Homework 3: Sections 5 & 6

Algebra

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Section 5

In Exercises 12 and 13, determine whether the given set of invertible $n \times n$ matrices with real number entries is a subgroup of $GL(n, \mathbb{R})$.
[Hint: Make use of Exercise 44. What must be the image of a generator under an automorphism?]

12. The $n \times n$ matrices with determinant -1 or 1

Solution.

13. The set of all $n \times n$ matrices A such that $(A^T)A = I_n$ [These matrices are called **orthogonal**. Recall that A^T, the *transpose* of A, is the matrix whose jth column is the jth row of A for $1 \leq j \leq n$, and that the transpose operation has the property $(AB)^T = (B^T)(A^T)$].

Solution.

In Exercise 34, find the order of the cyclic subgroup of the given group generated by the indicated element.

34. The subgroup of the multiplicative group G of invertible 4×4 matrices generated by

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Solution.

39. Mark each of the following true or false.

__✓__ **a.** The associative law holds in every group.
__✓__ **b.** There may be a group in which the cancellation law fails.
__✓__ **c.** Every group is a subgroup of itself.
__✓__ **d.** Every group has exactly two improper subgroups.
__✓__ **e.** In every cyclic group, every element has a generator.
__✓__ **f.** A cyclic group has a unique generator.
__✓__ **g.** Every set of numbers that is a group under addition is also a group under multiplication.
__✓__ **h.** A subgroup may be defined as a subset of a group.
__✓__ **i.** $\mathbb{Z}_4$ is a cyclic group.
__✓__ **j.** Every subset of every group is a subgroup under the induced operation.

53. Let H be a subgroup of a group G. For $a, b \in G$, let $a \sim b$ if and only if $ab^{-1} \in H$. Show that $\sim$ is an equivalence relation on G.

Solution.

Section 6

In Exercises 17, 18 and 19, find the number of elements in the indicated cyclic group.

17. The cyclic subgroup of $\mathbb{Z}_{30}$ generated by 25

Solution.

18. The cyclic subgroup of $\mathbb{Z}_{42}$ generated by 30

Solution.

19. The cyclic subgroup $\langle i \rangle$ of $\mathbb{C}^*$ of nonzero complex numbers under multiplication

Solution.

In Exercise 23, find all subgroups of the given group, and draw the subgroup diagram for the subgroups.

23. $\mathbb{Z}_{36}$

Solution.

46. Let a and b be elements of a group G. Show that if ab has finite order n, then ba also has order n.

Solution.