Multivariable Calculus Practice Set II

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1. (2 points) Write, in general equation form, an equation of the plane which contains the three points $P = (2, 7, 3)$, $Q = (-5, 0, 1)$, and $R = (-3, 1, 2)$.

   Solution. First, we find $\mathbf{PQ}$ and $\mathbf{PR}$:

   $$\mathbf{PQ} = \langle -7, -7, -2 \rangle \qquad \text{and} \qquad \mathbf{PR} = \langle -5, -6, -1 \rangle.$$

   With $\mathbf{PQ}$ and $\mathbf{PR}$, we can find $\mathbf{n}$ by solving for the cross product:

   $$\mathbf{n} = \mathbf{PQ} \times \mathbf{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -7 & -7 & -2 \\ -5 & -6, & -1 \end{vmatrix} = (7 - 12)\mathbf{i} - (7 - 10)\mathbf{j} + (42 - 35)\mathbf{k} = -5\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}.$$

   With $\mathbf{n}$, we get the general formula:

   $$\boxed{-5(x - 2) + 3(y - 7) + 7(z - 3) = 0.}$$

2. (2 points) Write, in scalar form, an equation of the plane which contains the point $(5, 2, 1)$ and the line given by $x + 2 = \dfrac{y}{4} = \dfrac{z - 5}{2}$.

   Solution. We start by parametrizing the line with common parameter t:

   - $x + 2 = t \quad \Rightarrow \quad x = t - 2$,
   - $\frac{y}{4} = t \quad \Rightarrow \quad y = 4t$, and
   - $\frac{z-5}{2} = t \quad \Rightarrow \quad z = 2t + 5$.

   This gives us the parametric form:

   $$(x, y, z) = (-2, 0, 5) + t(1, 4, 2)$$

   Thus, the line passes through the point $(-2, 0, 5)$ and has the direction vector

   $$\mathbf{v_1} = \langle 1, 4, 2 \rangle.$$

   Since the plane is two-dimensional, we need 2 independent directions within it. We got the first through our line, but we need another because there are infinitely many planes that contain the same line. Thus, we can form a second vector $\mathbf{v_2}$ by taking the difference between the given point and a point on the line:

   $$\mathbf{v_2} = (5, 2, 1) - (-2, 0, 5) = \langle 7, 2, -4 \rangle.$$

With $\mathbf{v}_1$ and $\mathbf{v}_2$, we find the normal vector:

$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 4 & 2 \\ 7 & 2 & -4 \end{vmatrix} = (-16 - 4)\mathbf{i} - (-4 - 14)\mathbf{j} + (2 - 28)\mathbf{k} = -20\mathbf{i} + 18\mathbf{j} - 26\mathbf{k}$$

Therefore, we find the scalar form to be:

$$\boxed{-20(x + 2) + 18y - 26(z - 5) = 0.}$$

3. (3 points) Determine the arc length parametrization for the curve $\mathbf{r}(t) = 3e^t \sin(t)\mathbf{i} + 3e^t \cos(t)\mathbf{j}$, where you start from $t = 0$.

Solution. From equation 3.11 from Theorem 3.4 in the book, (and this website) we know that we can rewrite the arc length parametrization as:

$$s = \int_0^t \|\mathbf{r}'(\tau)\| \, d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2} \, d\tau,$$

where $f(\tau) = 3e^\tau \sin(\tau)$ and $g(\tau) = 3e^\tau \cos(\tau)$. Thus, we find:

$$f'(\tau) = 3e^\tau (\sin(\tau) + \cos(\tau))$$
$$g'(\tau) = 3e^\tau (\cos(\tau) - \sin(\tau)).$$

Thus, we have:

$$s = \int_0^t \sqrt{\left[3e^\tau (\sin(\tau) + \cos(\tau))\right]^2 + \left[3e^\tau (\cos(\tau) - \sin(\tau))\right]^2} \, d\tau$$

$$= \int_0^t \sqrt{9e^{2\tau}\left(\sin^2(\tau) + 2\sin(\tau)\cos(\tau) + \cos^2(\tau)\right) + 9e^{2\tau}\left(\cos^2(\tau) - 2\sin(\tau)\cos(\tau) + \sin^2(\tau)\right)} \, d\tau$$

$$= \int_0^t \sqrt{9e^{2\tau}\left[2\left(\sin^2(\tau) + \cos^2(\tau)\right) + \left(2\sin(\tau)\cos(\tau) - 2\sin(\tau)\cos(\tau)\right)\right]}$$

$$= \int_0^t \sqrt{9e^{2\tau} \cdot [2(1 + 0)]}$$

$$= \int_0^t 3e^\tau \sqrt{2} \, d\tau$$

$$= 3\sqrt{2}\int_0^t e^\tau \, d\tau$$

$$= 3\sqrt{2}(e^t - 1).$$

With s, we know that $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so we need to find t in terms of s:

$$s = 3\sqrt{2}(e^t - 1)$$
$$e^t = \frac{s}{3\sqrt{2}} + 1$$
$$t = \ln\left(\frac{s}{3\sqrt{2}} + 1\right).$$

Finally, by replacing t with $t(s)$ in the original equation, we can get the arc length parametrization:

$$\boxed{\mathbf{r}(s) = \left(\frac{s}{\sqrt{2}} + 3\right)\sin\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right)\mathbf{i} + \left(\frac{s}{\sqrt{2}} + 3\right)\cos\left(\ln\left(\frac{s}{\sqrt{2}} + 3\right)\right)\mathbf{j}.}$$

4. (3 points) Use curvature to find the equation of the osculating circle at the planar curve $y = x^3 - 4x + 1$ at $x = 1$. Then, check your answer by graphing both the curve and its circle on the same axes. [you do not need to include the graph in your work turned in – but you should be able to tell if your work is correct.]

Solution. First, we need to find the curvature of the curve at $x = 1$. We start by finding the first and second derivatives of the function:

$$y(x) = x^3 - 4x + 1$$
$$y'(x) = 3x^2 - 4$$
$$y''(x) = 6x.$$

Then, we evaluate the point and the first and second derivatives at $x = 1$:

$$y(1) = 1 - 4 + 1 = -2$$
$$y'(1) = 3(1)^2 - 4 = -1$$
$$y''(1) = 6(1) = 6.$$

With these values, we can find the curvature:

$$\kappa = \frac{|y''(x)|}{\left(1 + y'(x)^2\right)^{3/2}} = \frac{6}{\left(1 + (-1)^2\right)^{3/2}} = \frac{6}{2^{3/2}} = \frac{3\sqrt{2}}{2}.$$

With the curvature, we can find the radius of the osculating circle:

$$R = \frac{1}{\kappa} = \frac{1}{\frac{3\sqrt{2}}{2}} = \frac{2\sqrt{2}}{6} = \frac{\sqrt{2}}{3}.$$

To find the center, we need the unit normal vector at $x = 1$:

$$\mathbf{N} = \frac{(-y', 1)}{\sqrt{1 + (y')^2}} = \frac{(-(-1), 1)}{\sqrt{1 + (-1)^2}} = \frac{(1, 1)}{\sqrt{2}}$$

The center C can be found by moving our point $P(1, -2)$ the distance R along the unit normal vector:

$$C = P + R\mathbf{N}$$
$$= (1, -2) + \frac{\sqrt{2}}{3}\frac{(1, 1)}{\sqrt{2}}$$
$$= (1, -2) + \left(\frac{1}{3}, \frac{1}{3}\right)$$
$$= \left(\frac{4}{3}, -\frac{5}{3}\right).$$

This gives the equation for the osculating circle:

$$\boxed{\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{5}{3}\right)^2 = \frac{2}{9}.}$$

5. **(3 points each)** Suppose the position of some particle is given by $\mathbf{r}(t) = \sin(t)\mathbf{i} + t\mathbf{j} + 3t\mathbf{k}$.

(a) Find the velocity vector, $\mathbf{v}(t)$.

> *Solution.*
> $$\mathbf{v}(t) = \mathbf{r}'(t) = \cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k}$$

(b) What total distance is travelled by the particle over the time period $[0, 3\pi]$? (You can set up the necessary internal, and calculate it using your calculator up to 3 decimal places.)

> *Solution.*
> $$\int_0^{3\pi} \|\mathbf{r}'(t)\|\, dt = \int_0^{3\pi} \sqrt{\cos^2(t) + 1 + 9}\, dt = 9.709$$

(c) Find the unit tangent vector $\mathbf{T}(t)$.

> *Solution.*
> $$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{\cos^2(t) + 10}}$$

(d) Find unit normal vector $\mathbf{N}(t)$.

> *Solution.* To find the unit normal vector, we need to find the derivative of the unit tangent vector. To avoid making mistakes (and making differentiating easier), lets break $\mathbf{T}(t)$ into separate functions $u(t)$ and $v(t)$:
> $$\mathbf{T}(t) = \underbrace{(\cos(t)\mathbf{i} + \mathbf{j} + 3\mathbf{k})}_{u(t)} \cdot \underbrace{(\cos^2(t) + 10)^{-1/2}}_{v(t)}$$
>
> Differentiating $u(t)$:
> $$u'(t) = -\sin(t)\mathbf{i},$$
>
> and differentiating $v(t)$ with the chain rule:
> $$v'(t) = -\frac{1}{2}(\cos^2 + 10)^{-3/2} \cdot 2\cos(t)(-\sin(t)) = \cos(t)\sin(t)(\cos^2(t) + 10)^{-3/2}.$$
>
> Thus, we apply the product rule for $\mathbf{T}'(t)$:
> $$\mathbf{T}'(t) = \left[-\sin(t)\,\mathbf{i}\right](\cos^2(t) + 10)^{-1/2} + \left[\cos(t)\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}\right]\left[\cos(t)\sin(t)\left(\cos^2(t) + 10\right)^{-3/2}\right].$$
>
> Notice that both terms contain a factor of $\sin(t)$ and a power of $\cos^2(t) + 10$, so we can factor them out:
> $$\mathbf{T}'(t) = \sin(t)\left(\cos^2(t) + 10\right)^{-3/2}\left\{-\left[\cos^2(t) + 10\right]\mathbf{i} + \cos(t)\left[\cos(t)\,\mathbf{i} + \mathbf{j} + 3\,\mathbf{k}\right]\right\}.$$

Inside the braces, multiply and combine terms:

$$\{\ldots\} = -\cos^2(t)\,\mathbf{i} - 10\,\mathbf{i} + \cos^2(t)\,\mathbf{i} + \cos(t)\,\mathbf{j} + 3\cos(t)\,\mathbf{k}$$
$$= -10\mathbf{i} + \cos(t)\mathbf{j} + 3\cos(t)\mathbf{k}.$$

This gives us:

$$\mathbf{T}'(t) = \frac{\sin(t)}{\left(\cos^2(t) + 10\right)^{3/2}} \left[-10\,\mathbf{i} + \cos(t)\,\mathbf{j} + 3\cos(t)\,\mathbf{k}\right].$$

Now we need to find the magnitude of $\mathbf{T}'(t)$:

$$\|\mathbf{T}'(t)\| = \frac{|\sin(t)|}{\left(\cos^2(t) + 10\right)^{3/2}} \sqrt{(-10)^2 + \left(\cos(t)\right)^2 + \left(3\cos(t)\right)^2}.$$

Simplify inside the square root and factor:

$$\|\mathbf{T}'(t)\| = \frac{|\sin(t)|}{\left(\cos^2(t) + 10\right)^{3/2}} \cdot \sqrt{10}\left(\cos^2(t) + 10\right)^{1/2}.$$

This simplifies to:

$$\|\mathbf{T}'(t)\| = \frac{|\sin(t)|\sqrt{10}}{\cos^2(t) + 10}.$$

Finally, we can find the unit normal vector:

$$\mathbf{N}(t) = \frac{\dfrac{\sin(t)}{\left(\cos^2(t)+10\right)^{3/2}} \left[-10\,\mathbf{i} + \cos(t)\,\mathbf{j} + 3\cos(t)\,\mathbf{k}\right]}{\dfrac{|\sin(t)|\sqrt{10}}{\cos^2(t)+10}}.$$

After further simplification, we see:

$$\boxed{\mathbf{N}(t) = \frac{-10\mathbf{i} + \cos(t)\mathbf{j} + 3\cos(t)\mathbf{k}}{\sqrt{10}\sqrt{\cos^2(t) + 10}}.}$$

(e) Find binormal vector, $\mathbf{B}(t)$.

Solution. The binomial vector is the cross product of the unit tangent and unit normal vectors:

$$\mathbf{T}(t) \times \mathbf{N}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\cos(t)}{\sqrt{\cos^2(t)+10}} & \dfrac{1}{\sqrt{\cos^2(t)+10}} & \dfrac{3}{\sqrt{\cos^2(t)+10}} \\ \dfrac{-10}{\sqrt{10}\sqrt{\cos^2(t)+10}} & \dfrac{\cos(t)}{\sqrt{10}\sqrt{\cos^2(t)+10}} & \dfrac{3\cos(t)}{\sqrt{10}\sqrt{\cos^2(t)+10}} \end{vmatrix}.$$

Thankfully, we can factor out the common term $\dfrac{1}{\sqrt{\cos^2(t)+10}}$ from each vector in the cross product:

$$\mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{\cos^2(t)+10}} \cdot \frac{1}{\sqrt{\cos^2(t)+10}} \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & 1 & 3 \\ \dfrac{-10}{\sqrt{10}} & \dfrac{\cos(t)}{\sqrt{10}} & \dfrac{3\cos(t)}{\sqrt{10}} \end{vmatrix}.$$

Now we can find the cross product:

$$\frac{1}{\cos^2(t)+10} \left[\left(1 \cdot \frac{3\cos(t)}{\sqrt{10}} \right) - \left(\frac{\cos(t)}{\sqrt{10}} \cdot 3 \right), \right.$$
$$\left. \left(3 \cdot \frac{-10}{\sqrt{10}} \right) - \left(\cos(t) \cdot \frac{3\cos(t)}{\sqrt{10}} \right), \left(\cos(t) \cdot \frac{\cos(t)}{\sqrt{10}} \right) - \left(1 \cdot \frac{-10}{\sqrt{10}} \right) \right].$$

Multiplying, we see that:

$$\frac{1}{\cos^2(t)+10} \left\langle 0, \ \frac{-3\left(\cos^2(t)+10\right)}{\sqrt{10}}, \ \frac{\cos^2(t)+10}{\sqrt{10}} \right\rangle.$$

Notice $\frac{1}{\cos^2(t)+10}$ cancels with the $\mathbf{j}^{\text{th}}$ and $\mathbf{k}^{\text{th}}$ terms. Thus, we can further simply this expression to:

$$\boxed{\left\langle 0, \ \frac{-3}{\sqrt{10}}, \ \frac{1}{\sqrt{10}} \right\rangle.}$$