Multivariable Calculus Practice Set III

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1. (3 points) Determine the absolute extrema for the function $f(x, y) = x^2 + 3y^2 - 2x - y - xy$ on the triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 1)$.

Solution. We first find the critical points of the function:

$$\begin{aligned}
\nabla f(x, y) &= \langle 2x - 2 - y,\, 6y - 1 - x \rangle = \mathbf{0} \\
&\implies y = 2x - 2 \quad \text{and} \quad x = 6(2x - 2) - 1 - x \\
&\implies y = \tfrac{4}{11} \quad \text{and} \quad x = \tfrac{13}{11}
\end{aligned}$$

This gives the critical point $\boxed{\left(\frac{13}{11}, \frac{4}{11}\right)}$. We also need to check the boundary of the region. Thus:

(ℓ_1): $y = 0$, $0 \leq x \leq 2 \implies f(x, y) = g(x) = x^2 + 3(0)^2 - 2x - (0) - x(0) = x^2 - 2x \implies g'(x) = 2x - 2$. Therefore, the critical points are $\boxed{(1, 0)}$.

(ℓ_2): $x = 0$, $0 \leq y \leq 1 \implies f(x, y) = h(y) = (0)^2 + 3y^2 - (0) - y - 0 = 3y^2 - y \implies h'(y) = 6y - 1$. Hence, the critical points are $\boxed{\left(0, \frac{1}{6}\right)}$.

(ℓ_3): $y = 1 - \frac{1}{2}x$, $0 \leq x \leq 2 \implies f(x, y) = k(x) = x^2 + 3\left(1 - \frac{1}{2}x\right)^2 - 2x - \left(1 - \frac{1}{2}x\right) - x\left(1 - \frac{1}{2}x\right)$. Solving this equation for x:

$$\begin{aligned}
k(x) &= x^2 + 3\left(1 - \tfrac{1}{2}x - \tfrac{1}{2}x + \tfrac{1}{4}x^2\right) - 2x - 1 + \tfrac{1}{2}x - x + \tfrac{1}{2}x^2 \\
&= x^2 + 3\left(1 - x + \tfrac{1}{4}x^2\right) - 2x - 1 + \tfrac{1}{2}x - x + \tfrac{1}{2}x^2 \\
&= \left[x^2 + \tfrac{3}{4}x^2 + \tfrac{1}{2}x^2\right] + \left[-3x - 2x - \tfrac{1}{2}x\right] + [3 - 1] \\
&= \tfrac{9}{4}x^2 - \tfrac{7}{2}x + 2 \\
&= \tfrac{1}{4}(9x^2 - 22x + 8) \\
\implies k'(x) &= \tfrac{1}{4} \cdot \tfrac{d}{dx}[9x^2 - 22x + 8] \\
0 &= \tfrac{1}{2}(9x - 11) \\
x &= \tfrac{11}{9}
\end{aligned}$$

Using this x-value, we plug it back into our equation for y to get the critical point $\boxed{\left(\frac{11}{9}, \frac{7}{18}\right)}$.

With our function's and lines' critical points found, we also need to find the vertices of the triangle:

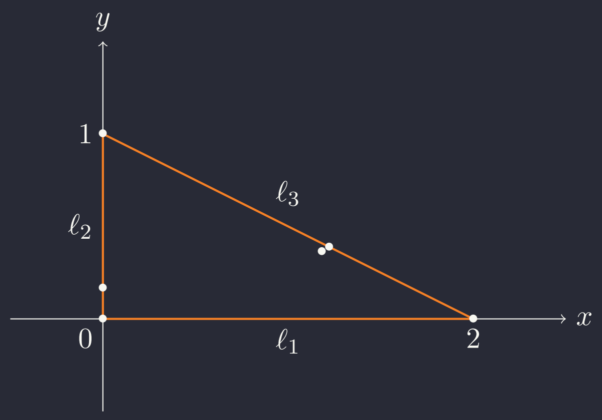
$$f(0,0) = 0^2 + 3(0)^2 - 2(0) - 0 - 0(0) = 0$$
$$f(2,0) = 2^2 + 3(0)^2 - 2(2) - 0 - 2(0) = 0$$
$$f(0,1) = 0^2 + 3(1)^2 - 2(0) - 1 - 0(1) = 2$$

Now that we have our critical points, we can evaluate the function at each of these points to determine the absolute extrema:

$$\begin{aligned} f\left(\tfrac{13}{11}, \tfrac{4}{11}\right) &= \left(\tfrac{13}{11}\right)^2 + 3\left(\tfrac{4}{11}\right)^2 - 2\left(\tfrac{13}{11}\right) - \tfrac{4}{11} - \tfrac{13}{11} \\ &\approx -1.363\ldots \\ f(1,0) &= 1^2 + 3(0)^2 - 2(1) - 0 - 1(0) \\ &= -1 \\ f\left(0, \tfrac{1}{6}\right) &= 0^2 + 3\left(\tfrac{1}{6}\right)^2 - 2(0) - \tfrac{1}{6} - 0 \\ &\approx -0.083\ldots \\ f\left(\tfrac{11}{9}, \tfrac{7}{18}\right) &= \left(\tfrac{11}{9}\right)^2 + 3\left(\tfrac{7}{18}\right)^2 - 2\left(\tfrac{11}{9}\right) - \tfrac{7}{18} - \tfrac{11}{9} \\ &\approx -1.361\ldots \end{aligned}$$

Thus, this gives us 7 critical points:

Point	$f(x,y)$	**Type**
$\left(\frac{13}{11}, \frac{4}{11}\right)$	-1.364	Interior CP
$(1,0)$	-1	ℓ_1
$\left(0, \frac{1}{6}\right)$	-0.083	ℓ_2
$\left(\frac{11}{9}, \frac{7}{18}\right)$	-1.361	ℓ_3
$(0,0)$	0	Vertex 1
$(2,0)$	-2	Vertex 2
$(0,1)$	2	Vertex 3



With these values, we can see that the absolute maximum is $\boxed{2}$, which occurs at the vertex $(0,1)$, and the absolute minimum is $\boxed{-1.364}$, which occurs at the critical point $\left(\frac{13}{11}, \frac{4}{11}\right)$.

Note that the points that I labeled as critical points should just be labeled as points. The critical points are the ones that are in the interior of the region.

2. (1 point each) Convert each as indicated; leave each answer as exact:

(a) Convert the rectangular point $(-5, 1)$ to polar coordinates.

Solution.

$$r = \sqrt{(-5)^2 + 1^2} = \sqrt{26},$$
$$\theta = \arctan\left(\frac{1}{-5}\right) = \arctan\left(-\frac{1}{5}\right) = \frac{7\pi}{6}.$$

The polar coordinates are $\boxed{\left(\sqrt{26}, \frac{7\pi}{6}\right)}$.

(b) Convert the cylindrical point $(5, \frac{7\pi}{6}, 2)$ to rectangular.

Solution.

$$x = 5\cos\left(\frac{7\pi}{6}\right) = 5\left(-\frac{\sqrt{3}}{2}\right) = -\frac{5\sqrt{3}}{2},$$
$$y = 5\sin\left(\frac{7\pi}{6}\right) = 5\left(-\frac{1}{2}\right) = -\frac{5}{2},$$
$$z = 2.$$

The rectangular coordinates are $\boxed{\left(-\frac{5\sqrt{3}}{2}, -\frac{5}{2}, 2\right)}$.

(c) Convert the rectangular point $(-2, 4, -1)$ to spherical.

Solution.

$$\rho = \sqrt{(-2)^2 + 4^2 + (-1)^2} = \sqrt{21}$$
$$\theta = \arctan\left(\frac{4}{-2}\right) = \arctan(-2)$$
$$\phi = \arccos\left(\frac{-1}{\sqrt{21}}\right) = \arccos\left(-\frac{1}{\sqrt{21}}\right).$$

Since the point $(-2, 4)$ is in the second quadrant, we add π to the arctan value. Hence, the spherical coordinates are $\boxed{\left(\sqrt{21}, \pi + \arctan(-2), \arccos\left(-\frac{1}{\sqrt{21}}\right)\right)}$.

(d) Convert the spherical point $(4, \frac{11\pi}{6}, \frac{3\pi}{4})$ to cylindrical.

Solution. The conversion from spherical to cylindrical follows the following equations:

$$r = \rho\sin\phi, \quad \theta = \theta, \quad \text{and} \quad z = \rho\cos\phi.$$

Thus, we have:

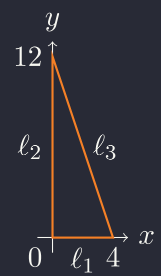
$$\begin{aligned}
r &= 4\sin\left(\tfrac{3\pi}{4}\right) = 4\left(\tfrac{\sqrt{2}}{2}\right) = 2\sqrt{2}\\
\theta &= \tfrac{11\pi}{6}\\
z &= 4\cos\left(\tfrac{3\pi}{4}\right) = 4\left(-\tfrac{\sqrt{2}}{2}\right) = -2\sqrt{2}.
\end{aligned}$$

Therefore, we get the cylindrical coordinates $\boxed{\left(2\sqrt{2}, \tfrac{11\pi}{6}, -2\sqrt{2}\right)}$.

3. (3 points) Determine the value of each given integral. You need to do the work here by hand, but of course can check any answers with technology.

(a) $\displaystyle\iint_D (x^2 + 6xy)\, dA$ where D is the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 12)$.

Solution. We can see that this triangle is bounded by three lines:



$$\begin{aligned}
\ell_1 &: y = 0\\
\ell_2 &: x = 0\\
\ell_3 &: y = -3x + 12
\end{aligned}$$

This gives us the limits of integration as follows:

$$\{(x, y)\colon 0 \le x \le 4, \quad 0 \le y \le -3x + 12\}.$$

Thus, we can write the double integral as:

$$\begin{aligned}
\iint_D (x^2 + 6xy)\, dA &= \int_0^4 \int_0^{-3x+12} (x^2 + 6xy)\, dy\, dx\\
&= \int_0^4 \left[x^2 y + 3xy^2\right]_0^{-3x+12} dx\\
&= \int_0^4 \left[x^2(-3x + 12) + 3x(-3x + 12)^2\right] dx\\
&= \int_0^4 \left[-3x^3 + 12x^2 + 3x(9x^2 - 72x + 144)\right] dx\\
&= \int_0^4 \left[24x^3 - 204x^2 + 432x\right] dx\\
&= 6\int_0^4 \left[4x^3 - 34x^2 + 72x\right] dx\\
&= 6\left[x^4 - \tfrac{34}{3}x^3 + 36x^2\right]_0^4\\
&= 48\left[32 - \tfrac{34}{3}(8) + 36(2)\right]\\
&= \boxed{640}
\end{aligned}$$

(b) $\displaystyle\int_0^2 \int_{x^2}^4 4x^3 \cos(y^3)\, dy\, dx$

Solution. To evaluate this integral, we must change the order of integration. The original region is:

$$\{(x,y)\colon 0 \le x \le 2, \quad x^2 \le y \le 4\}.$$

The new region is:

$$\{(x,y)\colon 0 \le y \le 4, \quad 0 \le x \le \sqrt{y}\}.$$

Thus, we can rewrite and solve the double integral:

$$\begin{aligned}
\int_0^2 \int_{x^2}^4 4x^3 \cos(y^3)\, dy\, dx &= \int_0^4 \int_0^{\sqrt{y}} 4x^3 \cos(y^3)\, dx\, dy \\
&= \int_0^4 \left[x^4 \cos(y^3)\right]_0^{\sqrt{y}} dy \\
&= \int_0^4 y^2 \cos(y^3)\, dy \\
&= \left[\tfrac{1}{3}\sin(y^3)\right]_0^4 \\
&= \tfrac{1}{3}\left[\sin(64) - 0\right] \\
&= \boxed{\tfrac{1}{3}\sin(64)}.
\end{aligned}$$

No problems like this where we have to switch the order of integration on the exam.

Convert to spherical coordinates. Gives

REST ON PAPER

Find the value of $\iint_D x^2 y\, dA$ where D is the region between the x-axis and the upper semicircle$y = \sqrt{9 - x^2}$.
and
Let E be the part of the unit sphere which lives in the first octant. Determine the exact value of $\iiint_E (x^2 z + y^2 z + z^3)\, dV$.
and
Consider the cylinder, with central axis along the y-axis, of radius 3 between the xy-plane and the plane $y = 10$. The temperature, T, at any point inside the cylinder is given by $T(x,y,z) = x^2 y z^2$. Determine the average temperature inside the cylinder.