



# HENDRIX

COLLEGE

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## Multivariable Calculus Notes

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### MATH 230

*Start*

JANUARY 22, 2025

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MAY 13, 2025

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## 1.1 Parametric Equations

### 1.1.1 Introduction

Most of your calculus experience has been single variable, so that the functions under consideration were typically  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Our course is divided into roughly 3 sections:

- Parametric Equations/Functions: Functions of the form  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  (Chapters 1 - 3)
- Scalar Functions: Functions of the form  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  (Chapters 4 - 5)
- Vector Fields: Functions of the form  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (Chapter 6)

### 1.1.2 Parametric Equations

A *parametric equation* (or, *sometimes parametric function* or *vector-valued function*) is a function of the form  $f : \mathbb{R} \rightarrow \mathbb{R}^n$ . We will typically consider  $n = 2$  or  $n = 3$  and call the input variable the parameter, usually denoted by  $t$ . We write them as

$$f(t) = \begin{cases} x(t) \\ y(t) \end{cases} \quad \text{or} \quad f(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}.$$

A *parametric curve* is the set of points  $(x(t), y(t))$  in  $\mathbb{R}^2$  or  $(x(t), y(t), z(t))$  in  $\mathbb{R}^3$  traced out. Note that in general, the curve may not be a function for  $y$  in terms of  $x$ , but is a function of the parameter  $t$ .

### 1.1.3 Graphing Parametric Curves in the Second Dimension

#### Elimination of the Parameter

In some cases, we can explicitly solve for  $t$  in terms of one of  $x$  or  $y$ . When this is possible, you can write  $y(x)$  or  $x(y)$  and use your “regular” algebraic knowledge. We call this process *eliminating the parameter*.

#### Using Technology

- Your TI-84 can graph this if you switch to **par** mode.
- Likewise, GeoGebra can do this, using the **curve** function.
  - In general, the syntax is: `curve(x(t), y(t), t, min, max)`



### 1.1.4 The Cycloid

A wheel of radius  $a$  is rolling along a flat road at a constant velocity. The curve generated by a point along the edge of the wheel traces out a shape called a *cycloid*. Let  $t$  represent the angle - in radians!!!! - rotated through, and that the point of interest starts at the origin. Before we find the equations for the point, let's find the location of the center of the circle:

$$f_{\text{center}}(t) = \begin{cases} x(t) = at \\ y(t) = a \end{cases}$$

Then, relative to the center, our point along the edge has equations

$$f(t) = \begin{cases} x(t) = -a \sin(t) \\ y(t) = -a \cos(t) \end{cases}$$

Thus, our point has parametric equations

$$f(t) = \begin{cases} x(t) = a(t - \sin(t)) \\ y(t) = a(1 - \cos(t)) \end{cases}$$

### 1.1.5 Final Notes

Next time, we'll start asking Calculus-y questions: What are the velocities in the  $x$ ,  $y$ , and total directions? What total distance does it travel? What is the area of the region under one period of the cycloid?

- The syllabus has a number of practice problems to work on. These are not required, and not to be turned in, but are for you to work before class next time.
- We will talk about them at the start of the next class. You should try them beforehand.
- The most common reason for a lack of success in this class is not spending time working problems on your own.

## 1.2 Calculus of Parametric Curves

For this section, we will have a parametric curve in  $\mathbb{R}^2$ , defined by  $f(t) = \begin{cases} x(t) \\ y(t) \end{cases}$ . In many cases, the curve does not describe  $y$  as a function of  $x$ . However, we can still carry over many ideas from single variable calculus.



### 1.2.1 Slope for a Parametric Curve

Given a point  $t_0$ , the *slope of the curve* in the  $xy$ -plane is given by

$$\left. \frac{dy}{dx} \right|_{t=t_0} = \left. \frac{dy/dt}{dx/dt} \right|_{t=t_0}.$$

Note that this is undefined when  $x'(t_0) = 0$ .

The *tangent line* at  $t_0$  is given by

$$y = \left( \left. \frac{dy}{dx} \right|_{t=t_0} \right) (x - x(t_0)) + y(t_0).$$

### 1.2.2 Second Derivative

The value of the second derivative for the curve at  $t_0$  is given by

$$\left. \frac{d^2y}{dx^2} \right|_{t=t_0} = \frac{d}{dt} \left( \left. \frac{dy}{dx} \right|_{t=t_0} \right) = \frac{d}{dt} \left( \frac{dy/dt}{dx/dt} \right) \bigg|_{t=t_0}.$$

Note the benefit of Leibnitz notation for each of these two derivatives!

### 1.2.3 Area Under a Curve

Suppose that a parametric curve is non-self intersecting. Then, the signed area of the region between the curve and the  $x$ -axis on the  $t$  interval  $[t_a, t_b]$  is given by

$$A = \int_{t_a}^{t_b} y(t) \frac{dx}{dt} dt = \int_{t_a}^{t_b} y(t) \frac{dx}{dt} dt.$$

### 1.2.4 Arc Length

The *arc length* of a parametric curve over the  $t$  interval  $[t_a, t_b]$  is given by

$$s = \int_{t_a}^{t_b} \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt.$$

### 1.2.5 Surface Area

The *surface area* of the region obtained by rotating a non-self intersecting parametric curve is given by

$$S = \int_{t_a}^{t_b} 2\pi y(t) \sqrt{\left( \frac{dx}{dt} \right)^2 + \left( \frac{dy}{dt} \right)^2} dt.$$



### 1.2.6 The Cycloid

We can apply each of the above to the cycloid:

- *Derivative:*  $\frac{dy}{dx} = \frac{dy}{dt} = \frac{\sin(t)}{1-\cos(t)}$ . Note that the slope is then independent of the radius of the wheel and that the slope is undefined at each of  $t = \dots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \dots$
- *Cartesian Equation:* With radius of 3 and when  $t = \frac{\pi}{3}$ , the point is found by solving for  $x(\frac{\pi}{3})$  and  $y(\frac{\pi}{3})$ :

$$\begin{aligned}x\left(\frac{\pi}{3}\right) &= 3\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right) = \pi - \frac{3\sqrt{3}}{2} \\y\left(\frac{\pi}{3}\right) &= 3\left(1 - \cos\left(\frac{\pi}{3}\right)\right) = \frac{3}{2} \\(x, y) &= \left(\pi - \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)\end{aligned}$$

Plugging in our  $t$  value into our derivative, we get a slope of

$$\frac{\sin(\pi/3)}{1 - \cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Now, we can write the equation of the tangent line as

$$y = \sqrt{3}\left(x - \pi + \frac{3\sqrt{3}}{2}\right) + \frac{3}{2}.$$



- *Concavity*:  $\frac{d^2y}{dx^2} = \frac{d}{dt} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( \frac{\sin(t)}{1-\cos(t)} \right)$ .

$$\begin{aligned}
 \frac{d^2y}{dx^2} &= \frac{d/dt(dy/dx)}{dx/dt} \\
 &= \frac{\frac{d}{dt} \left( \frac{\sin(t)}{1-\cos(t)} \right)}{a - a \cos(t)} \\
 &= \frac{\frac{\cos(t)(1-\cos(t)) - \sin(t)\sin(t)}{(1-\cos(t))^2}}{a - a \cos(t)} \\
 &= \frac{\cos(t) - \cos^2 - \sin^2(t)}{(1-\cos(t))^2 a (1-\cos(t))} \\
 &= \frac{\cos(t) - 1}{a(1-\cos(t))^2} \\
 &= -\frac{1}{a(1-\cos(t))^2} \\
 &= -\frac{a}{a^2(1-\cos(t))^2} \\
 &= -\frac{a}{y^2}
 \end{aligned}$$

After some work, we find that  $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$ , which shows that the cycloid is always concave down.

- *Area*: The area of one period of the cycloid  $A = 3\pi a^2$ , after some work:

$$\begin{aligned}
 A &= \int_0^{2\pi} y(t)x'(t)dt \\
 &= \int_0^{2\pi} (a - a \cos t)(a - a \cos t)dt \\
 &= a^2 \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t)dt \\
 &= a^2 \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t)dt \\
 &= a^2 \left( t + \frac{t}{2} + \frac{1}{4} \sin(2t) \right) \Big|_0^{2\pi} \\
 &= a^2 \left[ \left( 2\pi + \frac{2\pi}{2} + \frac{1}{4} \sin(2\pi) \right) - \left( 0 + \frac{0}{2} + \frac{1}{4} \sin(0) \right) \right] \\
 &= a^2 [2\pi + \pi] \\
 &= 3\pi a^2.
 \end{aligned}$$



- *Arc Length*: The arc length of one period of the cycloid is  $s = 8a$ , again after some work:

$$\begin{aligned}
 s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt \\
 &= \int_0^{2\pi} \sqrt{(a - a \cos t)^2 + (a \sin t)^2} dt \\
 &= a \int_0^{2\pi} \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} dt \\
 &= a \int_0^{2\pi} \sqrt{2 - 2 \cos t} dt \\
 &= \sqrt{2}a \int_0^{2\pi} \sqrt{1 - \cos t} dt \\
 &= \sqrt{2}a \int_0^{2\pi} \sqrt{2 \sin^2 \left(\frac{t}{2}\right)} dt \\
 &= \sqrt{2}a \cdot \sqrt{2} \int_0^{2\pi} \sin \left(\frac{t}{2}\right) dt \\
 &= 2a \left( -2 \cos \left(\frac{t}{2}\right) \right) \Big|_0^{2\pi} \\
 &= 8a.
 \end{aligned}$$

- *Surface Area*: The surface area of the solid obtained by rotating one period of the cycloid around the  $x$ -axis is  $S = \frac{64\pi a^2}{3}$ , after a lot of tedious work.

$$S = \int_0^{2\pi} 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$