Multivariable Calculus Exam 4

Terms and Formulas

- **Slope:** & **Equation** $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; & $y = \frac{dy}{dx}(x - x(t_0)) + y(t_0)$. For example, the curve defined by $x(t) = 3t^2 - 8t + 1$ and $y(t) = e^{-t^2}$, for $0 \le t \le 2$. Finding the equation at $t = 1$, we get:

  Solution. $\frac{dy/dt}{dx/dt} = \frac{-2te^{-t^2}}{6t-8}$, then we plug in $t = 1$: $x(1) = -4$, $y(1) = e^{-1}$, and our slope: $\frac{-2e^{-1}}{6-8} = e^{-1}$. Giving the equation $y = e^{-1}(x+4) + e^{-1}$.

- **Concavity:** $\frac{d^2y}{dx^2}\Big|_{t_0} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt}\Big|_{t_0}$. Remember, $+ \Rightarrow$ concave up.
- **Area Under a Curve:** $\int_{t_a}^{t_b} y(t)\frac{dx}{dt}\,dt$.
- **Arc Length:** $\int_{t_a}^{t_b} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$.
- **Surface Area:** $\int_{t_a}^{t_b} 2\pi y(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$.
- **Direction:** $P = (x_1, y_1)$ and $Q = (x_2, y_2)$: $\mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle$.
- **Vector Sum:** $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2\rangle$.
- **Magnitude:** $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}$.
- **Dot Product:** $\mathbf{u}\cdot\mathbf{v} = u_1v_1 + u_2v_2$.
- To **Normalize** a vector, divide it by its magnitude $\mathbf{v} = \langle x, y, z\rangle$, then $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left\langle \frac{x}{\|\mathbf{v}\|}, \frac{y}{\|\mathbf{v}\|}, \frac{z}{\|\mathbf{v}\|}\right\rangle$. $\therefore \mathbf{u} :=$ **Unit Vector** in direction of $\mathbf{v}$.
- **Projection:** $\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$.
- **Cross product:** $\mathbf{u}\times\mathbf{v} = \langle u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1\rangle$ ..
- **Symmetric Equation:** $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$. E.g., For the points: $R = (4, -6, 1)$ and $S = (1, 2, 3)$: $\frac{x-R_1}{S_1-R_1} = \frac{y-R_2}{S_2-R_2} = \frac{z-R_3}{S_3-R_3}$ or $\frac{x-4}{-3} = \frac{y+6}{8} = \frac{z-1}{2}$.
- If (x_0, y_0, z_0) is a point on a plane, the **Scalar Equation** would be: $\langle x - x_0, y - y_0, z - z_0\rangle\cdot\langle a, b, c\rangle = 0 \Longrightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.
- To **Parameterize** an equation such as $y = x^3 - 4x + 1$ we can let $x = t$ and $y = t^3 - 4t + 1$. This allows us to write the equation as $\mathbf{r}(t) = \langle t, t^3 - 4t + 1\rangle$.
- **Velocity Vector:** $\mathbf{v}(t) = \mathbf{r}'(t)$.
- **Unit Tangent Vector:** $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v(t)}\|}$.
- **Area:** $\iint_D 1\,dA$, where D is the region in the xy-plane over which we are integrating. $\int_a^b\int_c^d f(x,y)\,dy\,dx$.

Derivative Test, Curvature, and Practice Set # 1

- **Second derivative test:** $\Delta = f_{xx}f_{yy} - (f_{xy})^2$. If $\Delta > 0$ and $f_{xx} > 0$, then f has a local min. If $\Delta > 0$ and $f_{xx} < 0$, then f has a local max. If $\Delta < 0$, then f has a saddle point. If $\Delta = 0$, then the test is inconclusive.

Curvature: $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$; for $\mathbb{R}^3$: $\kappa = \frac{\|\mathbf{r}'(t)\times\mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$; if $y = f(x)$: $\kappa = \frac{|y''(x)|}{[1+(y'(x)^2)]^{3/2}}$

1. Find an equation in scalar form of the plane which passes through $(-2, 7, 1)$ and is perpendicular to the planes $3x + y - z = 0$ and $-2x - y + 5z + 1 = 0$ [Hint: Think about what the relationship among the various normal vectors must be.]

   Solution. For the plane to be perpendicular to a given plane, its normal vector must lie in that given plane. Hence, our normal vector must be orthogonal to both $\langle 3, 1, -1\rangle$ and $\langle -2, -1, 5\rangle$. Thus, we can take the cross product of these two vectors to get our normal vector: $\langle 3, 1, -1\rangle \times \langle -2, -1, 5\rangle = \langle 4, -13, -1\rangle$. With our normal vector found, we can plug in our point to get our scalar equation: $4(x+2) - 13(y-7) - (z-1) = 0$.

2. Write, in general equation form, an equation of the plane which contains the three points $P = (2, 7, 3)$, $Q = (-5, 0, 1)$, and $R = (-3, 1, 2)$.

   Solution. Find $\mathbf{PQ} = \langle -7, -7, -2\rangle$ and $\mathbf{PR} = \langle -5, -6, -1\rangle$. Then, we find $\mathbf{n}$ by solving for the cross product. With $\mathbf{n}$ ($\langle -5, 3, 7\rangle$), we get the general formula: $\boxed{-5(x-2) + 3(y-7) + 7(z-3) = 0.}$ where the numbers inside come from P. This can be directly translated to symmetric form by just plugging into the equation: $\frac{x-x_0}{n_x} = \frac{y-y_0}{n_y} = \frac{z-z_0}{n_z}$ where n_x, n_y, n_z are the components of the normal vector and x_0, y_0, z_0 are the coordinates of point P.

3. Find an equation, in symmetric form, of the line of intersection between the planes $2x + y - z + 4 = 0$ and $x - y + 3z = 1$.

   Solution. Choose parameter $t = z$. From $2x + y - z + 4 = 0 \Rightarrow y = -2x + z - 4$. Plug into $x - y + 3z = 1$: $x - (-2x + z - 4) + 3z = 1 \Longrightarrow 3x + 2z + 4 = 1 \Longrightarrow 3x + 2z = -3 \Longrightarrow x = -1 - \frac{2}{3}z$. Then, $x = -1 - \frac{2}{3}z$, $y = -2 + \frac{7}{3}z$, and $z = z$. Set $z = -3t$ (so the denominators clear): $x = -1 + 2t$, $y = -2 - 7t$, $z = -3t$. This gives: $\frac{x+1}{2} = \frac{y+2}{-7} = \frac{z}{-3}$.

Practice Set # 2

1. Write, in scalar form, an equation of the plane which contains the point $(5,2,1)$ and the line given by $x+2=\dfrac{y}{4}=\dfrac{z-5}{2}$.

   Solution. We start by parametrizing the line with common parameter t: $x+2=t \Rightarrow x=t-2$, $\frac{y}{4}=t \Rightarrow y=4t$, and $\frac{z-5}{2}=t \Rightarrow z=2t+5$. This yields $(x,y,z)=(-2,0,5)+t(1,4,2)$. Thus, the line passes through $(-2,0,5)$ and has the direction vector $\mathbf{v}_1=\langle 1,4,2\rangle$. We can form a second vector $\mathbf{v}_2$ by taking the difference between the given point and a point on the line: $\mathbf{v}_2=(5,2,1)-(-2,0,5)=\langle 7,2,-4\rangle$. Then, we find the cross product to get the normal vector. With the normal, we find the scalar form to be $-20(x+2)+18y-26(z-5)=0$.

2. Find total distance of a particle over a time period $[0,3\pi]$ for the position equation $\mathbf{r}(t)=\langle \sin(t),t,3t\rangle$.

   Solution. $\displaystyle\int_0^{3\pi}\|\mathbf{r}'(t)\|\,dt=\int_0^{3\pi}\sqrt{\cos^2(t)+1+9}\,dt=9.709.$

3. Use curvature to find the equation of the osculating circle at the planar curve $y=x^3-4x+1$ at $x=1$.

   Solution. First, we need to find the curvature of the curve at $x=1$. We start by finding the first and second derivatives of the function: $y'(x)=3x^2-4$, $y''(x)=6x$. Plugging in for $x=1$, we get $y(1)=-2$, $y'(1)=-1$, and $y''(1)=6$. With these values, we can find the curvature: $\kappa=\frac{6}{(1+(-1)^2)^{3/2}}=\frac{3\sqrt{2}}{3}$. With the curvature, we can find the radius of the osculating circle: $R=\frac{1}{\kappa}=\frac{\sqrt{2}}{3}$. Plug in $x=1$ to find the center with the unit normal vector: $\mathbf{N}=\frac{(-y',1)}{\sqrt{1+(y')^2}}=\frac{(1,1)}{\sqrt{2}}$. The center can be found by moving our point $P(1,-2)$ the distance R along the unit normal vector: $C=P+R\mathbf{N}=(1,-2)+\frac{\sqrt{2}}{3}\frac{(1,1)}{\sqrt{2}}=\left(\frac{4}{3},-\frac{5}{3}\right)$. This gives the equation for the osculating circle: $\left(x-\frac{4}{3}\right)^2+\left(y+\frac{5}{3}\right)^2=\frac{2}{9}$.

4. Find an equation of the tangent plane to $f(x,y)=x^2y-\sqrt{x}+y$ at the point $(3,1)$.

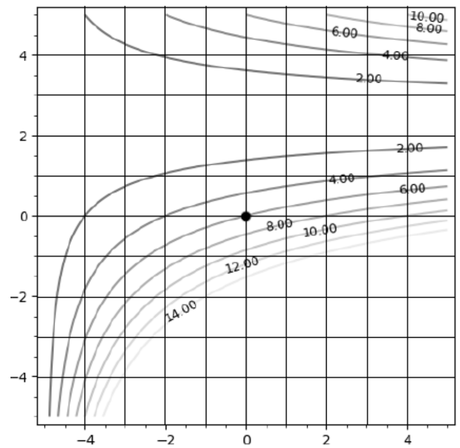
   Solution. Solve for $f_x(x,y)$, then $f_x(3,1)$, and $f_y(x,y)$, then $f_y(3,1)$ to get the values of the partial derivatives at the point $(3,1)$:

$$\begin{aligned} z &= f(3,1)+f_x(3,1)(x-3)+f_y(3,1)(y-1)\\ &= 7+\tfrac{23}{4}(x-3)+\tfrac{35}{4}(y-1). \end{aligned}$$

5. What is the angle between $\mathbf{u}=\langle 6,2,-5\rangle$ and $\mathbf{v}=\langle -4,1,-7\rangle$?

   Solution. $\theta=\cos^{-1}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)=\cos^{-1}\left(\frac{13}{\sqrt{65}\sqrt{66}}\right)=0.198$ rads.

Practice Set # 3



Determine the sign $(+,-,0)$ for each of the following partial derivatives.

1. $f_x(0,0)$
   We see $f(0,0)\approx 6$. As we move right (positive x), f increases, toward value 8. Thus, $+$.

2. $f_y(0,0)$
   As we move up f decreases toward 4. Thus, $-$.

3. $f_{xx}(0,0)$
   The contours are evenly spread in the x-direction through $(0,0)$. We are increasing at a constant rate. Hence, 0.

4. $f_{yy}(0,0)$
   As we move in positive y-direction, we decrease, but less rapidly. The amount by which we are changing is *increasing* (becoming less negative). Thus, $+$.

5. $f_{xy}(0,0)$
   If we move in positive x, the slope in the y-direction becomes more negative (i.e., decreases). Thus, $-$.

6. Consider the function $f(x,y)=x^2y-y^3$. Find the directional derivative for f at $(3,4)$ in the direction of $\mathbf{u}=\langle 5,-2\rangle$.
   Solution. Find partials for f_x and f_y. They are $2xy$ and x^2-3y^2, respectively. Plug in $(3,4)$ to get $f_x(3,4)=24$ and $f_y(3,4)=-39$. Then solve: $\frac{\langle 24,-39\rangle\cdot\langle 5,-2\rangle}{\sqrt{29}}=\frac{198}{\sqrt{29}}$

7. Find $\frac{\partial^2}{\partial x\partial y}\left(x^3y-y^3\tan(xy)\right)$
   Solution.

$$\frac{\partial^2}{\partial x\partial y}\left(x^3y-y^3\tan(xy)\right)=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}[x^3y]-\frac{\partial}{\partial y}\left[y^3\tan(xy)\right]\right]$$

   Splitting this into 3 partial derivatives:

$$\frac{\partial}{\partial x}[x^3]=3x^2,\quad -\frac{\partial}{\partial x}\left[3y^2\tan(xy)\right]=-3y^3\sec^2(xy),$$

   with the final derivative worked out:

$$\begin{aligned} -\frac{\partial}{\partial x}\left[xy^3\sec^2(xy)\right] &= y^3\sec^2(xy)+\left[(xy^3)\cdot 2y\sec^2(xy)\tan(xy)\right]\\ &= -y^3\sec^2(xy)-2xy^4\sec^2(xy)\tan(xy). \end{aligned}$$

   Combining these results, we have:

$$3x^2-3y^3\sec^2(xy)-y^3\sec^2(xy)-2xy^4\sec^2(xy)\tan(xy).$$

Practice Set # 4

1. (3 points) Determine the absolute extrema for the function $f(x,y) = x^2+3y^2-2x-y-xy$ on the triangular region with vertices $(0,0)$, $(2,0)$, and $(0,1)$.
We first find the critical points of the function:

$$\begin{aligned}\nabla f(x,y) &= \langle 2x-2-y,\, 6y-1-x\rangle = \mathbf{0}\\ \implies y &= 2x-2 \quad\text{and}\quad x = 6(2x-2)-1-x\\ \implies y &= \tfrac{4}{11} \quad\text{and}\quad x=\tfrac{13}{11}\end{aligned}$$

This gives the critical point $\boxed{\left(\frac{13}{11},\frac{4}{11}\right)}$. We also need to check the boundary of the region. Thus:

(ℓ_1): $y=0,\ 0\le x\le 2 \implies f(x,y)=g(x)=x^2+3(0)^2-2x-(0)-x(0) = x^2-2x \implies g'(x)=2x-2$. This gives $(1,0)$.

(ℓ_2): $x=0,\ 0\le y\le 1 \implies f(x,y)=h(y)=(0)^2+3y^2-(0)-y-0=3y^2-y \implies h'(y)=6y-1$. This gives $\left(0,\frac{1}{6}\right)$

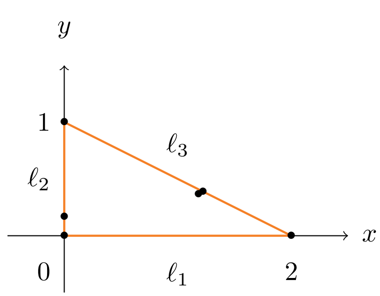
(ℓ_3): $y=1-\frac{1}{2}x,\ 0\le x\le 2 \implies f(x,y)=k(x)=x^2+3\left(1-\frac{1}{2}x\right)^2-2x-\left(1-\frac{1}{2}x\right)-x\left(1-\frac{1}{2}x\right)$.

$$\begin{aligned}k(x) &= x^2+3\left(1-\tfrac{1}{2}x-\tfrac{1}{2}x+\tfrac{1}{4}x^2\right)-2x-1+\tfrac{1}{2}x-x+\tfrac{1}{2}x^2\\ &= \tfrac{1}{4}(9x^2-22x+8)\\ \implies k'(x) &= \tfrac{1}{4}\cdot\tfrac{d}{dx}[9x^2-22x+8]\\ x &= \tfrac{11}{9}\end{aligned}$$

Using this x-value, we plug it back into our equation for y to get the critical point $\left(\frac{11}{9},\frac{7}{18}\right)$.

The vertices of the triangle give $f(0,0)=0$, $f(2,0)=-2$, and $f(0,1)=2$. We can do the same for the other points and add them to our table.

Point	$f(x,y)$	**Type**
$\left(\frac{13}{11},\frac{4}{11}\right)$	-1.364	Interior CP
$(1,0)$	-1	ℓ_1
$\left(0,\frac{1}{6}\right)$	-0.083	ℓ_2
$\left(\frac{11}{9},\frac{7}{18}\right)$	$\boxed{-1.361}$	ℓ_3
$(0,0)$	0	Vertex 1
$(2,0)$	-2	Vertex 2
$(0,1)$	$\boxed{2}$	Vertex 3



2. Convert the rectangular point $(-5,1)$ to polar coordinates.

$$\begin{aligned}r &= \sqrt{(-5)^2+1^2}=\sqrt{26}\\ \theta &= \arctan\left(\tfrac{1}{-5}\right) = \arctan\left(-\tfrac{1}{5}\right)=\tfrac{7\pi}{6}+\pi \text{ (2nd quadrant)}\end{aligned}$$

The polar coordinates are $\boxed{\left(\sqrt{26},\frac{7\pi}{6}+\pi\right)}$.

3. Convert the cylindrical point $(5,\frac{7\pi}{6},2)$ to rectangular.

$$\begin{aligned}x &= 5\cos\left(\tfrac{7\pi}{6}\right)=5\left(-\tfrac{\sqrt{3}}{2}\right)=-\tfrac{5\sqrt{3}}{2}\\ y &= 5\sin\left(\tfrac{7\pi}{6}\right)=5\left(-\tfrac{1}{2}\right)=-\tfrac{5}{2}\\ z &= 2\end{aligned}$$

The rectangular coordinates are $\boxed{\left(-\frac{5\sqrt{3}}{2},-\frac{5}{2},2\right)}$.

4. Convert the rectangular point $(-2,4,-1)$ to spherical.

$$\begin{aligned}\rho &= \sqrt{(-2)^2+4^2+(-1)^2}=\sqrt{21}\\ \theta &= \arctan\left(\tfrac{4}{-2}\right)=\arctan(-2)\\ \phi &= \arccos\left(\tfrac{-1}{\sqrt{21}}\right)=\arccos\left(-\tfrac{1}{\sqrt{21}}\right)\end{aligned}$$

Since the point $(-2,4)$ is in the second quadrant, we add π to the arctan value. Hence, the spherical coordinates are $\boxed{\left(\sqrt{21},\pi+\arctan(-2),\arccos\left(-\frac{1}{\sqrt{21}}\right)\right)}$.

5. Convert the spherical point $(4,\frac{11\pi}{6},\frac{3\pi}{4})$ to cylindrical.
The conversion from spherical to cylindrical follows the following equations:

$$r=\rho\sin\phi,\quad \theta=\theta,\quad\text{and}\quad z=\rho\cos\phi.$$

Thus, we have:

$$\begin{aligned}r &= 4\sin\left(\tfrac{3\pi}{4}\right)=4\left(\tfrac{\sqrt{2}}{2}\right)=2\sqrt{2}\\ \theta &= \tfrac{11\pi}{6}\\ z &= 4\cos\left(\tfrac{3\pi}{4}\right)=4\left(-\tfrac{\sqrt{2}}{2}\right)=-2\sqrt{2}\end{aligned}$$

Therefore, we get the cylindrical coordinates $\boxed{\left(2\sqrt{2},\frac{11\pi}{6},-2\sqrt{2}\right)}$.

Practice Set # 5

1. Determine $\displaystyle\int_C \frac{1}{x^2+y^2+z^2}\,ds$, where C is given by $\langle \cos t, \sin t, t\rangle$, $0 \le t \le \pi$.

   Solution. First, we need to find ds, which is given by finding the derivative of the vector function and taking the norm:

   $$ds = ||\mathbf{r}'(t)||\,dt = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}\,dt = \sqrt{1+1}\,dt = \sqrt{2}dt.$$

   Rewriting the integral in terms of t, we have:

   $$\int_C \frac{1}{x^2+y^2+z^2}\,ds = \int_0^\pi \frac{1}{\cos^2 t + \sin^2 t + t^2}\sqrt{2}\,dt = \sqrt{2}\int_0^\pi \frac{1}{1+t^2}\,dt.$$

   We know that $\int \frac{1}{1+t^2}dt = \tan^{-1}(t)$, so we can evaluate the integral as follows:

   $$\sqrt{2}\int_0^\pi \frac{1}{1+t^2}\,dt = \sqrt{2}\left[\tan^{-1}(t)\right]_0^\pi = \boxed{\sqrt{2}\tan^{-1}(\pi).}$$

2. Let $\mathbf{F}(x,y) = 3x^2y^2\mathbf{i} + (2x^3y+5)\mathbf{j}$. Find a scalar function f such that $\nabla f = \mathbf{F}$ and use this to determine $\displaystyle\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is given by $\mathbf{r}(t) = (t^3-2t)\mathbf{i} + (t^3+2t)\mathbf{j}$ for $0 \le t \le 1$.

   Solution. First, we need to check if $\mathbf{F}$ is conservative. We can do this by checking if the mixed partials are equal:

   $$\tfrac{\partial}{\partial y}(3x^2y^2) = 6x^2y, \quad \tfrac{\partial}{\partial x}(2x^3y+5) = 6x^2y.$$

   Since these are equal, we can conclude that $\mathbf{F}$ is conservative. Now, we need to find a scalar function f such that $\nabla f = \mathbf{F}$. We can do this by integrating the components of $\mathbf{F}$:

   $$f(x,y) = \int 3x^2y^2\,dx + h(y) = x^3y^2 + h(y).$$

   Then, we can differentiate f with respect to y and set it equal to the second component of $\mathbf{F}$:

   $$\tfrac{\partial}{\partial y}(x^3y^2 + h(y)) = 2x^3y + h'(y) \quad\Rightarrow\quad 2x^3y + h'(y) = 2x^3y + 5 \quad\Rightarrow\quad h'(y) = 5.$$

   This gives us $h'(y) = 5$, so we can integrate to find $h(y)$: $h(y) = 5y + K$. Thus, we have:

   $$f(x,y) = x^3y^2 + 5y + K.$$

   Finally, use the following (with t_a and t_b from $0 = t_a \le t \le t_b = 1$):

   $$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(t_b)) - f(\mathbf{r}(t_a)) = f(1,3) - f(0,0) = -9 + 15 + K - K = 6.$$

3. For what value(s), if any, of a is $(3x^2y + az)\mathbf{i} + x^3j + (3x + 3z^2)\mathbf{k}$ conservative?

   Solution. From Section 1, we know that for a 3-dimensional vector field to be conservative, the mixed partials must be equal. Thus, we must find each of the following:

   $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z},$$

   where $P = 3x^2y + az$, $Q = x^3$, and $R = 3x + 3z^2$. Starting with the first equality. (Skipping a lot) Since these are equal, we can move on to the last equality:

   $$\tfrac{\partial}{\partial x}(3x + 3z^2) = 3, \quad \tfrac{\partial}{\partial z}(3x^2y + az) = a.$$

   Thus, the only value of a for which the vector field is conservative is $\boxed{a = 3}$.

4. Find the work done by the force field $\mathbf{F} = x^2\mathbf{i} + y^3\mathbf{j}$ in moving an object from $(1,0)$ to $(2,2)$.

   Solution. From $(1,0)$ to $(2,2)$, we can parameterize and find its derivative for the line segment as follows: $\mathbf{r}(t) = \langle 1+t, 2t\rangle$, $0 \le t \le 1 \;\Rightarrow\; \mathbf{r}'(t) = \langle 1, 2\rangle$. Now, we can find $\mathbf{F}(\mathbf{r}(t))$ and its dot product with $\mathbf{r}'(t)$ as follows (then integrate):

   $$\mathbf{F}(\mathbf{r}(t)) = \langle (1+t)^2, 8t^3\rangle \Rightarrow \int_0^1 (\mathbf{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t))dt = \int_0^1 ((1+t)^2 + 16t^3)dt.$$

Green's Flux and Circulation Theorem

1. Calculate the circulation of $\mathbf{F} = \left\langle xy, x^2y^3\right\rangle$ along C, where C is the counter-clockwise oriented triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ with Green's Theorem.

   Solution. Using Green's Theorem for circulation, we get:

   $$\iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA = \int_0^1\int_0^{2x} \left(3x^2y^2 - 2xy\right)\,dy\,dx.$$

2. Find the flux of the same vector field as above across the boundary of the triangle.

   Solution. Using Green's Theorem for flux, we get:

   $$\iint_C \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)\,dA = \int_0^1\int_0^{2x} \left(2xy + 3x^2y^2\right)\,dy\,dx.$$

Practice Set # 6

1. Find $\iint_S x^2\,dS$, where S is the triangle with vertices $(1,0,0)$, $(0,-2,0)$, and $(0,0,4)$.

Solution. Parameterize the triangular surface:

$$\mathbf{r}(u,v) = (1,0,0)(1-u-v) + (0,-2,0)u + (0,0,4)v = (1-u-v, -2u, 4v)$$

where $0 \le u, v$ and $u+v \le 1$.

Computing the tangent vectors:

$$\mathbf{t}_u = (-1,-2,0) \quad \text{and} \quad \mathbf{t}_v = (-1,0,4)$$

The normal vector is:

$$\mathbf{n} = \mathbf{t}_u \times \mathbf{t}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 0 \\ -1 & 0 & 4 \end{vmatrix} = (-8,4,-2)$$

The magnitude of this normal vector gives the area element:

$$\|\mathbf{t}_u \times \mathbf{t}_v\| = \sqrt{64+16+4} = \sqrt{84} = 2\sqrt{21}$$

Now we can evaluate the surface integral:

$$\iint_S x^2\,dS = \iint_D (1-u-v)^2 \cdot 2\sqrt{21}\,du\,dv$$

Computing this double integral:

$$\begin{aligned} 2\sqrt{21}\int_0^1\int_0^{1-u}(1-u-v)^2\,dv\,du &= 2\sqrt{21}\int_0^1 \left[\frac{-(1-u-v)^3}{3}\right]_0^{1-u} du \\ &= 2\sqrt{21}\int_0^1 \left[\frac{-(0)^3}{3} + \frac{(1-u)^3}{3}\right] du \\ &= \frac{2\sqrt{21}}{3}\int_0^1 (1-u)^3\,du \end{aligned}$$

We make the substitution $v = 1-u$ to get $dv = -du$. Note this flips the limits of integration, but the negative sign cancels that out, so we can keep the limits as they are.

$$= \frac{2\sqrt{21}}{3}\int_0^1 v^3\,dv = \boxed{\frac{\sqrt{21}}{6}}.$$

2. Use Stokes' Theorem to find $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z) = \langle x^2z,\, xy^2,\, z^2\rangle$ and C is the curve of intersection between the plane $x+y+z=1$ and the cylinder $x^2+y^2=9$, oriented counter-clockwise when viewed from above.
Solution. Stokes' Theorem:

$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} \Rightarrow \iint_S \langle 0-0, x^2-0, y^2\rangle \cdot \mathbf{t}_r\times\mathbf{t}_\theta\,dS$$

The surface is $0\le r\le 3$, and because it's a cylinder, $0\le\theta\le 2\pi$. Then, we get $\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, 1-r\cos\theta-r\sin\theta\rangle$. This is from solving $x+y+z=1$ from the plane. Then, we find

$$\mathbf{t}_r\,\langle\cos\theta,\sin\theta,-\cos\theta-\sin\theta\rangle \quad\text{and}\quad \mathbf{t}_\theta = \langle -r\sin\theta, r\cos\theta, r\sin\theta - r\cos\theta\rangle.$$

Taking the cross product, $\mathbf{t}_r\times\mathbf{t}_\theta$ we get $\langle r,r,r\rangle$. Plugging into our integral:

$$\int_0^{2\pi}\int_0^3 \langle 0, x^2, y^2\rangle\cdot\langle r,r,r\rangle\,dr\,d\theta = \frac{81\pi}{2}.$$

3. Let $\mathbf{F}(x,y,z) = \langle x,y,z^2\rangle$ and S be the unit sphere with positive orientation. Find $\iint_S \mathbf{F}\cdot d\mathbf{S}$.
Solution. Since S is closed, we can use the Divergence Theorem. That is:

$$\begin{aligned} \iint_S \mathbf{F}\cdot d\mathbf{S} &= \iiint_V \nabla\cdot\mathbf{F}\,dV \\ &= \iiint_V (1+1+2z)\,dV \end{aligned}$$

Using spherical coordinates, we have:

$$= \int_0^\pi\int_0^{2\pi}\int_0^1 (2+2\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

Factor out 2, and do θ integral because none depend on it. Then do ρ and ϕ.

4. Find the flux of $\mathbf{F} = xy\mathbf{i} + x^2y^3\mathbf{j}$ over C, the same counter-clockwise oriented triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$ as in the previous problem (notice that the vector field is the same as well). Determine this by working three separate line integrals.

   Solution. To find the flux of $\mathbf{F}$ over C, we can use the following formula:

$$\int_C \mathbf{F}\cdot\mathbf{N}\,ds = \int_C \mathbf{F}\cdot\mathbf{n}(t)\,dt = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\langle y'(t), -x'(t)\rangle\,dt.$$

   We can reuse the parameterizations from the previous problem, but we need to find $\mathbf{n}(t)$ for each segment and then integrate each one. Thus:

   - **For C_1:** Our tangent vector is $\mathbf{r}_1'(t) = \langle 1, 0\rangle$. Using the formula above, we see that $\mathbf{n}_1(t) = \langle 0, -1\rangle$. Thus, we can evaluate the integral:

$$\int_{C_1} \mathbf{F}(\mathbf{r}_1(t))\cdot\mathbf{n}_1(t)\,dt = \int_0^1 \langle 0,0\rangle\cdot\langle 0,-1\rangle\;dt = 0.$$

   - **For C_2:** The tangent vector is $\mathbf{r}_2'(t) = \langle 0, 2\rangle$. Hence, the normal vector is $\mathbf{n}_2(t) = \langle 2, 0\rangle$. Now, we can evaluate the integral as follows:

$$\int_{C_2} \mathbf{F}(\mathbf{r}_2(t))\cdot\mathbf{n}_2(t)\,dt = \int_0^1 \left\langle 2t, 8t^3\right\rangle\cdot\langle 2,0\rangle\;dt = \int_0^1 4t\,dt = \left[2t^2\right]_0^1 = 2.$$

   - **For C_3:** From the tangent vector, we know the normal vector is $\mathbf{n}_3(t) = \langle -2, 1\rangle$. Evaluating the integral, we see:

$$\begin{aligned}\int_{C_3} \mathbf{F}(\mathbf{r}_3(t))\cdot\mathbf{n}_3(t)\,dt &= \int_0^1 \left\langle 2(1-t)^2, 8(1-t)^5\right\rangle\cdot\langle -2, 1\rangle \\ &= \frac{4}{3} - \frac{4}{3} = 0.\end{aligned}$$

   - Adding these up, we get 2.

5. Determine the value of $\int_0^2\int_{x^2}^4 4x^3\cos(y^3)\,dy\,dx$.
   Solution. Swap order of integration: $\int_0^4\int_0^{\sqrt{y}} 4x^3\cos(y^3)\,dx\,dy$

6. Determine the value of $\int_{-3}^3\int_0^{\sqrt{9-x^2}} \sin(5x^2+5y^2)\,dy\,dx$.
   Solution. Swap to polar coordinates: $\int_0^\pi\int_0^3 \sin(5r^2)\cdot r\,dr\,d\theta$.

7. Determine the value of $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx$.
   Solution. Switch to spherical coordinates: $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3\sin(\phi)\,d\rho\,d\theta\,d\phi$.

8. Find the volume of the solid described by $x^2+y^2\le 1$, $x\ge 0$, $0\le z\le 4-y$.
   Solution. Use cylindrical coordinates. This gives $0\le r\le 1$, $0\le\theta\le\frac{\pi}{2}$, and $0\le z\le 4-r\sin(\theta)$. Yielding: $\iiint dV \Rightarrow \int_{-\pi/2}^{\pi/2}\int_0^1\int_0^{4-r\sin(\theta)} r\,dz\,dr\,d\theta$

9. Find the average value of the function $f(x,y) = x\sin y$ over the region enclosed by $y=0$, $y=x^2$, and $x=1$.
   Solution. Average value is given by $\frac{1}{A}\int_R f(x,y)\,dA$. The area of the region is given by $\int_0^1\int_0^{x^2} dy\,dx = \int_0^1 x^2\,dx = \frac{1}{3}$. Thus, we have: $\int_0^1\int_0^{x^2} x\sin(y)\,dy\,dx = \int_0^1 x\left[-\cos(y)\right]_0^{x^2}\,dx = \int_0^1 x(1-\cos(x^2))\,dx$.

10. Find the volume of the solid that lives within both the cylinder $x^2+y^2=1$ and sphere $x^2+y^2+z^2=9$.
    Solution. Use cylindrical coordinates. This gives $0\le r\le 1$, $0\le\theta\le 2\pi$, and $-\sqrt{9-r^2}\le z\le\sqrt{9-r^2}$. Yielding: $\iiint dV \Rightarrow \int_0^{2\pi}\int_0^1\int_{-\sqrt{9-r^2}}^{\sqrt{9-r^2}} r\,dz\,dr\,d\theta$

When making choices about the order of integration, follow guiding principles:

1. **Keep inner limits constant or simple functions of outer variables.**
2. **Avoid limits that change form or sign within their interval.**
3. **Integrate the variable that appears most simply in the bounds first.**