

# Multivariable Calculus Practice Set III

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1. (3 points) Determine the absolute extrema for the function  $f(x, y) = x^2 + 3y^2 - 2x - y - xy$  on the triangular region with vertices  $(0, 0)$ ,  $(2, 0)$ , and  $(0, 1)$ .

*Solution.* We first find the critical points of the function:

$$\begin{aligned}\nabla f(x, y) &= \langle 2x - 2 - y, 6y - 1 - x \rangle = \mathbf{0} \\ \implies y &= 2x - 2 \quad \text{and} \quad x = 6(2x - 2) - 1 - x \\ \implies y &= \frac{4}{11} \quad \text{and} \quad x = \frac{13}{11}\end{aligned}$$

This gives the critical point  $\left(\frac{13}{11}, \frac{4}{11}\right)$ . We also need to check the boundary of the region. Thus:

$(\ell_1)$ :  $y = 0, 0 \leq x \leq 2 \implies f(x, y) = g(x) = x^2 + 3(0)^2 - 2x - (0) - x(0) = x^2 - 2x \implies g'(x) = 2x - 2$ .  
Therefore, the critical points are  $\boxed{(1, 0)}$ .

$(\ell_2)$ :  $x = 0, 0 \leq y \leq 1 \implies f(x, y) = h(y) = (0)^2 + 3y^2 - (0) - y - 0 = 3y^2 - y \implies h'(y) = 6y - 1$ .  
Hence, the critical points are  $\boxed{\left(0, \frac{1}{6}\right)}$ .

$(\ell_3)$ :  $y = 1 - \frac{1}{2}x, 0 \leq x \leq 2 \implies f(x, y) = k(x) = x^2 + 3\left(1 - \frac{1}{2}x\right)^2 - 2x - \left(1 - \frac{1}{2}x\right) - x\left(1 - \frac{1}{2}x\right)$ .  
Solving this equation for  $x$ :

$$\begin{aligned}k(x) &= x^2 + 3\left(1 - \frac{1}{2}x - \frac{1}{2}x + \frac{1}{4}x^2\right) - 2x - 1 + \frac{1}{2}x - x + \frac{1}{2}x^2 \\ &= x^2 + 3\left(1 - x + \frac{1}{4}x^2\right) - 2x - 1 + \frac{1}{2}x - x + \frac{1}{2}x^2 \\ &= \left[x^2 + \frac{3}{4}x^2 + \frac{1}{2}x^2\right] + [-3x - 2x - \frac{1}{2}x] + [3 - 1] \\ &= \frac{9}{4}x^2 - \frac{7}{2}x + 2 \\ &= \frac{1}{4}(9x^2 - 22x + 8) \\ \implies k'(x) &= \frac{1}{4} \cdot \frac{d}{dx}[9x^2 - 22x + 8] \\ 0 &= \frac{1}{2}(9x - 11) \\ x &= \frac{11}{9}\end{aligned}$$

Using this  $x$ -value, we plug it back into our equation for  $y$  to get the critical point  $\boxed{\left(\frac{11}{9}, \frac{7}{18}\right)}$ .

With our function's and lines' critical points found, we also need to find the vertices of the triangle:

$$f(0,0) = 0^2 + 3(0)^2 - 2(0) - 0 - 0(0) = 0$$

$$f(2,0) = 2^2 + 3(0)^2 - 2(2) - 0 - 2(0) = 0$$

$$f(0,1) = 0^2 + 3(1)^2 - 2(0) - 1 - 0(1) = 2$$

Now that we have our critical points, we can evaluate the function at each of these points to determine the absolute extrema:

$$\begin{aligned} f\left(\frac{13}{11}, \frac{4}{11}\right) &= \left(\frac{13}{11}\right)^2 + 3\left(\frac{4}{11}\right)^2 - 2\left(\frac{13}{11}\right) - \frac{4}{11} - \frac{13}{11} \\ &\approx -1.363\dots \end{aligned}$$

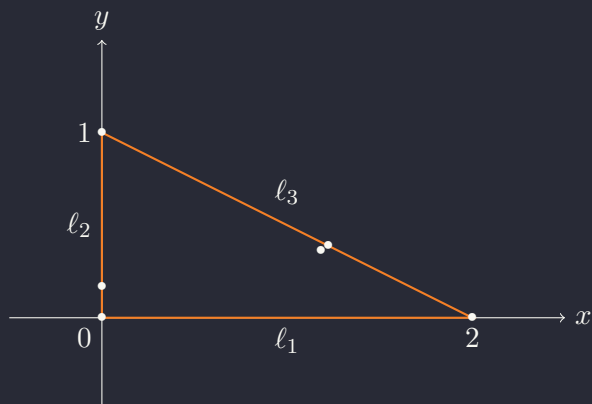
$$\begin{aligned} f(1,0) &= 1^2 + 3(0)^2 - 2(1) - 0 - 1(0) \\ &= -1 \end{aligned}$$

$$\begin{aligned} f\left(0, \frac{1}{6}\right) &= 0^2 + 3\left(\frac{1}{6}\right)^2 - 2(0) - \frac{1}{6} - 0 \\ &\approx -0.083\dots \end{aligned}$$

$$\begin{aligned} f\left(\frac{11}{9}, \frac{7}{18}\right) &= \left(\frac{11}{9}\right)^2 + 3\left(\frac{7}{18}\right)^2 - 2\left(\frac{11}{9}\right) - \frac{7}{18} - \frac{11}{9} \\ &\approx -1.361\dots \end{aligned}$$

Thus, this gives us 7 critical points:

Point	$f(x, y)$	Type
$\left(\frac{13}{11}, \frac{4}{11}\right)$	-1.364	Interior CP
$(1, 0)$	-1	$\ell_1$
$\left(0, \frac{1}{6}\right)$	-0.083	$\ell_2$
$\left(\frac{11}{9}, \frac{7}{18}\right)$	-1.361	$\ell_3$
$(0, 0)$	0	Vertex 1
$(2, 0)$	-2	Vertex 2
$(0, 1)$	2	Vertex 3



With these values, we can see that the absolute maximum is  $\boxed{2}$ , which occurs at the vertex  $(0,1)$ , and the absolute minimum is  $\boxed{-1.364}$ , which occurs at the critical point  $\left(\frac{13}{11}, \frac{4}{11}\right)$ .

*Note that the points that I labeled as critical points should just be labeled as points. The critical points are the ones that are in the interior of the region.*

2. (1 point each) Convert each as indicated; leave each answer as exact:

- (a) Convert the rectangular point  $(-5, 1)$  to polar coordinates.

*Solution.*

$$r = \sqrt{(-5)^2 + 1^2} = \sqrt{26}$$

$$\theta = \arctan\left(\frac{1}{-5}\right) = \arctan\left(-\frac{1}{5}\right) = \frac{7\pi}{6}$$

The polar coordinates are  $\boxed{(\sqrt{26}, \frac{7\pi}{6})}$ .

- (b) Convert the cylindrical point  $(5, \frac{7\pi}{6}, 2)$  to rectangular.

*Solution.*

$$x = 5 \cos\left(\frac{7\pi}{6}\right) = 5\left(-\frac{\sqrt{3}}{2}\right) = -\frac{5\sqrt{3}}{2}$$

$$y = 5 \sin\left(\frac{7\pi}{6}\right) = 5\left(-\frac{1}{2}\right) = -\frac{5}{2}$$

$$z = 2$$

The rectangular coordinates are  $\boxed{\left(-\frac{5\sqrt{3}}{2}, -\frac{5}{2}, 2\right)}$ .

- (c) Convert the rectangular point  $(-2, 4, -1)$  to spherical.

*Solution.*

$$\rho = \sqrt{(-2)^2 + 4^2 + (-1)^2} = \sqrt{21}$$

$$\theta = \arctan\left(\frac{4}{-2}\right) = \arctan(-2)$$

$$\phi = \arccos\left(\frac{-1}{\sqrt{21}}\right) = \arccos\left(-\frac{1}{\sqrt{21}}\right)$$

Since the point  $(-2, 4)$  is in the second quadrant, we add  $\pi$  to the arctan value. Hence, the spherical coordinates are  $\boxed{\left(\sqrt{21}, \pi + \arctan(-2), \arccos\left(-\frac{1}{\sqrt{21}}\right)\right)}$ .

- (d) Convert the spherical point  $(4, \frac{11\pi}{6}, \frac{3\pi}{4})$  to cylindrical.

*Solution.* The conversion from spherical to cylindrical follows the following equations:

$$r = \rho \sin \phi, \quad \theta = \theta, \quad \text{and} \quad z = \rho \cos \phi.$$

Thus, we have:

$$r = 4 \sin\left(\frac{3\pi}{4}\right) = 4 \left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

$$\theta = \frac{11\pi}{6}$$

$$z = 4 \cos\left(\frac{3\pi}{4}\right) = 4 \left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$$

Therefore, we get the cylindrical coordinates  $\boxed{\left(2\sqrt{2}, \frac{11\pi}{6}, -2\sqrt{2}\right)}$ .

3. (3 points) Determine the value of each given integral. You need to do the work here by hand, but of course can check any answers with technology.

(a)  $\iint_D (x^2 + 6xy) dA$  where  $D$  is the triangle with vertices  $(0, 0)$ ,  $(4, 0)$ , and  $(0, 12)$

*Solution.* We can see that this triangle is bounded by three lines:



$$\ell_1 : y = 0$$

$$\ell_2 : x = 0$$

$$\ell_3 : y = -3x + 12$$

This gives us the limits of integration as follows:

$$\{(x, y) : 0 \leq x \leq 4, \quad 0 \leq y \leq -3x + 12\}.$$

Thus, we can write the double integral as:

$$\begin{aligned} \iint_D (x^2 + 6xy) dA &= \int_0^4 \int_0^{-3x+12} (x^2 + 6xy) dy dx \\ &= \int_0^4 [x^2 y + 3xy^2]_0^{-3x+12} dx \\ &= \int_0^4 [x^2(-3x+12) + 3x(-3x+12)^2] dx \\ &= \int_0^4 [-3x^3 + 12x^2 + 3x(9x^2 - 72x + 144)] dx \\ &= \int_0^4 [24x^3 - 204x^2 + 432x] dx \\ &= 6 \int_0^4 [4x^3 - 34x^2 + 72x] dx \\ &= 6 [x^4 - \frac{34}{3}x^3 + 36x^2]_0^4 \\ &= 48 [32 - \frac{34}{3}(8) + 36(2)] \\ &= \boxed{640} \end{aligned}$$

(b)  $\int_0^2 \int_{x^2}^4 4x^3 \cos(y^3) dy dx$

*Solution.* To evaluate this integral, we must change the order of integration. The original region is:

$$\{(x, y) : 0 \leq x \leq 2, \quad x^2 \leq y \leq 4\}.$$

The new region is:

$$\{(x, y) : 0 \leq y \leq 4, \quad 0 \leq x \leq \sqrt{y}\}.$$

Thus, we can rewrite and solve the double integral:

$$\begin{aligned} \int_0^2 \int_{x^2}^4 4x^3 \cos(y^3) dy dx &= \int_0^4 \int_0^{\sqrt{y}} 4x^3 \cos(y^3) dx dy \\ &= \int_0^4 [x^4 \cos(y^3)]_0^{\sqrt{y}} dy \\ &= \int_0^4 y^2 \cos(y^3) dy \\ &= \left[ \frac{1}{3} \sin(y^3) \right]_0^4 \\ &= \frac{1}{3} [\sin(64) - 0] \\ &= \boxed{\frac{1}{3} \sin(64)}. \end{aligned}$$

*No problems like this where we have to switch the order of integration on the exam.*

*Convert to spherical coordinates. Gives*

## REST ON PAPER

Find the value of  $\iint_D x^2 y dA$  where  $D$  is the region between the  $x$ -axis and the upper semicircle  $y = \sqrt{9 - x^2}$ .  
and

Let  $E$  be the part of the unit sphere which lives in the first octant. Determine the exact value of  $\iiint_E (x^2 z + y^2 z + z^3) dV$ .

and

Consider the cylinder, with central axis along the  $y$ -axis, of radius 3 between the  $xy$ -plane and the plane  $y = 10$ . The temperature,  $T$ , at any point inside the cylinder is given by  $T(x, y, z) = x^2 y z^2$ . Determine the average temperature inside the cylinder.