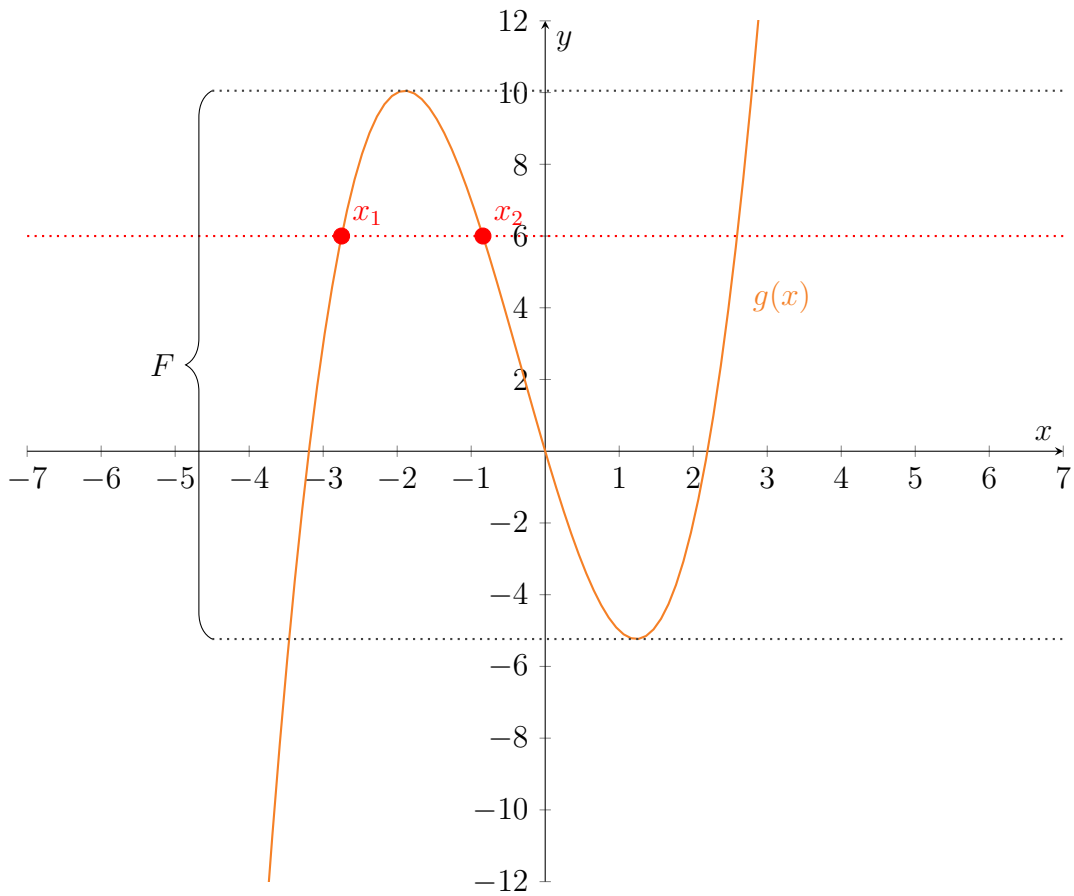


Figure 1: Graph of $g(x) = x^3 + x^2 - 7x$ (Non-empty)