Multivariable Calculus Exam 1

Derivation

Basic Derivatives

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$
$$\frac{d}{dx}\sin f(x) = \cos f(x)\cdot f'(x)$$
$$\frac{d}{dx}\cos f(x) = -\sin f(x)\cdot f'(x)$$
$$\frac{d}{dx}\tan f(x) = \sec^2 f(x)\cdot f'(x)$$
$$\frac{d}{dx}\cot f(x) = -\csc^2 f(x)\cdot f'(x)$$
$$\frac{d}{dx}\sec f(x) = \sec f(x)\tan f(x)\cdot f'(x)$$
$$\frac{d}{dx}\csc f(x) = -\csc f(x)\cot f(x)\cdot f'(x)$$
$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$$
$$\frac{d}{dx}\log_a f(x) = \frac{f'(x)}{f(x)\ln a}$$
$$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1}f'(x)$$
$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$$
$$\frac{d}{dx}a^x = a^x\ln a$$
$$\frac{d}{dx}b^{g(x)} = b^{g(x)}\ln b\cdot g'(x)$$

Inverse Trigonometric

$$\frac{d}{dx}\arcsin f(x) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$$
$$\frac{d}{dx}\arccos f(x) = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$$
$$\frac{d}{dx}\arctan f(x) = \frac{f'(x)}{1+(f(x))^2}$$
$$\frac{d}{dx}\operatorname{arccot} f(x) = -\frac{f'(x)}{1+(f(x))^2}$$
$$\frac{d}{dx}\operatorname{arcsec} f(x) = \frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$$
$$\frac{d}{dx}\operatorname{arccsc} f(x) = -\frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$$

Chain Rule

$$\frac{d}{dx}f(g(x)) = f'(g(x))\cdot g'(x)$$

Higher-Order Derivatives

$$\frac{d^2}{dx^2}e^x = e^x$$
$$\frac{d^3}{dx^3}\sin x = -\cos x$$
$$\frac{d^4}{dx^4}\cos x = \cos x$$

Product and Quotient

$$\frac{d}{dx}[u\cdot v] = u'\cdot v + u\cdot v'$$
$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'\cdot v - u\cdot v'}{v^2}$$

Integration

Trigonometric Integrals

$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sin^2 x = \frac{1}{2}(x-\sin x\cos x)+C$$
$$\int \cos^2 x = \frac{1}{2}(x+\sin x\cos x)+C$$
$$\int \tan x\,dx = -\ln|\cos x| + C$$
$$\int \cot x\,dx = \ln|\sin x| + C$$
$$\int \sec x\,dx = \ln|\sec x+\tan x| + C$$
$$\int \csc x\,dx = -\ln|\csc x+\cot x| + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \sec x\tan x\,dx = \sec x + C$$
$$\int \csc x\cot x\,dx = -\csc x + C$$

Inverse Trigonometric Integrals

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C$$
$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right)+C$$
$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right)+C$$

Regular Integrals and e

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}+C$$
$$\int \frac{1}{x}\,dx = \ln|x|+C$$
$$\int e^x\,dx = e^x + C$$
$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}+C$$
$$\int e^{f(x)}f'(x)\,dx = e^{f(x)}+C$$

Exponetials

$$\int_0^b e^x\,dx = e^b - 1$$
$$\int_0^b e^{-x}\,dx = 1-e^{-b}$$
$$\int_0^\infty e^{-x}\,dx = 1$$
$$\int_0^\infty x^n e^{-x}\,dx = n!$$

Reduction Formulas for Sine and Cosine

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

n	$\int_0^{\pi/2}\sin^n x\,dx$	$\int_0^{\pi/2}\cos^n x\,dx$	$\int_0^{\pi}\sin^n x\,dx$	$\int_0^{\pi}\cos^n x\,dx$	$\int_0^{2\pi}\sin^n x\,dx$	$\int_0^{2\pi}\cos^n x\,dx$
1	1	1	2	0	0	0
2	$\pi/4$	$\pi/4$	$\pi/2$	$\pi/2$	π	π
3	$2/3$	$2/3$	$4/3$	0	0	0
4	$3\pi/16$	$3\pi/16$	$3\pi/8$	$3\pi/8$	$3\pi/4$	$3\pi/4$
5	$8/15$	$8/15$	$16/15$	0	0	0
6	$5\pi/32$	$5\pi/32$	$5\pi/16$	$5\pi/16$	$5\pi/8$	$5\pi/8$

Radians	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
0	0	1	0
$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
$\pi/2$	1	0	–
π	0	-1	0
$3\pi/2$	-1	0	–

Trigonometric Identities

Pythagorean

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

Half Angle

$$\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos x}{2}$$
$$\cos^2\left(\frac{x}{2}\right) = \frac{1+\cos x}{2}$$
$$\tan^2\left(\frac{x}{2}\right) = \frac{1-\cos x}{1+\cos x}$$

Double Angle

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1-\tan^2 x}$$

Product to Sum

$$\sin x\sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos x\cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\sin x\cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$
$$\cos x\sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$$

Sum to Product

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Chapter 1: Parametric Equations and Polar Coordinates

- **Slope:** $\left.\dfrac{dy}{dx}\right|_{t=t_0} = \left.\dfrac{dy/dt}{dx/dt}\right|_{t=t_0}$.

  The ***tangent line*** at t_0 is given by

  $$y = \left(\left.\frac{dy}{dx}\right|_{t=t_0}\right)(x - x(t_0)) + y(t_0).$$

- **Concavity:** $\left.\dfrac{d^2y}{dx^2}\right|_{t=t_0} = \left.\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\right|_{t=t_0} = \left.\dfrac{d}{dt}\left(\dfrac{dy/dt}{dx/dt}\right)\right|_{t=t_0}$.

- **Area Under a Curve:** $\displaystyle\int_{t_a}^{t_b} y(t)\frac{dx}{dt}\,dt$.

- **Arc Length:** $\displaystyle\int_{t_a}^{t_b}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

- **Surface Area:** $\displaystyle\int_{t_a}^{t_b} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

Chapter 2: Vectors in Space

- ***Direction:*** $P = (x_1, y_1)$ and $Q = (x_2, y_2)$: $\mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$.
- ***Vector Sum:*** $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle$.
- ***Magnitude:*** $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2} = \sqrt{u}\cdot u$.
- ***Dot Product:*** $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$.
  - ***Angle:*** $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos(\theta)$, where $0 \le \theta \le \pi$ is between $\mathbf{u}$ & $\mathbf{v}$.
  - ***Self-Product:*** $\mathbf{u}\cdot\mathbf{u} = \|\mathbf{u}\|^2$.
  - ***Work:*** $W = \mathbf{F}\cdot\mathbf{PQ} = (\|\mathbf{F}\|)\,\|\mathbf{PQ}\|\cos(\theta)$.
- To ***Normalize*** a vector, divide it by its magnitude $\mathbf{v} = \langle x, y, z\rangle$, then $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left\langle \frac{x}{\|\mathbf{v}\|}, \frac{y}{\|\mathbf{v}\|}, \frac{z}{\|\mathbf{v}\|}\right\rangle$. $\therefore \mathbf{u} :=$ ***Unit Vector*** in direction of $\mathbf{v}$.
- ***Projection:*** $\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}$.
- ***Cross product:*** $\mathbf{u}\times\mathbf{v} = \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1\rangle$.
  - ***Angle:*** $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\|\sin(\theta)$, where $0 \le \theta \le \pi$ is between $\mathbf{u}$ & $\mathbf{v}$.
  - ***Torque:*** $\tau = \mathbf{r}\times\mathbf{F}$ or $\|\tau\| = \|\mathbf{r}\|\,\|\mathbf{F}\|\sin(\theta)$

Parametric Equations Revisted

- ***Vector Equation:*** $\mathbf{r}(t) = \mathbf{r_0} + \mathbf{tV}$.
- ***Parametric Equation:*** $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$.
- ***Symmetric Equation:*** $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.
- The ***Line Segment*** from P to Q: $\mathbf{r}(t) = (1-t)\mathbf{p} + t\mathbf{q}$ (where $\mathbf{p}, \mathbf{q}$ are the vector forms of P, Q and $0 \le t \le 1$).
- ***Direction Vector:*** $d = \left|\frac{\mathbf{PM}\times\mathbf{v}}{\|\mathbf{v}\|}\right|$.
  - ***Equal***: Same direction vector, share a point.
  - ***Parallel***: Same direction vector, do not share a point.
  - ***Intersecting***: Different direction vectors, share a point.
  - ***Skew***: Different direction vectors, do not share a point.
- If (x_0, y_0, z_0) is a point on a plane, the ***Scalar Equation*** would be: $\langle x - x_0, y - y_0, z - z_0\rangle\cdot\langle a, b, c\rangle = 0 \Longrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Chapter 3: Vector-Valued Functions

If each of $f_1, f_2, \dots, f_n : \mathbb{R} \to \mathbb{R}$ is a function we can then define the ***vector-valued function*** $\mathbf{r}: \mathbb{R} \to \mathbb{R}^n$ by $\mathbf{r}(t) = \langle f_1(t), f_2(t), \dots, f_n(t) \rangle$

- When $n = 2$, we might write $\mathbf{r} = \langle f(t), g(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath}$,
- and when $n = 3$, we might write $\mathbf{r} = \langle f(t), g(t), h(t) \rangle = f(t)\hat{\imath} + g(t)\hat{\jmath} + h(t)\hat{k}$.

Note: Deriving and integrating vector-valued functions follow the same rules as regular derivates.

- ***Principle unit tangent vector*** $\mathbf{T}(t)$: $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{||\mathbf{r}'(t)||}$.
  - This vector, of length 1, points in the tangent direction of the curve.
- ***Unit Normal Vector*** $\mathbf{N}$: $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{||\mathbf{T}'(t)||}$.
  - This vector points in the direction the curve is turning.
- ***Binormal Vector*** $\mathbf{B}$: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

Arc Length Parameterization

We can define the ***arc length parameterization*** of a curve C by:

- Define the arc length $s(t) = \int_0^t ||\mathbf{r}'(\tau)||\, d\tau = \int_0^t \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2}\, d\tau$. (Where $f(\tau), g(\tau)$ correspond to the x, y components of $\mathbf{r}(t)$).
- Solving, if possible, the resulting expression for t as a function of s.
- Rewriting $\mathbf{r}(t) = \mathbf{r}(t(s)) = \mathbf{r}$, so that the curve is written as a function of its length, from a given starting point.

Curvature (& Oscillating Circle)

Steps for Finding Oscillating Circle Equation

1. Get values for your curvature formulas.
2. Find the radius ($R = \frac{1}{\kappa}$).
3. Find the center by first finding the normal vector at $x = \dots$.

For all $\mathbf{r}$: $\kappa = \frac{||\mathbf{T}'(t)||}{||\mathbf{r}'(t)||}$; for $\mathbb{R}^3$: $\kappa = \frac{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}{||\mathbf{r}'(t)||^3}$; if $y = f(x)$: $\kappa = \frac{|y''(x)|}{[1+(y'(x)^2)]^{3/2}}$

2.6 Graph Theory

Definitions:

Graphs

Graph: $G = (V, E)$, is a pair of sets V, the *vertex set*, and E the *edge set*, so that each element of E has the form $\{v_i, v_j\}, v_1, v_j \in V$.
Degree: The number of edges which include v. Granted that $v \in V$.
Adjacent: Vertices u, v belong to $\{u, v\} \in E$.
Path: From vertex v_0 to vertex v_n is a sequence $v_0, v_1, v_2 \dots, v_n$, where each $v_i \in V$ and $\{v_i, v_{i+1}\} \in E$.
Simple: No edge occurs twice in a path.
Connected: If each pair of vertices are adjoined by an edge.

Circuits

Circuit: A path with the same starting and ending vertex.
Complete: On n vertices, K_n, is the connected graph where each vertex is adjacent to each other.

Bipartite Graphs

Bipartite: The vertex set $V = v_1 \cup v_2, v_1 \cap v_2 = \emptyset$, and no vertex in V_1 is adjacent to any other in V_1, and no vertex in V_2 is adjacent to any other in V_2.

Trees

Tree: A connected graph that has no circuit.
BiSTree: For every node, all elements in the left subtree are less than the node's value, and all elements in the right subtree are greater.
Lemma: If G is a tree, G has at least one vertex of degree 1.

Proof. For the sake of contradiction, suppose each vertex has degree ≥ 2. Pick a vertex, v_0. Since, $\deg(v_0) \geq 2$, it is adjacent to some v_j. Because $\deg(v_1) \geq 2$, it has an edge distinct from $\{v_0, v_1\}$, follow it to v_2. Then, v_2 has edge distinct from $\{v_1, v_2\}$, follow it to $v - 3$. If $v_3 = v_0(v_1, v_2)$ we have a circuit. v_3 has edge distinct from $\{v_2, v_3\}$. Go to v_4. Continue .... Either some v_j is visited again, or $v_0, v_1, v_2, \dots v_n$. Therefore, it must be a circuit.

Hence, G must have a vertex with degree of at least 1 such that $1 \leq$. ☐

Theorem 1: A tree with n vertices always has $n - 1$ edges.

Proof. By the Lemma, there exists a vertex of degree 1. Remove it and its edge. We still have a tree. This new tree has vertex of degree 1. Remove it and its edge. Continue until you get to k_2, then k_1. We stop with 1 vertex, 0 edges, we have removed $n - 1$ vertices. Each edge was removed. Thus, we threw out *all* $n - 1$ edges. ☐

2.6 Graph Theory (cont.)

Euler Circuits

Euler circuit:	A circuit graph which uses each edge only once.
Euler path:	A path which uses each edge – start and end vertices are distinct.

Note: G has an Euler circuit if, and only if, it is connected and each vertex has an even degree. Intuitively, if each vertex has an even degree, then if you come into the vertex through the entrance (first edge), and you leave through the exit (second edge) you have used up both openings.

G has an Euler path if, and only if, it is connected and has exactly two vertices of odd degree.

Isomorphisms

Two graphs, $G = (V, E)$ and $H = (W, f)$ are *isomorphic* if there is an $f \colon V \to W$ which is one-to-one, onto, and $\{v_i, v_j\} \in E \iff \{f(v_i), f(v_j)\} \in F$.

Vertex Colorings

If G contains a triangle (i.e., if it has a copy of k_3, we need at least 3 colors. If G contains a copy of K_n, we need at least n.

If a graph has no overlapping paths, the graph requires no more than 4 colors.

Hamilton Graphs

A graph has a *Hamilton Circuit* if there is a circuit that uses each vertex once.

Note: This is different from Euler, as Euler uses edges. This is specifically for vertices. If it has a vertex of degree 1, it cannot have a Hamilton circuit.

Estimating Big Θ

Sets

1. Suppose that the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the particular sets $A = \{2, 3, 5\}$ and $B = \{1, 3, 5, 7, 9\}$. Find each of the following:

   (a) $A \cup B = \{1, 2, 3, 5, 7, 9\}$

   (b) $A \cap B = \{3, 5\}$

   (c) $\{x \in U \colon x^2 < 10 \wedge x \in A\} = \{2, 3\}$

   (d) $B' \cap A = \{2\}$

   (e) $\{x \in A \colon x \geq 7\} = \emptyset$

   (f) $\mathcal{P}(A) = \{\emptyset, \{2\}, \{3\}, \{5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{2, 3, 5\}\}$