Homework 1: Section 2

Algebra

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Exercises **1** through **4** concern the binary operation $*$ defined on $S = \{a, b, c, d, e\}$ by means of Table 2.26.

Table 2.26*

$*$	a	b	c	d	e
a	a	b	a	e	c
b	c	a	b	b	a
c	c	a	b	b	a
d	b	e	b	e	d
e	d	b	a	d	c

*(Code partially from Tex StackExchange)

1. Compute $b * d, c * c$, and $[(a * c) * e] * a$.

Solution. From the table, we see that $b*d = b$ and $c*c = b$. For the last equation we work from the inside out:

$$[(a * c) * e] * a = [a * e] * a = c * a = c.$$

2. Compute $(a * b) * c$ and $a * (b * c)$. Can you say on the basis of this [sic] computations whether $*$ is associative?

Solution. For the first equation, we see that $(a * b) * c = b * c = b$. Then, for the second we see that $a * (b * c) = a * b = b$. Since these two computations yield the same answer, it shows that $*$ is associative for these elements, but not necessarily for the whole set, S.

3. Compute $(b*d)*c$ and $b*(d*c)$. Can you say on the basis of this computation whether $*$ is associative?

Solution.

(a) $(b * d) * c = b * c = b$, and

(b) $b * (d * c) = b * b = a$.

Since these computations are not equal, $*$ is not associative.

4. Is $*$ commutative? Why?

Solution. It is not commutative because $e * a = d$, but $a * e = c$.

In exercises **7** through **10**, determine whether the binary operation $*$ defined is commutative and whether $*$ is associative.

7. $*$ defined on $\mathbb{Z}$ by letting $a * b = a - b$.

Solution. This binary operation is not commutative or associative. Two counterexamples:

(a) Commutativity: $3 - 4 = -1 \quad \neq \quad 1 = 4 - 3$

(b) Associativity: $(3 - 4) - 2 = -3 \quad \neq \quad 1 = 3 - (4 - 2)$

8. $*$ defined on $\mathbb{Q}$ by letting $a * b = ab + 1$.

Solution. Since rational multiplication is commutative, $ab \in \mathbb{Q}$. Then, because addition is commutative and $1 \in \mathbb{Q}$, then

$$ab + 1 = ba + 1.$$

Therefore, $*$ is commutative. To check associativity, we compute the following equations:

$$(a * b) * c = (ab + 1) * c = (ab + 1)c + 1 = abc + c + 1,$$

and

$$a * (b * c) = a(bc + 1) = abc + a + 1.$$

Therefore, $*$ is not associative because $\boxed{abc + c + 1 \neq abc + a + 1}$ (for distinct elements $a, b, c \in \mathbb{Q}$).

10. $*$ defined on $\mathbb{Z}^+$ by letting $a * b = 2^{ab}$.

Solution. If $ab \in \mathbb{Z}^+$, then $ba \in \mathbb{Z}^+$ because the non-negative set of integers are commutative. Hence, $2^{ab} = 2^{ba}$, and $*$ is commutative. We can check associativity by working out the following equations:

$$(a * b) * c = 2^{ab} * c = 2^{2^{ab}c},$$

and

$$a(b * c) = a * 2^{bc} = 2^{a2^{bc}}.$$

Therefore, $*$ is not associative because $\boxed{2^{2^{ab}c} \neq 2^{a2^{bc}}}$

23. Let H be the subset of $M_2(\mathbb{R})$ consisting of all matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ for $a, b \in \mathbb{R}$. Is H closed under

(a) matrix addition?

(b) matrix multiplication?

Solution.

(a) The set H is closed under matrix addition because for any elements $a, b, c, d \in \mathbb{R}$:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix},$$

where we can see that the operations $a + c$ and $b + d$ results in some sum $e, f \in \mathbb{R}$:

$$\begin{bmatrix} e & -(f) \\ f & e \end{bmatrix} = \begin{bmatrix} e & -f \\ f & e \end{bmatrix}.$$

(b) The same can be shown for matrix multiplication:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ bc+ad & -bd+ac \end{bmatrix} = \begin{bmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{bmatrix}.$$

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26. Prove that if $*$ is an associative and commutative binary operation on a set S, then

$$(a * b) * (c * d) = [(d * c) * a] * b$$

for all $a, b, c, d \in S$. Assume the associative law only for triples as in the definition, that is, assume only

$$(x * y) * z = x * (y * z)$$

for all $x, y, z \in S$.

Proof. Suppose that $*$ is an associative and commutative binary operation on set S. Consider the equation

$$[(b * c) * a] * b.$$

We can rearrange the innermost parentheses using commutativity:

$$[(c * b) * a] * b.$$

Now, consider the substitution $x = (c*b)$, $y = a$, and $z = b$. By using associativity and substituting back, we can write this as

$$x * (y * z) = (c * b) * (a * b).$$

Finally, by employing commutativity once again, we get

$$(a * b) * (c * b).$$

Therefore, we have shown $\boxed{(a * b) * (c * d) = [(d * c) * a] * b}$ $\square$

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37. Suppose that $*$ is an associative and commutative binary operation on a set S. Show that $H = \{a \in S \mid a * a = a\}$ is closed under $*$. (The elements of H are **idempotents** of the binary operation $*$.)

Proof. Suppose that $*$ is an associative and commutative binary operation on set S. Let $a, b \in H$ and consider the following:

$$\begin{aligned}
(a * b) * (a * b) &= (b * a) * (a * b) && \text{commutative property,}\\
&= b * (a * a) * b && \text{associative property,}\\
&= b * a * b && \text{definition of } H,\\
&= b * b * a && \text{commutative property,}\\
&= (b * b) * a && \text{associative property,}\\
&= b * a && \text{property of } H,\\
&= a * b && \text{commutativity property.}
\end{aligned}$$

Therefore, H is closed because $a * b \in H$. $\square$