Real Analysis

MATH 350

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Chapter 2

Sequences and Series

2.1 Discussion: Rearrangement of Infinite Series

Questions:

What is a *sequence*?

A countable, ordered list of elements. An example could be $1, 2, 3, 4, 5, \ldots$. Note that this is *ordered*, therefore distinguishing it from a sequence like $3, 1, 2, 4, 5, 6, \ldots$. Hence, order matters.

A *sequence* is a function whose domain is $\mathbb{N}$. **Note:** The domain $\mathbb{N}$ refers to each element's position in the list. For example, $(a_n) = a_1, a_2, a_3, \ldots$.

We will focus on the *limit* of a sequence. We use sequences to approximate other things.

Example 2.1: Sequence

$3, 3.1, 3.14, 3.141, 3.1415, \cdots \approx \pi$.
$x, x - \frac{x^3}{6}, x - \frac{x^3}{6} + \frac{x^5}{120}, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}, \cdots \approx \sin(x)$

What is a *series*?

An infinite sum. We look at the sequence of partial sums. We ask, do the partial sums approach a limit?

Example 2.2: Alternating Harmonic Series

$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots$

We can rearrange these terms such that we can 'force' the series to converge to a specific number. Therefore, we will need to be careful with our definitions.

2.2 The Limit of a Sequence

Definition 2.2.1

A *sequence* is a function whose domain is $\mathbb{N}$. We write $(a_n) = a_1, a_2, a_3, \ldots$.

Definition 2.2.3

The sequence (a_n) *converges* to L if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq \mathbb{N}$, $|a_n - L| < \epsilon$. In other words, there exists $N \in \mathbb{N}$ such that

- **(In the interval)** $a_N \in (L - \epsilon, L + \epsilon)$.
- **(Stays in the interval)** $\forall n \geq N$, $a_n \in (L - \epsilon, L + \epsilon)$.

Example 2.3: Limit Proof 1

Let $a_n = \frac{1}{n}$. $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$

Proof. Our claim is $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. Thus, let $\epsilon = .01$. Does the sequence eventually get inside $(-.01, .01)$? We will set $N = 101$. So, for any $n \geq |0|$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{101} < .01.$$

From A_n and on, the sequence stayed within ϵ of 0. But what about $\epsilon = .001$, $\epsilon = .00001$ and so on?

Actual proof let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. Now, for any $n \geq N$,

$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{1/\epsilon}.$$

(Where $\frac{1}{1/\epsilon} = \epsilon$, but is in that form for demonstration purposes.) Therefore $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ $\square$

"To get close" means is that we are finding a bigger and bigger N as ϵ gets smaller. Note that the choice of N certainly depends on ϵ. This idea of "getting close" can be seen in the following definition:

Definition 2.2.3B

A sequence (a_n) converges to a if, given any ϵ-neighborhood $V_\epsilon(a)$ of a, there exists a point in the sequence after which all of the terms are in $V_\epsilon(a)$. In other words, every ϵ-neighborhood contains all but a finite number of the terms of (a_n).

2.2.1 Basic Structure of a Limit Proof

Claim: $\lim\limits_{n\to\infty} a_n = L$.
Proof: Let $\epsilon > 0$. There exists $N \in \mathbb{N}$ such that $\{\text{something involving } \epsilon\}$. Assume $n \geq N$. Then,

$$|a - n - L| \boxed{\ldots} < \epsilon$$

(Where $\boxed{\ldots}$ is going to be where the majority of the work is going to lie.

Example 2.4: Limit Proof 2

Claim: $\lim\limits_{n\to\infty} \dfrac{2n-3}{2n} = 1$

Proof. Let $\epsilon > 0$. *Scratch paper:* Solve for:

$$\left|\frac{2n-3}{2n} - 1\right| = \left|\frac{-3}{2n}\right| = \frac{3}{2n} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n.$$

By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{3}{2\epsilon}$. Assume $n \geq N$, (want to know what happens past this point)

$$\left|\frac{2n-3}{2n} - 1\right| \leq \frac{3}{2N} < \frac{3}{2 \cdot 3/2\epsilon} = \epsilon.$$

Therefore, $\lim\limits_{n\to\infty} \dfrac{2n-3}{2n} = 1$ □

Example 2.5: Limit Proof 3

Claim: $\lim_{n\to\infty} \frac{2n^2+1}{n^2} = 2$

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that [leave off] *Scratch paper:* Solve for

$$\left|\frac{2n^2+1}{n^2} - 2\right| = \frac{2n^2}{n^2} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n$$

[pick up] there exists $N \in \mathbb{N}$ such that

$$N > \frac{1}{\sqrt{\epsilon}}.$$

Assume $n \geq N$, then

$$\begin{aligned}\left|\frac{2n^2+1}{n^2} - 2\right| &= \frac{1}{n^2} \\ &\leq \frac{1}{N^2} \\ &< \frac{1}{(1/(\sqrt{\epsilon})^2} \\ &= \frac{1}{1/\epsilon} \\ &= \epsilon\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{2n^2+1}{n^2} = 2$ □

Example 2.6: Limit Proof 4

Claim: $\lim_{n\to\infty} \frac{7n+8}{3n+6} = \frac{7}{3}$

Proof.

$$\begin{aligned}\left|\frac{7n+8}{3n+6} - \frac{7}{3}\right| &= \left|\frac{21n+24}{3(3n+6)} - \frac{21n+42}{3(3n+6}\right| \\ &= \left|\frac{-18}{9n+18}\right| \\ &= \frac{18}{9n+18} < \epsilon ** \\ &= \frac{18}{3} < 9n+18 \\ &= \frac{18}{3} - 18 < 9n \\ &= \frac{18/\epsilon - 18}{9} < n\end{aligned}$$

$* * \frac{18}{9n+8} < \frac{18}{9n} < \epsilon \Rightarrow \frac{2}{\epsilon} < N$. $\exists N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Assume $n \geq N$,

$$\begin{aligned}\left|\frac{7n+8}{3n+6} - \frac{7}{3}\right| &= \frac{18}{9n+18} \\ &= \frac{2}{n+2} \\ &< \frac{2}{n} \\ &\leq \frac{2}{N} \\ &< \frac{2}{\epsilon/2} \\ &= \epsilon\end{aligned}$$

□

Does every sequence have a limit?

Theorem 2.2.4: Uniqueness of Limits

The limit when it exists, is unique.

Proof. Let (x_n) be a convergent sequence. Suppose L and M are limits of this sequence. Without the loss of generality, we are going to assume $M > L$ Let

$$\epsilon = \frac{M-L}{3}.$$

Since n_x converges to L, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N$, $|(x_n) - L| < \epsilon$. Since (x_n) converges to M, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|(x_n) - M| < \epsilon$. Consider $n = \max\{N_1, N_2\}$. Since $n \geq N_1$, $|(x_n) - L| < \epsilon$. Since $n \geq N_2$, $|(x_n) - M| < \epsilon$. Then $L - \epsilon < x_n < L + \epsilon$ and $M - \epsilon < x_n < M + \epsilon$. By our choice of ϵ, we now have

$$(x_n) < L + \epsilon < M - \epsilon < (x_n).$$

This is a contradiction. Thus, (x_n) cannot have two different limits. □

Example 2.7: Limit Proof 5

Let $(x_n) = \dfrac{\cos(n)}{3n}$. Claim: $\lim_{n\to\infty}(x_n) = 0$

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{3\epsilon}$ for all $n \geq N$,

$$\begin{aligned}\left|\frac{\cos(n)}{3n} - 0\right| &= \left|\frac{\cos(n)}{3n}\right| \\ &\leq \frac{1}{3n} \\ &\leq \frac{1}{3N} \\ &< \frac{1}{3(1/3\epsilon)} \\ &= \epsilon\end{aligned}$$

□

Example 2.8: Limit Proof 6

Let $(y_n) = \frac{4n-1}{n^2}$. Claim: $\lim_{n\to\infty} y_n = 0$.

Proof. Let $\epsilon > 0$. By the Archimedean Principle, there exists $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon}$. For all $n \geq N$,

$$\begin{aligned}\left|\frac{4n-1}{n^2} - 0\right| &= \left|\frac{4n-1}{n^2}\right| \\ &= \frac{4n-1}{n} \\ &< \frac{4n}{n^2} \\ &= \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \frac{4}{4/\epsilon} \\ &= \epsilon\end{aligned}$$

□

2.2.2 Exercises

Exercise: 2.1.1

What happens if we reverse the order of the quantifiers in Definition 2.2.3?
Definition: A sequence x_n *verconges* to x if there exists an $\epsilon > 0$ such that for all $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x - n - x| < \epsilon$.

(a) Give an example of a vercongent sequence.

(b) Is there an example of a vercongent sequence that is divergent?

(c) Can a sequence verconge to two different values?

(d) What exactly is being described in this strange definition?

Solution.

(a) Pick $\epsilon = 2$, $x_n = (-1)^n$ and $x = 0$. This sequence will stay within the bounds of $(-2, 2)$ for all $N \in \mathbb{N}$ and $n \geq N$.

(b) There cannot be a divergent vercongent sequence because vercongence wants us to be bounded, and divergence wants it to grow outside the bounds. These two ideas are mutually exclusive.

(c) Yes. For example, $x_n = 0$ and $x_n = 1$.

(d) This definition is describing a sequence that is bounded. It is a sequence that is not growing outside of a certain range.

Exercise: 2.2.2

Verify, using Definition 2.2.3, that the following sequences converge to the proposed limit.

(a) $\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) $\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$

Proof.

(a) Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $N > \dfrac{3}{25\epsilon}$. Then for all $n \geq N$,

$$\begin{aligned}\left|\frac{2n+1}{5n+4} - \frac{2}{5}\right| &= \left|\frac{-3}{5(5n+4)}\right| \\ &= \frac{3}{25n+20} \\ &\leq \frac{3}{25n} \\ &\leq \frac{3}{25N} \\ &< \epsilon\end{aligned}$$

Therefore, $\displaystyle\lim_{n\to\infty} \frac{2n+1}{5n+4} = \frac{2}{5}$.

(b) Let $\epsilon > 0$. By the Archimedean Principle, there exists an $N \in \mathbb{N}$ such that $N > \frac{2}{\epsilon}$. Then, for $n \geq N$,

$$\begin{aligned}\left|\frac{2n^2}{n^3+3} - 0\right| &= \left|\frac{2n^2}{n^3+3}\right| \\ &= \frac{2n^2}{n^3+3} \\ &< \frac{2n^2}{n^3} \\ &= \frac{2}{n} \\ &\leq \frac{2}{N} \\ &= \frac{2}{2/\epsilon} \\ &= \epsilon.\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{2n^2}{n^3+3} = 0$. $\square$

Exercise: 2.2.3

Describe what we would have to demonstrate in order to disprove each of the following statements.

(a) At every college in the United States, there is a student who is at least seven feet tall.

(b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.

(c) There exists a college in the United States where every student is at least six feet tall.

Solution.

(a) There is at least one college in the United States where all students are less than seven feet tall.

(b) There is at least one college in the United States where all professors give at least one student a grade of C or lower.

(c) For all colleges in the United States, there exists a student who is less than six feet tall.

Exercise: 2.2.4

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

(a) A sequence with an infinite number of ones that does not converge to one.

(b) A sequence with an infinite number of ones that converges to a limit not equal to one.

(c) A divergent sequence such that for every $n \in \mathbb{N}$ it is possible to find n consecutive ones somewhere in the sequence.

Solution.

(a) Possible. Consider the sequence $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$. This sequence has infinitely many ones but does not converge to one.

(b) Impossible. Suppose (a_n) is a sequence that converges to a limit $L \neq 1$ and has infinitely many ones. Since (a_n) converges to L, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - L| < \epsilon$. Choose $\epsilon = \frac{|1-L|}{2} > 0$. Then, for $n \geq N$,

$|a_n - L| < \epsilon$, which implies $a_n \neq 1$ beyond this N. This contradicts the existence of infinitely many ones. Therefore, such a sequence is impossible.

(c) Possible. Define a sequence by concatenating increasing blocks of ones separated by zeros: $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \dots)$. Specifically, the sequence consists of n ones followed by a zero for $n = 1, 2, 3, \dots$. For every $n \in \mathbb{N}$, there is a block of n consecutive ones somewhere in the sequence. The sequence does not converge, so it is divergent.

Exercise: 2.2.5

Let $[[x]]$ be the greatest integer less than or equal to x. For example, $[[\pi]] = 3$ and $[[3]] = 3$. For each sequence, find $\lim_{n\to\infty} a_n$ and verify it with the definition of convergence.

(a) $a_n = [[5/n]]$

(b) $a_n = [[(12 + 4n)/3n]]$

Reflecting on these examples, comment on the statement following Definition 2.2.3B that "the smaller the ϵ-neighborhood, the larger N may have to be."

Solution.

(a) We will show that $\lim\limits_{n\to\infty} a_n = 0$.

Proof. For $n \geq 6$, we have $\dfrac{5}{n} \leq \dfrac{5}{6} < 1$, so $a_n = [[5/n]] = 0$.

Let $\epsilon > 0$. Choose $N = 6$. Then for all $n \geq N$,

$$|a_n - 0| = |0 - 0| = 0 < \epsilon.$$

Therefore, by the definition of convergence, $\lim\limits_{n\to\infty} a_n = 0$. □

(b) We will show that $\lim\limits_{n\to\infty} a_n = 1$.

Proof. Observe that:

$$a_n = \left\lfloor \frac{12+4n}{3n} \right\rfloor = \left\lfloor \frac{4n+12}{3n} \right\rfloor = \left\lfloor \frac{4}{3} + \frac{4}{n} \right\rfloor.$$

As $n \to \infty$, $\frac{4}{n} \to 0$, so $\frac{4}{3} + \frac{4}{n} \to \frac{4}{3} \approx 1.333$.
For $n \geq 7$, we have:

$$\frac{4}{n} \leq \frac{4}{7} \approx 0.571, \quad \frac{4}{3} + \frac{4}{n} \leq 1.333 + 0.571 = 1.904.$$

Since $1 < \frac{4}{3} + \frac{4}{n} < 2$ for $n \geq 7$, we have:

$$a_n = \left\lfloor \frac{4}{3} + \frac{4}{n} \right\rfloor = 1.$$

Let $\epsilon > 0$. Choose $N = 7$. Then for all $n \geq N$,

$$|a_n - 1| = |1 - 1| = 0 < \epsilon.$$

Therefore, by the definition of convergence, $\lim_{n\to\infty} a_n = 1$. □

Reflection: In these examples, we see that once the sequence reaches a certain point (i.e., $n \geq N$), the terms remain constant. This means that for any $\epsilon > 0$, we can find a fixed N to satisfy the definition of convergence, regardless of how small ϵ is. However, in general, smaller ϵ-neighborhoods may require larger N because the sequence may not settle into its limit as neatly as it does in these cases.

Exercise: 2.2.6

Prove the Uniqueness of Limits theorem. To get started, assume $(a_n) \to a$ and $(a_n) \to b$. Now argue $a = b$.

Proof. Since $(a_n) \to a$, this means for all $\epsilon > 0$, there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \epsilon/2$. Similarly, since $(a_n) \to b$, this means for all $\epsilon > 0$, there exists an $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|a_n - b| < \epsilon/2$.

Now, let $N = \max\{N_1, N_2\}$ so that

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &\leq |(a_n - a) + (a_n - b)| \\ &< \epsilon/2 + \epsilon/2 \\ &< \epsilon \end{aligned}$$

Then, by Theorem 1.2.6, $a = b$. □

Exercise: 2.2.7

Here are two useful definitions:

(i) A sequence (a_n) is *eventually* in a set $A \subseteq \mathbb{R}$ if there exists an $N \in \mathbb{N}$ such that $a_n \in A$ for all $n \geq N$.

(ii) A sequence (a_n) is *frequently* in a set $A \subseteq \mathbb{R}$ if, for every $N \in \mathbb{N}$, there exists an $n \geq N$ such that $a_n \in A$.

(a) Is the sequence $(-1)^n$ eventually or frequently in the set $\{1\}$?

(b) Which definition is stronger? Does frequently imply eventually, or does eventually imply frequently?

(c) Give an alternate rephrasing of Definition 2.2.3B using either frequently or eventually. Which is the term we want?

(d) Suppose an infinite number of terms of a sequence (x_n) are equal to 2. Is (x_n) necessarily eventually in the interval $(1.9, 2.1)$? Is it frequently in $(1.9, 2.1)$?

Solution.

(a) The sequence $(-1)^n$ is *frequently* in the set $\{1\}$ because for every $N \in \mathbb{N}$, we can find an $n \geq N$ such that $(-1)^n = 1$.

(b) The definition of *eventually* is stronger because *eventually* implies *frequently*, but *frequently* does not imply *eventually*.

(c) An alternate rephrasing of Definition 2.2.3B using *eventually* is: A sequence (a_n) converges to a if, given any ϵ-neighborhood—$V_\epsilon(a)$ of a—(a_n) is *eventually* in $V_\epsilon(a)$. The term we want is eventually.

(d) If an infinite number of terms of a sequence (x_n) are equal to 2, (x_n) is not *eventually* in $(1.9, 2.1)$ because we can have a sequence (a_n) that will not settle in $(1.9, 2.1)$. For example, $(a_n) = (0, 2, 0, 2, \cdots)$ does not settle in $(1.9, 2.1)$. Whereas, (x_n) is *frequently* in the interval $(1.9, 2.1)$ because for every $N \in \mathbb{N}$ there exists an $n \geq N$ such that $x_n \in (1.9, 2.1)$ for all $n \geq N$. We can see an instance of this being true by examining the previous example.

2.3 The Algebraic and Order Limit Theorems

Definition 2.3.1

A sequence (x_n) is *bounded* if there exists some $M > 0$ such that every term in the sequence belongs to $[-M, M]$.

Theorem 2.3.2

Every convergent sequence is bounded.

Proof. Let (x_n) be a convergent sequence with limit L. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|(x_n) - L| < 1$. Equivalently, $(x_n) \in (L-1, L+1)$. Let

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |L+1|, |L-1|\}.$$

We claim that for all $n \in \mathbb{N}$, $|x_n| \leq M$.

1. This is true for $n < N$.
2. For $n \geq N$, we know $L - 1 < x_n < L + 1$, so $(x_n) \leq \max\{|L-1|, |L+1|\}$

Thus, every term is in $[-M, M]$. □

Theorem 2.3.3: Algebraic Limit Theorem

Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then,

(i) $\lim_{n\to\infty} ca_n = ca$ for all $c \in \mathbb{R}$;

(ii) $\lim_{n\to\infty}(a_n + b_n) = a + b$;

(iii) $\lim_{n\to\infty}(a_n b_n) = ab$;

(iv) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided $b \neq 0$.

Scratch Paper:

$$\begin{aligned} |ca_n - ca| = |c|\,|a_n - a| &< \epsilon \\ |a_n - a| &< \frac{\epsilon}{|c|} \end{aligned}$$

Leave off and go back to proof[1]

Proof. (i)
Let $\epsilon > 0$.[1] Since (a_n) converges to a, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < \frac{\epsilon}{|c|}$. Now, for any $n \geq N$ we have two case because we want to avoid dividing

by 0:

- If $c = 0$:
  then each $ca_n = 0$. So (ca_n) converges to 0, which can equal ca.

- If $c > 0$:
  $|ca_n - ca| = |c|\,|a_n - a| < |c|\,\frac{\epsilon}{|c|} = \epsilon$.

(ii)
Scratch paper:

$$\begin{aligned} |(a_n + b_n)| &= |(a_n - a) + (b_n - b)| && (2.1)\\ &\le |a_n - a| + |b_n - b| && (2.2)\\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} && (2.3) \end{aligned}$$

Note that (2.2) is from the triangle inequality. Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, $|a_n - a| < \frac{\epsilon}{2}$. Since (b_n) converges to b, there exists $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, $|b_n - b| < \frac{\epsilon}{2}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \ge N$, (refer back to scratch paper).

(iii)
Scratch paper:

$$\begin{aligned} |a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| && (2.4)\\ &= |a_n(b_n - b) + b(b_n - b)| && (2.5)\\ &\le |a_n|\,|b_n - b| + |b|\,|b_n - b| && (2.6)\\ &\le M\,|b_n - b| + M\,|a_n - a|\,. && (2.7)\\ &< M\left(\frac{\epsilon}{2M}\right) + M\left(\frac{\epsilon}{2M}\right) && (2.8)\\ &= \epsilon && (2.9) \end{aligned}$$

Note that: (2.4) is where we added 0, (2.5) is from the triangle inequality, and (2.6) is just factored. Additionally, we choose N to get the fractions in (2.8) Now, we will pick up to back at $\epsilon > 0$.

Let $\epsilon > 0$. Since convergent sequences are bounded, then there exists $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| \le M$. We can choose M so that $|b_n| \le M$ as well. Since (a_n) converges to a, there exists $N_1 \in \mathbb{N}$ such that for all $n \ge N_1$, $|a_n - a| < \frac{\epsilon}{2M}$. Since (b_n) converges to b, there exists $N_2 \in \mathbb{N}$ such that for all $n \ge N_2$, $|b_n - b| < \frac{\epsilon}{2M}$. Now, let $N = \max\{N_1, N_2\}$. Thus, for any $n \ge N$, (refer back to scratch paper, and change (2.4)'s sign from an '$=$' to '$\le$').

(iv)
Scratch paper:

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \left|\frac{a_n b - a b_n}{b_n b}\right| \\
&= \left|\frac{a_n b - a b_n + a b_n - ab}{b_n b}\right| \\
&= \left|\frac{a_n(b - b_n) + b(b_n - b)}{b_n b}\right| \\
&= \left|\frac{a_n(b - b_n)}{b_n b} + \frac{b(b_n - b)}{b_n b}\right| \\
&\leq \left|\frac{a_n}{b_n}\right| |b - b_n| + |b| \left|\frac{b_n - b}{b_n b}\right| \\
&< \epsilon
\end{aligned}$$

Let $\epsilon > 0$. Since (b_n) converges to b, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $|b_n| > \left|\frac{b}{2}\right|$. There also exists $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, $|b_n - b| < \frac{\epsilon |b|^2}{2}$. Now, let $N = \max\{N_1, N_2\}$. Let $n \geq N$, (refer back to scratch paper). □

Lemma 2.3.4

Let (a_n) and $c < a$. There exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > c$. Similarly, if $a < d$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n < d$.

2.3.1 Limits and Order

Theorem 2.3.5: Order Limit Theorem

Let (a_n) and (b_n) be sequences. If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then

(i) If $a_n \geq c$ for all $n \in \mathbb{N}$, then $a \geq c$.

(ii) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

(iii) If there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.

2.3.2 Exercises

Exercise: 2.3.1

(a) If $\lim_{n\to\infty} x_n = 0$, show that $\lim_{n\to\infty} \sqrt{x_n} = 0$.

(b) If $\lim_{n\to\infty} x_n = x$, show that $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

Proof.

(a) *Solution.* Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$|x_n| < \epsilon^2.$$
Then, for all $n \geq N$,
$$|\sqrt{x_n} - 0| = \sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon.$$
Therefore, $\lim_{n\to\infty} \sqrt{x_n} = 0$.

(b) *Solution.* Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, for any $\delta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$|x_n - x| < \delta.$$
We consider two cases:

Case 1: $x > 0$.

Since $x > 0$, choose $\delta = \min\left\{\epsilon\left(2\sqrt{x}\right), \frac{x}{2}\right\}$. Then for all $n \geq N$, we have $x_n > x - \frac{x}{2} = \frac{x}{2} > 0$. Thus,
$$\sqrt{x_n} + \sqrt{x} \geq \sqrt{\frac{x}{2}} + \sqrt{x} > 0.$$
Now,
$$\left|\sqrt{x_n} - \sqrt{x}\right| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{\delta}{\sqrt{\frac{x}{2}}} \leq \epsilon.$$

Case 2: $x = 0$.

From part (1), we have $\lim_{n\to\infty} \sqrt{x_n} = 0 = \sqrt{0}$.

Therefore, $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

$\square$

Exercise: 2.3.2

Using only Definition 2.2.3, prove that if $(x_n) \to 2$, then

(a) $\left(\dfrac{2x_n - 1}{3}\right) \to 1$;

(b) $(1/x_n) \to 1/2$.

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)

Solution.

(a) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - 2| < \epsilon$. Now, for any $n \geq N$,

$$\begin{aligned}\left|\frac{2x_n - 1}{3} - 1\right| &= \left|\frac{2x_n - 1 - 3}{3}\right| \\ &= \left|\frac{2x_n - 4}{3}\right| \\ &= \frac{2}{3}|x_n - 2| \\ &< |x_n - 2| \\ &< \epsilon\end{aligned}$$

Therefore, $\frac{2x_n - 1}{3} \to 1$ □

(b) *Proof.* Let $\epsilon > 0$. Since (x_n) converges to 2, there exists $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, $x_n \geq 1$. Then, we will choose N_2 so that $|x_n - 2| < \epsilon$ for all $n \geq N_2$. Afterwards, we take $N = \max\{N_1, N_2\}$. And note that for $n \geq N$,

$$\begin{aligned}\left|\frac{1}{x_n} - \frac{1}{2}\right| &= \left|\frac{2 - x_n}{2x_n}\right| \\ &< \frac{|2 - x_n|}{2} \\ &< \frac{\epsilon}{2} \\ &< \epsilon\end{aligned}$$

□

2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

Definition 2.4.1

A sequence a_n is *increasing* if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and *decreasing* if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is *monotone* if it is either increasing or decreasing.

Theorem 2.4.2: Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges.

Proof. Let (a_n) be an increasing and bounded sequence. Since (a_n) is bounded, the set $A = \{a_n \mid n \in \mathbb{N}\}$ is clearly also bounded. Since A is bounded, $\sup A$ exists. We claim that $\lim_{n\to\infty} a_n = \sup A$. Thus, for all $\epsilon > 0$ and by our definition of supremum, there exists $N \in \mathbb{N}$ such that $\sup A - \epsilon < a_N \leq \sup A$. Since (a_n) is increasing, for all $n \geq N$, $\sup A - \epsilon < a_N \leq a_n \leq \sup A$. It follows that $|a_n - \sup A| < \epsilon$. Therefore, $\lim_{n\to\infty} a_n = \sup A$. □

Example 2.9: MCT

Consider the recursively defined sequence x_n where $x_1 = 3$ and for all $n \in \mathbb{N}$, $x_{n+1} = \frac{1}{4-x_n}$. Show that x_n converges.

Proof. We will show that x_n is monotone and bounded.

- **Part 1: Monotone Decreasing**
  - Base case: $x_1 = 3$, $x_2 = 1$.
  - Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq x_{n+1}$. It follows that

$$\begin{aligned} x_n &\geq x_{n+1} \\ 4 - x_n &\leq 4 - x_{n+1} \\ \frac{1}{4-x_n} &\geq \frac{1}{4-x_{n+1}} \\ x_{n+1} &\geq x_{n+2} \end{aligned}$$

- **Part 2: Bounded Below** Claim: Sequence is bounded below by 0.
  - Base case: $x_1 = 3 > 0$.
  - Induction step: Assume for some $n \in \mathbb{N}$, $x_n \geq 0$. It follows that $4 - x_n \leq 4$, and when we take the reciprocal, we get

$$\begin{aligned} \frac{1}{4-x_n} &\leq \frac{1}{4} \\ x_{n+1} &\geq 1/4 \\ &> 0 \end{aligned}$$

    By math induction, x_n is bounded below by 0.

By the Monotone Convergence Theorem, x_n converges.

So, what is the limit? We know (x_n) converges so let $L = \lim_{n\to\infty} x_n$. Then, $\lim_{n\to\infty} x_{n+1} = L$. We also know $x_{n+1} = \frac{1}{4-x_n}$. So $L = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{4-x_n} =$

$\frac{1}{4-L}$. It must be true that $L = \frac{1}{4-L}$. Solving for L, we get

$$L(4-L) = 1$$
$$4L - L^2 = 1$$
$$L^2 - 4L + 1 = 0$$

Hence, $L = 2 - \sqrt{3}$ or $L = 2 + \sqrt{3}$. Notice that it cannot be the latter because it is bigger than 3. □

2.4.1 Recap and Summary

We use limits to define multiple things in calculus. This is why we are focusing so heavily upon it. For example,

1. Derivatives: $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
2. Integrals: $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$
3. Infinite Series: $\lim_{n\to\infty} \sum_{i=1}^{n} a_i$ Consider geometric series, C_a such that each term is multiplied by a ratio r. This is represented as $\sum_{n=0}^{\infty} ar^n = 1 + r + r^2 + r^3 \dots$. When we look at partial sums, we get $S_n = 1 + r + r^2 + r^3 + \cdots + r^n$. We can then multiply by r to get $rS_n = r + r^2 + r^3 + r^4 + \cdots + r^{n+1}$. Subtracting the two, we get $(1-r)S_n = 1 - r^{n+1}$. Thus,
$$S_n = \frac{1-r^{n+1}}{1-r}.$$
If $|r| < 1$, then $\lim_{n\to\infty} r^n = 0$. Thus, $\lim_{n\to\infty} S_n = \frac{1}{1-r}$.

Looking to the future, we are going to use functions and summations together. For example, when we have $f(x) = \sum_{n=0}^{\infty}(a_n)x^n$ such that $f'(x) = \sum_{n=0}^{\infty}(a_n)x^{n-1}$.

Definition 2.4.3

Let (x_n) be a bounded sequence. Then the *limit inferior* is $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf\{x_k \mid k \geq n\}$. This is the largest a limit can get. The *limit superior* is $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k \mid k \geq n\}$. This is the smallest a limit can get.

See Exercise 2.4.7 in the book for more information.

Example 2.10: Monotone Decreasing Sequence

The following sequence is an example of a monotone decreasing sequence.

$$x_1, x_2, x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 1\} = S.$$
$$x_2, x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 2\} = S.$$
$$x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 3\} = S.$$
$$x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 4\} = S.$$

$\limsup_{n\to\infty} x_n$ is guaranteed to exist by the Monotone Convergence Theorem.

Example 2.11: liminf

Let $x_n = (-1)^n(1+\frac{1}{n})$. Thus, $x_{1,2,3} = -2, 1\frac{1}{2}, -1\frac{1}{3}\ldots$

Example 2.12: Convergence Towards 0

Let $x_n = (-1)^n\frac{1}{n}$. Thus, $x_{1,2,3} = -1, \frac{1}{2}, -\frac{1}{3}\ldots.$

Theorem 2.4.4

A sequence x_n is convergent if, and only if, $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$.

See Theorem 2.4.6 in the book for another view.

2.4.2 Exercise

Exercise: 2.4.7 (Limit Superior)

Let (a_n) be a bounded sequence.

(a) Prove that the sequence defined by $y_n = \sup\{a_k \colon k \geq n\}$ converges.

(b) The *limit superior* of (a_n) or $\limsup_{n\to\infty} a_n$, is defined by

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} y_n,$$

where y_n is the sequence from part (a) of this exercise. Provide a reasonable definition for $\liminf_{n\to\infty} a_n$ and briefly explain why it always exists for any bounded sequence.

(c) Prove that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence, and give an example of a sequence for which the inequality is strict.

(d) Show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Solution.

(a) We will show that (y_n) converges.

Proof. Since (a_n) is bounded, there exists $M > 0$ such that $|a_n| \leq M$ for all n.

For each n, define $y_n = \sup\{a_k : k \geq n\}$. As n increases, the set $\{a_k : k \geq n\}$ becomes smaller, so the supremum cannot increase. Therefore, the sequence (y_n) is non-increasing:

$$y_{n+1} \leq y_n \quad \text{for all } n.$$

Additionally, since (a_n) is bounded below, so is (y_n). Therefore, (y_n) is a bounded, non-increasing sequence.

By the Monotone Convergence Theorem, every bounded, monotonic sequence converges. Thus, (y_n) converges. □

(b) A reasonable definition for $\liminf_{n\to\infty} a_n$ is to define $z_n = \inf\{a_k : k \geq n\}$ for each n. Then, the *limit inferior* of (a_n) is defined by:

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} z_n.$$

Since (a_n) is bounded, each z_n exists and the sequence (z_n) is non-decreasing. As n increases, the set $\{a_k : k \geq n\}$ becomes smaller, so the infimum cannot decrease. Therefore, (z_n) is a bounded, non-decreasing sequence, which converges by the Monotone Convergence Theorem. Hence, $\liminf_{n\to\infty} a_n$ always exists for any bounded sequence.

(c) We will show that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$ for every bounded sequence.

Proof. For each n, we have $z_n = \inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\} = y_n$. This implies:

$$z_n \leq y_n \quad \text{for all } n.$$

Taking limits as $n \to \infty$, we get:

$$\lim_{n\to\infty} z_n \leq \lim_{n\to\infty} y_n,$$

which means:

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n.$$

For an example where the inequality is strict, consider the sequence $a_n = (-1)^n$. Then:

$$y_n = \sup\{(-1)^k : k \geq n\} = 1, \quad z_n = \inf\{(-1)^k : k \geq n\} = -1.$$

Therefore:

$$\limsup_{n\to\infty} a_n = 1, \quad \liminf_{n\to\infty} a_n = -1, \quad \liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n.$$

□

(d) We will show that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ if and only if $\lim a_n$ exists. In this case, all three share the same value.

Proof. We show this by proving both implications:

($\Rightarrow$) Suppose $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$. We will show that $\lim a_n$ exists and equals L.
Let $\epsilon > 0$. Since $\limsup_{n\to\infty} a_n = L$, there exists N_1 such that for all $n \geq N_1$:

$$y_n = \sup\{a_k : k \geq n\} < L + \epsilon.$$

Similarly, since $\liminf_{n\to\infty} a_n = L$, there exists N_2 such that for all $n \geq N_2$:

$$z_n = \inf\{a_k : k \geq n\} > L - \epsilon.$$

Let $N = \max\{N_1, N_2\}$. Then, for all $n \geq N$:

$$L - \epsilon < z_n \leq a_n \leq y_n < L + \epsilon,$$

which implies:

$$|a_n - L| < \epsilon.$$

Therefore, $\lim a_n = L$.

($\Leftarrow$) Conversely, suppose $\lim a_n = L$. Then, for every $\epsilon > 0$, there exists N such that for all $n \geq N$:

$$|a_n - L| < \epsilon.$$

This implies that for all $n \geq N$, the set $\{a_k : k \geq n\}$ is contained in $(L - \epsilon, L + \epsilon)$. Therefore:

$$y_n = \sup\{a_k : k \geq n\} \leq L + \epsilon, \quad z_n = \inf\{a_k : k \geq n\} \geq L - \epsilon.$$

Taking limits, we get:

$$\limsup_{n\to\infty} a_n \leq L + \epsilon, \quad \liminf_{n\to\infty} a_n \geq L - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, it follows that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = L$. □

2.5 Subsequences and the Bolzano-Weierstrass Theorem

Definition 2.5.1

Let a_n be a sequence of real numbers, and let $n_1 < n_2 < n_3 < \ldots$ be an increasing sequence of natural numbers. Then, the sequence $a_{n_1}, a_{n_2}, a_{n_3}, \ldots$ is called a *subsequence* of a_n and is denoted by a_{n_k}, where $k \in \mathbb{N}$ indexes the subsequence.

Theorem 2.5.2

Subsequences of a convergent sequence converge to the same limit as the original sequence.

Proof. Let x_{n_k} be a subsequence of x_n, and let $L = \lim_{n\to\infty} x_n$. We want to show that $\lim_{n\to\infty} x_{n_k} = L$. Let $\epsilon > 0$. Since x_n converges to L, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$. Since n_k is increasing, there exists $M \in \mathbb{N}$ such that $n_k \geq N$ for all $k \geq M$. Thus, for all $k \geq M$, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n\to\infty} x_{n_k} = L$.

Let x_{n_k} be a subsequence of x_n. Let $\epsilon > 0$. Since $(x_n) \to L$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $|x_n - L| < \epsilon$.

Now, looking at x_{n_k}, notice that $n_k \geq k$ for all k. Consider $k = N$. For any $n \geq N$, $n \geq N \geq k$. Thus, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n\to\infty} x_{n_k} = L$. □

Theorem 2.5.3: Divergence Criterion

If x_n has two subsequences that converge to different limits, then x_n diverges.

Building upon this idea of Divergence, we can list some other ways a sequence can diverge:

1. Find one subsequence that diverges.
2. Find tow subsequences that converge to separate limits.
3. Negate the definition of convergence.
   - For example, a sequence converges to L if there exists $\epsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n - a| \geq \epsilon$. There exists a subsequence (a_{n_k}) such that for all $k \in \mathbb{N}$, $|a_{n_k} - L| \geq \epsilon$.

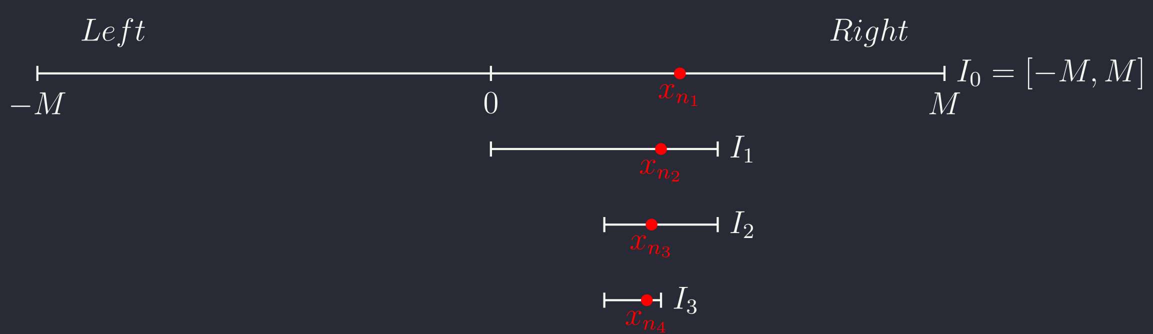
Theorem 2.5.4: Bolzano-Weierstrass Theorem

Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof. Let x_n be a bounded sequence. There exists an $M > 0$ such that every term x_n belongs to $[-M, M]$. To prove this theorem, we will be utilizing a recursive argument style. Thus, let $I_0 = [-M, M]$. I_0 has length $2M$. Cut I_0 in half with I_1 and I_2 both being half as long as I_0. Since x_n is bounded, there exists an I_L or I_R that contains infinitely many terms of x_n. We will pick one, call it I_1 that is contained in I_L. I_1 has length M. Pick one of those terms inside I_1 and call it x_{n_1}. Now, cut I_1 in half with equal length in intervals. One of them contains infinitely many terms. Call that interval I_2. I_2 has length $\frac{M}{2}$. Pick one of those terms inside I_2 and call it x_{n_2}. Continue this process indefinitely for all $n \geq \mathbb{N}$ with $n_1 > n_2$. Continue this process, and we get

- a sequence of closed intervals I_n.
  - I_n has length $\frac{2M}{2^n}$.
  - They are nested, $I_n \subseteq I_{n-1}$.
- a subsequence x_{n_k}
  - for all k_1, $x_{n_k} \in I_k$.

The Nested Interval Property states that $\bigcup_{n=1}^{\infty} I_n$ is non empty. Let L be a point in $\bigcup_{n=1}^{\infty} I_n$. We claim $\lim_{n\to\infty} x_{n_k} = L$. Let $\epsilon > 0$. There exists an $N \in \mathbb{N}$ such that $\frac{2M}{2^n} < \epsilon$. (Since $\lim_{n\to\infty} \frac{2M}{2^n} = 0$. See Theorem 2.5.5) For any $k \geq N$, recall that x_{n_k}, $L \in I_k$. Since I_k has length $\frac{2M}{2^n}$. Thus, $|x_{n_k} - L| < \epsilon$. Therefore, $\lim_{n\to\infty} x_{n_k} = L$ and (x_n) has a convergence subsequence. □



Theorem 2.5.5

Let $b \in (0, 1)$. Then $\lim_{n\to\infty} b^n = 0$.

Proof. The sequence (b^n) is monotone decreasing. This is because $b^{n+1} = b^n b < b^n$. This sequence is also bounded by 0. Hence, by the Monotone Convergence Theorem, (b^n) converges. Now, let $L = \lim_{n\to\infty} b^n$. Consider the subsequence b^{2n}. This sequence also converges to L. Thus,

$$\begin{aligned} L &= \lim_{n\to\infty} b^{2n} \\ &= \lim_{n\to\infty} b^n b^n \\ &= \lim_{n\to\infty} b^n \lim_{n\to\infty} b^n \\ &= L^2. \end{aligned}$$

Thus, $L = 0$ or $L = 1$. The limit cannot be 1 because b^n is decreasing away from 1. Therefore, $L = 0$. □

2.5.1 Exercises

Exercise: 2.5.1

Give an example of each of the following, or argue that such a request is impossible.

(a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

(b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

Solution.

(a) **Impossible.** This violates the Bolzano-Weierstrass Theorem. It assures us that every bounded sequence has a convergent subsequence. If a subsequence is bounded, then it must have a convergent subsequence.

(b) Consider the sequence $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \cdots \frac{1}{n}, \frac{(n-1)}{n})$. From this, you can have a subsequence $(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n})$ which converges to 0, and also a subsequence $(\frac{1}{2}, \frac{2}{3}, \ldots \frac{n-1}{n})$, which converges to 1.

Exercise: 2.5.2

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

(a) If every proper subsequence of (x_n) converges, then (x_n) converges as well.

(b) If (x_n) contains a divergent subsequence, then (x_n) diverges.

(c) If (x_n) is bounded and diverges, then there exist two subsequences of (x_n) that converge to different limits.

Solution.

(a) **True.** If every proper subsequence of (x_n) converges, then (x_n) must converge to the same limit. If (x_n) did not converge, there would exist at least one divergent subsequence or two subsequences converging to different limits, contradicting the assumption.

(b) **True.** If (x_n) contained a divergent subsequence, then (x_n) cannot converge. A convergent sequence has all its subsequences converging to the same limit, so the existence of a divergent subsequence implies that (x_n) diverges (contrapositive).

(c) **True.** Since (x_n) is bounded and diverges, the Bolzano-Weierstrass Theorem guarantees the existence of at least one convergent subsequence. Let this subsequence converge to L_1. Because (x_n) does not converge to L_1, there is an $\epsilon > 0$ and infinitely many terms of (x_n) such that $|x_n - L_1| \geq \epsilon$. Extracting a subsequence from these terms, the Bolzano-Weierstrass Theorem ensures a further subsequence converging to a limit $L_2 \neq L_1$. Thus, (x_n) has two subsequences converging to different limits.

Exercise: 2.5.5

Assume (a_n) is a bounded sequence with the property that every convergent subsequence of (a_n) converges to the same limit $a \in \mathbb{R}$. Show that (a_n) must converge to a.

Proof. Suppose that (a_n) does not converge to $a \in \mathbb{R}$. By the definition of convergence, this means there is a positive real number ϵ_0 such that no matter how large we choose $N \in \mathbb{N}$, there will always exist some $n > N$ where $|a_n - a| \geq \epsilon_0$. In a formal way, this shows that (a_n) does not converge to a within the ϵ_0-neighborhood.

We aim to demonstrate that this leads to a contradiction by constructing a subsequence of (a_n) that stays outside this neighborhood. Begin by selecting n_1 such that $|a_{n_1} - a| \geq \epsilon_0$. Next, since the condition holds for all $N \in \mathbb{N}$, we can find another index $n_2 > n_1$ such that $|a_{n_2} - a| \geq \epsilon_0$. Continuing this process, we generate an increasing sequence of indices $n_1 < n_2 < n_3 < \dots$ such that for each $i \in \mathbb{N}$, $|a_{n_i} - a| \geq \epsilon_0$.

Now consider the subsequence (a_{n_i}) we have built. Since (a_n) is bounded by assumption, its subsequence (a_{n_i}) is also bounded. By the Bolzano-Weierstrass Theorem, every bounded sequence has a convergent subsequence. Let $(a_{n_{i_k}})$ denote a convergent subsequence of (a_{n_i}). According to our assumption, any convergent subsequence of (a_n) must converge to a.

However, each term of $(a_{n_{i_k}})$ remains outside the ϵ_0-neighborhood of a. Thus, it is impossible for $(a_{n_{i_k}})$ to converge to a. This contradiction implies that our initial assumption—that (a_n) does not converge to a—is false. Therefore, the sequence (a_n) must converge to a. □

Exercise: 2.5.6

Use a similar strategy to the one in Theorem 2.5.5 to show

$$\lim b^{1/n} \text{ exists for all } b \geq 0$$

and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

Proof. We will show that $\lim_{n\to\infty} b^{1/n}$ exists for all $b \geq 0$ and find its value.

- **Case 1:** $b = 0$.

  When $b = 0$, the sequence becomes $a_n = 0^{1/n} = 0$ for all n. Thus, $\lim_{n\to\infty} b^{1/n} = 0$.

- **Case 2:** $b > 0$.

  Suppose, for contradiction, that $\lim_{n\to\infty} b^{1/n} \neq 1$. Then there exists $\epsilon > 0$ and infinitely many n such that $|b^{1/n} - 1| \geq \epsilon$. Extract a subsequence (b^{1/n_k}) where this inequality holds for all k.

  Since $b^{1/n} > 0$ and bounded, by the Bolzano-Weierstrass Theorem, the subsequence (b^{1/n_k}) has a further subsequence that converges to a limit L. According to Exercise 2.3.1, any convergent subsequence of $(b^{1/n})$ must have its limit equal to $\lim_{n\to\infty} b^{1/n}$.

  Consider $\ln b^{1/n} = \frac{\ln b}{n}$. As $n \to \infty$, $\frac{\ln b}{n} \to 0$, so $\ln b^{1/n} \to 0$, which implies $b^{1/n} \to e^0 = 1$.

  This contradicts the assumption that $|b^{1/n_k} - 1| \geq \epsilon$, so $\lim_{n\to\infty} b^{1/n} = 1$.

Conclusion:

$$\lim_{n\to\infty} b^{1/n} = \begin{cases} 0, & \text{if } b = 0, \\ 1, & \text{if } b > 0. \end{cases} \qquad \square$$

2.6 The Cauchy Criterion

Recall

How do we prove x_n converges?

1. We know and prove the limit $\rightarrow$ claim L, show terms get close to L.
2. Monotone Convergence Theorem.

Definition 2.6.1

A sequence x_n is a *Cauchy sequence* if for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $|x_m - x_n| < \epsilon$.

This says that as terms get close to each other and stay close together, there's some value they're all getting close to.

Geometric Series Review

Remember that geometric series consist of terms that are multiplied by a ratio r. For example, that could look like $1 + r + r^2 + r^3 + \cdots$.

We are most interested in **partial sums**. That is,

$$1 + r + r^2 + \cdots r^{n-1} + r^n = S_n.$$

From here, we we would multiply both sides by r. This gives

$$r + r^2 + \cdots r^n + r^{n+1} = rS_n.$$

When we subtract these two from each other, we get

$$1 - r^{n+1} = S_n - rS_n.$$

This yields the identity

$$1 + r + r^2 + \cdots r^n = \frac{1 - r^{n+1}}{1 - r}.$$

Example 2.13: Cauchy Sequence

Consider the sequence $a_1 = 1, a_2 = 2$, where

$$a_n = \frac{a_{n-1} + a_{n-2}}{2} \text{ for all } n \geq 2.$$

Show this sequence is Cauchy.

Proof. Look at the differences of consequtive terms, $|a_1 - a_2| = 1$, $|a_2 - a_3| = 1/2$, we can see a formula $a_n - a_{n+1} = 1/2^{n-1}$. Assume $|a_n - a_m| = |a_n - a_{n+1} - a_{n+2}| - \cdots -$

$a_{m-1} - a_m$ with $n < m$. From the Triangle Inequality,

$$|a_n - a_m| \le |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \tag{2.10}$$
$$= \frac{1}{2^{n-1}} + \frac{1}{2}^n + \cdots + \frac{1}{2}^{m-2} \tag{2.11}$$
$$= \frac{1}{2^{n-1}} \left(1 + \frac{1}{2} + \cdots + \frac{1}{2}^{m-n-1}\right) \tag{2.12}$$
$$= \frac{1}{2^{n-1}} \left(\frac{1 - \frac{1}{2}^{m-n}}{1 - \frac{1}{2}}\right) \tag{2.13}$$
$$= \frac{1}{2^n} \left(1 - \frac{1}{2^{m-n}}\right) \tag{2.14}$$
$$< \frac{1}{2^n}. \tag{2.15}$$

Notice that we were able to pull out the 1/2 and use the geometric series formula at step 2.12. From here we know that $|a_n - a_m| < \frac{1}{2^n}$.

Now, conclude the proof by letting $\epsilon > 0$. We know $(1/2^n) \to 0$. Thus, there exists $N \in \mathbb{N}$ such that $\frac{1}{2^N} < \epsilon$. For all $n, m \ge N$, (without loss of generality $n < m$) $|a_n - a_m| < \frac{1}{2^n} \le \frac{1}{2^N} < \epsilon$. Therefore, a_n is Cauchy and it converges. □

Note: To find the limit of this series, a proof strategy is finding subsequences that are odd and even, and show the converge to the same limit.

Theorem 2.6.2: Cauchy Criterion

A sequence x_n converges if, and only if, it is a Cauchy sequence.

Proof. We show this by proving both implications:

($\Rightarrow$) Assume (x_n) is a convergent seqeuence in $\mathbb{R}$. Given $\epsilon > 0$. Let $L = \lim_{n\to\infty} x_n$. Since $(x_n) \to L$, there exists an $N \in \mathbb{N}$ such that for all $n \ge N$, $|x_n - L| < \frac{\epsilon}{2}$. For all $n, m \ge N$,

$$\begin{aligned} |x_m - x_n| &= |x_m - L + L - x_n| \\ &\le |x_m - L| + |L - x_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, x_n is a Cauchy sequence.

(⇐) Assume x_n is a Cauchy sequence.

- **Step 1:** Show that x_n is bounded.
  Since x_n is Cauchy, there exists $N \in \mathbb{N}$ such that for all $n, m \geq N$, $|x_n - x_m| < l$. It follows that for all $n \geq N$, we need to account for $x_1 \ldots, x_{n-1}$. Thus, let $M = \max\{|x_1|, |x_2|, \ldots, |x_{n-1}|, |x_n| + 1\}$. Then for all $n \in \mathbb{N}$, $|x_m| < M$.
- **Step 2:** Since x_n is bounded, there exits a convergent subsequence x_{n_k} by the Bolzano-Weierstrass Theorem. Let L be the limit of the subsequence.
- **Step 3:** Show that x_n converges to L.
  If some get close to L and all get close to each other, they all get close to L. Let $\epsilon > 0$. Since x_{n_k} converges to L, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $|x_{n_k} - L| < \frac{\epsilon}{2}$. Since x_n is Cauchy, there exists $M \in \mathbb{N}$ such that for all $n, m \geq M$, $|x_n - x_m| < \frac{\epsilon}{2}$. Let $M_0 = \max\{N_1, n_k\}$. By the Archimedean Principle, there exists N_0 such that $n_{k_0} \geq M_0$. Then, from the Triangle Inequality, we say that for all $n \geq N_0$,

$$\begin{aligned} |x_n - L| &\leq \left|x_n - x_{n_{k_0}}\right| + \left|x_{n_{k_0}} - L\right| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

  Therefore, $(x_n) \to L$.

By proving both directions of the inequality, we found that a sequence (x_n) converges if, and only if, it is a Cauchy sequence. □

Definition 2.6.3

A sequence is called *contracting* is there exists $0 < C < 1$ such that for all $n \in \mathbb{N}$, $|x_{n+1} - x_n| \leq C\,|x_n - x_{n-1}|$.

How this works: we take a sequence $a_1, a_2, \ldots$ and subtract $a_1 - a_2$. Then, we have the inequality:

$$\begin{aligned} &|a_2 - a_1| \leq C\,|a_1 - a_0| \\ &|a_3 - a_2| \leq C\,|a_2 - a_1| \leq C^2\,|a_1 - a_2| \\ &|a_4 - a_3| \leq C\,|a_3 - a_2| \leq C^3\,|a_1 - a_2| \\ &\qquad\vdots \end{aligned}$$

From this, a theorem emerges:

Theorem 2.6.4

If a sequence is contracting, then it is Cauchy, and thus converges.

Proof. Let (a_n) be a contracting sequence; that is, there exists a constant $0 < C < 1$ such that for all $n \in \mathbb{N}$,

$$|a_{n+1} - a_n| \leq C|a_n - a_{n-1}|.$$

We will show that (a_n) is a Cauchy sequence.

First, we observe by induction that for all $k \geq 1$,

$$|a_{n+k} - a_{n+k-1}| \leq C^k|a_n - a_{n-1}|.$$

Proof by induction:

Base case $(k = 1)$**:**

$$|a_{n+1} - a_n| \leq C|a_n - a_{n-1}|.$$

Inductive step: Assume that for some $k \geq 1$,

$$|a_{n+k} - a_{n+k-1}| \leq C^k|a_n - a_{n-1}|.$$

Then,

$$\begin{aligned}|a_{n+k+1} - a_{n+k}| &\leq C|a_{n+k} - a_{n+k-1}| \\ &\leq C\left(C^k|a_n - a_{n-1}|\right) \\ &= C^{k+1}|a_n - a_{n-1}|.\end{aligned}$$

Thus, the inequality holds for $k + 1$, completing the induction.

Next, for any integers $m > n$, we have:

$$|a_m - a_n| = \left|\sum_{j=n}^{m-1}(a_{j+1} - a_j)\right| \leq \sum_{j=n}^{m-1}|a_{j+1} - a_j|.$$

Applying the inequality obtained from the induction,

$$|a_{j+1} - a_j| \leq C^{j-n+1}|a_n - a_{n-1}|.$$

Therefore,

$$\begin{aligned}
|a_m - a_n| &\leq |a_n - a_{n-1}| \sum_{j=n}^{m-1} C^{j-n+1} \\
&= |a_n - a_{n-1}| \sum_{k=1}^{m-n} C^k \quad (\text{Let } k = j - n + 1) \\
&= |a_n - a_{n-1}| \left(\frac{C(1 - C^{m-n})}{1 - C} \right).
\end{aligned}$$

Since $C^{m-n} \geq 0$, we have:

$$|a_m - a_n| \leq |a_n - a_{n-1}| \left(\frac{C}{1 - C} \right).$$

As $n \to \infty$, the term $|a_n - a_{n-1}|$ tends to zero because:

$$|a_n - a_{n-1}| \leq C^{n-1} |a_1 - a_0| \to 0 \quad \text{as } n \to \infty.$$

Therefore, for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$|a_n - a_{n-1}| < \epsilon \left(\frac{1 - C}{C} \right).$$

Then, for all $m, n \geq N$ (with $m > n$),

$$|a_m - a_n| \leq |a_n - a_{n-1}| \left(\frac{C}{1 - C} \right) < \epsilon.$$

This shows that (a_n) is a Cauchy sequence. Since every Cauchy sequence in $\mathbb{R}$ converges, the sequence (a_n) converges. □

Chapter 3

Basic Topology of Real Numbers

3.1 Discussion: The Cantor Set

We will build this set through an iterative process. Start with a number line C_0 that stretches from 0 to 1. Remove the middle third of the interval, leaving two intervals of length $\frac{1}{3}$. We will call the set of points removed from C_0 C_1. Next, remove the middle third of each of the two intervals, leaving four intervals of length $\frac{1}{9}$. We will call the set of points removed from C_1 C_2. Continue this process indefinitely.

Definition 3.1.1

The *Cantor set*, C, is defined as $C = \bigcap_{n=0}^{\infty} C_n$. This set is

1. non-empty. All end points stay within the interval.
2. uncountable.

The second part of that definition is a bit tricky to prove, but a visual will do for now. We can put all elements of the Cantor set in a one-to-one correspondence with the set of all 0s and 1s. This shows that not only is it uncountable, but it also has the same cardinality as $[0, 1]$.

The total length of removed elements, $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots = \frac{1}{3}(1 + \frac{2}{3} + \frac{4}{9})$. Notice the resemblance to the geometric series? We can write this as

$$\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1.$$

In summary,

1. Start with intercal $[0, 1]$.
2. Remove countably disjoint intervals.
3. Uncountably many points between these intervals, all isolated from each other.
4. The space taken up by the leftover points has “length” 0.

When we review the properties of a fractal, we see that the Cantor set is a fractal. It is self-similar, and the dimension of the Cantor set is $\log_3 2$.

For a cool look at the cantor set as a fractal, check out `https://en.wikipedia.org/wiki/File:Cantor_Set_Expansion.gif`.

3.2 Open and closed Sets

Definition 3.2.1

For a point $x \in \mathbb{R}$, and $\epsilon > 0$, we define the *epsilon-neighborhood* of x to be $V_\epsilon(x) = \{y \in \mathbb{R} \mid |x - y| < \epsilon\}$.

In other words, $V_\epsilon(x)$ is the open interval $(x - \epsilon, x + \epsilon)$, centered at x with radius ϵ.

3.2.1 Open Sets

Definition 3.2.2

A set $A \subseteq \mathbb{R}$, is called an *open set* if for every $x \in A$, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq A$.

Some Examples of Open Sets

- All open intervals are also open sets.
- $\mathbb{R}$ is open.
- $\emptyset$ is open.
- $\{1\}$ is not open.
- $[0, 2]$ is not open.
- $\mathbb{Q}$ is not open.
- $[4, 6)$ is not open.
- $(0, 1) \cup (1, 3) \cup (5, 10)$ is open.
- $(0, 3] \cap [2, 4)$ is open.
- Cantor set is not open.

Theorem 3.2.3

(i) The union of an arbitrary collection of open sets is open.

(ii) The intersection of a finite collection of open sets is open.

Proof. (i) Let $\{O_\lambda \colon \lambda \in A\}$ be a collection of open sets. Then, let $O = \bigcup_{\lambda \in A} O_\lambda$. Let a be an element of O. To show that O is open, we need to find an ϵ-neighborhood that is completely contained within O to satisfy Definition 3.2.1. But $a \in O$ implies that a is an element of at least one particular $O_{\lambda'}$. Because we are assuming $O_{\lambda'}$ to be open, then we can use Definition 3.2.1 to assert that there exists $V_\epsilon(a) \subseteq O_{\lambda'}$. The fact that $O_{\lambda'} \subseteq O$ confirms that $V_\epsilon(a) \subseteq O$.

(ii) Let $\{O_1, O_2, \ldots, O_N\}$ be a finite collection of open sets. Then, let $a \in \bigcap_{k=1}^{N} O_k$. This means a is an element of every open set. Definition 3.2.1 tells us that for $1 \leq k \leq K$, there exists an $V_\epsilon(a) \subseteq O_k$. From this set, we are in search of one ϵ-neighborhood of a that is contained in every O_k, so the trick is to pick the smallest one. Letting $\epsilon = \min\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$, it follows that $V_\epsilon(a) \subseteq V_{\epsilon_k}(a)$ for all k, and hence $V_\epsilon(a) \subseteq \bigcap_{k=1}^{N} O_k$. $\square$

Note that we cannot use this for cases with infinity. For example, consider $A_n = (-\frac{1}{n}, \frac{1}{n})$. This is open, but $\bigcap_{n=1}^{\infty} A_n = \{0\}$, which is not open.

3.2.2 Closed Sets

Definition 3.2.4

Let $A \subseteq \mathbb{R}$. We say x is a *limit point* of A if for all $\epsilon > 0$, there exists $a \in A$ such that $a \in V_\epsilon(x)$ that is not x. Additionally, a point $x \in \mathbb{R}$ is a *limit point* if, and only if, there exists a sequence (a_n) of points from A that are not x. And $\lim_{n\to\infty}(a_n) = x$.

Definition 3.2.5

A set $B \subseteq \mathbb{R}$ is called a *closed set* if B contains all its limit points.

Important note: Limit points could be outside a set. Consider $(0, 1)$. Even though 0 and 1 do not belong to the set, they are considered limit points that are outside the set.

Some Examples of Closed Sets

- $[0, 1]$ is closed.
- $(0, 1)$ is not closed.
- $\mathbb{R}$ is closed.
- $\emptyset$ is closed.
- $\mathbb{Q}$ is not closed.
- $[3, \infty)$ is closed.
- $\frac{1}{n} \mid n \in \mathbb{N}$ not closed. (Because of 0)
- $[1, 4] \cup \{8\}$ is closed.
- $\{1\}$ is closed.
- $[1, 2)$ is not closed. Note that this set is neither open or closed.

Theorem 3.2.6

A set $B \subseteq \mathbb{R}$ is closed if, and only if, its complement is open. Similarly, a set $A \subseteq \mathbb{R}$ is open if, and only if, its complement is closed.

Proof. We show this by proving both implications:

($\Rightarrow$) Assume $B \subseteq \mathbb{R}$ is a closed set. We will show that B^c is open. Let $x \in B^c$. So, $x \notin B$. This means x is not a limit point. (From the negated definition of limit point:) There must exist $\epsilon > 0$ such that no elements of B belong to $V_\epsilon(x)$. Then, $V_\epsilon(x) \subseteq B^c$. Therefore, B^c is open.

($\Leftarrow$) Assume B^c is open. We will show that B is closed. Let x be a limit point of B. For all $\epsilon > 0$, there exists a $b \in B$ such that $b \subseteq V_\epsilon(x)$. So, $V_\epsilon(x)$ is not a subset of B^c. This is true for every ϵ. Since B^c is open, it must be that $x \notin B^c$. Thus, $x \in B$. So, B contains all its limit points. Therefore, B is closed. $\square$

Definition 3.2.7

Let $A \subseteq \mathbb{R}$ and let L be the set of limit points of A. The *closure* of A is defined as $\bar{A} = A \cup L$.

Theorem 3.2.8

(i) The intersection of any collection of closed sets is closed.

(ii) The union of finitely many closed sets is closed.

Proof. De Morgan's Laws state that for any collection of sets $\{E_\lambda : \lambda \in A\}$ it is true that

$$\left(\bigcup_{\lambda \in A} E_\lambda\right)^c = \bigcap_{\lambda \in A} E_\lambda^c \qquad \text{and} \qquad \left(\bigcap_{\lambda \in A} E_\lambda\right)^c = \bigcup_{\lambda \in A} E_\lambda^c.$$

The result follows directly from these statements and Theorem 3.2.3. $\square$

Theorem 3.2.9

The closure of a set is a closed set.

Note: This theorem may seem trivial, but it answers the question of "Are there limit points in L that are not accounted for?"

Proof. We need to show that $\bar{A}$ contains all the limit points of $\bar{A}$. Let L be the limit points of A. Thus, $\bar{A} = A \cup L$. Let x be a limit point of $\bar{A}$. There exists a sequence of points (x_n) coming from $\bar{A}$ such that $(x_n) \to x$. Then, for all $n \in \mathbb{N}$, either $x_n \in A$ or $x_n \in L$.

- **Case 1:** $x_n \in A$

There exists a subsequence (x_{n_k}) where each $x_{n_k} \in A$. This subsequence also converges to x, and we know the limit belongs to L, so $x \in L \subseteq \bar{A}$.

- **Case 2:** $x_n \in L$
  x_n belongs to A for only finitely many $n \in \mathbb{N}$. Thus, a tail-end of the sequence is comprised entirely of points from L. To simplify things, we will assume the entire sequence (x_n) comes from L. (We know that (x_n) converges to x, but we cannot assume those limit points converge as well.) Let $n \in \mathbb{N}$. Since $x_n \in L$, there exists $a_n \in A$ such that $|x_n - a_n| < \frac{1}{n}$. We now have $(x_n) \to x$ and $(x_n - a_n) \to 0$. Then, $(a_n) \to x$. Thus, $x \in L \subseteq \bar{A}$.

Now that we have shown that either cases leads to the same conclusion, we know that $x \in L \subseteq \bar{A}$, and therefore $\bar{A}$ contains all its limit points. □

Theorem 3.2.10

The closure set $\overline{A}$ is the *smallest* closed set containing A. (Where "smallest" refers to a subset of any other closed set containing A.)

Proof. If B is a closed set containing A, then $A \subseteq B$ and $L \subseteq B$. Thus, $\bar{A} = A \cup L \subseteq B$. □

Example 3.1: Closed Sets 1

Generate countably many closed sets where the union is not closed.

Solution. $B_n = [\frac{1}{n}, 1 - \frac{1}{n}]$. Therefore, $\bigcup_{n=3}^{\infty} B_n = (0, 1)$. For example, that would look like: $\{\frac{1}{2}\} \cup \{\frac{1}{3}\} \cup \ldots$

Example 3.2: Closed Sets 2

What is the closure of the following sets?

$$\text{(a) } (0,1), \quad \text{(b) } \mathbb{R}, \quad \text{(c) } \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}, \quad \text{(d) } [0,1) \cup (1,3], \quad \text{(e) } \mathbb{Q}$$

Solution. (a) $\bar{A} = [0, 1]$, (b) $\bar{A} = \mathbb{R}$, (c) $\bar{A} = A \cup \{0\}$, (d) $\bar{A} = [0, 3]$, (e) $\bar{A} = \mathbb{R}$.

3.2.3 Exercises

Exercise: 3.2.4

Let A be a nonempty and bounded above set so that $s = \sup(A)$ exists. (See Definition 1.3.2 and Definition 3.2.7)

(a) Show that $s \in \bar{A}$.

(b) Can an open set contain its supremum?

Solution.

(a) We need to show that $s = \sup(A) \in \bar{A}$, where $\bar{A} = A \cup L$, and L is the set of limit points of A.

Since A is nonempty and bounded above, $s = \sup(A)$ exists.

If $s \in A$, then $s \in \bar{A}$ trivially.

Suppose $s \notin A$. We will show that s is a limit point of A, so $s \in L \subseteq \bar{A}$.

By definition, x is a limit point of A if for all $\epsilon > 0$, there exists $a \in A$ such that $a \in V_\epsilon(x)$ and $a \neq x$.

Fix any $\epsilon > 0$. Since $s = \sup(A)$, for this ϵ, $s - \epsilon$ is not an upper bound of A. Therefore, there exists $a \in A$ such that

$$s - \epsilon < a \leq s.$$

Since $a \leq s$ and $a > s - \epsilon$, we have $|a - s| < \epsilon$, so $a \in V_\epsilon(s)$ and $a \neq s$.

Therefore, s is a limit point of A, and hence $s \in \bar{A}$.

(b) An open set cannot contain its supremum if the supremum is finite.

Assume A is an open set containing its supremum s.

Since A is open and $s \in A$, there exists $\epsilon > 0$ such that

$$V_\epsilon(s) = \{x \in \mathbb{R} \mid |x - s| < \epsilon\} \subseteq A.$$

This means $s + \frac{\epsilon}{2} \in A$.

However, s is an upper bound of A, so no element of A can be greater than s.

This is a contradiction.

Therefore, an open set cannot contain its supremum.

Exercise: 3.2.6

Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.

(b) The Nested Interval Property remains true if the term "closed interval" is replaced by "closed set."

(c) Every nonempty open set contains a rational number.

(d) Every bounded infinite closed set contains a rational number.

(e) The Cantor set is closed.

Solution.

(a) **False.**

Counterexample: Consider the set $U = \bigcup_{n=1}^{\infty} \left(q_n - \frac{1}{n},\ q_n + \frac{1}{n}\right)$, where (q_n) is an enumeration of all rational numbers.

Each interval $\left(q_n - \frac{1}{n},\ q_n + \frac{1}{n}\right)$ is open, and the union U is open. Since every rational number is included in some interval, U contains all rationals. However, $U \neq \mathbb{R}$ because there are irrational numbers not covered by these intervals. Therefore, an open set can contain all rational numbers without being all of $\mathbb{R}$.

(b) **True.**

Proof. The Nested Interval Property holds for any nested sequence of nonempty closed and bounded sets in $\mathbb{R}$. If $\{F_n\}$ is such a sequence with $F_{n+1} \subseteq F_n$ for all n, then the intersection $\bigcap_{n=1}^{\infty} F_n$ is nonempty. This follows from the completeness of $\mathbb{R}$, as every decreasing sequence of nonempty closed and bounded sets has a nonempty intersection. Therefore, replacing "closed interval" with "closed set" does not invalidate the property. □

(c) **True.**

Proof. Let U be a nonempty open set. Then there exists $x \in U$ and $\epsilon > 0$ such that $(x - \epsilon,\ x + \epsilon) \subseteq U$. Since the rationals are dense in $\mathbb{R}$, there exists a rational number $q \in (x - \epsilon,\ x + \epsilon)$. Therefore, U contains a rational number. □

(d) **True.**

Proof. Let F be a bounded infinite closed set. Since F is infinite and bounded, it must contain limit points. As F is closed, it contains its limit points. The real numbers are densely ordered with rationals between any two real numbers. Therefore, F must contain a rational number. $\square$

(e) **True.**

Proof. The Cantor set C is constructed as the intersection of a decreasing sequence of closed sets (finite unions of closed intervals). Since each of these sets is closed and the intersection of closed sets is closed, C is closed. $\square$

Exercise: 3.2.8

Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

(a) $\overline{A \cup B}$

(b) $A \setminus B = \{x \in A \mid x \notin B\}$

(c) $(A^c \cup B)^c$

(d) $(A \cap B) \cup (A^c \cap B)$

(e) $\overline{A}^c \cap \overline{A^c}$

Solution.

(a) $\overline{A \cup B}$

The closure of any set is closed by Theorem 3.2.9. Therefore, $\overline{A \cup B}$ is closed.

Conclusion: Closed.

(b) $A \setminus B = \{x \in A \mid x \notin B\}$

Since B is closed, its complement B^c is open. The set A is open by assumption. The intersection of two open sets is open. Note that $A \setminus B = A \cap B^c$. Therefore, $A \setminus B$ is open.

Conclusion: Open.

(c) $(A^c \cup B)^c$

By De Morgan's Law, $(A^c \cup B)^c = A \cap B^c$. Both A and B^c are open sets. The intersection of open sets is open. Thus, $(A^c \cup B)^c$ is open.

Conclusion: Open.

(d) $(A \cap B) \cup (A^c \cap B)$

We can factor out B:

$$(A \cap B) \cup (A^c \cap B) = [(A \cup A^c) \cap B] = \mathbb{R} \cap B = B.$$

Therefore, the set equals B, which is closed.

Conclusion: Closed.

(e) $\overline{A}^c \cap \overline{A^c}$

Since A is open, its closure $\overline{A}$ is a closed set containing all limit points of A. Therefore, the complement $\overline{A}^c$ is open.

Similarly, A^c is closed (being the complement of an open set), so its closure is $\overline{A^c} = A^c$, which is closed.

Now, consider the intersection:

$$\overline{A}^c \cap \overline{A^c} = \overline{A}^c \cap A^c$$

Since $\overline{A} \supseteq A$, we have $\overline{A}^c \subseteq A^c$. Thus, the intersection simplifies to $\overline{A}^c$.

However, any point not in $\overline{A}$ cannot be a limit point of A or belong to A. In $\mathbb{R}$, this set is empty unless A is either $\emptyset$ or $\mathbb{R}$.

Therefore,

$$\overline{A}^c \cap \overline{A^c} = \emptyset$$

The empty set is both open and closed.

Conclusion: Both open and closed (since the set is empty).

Exercise: 3.2.11

(a) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Does this result about closures extend to infinite unions of sets?

Solution.

(a) We will prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof. Recall that the closure of a set A is defined as $\overline{A} = A \cup L_A$, where L_A is the set of limit points of A.

Proof of $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$:

Let $x \in \overline{A \cup B}$. Then $x \in A \cup B$ or x is a limit point of $A \cup B$.

- If $x \in A \cup B$, then $x \in A$ or $x \in B$, so $x \in \overline{A}$ or $x \in \overline{B}$, thus $x \in \overline{A} \cup \overline{B}$.

- If x is a limit point of $A \cup B$, then every neighborhood $V_\epsilon(x)$ contains a point $y \neq x$ such that $y \in A \cup B$. Therefore, $y \in A$ or $y \in B$, so x is a limit point of A or B. Hence, $x \in \overline{A}$ or $x \in \overline{B}$, so $x \in \overline{A} \cup \overline{B}$.

Therefore, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

Proof of $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$:

Let $x \in \overline{A} \cup \overline{B}$. Then $x \in \overline{A}$ or $x \in \overline{B}$.

- If $x \in \overline{A}$, then $x \in A$ or x is a limit point of A. Since $A \subseteq A \cup B$, $x \in A \cup B$ or x is a limit point of $A \cup B$. Thus, $x \in \overline{A \cup B}$.

- Similarly, if $x \in \overline{B}$, then $x \in \overline{A \cup B}$.

Therefore, $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Hence, $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

□

(b) The result does not necessarily extend to infinite unions of sets.

Consider the sets $A_n = \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ for $n \in \mathbb{N}$. Then $\overline{A_n} = \left[\frac{1}{n}, 1 - \frac{1}{n}\right]$.

The infinite union is $A = \bigcup_{n=1}^{\infty} A_n = (0,1)$, so $\overline{A} = [0,1]$.

The union of the closures is $\bigcup_{n=1}^{\infty} \overline{A_n} = (0,1)$, since none of the closed intervals $\left[\frac{1}{n}, 1 - \frac{1}{n}\right]$ include the endpoints 0 or 1.

Therefore, $\overline{A} \neq \bigcup_{n=1}^{\infty} \overline{A_n}$.

Hence, the equality does not hold for infinite unions.

3.3 Compact Sets

Definition 3.3.1

A set $K \subseteq \mathbb{R}$ is a *compact set* if every sequence from K has a convergent subsequence where the limit is also K.

Theorem 3.3.2

A set K is compact if, and only if, it is closed and bounded.

Proof. We show this by proving both implications:

(⇒) Assume a set $A \subseteq \mathbb{R}$ is closed and bounded. Thus, there exists a convergent subsequence by Bolzano-Weierstrass Theorem. Because A is closed, the limit is in the set by Definition 3.2.5.

(⇐) Assume a set A is compact. If it is not bounded, then there exists an (a_n) that heads toward infinity. This contradicts Definition 3.3.1, so it must be bounded. Then, by the same definition, the limit points belong in the set, so it is closed. □

Definition 3.3.3

Let $A \in \mathbb{R}$. An *open cover* for A is a collection of open sets $\{O_\lambda \mid \lambda \in A\}$ whose union contains the set A; that is $A \subseteq \bigcup_{\lambda \in A} O_\lambda$. Given an open cover for A, a *finite subcover* is a finite sub-collection of open sets from the original open cover whose union still manages to completely contain A.

Theorem 3.3.4: Heine-Borel Theorem

Let K be a subset of $\mathbb{R}$. All the following statements are equivalent in the sense that any of them implies the two others:

(i) K is compact.

(ii) K is closed and bounded.

(iii) Every open cover of K has a finite subcover.

Proof. The first set of "if and only if proofs" will be to prove (i) and (ii) are equivalent. Then, we will prove (ii) and (iii) are equivalent.

(⇒) Assume K is compact. We need to show that K is closed and bounded. To show K is bounded, consider the open cover $\mathcal{U} = \{(-n, n) \mid n \in \mathbb{N}\}$. $\mathcal{U}$ covers all of $\mathbb{R}$, so it certainly covers K. Thus, there must exist a finite subcover. Consider the longest interval in the subcover. Then, K is a subset of this interval, so K is bounded. To show K is closed, we need to show every limit point belongs to K. Assume x is a limit point of K. From Definition 3.2.4 for every $a \in K$, let $\epsilon_a = \frac{1}{2}|a - x|$. Consider the open cover $\mathcal{U} = \{V_{\epsilon_a}(a) \mid a \in K\}$. This covers every point on K.

(⇐) Because it is closed and bounded, by Theorem 3.3.2, K is compact.

Now for the second part of the proof:

($\Rightarrow$) Let x be a limit point of K. This means there must exist a sequence (x_n) in K with $\lim_{n\to\infty} x_n = x$. Suppose $x \notin K$. For every $y \in K$. Let $\epsilon_y = \frac{1}{2}|y-x|$. Consider the open neighborhood $V_{\epsilon_y}(y)$. Notice $x \in V_{\epsilon_y}(y)$. Now, we will work with the collection of all such neighborhoods $\mathcal{U} = \{V_{\epsilon_y}(y) \mid y \in K\}$. This $\mathcal{U}$ is an open cover of K. By our hypothesis there exists a finite subcover. There are some $y_1, y_2, \ldots, y_m$ such that $K \subset \bigcup_{i=1}^{m} V_{\epsilon_y}(y_i)$. Look at the distance from x to each y_i: $(x - \epsilon_{y_i},\, x + \epsilon_{y_i}) \cap V_{\epsilon_{y_i}}(y_i) = \emptyset$. Similar statements are for every y_i. Let $\epsilon = \min\{\epsilon_{y_1}, \epsilon_{y_2}, \ldots \epsilon_{y_M}\}$. Since there are infinitely many $\epsilon > 0$, we see that $V_\epsilon(x) \cap V_{\epsilon_y}(y_i) = \emptyset$ for every $i \leq M$. So $V_\epsilon(x) \cap K = \emptyset$. This gives us an ϵ-neighborhood around x that does not intersect K. Since (x_n) approaches x, there must be elements from the sequence that are inside of $V_\epsilon(x)$. This creates a contradiction because we said x was a limit point. Therefore $x \in u$ and K must be closed.

($\Leftarrow$) Let $\mathcal{U}$ be an open cover of K. Suppose there is no finite subcover. Since K is bounded there exists a closed interval I_0 that contains K. Bisect I_0 and look at the two sub intervals A and B. My claim is at least one of $A \cap K$ and $B \cap K$ does not have a finite subcover from $\mathcal{U}$. If not, then we would have a finite subcover of all of K. Whichever half does not have a finites of cover will be called I_1. Repeat this process. We get a sequence of nested closed intervals $I_0 \supseteq I_1 \supseteq I_2 \supseteq \cdots$ such that for all $j \in \mathbb{N}$, $I_j \cap K$ does not have a finite subcover from $\mathcal{U}$. Also, as the length of I_j approach 0, by the Nested Interval Property, there exists $x \in \bigcap_{j=1}^{\infty} I_j$. Since each I_j contains an element of K and the interval approaches 0, x must be a limit point of K. Thus, since K is closed, $x \in K$. There must be an open set $U \in \mathcal{U}$ such that $x \in U$. Since U is open and $x \in U$, there exists an $\epsilon > 0$ such that $V_\epsilon(x) \subseteq U$. There is an I_j whose length is smaller than ϵ. This means $I_j \subseteq V_\epsilon \subset U$. So $\{U\}$ is a finite subcover of $I_j \cap K$. This contradicts how we defined I_j, therefore there must be a finite subcover from $\mathcal{U}$. $\square$

This allows us to take an infinite amount of ϵ-neighborhoods and turn them into finite subcovers.

Example 3.3: Compactness

Let $A = (0, 1)$. Construct a set $\mathcal{U}$ that is an open cover of $(0, 1)$, but does not have a finite subcover.

Solution. Consider $\mathcal{U} = \{(0, t) \mid 0 < t < 1\}$. Thus, $\mathcal{U}$ is an open cover, but does not contain a finite amount of subcovers because there will always be a point not covered.

3.3.1 Exercises

Exercise: 3.3.4

Assume K is compact and F is closed. Decide if the following sets are definitely compact, definitely closed, both, or neither.

(a) $K \cap F$

(b) $\overline{F^c \cup K^c}$

(c) $K \setminus F = \{x \in K \mid x \notin F\}$

(d) $\overline{K \cap F^c}$

Solution.

(a) Since K and F are closed, their intersection $K \cap F$ is closed (Theorem 3.2.8).

To show that $K \cap F$ is compact, let $\mathcal{U}$ be any open cover of $K \cap F$. Our goal is to extract a finite subcover from $\mathcal{U}$. We can then use the Bolzano-Weierstrass Theorem (iii) to show that $K \cap F$ is compact.

Since F is closed, its complement F^c is open (Theorem 3.2.6). Then $K \setminus F = K \cap F^c$ is open as the intersection of an open set and K.

Consider the open cover $\mathcal{U}' = \mathcal{U} \cup \{K \setminus F\}$ of K. Every point in K is either in $K \cap F$ (covered by $\mathcal{U}$) or in $K \setminus F$ (covered by $K \setminus F$).

Since K is compact, there exists a finite subcover $\mathcal{U}'' \subseteq \mathcal{U}'$ that covers K.

If $K \setminus F$ is in $\mathcal{U}''$, remove it to obtain a finite subcollection of $\mathcal{U}$ that still covers $K \cap F$. If $K \setminus F$ is not in $\mathcal{U}''$, then $\mathcal{U}'' \subseteq \mathcal{U}$ already covers $K \cap F$.

Therefore, $K \cap F$ is compact.

Conclusion: Both compact and closed.

(b) Since F and K are closed, F^c and K^c are open. The union $F^c \cup K^c$ is open (Theorem 3.2.3), so its closure $\overline{F^c \cup K^c}$ is closed by Theorem 3.2.9.

This set may not be bounded, so it's not necessarily compact.

Conclusion: Definitely closed.

(c) The set $K \setminus F = K \cap F^c$ is the intersection of a compact set K and an open set F^c. This set is open in K but not necessarily open or closed in $\mathbb{R}$.

Since $K \setminus F$ is not necessarily closed, it may not be compact.

Conclusion: Neither compact nor closed.

(d) The set $K \cap F^c$ is open in K, so its closure $\overline{K \cap F^c}$ is closed by Theorem 3.2.9.

To show that $\overline{K \cap F^c}$ is compact, let $\mathcal{U}$ be any open cover of $\overline{K \cap F^c}$.

Since $\overline{K \cap F^c} \subseteq K$ and K is compact, we can consider $\mathcal{U}$ as an open cover of a subset of K.

By the definition of open cover, there exists a finite subcover of $\mathcal{U}$ that covers $\overline{K \cap F^c}$.

Therefore, $\overline{K \cap F^c}$ is compact.

Conclusion: Both compact and closed.

Exercise: 3.2.8

Assume A is an open set and B is a closed set. Determine if the following sets are definitely open, definitely closed, both, or neither.

(a) $\overline{A \cup B}$

(b) $A \setminus B = \{x \in A \mid x \notin B\}$

(c) $(A^c \cup B)^c$

(d) $(A \cap B) \cup (A^c \cap B)$

(e) $\overline{A}^c \cap \overline{A^c}$

Solution.

(a) $\overline{A \cup B}$

The closure of any set is closed by definition. Therefore, $\overline{A \cup B}$ is definitely closed.

Conclusion: Closed.

(b) $A \setminus B = \{x \in A \mid x \notin B\}$

Since B is closed, its complement B^c is open. Since A is open, the intersection $A \cap B^c = A \setminus B$ is the intersection of two open sets, which is open.

Conclusion: Open.

(c) $(A^c \cup B)^c$

Applying De Morgan's Law:

$$(A^c \cup B)^c = A \cap B^c$$

Since A is open and B^c is open (because B is closed), their intersection $A \cap B^c$ is open.

Conclusion: Open.

(d) $(A \cap B) \cup (A^c \cap B)$

Simplify the expression:

$$(A \cap B) \cup (A^c \cap B) = [A \cup A^c] \cap B = \mathbb{R} \cap B = B$$

Thus, the set equals B, which is closed.

Conclusion: Closed.

(e) $\overline{A}^c \cap \overline{A^c}$

Since A is open, its closure $\overline{A}$ is closed, so $\overline{A}^c$ is open.

Since A^c is closed (being the complement of an open set), $\overline{A^c} = A^c$ is closed.

Therefore, $\overline{A}^c \cap \overline{A^c}$ is the intersection of an open set and a closed set, which is generally open but not necessarily closed.

For example, let $A = (0, 1)$. Then:

$$\overline{A} = [0, 1], \quad \overline{A}^c = (-\infty, 0) \cup (1, \infty)$$

and

$$\overline{A^c} = A^c = (-\infty, 0] \cup [1, \infty)$$

Then:

$$\overline{A}^c \cap \overline{A^c} = [(-\infty, 0) \cup (1, \infty)] \cap [(-\infty, 0] \cup [1, \infty)] = (-\infty, 0) \cup (1, \infty)$$

Which is an open set.

Conclusion: Open.

Chapter 4

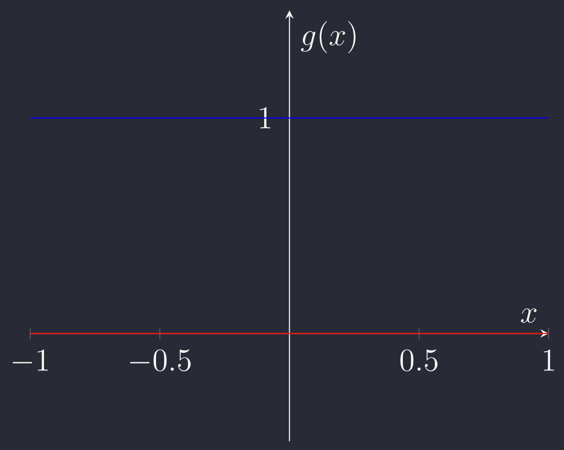
Functional Limits and Continuity

4.1 Discussion: Examples of Dirichlet and Thomae

Definition 4.1.1

The *Dirichlet function* $\lim_{x \to c} g(x)$ does not exist for any $c \in \mathbb{R}$.

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$



Definition 4.1.2

The *Thomae function* is defined as

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is in lowest terms}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Thomae's function, $t(x)$ is continuous at all $x \notin \mathbb{Q}$. It is not continuous at any $x \in \mathbb{Q}$.

4.2 Functional Limits

Recall from calculus I, that a function $f(x)$ is continuous at $x = c$ if $\lim_{x\to c} f(x) = f(c)$.

Definition 4.2.1

Let $f\colon A \to \mathbb{R}$ be a function and let c be a limit point of A. We say $\lim_{x\to c} f(x) = L$, if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Example 4.1: Functional Limit (From book) 1

Let $f(x) = 3x + 1$. Claim: $\lim_{x\to 2} f(x) = 7$.

Proof. Let $\epsilon > 0$. After we have done our scratch work, we can choose $\delta = \epsilon/3$, then $0 < |x - 2| < \delta$ implies $|f(x) - 7| < 3(\epsilon/3) = 3$. □

Scratch Paper. Definition 4.2.1 requires that we produce a $\delta > 0$ so that $0 < |x - 2| < \delta$ leads to the conclusion that $|f(x) - 7| < \epsilon$. Notice that

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3\,|x - 2|\,.$$

Example 4.2: Functional Limit (From book) 2

Let $g(x) = x^2$. Claim: $\lim_{x\to 2} g(x) = 4$.

Proof. Let $\epsilon > 0$. Choose $\delta = \min\{1, \epsilon/5\}$. If $0 < |x - 2| < \delta$, then

$$\begin{aligned} |g(x) - 4| &= \left|x^2 - 4\right| \\ &= |x - 2|\,|x + 2| \\ &< 5\delta \\ &= (5)\frac{\epsilon}{5} \\ &= \epsilon. \end{aligned}$$

□

Scratch Paper. Our goal this time is to make $|g(x) - 4| < \epsilon$ by restricting $|x - 2|$ to be smaller than some carefully chosen δ. As in the previous example, a little algebra reveals

$$|g(x) - 4| = \left|x^2 - 4\right| = |x - 2|\,|x + 2|\,.$$

We can make $|x + 2|$ as small as we like, but we need an upper bound on $|x + 2|$ in order to know how small to choose δ. The presence of the variable x causes some initial confusion, but keep in mind that we are discussing the limt as x approaches 2. If we agree that our $\delta-$neighborhood around $c = 2$ must have radius no bigger than $\delta = 1$, then we get the upper bound $|x + 2| < |3 + 2| = 5$ for all $x \in V_\delta(c)$.

Example 4.3: Functional Limit 1

Let $f(x) = 3x + 1$. Show that $\lim_{x\to 2} f(x) = 7$.

Proof. Let $\epsilon > 0$. Set $\delta = \frac{\epsilon}{3}$. Assume $0 < |x - 2| < \delta$. Since $\delta > 0$, $2 - \delta < x < 2 + \delta$. Then,

$$\begin{aligned} |x-2| &< \delta, \\ |f(x) - 7| &= |3x + 1 - 7| \\ &= |3x - 6| \\ &= 3\,|x-2| \\ &< 3\delta \\ &= \epsilon. \end{aligned}$$

Therefore, $\lim_{x\to 2} f(x) = 7$. □

Example 4.4: Functional Limit 3

Let $f(x) = x^2$. Claim: $\lim_{x\to 7} f(x) = 49$

Proof. Let $\epsilon > 0$. Set $\delta = \min\{\frac{\epsilon}{8}, 1\}$. If $0 < |x - 7| < \delta$, then

$$\begin{aligned} |f(x) - 49| &= \left|x^2 - 49\right| \\ &= |x-7|\,|x+7| \\ &< 8\delta \\ &= 8\left(\frac{\epsilon}{8}\right) \\ &= \epsilon. \end{aligned}$$

□

Scratch Paper. Always start with the goal statement: $|f(x) - 49| = |x^2 - 49|$. This factors into $|x - 7|\,|x + 7|$. Then, if $\delta < 1$, $|x - 7| < \delta$ and $|x + 7| < 8$. All together, we have $8\delta < \epsilon < \frac{\epsilon}{8}$.

Example 4.5: Functional Limit 4

Claim: $\lim_{x\to 3} \frac{1}{x+1} = \frac{1}{4}$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{12\epsilon, 1\}$. If $0 < |x-3| < \delta$, then

$$\begin{aligned}\left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{4-(x+1)}{4(x+1)}\right| \\ &= \left|\frac{3-x}{4(x+1)}\right| \\ &< \frac{\delta}{4(3)} \\ &= \frac{12\epsilon}{12} \\ &= \epsilon.\end{aligned}$$

Therefore, $\lim_{x\to 3} \frac{1}{x+1} = \frac{1}{4}$ □

Scratch Paper. Goal: $\left|\frac{1}{x+1} - \frac{1}{4}\right|$. Hence,

$$\begin{aligned}\left|\frac{1}{x+1} - \frac{1}{4}\right| &= \left|\frac{4-(x+1)}{4(x+1)}\right| \\ &= \left|\frac{3-x}{4(x+1)}\right| \\ &< \frac{\delta}{4\,|x+1|} \\ &< \frac{\delta}{4(3)} \\ &= \frac{\delta}{12} \\ &< \epsilon.\end{aligned}$$

Thus, we need a $\delta < 1$, and we can choose $\delta = \min\{12\epsilon, 1\}$. Note: When we are determining the value for $|x+2|$, we solve for $\delta = 3 \pm 1 \Rightarrow x \in (2,4)$. Then, we find $x + 1 = (3,5)$. We choose 3 rather than 5 because of division. We want to be as close as possible.

Example 4.6: Functional Limit 5

Claim: $\lim_{x\to 3}(x^2 + 7x) = 30$.

Proof. Let $\epsilon > 0$ and set $\delta = \min\{\frac{\epsilon}{14}, 1\}$. If $0 < |x-3| < \delta$, then

$$\begin{aligned}\left|x^2 + 7x - 30\right| &= |x-3|\,|x+10| \\ &< 14\delta \\ &= 14\left(\frac{\epsilon}{14}\right) \\ &= \epsilon.\end{aligned}$$

□

Example 4.7: Functional Limit 6

Claim: $\lim_{x\to 3} \frac{2x+3}{4x-9} = 3$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{\frac{\epsilon}{10}, \frac{1}{2}\}$. (Note: We are choosing $\frac{1}{2}$ because we want to avoid having 0 anywhere in the interval.) Assume $0 < |x-3| < \delta$. Since $\delta < \frac{1}{2}$, $\frac{5}{2} < x < \frac{7}{2}$, then $1 < |4x-9| < 5$. (Thus, 0 can not possibly be in the denominator.) $\square$

Scratch Paper.

$$\begin{aligned}\left|\frac{2x+3}{4x+9} - 3\right| &= \left|\frac{2x+3-3(4x+9)}{4x+9}\right| \\ &= \left|\frac{2x+3-12x-27}{4x+9}\right| \\ &= 10\left|\frac{x-3}{4x-4}\right| \\ &< 10\frac{\epsilon/10}{1} \\ &= \epsilon.\end{aligned}$$

Example 4.8: Functional Limit 7

Claim: $\lim_{x\to 4} \sqrt{x} = 2$.

Proof. Let $\epsilon > 0$. Set $\delta = \min\{1, 3\epsilon\}$. Assume $0 < |x-4| < \delta$. Then (refer to scratch work). $\square$

Scratch Paper.

$$\begin{aligned}\left|\sqrt{x} - 2\right| &= \left|\sqrt{x} - 2\right| \\ &= \left|\frac{(\sqrt{x}-2)\cdot(\sqrt{x}+2)}{\sqrt{x}+2}\right| \\ &= \left|\frac{x-4}{\sqrt{x}+2}\right| \\ &< \frac{\delta}{3} \\ &< \frac{3\epsilon}{3} \\ &= \epsilon.\end{aligned}$$

Notice that we picked $\delta < 1$ such that $3 < x < 4$ so $1 < \sqrt{x} < 2$ and $3 < \sqrt{x} + 2 < 4$.