Multivariable Calculus Practice Set III

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1. (3 points) Determine the absolute extrema for the function $f(x,y) = x^2 + 3y^2 - 2x - y - xy$ on the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$.

Solution. We first find the critical points of the function:

$$\begin{aligned}
\nabla f(x,y) &= \langle 2x - 2 - y,\, 6y - 1 - x \rangle = \mathbf{0} \\
&\implies y = 2x - 2 \quad \text{and} \quad x = 6(2x-2) - 1 - x \\
&\implies y = \tfrac{4}{11} \quad \text{and} \quad x = \tfrac{13}{11}
\end{aligned}$$

This gives the critical point $\boxed{\left(\dfrac{13}{11}, \dfrac{4}{11}\right)}$. We also need to check the boundary of the region. Thus:

(ℓ_1): $y = 0$, $0 \le x \le 2 \implies f(x,y) = g(x) = x^2 + 3(0)^2 - 2x - (0) - x(0) = x^2 - 2x \implies g'(x) = 2x - 2$. Therefore, the critical points are $\boxed{(1,0)}$.

(ℓ_2): $x = 0$, $0 \le y \le 1 \implies f(x,y) = h(y) = (0)^2 + 3y^2 - (0) - y - 0 = 3y^2 - y \implies h'(y) = 6y - 1$. Hence, the critical points are $\boxed{\left(0, \dfrac{1}{6}\right)}$.

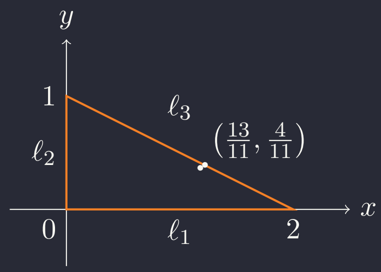
(ℓ_3): $y = 1 - \frac{1}{2}x$, $0 \le x \le 2 \implies f(x,y) = k(x) = x^2 + 3\left(1 - \frac{1}{2}x\right)^2 - 2x - \left(1 - \frac{1}{2}x\right) - x\left(1 - \frac{1}{2}x\right)$. Solving this equation for x:

$$\begin{aligned}
k(x) &= x^2 + 3\left(1 - \tfrac{1}{2}x - \tfrac{1}{2}x + \tfrac{1}{4}x^2\right) - 2x - 1 + \tfrac{1}{2}x - x + \tfrac{1}{2}x^2 \\
&= x^2 + 3\left(1 - x + \tfrac{1}{4}x^2\right) - 2x - 1 + \tfrac{1}{2}x - x + \tfrac{1}{2}x^2 \\
&= \left[x^2 + \tfrac{3}{4}x^2 + \tfrac{1}{2}x^2\right] + \left[-3x - 2x - \tfrac{1}{2}x\right] + [3 - 1] \\
&= \tfrac{9}{4}x^2 - \tfrac{7}{2}x + 2 \\
&= \tfrac{1}{4}(9x^2 - 22x + 8) \\
\implies k'(x) &= \tfrac{1}{4} \cdot \tfrac{d}{dx}[9x^2 - 22x + 8] \\
0 &= \tfrac{1}{2}(9x - 11) \\
x &= \tfrac{11}{9}
\end{aligned}$$

Using this x-value, we plug it back into our equation for y to get the critical point $\boxed{\left(\dfrac{11}{9}, \dfrac{7}{18}\right)}$.

Now that we have our critical points, we can evaluate the function at each of these points to determine the absolute extrema:

Critical Point	Evaluation		Solution
$\left(\frac{13}{11}, \frac{4}{11}\right)$	$\frac{169}{121} + \frac{48}{121} - \frac{26}{11} - \frac{4}{11} - \frac{52}{121}$	$=$	$-\frac{15}{11} \approx -1.363\ldots$
$(1, 0)$	$1 + 0 - 2 - 0 - 0$	$=$	-1
$\left(0, \frac{1}{6}\right)$	$0 + \frac{1}{12} - 0 - \frac{1}{6} - 0$	$=$	$-\frac{1}{12} \approx -0.083\ldots$
$\left(\frac{11}{9}, \frac{7}{18}\right)$	$\frac{121}{81} + \frac{49}{108} - \frac{22}{9} - \frac{7}{18} - \frac{77}{162}$	$=$	$-\frac{49}{36} \approx -1.361\ldots$



With these values, we can see that the absolute maximum is $\boxed{-0.083}$ at the point $\boxed{\left(0, \frac{1}{6}\right)}$ and the absolute minimum is $\boxed{-1.363}$ at the point $\boxed{\left(\frac{13}{11}, \frac{4}{11}\right)}$.

2. (1 point each) Convert each as indicated; leave each answer as exact:

   (a) Convert the rectangular point $(-5, 1)$ to polar coordinates.

   (b) Convert the cylindrical point $(5, \frac{7\pi}{6}, 2)$ to rectangular.

   (c) Convert the rectangular point $(-2, 4, -1)$ to spherical.

   (d) Convert the spherical point $(4, \frac{11\pi}{6}, \frac{3\pi}{4})$ to cylindrical.

3. (3 points) Determine the value of each given integral. You need to do the work here by hand, but of course can check any answers with technology.

   (a) $\displaystyle\iint_D (x^2 + 6xy)\, dA$ where D is the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 12)$.

   (b) $\displaystyle\int_0^2 \int_{x^2}^4 4x^3 \cos(y^3)\, dy\, dx$

   (c) $\displaystyle\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sin(5x^2 + 5y^2)\, dy\, dx$

   (d) $\displaystyle\int_0^1 \int_0^{\sqrt{1-x}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx$

4. (3 points) Find the volume of the solid described by $x^2 + y^2 \leq 1, \quad x \geq 0, \quad 0 \leq z \leq 4 - y$.

5. (3 points) Find the average value of the function $f(x, y) = x\sin(y)$ over the region enclosed by $y = 0$, $y = x^2$, and $x = 1$.

6. (3 points) Find the volume of the solid that lives within both the cylinder $x^2 + y^2 = 1$ and sphere $x^2 + y^2 + z^2 = 9$.

Solution. Use cylindrical coordinates.