Multivariable Calculus Practice Set I

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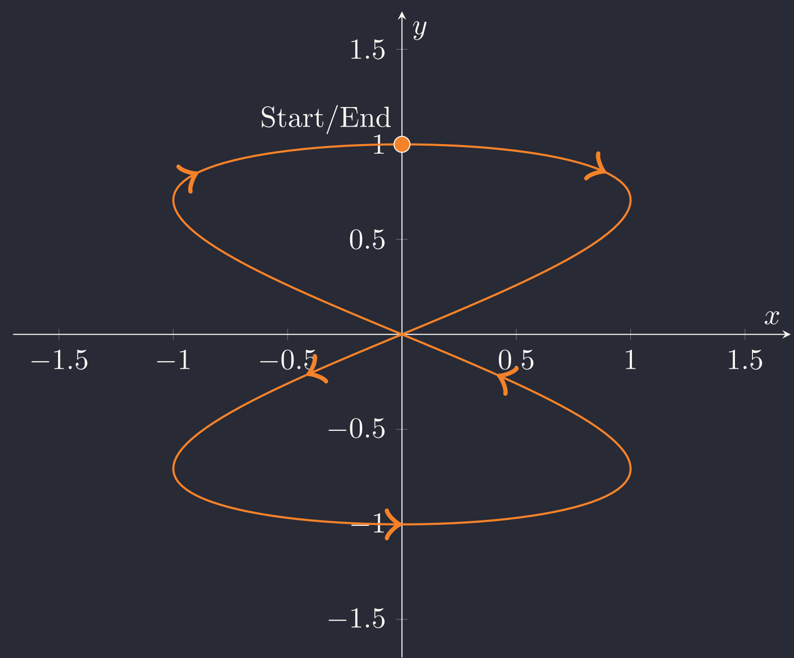
1. Consider the curve defined by the parametric equations $x(t) = \sin(2t)$, $y(t) = \cos(t)$, for $0 \le t \le 2\pi$.

   (a) (1 point) Use GeoGebra to graph this curve. Then, hand-draw what you see here, including a well-labeled start and stop and arrows to indicate the trajectory.



Solution.

   (b) (2 points) Determine the exact value of the equation of the tangent line when $t = \pi/3$.

Solution. To get the equation of the tangent line, we need to solve for $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Solving for $\frac{dx}{dt}$:

$$\begin{aligned} y(t) &= \cos t \\ \frac{dy}{dt} &= -\sin t. \end{aligned}$$

Then, for $\frac{dy}{dt}$:

$$\begin{aligned} x(t) &= \sin 2t \\ \frac{dx}{dt} &= 2\cos 2t. \end{aligned}$$

Now, we have our slope:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{2\cos 2t}.$$

Evaluated at $t = \pi/3$:

$$\frac{-\sin t}{2\cos 2t} = \frac{-\sqrt{3}/2}{-1} = \frac{\sqrt{3}}{2}.$$

Now that we have our slope, we need our points:

$$x(\pi/3) = \sin(2(\pi/3)) = \sin(\pi/3) = \sqrt{3}/2 \qquad \text{and} \qquad y(\pi/3) = \cos(\pi/3) = 1/2.$$

Putting it all together, we have:

$$\begin{aligned} y &= \frac{\sqrt{3}}{2}(x - \sqrt{3}/2) + 1/2 \\ y &= \frac{\sqrt{3}}{2}x - \frac{3}{4} + \frac{1}{2} \\ y &= \frac{\sqrt{3}}{2}x - \frac{1}{4}. \end{aligned}$$

(c) (2 points) Determine the exact value of the geometric area of the region enclosed by curve define above. (Please notice that this area is certainly positive. You might need to think carefully about how to use symmetry to answer this question.) You may find the trigonometric identity $\cos(2\theta) = 1 - 2\sin^2(\theta)$ useful. You must work out any integrals completely "by-hand," showing steps to receive credit for this problem.

Solution. Our set of parametric equations maps out an hourglass shape that has 4 quadrants: 2 positive and 2 negative. Therefore, if we can map one quadrant and multiply it by 4, we can get the total geometric area for the whole shape. Thus, we can restrict our interval such that $t \in [0, \pi/2]$.

$$\begin{aligned} 4\int_0^{\pi/2} y(t)\frac{dx}{dt}\ dt &= 4\int_0^{\pi/2} \cos t(2\cos 2t)\ dt \\ &= 8\int_0^{\pi/2} \cos t(1 - 2\sin^2 t)\ dt \\ &= 8\int_0^{\pi/2} \cos t - 2\cos t\sin^2 t\ dt \\ &= 8\left[\int_0^{\pi/2} \cos t\ dt - 2\int_0^{\pi/2} \cos t \sin^2 t\ dt\right]. \end{aligned}$$

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Now, we can employ u-substitution to further simplify the intergrand. Thus, let $u = \sin t$ such that $\dfrac{du}{\cos t} = dt$. Hence,

$$\begin{aligned} 8\left[\int_0^{\pi/2} \cos t\ dt - 2\int_0^{\pi/2} \cos t \sin^2 t\ dt\right] &= 8\left[\int_0^{\pi/2} \cos t\ dt - 2\int_0^{\pi/2} u^2\ du\right] \\ &= 8\left[\sin t - \frac{2}{3}\sin^3 t\right]_0^{\pi/2} \\ &= 8\left[1 - \frac{2}{3}\right] \\ &= \frac{8}{3}. \end{aligned}$$

(d) (2 points) Set up – but do not evaluate! – the integral necessary to determine the arc length for this curve. Then, use your calculator to approximate this integral to 3 decimal places.

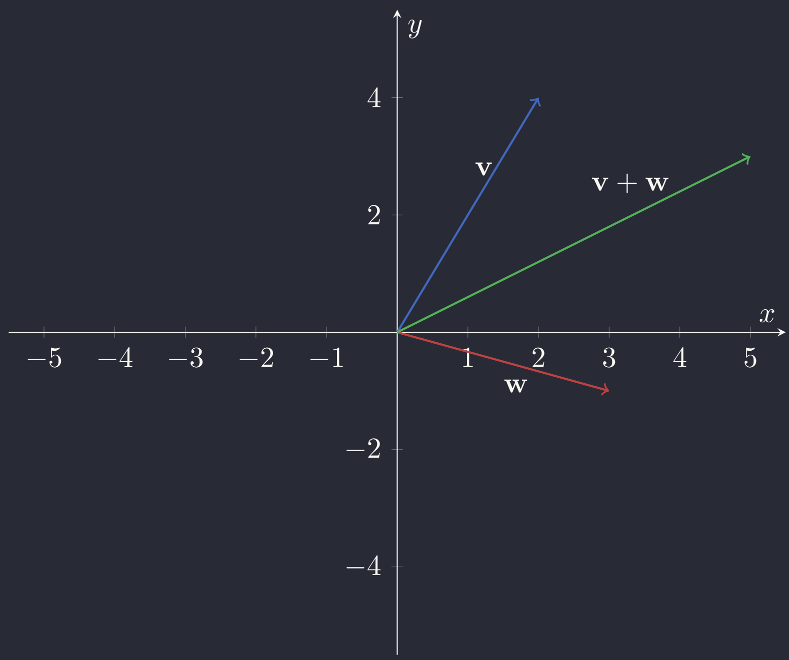
Solution. The integral necessary to determine the arc length would be:

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt = \int_0^{2\pi} \sqrt{(2\cos 2t)^2 + (-\sin t)^2}\ dt = \boxed{\pi}$$

If we approximate π to 3 decimal places, we get 3.142.

2. (1 point each) Let $\mathbf{v} = 3\mathbf{i} - \mathbf{j}$ and $\mathbf{w} = 2\mathbf{i} + 4\mathbf{j}$.

(a) On the axes provided, draw in both $\mathbf{v}$ and $\mathbf{w}$.



Solution.

(b) Find the value of $\mathbf{v}+\mathbf{w}$ and draw it on the above axes as well.

Solution.

$$\mathbf{v}+\mathbf{w} = \langle 3+2, -1+4 \rangle = \langle 5, 3 \rangle .$$

The graph of $\mathbf{v}+\mathbf{w}$ is pictured above.

(c) Find $4\mathbf{v}-2\mathbf{w}$. (It is not necessary to draw this one.)

Solution.

$$4\mathbf{v}-2\mathbf{w} = \langle 12-4, -4-8 \rangle = \langle 8, -12 \rangle .$$

(d) Find the exact value of $\|\mathbf{v}\|$.

Solution.

$$\|\mathbf{v}\| = \sqrt{3^2+(-1)^2} = \sqrt{9+1} = \sqrt{10}.$$

(e) Find a unit vector which points in the same direction as $\mathbf{v}$.

Solution.

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 3, -1 \rangle}{\sqrt{10}} = \left\langle \frac{3}{\sqrt{10}}, \frac{-1}{\sqrt{10}} \right\rangle .$$

(f) Suppose that we find scalars c and d such that $c\mathbf{v}+d\mathbf{w}=\mathbf{0}$. Show that $c=0$ and $d=0$.

Solution.

$$c\mathbf{v}+d\mathbf{w} = \langle 3c+2d, -c+4d \rangle = \langle 0, 0 \rangle .$$

This implies that $3c+2d=0$ and $-c+4d=0$. Solving the system of equations,

$$\begin{bmatrix} 3 & 2 & 0 \\ -1 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

we find that $c=0$ and $d=0$.

Without using Linear Algebra:

From the system of equations, we can solve for c in terms of d:

$$3c+2d=0 \implies 3c=-2d \implies c=-\frac{2}{3}d.$$

Substituting this into the second equation:

$$-\frac{2}{3}d-d=0 \implies -\frac{5}{3}d=0 \implies d=0.$$

Substituting this back into the first equation:

$$3c+2(0)=0 \implies 3c=0 \implies c=0.$$

3. (1 point each) Suppose that $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + \mathbf{j} - 7\mathbf{k}$.

   (a) Determine $\mathbf{u} \cdot \mathbf{v}$.

   Solution.
$$\mathbf{u} \cdot \mathbf{v} = (6)(-4) + (2)(1) + (-5)(-7) = -24 + 2 + 35 = 13.$$

   (b) Determine $\mathbf{u} \times \mathbf{v}$.

   Solution.
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 2 & -5 \\ -4 & 1 & -7 \end{vmatrix} = \langle [2(-7) - (-5)(1)],\ -[6(-7) - (-5)(-4)],\ [6(1) - 2(-4)] \rangle = \langle -9, 62, 14 \rangle .$$

   (c) What is the angle between $\mathbf{u}$ and $\mathbf{v}$? [An answer, in radians rounded to 3 decimal places, is appropriate here.]

   Solution. First, we need to find the magnitudes of $\mathbf{u}$ and $\mathbf{v}$:
$$\|\mathbf{u}\| = \sqrt{6^2 + 2^2 + (-5)^2} = \sqrt{65} \qquad \text{and} \qquad \|\mathbf{v}\| = \sqrt{(-4)^2 + 1^2 + (-7)^2} = \sqrt{66}.$$
   With the magnitudes, we can find the angle between the two vectors:
$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \, \|\mathbf{v}\|} = \frac{13}{\sqrt{65}\sqrt{66}} \approx 0.198.$$
   Thus, $\theta \approx \cos^{-1}(0.198) \approx 1.371$ radians.

   (d) Find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

   Solution.
$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v} = \frac{13}{66} \langle -4, 1, -7 \rangle = \left\langle -\frac{26}{33}, \frac{13}{66}, -\frac{91}{66} \right\rangle .$$

4. (2 points) Determine a parametric equation for the line *segment* that goes from the point $P = (6, 1, -2)$ to $Q = (-2, 0, 5)$.

   Solution. The parametric equation for the line segment is given by:
$$f(t) = \begin{cases} x(t) = 6 - 8t \\ y(t) = 1 - t \\ z(t) = -2 + 7t \end{cases}$$

5. (2 points) Find a symmetric equation for the line which contains the points $R = (4, -6, 1)$ and $S = (1, 2, 3)$.

   Solution. The symmetric equation for the line is given by:
$$\frac{x - 4}{-3} = \frac{y + 6}{8} = \frac{z - 1}{-2}.$$

6. (2 points) Find the general form of an equation of the plane which contain the three points $P = (3, 1, -4)$, $Q = (-2, 0, 5)$ and $R = (4, -6, 1)$.

Solution. To find the general form of the equation of the plane, we can use the cross product of two vectors that lie in the plane. Let $\mathbf{PQ} = \langle -2-3, 0-1, 5+4\rangle = \langle -5, -1, 9\rangle$ and $\mathbf{QR} = \langle 4+2, -6-0, 1-5\rangle = \langle 6, -6, -4\rangle$. Then, the normal vector to the plane is given by

$$\mathbf{PQ}\times\mathbf{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -1 & 9 \\ 6 & -6 & 4 \end{vmatrix} = \langle [(-1)(4) - 9(-6)],\ -[(-5)(4) - 9(6)],\ [(-5)(-6) - (-1)(6)]\rangle = \langle 50, 74, 24\rangle\,.$$

Thus, $\mathbf{n} = \langle 50, 74, 24\rangle$ and we can choose the following points to make the equation of the plane:

$$50(x-3) + 74(y-1) + 24(z+4) = 0.$$

7. (2 points) Find an equation, in symmetric form, of the line of intersection between the planes $2x+y-z+4 = 0$ and $x - y + 3z = 1$.

Solution. To find an equation of the line of intersection, we need to add the plane equations to eliminate y:

$$\begin{array}{ccccccc} 2x & + & y & - & z & = & -4 \\ x & - & y & + & 3z & = & 1 \\ \hline 3x & & & + & 2z & = & -3 \end{array}$$

Thus, $x = -1 - \frac{2}{3}z$. Substitute this equation into the first equation to express y in terms of z:

$$\begin{aligned} 2x + y - z &= -4 \\ 2(-1 - \frac{2}{3}z) + y - z &= -4 \\ -\frac{7}{3}z + y &= -2 \\ y &= -2 + \frac{7}{3}z. \end{aligned}$$

With x and y in terms of z, we need to define z in terms of t. Choose parameter t as $t = -\frac{1}{3}z$. This gives $z = -3t$. When we substitute our value t back into the previous two equations, we see that the parametric equations for the line of intersection are $x = -1 + 2t$, $y = -2 - 7t$, and $z = -3t$. Therefore, the symmetric equations for the line are:

$$\frac{x+1}{2} = \frac{y+2}{-7} = \frac{z}{-3}.$$