HENDRIX
COLLEGE

Multivariable Calculus Notes

MATH 230

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CHAPTER 1

PARAMETRIC EQS AND POLAR COORDS

1.1 Parametric Equations

1.1.1 Introduction

Most of your calculus experience has been single variable, so that the functions under consideration were typically $f : \mathbb{R} \to \mathbb{R}$. Our course is divided into roughly 3 sections:

- Parametric Equations/Functions: Functions of the form $f : \mathbb{R} \to \mathbb{R}^n$ (Chapters 1 - 3)
- Scalar Functions: Functions of the form $f : \mathbb{R}^n \to \mathbb{R}$ (Chapters 4 - 5)
- Vector Fields: Functions of the form $f : \mathbb{R}^n \to \mathbb{R}^n$ (Chapter 6)

1.1.2 Parametric Equations

A *parametric equation* (or, *sometimes parametric function* or *vector-valued function*) is a function of the form $f : \mathbb{R} \to \mathbb{R}^n$. We will typically consider $n = 2$ or $n = 3$ and call the input variable the parameter, usually denoted by t. We write them as

$$f(t) = \begin{cases} x(t) \\ y(t) \end{cases} \qquad \text{or} \qquad f(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}.$$

A *parametric curve* is the set of points $(x(t), y(t))$ in $\mathbb{R}^2$ or $(x(t), y(t), z(t))$ in $\mathbb{R}^3$ traced out. Note that in general, the curve may not be a function for y in terms of x, but is a function of the parameter t.

1.1.3 Graphing Parametric Curves in the Second Dimension

Elimination of the Parameter

In some cases, we can explicitly solve for t in terms of one of x or y. When this is possible, you can write $y(x)$ or $x(y)$ and use your "regular" algebraic knowledge. We call this process *eliminating the parameter*.

Using Technology

- Your TI-84 can graph this if you switch to `par` mode.
- Likewise, GeoGebra can do this, using the `curve` function.
  - In general, the syntax is: `curve(x(t), y(t), t, min, max)`

1.1.4 The Cycloid

A wheel of radius a is rolling along a flat road at a constant velocity. The curve generated by a point along the edge of the wheel traces out a shape called a *cycloid*. Let t represent the angle - in radians!!!! - rotated through, and that the point of interest starts at the origin. Before we find the equations for the point, let's find the location of the center of the circle:

$$f_{\text{center}}(t) = \begin{cases} x(t) = at \\ y(t) = a \end{cases}$$

Then, relative to the center, our point along the edge has equations

$$f(t) = \begin{cases} x(t) = -a\sin t \\ y(t) = -a\cos t \end{cases}$$

Thus, our point has parametric equations

$$f(t) = \begin{cases} x(t) = a(t - \sin t) \\ y(t) = a(1 - \cos t) \end{cases}$$

1.1.5 Final Notes

Next time, we'll start asking Calculus-y questions: What are the velocities in the x, y, and total directions? What total distance does it travel? What is the area of the region under one period of the cycloid?

- The syllabus has a number of practice problems to work on. These are not required, and not to be turned in, but are for you to work before class next time.
- We will talk about them at the start of the next class. You should try them beforehand.
- The most common reason for a lack of success in this class is not spending time working problems on your own.

1.2 Calculus of Parametric Curves

For this section, we will have a parametric curve in R2, defined by $f(t) = \begin{cases} x(t) \\ y(t) \end{cases}$. In many cases, the curve does not describe y as a function of x. However, we can still carry over many ideas from single variable calculus.

1.2.1 Slope for a Parametric Curve

Given a point t_0, the *slope of the curve* in the xy-plane is given by

$$\left.\frac{dy}{dx}\right|_{t=t_0} = \left.\frac{dy/dt}{dx/dt}\right|_{t=t_0}.$$

Note that this is undefined when $x'(t_0) = 0$.

The *tangent line* at t_0 is given by

$$y = \left(\left.\frac{dy}{dx}\right|_{t=t_0}\right)(x - x(t_0)) + y(t_0).$$

1.2.2 Second Derivative

The value of the second derivative for the curve at t_0 is given by

$$\left.\frac{d^2y}{dx^2}\right|_{t=t_0} = \left.\frac{d}{dt}\left(\frac{dy}{dx}\right)\right|_{t=t_0} = \left.\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\right|_{t=t_0}.$$

Note the benefit of Leibnitz notation for each of these two derivatives!

1.2.3 Area Under a Curve

Suppose that a parametric curve is non-self intersecting. Then, the signed area of the region between the curve and the x-axis on the t interval $[t_a, t_b]$ is given by

$$A = \int_{t_a}^{t_b} y(t)\frac{dx}{dt}\,dt.$$

1.2.4 Arc Length

The *arc length* of a parametric curve over the t interval $[t_a, t_b]$ is given by

$$s = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

1.2.5 Surface Area

The *surface area* of the region obtained by rotating a non-self intersecting parametric curve is given by

$$S = \int_{t_a}^{t_b} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

1.2.6 The Cycloid

We can apply each of the above to the cycloid:

- *Derivative*: $\frac{dy}{dx} = \frac{dy}{dx} = \frac{\sin t}{1-\cos t}$. Note that the slope is then independent of the radius of the wheel and that the slope is undefined at each of $t = \ldots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$.

- *Cartesian Equation*: With radius of 3 and when $t = \frac{\pi}{3}$, the point is found by solving for $x(\frac{\pi}{3})$ and $y(\frac{\pi}{3})$:

$$x\left(\frac{\pi}{3}\right) = 3\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right) = \pi - \frac{3\sqrt{3}}{2}$$
$$y\left(\frac{\pi}{3}\right) = 3\left(1 - \cos\left(\frac{\pi}{3}\right)\right) = \frac{3}{2}$$
$$(x, y) = \left(\pi - \frac{3\sqrt{3}}{2}, \frac{3}{2}\right)$$

Plugging in our t value into our derivative, we get a slope of

$$\frac{\sin(\pi/3)}{1-\cos(\pi/3)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Now, we can write the equation of the tangent line as

$$y = \sqrt{3}\left(x - \pi + \frac{3\sqrt{3}}{2}\right) + \frac{3}{2}.$$

- *Concavity*: $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)$.

$$\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d/dt(dy/dx)}{dx/dt} \\
&= \frac{\frac{d}{dt}\left(\frac{\sin t}{1-\cos t}\right)}{a - a\cos t} \\
&= \frac{\frac{\cos t(1-\cos t)-\sin t\sin t}{(1-\cos t)^2}}{a - a\cos t} \\
&= \frac{\cos t - \cos^2 - \sin^2(t)}{(1-\cos t)^2 a(1-\cos t)} \\
&= \frac{\cos t - 1}{a(1-\cos t)^2} \\
&= -\frac{1}{a(1-\cos t)^2} \\
&= -\frac{a}{a^2(1-\cos t)^2} \\
&= -\frac{a}{y^2}
\end{aligned}$$

  After some work, we find that $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$, which shows that the cycloid is always concave down.

- *Area*: The area of one period of the cycloid $A = 3\pi a^2$, after some work:

$$\begin{aligned}
A &= \int_{t_a}^{t_b} y(t)x'(t)dt \\
&= \int_0^{2\pi} (a - a\cos t)(a - a\cos t)dt \\
&= a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)dt \\
&= a^2 \left(t + \frac{t}{2} + \frac{1}{4}\sin(2t)\right)\Bigg|_0^{2\pi} \\
&= a^2 \left[\left(2\pi + \frac{2\pi}{2} + \frac{1}{4}\sin(2\pi)\right) - \left(0 + \frac{0}{2} + \frac{1}{4}\sin(0)\right)\right] \\
&= a^2[2\pi + \pi] \\
&= 3\pi a^2.
\end{aligned}$$

- *Arc Length*: The arc length of one period of the cycloid is $s = 8a$, again after some work:

$$\begin{aligned}
s &= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{(a - a\cos t)^2 + (a\sin t)^2}\, dt \\
&= a\int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt \\
&= a\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt \\
&= \sqrt{2}a\int_0^{2\pi} \sqrt{1 - \cos t}\, dt \\
&= \sqrt{2}a\int_0^{2\pi} \sqrt{2\sin^2\left(\frac{t}{2}\right)}\, dt \\
&= \sqrt{2}a \cdot \sqrt{2}\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\, dt \\
&= 2a\left(-2\cos\left(\frac{t}{2}\right)\right)\Big|_0^{2\pi} \\
&= 8a.
\end{aligned}$$

- *Surface Area*: The surface area of the solid obtained by rotating one period of the cycloid around the x-axis is $S = \frac{64\pi a^2}{3}$, after a lot of tedious work.

$$S = \int_0^{2\pi} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

CHAPTER 2

VECTORS IN SPACE

2.1 Vectors in the Plane

2.1.1 Notation

In print, we write vectors in bold like: $\mathbf{v}$, $\mathbf{w}$, $\mathbf{u}$, ... In handwriting, we often write vectors with an arrow over the top: $\vec{v}$, $\vec{w}$, $\vec{u}$, ...

2.1.2 Vectors

A *vector* is a quantity with both *magnitude* (size, length, strength, ...) and *direction*. Given two points in the plane $P = (x_1, y_1)$ and $Q = (x_2, y_2)$, the vector from P to Q, denoted $\overrightarrow{PQ} = \mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

We can also simply state components (known as *component form*): $\mathbf{v} = \langle x, y \rangle$.

The *zero vector*, denoted $\mathbf{0}$, is $\mathbf{0} = \langle 0, 0 \rangle$. Note that $\mathbf{0} \neq 0$.

A *scalar* is a real number (or a magnitude), without direction.

If c is a scalar and $\mathbf{v} = \langle x, y \rangle$, then

$$c\mathbf{v} = c\langle x, y \rangle = \langle cx, cy \rangle.$$

This operation is called *scalar multiplication*. Scalar multiplication changes the magnitude of a vector, but not its direction.

Note that the individual components of a vector are themselves *scalars*. You need to keep track of which is which.

If $\mathbf{v} = \langle x_1, y_1 \rangle$ and $\mathbf{w} = \langle x_2, y_2 \rangle$, then the *vector sum*

$$\mathbf{v} + \mathbf{w} = \langle x_1 + x_2, y_1 + y_2 \rangle.$$

That is, we add component wise.

If $\mathbf{v} = \langle x_1, y_1 \rangle$, then the *magnitude* of $\mathbf{v}$ is given by

$$\|\mathbf{v}\| = \sqrt{x_1^2 + y_1^2}.$$

This is really just the Pythagorean theorem.

2.2 Vectors in Space

In $\mathbb{R}^3$, we have three axes, x, y, and z, which follow the *right-hand rule*: point the fingers of the right hand in the direction of the positive x-axis, curl them towards the positive y-axis, and the thumb points in the direction of the positive z-axis.

Since the distance formula in $\mathbb{R}^3$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$, then $\mathbf{u} = \langle x, y, z \rangle$ we have $\|\mathbf{u}\| = \sqrt{x^2 + y^2 + z^2}$.

To *normalize* a vector, we divide by its magnitude: $\mathbf{v} = \langle x, y, z \rangle$, then $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left\langle \frac{x}{\|\mathbf{v}\|}, \frac{y}{\|\mathbf{v}\|}, \frac{z}{\|\mathbf{v}\|} \right\rangle$. This gives us a *unit vector* in the direction of $\mathbf{v}$.

Everything else is basically the same.

2.2.1 Vector Properties

Suppose that each of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and r and s are scalars. Then the following properties hold:

- *Additive Commutativity*: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
- *Additive Associativity*: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- *Additive Identity*: $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
- *Additive Inverse*: $-\mathbf{v} = (-1)\mathbf{v}$ and $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
- *Scalar Associativity*: $r(s\mathbf{u}) = (rs)\mathbf{u}$.
- *Scalars Distributive over Vectors*: $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$.
- *Vectors Distributive over Scalars*: $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$.
- *Multiplicative Identity*: $1\mathbf{u} = \mathbf{u}$.
- *Zero Scalar*: $0\mathbf{u} = \mathbf{0}$.

2.2.2 Special Vectors

A *unit vector* is a vector $\mathbf{u}$ such that $\|\mathbf{u}\| = 1$.

In $\mathbb{R}^2$ the *standard unit vectors* are $\hat{\imath} = \mathbf{i} = \langle 1, 0 \rangle$ and $\hat{\jmath} = \mathbf{j} = \langle 0, 1 \rangle$. This allows us to write $\mathbf{v} = \langle 2, 3 \rangle = 2\mathbf{i} + 3\mathbf{j}$, for example.

In $\mathbb{R}^3$, we have three stand unit vectors, $\hat{\imath} = \mathbf{i} = \langle 1, 0, 0 \rangle$, $\hat{\jmath} = \mathbf{j} = \langle 0, 1, 0 \rangle$, and $\hat{k} = \mathbf{k} = \langle 0, 0, 1 \rangle$.

It is a picky detail, but $\mathbf{i} \in \mathbb{R}^2 \neq \mathbf{i} \in \mathbb{R}^3$.

2.3 The Dot Product

Suppose $\mathbf{u} = \langle u_1, u_2, \ldots u_n \rangle$ and $\mathbf{v} = \langle v_1, v_2, \ldots v_n \rangle$ are vectors in $\mathbb{R}^n$. Then the *dot product* of $\mathbf{u}$ and $\mathbf{v}$ is given by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n.$$

That is, we multiply the corresponding components and sum the results.

It should be clear that $\mathbf{u} \cdot \mathbf{v}$ results in a scalar. The dot product is a special type of inner product.

Think of the dot product as a way to measure how much of one vector points in the same direction as another.

2.3.1 Properties of the Dot Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and c be a scalar. Then the following properties hold:

- *Commutativity*: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
- *Distributive Property*: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
- *Scalar Associativity*: $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) = \mathbf{u} \cdot (c\mathbf{v})$.
- *Self-Product*: $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.
- *Magnitude*: $\|\mathbf{v}\| = \sqrt{v} \cdot v$
- *Angle*: $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos(\theta)$, where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$. (Law of Cosines.)
- *Orthogonality*: $\mathbf{u} \cdot \mathbf{v} = 0$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

2.3.2 Projections

The *projection* of $\mathbf{u}$ onto $\mathbf{v}$ is given by

$$\mathrm{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}.$$

This is a vector parallel to $\mathbf{v}$, which has length equal to the amount of $\mathbf{u}$ which points in the same direction as $\mathbf{v}$.

Think of a projection as a measure of how much of one vector points in the same direction as another.

2.3.3 Work

If a constant force $\mathbf{F}$ moved an object from P to Q, the *work* done is given by

$$W = \mathbf{F} \cdot \overrightarrow{PQ}.$$

Thus, if that force acts at an angle θ to the line of motion, the work is:

$$W = (\|\mathbf{F}\|)\,\|\mathbf{PQ}\|\cos(\theta).$$

Later this semester, we will learn how to compensate for a non-constant force, and over a non-linear path.

2.4 The Cross Product

Suppose that $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. Then, the *cross product* of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u} \times \mathbf{v}$, is the unique right-hand rule vector orthogonal to each of $\mathbf{u}$ and v whose magnitude is equal to the area of the parallelogram spanned by $\mathbf{u}$ and $\mathbf{v}$.

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then,

$$\mathbf{u} \times \mathbf{v} = \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1 \rangle .$$

NOTE: You will never multiply an v_1-coordinate by an u_1-coordinate. This is true for all v_n and u_n coordinates.

You can show by working the algebra that $\mathbf{u} \cdot (\mathbf{u} \times \mathbf{v}) = 0$ and $\mathbf{v} \cdot (\mathbf{u} \times \mathbf{v}) = 0$.

With determinants, you can do this in one step:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Oddly, we can only define a cross-product in $\mathbb{R}$, $\mathbb{R}^3$, and $\mathbb{R}^7$, while the dot product is *always* defined.

Example

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \langle 2, 1, 4 \rangle \cdot \langle 1, -3, 1 \rangle \\ &= \langle (1)(1) - 4(-3), 4(1) - 2(1), 2(-3) - 1(1) \rangle \\ &= \langle 13, 2, -7 \rangle . \end{aligned}$$

2.4.1 Properties of the Cross Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and c be a scalar. Then the following properties hold:

- *Anticommutativity*: $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.
- *Distributive Property*: $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$.
- *Scalar Associativity*: $(c\mathbf{u}) \times \mathbf{v} = c(\mathbf{u} \times \mathbf{v}) = \mathbf{u} \times (c\mathbf{v})$.
- *Zero*: $\mathbf{u} \times \mathbf{u} = \mathbf{0}$.
- *Nilpotence*: $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- *Scalar Triple Product*: $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.
- *Angle*: $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$, where $0 \leq \theta \leq \pi$ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

2.4.2 Standard Unit Vectors and the Cross Product

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$.

- $\mathbf{i} \times \mathbf{j} = \mathbf{k}$
- $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$
- $\mathbf{j} \times \mathbf{k} = \mathbf{i}$
- $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$
- $\mathbf{k} \times \mathbf{i} = \mathbf{j}$
- $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$

2.4.3 Torque

Torque, denoted by τ, measures the tendency to produce a rotation about an axis.

If $\mathbf{r}$ is a radial vector from an axis to a force and $\mathbf{F}$ is the force, then the torque induced on the axis by the force is given by:

$$\tau = \mathbf{r} \times \mathbf{F} \qquad \text{or} \qquad \|\tau\| = \|\mathbf{r}\| \|\mathbf{F}\| \sin(\theta),$$