Real Analysis

MATH 350

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CHAPTER 1

THE REAL NUMBERS

1.1 Types of Numbers

The book, *Understanding Analysis* by Stephen Abbott, can be found at this link

The natural numbers, $\mathbb{N}$:

- No additive inverse.
- You can:
  - Add,
  - Multiply

The integers, $\mathbb{Z}$ are known as a Group (more specifically, a "ring").

- You can:
  - Add,
  - Multiply,
  - Subtract

The rational numbers, $\mathbb{Q}$ are known as a "Field."

- You can:
  - Add,
  - Subtract,
  - Multiply,
  - Divide

A problem that rational numbers could not explain: The $45, 45, 90$ triangle had a hypotenuse of $\sqrt{2}$. This did not exist at the time, so it was simply $c^2 = 2$. Therefore, new numbers needed to be invented.

Theorem 1.1.1

There does not exist a rational number r such that $r^2 = 2$.

Proof. Suppose there exists a rational number r such that $r^2 = 2$. Since r is rational, there exists $p, q \in \mathbb{Z}$ such that $r = \frac{p}{q}$. We can assume the p and q have no common factors. (If not, we can factor out the common factor.) By our assumption,

$$\begin{aligned} r^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \end{aligned}$$

It follows that,

$$p^2 = 2q^2$$

Such that p^2 is an even number because if p were odd, then p^2 would be odd. There exists $x \in \mathbb{Z}$ such that $p = 2x$. Recall that $p^2 = 2q^2$. Thus

$$\begin{aligned} (2x)^2 &= 2q^2 \\ 4x^2 &= 2q^2 \\ 2x^2 &= q \end{aligned}$$

Thus, q^2 is even. Hence q is also even. So p and q are both divisible by 2. This contradicts that p and q have no common factors. Thus, our supposition is false. Therefore, there does not exist a rational number r such that $r^2 = 2$ □

So we are going to work with a larger set called the real numbers, $\mathbb{R}$.

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- You can:
  - Add,
  - Subtract,
  - Multiply,
  - Divide
- In other words, all field axioms apply.
- Totally ordered set for any $x, y \in \mathbb{R}$. Thus, one of these are true:
  (a) $x < y$,
  (b) $x > y$,
  (c) $x = y$
- Think of it as a number line.
- $\mathbb{Q}$ is dense:
  If $a, b \in \mathbb{Q}$ with $a \neq b$, there exists $c \in \mathbb{Q}$ which is between a and b such that $a < c < b$. One example is $\frac{a+b}{2}$.
- $\mathbb{Q}$ is not *complete*, but $\mathbb{R}$ is.
  - *Complete:* Think, "no gaps."

1.2 Preliminaries

Things to remember from Intro and Discrete.

Set Notation	Complement
$x \in A$	A^c (not $\overline{A}$)
$A \cup B$	$\mathbb{R} \setminus A$
$A \cap B$	

- $\displaystyle\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \dots$.
- $\displaystyle\bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \dots$.
- **De Morgan's Laws**

1.2.1 Infinite Unions and Intersections

For each $n \in \mathbb{N}$, define $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$. In other words, each subsequent element in the subset will start at n. For example, $A_1 = \{1, 2, \dots\}$, whereas $A_5 = \{5, 6, \dots\}$.

$\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$. To show a number $\in \mathbb{N}$ belongs in the set A_n, we can start with that, $k \in \mathbb{N}$. Then $k \in A_k$. Thus, $k \in A_k \subseteq \bigcup_{n=1}^{\infty} A_n$. therefore, $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$.

$\bigcap_{n=1}^{\infty} A_n = \emptyset$. Obviously, we know that the empty set is a subset of A_n, but to prove that $\bigcap_{n=1}^{\infty} A_n$ is a subset of the empty set, we should suppose a $k \in \mathbb{N}$ such that $k \in \bigcap_{n=1}^{\infty} A_n$. Notice that $k \notin \bigcap_{n=1}^{\infty} A_n$. So, $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

1.2.2 Functions and Notation

$f\colon A \to B$ where f is a function, A is a domain, and B is the co-domain. Thus, $f(x) = y$ such that $x \in A$ and $y \in B$.

Some definitions to keep in mind:

The Dirichlet Function

[Refer to notepaper for these following definitions]

Image

Example: $g : \mathbb{R} \to \mathbb{R}$, when we say $y \in g(A)$ implies $\exists x$ such that $g(x) = y$
Pre-image

Triangle inequality:
The most common application: For any $a, b, c \in \mathbb{R}$, $|a - b| \leq |a - c| + |c - b|$, with the intermediate step of $a - b = (a - c) + (c - b)$.

1.2.3 Common Strategies for Analysis Proofs

Theorem 1.2.6

Let $a, b \in \mathbb{R}$. Then,

$$a = b \text{ if and only if for all } \epsilon > 0, |a - b| < \epsilon.$$

Proof. We show this by proving both implications:

($\Rightarrow$) Assume $a = b$. Let $\epsilon > 0$. Then $|a - b| = 0 < \epsilon$

($\Leftarrow$) Assume for all $\epsilon > 0$, $|a - b| < \epsilon$. Suppose $a \neq b$. Then $a - b \neq 0$. So, $|a - b| \neq 0$. Now, Consider $\epsilon_0 = |a - b|$. By our assumption we know that $|a - b| < \epsilon_0$. It is not true that $|a - b| < |a - b|$. Therefore, it must be the case that $a = b$.

Therefore, by showing both sides of the implication accomplish the same thing as the other side, we know that $a = b$ if and only if for all $\epsilon > 0, |a - b| < \epsilon$. ☐

1.2.4 Mathematical Induction

Inductive Hypothesis: Let $x_1 = 1$. For all $n \in \mathbb{N}$, let $x_{n+1} = \frac{1}{2}x_n + 1$.

Inductive Step: $x_1 = 1, x_2 = 1.5, x_3 = 1.75, x_4 = 1.875$.

Example 1.1: Induction

The sequence (x_n) is increasing. In other words, for all $n \in \mathbb{N}$, $x_n \leq x_{n+1}$.

Proof. Suppose the sequence (x_n) is increasing. We will prove this point by using induction.

Base Case: We see that $x_1 = 1$ and $x_2 = 1.5$. Thus, $x_1 \leq x_2$.

Inductive Hypothesis: For $n \in \mathbb{N}$, assume $x_n \leq x_{n+1}$.

Scratch work: We want: $x_{n+1} \leq x_{n+2}$. We know: $x_{n+1} = \frac{1}{2}x_{n+1} + 1$.

Inductive Step: Then $\frac{1}{2}x_n \leq \frac{1}{2}x_{n+1}$. Hence, $\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1$. Therefore we have proven through induction that, $x_{n+1} \leq x_{n+2}$. □

Exercises

Exercise: 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

(a) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all sets containing an infinite number of elements, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is infinite as well.

(b) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \ldots$ are all finite, nonempty sets of real numbers, then the intersection $\bigcap_{n=1}^{\infty} A_n$ is finite and nonempty.

(c) $A \cap (B \cup C) = (A \cap B) \cup C$

(d) $A \cap (B \cap C) = (A \cap B) \cap C$

(e) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Solution.

(a) This is false. Consider the following as a counterexample: If we define A_1 as $A_n = \{n, n+1, n+2, \ldots\} = \{k \in \mathbb{N} \mid k \geq n\}$, we can see why the intersection of these sets of infinite numbers are actually empty. Consider a number m that actually satisfies $m \in A_n$ for every A_n in our collection of sets. Because m is not an element of A_{m+1}, no such m exists and the intersection is empty.

(b) This is true.

(c) False. Consider sets $A = \{1, 2, 3\}$, $B = \{3, 6, 7\}$ and $C = \{5\}$. Note that $A \cap (B \cup C) = \{3\}$ is not equal to $(A \cap B) \cup C = \{3, 5\}$.

(d) This is true. A proof would start with $x \in A \cap (B \cap C)$.

(e) This is true. A proof would start with $x \in A \cap (B \cup C)$.

Exercise: 1.2.5

De Morgan's Laws. Let A and B be subsets of $\mathbb{R}$.

(a) If $x \in (A \cap B)^c$, explain why $x \in A^c \cup B^c$. This shows that $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Prove the reverse inclusion $(A \cap B)^c \supseteq A^c \cup B^c$, and conclude that $(A \cap B)^c = A^c \cup B^c$.

(c) Show $(A \cup B)^c = A^c \cap B^c$ by demonstrating inclusion both ways.

Solution.

(a) If $x \in (A \cap B)^c$, and we know that $A^c = \{x \in \mathbb{R} : x \notin A\}$, then we know x must cannot exist in A^c and B^c because $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$. Thus, x is in either A^c or B^c. Put another way $x \in A^c \cup B^c$. Since we have shown that an element that started in $(A \cap B)^c$ ended up in $A^c \cup B^c$, then we know $(A \cap B)^c \subseteq A^c \cup B^c$.

(b) Assume there exists a $y \in A^c \cup B^c$. Thus, it must be the case that $y \notin A$ or $y \notin B$. Hence, y cannot be exist in both sets at the same time, so $y \in (A \cap B)^c$. Because we have taken an element that started in $A^c \cup B^c$ and have shown that it exists in $(A \cap B)^c$, we have proven $A^c \cup B^c \subseteq (A \cap B)^c$.

(c)

Proof. We need to show these expressions are subsets of each other in order to prove they are equivalent.

($\subseteq$) Assume there exists $x \in (A \cap B)^c$, and we know that $A^c = \{x \in \mathbb{R} : x \notin A\}$, then we know x must cannot exist in A^c and B^c because $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$. Thus, x is in either A^c or B^c. Put another way $x \in A^c \cup B^c$. Since we have shown that an element that started in $(A \cap B)^c$ ended up in $A^c \cup B^c$, then we know $(A \cap B)^c \subseteq A^c \cup B^c$.

($\supseteq$) Now assume there exists a $y \in A^c \cup B^c$. Thus, it must be the case that $y \notin A$ or $y \notin B$. Hence, y cannot be exist in both sets at the same time, so $y \in (A \cap B)^c$. Because we have taken an element that started in $A^c \cup B^c$ and have shown that it exists in $(A \cap B)^c$, we have proven $A^c \cup B^c \subseteq (A \cap B)^c$.

Therefore, we have shown through proving both sides of the implication, that these two statements are logically equivalent. In that, all elements of $A^c \cup B^c$ are the same elements that are in $(A \cap B)^c$ $\square$

Exercise: 1.2.7

Given a function f and a subset A of its domain, let $f(A)$ represent the range of f over the set A; that is, $f(a) = \{f(x)\colon x \in A\}$.

(a) Let $f(x) = x^2$. if $A = [0, 2]$ (the closed interval $\{x \in \mathbb{R}\colon 0 \leq x \leq 2\}$) and $B = [1, 4]$, find $f(A)$ and $f(B)$. Does $f(A \cap B) = f(A) \cap f(B)$ in this case? Does $f(A \cup B) = f(A) \cup f(B)$?

(b) Find two sets A and B for which $f(A \cap B) \neq f(A) \cap f(B)$.

(c) Show that, for an arbitrary function $g : \mathbb{R} \to \mathbb{R}$, it is always true that $g(A \cap B) \subseteq g(A) \cap g(B)$ for all sets $A, B \subseteq \mathbb{R}$.

(d) Form and prove a conjecture about the relationship between $g(A \cup B)$ and $g(A) \cup g(B)$ for an arbitrary function g.

Solution.

(a) Since $f(x) = x^2$, the intervals of $f(A)$ would be $[0, 4]$ and $f(B)$ would be $[1, 16]$. The interval of the intersection of $A \cap B$ is $[1, 2]$. Take this through our function, we get $f(A \cap B) = [1, 4]$. On the other side of the equation, we already know the intervals of $f(A)$ and $f(B)$, and the intersection of theirs would be $[1, 4]$. So they do equal each other. We know $f(A \cup B)$ and $f(A) \cup f(B)$ will be equivalent because $f(A \cup B)$ has an interval of $[0, 16]$, and $f(A) \cup f(B)$ also has an interval of $[0, 16]$ because taking the union of $[0, 4] \cup [1, 16]$ is $[0, 16]$.

(b) Two sets could be $A = [5, 6]$ and $B = [0, 0]$. Because the sets have nothing in common even after taking their function, they do not equal each other.

(c)

Proof. Let $x \in g(A \cap B)$. Using the definition of function, we know there exists a $y \in A \cap B$ to which that y is mapped to as $g(y) = x$. From the definition of intersection, we know $y \in A$ and $y \in B$ such that $x = g(y) \in g(A)$ and $x = g(y) \in g(B)$ because $y \in A \cap B$. Putting it together, we have $x \in g(A) \cap g(B)$ thus proving $g(A \cap B) \subseteq g(A) \cap g(B)$ □

(d) Conjecture: For any function g defined as $g : \mathbb{R} \to \mathbb{R}$ and for any subsets $A, B \subseteq \mathbb{R}$, the following holds:

$$g(A \cup B) = g(A) \cup g(B)$$

Proof. We need to show these expressions are subsets of each other in order to prove they are equivalent.

($\subseteq$) Take any element $x \in g(A \cap B)$. By definition of function, we know there exists some $y \in A \cup B$ such that $g(x) = y$. From the definition of union, we know $y \in A$ or $y \in B$ such that $x = g(y) \in g(A)$ or $x = g(y) \in g(B)$ or both. Putting it together, we have $x \in g(A) \cup g(B)$ thus proving $g(A \cup B) \subseteq g(A) \cup g(B)$.

($\supseteq$) Take any element $p \in g(A) \cap g(B)$. By definition of union, we know p is either in $g(A)$ or $g(B)$ or both. From the definition of function, we know that if $p \in g(A)$ or $p \in g(B)$ then there exists some $q \in A$ or $q \in B$ such that $g(q) = p$. Putting it together, we have $q \in A \cup B$. Moreover, this means $p = g(x) \in g(A \cup B)$. And since $p \in g(A) \cup g(B)$ implies $p \in g(A \cup B)$, we know $g(A) \cup g(B) \subseteq g(A \cup B)$.

Therefore, since we have proven that both expressions are functions of each other, we have proved that they are equal. □

Exercise: 1.2.8

Given a function $f : A \to B$ can be defined as either one-to-one or onto, give an example of each or state that the request is impossible:

(a) $f : \mathbb{N} \to \mathbb{N}$ that is 1-1 but not onto.

(b) $f : \mathbb{N} \to \mathbb{N}$ that is onto but not 1-1.

(c) $f : \mathbb{N} \to \mathbb{Z}$ that is 1-1 and onto.

Solution.

(a) The function $f(a) + 1$ is 1-1 because when

$$\begin{aligned} f(a_1) &= f(a_2) \\ a_1 + 1 &= a_2 + 1 \\ a_1 &= a_2 \end{aligned}$$

However, the function is not onto because the entire co-domain is not covered. That being 1.

(b) *We need to find a function that will cover every entry in the co-domain, while also* *avoiding a scenario where* $a_1 = a_2 \ldots$ Consider the function,

$$f(a) = \begin{cases} a & \text{if } a \text{ is odd,} \\ a-1 & \text{if } a \text{ is even} \end{cases}$$

This function is onto because every natural number is covered, but it is not 1-1 because $a_1 \neq a_2 - 1$.

(c) This request is not possible. There is no way to map every natural number to every integer because we are simply missing 0! (Not 0 factorial, we do have the number 1, I just mean the number 0 in a exclamatory sense.)

1.3 Axiom of Completeness

Think about $\mathbb{Q}$ and $\mathbb{R}$.

- Both are fields.
  - Both have $+, -, \times, \div$ operations.
- Both are totally ordered
  - $a < b$,
  - $a > b$,
  - or $a = b$
- $\mathbb{R}$ is complete. $\mathbb{Q}$ is not.

Axiom of Completeness

Every nonempty set of real numbers that is bounded has a least upper bound.

Note that upper bounds are not unique! For example, consider the line, A, from 0 to 1. There are infinitely many upper bounds past 1 because A is bounded.

We often call the least upper bound the *supremum* of a set. Example:

Imagine a number line from $(1, 8)$. Note that parenthesis mean $<$ and not $\leq$. Hence, the supremum is 8. Wrote simply as $\sup A$.

Example 1.2: Supremum

Consider a set, $B = [-5, -2] \cup (3, 6) \cup \{13\}$. What is the supremum?

Solution. $\sup B = 13$

At the other end of the set, we have the following:

- lower bounds,
- greatest lower bound
- often called infimum.

The infimum of the previous example would be $\inf B = -5$.

Example 1.3:

Consider the set, $\mathbb{C} = \{\frac{1}{n} : n \in \mathbb{N}\}$. What is the supremum and the infimum?

Solution. $\sup \mathbb{C} = 1$, $\inf \mathbb{C} = 0$.

Example 1.4: L

et $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. Define the set $c + A$ by

$$c + A = \{c + a : a \in A\}$$

Then $\sup(c + A) = c + \sup A$.

Solution. To properly verify this we focus separately on each part of Definition 1.3.2. Setting $s = \sup A$, we see that $a \leq s$ for all $a \in A$, which implies $c + a \leq c + s$ for all $a \in A$. Thus, $c + s$ is an upper bound for $c + A$ and condition (i) is verified. For (ii), let b be an arbitrary upper bound for $c + A$; i.e., $c + a \leq b$ for all $a \in A$. This is equivalent to $a \leq b - c$ for all $a \in A$, from which we conclude that $b - c$ is an upper bound for A. Because s is the least upper bound of A, $s \leq b - c$, which can be rewritten as $c + s \leq b$. This verifies part (ii) of Definition 1.3.2, and we conclude $\sup(c + A) = c + \sup A$.

Why do we need to include infimum and supremum? Don't we have the max and min of a set already? Well, what exactly do we mean by the maximum value of a set?

We say $m \in \mathbb{R}$ is the *maximum* of A if $m \in A$ and for all $x \in A$, $x \leq m$. Note that some sets have a maximum and some sets do not. You cannot refer to a maximum without first knowing it exists. This is the same with minimums.

Lemma 1.3.1

1.3.8Assume s is an upper bound for a set $A \subseteq \mathbb{R}$. Then, s is the supremum of A if and only if for every $\epsilon > 0$, there exists $x \in A$ such that $s - \epsilon < x$.

This lemma allows us to take any positive number and take a "step back." In essence, you can verify something as an upper bound if you continuously back up over and over until you cannot back up any longer.

Proof. We show this by proving both implications:

($\Rightarrow$) Assume $s = \sup A$. Let $\epsilon > 0$. Suppose there are no elements x of A such that $s - \epsilon < x$. Then $s - \epsilon$ would be an upper bound. This contradicts that s is the least upper bound. Therefore, there must exist an element $x \in A$ such that $s - \epsilon < x$.

($\Leftarrow$) Assume for every $\epsilon > 0$, there exists $x \in A$ such that $s - \epsilon < x$. Let t be an upper bound of A. Suppose $t < s$. Consider $\epsilon_0 = s - t > 0$. By our assumption, there exists $x \in A$ such that $s - \epsilon_0 < x$. So, $t < x$. This contradicts that t is an upper bound of A. So, $t \geq s$. Thus, s is the least upper bound

Therefore, by proving both the right and left implication, we have shown the statement to be true. $\square$

Analogous statement about infimums: Assume z is a lower bound of a set $A \subseteq \mathbb{R}$. Then $z = \inf A \iff$ for all $\epsilon > 0$, there exists $y \in A$ such that $y < z + \epsilon$.

Exercises

Exercise: 1.3.4

Let $A_1, A_2, A_3 \ldots$ be a collection of nonempty sets each of which is bounded above.

(a) Find a formula for $\sup(A_1 \cup A_2)$. Extend this to $\sup(\bigcup_{k=1}^{n} A_k)$.

(b) Consider $\sup(\bigcup_{k=1}^{\infty} A_k)$. Does the formula in (a) extend to the infinite case?

Solution.

(a) Let A_1 and A_2 be nonempty sets, each bounded above. To find the largest of the two suprema, we can use the following: $\sup(A_1 \cap A_2) = \max\{\sup A_1, \sup A_2\}$. If we extend this notion to $\sup(\bigcup_{k=1}^{n} A_k)$, we can use the same idea from before and write it as $\sup(\bigcup_{k=1}^{n} A_k) = \max\{\sup A_1, \sup A_2, \ldots, \sup A_n\}$.

(b) The formula does not extend to the infinite case. Consider the counterexample $\bigcup_{k=1}^{\infty} A_k$ where $A_k := [k, k+1]$. Even though these sets are bounded above, when we take the union of them, we approach infinity, which is not bounded: $\bigcup_{k=1}^{\infty} A_k = [1,2] \cup [2,3] \cup \cdots = [1, \infty)$.

Exercise: 1.3.5

As in Example 1.3.7, let $A \subseteq \mathbb{R}$ be nonempty and bounded above, and let $c \in \mathbb{R}$. This time define the set $cA = \{ca : a \in A\}$.

(a) If $c \geq 0$, show that $\sup(cA) = c \sup A$.

(b) Postulate a similar type of statement for $\sup(cA)$ for the case $c < 0$.

Solution.

(a) Let $A \subseteq \mathbb{R}$ be nonempty and bounded above. Define the set $cA := \{ca : a \in A\}$. From the axiom of completeness, because A is bounded above, we know there is a least upper bound, $s = \sup A$. Following from Example 1.3.7, we see that $a \leq s$ for all $a \in A$ which implies $ca \leq cs$ for all $a \in A$. Thus, cs is an upper bound for cA, and the first condition of Definition 1.3.2 is satisfied. For the second condition, we need to look at both $c = 0$ and $c > 0$ to avoid dividing by zero. So, we have two cases:

- $c = 0$**:** If $c = 0$, then $cA = \{0 \colon\ a \in A\} = \{0\}$. Since the only element in cA is 0, $\sup(cA) = 0$. Similarly, because $c = 0$, $c \sup A = 0 \cdot \sup A = 0$. Therefore, $\sup(cA) = c \sup(A)$.
- $c > 0$**:** Let b be an arbitrary upper bound for cA and $c > 0$. In other words, $ca \leq b$ for all $a \in A$. This is equivalent to $a \leq b/c$ where $c \neq 0$, from which we can see that b/c is an upper bound for A. Because s is the least upper bound of A, $s \leq b/c$, which can be rewritten as $cs \leq b$. This verifies the second part of Definition 1.3.2, and we conclude $\sup(cA) = c \sup A$.

(b) Postulate: If $c < 0$, then $\sup(cA) = c \inf(A)$.

Exercise: 1.3.8

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

(a) $\left\{\frac{m}{n} : m, n \in \mathbb{N} \text{ with } m < n\right\}$.

(b) $\left\{\frac{(-1)^m}{n} : m, n \in \mathbb{N}\right\}$.

(c) $\left\{\frac{n}{3n+1} : n \in \mathbb{N}\right\}$.

(d) $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$.

Solution. To avoid writing out every set definition, I am going to denote each set as A_n where n corresponds to the numerical value of the list from (a) - (d).

(a) $\sup A_1 = 1$, $\inf A_1 = 0$

(b) $\sup A_2 = 1$, $\inf A_2 = -1$

(c) $\sup A_3 = \frac{1}{3}$, $\inf A_3 = \frac{1}{4}$

(d) $\sup A_4 = 1$, $\inf A_3 = 0$

1.4 Consequences of Completeness

Theorem 1.4.1: Nested Interval Property

For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = (a_n, b_n)$. Assume I_n contains I_{n+1}. This results in a nested sequence of intervals.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \dots$$

Then, $\bigcap_{n=1}^{\infty} \neq \emptyset$.

tl;dr there has to be something that is common to all of the sets.

Proof. Notice that the sequence, $a_1, a_2, a_3, \dots$ is increasing. In other words, for each $n \in \mathbb{N}$, since $I_n \supset I_{n+1}$ we have $a_n \leq a_{n+1}$. If we consider the set $A = \{a_n : n \in \mathbb{N}\}$. The element b_1 is an upper bound of A. (Note that b_1 and a_1 corresponds to the end-points of the first set, I_1. Think of this as a tornado looking structure where the larger the I_n, the smaller the number line.) For each $n \in \mathbb{N}$, $a_n \leq b_n \leq b_1$.

Since A has an upper bound, it must have a least upper bound. Hence, let $\alpha = \sup A$. We claim that $\alpha \in \bigcap_{n=1}^{\infty} I_n$. We said b_1 was an upper bound. In fact, every b_n is an upper bound of A. Choose any $n, m \in \mathbb{N}$. We want to show that $a_n \leq b_m$. Consider the following cases:

Case 1: If $n < m$, then $a_n \leq a_m \leq b_m$. (Think: two number lines stacked on top of each other. The top number line is larger, call it I_n and it has a_n and b_n as endpoints. Consider a contained line ($I_n \supseteq I_m$) that is smaller, and has endpoints a_m and b_m.)

Case 2: If $n > m$, then $a_n \leq b_n \leq b_m$. So every b_n is an upper bound of A.

Hence,

- Because $\alpha = \sup A$, we have $\alpha \geq a_n$.
- Since b_n is an upper bound of A, we have $\alpha \leq b_n$.

so, $\alpha \in [a_n, b_n] = I_n$. Thus, $\alpha \in \bigcap_{n=1}^{\infty} I_n$. □

Nested, closed, Bounded Intervals ⇒ non-empty intersection.

Theorem 1.4.2: Archimedean Principle

(a) Given any number $x \in R$, there exists an $n \in N$ satisfying $n > x$.

(b) Given any real number $y > 0$, there exists an $n \in N$ satisfying $1/n < y$.

Proof. (a) If $\mathbb{N}$ was bounded, then we can let $s \in \mathbb{N} = \sup \mathbb{N}$. However, we know that there is always a higher number (e.g., $n+1$) for any $n \in \mathbb{N}$ that is given. Thus, by contradiction, there must exist $n \geq x$.

(b) For any $x > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < x$.

□

Theorem 1.4.3: Density of the Rationals in the Reals

For any $a, b \in \mathbb{R}$ with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Proof. Since $b - a > 0$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. From the Archimedean Principle, since $a \times n \in \mathbb{R}$, there exists $m \in \mathbb{N}$ such that $a \times n < m$. Let m be there smallest such natural numbers (by the well ordered principle). Since m is the smallest such natural number, it follows that $m - 1 \leq a \times n < m$. We then see that $a < \frac{m}{n}$. Now, we need to find some $\frac{m}{n} < b$.

$$\begin{aligned} m - 1 &\leq a \times n \\ m &\leq a \times n + 1 \\ \frac{m}{n} &\leq a + \frac{1}{n} \\ \frac{m}{n} &< a + (b - a) \\ \frac{m}{n} &< b \end{aligned}$$

We now have that $a < \frac{m}{n} < b$ so $\frac{m}{n}$ is a rational number in (a, b) □

Exercise: 1.4.1

Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a + b$ are elements of $\mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a + t \in \mathbb{I}$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s + t$ and st? In other words, are there two irrational numbers that can be added and multiplied such that you get a number x such that $x \notin \mathbb{I}$.

Solution.

(a) Let $a, b \in \mathbb{Q}$. This means there exists some $p, q, a, b \in \mathbb{Z}$ such that

$$a = \frac{p}{q}$$

and

$$b = \frac{a}{b}$$

where $q, b \neq 0$. The product of these numbers is

$$ab = \frac{p}{q} \cdot \frac{a}{b} = \frac{pa}{qb}.$$

Since $pa, qb \in \mathbb{Z}$, $ab \in \mathbb{Q}$. The sum of these numbers is

$$a + b = \frac{p}{q} + \frac{a}{b} = \frac{pb + aq}{qb}.$$

Since $pb + aq, qb \in \mathbb{Z}$, $a + b \in \mathbb{Q}$.

(b) Let $a \in \mathbb{Q}$ and $t \in \mathbb{I}$. Assume, for contradiction, that $a + t \in \mathbb{Q}$. This would imply $t = (a + t) - a$ (because we can subtract $t + a$ from the original equation and rearrange terms). Since $a + t, a \in \mathbb{Q}$ their sum would be rational because the rational numbers are closed under addition. However, that would contradict the assumption that $t \in \mathbb{I}$. Hence, $a + t \in \mathbb{I}$.

(c) For $\mathbb{I}$, it is not closed under addition and multiplication. Consider the following counterexample: $\sqrt{2} + (-\sqrt{2}) = 0$ which is not in the irrationals. For multiplication, consider $\sqrt{2} \cdot \sqrt{2} = 2$, which is also not in the irrationals.

1.5 Cardinality

Two sets have the same *cardinality* if there exists a bijection between them. Thus, the natural numbers, the integers, and the rational numbers have the same cardinality. A set is *countably infinite* if it has the same cardinality as $\mathbb{N}$. (If it can be put into one-to-one correspondence with $\mathbb{N}$.) A set is *countable* if it is countably infinite or finite.

Theorem 1.5.6

$\mathbb{R}$ is not countable.

Proof. 1 (most common)
Suppose $\mathbb{R}$ is countable. Then we can list them all, or we can enumerate them. $\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\}$. We can write the decimal expansion of each of these. Consider the

following table:

$x_1 =$	$\boxed{a_{10}}$	a_{11}	a_{12}	a_{13}	a_{14}	$\dots$
$x_2 =$	a_{20}	$\boxed{a_{21}}$	a_{22}	a_{23}	a_{24}	$\dots$
$x_3 =$	a_{30}	a_{31}	$\boxed{a_{32}}$	a_{33}	a_{34}	$\dots$
$x_4 =$	a_{40}	a_{41}	a_{42}	$\boxed{a_{43}}$	a_{44}	$\dots$
$x_5 =$	a_{50}	a_{51}	a_{52}	a_{53}	$\boxed{a_{54}}$	$\dots$
$x_6 =$	a_{60}	a_{61}	a_{62}	a_{63}	a_{64}	$\dots$

We will now construct a number that is not in this list. Focus on diagonal entries. For each $n \in \mathbb{N}$, let b_n be a digit that is different fron a_{nn}. Now consider the number $y = 0.b_1b_2b_3b_4b_5\dots$. This number y is not in our list. So our list did not include all of $\mathbb{R}$. Avoid repeating 9s. $\square$

Proof. 2 (uses nested interval theorem)
Suppose $\mathbb{R}$ is countable. Then we can enumerate $\mathbb{R}$ $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$. Let I_1 be any closed interval that does not contain x_1. Next, we will find another closed interval I_2 that:

- $I_2 \subseteq I_1$
- $x_2 \notin I_2$

Continue in this fashion creating a sequence of nested closed intervals: $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ such that for all $k \in \mathbb{N}$, $x_k \notin I_k$. Now consider:

$$\bigcap_{n=1}^{\infty} I_n$$

- For each $k \in \mathbb{N}$, since $x_k \notin I_k$, we see $x_k \notin \bigcap_{n=1}^{\infty} I_n$.
- By the nested interval theorem, there exists $x \in \mathbb{R}$ such that $x \in \bigcap_{n=1}^{\infty} I_n$. So x is a real number that is not included in our list.

$\square$

Theorem 1.5.7

A countable collection of finite sets is *countable.*

Theorem 1.5.8

(i) The union of two countable sets is *countable.*

(ii) A countable union of countable sets is *countable.*

From Theorem 1.5.6, we know that $\mathbb{R}$ is uncountable, but what about $(0, 1)$? It does have the same cardinality of $\mathbb{R}$ because we can make a one-to-one and onto function between both the sets. Similarly, (a, b) also has the same cardinality. What about $[a, b]$?

Recap: $\mathbb{N}$ is countable, and $\mathbb{R}$ is uncountable and has a different cardinality than $\mathbb{N}$. Thus, the question is, do all uncountable sets have the same cardinality as $\mathbb{R}$? The answer is **no**.

Theorem 1.5.9: Canter's Theorem

For any set A, there does not exist an onto map from A into $\mathcal{P}$.

Proof. Suppose there exists an onto function, $f : A \to \mathcal{P}(A)$. So each $a \in A$ is mapped to an element $f(a) \in \mathcal{P}(A)$. Then, $f(a) \subseteq A$. We are going to construct an element of $\mathcal{P}(A)$ which is not mapped to by f.

Consider $B = \{a \in A : a \notin f(a)\}$. Since f is onto there exists $a' \in A$ such that $B = f(a')$. Thus, there are two cases to consider:

- **Case 1:** If $a' \in B = f(a')$, then $a' \notin B$.
- **Case 2:** If $a' \notin B = f(a')$, then $a' \in B$.

As evidenced, both cases lead to contradictions, so B is not the image of any $a \in A$. Therefore f is not onto. $\square$

Example 1.5: Set and Power Set Matching

$A = \{a, b, c\}$.

Solution. $\mathcal{P}(A) = \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$. Note that you can map $\{a\}$, $\{b\}$, $\{c\}$, to elements such as $\emptyset$, $\{a, b\}$, $\{a, b, c\}$, but there are still more elements that are left unmapped. We can extrapolate from our proof a set B such that $B = \{a, c\}$ because those elements are not mapped to.

All of this is to show $\mathcal{P}(\mathbb{R})$ has a larger cardinality than $\mathbb{R}$. Then $\mathcal{P}(\mathcal{P}(\mathbb{R}))$ has a larger cardinality than $\mathcal{P}(\mathbb{R})$.

CHAPTER 2

SEQUENCES AND SERIES

2.1 Discussion: Rearrangement of Infinite Series

Questions:

What is a *sequence*?

A countable, ordered list of elements. An example could be $1, 2, 3, 4, 5, \ldots$. Note that this is *ordered*, therefore distinguishing it from a sequence like $3, 1, 2, 4, 5, 6, \ldots$. Hence, order matters.

A *sequence* is a function whose domain is $\mathbb{N}$. **Note:** The domain $\mathbb{N}$ refers to each element's position in the list. For example, $(a_n) = a_1, a_2, a_3, \ldots$.

We will focus on the *limit* of a sequence. We use sequences to approximate other things.

Example 2.1: Sequence

$3, 3.1, 3.14, 3.141, 3.1415, \cdots \approx \pi$.
$x, x - \frac{x^3}{6}, x - \frac{x^3}{6} + \frac{x^5}{120}, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}, \cdots \approx \sin(x)$

What is a *series*?

An infinite sum. We look at the sequence of partial sums. We ask, do the partial sums approach a limit?

Example 2.2: Alternating Harmonic Series

$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots$

We can rearrange these terms such that we can 'force' the series to converge to a specific number. Therefore, we will need to be careful with our definitions.

2.2 The Limit of a Sequence

Definition 2.2.1

A *sequence* is a function whose domain is $\mathbb{N}$. We write $(a_n) = a_1, a_2, a_3, \ldots$.

Definition 2.2.3

The sequence (a_n) converges to L if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$, $|a_n - L| < \epsilon$. In other words, there exists $N \in \mathbb{N}$ such that

- **(In the interval)** $a_N \in (L - \epsilon, L + \epsilon)$.
- **(Stays in the interval)** $\forall n \geq N$, $a_n \in (L - \epsilon, L + \epsilon)$.

Example 2.3: In-class

Let $a_n = \frac{1}{n}$. $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

Proof. Our claim is $\lim_{n\to\infty} \frac{1}{n} = 0$. Thus, let $\epsilon = .01$. Does the sequence eventually get inside $(-.01, .01)$? We will set $N = 101$. So, for any $n \geq |0|$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{101} < .01.$$

From A_n and on, the sequence stayed within ϵ of 0. But what about $\epsilon = .001$, $\epsilon = .00001$ and so on?

Actual proof let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Now, for any $n \geq N$,

$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{1/\epsilon}.$$

(Where $\frac{1}{1/\epsilon} = \epsilon$, but is in that form for demonstration purposes.) Therefore $\lim_{n\to\infty} \frac{1}{n} = 0$ $\square$

"To get close" means is that we are finding a bigger and bigger N as ϵ gets smaller. Note that the choice of N certainly depends on ϵ.

2.2.1 Basic Structure of a Limit Proof

Claim: $\lim_{n\to\infty} a_n = L$.
Proof: Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that {something involving ϵ}. Assume $n \geq N$. Then,

$$|a - n - L| \boxed{\dots} < \epsilon$$

(Where $\boxed{\dots}$ is going to be where the majority of the work is going to lie.

Example 2.4: In-class

Claim: $\lim_{n\to\infty} \frac{2n-3}{2n} = 1$

Proof. Let $\epsilon > 0$. *Scratch paper:* Solve for:

$$\left|\frac{2n-3}{2n} - 1\right| = \left|\frac{-3}{2n}\right| = \frac{3}{2n} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n.$$

By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{3}{2\epsilon}$. Assume $n \geq N$, (want to know what happens past this point)

$$\left|\frac{2n-3}{2n} - 1\right| \leq \frac{3}{2N} < \frac{3}{2 \cdot 3/2\epsilon} = \epsilon.$$

Therefore, $\lim_{n\to\infty} \frac{2n-3}{2n} = 1$ □

Example 2.5: C

laim: $\lim_{n\to\infty} \frac{2n^2+1}{n^2} = 2$

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that [leave off] *Scratch paper:* Solve for

$$\left|\frac{2n^2+1}{n^2} - 2\right| = \frac{2n^2}{n^2} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n$$

[pick up] there exists $N \in \mathbb{N}$ such that

$$N > \frac{1}{\sqrt{\epsilon}}.$$

Assume $n \geq N$, then

$$\begin{aligned}\left|\frac{2n^2+1}{n^2}-2\right| &= \frac{1}{n^2} \\ &\leq \frac{1}{N^2} \\ &< \frac{1}{(1/(\sqrt{\epsilon})^2} \\ &= \frac{1}{1/\epsilon} \\ &= \epsilon\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{2n^2+1}{n^2} = 2$ □

Example 2.6: In-class

Claim: $\lim_{n\to\infty} \frac{7n+8}{3n+6} = \frac{7}{3}$

Proof.

$$\begin{aligned}\left|\frac{7n+8}{3n+6}-\frac{7}{3}\right| &= \left|\frac{21n+24}{3(3n+6)}-\frac{21n+42}{3(3n+6}\right| \\ &= \left|\frac{-18}{9n+18}\right| \\ &= \frac{18}{9n+18} < \epsilon ** \\ &= \frac{18}{3} < 9n+18 \\ &= \frac{18}{3} - 18 < 9n \\ &= \frac{18/\epsilon - 18}{9} < n\end{aligned}$$

$* * \frac{18}{9n+8} < \frac{18}{9n} < \epsilon \Rightarrow \frac{2}{\epsilon} < N$. $\exists N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Assume $n \geq N$,

$$\begin{aligned}
\left|\frac{7n+8}{3n+6} - \frac{7}{3}\right| &= \frac{18}{9n+18} \\
&= \frac{2}{n+2} \\
&< \frac{2}{n} \\
&\leq \frac{2}{N} \\
&< \frac{2}{\epsilon/2} \\
&= \epsilon
\end{aligned}$$

□

Does every sequence have a limit?

Theorem 2.2.4: Uniqueness of Limits

The limit when it exists, is unique.

Proof. Let (x_n) be a convergent sequence. Suppose L and M are limits of this sequence. Without the loss of generality, we are going to assume $M > L$ Let

$$\epsilon = \frac{M-L}{3}.$$

Since n_x converges to L, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N$, $|(x_n) - L| < \epsilon$. Since (x_n) converges to M, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|(x_n) - M| < \epsilon$. Consider $n = \max\{N_1, N_2\}$. Since $n \geq N_1$, $|(x_n) - L| < \epsilon$. Since $n \geq N_2$, $|(x_n) - M| < \epsilon$. Then $L - \epsilon < x_n < L + \epsilon$ and $M - \epsilon < x_n < M + \epsilon$. By our choice of ϵ, we now have

$$(x_n) < L + \epsilon < M - \epsilon < (x_n).$$

This is a contradiction. Thus, (x_n) cannot have two different limits. □

Example 2.7:

Let $(x_n) = \dfrac{\cos(n)}{3n}$. Claim: $\lim\limits_{n\to\infty}(x_n) = 0$

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{3\epsilon}$

for all $n \geq N$,

$$\begin{aligned}\left|\frac{\cos(n)}{3n} - 0\right| &= \left|\frac{\cos(n)}{3n}\right| \\ &\leq \frac{1}{3n} \\ &\leq \frac{1}{3N} \\ &< \frac{1}{3(1/3\epsilon)} \\ &= \epsilon\end{aligned}$$

□

Example 2.8:

Let $(y_n) = \frac{4n-1}{n^2}$. Claim: $\lim\limits_{n\to\infty} y_n = 0$.

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. For all $n \geq N$,

$$\begin{aligned}\left|\frac{4n-1}{n^2} - 0\right| &= \left|\frac{4n-1}{n^2}\right| \\ &= \frac{4n-1}{n} \\ &< \frac{4n}{n^2} \\ &= \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \frac{4}{4/\epsilon} \\ &= \epsilon\end{aligned}$$

□

Exercises

Exercise: 2.2.2(b)

Verify, using Definition 2.2.3, that the following sequences converge to the proposed limit.

(b) $\displaystyle\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$

Proof.

(b) Let $\epsilon > 0$. By the Archimedean Principle, there exists an $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Then, for $n \geq N$,

$$\begin{aligned}\left|\frac{2n^2}{n^3+3} - 0\right| &= \left|\frac{2n^2}{n^3+3}\right| \\ &= \frac{2n^2}{n^3+3} \\ &< \frac{2n^2}{n^3} \\ &= \frac{2}{n} \\ &\leq \frac{2}{N} \\ &= \frac{2}{2/\epsilon} \\ &= \epsilon.\end{aligned}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$. □

Exercise: 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution.

(a) There is at least one college in the United States where all students are less than seven feet tall.

(b) There is at least one college in the United States where all professors give at least one student a grade of C or lower.

(c) For all colleges in the United States, there exists a student who is less than six feet tall.

Exercise: 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution.

(a) Possible. Consider the piecewise function: $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

(b) Impossible. A sequence that converges must have its terms approach a specific value (the limit). If the sequence has an infinite number of ones, it must have subsequences of ones arbitrarily far out. For the sequence to converge to a limit different from 1, the terms would have to approach that different limit, say $L \neq 1$, meaning the ones must become rare or eventually stop appearing, contradicting the infinite number of ones. Therefore, such a sequence is impossible.

(c) Possible. $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \dots)$

2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1

A sequence (x_n) is *bounded* if there exists some $M > 0$ such that every term in the sequence belongs to $[-M, M]$.

Theorem 2.3.2

Every convergent sequence is bounded.

Proof. Let (x_n) be a convergent sequence with limit L. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|(x_n) - L| < 1$. Equivalently, $(x_n) \in (L-1, L+1)$. Let

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |L+1|, |L-1|\}.$$

We claim that for all $n \in \mathbb{N}$, $|x_n| \leq M$.

(a) This is true for $n < N$.

(b) For $n \geq N$, we know $L - 1 < x_n < L + 1$, so $(x_n) \leq \max\{|L-1|, |L+1|\}$

Thus, every term is in $[-M, M]$. □

Theorem 2.3.3: Algebraic Limit Theorem

Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then,

(i) $\lim_{n\to\infty} ca_n = ca$ for all $c \in \mathbb{R}$;

(ii) $\lim_{n\to\infty} (a_n + b_n) = a + b$;

(iii) $\lim_{n\to\infty} (a_n b_n) = ab$;

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided $b \neq 0$.

Scratch Paper:

$$|ca_n - ca| = |c|\,|a_n - a| < \epsilon$$
$$|a_n - a| < \frac{\epsilon}{|c|}$$

Leave off and go back to proof[1]

Proof. (i)
Let $\epsilon > 0$.[1] Since (a_n) converges to a, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \frac{\epsilon}{|c|}$. Now, for any $n \geq N$ we have two case because we want to avoid dividing by 0:

- If $c = 0$:
  then each $ca_n = 0$. So (ca_n) converges to 0, which can equal ca.

- If $c > 0$:

$$|ca_n - ca| = |c|\,|a_n - a| < |c|\,\frac{\epsilon}{|c|} = \epsilon.$$ □

(ii)
Scratch paper:

$$\begin{aligned}
|(a_n + b_n)| &= |(a_n - a) + (b_n - b)| && (2.1)\\
&\leq |a_n - a| + |b_n - b| && (2.2)\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} && (2.3)
\end{aligned}$$

Note that (2.2) is from the triangle inequality. Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \frac{\epsilon}{2}$. Since (b_n) converges to b, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon}{2}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \geq N$, (refer back to scratch paper).

(iii)
Scratch paper:

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| && (2.4)\\
&= |a_n(b_n - b) + b(b_n - b)| && (2.5)\\
&\leq |a_n|\,|b_n - b| + |b|\,|b_n - b| && (2.6)\\
&\leq M\,|b_n - b| + M\,|a_n - a|\,. && (2.7)\\
&< M\left(\frac{\epsilon}{2M}\right) + M\left(\frac{\epsilon}{2M}\right) && (2.8)\\
&= \epsilon && (2.9)
\end{aligned}$$

Note that: (2.4) is where we added 0, (2.5) is from the triangle inequality, and (2.6) is just factored. Additionally, we choose N to get the fractions in (2.8) Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since convergent sequences are bounded, then there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \leq M$. We can choose M so that $|b_n| \leq M$ as well. Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|a_n - a| < \frac{\epsilon}{2M}$. Since (b_n) converges to b, there exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon}{2M}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \geq N$, (refer back to scratch paper, and change (2.4)'s sign from an '$=$' to '$\leq$').

(iv)

Scratch paper:

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - a b_n}{b_n b}\right| \\
&= \left|\frac{a_n b - a b_n + a b_n - ab}{b_n b}\right| \\
&= \left|\frac{a_n(b - b_n) + b(b_n - b)}{b_n b}\right| \\
&= \left|\frac{a_n(b - b_n)}{b_n b} + \frac{b(b_n - b)}{b_n b}\right| \\
&\leq \left|\frac{a_n}{b_n}\right| |b - b_n| + |b| \left|\frac{b_n - b}{b_n b}\right| \\
&< \epsilon
\end{aligned}$$

Let $\epsilon > 0$. Since (b_n) converges to b, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|b_n| > \left|\frac{b}{2}\right|$. There also exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon |b|^2}{2}$. Now, let $N = \max\{N_1, N_2\}$. Let $n \geq N$, (refer back to scratch paper).

Lemma 2.3.4

Let (a_n) and $c < a$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > c$. Similarly, if $a < d$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n < d$.

2.3.1 Limits and Order

Theorem 2.3.5: Order Limit Theorem

Let (a_n) and (b_n) be sequences. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

(i) If $a_n \geq c$ for all $n \in \mathbb{N}$, then $a \geq c$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(iii) If there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

Exercises

Exercise: 2.3.1

(a) Assume $\lim_{n\to\infty} x_n = 0$ with $x_n \geq 0$. Show that $\lim_{n\to\infty} \sqrt{x_n} = 0$.

(b) Assume $\lim_{n\to\infty} x_n = 49$ with $x_n \geq 0$. Show that $\lim_{n\to\infty} \sqrt{x_n} = 7$

Proof.

(a) Let $\epsilon > 0$. Since (x_n) converges to 0, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - 0| < \epsilon^2$. Now, for any $n \geq N$, $|\sqrt{x_n} - 0| = \sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon$. Therefore, $\lim\limits_{n\to\infty} \sqrt{x_n} = 0$.

(b) Let $\epsilon > 0$. Since (x_n) converges to 49, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\begin{aligned} |x_n - 7| &= \left|\frac{(x_n - 7)(x_n + 7)}{\sqrt{x_n} + 7}\right| \\ &= \left|\frac{x_n - 49}{\sqrt{x_n} + 7}\right| \\ &\leq \frac{|x_n - 49|}{7} \end{aligned}$$

□

Exercise: 2.3.2

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\dfrac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution.

(a) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - 2| < \epsilon$. Now, for any $n \geq N$,

$$\begin{aligned} \left|\frac{2x_n - 1}{3} - 1\right| &= \left|\frac{2x_n - 1 - 3}{3}\right| \\ &= \left|\frac{2x_n - 4}{3}\right| \\ &= \frac{2}{3}|x_n - 2| \\ &< |x_n - 2| \\ &< \epsilon \end{aligned}$$

Therefore, $\frac{2x_n - 1}{3} \to 1$ □

(b) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $x_n \geq 1$. Then, we will choose N_2 so that $|x_n - 2| < \epsilon$ for all $n \geq N_2$.

Afterwards, we take $N = \max\{N_1, N_2\}$. And note that for $n \geq N$,

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{2}\right| &= \left|\frac{2 - x_n}{2x_n}\right| \\ &< \frac{|2 - x_n|}{2} \\ &< \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

□

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Definition 2.4.1

A sequence a_n is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.2: Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges.

Proof. Let (a_n) be an increasing and bounded sequence. Since (a_n) is bounded, the set $A = \{a_n \mid n \in \mathbb{N}\}$ is clearly also bounded. Since A is bounded, $\sup A$ exists. We claim that $\lim\limits_{n\to\infty} a_n = \sup A$. Thus, for all $\epsilon > 0$ and by our definition of supremum, there exists $N \in \mathbb{N}$ such that $\sup A - \epsilon < a_N \leq \sup A$. Since (a_n) is increasing, for all $n \geq N$, $\sup A - \epsilon < a_N \leq a_n \leq \sup A$. It follows that $|a_n - \sup A| < \epsilon$. Therefore, $\lim\limits_{n\to\infty} a_n = \sup A$. □

Example 2.9: MCT

Consider the recursively defined sequence x_n where $x_1 = 3$ and for all $n \in \mathbb{N}$, $x_{n+1} = \frac{1}{4 - x_n}$. Show that x_n converges.

Proof. We will show that x_n is monotone and bounded.

- **Part 1: Monotone Decreasing**
  - Base case: $x_1 = 3$, $x_2 = 1$.

– Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq x_{n+1}$. It follows that

$$\begin{aligned} x_n &\geq x_{n+1} \\ 4 - x_n &\leq 4 - x_{n+1} \\ \frac{1}{4-x_n} &\geq \frac{1}{4-x_{n+1}} \\ x_{n+1} &\geq x_{n+2} \end{aligned}$$

- **Part 2: Bounded Below** Claim: Sequence is bounded below by 0.
  – Base case: $x_1 = 3 > 0$.
  – Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq 0$. It follows that $4 - x_n \leq 4$, and when we take the reciprocal, we get

$$\begin{aligned} \frac{1}{4-x_n} &\leq \frac{1}{4} \\ x_{n+1} &\geq 1/4 \\ &> 0 \end{aligned}$$

By math induction, x_n is bounded below by 0.

By the Monotone Convergence Theorem, x_n converges.

So, what is the limit? We know (x_n) converges so let $L = \lim_{n\to\infty} x_n$. Then, $\lim_{n\to\infty} x_{n+1} = L$. We also know $x_{n+1} = \frac{1}{4-x_n}$. So $L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \dfrac{1}{4-x_n} = \dfrac{1}{4-L}$. It must be true that $L = \frac{1}{4-L}$. Solving for L, we get

$$\begin{aligned} L(4-L) &= 1 \\ 4L - L^2 &= 1 \\ L^2 - 4L + 1 &= 0 \end{aligned}$$

Hence, $L = 2 - \sqrt{3}$ or $L = 2 + \sqrt{3}$. Notice that it cannot be the latter because it is bigger than 3. $\square$

Exercise: 1.2.13

For this exercise, assume Exercise 1.2.5 has been successfully completed.

(a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^c = A_1^c \cap A_2^c \cap \cdots \cap A_n^c$$

for any finite $n \in \mathbb{N}$.

(b) It is tempting to appeal to induction to conclude

$$\left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of $n \in \mathbb{N}$, but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets $B_1, B_2, B_3, \ldots$ where

$$\bigcap_{i=1}^{n} B_i \neq \emptyset \quad \text{is true for every } n \in \mathbb{N},$$

but

$$\bigcap_{i=1}^{\infty} B_i = \emptyset$$

fails.

(c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.

Proof. In this proof, we plan to prove (c). Thus, we need to show that:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$$

and

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

($\subseteq$) Let $x \in (\bigcup_{i=1}^{\infty} A_i)^c$. This means x is in the union set of A_i for all $i \in \mathbb{N}$. Then, because we are taking the complement of $(\bigcup_{i=1}^{\infty} A_i)$, that means $x \notin A_i$ for all $i \in \mathbb{N}$. Hence, x is in the complement of each A_i. Thus, we can use the definition of intersection to assert $x \in \bigcap_{i=1}^{\infty} A_i^c$.
Therefore, we have shown:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c.$$

($\supseteq$) Similar to before, let $x \in \bigcap_{i=1}^{\infty} A_i^c$. Because $x \in A_i^c$ for all $i \in \mathbb{N}$ we know $x \notin A_i$. Hence, $x \notin (\bigcup_{i=1}^{\infty} A_i)$, which means $x \in (\bigcup_{i=1}^{\infty} A_i)^c$.
Therefore, we have shown:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

By showing both inclusions, we see that:

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c.$$

$\square$