

Multivariable Calculus Practice Set III

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March 17, 2025

1. (3 points) Determine the absolute extrema for the function $f(x, y) = x^2 + 3y^2 - 2x - y - xy$ on the triangular region with vertices $(0, 0)$, $(2, 0)$, and $(0, 1)$.

Solution. We first find the critical points of the function:

$$\begin{aligned}\nabla f(x, y) &= \langle 2x - 2 - y, 6y - 1 - x \rangle = \mathbf{0} \\ \implies y &= 2x - 2 \quad \text{and} \quad x = 6(2x - 2) - 1 - x \\ \implies y &= \frac{4}{11} \quad \text{and} \quad x = \frac{13}{11}\end{aligned}$$

This gives the critical point $\left(\frac{13}{11}, \frac{4}{11}\right)$. We also need to check the boundary of the region. Thus:

(ℓ_1) : $y = 0, 0 \leq x \leq 2 \implies f(x, y) = g(x) = x^2 + 3(0)^2 - 2x - (0) - x(0) = x^2 - 2x \implies g'(x) = 2x - 2$.
Therefore, the critical points are $\boxed{(1, 0)}$.

(ℓ_2) : $x = 0, 0 \leq y \leq 1 \implies f(x, y) = h(y) = (0)^2 + 3y^2 - (0) - y - 0 = 3y^2 - y \implies h'(y) = 6y - 1$.
Hence, the critical points are $\boxed{\left(0, \frac{1}{6}\right)}$.

(ℓ_3) : $y = 1 - \frac{1}{2}x, 0 \leq x \leq 2 \implies f(x, y) = k(x) = x^2 + 3\left(1 - \frac{1}{2}x\right)^2 - 2x - \left(1 - \frac{1}{2}x\right) - x\left(1 - \frac{1}{2}x\right)$.
Solving this equation for x :

$$\begin{aligned}k(x) &= x^2 + 3\left(1 - \frac{1}{2}x - \frac{1}{2}x + \frac{1}{4}x^2\right) - 2x - 1 + \frac{1}{2}x - x + \frac{1}{2}x^2 \\ &= x^2 + 3\left(1 - x + \frac{1}{4}x^2\right) - 2x - 1 + \frac{1}{2}x - x + \frac{1}{2}x^2 \\ &= \left[x^2 + \frac{3}{4}x^2 + \frac{1}{2}x^2\right] + [-3x - 2x - \frac{1}{2}x] + [3 - 1] \\ &= \frac{9}{4}x^2 - \frac{7}{2}x + 2 \\ &= \frac{1}{4}(9x^2 - 22x + 8) \\ \implies k'(x) &= \frac{1}{4} \cdot \frac{d}{dx}[9x^2 - 22x + 8] \\ 0 &= \frac{1}{2}(9x - 11) \\ x &= \frac{11}{9}\end{aligned}$$

Using this x -value, we plug it back into our equation for y to get the critical point $\boxed{\left(\frac{11}{9}, \frac{7}{18}\right)}$.

With our function's and lines' critical points found, we also need to find the vertices of the triangle:

$$f(0,0) = 0^2 + 3(0)^2 - 2(0) - 0 - 0(0) = 0$$

$$f(2,0) = 2^2 + 3(0)^2 - 2(2) - 0 - 2(0) = 0$$

$$f(0,1) = 0^2 + 3(1)^2 - 2(0) - 1 - 0(1) = 2$$

Now that we have our critical points, we can evaluate the function at each of these points to determine the absolute extrema:

$$\begin{aligned} f\left(\frac{13}{11}, \frac{4}{11}\right) &= \left(\frac{13}{11}\right)^2 + 3\left(\frac{4}{11}\right)^2 - 2\left(\frac{13}{11}\right) - \frac{4}{11} - \frac{13}{11} \\ &\approx -1.363\dots \end{aligned}$$

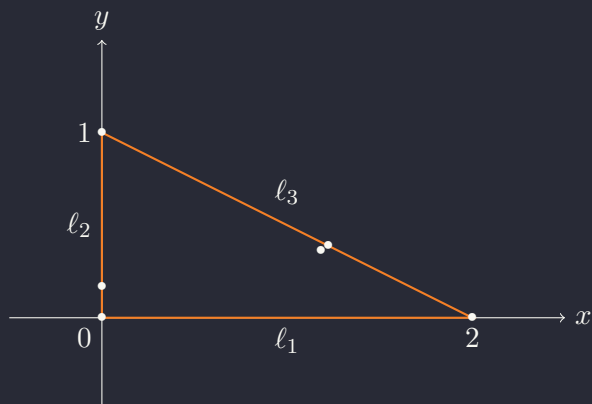
$$\begin{aligned} f(1,0) &= 1^2 + 3(0)^2 - 2(1) - 0 - 1(0) \\ &= -1 \end{aligned}$$

$$\begin{aligned} f\left(0, \frac{1}{6}\right) &= 0^2 + 3\left(\frac{1}{6}\right)^2 - 2(0) - \frac{1}{6} - 0 \\ &\approx -0.083\dots \end{aligned}$$

$$\begin{aligned} f\left(\frac{11}{9}, \frac{7}{18}\right) &= \left(\frac{11}{9}\right)^2 + 3\left(\frac{7}{18}\right)^2 - 2\left(\frac{11}{9}\right) - \frac{7}{18} - \frac{11}{9} \\ &\approx -1.361\dots \end{aligned}$$

Thus, this gives us 7 critical points:

Point	$f(x, y)$	Type
$\left(\frac{13}{11}, \frac{4}{11}\right)$	-1.364	Interior CP
$(1, 0)$	-1	ℓ_1
$\left(0, \frac{1}{6}\right)$	-0.083	ℓ_2
$\left(\frac{11}{9}, \frac{7}{18}\right)$	-1.361	ℓ_3
$(0, 0)$	0	Vertex 1
$(2, 0)$	-2	Vertex 2
$(0, 1)$	2	Vertex 3



With these values, we can see that the absolute maximum is $\boxed{2}$, which occurs at the vertex $(0,1)$, and the absolute minimum is $\boxed{-1.364}$, which occurs at the critical point $\left(\frac{13}{11}, \frac{4}{11}\right)$.

2. (1 point each) Convert each as indicated; leave each answer as exact:

- (a) Convert the rectangular point $(-5, 1)$ to polar coordinates.

Solution.

$$r = \sqrt{(-5)^2 + 1^2} = \sqrt{26}$$

$$\theta = \arctan\left(\frac{1}{-5}\right) = \arctan\left(-\frac{1}{5}\right) = \frac{7\pi}{6}$$

Thus, the polar coordinates are $\boxed{\left(\sqrt{26}, \frac{7\pi}{6}\right)}$.

- (b) Convert the cylindrical point $(5, \frac{7\pi}{6}, 2)$ to rectangular.

Solution.

$$x = 5 \cos\left(\frac{7\pi}{6}\right) = 5\left(-\frac{\sqrt{3}}{2}\right) = -\frac{5\sqrt{3}}{2}$$

$$y = 5 \sin\left(\frac{7\pi}{6}\right) = 5\left(-\frac{1}{2}\right) = -\frac{5}{2}$$

$$z = 2$$

Thus, the rectangular coordinates are $\boxed{\left(-\frac{5\sqrt{3}}{2}, -\frac{5}{2}, 2\right)}$.

- (c) Convert the rectangular point $(-2, 4, -1)$ to spherical.

Solution.

$$\rho = \sqrt{(-2)^2 + 4^2 + (-1)^2} = \sqrt{4 + 16 + 1} = \sqrt{21}$$

$$\theta = \arctan\left(\frac{4}{-2}\right) = \arctan(-2)$$

$$\phi = \arccos\left(\frac{-1}{\sqrt{21}}\right) = \arccos\left(-\frac{1}{\sqrt{21}}\right)$$

Since the point $(-2, 4)$ is in the second quadrant, we add π to the arctan value. Thus, the spherical coordinates are $\boxed{\left(\sqrt{21}, \pi + \arctan(-2), \arccos\left(-\frac{1}{\sqrt{21}}\right)\right)}$.

- (d) Convert the spherical point $(4, \frac{11\pi}{6}, \frac{3\pi}{4})$ to cylindrical.

Solution. The conversion from spherical to cylindrical follows the following equations:

$$r = \rho \sin \phi, \quad \theta = \theta, \quad \text{and} \quad z = \rho \cos \phi.$$

Thus, we have:

$$r = 4 \sin\left(\frac{3\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

$$\theta = \frac{11\pi}{6}$$

$$z = 4 \cos\left(\frac{3\pi}{4}\right) = 4\left(-\frac{\sqrt{2}}{2}\right) = -2\sqrt{2}$$

Therefore, we get the cylindrical coordinates $\boxed{\left(2\sqrt{2}, \frac{11\pi}{6}, -2\sqrt{2}\right)}$.

3. (3 points) Determine the value of each given integral. You need to do the work here by hand, but of course can check any answers with technology.

(a) $\iint_D (x^2 + 6xy) dA$ where D is the triangle with vertices $(0, 0)$, $(4, 0)$, and $(0, 12)$.

Solution.

(b) $\int_0^2 \int_{x^2}^4 4x^3 \cos(y^3) dy dx.$

Solution.

(c) $\int_{-3}^3 \int_0^{\sqrt{9-x^2}} \sin(5x^2 + 5y^2) dy dx.$

Solution.

(d) $\int_0^1 \int_0^{\sqrt{1-x}} \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2 + y^2 + z^2} dz dy dx.$

Solution.

4. (3 points) Find the volume of the solid described by $x^2 + y^2 \leq 1$, $x \geq 0$, $0 \leq z \leq 4 - y$.

Solution.

5. (3 points) Find the average value of the function $f(x, y) = x \sin(y)$ over the region enclosed by $y = 0$, $y = x^2$, and $x = 1$.

Solution.

6. (3 points) Find the volume of the solid that lives within both the cylinder $x^2 + y^2 = 1$ and sphere $x^2 + y^2 + z^2 = 9$.

Solution. Use cylindrical coordinates.