



# HENDRIX

COLLEGE

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## Real Analysis

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MATH 350

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## 1.1 Types of Numbers

### Definition 1.1.1

The *natural numbers* contain all positive, non-zero, and non-fractional numbers. Expressed as  $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ . They do not have an additive inverse, but you can add and multiply them.

### Definition 1.1.2

The *integers* contains all non-fractional numbers. Expressed as:  $\mathbb{Z} = \{\dots - 2, -1, 0, 1, 2, \dots\}$ —are known as a Group (more specifically, a “ring”). You can add, multiply, and subtract these numbers.

### Definition 1.1.3

The *rational numbers* contain all numbers, except irrational numbers. Expressed as:  $\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0\}$ —are known as a “Field.” You can add, subtract, multiply, and divide these numbers.

A problem that rational numbers could not explain: The 45, 45, 90 triangle had a hypotenuse of  $\sqrt{2}$ . This did not exist at the time, so it was simply  $c^2 = 2$ . Therefore, new numbers needed to be invented.

### Theorem 1.1.4

There does not exist a rational number  $r$  such that  $r^2 = 2$ .

*Proof.* Suppose there exists a rational number  $r$  such that  $r^2 = 2$ . Since  $r$  is rational, there exists  $p, q \in \mathbb{Z}$  such that  $r = \frac{p}{q}$ . We can assume the  $p$  and  $q$  have no common factors. (If not, we can factor out the common factor.) By our assumption,

$$\begin{aligned} r^2 &= 2 \\ \frac{p^2}{q^2} &= 2 \end{aligned}$$

It follows that,

$$p^2 = 2q^2$$



Such that  $p^2$  is an even number because if  $p$  were odd, then  $p^2$  would be odd. There exists  $x \in \mathbb{Z}$  such that  $p = 2x$ . Recall that  $p^2 = 2q^2$ . Thus

$$\begin{aligned}(2x)^2 &= 2q^2 \\ 4x^2 &= 2q^2 \\ 2x^2 &= q\end{aligned}$$

Thus,  $q^2$  is even. Hence  $q$  is also even. So  $p$  and  $q$  are both divisible by 2. This contradicts that  $p$  and  $q$  have no common factors. Thus, our supposition is false. Therefore, there does not exist a rational number  $r$  such that  $r^2 = 2$   $\square$

So we are going to work with a larger set called the real numbers,  $\mathbb{R}$ .

- $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
- You can:
  - Add,
  - Subtract,
  - Multiply,
  - Divide
- In other words, all field axioms apply.
- Totally ordered set for any  $x, y \in \mathbb{R}$ . Thus, one of these are true:
  1.  $x < y$ ,
  2.  $x > y$ ,
  3.  $x = y$
- Think of it as a number line.
- $\mathbb{Q}$  is dense:
 

If  $a, b \in \mathbb{Q}$  with  $a \neq b$ , there exists  $c \in \mathbb{Q}$  which is between  $a$  and  $b$  such that  $a < c < b$ . One example is  $\frac{a+b}{2}$ .
- $\mathbb{Q}$  is not *complete*, but  $\mathbb{R}$  is.
  - *Complete*: Think, “no gaps.”

## 1.2 Preliminaries

Things to remember from Intro and Discrete.

- $\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup A_3 \cup \dots$



Set Notation	Complement
$x \in A$	$A^c$ (not $\overline{A}$ )
$A \cup B$	$\mathbb{R} \setminus A$
$A \cap B$	

$$\bullet \bigcap_{n=1}^{\infty} A_n = A_1 \cap A_2 \cap \dots$$

### Definition 1.2.1

*De Morgan's Laws* are defined as  $(A \cup B)^c = A^c \cap B^c$  and  $(A \cap B)^c = A^c \cup B^c$ .

## 1.2.1 Infinite Unions and Intersections

For each  $n \in \mathbb{N}$ , define  $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$ . In other words, each subsequent element in the subset will start at  $n$ . For example,  $A_1 = \{1, 2, \dots\}$ , whereas  $A_5 = \{5, 6, \dots\}$ .

$\bigcup_{n=1}^{\infty} A_n = \mathbb{N}$ . To show a number  $\in \mathbb{N}$  belongs in the set  $A_n$ , we can start with that,  $k \in \mathbb{N}$ . Then  $k \in A_k$ . Thus,  $k \in A_k \subseteq \bigcup_{n=1}^{\infty} A_n$ . Therefore,  $\mathbb{N} \subseteq \bigcup_{n=1}^{\infty} A_n$ .

$\bigcap_{n=1}^{\infty} A_n = \emptyset$ . Obviously, we know that the empty set is a subset of  $A_n$ , but to prove that  $\bigcap_{n=1}^{\infty} A_n$  is a subset of the empty set, we should suppose a  $k \in \mathbb{N}$  such that  $k \in \bigcap_{n=1}^{\infty} A_n$ . Notice that  $k \notin \bigcap_{n=1}^{\infty} A_n$ . So,  $\bigcap_{n=1}^{\infty} A_n = \emptyset$ .

## 1.2.2 Functions and Notation

$f: A \rightarrow B$  where  $f$  is a function,  $A$  is a domain, and  $B$  is the co-domain. Thus,  $f(x) = y$  such that  $x \in A$  and  $y \in B$ .

Some definitions to keep in mind

### Definition 1.2.2

The *Dirichlet Function* is defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

### Definition 1.2.3

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$ . If  $E \subseteq \mathbb{R}$ , then  $f(E) = \{f(x) \mid x \in E\}$ .



Example:  $g : \mathbb{R} \rightarrow \mathbb{R}$ , when we say  $y \in g(A)$  implies there exists an  $x$  such that  $g(x) = y$

### Definition 1.2.4

The *Triangle Inequality* is defined as: For any  $a, b \in \mathbb{R}$ ,  $|a + b| \leq |a| + |b|$ .

The most common application: For any  $a, b, c \in \mathbb{R}$ ,  $|a - b| \leq |a - c| + |c - b|$ , with the intermediate step of  $a - b = (a - c) + (c - b)$ .

### Definition 1.2.5

A function  $f$  is *injective* (or *one-to-one*) if  $a_1 \neq a_2$  in  $A$  implies that  $f(a_1) \neq f(a_2)$  in  $B$ . Note the contrapositive of this definition: If  $f(a_1) = f(a_2)$ , then  $a_1 = a_2$ .

### Definition 1.2.6

A function  $f$  is *surjective* (or *onto*) if for every  $b \in B$ , there exists an  $a \in A$  such that  $f(a) = b$ . Note the contrapositive of this definition: If there exists a  $b \in B$  such that there is no  $a \in A$  such that  $f(a) = b$ , then the function is not surjective.

## 1.2.3 Common Strategies for Analysis Proofs

### Theorem 1.2.6

Let  $a, b \in \mathbb{R}$ . Then,

$$a = b \text{ if and only if for all } \epsilon > 0, |a - b| < \epsilon.$$

*Proof.* We will show this by proving both implications:

|  $(\Rightarrow)$  Assume  $a = b$ . Let  $\epsilon > 0$ . Then  $|a - b| = 0 < \epsilon$

|  $(\Leftarrow)$  Assume for all  $\epsilon > 0$ ,  $|a - b| < \epsilon$ . Suppose  $a \neq b$ . Then  $a - b \neq 0$ . So,  $|a - b| \neq 0$ . Now, Consider  $\epsilon_0 = |a - b|$ . By our assumption we know that  $|a - b| < \epsilon_0$ . It is not true that  $|a - b| < |a - b|$ . Therefore, it must be the case that  $a = b$ .

Therefore, by showing both sides of the implication accomplish the same thing as the other side, we know that  $a = b$  if and only if for all  $\epsilon > 0$ ,  $|a - b| < \epsilon$ .  $\square$

## 1.2.4 Mathematical Induction

*Inductive Hypothesis:* Let  $x_1 = 1$ . For all  $n \in \mathbb{N}$ , let  $x_{n+1} = \frac{1}{2}x_n + 1$ .



*Inductive Step:*  $x_1 = 1, x_2 = 1.5, x_3 = 1.75, x_4 = 1.875$ .

### Example 1.1: Induction

The sequence  $(x_n)$  is increasing. In other words, for all  $n \in \mathbb{N}$ ,  $x_n \leq x_{n+1}$ .

*Proof.* Suppose the sequence  $(x_n)$  is increasing. We will prove this point by using induction.

**Base Case:** We see that  $x_1 = 1$  and  $x_2 = 1.5$ . Thus,  $x_1 \leq x_2$ .

**Inductive Hypothesis:** For  $n \in \mathbb{N}$ , assume  $x_n \leq x_{n+1}$ .

*Scratch work:* We want:  $x_{n+1} \leq x_{n+2}$ . We know:  $x_{n+1} = \frac{1}{2}x_{n+1} + 1$ .

**Inductive Step:** Then  $\frac{1}{2}x_n \leq \frac{1}{2}x_{n+1}$ . Hence,  $\frac{1}{2}x_n + 1 \leq \frac{1}{2}x_{n+1} + 1$ . Therefore we have proven through induction that,  $x_{n+1} \leq x_{n+2}$ .  $\square$

## 1.2.5 Exercises

### Exercise: 1.2.3

Decide which of the following represent true statements about the nature of sets. For any that are false, provide a specific example where the statement in question does not hold.

- (a) If  $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$  are all sets containing an infinite number of elements, then the intersection  $\bigcap_{n=1}^{\infty} A_n$  is infinite as well.
- (b) If  $A_1 \supseteq A_2 \supseteq A_3 \supseteq A_4 \dots$  are all finite, nonempty sets of real numbers, then the intersection  $\bigcap_{n=1}^{\infty} A_n$  is finite and nonempty.
- (c)  $A \cap (B \cup C) = (A \cap B) \cup C$
- (d)  $A \cap (B \cap C) = (A \cap B) \cap C$
- (e)  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

*Solution.*

- (a) This is false. Consider the following as a counterexample: If we define  $A_1$  as  $A_n = \{n, n+1, n+2, \dots\} = \{k \in \mathbb{N} \mid k \geq n\}$ , we can see why the intersection of these sets of infinite numbers are actually empty. Consider a number  $m$  that actually satisfies  $m \in A_n$  for every  $A_n$  in our collection of sets. Because  $m$  is not



an element of  $A_{m+1}$ , no such  $m$  exists and the intersection is empty.

- (b) This is true.
- (c) False. Consider sets  $A = \{1, 2, 3\}$ ,  $B = \{3, 6, 7\}$  and  $C = \{5\}$ . Note that  $A \cap (B \cup C) = \{3\}$  is not equal to  $(A \cap B) \cup C = \{3, 5\}$ .
- (d) This is true. A proof would start with  $x \in A \cap (B \cap C)$ .
- (e) This is true. A proof would start with  $x \in A \cap (B \cup C)$ .





### Exercise: 1.2.5

**De Morgan's Laws** Let  $A$  and  $B$  be subsets of  $\mathbb{R}$ .

- (a) If  $x \in (A \cap B)^c$ , explain why  $x \in A^c \cup B^c$ . This shows that  $(A \cap B)^c \subseteq A^c \cup B^c$ .
- (b) Prove the reverse inclusion  $(A \cap B)^c \supseteq A^c \cup B^c$ , and conclude that  $(A \cap B)^c = A^c \cup B^c$ .
- (c) Show  $(A \cup B)^c = A^c \cap B^c$  by demonstrating inclusion both ways.

*Solution.*

- (a) If  $x \in (A \cap B)^c$ , and we know that  $A^c = \{x \in \mathbb{R} : x \notin A\}$ , then we know  $x$  must not exist in  $A$  and  $B$  because  $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$ . Thus,  $x$  is in either  $A^c$  or  $B^c$ . Put another way  $x \in A^c \cup B^c$ . Since we have shown that an element that started in  $(A \cap B)^c$  ended up in  $A^c \cup B^c$ , then we know  $(A \cap B)^c \subseteq A^c \cup B^c$ .
- (b) Assume there exists a  $y \in A^c \cup B^c$ . Thus, it must be the case that  $y \notin A$  or  $y \notin B$ . Hence,  $y$  cannot exist in both sets at the same time, so  $y \in (A \cap B)^c$ . Because we have taken an element that started in  $A^c \cup B^c$  and have shown that it exists in  $(A \cap B)^c$ , we have proven  $A^c \cup B^c \subseteq (A \cap B)^c$ .
- (c) *Proof.* We need to show these expressions are subsets of each other in order to prove they are equivalent.

( $\subseteq$ ) Assume there exists  $x \in (A \cap B)^c$ , and we know that  $A^c = \{x \in \mathbb{R} : x \notin A\}$ , then we know  $x$  must not exist in  $A$  and  $B$  because  $(A \cap B)^c = \{x \in \mathbb{R} : x \notin (A \cap B)\}$ . Thus,  $x$  is in either  $A^c$  or  $B^c$ . Put another way  $x \in A^c \cup B^c$ . Since we have shown that an element that started in  $(A \cap B)^c$  ended up in  $A^c \cup B^c$ , then we know  $(A \cap B)^c \subseteq A^c \cup B^c$ .

( $\supseteq$ ) Now assume there exists a  $y \in A^c \cup B^c$ . Thus, it must be the case that  $y \notin A$  or  $y \notin B$ . Hence,  $y$  cannot exist in both sets at the same time, so  $y \in (A \cap B)^c$ . Because we have taken an element that started in  $A^c \cup B^c$  and have shown that it exists in  $(A \cap B)^c$ , we have proven  $A^c \cup B^c \subseteq (A \cap B)^c$ .

Therefore, we have shown through proving both sides of the implication, that these two statements are logically equivalent. In that, all elements of  $A^c \cup B^c$  are the same elements that are in  $(A \cap B)^c$   $\square$


**Exercise: 1.2.7**

Given a function  $f$  and a subset  $A$  of its domain, let  $f(A)$  represent the range of  $f$  over the set  $A$ ; that is,  $f(A) = \{f(x) : x \in A\}$ .

- Let  $f(x) = x^2$ . If  $A = [0, 2]$  (the closed interval  $\{x \in \mathbb{R} : 0 \leq x \leq 2\}$ ) and  $B = [1, 4]$ , find  $f(A)$  and  $f(B)$ . Does  $f(A \cap B) = f(A) \cap f(B)$  in this case? Does  $f(A \cup B) = f(A) \cup f(B)$ ?
- Find two sets  $A$  and  $B$  for which  $f(A \cap B) \neq f(A) \cap f(B)$ .
- Show that, for an arbitrary function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , it is always true that  $g(A \cap B) \subseteq g(A) \cap g(B)$  for all sets  $A, B \subseteq \mathbb{R}$ .
- Form and prove a conjecture about the relationship between  $g(A \cup B)$  and  $g(A) \cup g(B)$  for an arbitrary function  $g$ .

*Solution.*

- Since  $f(x) = x^2$ , the intervals of  $f(A)$  would be  $[0, 4]$  and  $f(B)$  would be  $[1, 16]$ . The interval of the intersection of  $A \cap B$  is  $[1, 2]$ . Take this through our function, we get  $f(A \cap B) = [1, 4]$ . On the other side of the equation, we already know the intervals of  $f(A)$  and  $f(B)$ , and the intersection of theirs would be  $[1, 4]$ . So they do equal each other. We know  $f(A \cup B)$  and  $f(A) \cup f(B)$  will be equivalent because  $f(A \cup B)$  has an interval of  $[0, 16]$ , and  $f(A) \cup f(B)$  also has an interval of  $[0, 16]$  because taking the union of  $[0, 4] \cup [1, 16]$  is  $[0, 16]$ .
- Two sets could be  $A = [5, 6]$  and  $B = [0, 0]$ . Because the sets have nothing in common even after taking their function, they do not equal each other.
- Proof.* Let  $x \in g(A \cap B)$ . Using the definition of function, we know there exists a  $y \in A \cap B$  to which that  $y$  is mapped to as  $g(y) = x$ . From the definition of intersection, we know  $y \in A$  and  $y \in B$  such that  $x = g(y) \in g(A)$  and  $x = g(y) \in g(B)$  because  $y \in A \cap B$ . Putting it together, we have  $x \in g(A) \cap g(B)$  thus proving  $g(A \cap B) \subseteq g(A) \cap g(B)$  □
- Conjecture: For any function  $g$  defined as  $g : \mathbb{R} \rightarrow \mathbb{R}$  and for any subsets  $A, B \subseteq \mathbb{R}$ , the following holds:

$$g(A \cup B) = g(A) \cup g(B)$$



*Proof.* We need to show these expressions are subsets of each other in order to prove they are equivalent.

( $\subseteq$ ) Take any element  $x \in g(A \cap B)$ . By definition of function, we know there exists some  $y \in A \cup B$  such that  $g(y) = x$ . From the definition of union, we know  $y \in A$  or  $y \in B$  such that  $x = g(y) \in g(A)$  or  $x = g(y) \in g(B)$  or both. Putting it together, we have  $x \in g(A) \cup g(B)$  thus proving  $g(A \cap B) \subseteq g(A) \cup g(B)$ .

( $\supseteq$ ) Take any element  $p \in g(A) \cup g(B)$ . By definition of union, we know  $p$  is either in  $g(A)$  or  $g(B)$  or both. From the definition of function, we know that if  $p \in g(A)$  or  $p \in g(B)$  then there exists some  $q \in A$  or  $q \in B$  such that  $g(q) = p$ . Putting it together, we have  $q \in A \cup B$ . Moreover, this means  $p = g(q) \in g(A \cup B)$ . And since  $p \in g(A) \cup g(B)$  implies  $p \in g(A \cup B)$ , we know  $g(A) \cup g(B) \subseteq g(A \cup B)$ .

Therefore, since we have proven that both expressions are functions of each other, we have proved that they are equal.  $\square$

### Exercise: 1.2.8

Given a function  $f : A \rightarrow B$  can be defined as either **injective** or **surjective**, give an example of each or state that the request is impossible:

- (a)  $f : \mathbb{N} \rightarrow \mathbb{N}$  that is 1-1 but not onto.
- (b)  $f : \mathbb{N} \rightarrow \mathbb{N}$  that is onto but not 1-1.
- (c)  $f : \mathbb{N} \rightarrow \mathbb{Z}$  that is 1-1 and onto.

*Solution.*

- (a) The function  $f(a) = a + 1$  is 1-1 because when

$$\begin{aligned} f(a_1) &= f(a_2) \\ a_1 + 1 &= a_2 + 1 \\ a_1 &= a_2 \end{aligned}$$

However, the function is not onto because the entire co-domain is not covered. That being 1.

- (b) We need to find a function that will cover every entry in the co-domain, while also



avoiding a scenario where  $a_1 = a_2 \dots$ . Consider the function,

$$f(a) = \begin{cases} a & \text{if } a \text{ is odd,} \\ a - 1 & \text{if } a \text{ is even} \end{cases}$$

This function is onto because every natural number is covered, but it is not 1-1 because  $a_1 \neq a_2 - 1$ .

- (c) This request is not possible. There is no way to map every natural number to every integer because we are simply missing 0! (Not 0 factorial, we do have the number 1, I just mean the number 0 in an exclamatory sense.)

### Exercise: 1.2.13

For this exercise, assume Exercise 1.2.5 has been successfully completed.

- (a) Show how induction can be used to conclude that

$$(A_1 \cup A_2 \cup \dots \cup A_n)^c = A_1^c \cap A_2^c \cap \dots \cap A_n^c$$

for any finite  $n \in \mathbb{N}$ .

- (b) It is tempting to appeal to induction to conclude

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c,$$

but induction does not apply here. Induction is used to prove that a particular statement holds for every value of  $n \in \mathbb{N}$ , but this does not imply the validity of the infinite case. To illustrate this point, find an example of a collection of sets  $B_1, B_2, B_3, \dots$  where

$$\bigcap_{i=1}^n B_i \neq \emptyset \quad \text{is true for every } n \in \mathbb{N},$$

but

$$\bigcap_{i=1}^{\infty} B_i = \emptyset$$

fails.

- (c) Nevertheless, the infinite version of De Morgan's Law stated in (b) is a valid statement. Provide a proof that does not use induction.



*Proof.* In this proof, we plan to prove (c). Thus, we need to show that:

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c$$

and

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

( $\subseteq$ ) Let  $x \in \left( \bigcup_{i=1}^{\infty} A_i \right)^c$ . This means  $x$  is in the union set of  $A_i$  for all  $i \in \mathbb{N}$ . Then, because we are taking the complement of  $\left( \bigcup_{i=1}^{\infty} A_i \right)$ , that means  $x \notin A_i$  for all  $i \in \mathbb{N}$ . Hence,  $x$  is in the complement of each  $A_i$ . Thus, we can use the definition of intersection to assert  $x \in \bigcap_{i=1}^{\infty} A_i^c$ . Therefore, we have shown:

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c \subseteq \bigcap_{i=1}^{\infty} A_i^c.$$

( $\supseteq$ ) Similar to before, let  $x \in \bigcap_{i=1}^{\infty} A_i^c$ . Because  $x \in A_i^c$  for all  $i \in \mathbb{N}$  we know  $x \notin A_i$ . Hence,  $x \notin \left( \bigcup_{i=1}^{\infty} A_i \right)$ , which means  $x \in \left( \bigcup_{i=1}^{\infty} A_i \right)^c$ . Therefore, we have shown:

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c \supseteq \bigcap_{i=1}^{\infty} A_i^c.$$

By showing both inclusions, we see that:

$$\left( \bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c. \quad \square$$

## 1.3 Axiom of Completeness

### Axiom of Completeness

Every nonempty set of real numbers that is **bounded** has a **least upper bound**.

Think about  $\mathbb{Q}$  and  $\mathbb{R}$ .

- Both are fields.
  - Both have  $+$ ,  $-$ ,  $\times$ ,  $\div$  operations.
- Both are totally ordered



- $a < b$ ,
- $a > b$ ,
- or  $a = b$

- $\mathbb{R}$  is complete.  $\mathbb{Q}$  is not.

### 1.3.1 Least Upper Bounds and Greatest Lower Bounds

#### Definition 1.3.1

A set  $A \subseteq \mathbb{R}$  is *bounded above* if there exists a number  $b \in \mathbb{R}$  such that  $a \leq b$  for all  $a \in A$ . The number  $b$  is called an *upper bound* of  $A$ . Similarly, a set  $A \subseteq \mathbb{R}$  is *bounded below* if there exists a *lower bound*  $l \in \mathbb{R}$  satisfying  $l \leq a$  for every  $a \in A$ .

**Note that upper bounds are not unique!** For example, consider the line,  $A$ , from 0 to 1. There are infinitely many upper bounds past 1 because  $A$  is bounded.

#### Definition 1.3.2

A number  $s$  is a *least upper bound* for a set  $A \subseteq \mathbb{R}$  if it meets the following two criteria:

- (i)  $s$  is an upper bound for  $A$ ;
- (ii) if  $b$  is any upper bound for  $A$ , then  $s \leq b$ .

We often call the least upper bound the *supremum* of a set.

#### Example 1.2: Supremum

Imagine a number line from  $(1, 8)$ . Note that parenthesis mean  $<$  and not  $\leq$ . Hence, the supremum is 8. Wrote simply as  $\sup A$ .

#### Example 1.3: Supremum and Infimum 1

Consider a set,  $B = [-5, -2] \cup (3, 6) \cup \{13\}$ . What is the supremum and the infimum?

*Solution.*  $\sup B = 13$ ;  $\inf B = -5$  because  $-5$  is the greatest lower bound.

#### Example 1.4: Supremum and Infimum 2

Consider the set,  $\mathbb{C} = \{\frac{1}{n} : n \in \mathbb{N}\}$ . What is the supremum and the infimum?



*Solution.*  $\sup \mathbb{C} = 1, \inf \mathbb{C} = 0.$

### Example 1.5: L

Let  $A \subseteq \mathbb{R}$  be nonempty and bounded above, and let  $c \in \mathbb{R}$ . Define the set  $c + A$  by

$$c + A = \{c + a : a \in A\}$$

Then  $\sup(c + A) = c + \sup A$ .

*Solution.* To properly verify this we focus separately on each part of [Definition 1.3.2](#). Setting  $s = \sup A$ , we see that  $a \leq s$  for all  $a \in A$ , which implies  $c + a \leq c + s$  for all  $a \in A$ . Thus,  $c + s$  is an upper bound for  $c + A$  and condition (i) is verified. For (ii), let  $b$  be an arbitrary upper bound for  $c + A$ ; i.e.,  $c + a \leq b$  for all  $a \in A$ . This is equivalent to  $a \leq b - c$  for all  $a \in A$ , from which we conclude that  $b - c$  is an upper bound for  $A$ . Because  $s$  is the least upper bound of  $A$ ,  $s \leq b - c$ , which can be rewritten as  $c + s \leq b$ . This verifies part (ii) of [Definition 1.3.2](#), and we conclude  $\sup(c + A) = c + \sup A$ .

### Definition 1.3.4

A real number  $a_0$  is a *maximum* of the set  $A$  if  $a_0$  is an element of  $A$  and  $a_0 \geq a$  for all  $a \in A$ . Similarly, a number  $a_1$  is a *minimum* of  $A$  if  $a_1$  is an element of  $A$  and  $a_1 \leq a$  for all  $a \in A$ .

Note that some sets have a maximum and some sets do not. You cannot refer to a maximum without first knowing it exists. This is the same with minimums.

### Lemma 1.3.5

Assume  $s$  is an [upper bound](#) for a set  $A \subseteq \mathbb{R}$ . Then,  $s$  is the supremum of  $A$  if and only if for every  $\epsilon > 0$ , there exists  $x \in A$  such that  $s - \epsilon < x$ .

This lemma allows us to take any positive number and take a “step back.” In essence, you can verify something as an upper bound if you continuously back up over and over until you cannot back up any longer.

*Proof.* We show this by proving both implications:

( $\Rightarrow$ ) Assume  $s = \sup A$ . Let  $\epsilon > 0$ . Suppose there are no elements  $x$  of  $A$  such that  $s - \epsilon < x$ . Then  $s - \epsilon$  would be an upper bound. This contradicts that  $s$  is the least upper bound. Therefore, there must exist an element  $x \in A$  such that  $s - \epsilon < x$ .



( $\Leftarrow$ ) Assume for every  $\epsilon > 0$ , there exists  $x \in A$  such that  $s - \epsilon < x$ . Let  $t$  be an upper bound of  $A$ . Suppose  $t < s$ . Consider  $\epsilon_0 = s - t > 0$ . By our assumption, there exists  $x \in A$  such that  $s - \epsilon_0 < x$ . So,  $t < x$ . This contradicts that  $t$  is an upper bound of  $A$ . So,  $t \geq s$ . Thus,  $s$  is the least upper bound

Therefore, by proving both the right and left implication, we have shown the statement to be true.  $\square$

Analogous statement about infimums: Assume  $z$  is a lower bound of a set  $A \subseteq \mathbb{R}$ . Then  $z = \inf A \iff$  for all  $\epsilon > 0$ , there exists  $y \in A$  such that  $y < z + \epsilon$ .

### 1.3.2 Exercises

#### Exercise: 1.3.4

Let  $A_1, A_2, A_3 \dots$  be a collection of nonempty sets each of which is bounded above.

- Find a formula for  $\sup(A_1 \cup A_2)$ . Extend this to  $\sup(\bigcup_{k=1}^n A_k)$ .
- Consider  $\sup(\bigcup_{k=1}^{\infty} A_k)$ . Does the formula in (a) extend to the infinite case?

*Solution.*

- Let  $A_1$  and  $A_2$  be nonempty sets, each bounded above. To find the largest of the two suprema, we can use the following:  $\sup(A_1 \cap A_2) = \max\{\sup A_1, \sup A_2\}$ . If we extend this notion to  $\sup(\bigcup_{k=1}^n A_k)$ , we can use the same idea from before and write it as  $\sup(\bigcup_{k=1}^n A_k) = \max\{\sup A_1, \sup A_2, \dots, \sup A_n\}$ .
- The formula does not extend to the infinite case. Consider the counterexample  $\bigcup_{k=1}^{\infty} A_k$  where  $A_k := [k, k + 1]$ . Even though these sets are bounded above, when we take the union of them, we approach infinity, which is not bounded:  $\bigcup_{k=1}^{\infty} A_k = [1, 2] \cup [2, 3] \cup \dots = [1, \infty)$ .

#### Exercise: 1.3.5

As in Example 1.3.7, let  $A \subseteq \mathbb{R}$  be nonempty and bounded above, and let  $c \in \mathbb{R}$ . This time define the set  $cA = \{ca : a \in A\}$ .

- If  $c \geq 0$ , show that  $\sup(cA) = c \sup A$ .
- Postulate a similar type of statement for  $\sup(cA)$  for the case  $c < 0$ .

*Solution.*

- Let  $A \subseteq \mathbb{R}$  be nonempty and bounded above. Define the set  $cA := \{ca : a \in A\}$ .





From the axiom of completeness, because  $A$  is bounded above, we know there is a least upper bound,  $s = \sup A$ . Following from Example 1.3.7, we see that  $a \leq s$  for all  $a \in A$  which implies  $ca \leq cs$  for all  $a \in A$ . Thus,  $cs$  is an upper bound for  $cA$ , and the first condition of Definition 1.3.2 is satisfied. For the second condition, we need to look at both  $c = 0$  and  $c > 0$  to avoid dividing by zero. So, we have two cases:

- $c = 0$ : If  $c = 0$ , then  $cA = \{0 : a \in A\} = \{0\}$ . Since the only element in  $cA$  is 0,  $\sup(cA) = 0$ . Similarly, because  $c = 0$ ,  $c \sup A = 0 \cdot \sup A = 0$ . Therefore,  $\sup(cA) = c \sup(A)$ .
- $c > 0$ : Let  $b$  be an arbitrary upper bound for  $cA$  and  $c > 0$ . In other words,  $ca \leq b$  for all  $a \in A$ . This is equivalent to  $a \leq b/c$  where  $c \neq 0$ , from which we can see that  $b/c$  is an upper bound for  $A$ . Because  $s$  is the least upper bound of  $A$ ,  $s \leq b/c$ , which can be rewritten as  $cs \leq b$ . This verifies the second part of Definition 1.3.2, and we conclude  $\sup(cA) = c \sup A$ .

(b) Postulate: If  $c < 0$ , then  $\sup(cA) = c \inf(A)$ .

### Exercise: 1.3.8

Compute, without proofs, the suprema and infima (if they exist) of the following sets:

- (a)  $\left\{ \frac{m}{n} : m, n \in \mathbb{N} \text{ with } m < n \right\}$ .
- (b)  $\left\{ \frac{(-1)^m}{n} : m, n \in \mathbb{N} \right\}$ .
- (c)  $\left\{ \frac{n}{3n+1} : n \in \mathbb{N} \right\}$ .
- (d)  $\left\{ \frac{m}{m+n} : m, n \in \mathbb{N} \right\}$ .

*Solution.* To avoid writing out every set definition, I am going to denote each set as  $A_n$  where  $n$  corresponds to the numerical value of the list from (a) - (d).

- (a)  $\sup A_1 = 1, \inf A_1 = 0$
- (b)  $\sup A_2 = 1, \inf A_2 = -1$
- (c)  $\sup A_3 = \frac{1}{3}, \inf A_3 = \frac{1}{4}$
- (d)  $\sup A_4 = 1, \inf A_4 = 0$



## 1.4 Consequences of Completeness

### Theorem 1.4.1: Nested Interval Property

For each  $n \in \mathbb{N}$ , assume we are given a closed interval  $I_n = [a_n, b_n]$ . Assume  $I_n$  contains  $I_{n+1}$ . This results in a nested sequence of intervals.

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \dots$$

Then,  $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$ .

**tl;dr** there has to be something that is common to all of the sets.

*Proof.* Notice that the sequence,  $a_1, a_2, a_3, \dots$  is increasing. In other words, for each  $n \in \mathbb{N}$ , since  $I_n \supset I_{n+1}$  we have  $a_n \leq a_{n+1}$ . If we consider the set  $A = \{a_n : n \in \mathbb{N}\}$ . The element  $b_1$  is an upper bound of  $A$ . (Note that  $b_1$  and  $a_1$  corresponds to the endpoints of the first set,  $I_1$ . Think of this as a tornado looking structure where the larger the  $I_n$ , the smaller the number line.) For each  $n \in \mathbb{N}$ ,  $a_n \leq b_n \leq b_1$ .

Since  $A$  has an upper bound, it must have a least upper bound. Hence, let  $\alpha = \sup A$ . We claim that  $\alpha \in \bigcap_{n=1}^{\infty} I_n$ . We said  $b_1$  was an upper bound. In fact, every  $b_n$  is an upper bound of  $A$ . Choose any  $n, m \in \mathbb{N}$ . We want to show that  $a_n \leq b_m$ . Consider the following cases:

**Case 1:** If  $n < m$ , then  $a_n \leq a_m \leq b_m$ . (Think: two number lines stacked on top of each other. The top number line is larger, call it  $I_n$  and it has  $a_n$  and  $b_n$  as endpoints. Consider a contained line ( $I_n \supseteq I_m$ ) that is smaller, and has endpoints  $a_m$  and  $b_m$ .)

**Case 2:** If  $n > m$ , then  $a_n \leq b_n \leq b_m$ . So every  $b_n$  is an upper bound of  $A$ .

Hence,

- Because  $\alpha = \sup A$ , we have  $\alpha \geq a_n$ .
- Since  $b_n$  is an upper bound of  $A$ , we have  $\alpha \leq b_n$ .

so,  $\alpha \in [a_n, b_n] = I_n$ . Thus,  $\alpha \in \bigcap_{n=1}^{\infty} I_n$ . □

Nested, closed, Bounded Intervals  $\Rightarrow$  non-empty intersection.

### Theorem 1.4.2: Archimedean Principle

1. Given any number  $x \in \mathbb{R}$ , there exists an  $n \in \mathbb{N}$  satisfying  $n > x$ .
2. Given any real number  $y > 0$ , there exists an  $n \in \mathbb{N}$  satisfying  $1/n < y$ .



*Proof.* 1. If  $\mathbb{N}$  was bounded, then we can let  $s \in \mathbb{N} = \sup \mathbb{N}$ . However, we know that there is always a higher number (e.g.,  $n + 1$ ) for any  $n \in \mathbb{N}$  that is given. Thus, by contradiction, there must exist  $n \geq x$ .

2. For any  $x > 0$ , there exists  $n \in \mathbb{N}$  such that  $\frac{1}{n} < x$ .

□

### Theorem 1.4.3: Density of the Rationals in the Reals

For any  $a, b \in \mathbb{R}$  with  $a < b$ , there exists  $q \in \mathbb{Q}$  such that  $a < q < b$ .

*Proof.* Since  $b - a > 0$ , there exists  $n \in \mathbb{N}$  such that  $\frac{1}{n} < b - a$ . From the **Archimedean Principle**, since  $a \times n \in \mathbb{R}$ , there exists  $m \in \mathbb{N}$  such that  $a \times n < m$ . Let  $m$  be the smallest such natural numbers (by the well ordered principle). Since  $m$  is the smallest such natural number, it follows that  $m - 1 \leq a \times n < m$ . We then see that  $a < \frac{m}{n}$ . Now, we need to find some  $\frac{m}{n} < b$ .

$$\begin{aligned} m - 1 &\leq a \times n \\ m &\leq a \times n + 1 \\ \frac{m}{n} &\leq a + \frac{1}{n} \\ \frac{m}{n} &< a + (b - a) \\ \frac{m}{n} &< b \end{aligned}$$

We now have that  $a < \frac{m}{n} < b$  so  $\frac{m}{n}$  is a rational number in  $(a, b)$

□

### Exercise: 1.4.1

Recall that  $\mathbb{I}$  stands for the set of irrational numbers.

1. Show that if  $a, b \in \mathbb{Q}$ , then  $ab$  and  $a + b$  are elements of  $\mathbb{Q}$  as well.
2. Show that if  $a \in \mathbb{Q}$  and  $t \in \mathbb{I}$ , then  $a + t \in \mathbb{I}$  and  $at \in \mathbb{I}$  as long as  $a \neq 0$ .
3. Part (a) can be summarized by saying that  $\mathbb{Q}$  is closed under addition and multiplication. Is  $\mathbb{I}$  closed under addition and multiplication? Given two irrational numbers  $s$  and  $t$ , what can we say about  $s + t$  and  $st$ ? In other words, are there two irrational numbers that can be added and multiplied such that you get a number  $x$  such that  $x \notin \mathbb{I}$ .

*Solution.*



1. Let  $a, b \in \mathbb{Q}$ . This means there exists some  $p, q, a, b \in \mathbb{Z}$  such that

$$a = \frac{p}{q}$$

and

$$b = \frac{a}{b}$$

where  $q, b \neq 0$ . The product of these numbers is

$$ab = \frac{p}{q} \cdot \frac{a}{b} = \frac{pa}{qb}.$$

Since  $pa, qb \in \mathbb{Z}$ ,  $ab \in \mathbb{Q}$ . The sum of these numbers is

$$a + b = \frac{p}{q} + \frac{a}{b} = \frac{pb + aq}{qb}.$$

Since  $pb + aq, qb \in \mathbb{Z}$ ,  $a + b \in \mathbb{Q}$ .

2. Let  $a \in \mathbb{Q}$  and  $t \in \mathbb{I}$ . Assume, for contradiction, that  $a + t \in \mathbb{Q}$ . This would imply  $t = (a + t) - a$  (because we can subtract  $t + a$  from the original equation and rearrange terms). Since  $a + t, a \in \mathbb{Q}$  their sum would be rational because the rational numbers are closed under addition. However, that would contradict the assumption that  $t \in \mathbb{I}$ . Hence,  $a + t \in \mathbb{I}$ .
3. For  $\mathbb{I}$ , it is not closed under addition and multiplication. Consider the following counterexample:  $\sqrt{2} + (-\sqrt{2}) = 0$  which is not in the irrationals. For multiplication, consider  $\sqrt{2} \cdot \sqrt{2} = 2$ , which is also not in the irrationals.

## 1.5 Cardinality

Two sets have the same *cardinality* if there exists a bijection between them. Thus, the natural numbers, the integers, and the rational numbers have the same cardinality. A set is *countably infinite* if it has the same cardinality as  $\mathbb{N}$ . (If it can be put into one-to-one correspondence with  $\mathbb{N}$ .) A set is *countable* if it is countably infinite or finite.

### Theorem 1.5.6

$\mathbb{R}$  is not countable.

*Proof.* 1 (most common)

Suppose  $\mathbb{R}$  is countable. Then we can list them all, or we can enumerate them.  $\mathbb{R} = \{x_1, x_2, x_3, x_4, \dots\}$ . We can write the decimal expansion of each of these. Consider the following table:



$x_1 =$	$\boxed{a_{10}}$	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	$\dots$
$x_2 =$	$a_{20}$	$\boxed{a_{21}}$	$a_{22}$	$a_{23}$	$a_{24}$	$\dots$
$x_3 =$	$a_{30}$	$a_{31}$	$\boxed{a_{32}}$	$a_{33}$	$a_{34}$	$\dots$
$x_4 =$	$a_{40}$	$a_{41}$	$a_{42}$	$\boxed{a_{43}}$	$a_{44}$	$\dots$
$x_5 =$	$a_{50}$	$a_{51}$	$a_{52}$	$a_{53}$	$\boxed{a_{54}}$	$\dots$
$x_6 =$	$a_{60}$	$a_{61}$	$a_{62}$	$a_{63}$	$a_{64}$	$\dots$

We will now construct a number that is not in this list. Focus on diagonal entries. For each  $n \in \mathbb{N}$ , let  $b_n$  be a digit that is different from  $a_{nn}$ . Now consider the number  $y = 0.b_1b_2b_3b_4b_5\dots$ . This number  $y$  is not in our list. So our list did not include all of  $\mathbb{R}$ . Avoid repeating 9s.  $\square$

*Proof. 2* (uses nested interval theorem)

Suppose  $\mathbb{R}$  is countable. Then we can enumerate  $\mathbb{R}$   $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$ . Let  $I_1$  be any closed interval that does not contain  $x_1$ . Next, we will find another closed interval  $I_2$  that:

- $I_2 \subseteq I_1$
- $x_2 \notin I_2$

Continue in this fashion creating a sequence of nested closed intervals:  $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$  such that for all  $k \in \mathbb{N}$ ,  $x_k \notin I_k$ . Now consider:

$$\bigcap_{n=1}^{\infty} I_n$$

- For each  $k \in \mathbb{N}$ , since  $x_k \notin I_k$ , we see  $x_k \notin \bigcap_{n=1}^{\infty} I_n$ .
- By the nested interval theorem, there exists  $x \in \mathbb{R}$  such that  $x \in \bigcap_{n=1}^{\infty} I_n$ . So  $x$  is a real number that is not included in our list.  $\square$

### Theorem 1.5.7

A countable collection of finite sets is *countable*.

### Theorem 1.5.8

- The union of two countable sets is *countable*.
- A countable union of countable sets is *countable*.

From Theorem 1.5.6, we know that  $\mathbb{R}$  is uncountable, but what about  $(0, 1)$ ? It does



have the same cardinality of  $\mathbb{R}$  because we can make a one-to-one and onto function between both the sets. Similarly,  $(a, b)$  also has the same cardinality. What about  $[a, b]$ ?

**Recap:**  $\mathbb{N}$  is countable, and  $\mathbb{R}$  is uncountable and has a different cardinality than  $\mathbb{N}$ . Thus, the question is, do all uncountable sets have the same cardinality as  $\mathbb{R}$ ? The answer is **no**.

### Theorem 1.5.9: Cantor's Theorem

For any set  $A$ , there does not exist an onto map from  $A$  into  $\mathcal{P}$ .

*Proof.* Suppose there exists an onto function,  $f : A \rightarrow \mathcal{P}(A)$ . So each  $a \in A$  is mapped to an element  $f(a) \in \mathcal{P}(A)$ . Then,  $f(a) \subseteq A$ . We are going to construct an element of  $\mathcal{P}(A)$  which is not mapped to by  $f$ .

Consider  $B = \{a \in A : a \notin f(a)\}$ . Since  $f$  is onto there exists  $a' \in A$  such that  $B = f(a')$ . Thus, there are two cases to consider:

- **Case 1:** If  $a' \in B = f(a')$ , then  $a' \notin B$ .
- **Case 2:** If  $a' \notin B = f(a')$ , then  $a' \in B$ .

As evidenced, both cases lead to contradictions, so  $B$  is not the image of any  $a \in A$ . Therefore  $f$  is not onto.  $\square$

### Example 1.6: Set and Power Set Matching

$A = \{a, b, c\}$ .

*Solution.*  $\mathcal{P}(A) = \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$ . Note that you can map  $\{a\}, \{b\}, \{c\}$ , to elements such as  $\emptyset, \{a, b\}, \{a, b, c\}$ , but there are still more elements that are left unmapped. We can extrapolate from our proof a set  $B$  such that  $B = \{a, c\}$  because those elements are not mapped to.

All of this is to show  $\mathcal{P}(\mathbb{R})$  has a larger cardinality than  $\mathbb{R}$ . Then  $\mathcal{P}(\mathcal{P}(\mathbb{R}))$  has a larger cardinality than  $\mathcal{P}(\mathbb{R})$ .

## 2.1 Discussion: Rearrangement of Infinite Series

### Questions:

What is a *sequence*?

A countable, ordered list of elements. An example could be  $1, 2, 3, 4, 5, \dots$ . Note that this is *ordered*, therefore distinguishing it from a sequence like  $3, 1, 2, 4, 5, 6, \dots$ . Hence, order matters.

A *sequence* is a function whose domain is  $\mathbb{N}$ . **Note:** The domain  $\mathbb{N}$  refers to each element's position in the list. For example,  $(a_n) = a_1, a_2, a_3, \dots$

We will focus on the *limit* of a sequence. We use sequences to approximate other things.

### Example 2.1: Sequence

$$3, 3.1, 3.14, 3.141, 3.1415, \dots \approx \pi.$$

$$x, x - \frac{x^3}{6}, x - \frac{x^3}{6} + \frac{x^5}{120}, x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{7!}, \dots \approx \sin(x)$$

### What is a *series*?

An infinite sum. We look at the sequence of partial sums. We ask, do the partial sums approach a limit?

### Example 2.2: Alternating Harmonic Series

$$1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \dots$$

We can rearrange these terms such that we can ‘force’ the series to converge to a specific number. Therefore, we will need to be careful with our definitions.

## 2.2 The Limit of a Sequence

### Definition 2.2.1

A *sequence* is a function whose domain is  $\mathbb{N}$ . We write  $(a_n) = a_1, a_2, a_3, \dots$



### Definition 2.2.3

The sequence  $(a_n)$  *converges* to  $L$  if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - L| < \epsilon$ . In other words, there exists  $N \in \mathbb{N}$  such that

- **(In the interval)**  $a_N \in (L - \epsilon, L + \epsilon)$ .
- **(Stays in the interval)**  $\forall n \geq N, a_n \in (L - \epsilon, L + \epsilon)$ .

### Example 2.3: Limit Proof 1

Let  $a_n = \frac{1}{n}$ .  $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

*Proof.* Our claim is  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ . Thus, let  $\epsilon = .01$ . Does the sequence eventually get inside  $(-.01, .01)$ ? We will set  $N = 101$ . So, for any  $n \geq |0|$ ,

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{101} < .01.$$

From  $A_n$  and on, the sequence stayed within  $\epsilon$  of 0. But what about  $\epsilon = .001$ ,  $\epsilon = .00001$  and so on?

Actual proof let  $\epsilon > 0$ . By the Archimedean Principle, there exists  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon}$ . Now, for any  $n \geq N$ ,

$$\left| \frac{1}{n} \right| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{1/\epsilon}.$$

(Where  $\frac{1}{1/\epsilon} = \epsilon$ , but is in that form for demonstration purposes.) Therefore  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$   $\square$

“To get close” means is that we are finding a bigger and bigger  $N$  as  $\epsilon$  gets smaller. Note that the choice of  $N$  certainly depends on  $\epsilon$ . This idea of “getting close” can be seen in the following definition:

### Definition 2.2.3B

A sequence  $(a_n)$  converges to  $a$  if, given any  $\epsilon$ -neighborhood  $V_\epsilon(a)$  of  $a$ , there exists a point in the sequence after which all of the terms are in  $V_\epsilon(a)$ . In other words, every  $\epsilon$ -neighborhood contains all but a finite number of the terms of  $(a_n)$ .





### 2.2.1 Basic Structure of a Limit Proof

Claim:  $\lim_{n \rightarrow \infty} a_n = L$ .

Proof: Let  $\epsilon > 0$ . There exists  $N \in \mathbb{N}$  such that {something involving  $\epsilon$ }. Assume  $n \geq N$ . Then,

$$|a_n - L| \boxed{\dots} < \epsilon$$

(Where  $\boxed{\dots}$  is going to be where the majority of the work is going to lie.)

#### Example 2.4: Limit Proof 2

Claim:  $\lim_{n \rightarrow \infty} \frac{2n-3}{2n} = 1$

*Proof.* Let  $\epsilon > 0$ . *Scratch paper:* Solve for:

$$\left| \frac{2n-3}{2n} - 1 \right| = \left| \frac{-3}{2n} \right| = \frac{3}{2n} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n.$$

By the **Archimedean Principle**, there exists  $N \in \mathbb{N}$  such that  $N > \frac{3}{2\epsilon}$ . Assume  $n \geq N$ , (want to know what happens past this point)

$$\left| \frac{2n-3}{2n} - 1 \right| \leq \frac{3}{2N} < \frac{3}{2 \cdot 3/2\epsilon} = \epsilon.$$

Therefore,  $\lim_{n \rightarrow \infty} \frac{2n-3}{2n} = 1$

□

#### Example 2.5: Limit Proof 3

Claim:  $\lim_{n \rightarrow \infty} \frac{2n^2+1n^2}{=}$

*Proof.* Let  $\epsilon > 0$ . By the **Archimedean Principle**, there exists  $N \in \mathbb{N}$  such that [leave off] *Scratch paper:* Solve for

$$\left| \frac{2n^2+1}{n^2} - 2 \right| = \frac{2n^2}{n^2} < \epsilon \Rightarrow \frac{3}{2\epsilon} < n$$

[pick up] there exists  $N \in \mathbb{N}$  such that

$$N > \frac{1}{\sqrt{\epsilon}}.$$



Assume  $n \geq N$ , then

$$\begin{aligned}
 \left| \frac{2n^2 + 1}{n^2} - 2 \right| &= \frac{1}{n^2} \\
 &\leq \frac{1}{N^2} \\
 &< \frac{1}{(1/(\sqrt{\epsilon})^2)} \\
 &= \frac{1}{1/\epsilon} \\
 &= \epsilon
 \end{aligned}$$

Therefore,  $\lim_{n \rightarrow \infty} \frac{2n^2 + 1}{n^2} = 2$

□

### Example 2.6: Limit Proof 4

Claim:  $\lim_{n \rightarrow \infty} \frac{7n+8}{3n+6} = \frac{7}{3}$

*Proof.*

$$\begin{aligned}
 \left| \frac{7n+8}{3n+6} - \frac{7}{3} \right| &= \left| \frac{21n+24}{3(3n+6)} - \frac{21n+42}{3(3n+6)} \right| \\
 &= \left| \frac{-18}{9n+18} \right| \\
 &= \frac{18}{9n+18} < \epsilon * * \\
 &= \frac{18}{3} < 9n+18 \\
 &= \frac{18}{3} - 18 < 9n \\
 &= \frac{18/\epsilon - 18}{9} < n
 \end{aligned}$$



\* \*  $\frac{18}{9n+8} < \frac{18}{9n} < \epsilon \Rightarrow \frac{2}{\epsilon} < N$ .  $\exists N \in \mathbb{N}$  such that  $N > \frac{2}{\epsilon}$ . Assume  $n \geq N$ ,

$$\begin{aligned} \left| \frac{7n+8}{3n+6} - \frac{7}{3} \right| &= \frac{18}{9n+18} \\ &= \frac{2}{n+2} \\ &< \frac{2}{n} \\ &\leq \frac{2}{N} \\ &< \frac{2}{\epsilon/2} \\ &= \epsilon \end{aligned}$$

□

Does every sequence have a limit?

### Theorem 2.2.4: Uniqueness of Limits

The limit when it exists, is unique.

*Proof.* Let  $(x_n)$  be a convergent sequence. Suppose  $L$  and  $M$  are limits of this sequence. Without the loss of generality, we are going to assume  $M > L$ . Let

$$\epsilon = \frac{M - L}{3}.$$

Since  $x_n$  converges to  $L$ , there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|x_n - L| < \epsilon$ . Since  $(x_n)$  converges to  $M$ , there exists an  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,  $|x_n - M| < \epsilon$ . Consider  $n = \max\{N_1, N_2\}$ . Since  $n \geq N_1$ ,  $|x_n - L| < \epsilon$ . Since  $n \geq N_2$ ,  $|x_n - M| < \epsilon$ . Then  $L - \epsilon < x_n < L + \epsilon$  and  $M - \epsilon < x_n < M + \epsilon$ . By our choice of  $\epsilon$ , we now have

$$(x_n) < L + \epsilon < M - \epsilon < (x_n).$$

This is a contradiction. Thus,  $(x_n)$  cannot have two different limits. □

### Example 2.7: Limit Proof 5

Let  $(x_n) = \frac{\cos(n)}{3n}$ . Claim:  $\lim_{n \rightarrow \infty} (x_n) = 0$



*Proof.* Let  $\epsilon > 0$ . By the **Archimedean Principle**, there exists  $N \in \mathbb{N}$  such that  $N > \frac{1}{3\epsilon}$  for all  $n \geq N$ ,

$$\begin{aligned} \left| \frac{\cos(n)}{3n} - 0 \right| &= \left| \frac{\cos(n)}{3n} \right| \\ &\leq \frac{1}{3n} \\ &\leq \frac{1}{3N} \\ &< \frac{1}{3(1/3\epsilon)} \\ &= \epsilon \end{aligned}$$

□

### Example 2.8: Limit Proof 6

Let  $(y_n) = \frac{4n-1}{n^2}$ . Claim:  $\lim_{n \rightarrow \infty} y_n = 0$ .

*Proof.* Let  $\epsilon > 0$ . By the **Archimedean Principle**, there exists  $N \in \mathbb{N}$  such that  $N > \frac{1}{\epsilon}$ . For all  $n \geq N$ ,

$$\begin{aligned} \left| \frac{4n-1}{n^2} - 0 \right| &= \left| \frac{4n-1}{n^2} \right| \\ &= \frac{4n-1}{n} \\ &< \frac{4n}{n^2} \\ &= \frac{4}{n} \\ &\leq \frac{4}{N} \\ &< \frac{4}{4/\epsilon} \\ &= \epsilon \end{aligned}$$

□



## 2.2.2 Exercises

### Exercise: 2.1.1

What happens if we reverse the order of the quantifiers in [Definition 2.2.3](#)?

*Definition:* A sequence  $x_n$  *verconges* to  $x$  if there exists an  $\epsilon > 0$  such that for all  $N \in \mathbb{N}$  it is true that  $n \geq N$  implies  $|x - n - x| < \epsilon$ .

- (a) Give an example of a vercongent sequence.
- (b) Is there an example of a vercongent sequence that is divergent?
- (c) Can a sequence verconge to two different values?
- (d) What exactly is being described in this strange definition?

*Solution.*

- (a) Pick  $\epsilon = 2$ ,  $x_n = (-1)^n$  and  $x = 0$ . This sequence will stay within the bounds of  $(-2, 2)$  for all  $N \in \mathbb{N}$  and  $n \geq N$ .
- (b) There cannot be a divergent vercongent sequence because vercongence wants us to be bounded, and divergence wants it to grow outside the bounds. These two ideas are mutually exclusive.
- (c) Yes. For example,  $x_n = 0$  and  $x_n = 1$ .
- (d) This definition is describing a sequence that is bounded. It is a sequence that is not growing outside of a certain range.

### Exercise: 2.2.2

Verify, using [Definition 2.2.3](#), that the following sequences converge to the proposed limit.

- (a)  $\lim_{n \rightarrow \infty} \frac{2n+15n+4}{5} = \frac{2}{5}$ .
- (b)  $\lim_{n \rightarrow \infty} \frac{2n^2n^3+3}{0} = 0$

*Proof.*



- (a) Let  $\epsilon > 0$ . Choose  $N \in \mathbb{N}$  such that  $N > \frac{3}{25\epsilon}$ . Then for all  $n \geq N$ ,

$$\begin{aligned} \left| \frac{2n+1}{5n+4} - \frac{2}{5} \right| &= \left| \frac{-3}{5(5n+4)} \right| \\ &= \frac{3}{25n+20} \\ &\leq \frac{3}{25n} \\ &\leq \frac{3}{25N} \\ &< \epsilon \end{aligned}$$

Therefore,  $\lim_{n \rightarrow \infty} \frac{2n+1}{5n+4} = \frac{2}{5}$ .

- (b) Let  $\epsilon > 0$ . By the **Archimedean Principle**, there exists an  $N \in \mathbb{N}$  such that  $N > \frac{2}{\epsilon}$ . Then, for  $n \geq N$ ,

$$\begin{aligned} \left| \frac{2n^2}{n^3+3} - 0 \right| &= \left| \frac{2n^2}{n^3+3} \right| \\ &= \frac{2n^2}{n^3+3} \\ &< \frac{2n^2}{n^3} \\ &= \frac{2}{n} \\ &\leq \frac{2}{N} \\ &= \frac{2}{2/\epsilon} \\ &= \epsilon. \end{aligned}$$

Therefore,  $\lim_{n \rightarrow \infty} \frac{2n^2}{n^3+3} = 0$ .

□

**Exercise: 2.2.3**

Describe what we would have to demonstrate in order to disprove each of the following statements.

- (a) At every college in the United States, there is a student who is at least seven feet tall.
- (b) For all colleges in the United States, there exists a professor who gives every student a grade of either A or B.
- (c) There exists a college in the United States where every student is at least six feet tall.

*Solution.*

- (a) There is at least one college in the United States where all students are less than seven feet tall.
- (b) There is at least one college in the United States where all professors give at least one student a grade of C or lower.
- (c) For all colleges in the United States, there exists a student who is less than six feet tall.

**Exercise: 2.2.4**

Give an example of each or state that the request is impossible. For any that are impossible, give a compelling argument for why that is the case.

- (a) A sequence with an infinite number of ones that does not converge to one.
- (b) A sequence with an infinite number of ones that converges to a limit not equal to one.
- (c) A divergent sequence such that for every  $n \in \mathbb{N}$  it is possible to find  $n$  consecutive ones somewhere in the sequence.

*Solution.*

- (a) Possible. Consider the sequence  $a_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ . This sequence has infinitely many ones but does not converge to one.
- (b) Impossible. Suppose  $(a_n)$  is a sequence that converges to a limit  $L \neq 1$  and has infinitely many ones. Since  $(a_n)$  converges to  $L$ , for any  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - L| < \epsilon$ . Choose  $\epsilon = \frac{|1-L|}{2} > 0$ . Then, for  $n \geq N$ ,



$|a_n - L| < \epsilon$ , which implies  $a_n \neq 1$  beyond this  $N$ . This contradicts the existence of infinitely many ones. Therefore, such a sequence is impossible.

- (c) Possible. Define a sequence by concatenating increasing blocks of ones separated by zeros:  $(0, 1, 0, 1, 1, 0, 1, 1, 1, 0, \dots)$ . Specifically, the sequence consists of  $n$  ones followed by a zero for  $n = 1, 2, 3, \dots$ . For every  $n \in \mathbb{N}$ , there is a block of  $n$  consecutive ones somewhere in the sequence. The sequence does not converge, so it is divergent.

### Exercise: 2.2.5

Let  $\llbracket x \rrbracket$  be the greatest integer less than or equal to  $x$ . For example,  $\llbracket \pi \rrbracket = 3$  and  $\llbracket 3 \rrbracket = 3$ . For each sequence, find  $\lim_{n \rightarrow \infty} a_n$  and verify it with the definition of convergence.

(a)  $a_n = \llbracket 5/n \rrbracket$

(b)  $a_n = \llbracket (12 + 4n)/3n \rrbracket$

Reflecting on these examples, comment on the statement following [Definition 2.2.3B](#) that “the smaller the  $\epsilon$ -neighborhood, the larger  $N$  may have to be.”

*Solution.*

- (a) We will show that  $\lim_{n \rightarrow \infty} a_n = 0$ .

*Proof.* For  $n \geq 6$ , we have  $\frac{5}{n} \leq \frac{5}{6} < 1$ , so  $a_n = \llbracket 5/n \rrbracket = 0$ .  
Let  $\epsilon > 0$ . Choose  $N = 6$ . Then for all  $n \geq N$ ,

$$|a_n - 0| = |0 - 0| = 0 < \epsilon.$$

Therefore, by the definition of convergence,  $\lim_{n \rightarrow \infty} a_n = 0$ . □

- (b) We will show that  $\lim_{n \rightarrow \infty} a_n = 1$ .





*Proof.* Observe that:

$$a_n = \left\lceil \frac{12 + 4n}{3n} \right\rceil = \left\lceil \frac{4n + 12}{3n} \right\rceil = \left\lceil \frac{4}{3} + \frac{4}{n} \right\rceil.$$

As  $n \rightarrow \infty$ ,  $\frac{4}{n} \rightarrow 0$ , so  $\frac{4}{3} + \frac{4}{n} \rightarrow \frac{4}{3} \approx 1.333$ .

For  $n \geq 7$ , we have:

$$\frac{4}{n} \leq \frac{4}{7} \approx 0.571, \quad \frac{4}{3} + \frac{4}{n} \leq 1.333 + 0.571 = 1.904.$$

Since  $1 < \frac{4}{3} + \frac{4}{n} < 2$  for  $n \geq 7$ , we have:

$$a_n = \left\lceil \frac{4}{3} + \frac{4}{n} \right\rceil = 1.$$

Let  $\epsilon > 0$ . Choose  $N = 7$ . Then for all  $n \geq N$ ,

$$|a_n - 1| = |1 - 1| = 0 < \epsilon.$$

Therefore, by the definition of convergence,  $\lim_{n \rightarrow \infty} a_n = 1$ . □

**Reflection:** In these examples, we see that once the sequence reaches a certain point (i.e.,  $n \geq N$ ), the terms remain constant. This means that for any  $\epsilon > 0$ , we can find a fixed  $N$  to satisfy the definition of convergence, regardless of how small  $\epsilon$  is. However, in general, smaller  $\epsilon$ -neighborhoods may require larger  $N$  because the sequence may not settle into its limit as neatly as it does in these cases.

### Exercise: 2.2.6

Prove the **Uniqueness of Limits** theorem. To get started, assume  $(a_n) \rightarrow a$  and  $(a_n) \rightarrow b$ . Now argue  $a = b$ .

*Proof.* Since  $(a_n) \rightarrow a$ , this means for all  $\epsilon > 0$ , there exists an  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|a_n - a| < \epsilon/2$ . Similarly, since  $(a_n) \rightarrow b$ , this means for all  $\epsilon > 0$ , there exists an  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,  $|a_n - b| < \epsilon/2$ .

Now, let  $N = \max\{N_1, N_2\}$  so that

$$\begin{aligned} |a - b| &= |a - a_n + a_n - b| \\ &\leq |(a_n - a) + (a_n - b)| \\ &< \epsilon/2 + \epsilon/2 \\ &< \epsilon \end{aligned}$$



Then, by [Theorem 1.2.6](#),  $a = b$ . □

### Exercise: 2.2.7

Here are two useful definitions:

- (i) A sequence  $(a_n)$  is *eventually* in a set  $A \subseteq \mathbb{R}$  if there exists an  $N \in \mathbb{N}$  such that  $a_n \in A$  for all  $n \geq N$ .
- (ii) A sequence  $(a_n)$  is *frequently* in a set  $A \subseteq \mathbb{R}$  if, for every  $N \in \mathbb{N}$ , there exists an  $n \geq N$  such that  $a_n \in A$ .
  - (a) Is the sequence  $(-1)^n$  eventually or frequently in the set  $\{1\}$ ?
  - (b) Which definition is stronger? Does frequently imply eventually, or does eventually imply frequently?
  - (c) Give an alternate rephrasing of [Definition 2.2.3B](#) using either frequently or eventually. Which is the term we want?
  - (d) Suppose an infinite number of terms of a sequence  $(x_n)$  are equal to 2. Is  $(x_n)$  necessarily eventually in the interval  $(1.9, 2.1)$ ? Is it frequently in  $(1.9, 2.1)$ ?

*Solution.*

- (a) The sequence  $(-1)^n$  is *frequently* in the set  $\{1\}$  because for every  $N \in \mathbb{N}$ , we can find an  $n \geq N$  such that  $(-1)^n = 1$ .
- (b) The definition of *eventually* is stronger because *eventually* implies *frequently*, but *frequently* does not imply *eventually*.
- (c) An alternate rephrasing of Definition 2.2.3B using *eventually* is: A sequence  $(a_n)$  converges to  $a$  if, given any  $\epsilon$ -neighborhood— $V_\epsilon(a)$  of  $a$ — $(a_n)$  is *eventually* in  $V_\epsilon(a)$ . The term we want is eventually.
- (d) If an infinite number of terms of a sequence  $(x_n)$  are equal to 2,  $(x_n)$  is not *eventually* in  $(1.9, 2.1)$  because we can have a sequence  $(a_n)$  that will not settle in  $(1.9, 2.1)$ . For example,  $(a_n) = (0, 2, 0, 2, \dots)$  does not settle in  $(1.9, 2.1)$ . Whereas,  $(x_n)$  is *frequently* in the interval  $(1.9, 2.1)$  because for every  $N \in \mathbb{N}$  there exists an  $n \geq N$  such that  $x_n \in (1.9, 2.1)$  for all  $n \geq N$ . We can see an instance of this being true by examining the previous example.



## 2.3 The Algebraic and Order Limit Theorems

### Definition 2.3.1

A sequence  $(x_n)$  is *bounded* if there exists some  $M > 0$  such that every term in the sequence belongs to  $[-M, M]$ .

### Theorem 2.3.2

Every convergent sequence is bounded.

*Proof.* Let  $(x_n)$  be a convergent sequence with limit  $L$ . There exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|(x_n) - L| < 1$ . Equivalently,  $(x_n) \in (L - 1, L + 1)$ . Let

$$M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |L + 1|, |L - 1|\}.$$

We claim that for all  $n \in \mathbb{N}$ ,  $|x_n| \leq M$ .

1. This is true for  $n < N$ .
2. For  $n \geq N$ , we know  $L - 1 < x_n < L + 1$ , so  $(x_n) \leq \max\{|L - 1|, |L + 1|\}$

Thus, every term is in  $[-M, M]$ . □

### Theorem 2.3.3: Algebraic Limit Theorem

Let  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ . Then,

- (i)  $\lim_{n \rightarrow \infty} ca_n = ca$  for all  $c \in \mathbb{R}$ ;
- (ii)  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ ;
- (iii)  $\lim_{n \rightarrow \infty} (a_n b_n) = ab$ ;
- (iv)  $\lim_{n \rightarrow \infty} \frac{a_n b_n}{b} = \frac{a}{b}$  provided  $b \neq 0$ .

*Scratch Paper:*

$$|ca_n - ca| = |c| |a_n - a| < \epsilon$$

$$|a_n - a| < \frac{\epsilon}{|c|}$$

Leave off and go back to proof<sup>1</sup>

*Proof.* (i)

Let  $\epsilon > 0$ .<sup>1</sup> Since  $(a_n)$  converges to  $a$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|a_n - a| < \frac{\epsilon}{|c|}$ . Now, for any  $n \geq N$  we have two case because we want to avoid dividing



by 0:

- If  $c = 0$ :  
then each  $ca_n = 0$ . So  $(ca_n)$  converges to 0, which can equal  $ca$ .
- If  $c > 0$ :  
 $|ca_n - ca| = |c| |a_n - a| < |c| \frac{\epsilon}{|c|} = \epsilon$ .

(ii)

*Scratch paper:*

$$|(a_n + b_n)| = |(a_n - a) + (b_n - b)| \quad (2.1)$$

$$\leq |a_n - a| + |b_n - b| \quad (2.2)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad (2.3)$$

Note that (2.2) is from the triangle inequality. Now, we will pick up to back at  $\epsilon > 0$ .

Let  $\epsilon > 0$ . Since  $(a_n)$  converges to  $a$ , there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|a_n - a| < \frac{\epsilon}{2}$ . Since  $(b_n)$  converges to  $b$ , there exists  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,  $|b_n - b| < \frac{\epsilon}{2}$ . Now, let  $N = \max\{N_1, N_2\}$ . Thus, for any  $n \geq N$ , (refer back to scratch paper).

(iii)

*Scratch paper:*

$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \quad (2.4)$$

$$= |a_n(b_n - b) + b(b_n - b)| \quad (2.5)$$

$$\leq |a_n| |b_n - b| + |b| |b_n - b| \quad (2.6)$$

$$\leq M |b_n - b| + M |a_n - a|. \quad (2.7)$$

$$< M \left( \frac{\epsilon}{2M} \right) + M \left( \frac{\epsilon}{2M} \right) \quad (2.8)$$

$$= \epsilon \quad (2.9)$$

Note that: (2.4) is where we added 0, (2.5) is from the triangle inequality, and (2.6) is just factored. Additionally, we choose  $N$  to get the fractions in (2.8). Now, we will pick up to back at  $\epsilon > 0$ .

Let  $\epsilon > 0$ . Since convergent sequences are bounded, then there exists  $M > 0$  such that for all  $n \in \mathbb{N}$ ,  $|a_n| \leq M$ . We can choose  $M$  so that  $|b_n| \leq M$  as well. Since  $(a_n)$  converges to  $a$ , there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|a_n - a| < \frac{\epsilon}{2M}$ . Since  $(b_n)$  converges to  $b$ , there exists  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,  $|b_n - b| < \frac{\epsilon}{2M}$ . Now, let  $N = \max\{N_1, N_2\}$ . Thus, for any  $n \geq N$ , (refer back to scratch paper, and change (2.4)'s sign from an '=' to '<').



(iv)

*Scratch paper:*

$$\begin{aligned}
 \left| \frac{a_n}{b_n} - \frac{a}{b} \right| &= \left| \frac{a_nb - ab_n}{b_nb} \right| \\
 &= \left| \frac{a_nb - ab_n + ab_n - ab}{b_nb} \right| \\
 &= \left| \frac{a_n(b - b_n) + b(b_n - b)}{b_nb} \right| \\
 &= \left| \frac{a_n(b - b_n)}{b_nb} + \frac{b(b_n - b)}{b_nb} \right| \\
 &\leq \left| \frac{a_n}{b_n} \right| |b - b_n| + |b| \left| \frac{b_n - b}{b_nb} \right| \\
 &< \epsilon
 \end{aligned}$$

Let  $\epsilon > 0$ . Since  $(b_n)$  converges to  $b$ , there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|b_n| > \left|\frac{b}{2}\right|$ . There also exists  $N_2 \in \mathbb{N}$  such that for all  $n \geq N_2$ ,  $|b_n - b| < \frac{\epsilon|b|^2}{2}$ . Now, let  $N = \max\{N_1, N_2\}$ . Let  $n \geq N$ , (refer back to scratch paper).  $\square$

### Lemma 2.3.4

Let  $(a_n)$  and  $c < a$ . There exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a_n > c$ . Similarly, if  $a < d$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $a_n < d$ .

## 2.3.1 Limits and Order

### Theorem 2.3.5: Order Limit Theorem

Let  $(a_n)$  and  $(b_n)$  be sequences. If  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$ , then

- (i) If  $a_n \geq c$  for all  $n \in \mathbb{N}$ , then  $a \geq c$ .
- (ii) If  $a_n \leq b_n$  for all  $n \in \mathbb{N}$ , then  $a \leq b$ .
- (iii) If there exists  $c \in \mathbb{R}$  for which  $c \leq b_n$  for all  $n \in \mathbb{N}$ , then  $c \leq b$ . Similarly, if  $a_n \leq c$  for all  $n \in \mathbb{N}$ , then  $a \leq c$ .

## 2.3.2 Exercises

### Exercise: 2.3.1

- (a) If  $\lim_{n \rightarrow \infty} x_n = 0$ , show that  $\lim_{n \rightarrow \infty} \sqrt{x_n} = 0$ .
- (b) If  $\lim_{n \rightarrow \infty} x_n = x$ , show that  $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{x}$ .



*Proof.*

- (a) *Solution.* Let  $\epsilon > 0$ . Since  $\lim_{n \rightarrow \infty} x_n = 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$|x_n| < \epsilon^2.$$

Then, for all  $n \geq N$ ,

$$|\sqrt{x_n} - 0| = \sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon.$$

Therefore,  $\lim_{n \rightarrow \infty} \sqrt{x_n} = 0$ .

- (b) *Solution.* Let  $\epsilon > 0$ . Since  $\lim_{n \rightarrow \infty} x_n = x$ , for any  $\delta > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$|x_n - x| < \delta.$$

We consider two cases:

**Case 1:**  $x > 0$ .

Since  $x > 0$ , choose  $\delta = \min \left\{ \epsilon(2\sqrt{x}), \frac{x}{2} \right\}$ . Then for all  $n \geq N$ , we have  $x_n > x - \frac{x}{2} = \frac{x}{2} > 0$ . Thus,

$$\sqrt{x_n} + \sqrt{x} \geq \sqrt{\frac{x}{2}} + \sqrt{x} > 0.$$

Now,

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{\delta}{\sqrt{\frac{x}{2}}} \leq \epsilon.$$

**Case 2:**  $x = 0$ .

From part (1), we have  $\lim_{n \rightarrow \infty} \sqrt{x_n} = 0 = \sqrt{0}$ .

Therefore,  $\lim_{n \rightarrow \infty} \sqrt{x_n} = \sqrt{x}$ .

□

### Exercise: 2.3.2

Using only [Definition 2.2.3](#), prove that if  $(x_n) \rightarrow 2$ , then

(a)  $\left( \frac{2x_n - 1}{3} \right) \rightarrow 1;$

(b)  $(1/x_n) \rightarrow 1/2.$

(For this exercise the Algebraic Limit Theorem is off-limits, so to speak.)



*Solution.*

- (a) *Proof.* Let  $\epsilon > 0$ . Since  $(x_n)$  converges to 2, there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - 2| < \epsilon$ . Now, for any  $n \geq N$ ,

$$\begin{aligned} \left| \frac{2x_n - 1}{3} - 1 \right| &= \left| \frac{2x_n - 1 - 3}{3} \right| \\ &= \left| \frac{2x_n - 4}{3} \right| \\ &= \frac{2}{3} |x_n - 2| \\ &< |x_n - 2| \\ &< \epsilon \end{aligned}$$

Therefore,  $\frac{2x_n - 1}{3} \rightarrow 1$  □

- (b) *Proof.* Let  $\epsilon > 0$ . Since  $(x_n)$  converges to 2, there exists  $N_1 \in \mathbb{N}$  such that for all  $n \geq N_1$ ,  $|x_n - 2| < \epsilon$ . Then, we will choose  $N_2$  so that  $|x_n - 2| < \epsilon$  for all  $n \geq N_2$ . Afterwards, we take  $N = \max\{N_1, N_2\}$ . And note that for  $n \geq N$ ,

$$\begin{aligned} \left| \frac{1}{x_n} - \frac{1}{2} \right| &= \left| \frac{2 - x_n}{2x_n} \right| \\ &< \frac{|2 - x_n|}{2} \\ &< \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

□

## 2.4 The Monotone Convergence Theorem and a First Look at Infinite Series

### Definition 2.4.1

A sequence  $a_n$  is *increasing* if  $a_n \leq a_{n+1}$  for all  $n \in \mathbb{N}$  and *decreasing* if  $a_n \geq a_{n+1}$  for all  $n \in \mathbb{N}$ . A sequence is *monotone* if it is either increasing or decreasing.

### Theorem 2.4.2: Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges.



*Proof.* Let  $(a_n)$  be an increasing and bounded sequence. Since  $(a_n)$  is bounded, the set  $A = \{a_n \mid n \in \mathbb{N}\}$  is clearly also bounded. Since  $A$  is bounded,  $\sup A$  exists. We claim that  $\lim_{n \rightarrow \infty} a_n = \sup A$ . Thus, for all  $\epsilon > 0$  and by our definition of supremum, there exists  $N \in \mathbb{N}$  such that  $\sup A - \epsilon < a_N \leq \sup A$ . Since  $(a_n)$  is increasing, for all  $n \geq N$ ,  $\sup A - \epsilon < a_N \leq a_n \leq \sup A$ . It follows that  $|a_n - \sup A| < \epsilon$ . Therefore,  $\lim_{n \rightarrow \infty} a_n = \sup A$ .  $\square$

### Example 2.9: MCT

Consider the recursively defined sequence  $x_n$  where  $x_1 = 3$  and for all  $n \in \mathbb{N}$ ,  $x_{n+1} = \frac{1}{4-x_n}$ . Show that  $x_n$  converges.

*Proof.* We will show that  $x_n$  is monotone and bounded.

- **Part 1: Monotone Decreasing**

- Base case:  $x_1 = 3, x_2 = 1$ .
- Induction step: Assume for some  $n \in \mathbb{N}$ ,  $x_n \geq x_{n+1}$ . It follows that

$$\begin{aligned} x_n &\geq x_{n+1} \\ 4 - x_n &\leq 4 - x_{n+1} \\ \frac{1}{4 - x_n} &\geq \frac{1}{4 - x_{n+1}} \\ x_{n+1} &\geq x_{n+2} \end{aligned}$$

- **Part 2: Bounded Below Claim:** Sequence is bounded below by 0.

- Base case:  $x_1 = 3 > 0$ .
- Induction step: Assume for some  $n \in \mathbb{N}$ ,  $x_n \geq 0$ . It follows that  $4 - x_n \leq 4$ , and when we take the reciprocal, we get

$$\begin{aligned} \frac{1}{4 - x_n} &\leq \frac{1}{4} \\ x_{n+1} &\geq 1/4 \\ &> 0 \end{aligned}$$

By math induction,  $x_n$  is bounded below by 0.

By the Monotone Convergence Theorem,  $x_n$  converges.

So, what is the limit? We know  $(x_n)$  converges so let  $L = \lim_{n \rightarrow \infty} x_n$ . Then,  $\lim_{n \rightarrow \infty} x_{n+1} = L$ . We also know  $x_{n+1} = \frac{1}{4-x_n}$ . So  $L = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \frac{1}{4-x_n} =$





$\frac{1}{4-L}$ . It must be true that  $L = \frac{1}{4-L}$ . Solving for  $L$ , we get

$$\begin{aligned} L(4-L) &= 1 \\ 4L - L^2 &= 1 \\ L^2 - 4L + 1 &= 0 \end{aligned}$$

Hence,  $L = 2 - \sqrt{3}$  or  $L = 2 + \sqrt{3}$ . Notice that it cannot be the latter because it is bigger than 3.  $\square$

### 2.4.1 Recap and Summary

We use limits to define multiple things in calculus. This is why we are focusing so heavily upon it. For example,

1. Derivatives:  $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$
2. Integrals:  $\lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x$
3. Infinite Series:  $\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i$  Consider geometric series,  $C_a$  such that each term is multiplied by a ratio  $r$ . This is represented as  $\sum_{n=0}^{\infty} ar^n = 1 + r + r^2 + r^3 \dots$ . When we look at partial sums, we get  $S_n = 1 + r + r^2 + r^3 + \dots + r^n$ . We can then multiply by  $r$  to get  $rS_n = r + r^2 + r^3 + r^4 + \dots + r^{n+1}$ . Subtracting the two, we get  $(1-r)S_n = 1 - r^{n+1}$ . Thus,

$$S_n = \frac{1 - r^{n+1}}{1 - r}.$$

If  $|r| < 1$ , then  $\lim_{n \rightarrow \infty} r^n = 0$ . Thus,  $\lim_{n \rightarrow \infty} S_n = \frac{1}{1-r}$ .

Looking to the future, we are going to use functions and summations together. For example, when we have  $f(x) = \sum_{n=0}^{\infty} (a_n)x^n$  such that  $f'(x) = \sum_{n=0}^{\infty} (a_n)x^{n-1}$ .

#### Definition 2.4.3

Let  $(x_n)$  be a bounded sequence. Then the *limit inferior* is  $\liminf_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \inf\{x_k \mid k \geq n\}$ . This is the largest a limit can get. The *limit superior* is  $\limsup_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \sup\{x_k \mid k \geq n\}$ . This is the smallest a limit can get.

See Exercise 2.4.7 in the book for more information.

#### Example 2.10: Monotone Decreasing Sequence

The following sequence is an example of a monotone decreasing sequence.

$$\begin{aligned} x_1, x_2, x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 1\} &= S. \\ x_2, x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 2\} &= S. \\ x_3, x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 3\} &= S. \\ x_4, x_5, x_6, \dots \sup\{x_k \mid k \geq 4\} &= S. \end{aligned}$$



$\limsup_{n \rightarrow \infty} x_n$  is guaranteed to exist by the **Monotone Convergence Theorem**.

### Example 2.11: liminf

Let  $x_n = (-1)^n(1 + \frac{1}{n})$ . Thus,  $x_{1,2,3} = -2, 1\frac{1}{2}, -1\frac{1}{3} \dots$

### Example 2.12: Convergence Towards 0

Let  $x_n = (-1)^n \frac{1}{n}$ . Thus,  $x_{1,2,3} = -1, \frac{1}{2}, -\frac{1}{3} \dots$

### Theorem 2.4.4

A sequence  $x_n$  is convergent if, and only if,  $\liminf_{n \rightarrow \infty} x_n = \limsup_{n \rightarrow \infty} x_n$ .

See Theorem 2.4.6 in the book for another view.

## 2.4.2 Exercises

### Exercise: 2.4.7 (Limit Superior)

Let  $(a_n)$  be a bounded sequence.

- (a) Prove that the sequence defined by  $y_n = \sup\{a_k : k \geq n\}$  converges.
- (b) The *limit superior* of  $(a_n)$  or  $\limsup_{n \rightarrow \infty} a_n$ , is defined by

$$\limsup_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} y_n,$$

where  $y_n$  is the sequence from part (a) of this exercise. Provide a reasonable definition for  $\liminf_{n \rightarrow \infty} a_n$  and briefly explain why it always exists for any bounded sequence.

- (c) Prove that  $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$  for every bounded sequence, and give an example of a sequence for which the inequality is strict.
- (d) Show that  $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$  if and only if  $\lim a_n$  exists. In this case, all three share the same value.

*Solution.*



(a) We will show that  $(y_n)$  converges.

*Proof.* Since  $(a_n)$  is bounded, there exists  $M > 0$  such that  $a_n \leq M$  for all  $n$ .

For each  $n$ , define  $y_n = \sup\{a_k : k \geq n\}$ . As  $n$  increases,  $\{k \geq n\}$  becomes smaller, so the supremum cannot increase. Thus, the sequence  $(y_n)$  is non-increasing:

$$y_{n+1} \leq y_n \quad \text{for all } n.$$

Additionally, since  $(a_n)$  is bounded below, so is  $(y_n)$ . Therefore,  $(y_n)$  is a bounded, non-increasing sequence.

By the Monotone Convergence Theorem, every bounded, non-increasing sequence converges. Thus,  $(y_n)$  converges.

(b) A reasonable definition for  $\liminf_{n \rightarrow \infty} a_n$  is to define  $z_n = \inf\{a_k : k \geq n\}$  for each  $n$ . Then, the *limit inferior* of  $(a_n)$  is defined by:

$$\liminf_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} z_n.$$

Since  $(a_n)$  is bounded, each  $z_n$  exists and the sequence  $(z_n)$  is non-decreasing. As  $n$  increases, the set  $\{a_k : k \geq n\}$  becomes smaller, so the infimum cannot decrease. Therefore,  $(z_n)$  is a bounded, non-decreasing sequence, which converges by the **Monotone Convergence Theorem**. Hence,  $\liminf_{n \rightarrow \infty} a_n$  always exists for any bounded sequence.

(c) We will show that  $\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n$  for every bounded sequence.



*Proof.* For each  $n$ , we have  $z_n = \inf\{a_k : k \geq n\} \leq a_n \leq \sup\{a_k : k \geq n\} = y_n$ . This implies:

$$z_n \leq y_n \quad \text{for all } n.$$

Taking limits as  $n \rightarrow \infty$ , we get:

$$\lim_{n \rightarrow \infty} z_n \leq \lim_{n \rightarrow \infty} y_n,$$

which means:

$$\liminf_{n \rightarrow \infty} a_n \leq \limsup_{n \rightarrow \infty} a_n.$$

For an example where the inequality is strict, consider the sequence  $a_n = (-1)^n$ . Then:

$$y_n = \sup\{(-1)^k : k \geq n\} = 1, \quad z_n = \inf\{(-1)^k : k \geq n\} = -1.$$

Therefore:

$$\limsup_{n \rightarrow \infty} a_n = 1, \quad \liminf_{n \rightarrow \infty} a_n = -1, \quad \liminf_{n \rightarrow \infty} a_n < \limsup_{n \rightarrow \infty} a_n.$$

□

(d) We will show that  $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n$  if and only if  $\lim a_n$  exists. In this



case, all three share the same value.

*Proof.* We will show this by proving both implications:

( $\Rightarrow$ ) Suppose  $\liminf_{n \rightarrow \infty} a_n = \limsup_{n \rightarrow \infty} a_n = L$ . We will show  $\lim a_n$  exists and equals  $L$ .  
Let  $\epsilon > 0$ . Since  $\limsup_{n \rightarrow \infty} a_n = L$ , there exists  $N_1$  such that for all  $n \geq N_1$ :

$$y_n = \sup\{a_k : k \geq n\} < L + \epsilon.$$

Similarly, since  $\liminf_{n \rightarrow \infty} a_n = L$ , there exists  $N_2$  such that for all  $n \geq N_2$ :

$$z_n = \inf\{a_k : k \geq n\} > L - \epsilon.$$

Let  $N = \max\{N_1, N_2\}$ . Then, for all  $n \geq N$ :

$$L - \epsilon < z_n \leq a_n \leq y_n < L + \epsilon,$$

which implies:

$$|a_n - L| < \epsilon.$$

Therefore,  $\lim a_n = L$ .

( $\Leftarrow$ ) Conversely, suppose  $\lim a_n = L$ . Then, for every  $\epsilon > 0$ , there exists  $N$  such that for all  $n \geq N$ :

$$|a_n - L| < \epsilon.$$

This implies that for all  $n \geq N$ , the set  $\{a_k : k \geq n\}$  is contained in  $(L - \epsilon, L + \epsilon)$ . Therefore:

$$y_n = \sup\{a_k : k \geq n\} \leq L + \epsilon, \quad z_n = \inf\{a_k : k \geq n\} \geq L - \epsilon.$$

Taking limits, we get:

$$\limsup_{n \rightarrow \infty} a_n \leq L + \epsilon, \quad \liminf_{n \rightarrow \infty} a_n \geq L - \epsilon.$$

Since  $\epsilon > 0$  is arbitrary, it follows that  $\limsup_{n \rightarrow \infty} a_n = \liminf_{n \rightarrow \infty} a_n = L$ .



### Exercise: 2.4.10 (Infinite Products)

A close relative of infinite series is the infinite product

$$\prod_{n=1}^{\infty} b_n = b_1 b_2 b_3 \cdots,$$

which is understood in terms of its sequence of partial products

$$p_m = \prod_{n=1}^m b_n = b_1 b_2 b_3 \cdots b_m.$$

Consider the special class of infinite products of the form

$$\prod_{n=1}^{\infty} (1 + a_n) = (1 + a_1)(1 + a_2)(1 + a_3) \cdots, \quad \text{where } a_n \geq 0.$$

- (a) Find an explicit formula for the sequence of partial products in the case where  $a_n = \frac{1}{n}$  and decide whether the sequence converges. Write out the first few terms in the sequence of partial products in the case where  $a_n = \frac{1}{n^2}$  and make a conjecture about the convergence of this sequence.
- (b) Show, in general, the sequence of partial products converges if and only if  $\sum_{n=1}^{\infty} a_n$  converges. (The inequality  $1 + x \leq 3^x$  for positive  $x$  will be useful in one direction.)

*Solution.*

- (a) For  $a_n = \frac{1}{n}$ :

The sequence of partial products is:

$$p_m = \prod_{n=1}^m \left(1 + \frac{1}{n}\right) = \prod_{n=1}^m \frac{n+1}{n}$$

This telescopes:

$$p_m = \frac{2}{1} \times \frac{3}{2} \times \frac{4}{3} \times \cdots \times \frac{m+1}{m} = \frac{m+1}{1} = m+1$$

Therefore, the sequence diverges as  $m \rightarrow \infty$ .

- For  $a_n = \frac{1}{n^2}$ :



Compute the first few terms:

$$\begin{aligned}
 p_1 &= 1 + \frac{1}{1^2} = 2 \\
 p_2 &= \left(1 + \frac{1}{1^2}\right) \left(1 + \frac{1}{2^2}\right) = 2 \times \frac{5}{4} = \frac{5}{2} \\
 p_3 &= p_2 \times \left(1 + \frac{1}{3^2}\right) = \frac{5}{2} \times \frac{10}{9} = \frac{25}{9} \\
 p_4 &= p_3 \times \left(1 + \frac{1}{4^2}\right) = \frac{25}{9} \times \frac{17}{16} = \frac{425}{144}
 \end{aligned}$$

The sequence increases slowly, suggesting that the infinite product is monotone increasing, and thus it converges.

(b) We will provide an if and only if proof below.

*Proof.* We will show that the infinite product  $\prod_{n=1}^{\infty} (1 + a_n)$  converges if and only if the series  $\sum_{n=1}^{\infty} a_n$  converges.

( $\Rightarrow$ ) If  $\sum_{n=1}^{\infty} a_n$  converges, then  $a_n \rightarrow 0$ . For  $a_n \geq 0$ , we have  $\ln(1 + a_n) \leq a_n$ . Thus,

$$\sum_{n=1}^{\infty} \ln(1 + a_n) \leq \sum_{n=1}^{\infty} a_n < \infty$$

So the series  $\sum_{n=1}^{\infty} \ln(1 + a_n)$  converges, which implies that the product  $\prod_{n=1}^{\infty} (1 + a_n)$  converges.

( $\Leftarrow$ ) Conversely, if  $\prod_{n=1}^{\infty} (1 + a_n)$  converges, then the partial products are bounded. For  $a_n \geq 0$  and  $1 + x \geq e^{x/2}$  for small  $x$ , we have

$$\ln(1 + a_n) \geq \frac{a_n}{2}$$

For sufficiently large  $n$ , this gives

$$\sum_{n=1}^{\infty} a_n \leq 2 \sum_{n=1}^{\infty} \ln(1 + a_n)$$

Since  $\sum_{n=1}^{\infty} \ln(1 + a_n)$  converges, so does  $\sum_{n=1}^{\infty} a_n$ .  $\square$



## 2.5 Subsequences and the Bolzano-Weierstrass Theorem

### Definition 2.5.1

Let  $a_n$  be a sequence of real numbers, and let  $n_1 < n_2 < n_3 < \dots$  be an increasing sequence of natural numbers. Then, the sequence  $a_{n_1}, a_{n_2}, a_{n_3}, \dots$  is called a *subsequence* of  $a_n$  and is denoted by  $a_{n_k}$ , where  $k \in \mathbb{N}$  indexes the subsequence.

### Theorem 2.5.2

Subsequences of a convergent sequence converge to the same limit as the original sequence.

*Proof.* Let  $x_{n_k}$  be a subsequence of  $x_n$ , and let  $L = \lim_{n \rightarrow \infty} x_n$ . We want to show that  $\lim_{n \rightarrow \infty} x_{n_k} = L$ . Let  $\epsilon > 0$ . Since  $x_n$  converges to  $L$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - L| < \epsilon$ . Since  $n_k$  is increasing, there exists  $M \in \mathbb{N}$  such that  $n_k \geq N$  for all  $k \geq M$ . Thus, for all  $k \geq M$ ,  $|x_{n_k} - L| < \epsilon$ . Therefore,  $\lim_{n \rightarrow \infty} x_{n_k} = L$ .

Let  $x_{n_k}$  be a subsequence of  $x_n$ . Let  $\epsilon > 0$ . Since  $(x_n) \rightarrow L$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - L| < \epsilon$ .

Now, looking at  $x_{n_k}$ , notice that  $n_k \geq k$  for all  $k$ . Consider  $k = N$ . For any  $n \geq N$ ,  $n \geq N \geq k$ . Thus,  $|x_{n_k} - L| < \epsilon$ . Therefore,  $\lim_{n \rightarrow \infty} x_{n_k} = L$ .  $\square$

### Theorem 2.5.3: Divergence Criterion

If  $x_n$  has two subsequences that converge to different limits, then  $x_n$  diverges.

Building upon this idea of Divergence, we can list some other ways a sequence can diverge:

1. Find one subsequence that diverges.
2. Find two subsequences that converge to separate limits.
3. Negate the **definition of convergence**.
  - For example, a sequence converges to  $L$  if there exists  $\epsilon > 0$  such that for all  $N \in \mathbb{N}$  there exists  $n \geq N$  such that  $|a_n - L| \geq \epsilon$ . There exists a subsequence  $(a_{n_k})$  such that for all  $k \in \mathbb{N}$ ,  $|a_{n_k} - L| \geq \epsilon$ .

### Theorem 2.5.4: Bolzano-Weierstrass Theorem

Every bounded sequence in  $\mathbb{R}$  has a convergent subsequence.

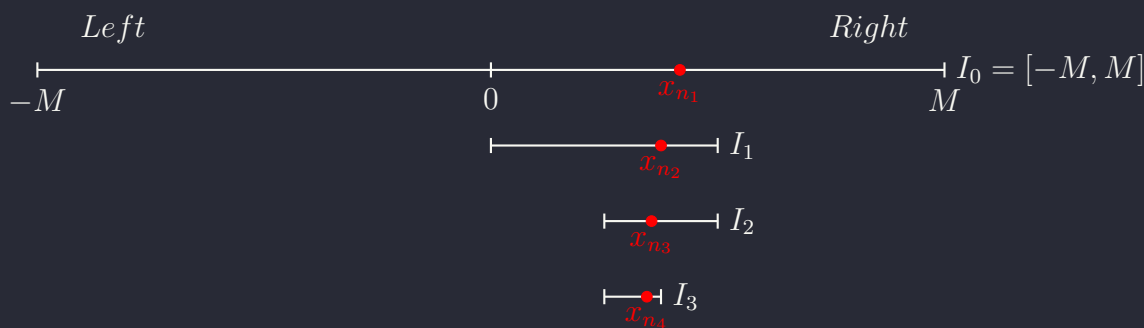




*Proof.* Let  $x_n$  be a bounded sequence. There exists an  $M > 0$  such that every term  $x_n$  belongs to  $[-M, M]$ . To prove this theorem, we will be utilizing a recursive argument style. Thus, let  $I_0 = [-M, M]$ .  $I_0$  has length  $2M$ . Cut  $I_0$  in half with  $I_1$  and  $I_2$  both being half as long as  $I_0$ . Since  $x_n$  is bounded, there exists an  $I_L$  or  $I_R$  that contains infinitely many terms of  $x_n$ . We will pick one, call it  $I_1$  that is contained in  $I_L$ .  $I_1$  has length  $M$ . Pick one of those terms inside  $I_1$  and call it  $x_{n_1}$ . Now, cut  $I_1$  in half with equal length in intervals. One of them contains infinitely many terms. Call that interval  $I_2$ .  $I_2$  has length  $\frac{M}{2}$ . Pick one of those terms inside  $I_2$  and call it  $x_{n_2}$ . Continue this process indefinitely for all  $n \geq \mathbb{N}$  with  $n_1 > n_2$ . Continue this process, and we get

- a sequence of closed intervals  $I_n$ .
  - $I_n$  has length  $\frac{2M}{2^n}$ .
  - They are nested,  $I_n \subseteq I_{n-1}$ .
- a subsequence  $x_{n_k}$ 
  - for all  $k_1, x_{n_k} \in I_k$ .

The **Nested Interval Property** states that  $\bigcup_{n=1}^{\infty} I_n$  is non empty. Let  $L$  be a point in  $\bigcup_{n=1}^{\infty} I_n$ . We claim  $\lim_{n \rightarrow \infty} x_{n_k} = L$ . Let  $\epsilon > 0$ . There exists an  $N \in \mathbb{N}$  such that  $\frac{2M}{2^N} < \epsilon$ . (Since  $\lim_{n \rightarrow \infty} \frac{2M}{2^n} = 0$ . See **Theorem 2.5.5**) For any  $k \geq N$ , recall that  $x_{n_k}, L \in I_k$ . Since  $I_k$  has length  $\frac{2M}{2^k}$ . Thus,  $|x_{n_k} - L| < \epsilon$ . Therefore,  $\lim_{n \rightarrow \infty} x_{n_k} = L$  and  $(x_n)$  has a convergence subsequence.  $\square$



### Theorem 2.5.5

Let  $b \in (0, 1)$ . Then  $\lim_{n \rightarrow \infty} b^n = 0$ .

*Proof.* The sequence  $(b^n)$  is monotone decreasing. This is because  $b^{n+1} = b^n b < b^n$ . This sequence is also bounded by 0. Hence, by the **Monotone Convergence Theorem**,  $(b^n)$  converges. Now, let  $L = \lim_{n \rightarrow \infty} b^n$ . Consider the subsequence  $b^{2n}$ . This sequence also



converges to  $L$ . Thus,

$$\begin{aligned} L &= \lim_{n \rightarrow \infty} b^{2n} \\ &= \lim_{n \rightarrow \infty} b^n b^n \\ &= \lim_{n \rightarrow \infty} b^n \lim_{n \rightarrow \infty} b^n \\ &= L^2. \end{aligned}$$

Thus,  $L = 0$  or  $L = 1$ . The limit cannot be 1 because  $b^n$  is decreasing away from 1. Therefore,  $L = 0$ .  $\square$

### 2.5.1 Exercises

#### Exercise: 2.5.1

Give an example of each of the following, or argue that such a request is impossible.

- (a) A sequence that has a subsequence that is bounded but contains no subsequence that converges.
- (b) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

*Solution.*

- (a) **Impossible.** This violates the **Bolzano-Weierstrass Theorem**. It assures us that every bounded sequence has a convergent subsequence. If a subsequence is bounded, then it must have a convergent subsequence.
- (b) Consider the sequence  $(\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots, \frac{1}{n}, \frac{(n-1)}{n})$ . From this, you can have a subsequence  $(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n})$  which converges to 0, and also a subsequence  $(\frac{1}{2}, \frac{2}{3}, \dots, \frac{n-1}{n})$ , which converges to 1.

#### Exercise: 2.5.2

Decide whether the following propositions are true or false, providing a short justification for each conclusion.

- (a) If every proper subsequence of  $(x_n)$  converges, then  $(x_n)$  converges as well.
- (b) If  $(x_n)$  contains a divergent subsequence, then  $(x_n)$  diverges.
- (c) If  $(x_n)$  is bounded and diverges, then there exist two subsequences of  $(x_n)$  that converge to different limits.



*Solution.*

- (a) **True.** If every proper subsequence of  $(x_n)$  converges, then  $(x_n)$  must converge to the same limit. If  $(x_n)$  did not converge, there would exist at least one divergent subsequence or two subsequences converging to different limits, contradicting the assumption.
- (b) **True.** If  $(x_n)$  contained a divergent subsequence, then  $(x_n)$  cannot converge. A convergent sequence has all its subsequences converging to the same limit, so the existence of a divergent subsequence implies that  $(x_n)$  diverges (contrapositive).
- (c) **True.** Since  $(x_n)$  is bounded and diverges, the **Bolzano-Weierstrass Theorem** guarantees the existence of at least one convergent subsequence. Let this subsequence converge to  $L_1$ . Because  $(x_n)$  does not converge to  $L_1$ , there is an  $\epsilon > 0$  and infinitely many terms of  $(x_n)$  such that  $|x_n - L_1| \geq \epsilon$ . Extracting a subsequence from these terms, the Bolzano-Weierstrass Theorem ensures a further subsequence converging to a limit  $L_2 \neq L_1$ . Thus,  $(x_n)$  has two subsequences converging to different limits.

### Exercise: 2.5.5

Assume  $(a_n)$  is a bounded sequence with the property that every convergent subsequence of  $(a_n)$  converges to the same limit  $a \in \mathbb{R}$ . Show that  $(a_n)$  must converge to  $a$ .

*Proof.* Suppose that  $(a_n)$  does not converge to  $a \in \mathbb{R}$ . By the definition of convergence, this means there is a positive real number  $\epsilon_0$  such that no matter how large we choose  $N \in \mathbb{N}$ , there will always exist some  $n > N$  where  $|a_n - a| \geq \epsilon_0$ . In a formal way, this shows that  $(a_n)$  does not converge to  $a$  within the  $\epsilon_0$ -neighborhood.

We aim to demonstrate that this leads to a contradiction by constructing a subsequence of  $(a_n)$  that stays outside this neighborhood. Begin by selecting  $n_1$  such that  $|a_{n_1} - a| \geq \epsilon_0$ . Next, since the condition holds for all  $N \in \mathbb{N}$ , we can find another index  $n_2 > n_1$  such that  $|a_{n_2} - a| \geq \epsilon_0$ . Continuing this process, we generate an increasing sequence of indices  $n_1 < n_2 < n_3 < \dots$  such that for each  $i \in \mathbb{N}$ ,  $|a_{n_i} - a| \geq \epsilon_0$ .

Now consider the subsequence  $(a_{n_i})$  we have built. Since  $(a_n)$  is bounded by assumption, its subsequence  $(a_{n_i})$  is also bounded. By the **Bolzano-Weierstrass Theorem**, every bounded sequence has a convergent subsequence. Let  $(a_{n_{i_k}})$  denote a convergent subsequence of  $(a_{n_i})$ . According to our assumption, any convergent subsequence of  $(a_n)$  must converge to  $a$ .

However, each term of  $(a_{n_{i_k}})$  remains outside the  $\epsilon_0$ -neighborhood of  $a$ . Thus, it is impossible for  $(a_{n_{i_k}})$  to converge to  $a$ . This contradiction implies that our initial assumption—that  $(a_n)$  does not converge to  $a$ —is false. Therefore, the sequence  $(a_n)$  must converge to  $a$ .  $\square$



### Exercise: 2.5.6

Use a similar strategy to the one in [Theorem 2.5.5](#) to show

$$\lim b^{1/n} \text{ exists for all } b \geq 0$$

and find the value of the limit. (The results in Exercise 2.3.1 may be assumed.)

*Proof.* We will show that  $\lim_{n \rightarrow \infty} b^{1/n}$  exists for all  $b \geq 0$  and find its value.

- **Case 1:**  $b = 0$ .

When  $b = 0$ , the sequence becomes  $a_n = 0^{1/n} = 0$  for all  $n$ . Thus,  $\lim_{n \rightarrow \infty} b^{1/n} = 0$ .

- **Case 2:**  $b > 0$ .

Suppose, for contradiction, that  $\lim_{n \rightarrow \infty} b^{1/n} \neq 1$ . Then there exists  $\epsilon > 0$  and infinitely many  $n$  such that  $|b^{1/n} - 1| \geq \epsilon$ . Extract a subsequence  $(b^{1/n_k})$  where this inequality holds for all  $k$ .

Since  $b^{1/n} > 0$  and bounded, by the [Bolzano-Weierstrass Theorem](#), the subsequence  $(b^{1/n_k})$  has a further subsequence that converges to a limit  $L$ . According to Exercise 2.3.1, any convergent subsequence of  $(b^{1/n})$  must have its limit equal to  $\lim_{n \rightarrow \infty} b^{1/n}$ .

Consider  $\ln b^{1/n} = \frac{\ln b}{n}$ . As  $n \rightarrow \infty$ ,  $\frac{\ln b}{n} \rightarrow 0$ , so  $\ln b^{1/n} \rightarrow 0$ , which implies  $b^{1/n} \rightarrow e^0 = 1$ .

This contradicts the assumption that  $|b^{1/n_k} - 1| \geq \epsilon$ , so  $\lim_{n \rightarrow \infty} b^{1/n} = 1$ .

**Conclusion:**

$$\lim_{n \rightarrow \infty} b^{1/n} = \begin{cases} 0, & \text{if } b = 0, \\ 1, & \text{if } b > 0. \end{cases} \quad \square$$

## 2.6 The Cauchy Criterion

### Recall

How do we prove  $x_n$  converges?

1. We know and prove the limit  $\rightarrow$  claim  $L$ , show terms get close to  $L$ .
2. [Monotone Convergence Theorem](#).

### Definition 2.6.1

A sequence  $(x_n)$  is a *Cauchy sequence* if for all  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $m, n \geq N$ ,  $|x_m - x_n| < \epsilon$ .



This says that as terms get close to each other and stay close together, there's some value they're all getting close to.

### Geometric Series Review

Remember that geometric series consist of terms that are multiplied by a ratio  $r$ . For example, that could look like  $1 + r + r^2 + r^3 + \dots$ .

We are most interested in **partial sums**. That is,

$$1 + r + r^2 + \dots + r^{n-1} + r^n = S_n.$$

From here, we would multiply both sides by  $r$ . This gives

$$r + r^2 + \dots + r^n + r^{n+1} = rS_n.$$

When we subtract these two from each other, we get

$$1 - r^{n+1} = S_n - rS_n.$$

This yields the identity

$$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

### Example 2.13: Cauchy Sequence

Consider the sequence  $a_1 = 1, a_2 = 2$ , where

$$a_n = \frac{a_{n-1} + a_{n-2}}{2} \text{ for all } n \geq 2.$$

Show this sequence is Cauchy.

*Proof.* Look at the differences of consecutive terms,  $|a_1 - a_2| = 1$ ,  $|a_2 - a_3| = 1/2$ , we can see a formula  $a_n - a_{n+1} = 1/2^{n-1}$ . Assume  $|a_n - a_m| = |a_n - a_{n+1} - a_{n+2}| - \dots -$



$a_{m-1} - a_m$  with  $n < m$ . From the **Triangle Inequality**,

$$|a_n - a_m| \leq |a_n - a_{n+1}| + |a_{n+1} - a_{n+2}| + \cdots + |a_{m-1} - a_m| \quad (2.10)$$

$$= \frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{m-2}} \quad (2.11)$$

$$= \frac{1}{2^{n-1}} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}} \right) \quad (2.12)$$

$$= \frac{1}{2^{n-1}} \left( \frac{1 - \frac{1}{2^{m-n}}}{1 - \frac{1}{2}} \right) \quad (2.13)$$

$$= \frac{1}{2^n} \left( 1 - \frac{1}{2^{m-n}} \right) \quad (2.14)$$

$$< \frac{1}{2^n}. \quad (2.15)$$

Notice that we were able to pull out the  $1/2$  and use the geometric series formula at step 2.12. From here we know that  $|a_n - a_m| < \frac{1}{2^n}$ .

Now, conclude the proof by letting  $\epsilon > 0$ . We know  $(1/2^n) \rightarrow 0$ . Thus, there exists  $N \in \mathbb{N}$  such that  $\frac{1}{2^N} < \epsilon$ . For all  $n, m \geq N$ , (without loss of generality  $n < m$ )  $|a_n - a_m| < \frac{1}{2^n} \leq \frac{1}{2^N} < \epsilon$ . Therefore,  $a_n$  is **Cauchy** and it converges.  $\square$

**Note:** To find the limit of this series, a proof strategy is finding subsequences that are odd and even, and show the converge to the same limit.

### Theorem 2.6.2: Cauchy Criterion

A sequence  $x_n$  converges if, and only if, it is a Cauchy sequence.



*Proof.* We will show this by proving both implications:

( $\Rightarrow$ ) Assume  $(x_n)$  is a convergent sequence in  $\mathbb{R}$ . Given  $\epsilon > 0$ . Let  $L = \lim_{n \rightarrow \infty} x_n$ . Since  $(x_n) \rightarrow L$ , there exists an  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - L| < \frac{\epsilon}{2}$ . For all  $n, m \geq N$ ,

$$\begin{aligned} |x_m - x_n| &= |x_m - L + L - x_n| \\ &\leq |x_m - L| + |L - x_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore,  $x_n$  is a Cauchy sequence.

( $\Leftarrow$ ) Assume  $x_n$  is a Cauchy sequence.

- **Step 1:** Show that  $x_n$  is bounded.

Since  $x_n$  is Cauchy, there exists  $N \in \mathbb{N}$  such that for all  $n, m \geq N$ ,  $|x_n - x_m| < 1$ . It follows that for all  $n \geq N$ , we need to account for  $x_1, \dots, x_{N-1}$ . Thus, let  $M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |x_N| + 1\}$ . Then for all  $n \in \mathbb{N}$ ,  $|x_n| < M$ .

- **Step 2:** Since  $x_n$  is bounded, there exists a convergent subsequence  $x_{n_k}$  by the **Bolzano-Weierstrass Theorem**. Let  $L$  be the limit of the subsequence.

- **Step 3:** Show that  $x_n$  converges to  $L$ .

If some get close to  $L$  and all get close to each other, they all get close to  $L$ . Let  $\epsilon > 0$ . Since  $x_{n_k}$  converges to  $L$ , there exists  $N \in \mathbb{N}$  such that for all  $k \geq N$ ,  $|x_{n_k} - L| < \frac{\epsilon}{2}$ . Since  $x_n$  is Cauchy, there exists  $M \in \mathbb{N}$  such that for all  $n, m \geq M$ ,  $|x_n - x_m| < \frac{\epsilon}{2}$ . Let  $M_0 = \max\{N, n_k\}$ . By the **Archimedean Principle**, there exists  $N_0$  such that  $n_{k_0} \geq M_0$ . Then, from the **Triangle Inequality**, we say that for all  $n \geq N_0$ ,

$$\begin{aligned} |x_n - L| &\leq |x_n - x_{n_{k_0}}| + |x_{n_{k_0}} - L| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore,  $(x_n) \rightarrow L$ .

By proving both directions of the inequality, we found that a sequence  $(x_n)$  converges if, and only if, it is a Cauchy sequence.  $\square$



### Definition 2.6.3

A sequence is called *contracting* if there exists  $0 < C < 1$  such that for all  $n \in \mathbb{N}$ ,  $|x_{n+1} - x_n| \leq C |x_n - x_{n-1}|$ .

How this works: we take a sequence  $a_1, a_2, \dots$  and subtract  $a_1 - a_2$ . Then, we have the inequality:

$$\begin{aligned} |a_2 - a_1| &\leq C |a_1 - a_0| \\ |a_3 - a_2| &\leq C |a_2 - a_1| \leq C^2 |a_1 - a_0| \\ |a_4 - a_3| &\leq C |a_3 - a_2| \leq C^3 |a_1 - a_0| \\ &\vdots \end{aligned}$$

From this, a theorem emerges:

### Theorem 2.6.4

If a sequence is contracting, then it is Cauchy, and thus converges.

*Proof.* Let  $(a_n)$  be a contracting sequence; that is, there exists a constant  $0 < C < 1$  such that for all  $n \in \mathbb{N}$ ,

$$|a_{n+1} - a_n| \leq C |a_n - a_{n-1}|.$$

We will show that  $(a_n)$  is a Cauchy sequence.

First, we observe by induction that for all  $k \geq 1$ ,

$$|a_{n+k} - a_{n+k-1}| \leq C^k |a_n - a_{n-1}|.$$

*Proof by induction:*

**Base case** ( $k = 1$ ):

$$|a_{n+1} - a_n| \leq C |a_n - a_{n-1}|.$$

**Inductive step:** Assume that for some  $k \geq 1$ ,

$$|a_{n+k} - a_{n+k-1}| \leq C^k |a_n - a_{n-1}|.$$

Then,

$$\begin{aligned} |a_{n+k+1} - a_{n+k}| &\leq C |a_{n+k} - a_{n+k-1}| \\ &\leq C (C^k |a_n - a_{n-1}|) \\ &= C^{k+1} |a_n - a_{n-1}|. \end{aligned}$$

Thus, the inequality holds for  $k + 1$ , completing the induction.





Next, for any integers  $m > n$ , we have:

$$|a_m - a_n| = \left| \sum_{j=n}^{m-1} (a_{j+1} - a_j) \right| \leq \sum_{j=n}^{m-1} |a_{j+1} - a_j|.$$

Applying the inequality obtained from the induction,

$$|a_{j+1} - a_j| \leq C^{j-n+1} |a_n - a_{n-1}|.$$

Therefore, Since  $C^{m-n} \geq 0$ , we have:

$$|a_m - a_n| \leq |a_n - a_{n-1}| \left( \frac{C}{1-C} \right).$$

As  $n \rightarrow \infty$ , the term  $|a_n - a_{n-1}|$  tends to zero because:

$$|a_n - a_{n-1}| \leq C^{n-1} |a_1 - a_0| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, for any  $\epsilon > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,

$$|a_n - a_{n-1}| < \epsilon \left( \frac{1-C}{C} \right).$$

Then, for all  $m, n \geq N$  (with  $m > n$ ),

$$|a_m - a_n| \leq |a_n - a_{n-1}| \left( \frac{C}{1-C} \right) < \epsilon.$$

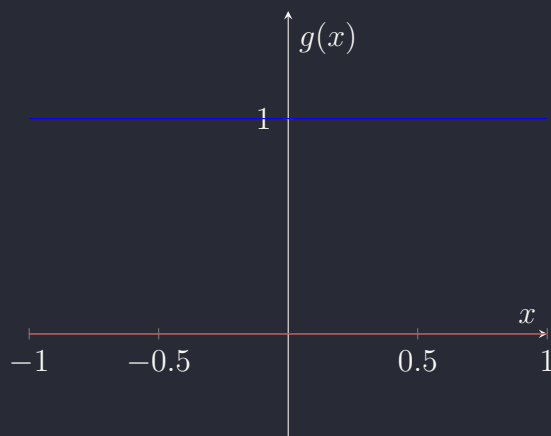
This shows that  $(a_n)$  is a Cauchy sequence. Since every Cauchy sequence in  $\mathbb{R}$  converges, the sequence  $(a_n)$  converges.  $\square$

## 4.1 Discussion: Examples of Dirichlet and Thomae

## Definition 4.1.1

The *Dirichlet function*  $\lim_{x \rightarrow c} g(x)$  does not exist for any  $c \in \mathbb{R}$ .

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$



## Definition 4.1.2

The *Thomae function* is defined as

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ is in lowest terms,} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

*Thomae's function*,  $t(x)$  is continuous at all  $x \notin \mathbb{Q}$ . It is not continuous at any  $x \in \mathbb{Q}$ .



## 4.2 Functional Limits

Recall from calculus I, that a function  $f(x)$  is continuous at  $x = c$  if  $\lim_{x \rightarrow c} f(x) = f(c)$ .

### Definition 4.2.1

Let  $f: A \rightarrow \mathbb{R}$  be a function and let  $c$  be a limit point of  $A$ . We say  $\lim_{x \rightarrow c} f(x) = L$ , if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $0 < |x - c| < \delta$ , then  $|f(x) - L| < \epsilon$ .

### Example 4.1: Functional Limit (From book) 1

Let  $f(x) = 3x + 1$ . Claim:  $\lim_{x \rightarrow 2} f(x) = 7$ .

*Proof.* Let  $\epsilon > 0$ . After we have done our scratch work, we can choose  $\delta = \epsilon/3$ , then  $0 < |x - 2| < \delta$  implies  $|f(x) - 7| < 3(\epsilon/3) = \epsilon$ .  $\square$

*Scratch Paper.* Definition 4.2.1 requires that we produce a  $\delta > 0$  so that  $0 < |x - 2| < \delta$  leads to the conclusion that  $|f(x) - 7| < \epsilon$ . Notice that

$$|f(x) - 7| = |3x + 1 - 7| = |3x - 6| = 3|x - 2|.$$

### Example 4.2: Functional Limit (From book) 2

Let  $g(x) = x^2$ . Claim:  $\lim_{x \rightarrow 2} g(x) = 4$ .

*Proof.* Let  $\epsilon > 0$ . Choose  $\delta = \min\{1, \epsilon/5\}$ . If  $0 < |x - 2| < \delta$ , then

$$\begin{aligned} |g(x) - 4| &= |x^2 - 4| \\ &= |x - 2| |x + 2| \\ &< 5\delta \\ &= (5) \frac{\epsilon}{5} \\ &= \epsilon. \end{aligned}$$

$\square$

*Scratch Paper.* Our goal this time is to make  $|g(x) - 4| < \epsilon$  by restricting  $|x - 2|$  to be smaller than some carefully chosen  $\delta$ . As in the previous example, a little algebra reveals

$$|g(x) - 4| = |x^2 - 4| = |x - 2| |x + 2|.$$

We can make  $|x + 2|$  as small as we like, but we need an upper bound on  $|x + 2|$  in order to know how small to choose  $\delta$ . The presence of the variable  $x$  causes some initial confusion, but keep in mind that we are discussing the limit as  $x$  approaches 2. If we agree that our  $\delta$ -neighborhood around  $c = 2$  must have radius no bigger than  $\delta = 1$ , then we get the upper bound  $|x + 2| < |3 + 2| = 5$  for all  $x \in V_\delta(c)$ .



### Example 4.3: Functional Limit 1

Let  $f(x) = 3x + 1$ . Show that  $\lim_{x \rightarrow 2} f(x) = 7$ .

*Proof.* Let  $\epsilon > 0$ . Set  $\delta = \frac{\epsilon}{3}$ . Assume  $0 < |x - 2| < \delta$ . Since  $\delta > 0$ ,  $2 - \delta < x < 2 + \delta$ . Then,

$$\begin{aligned} |x - 2| &< \delta, \\ |f(x) - 7| &= |3x + 1 - 7| \\ &= |3x - 6| \\ &= 3|x - 2| \\ &< 3\delta \\ &= \epsilon. \end{aligned}$$

Therefore,  $\lim_{x \rightarrow 2} f(x) = 7$ . □

### Example 4.4: Functional Limit 3

Let  $f(x) = x^2$ . Claim:  $\lim_{x \rightarrow 7} f(x) = 49$

*Proof.* Let  $\epsilon > 0$ . Set  $\delta = \min\{\frac{\epsilon}{8}, 1\}$ . If  $0 < |x - 7| < \delta$ , then

$$\begin{aligned} |f(x) - 49| &= |x^2 - 49| \\ &= |x - 7| |x + 7| \\ &< 8\delta \\ &< 8 \left( \frac{\epsilon}{8} \right) \\ &= \epsilon. \end{aligned}$$

*Scratch Paper.* Always start with the goal statement:  $|f(x) - 49| = |x^2 - 49|$ . This factors into  $|x - 7| |x + 7|$ . Then, if  $\delta < 1$ ,  $|x - 7| < \delta$  and  $|x + 7| < 8$ . All together, we have  $8\delta < \epsilon < \frac{\epsilon}{8}$ .

□

### Example 4.5: Functional Limit 4

Claim:  $\lim_{x \rightarrow 3} \frac{1}{x+1} = \frac{1}{4}$ .



*Proof.* Let  $\epsilon > 0$ . Set  $\delta = \min\{12\epsilon, 1\}$ .  
If  $0 < |x - 3| < \delta$ , then

$$\begin{aligned} \left| \frac{1}{x+1} - \frac{1}{4} \right| &= \left| \frac{4 - (x+1)}{4(x+1)} \right| \\ &= \left| \frac{3-x}{4(x+1)} \right| \\ &< \frac{\delta}{4(3)} \\ &= \frac{12\epsilon}{12} \\ &= \epsilon. \end{aligned}$$

*Scratch Paper.* Goal:  $\left| \frac{1}{x+1} - \frac{1}{4} \right|$ . Hence,

$$\begin{aligned} \left| \frac{1}{x+1} - \frac{1}{4} \right| &= \left| \frac{4 - (x+1)}{4(x+1)} \right| \\ &= \left| \frac{3-x}{4(x+1)} \right| \\ &< \frac{\delta}{4|x+1|} \\ &< \frac{\delta}{4(3)} \\ &= \frac{\delta}{12} \\ &< \epsilon. \end{aligned}$$

Therefore,  $\lim_{x \rightarrow 3} \frac{1}{x+1} = \frac{1}{4}$

□

Thus, we need a  $\delta < 1$ , and we can choose  $\delta = \min\{12\epsilon, 1\}$ . Note: When we are determining the value for  $|x + 2|$ , we solve for  $\delta = 3 \pm 1 \Rightarrow x \in (2, 4)$ . Then, we find  $x + 1 = (3, 5)$ . We choose 3 rather than 5 because of division. We want to be as close as possible.

### Example 4.6: Functional Limit 5

Claim:  $\lim_{x \rightarrow 3} (x^2 + 7x) = 30$ .

*Proof.* Let  $\epsilon > 0$  and set  $\delta = \min\{\frac{\epsilon}{14}, 1\}$ . If  $0 < |x - 3| < \delta$ , then

$$\begin{aligned} |x^2 + 7x - 30| &= |x - 3| |x + 10| \\ &< 14\delta \\ &= 14 \left( \frac{\epsilon}{14} \right) \\ &= \epsilon. \end{aligned}$$

□

### Example 4.7: Functional Limit 6

Claim:  $\lim_{x \rightarrow 3} \frac{2x+3}{4x-9} = 3$ .



*Proof.* Let  $\epsilon > 0$ . Set  $\delta = \min\{\frac{\epsilon}{10}, \frac{1}{2}\}$ . (Note: We are choosing  $\frac{1}{2}$  because we want to avoid having 0 anywhere in the interval.) Assume  $0 < |x - 3| < \delta$ . Since  $\delta < \frac{1}{2}$ ,  $\frac{5}{2} < x < \frac{7}{2}$ , then  $1 < |4x - 9| < 5$ . (Thus, 0 can not possibly be in the denominator.)  $\square$

*Scratch Paper.*

$$\begin{aligned} \left| \frac{2x+3}{4x+9} - 3 \right| &= \left| \frac{2x+3-3(4x+9)}{4x+9} \right| \\ &= \left| \frac{2x+3-12x-27}{4x+9} \right| \\ &= 10 \left| \frac{x-3}{4x-4} \right| \\ &< 10 \frac{\epsilon/10}{1} \\ &= \epsilon. \end{aligned}$$

### Example 4.8: Functional Limit 7

Claim:  $\lim_{x \rightarrow 4} \sqrt{x} = 2$ .

*Proof.* Let  $\epsilon > 0$ . Set  $\delta = \min\{1, 3\epsilon\}$ . Assume  $0 < |x - 4| < \delta$ . Then (refer to scratch work).  $\square$

*Scratch Paper.*

$$\begin{aligned} |\sqrt{x} - 2| &= |\sqrt{x} - 2| \\ &= \left| \frac{(\sqrt{x} - 2) \cdot (\sqrt{x} + 2)}{\sqrt{x} + 2} \right| \\ &= \left| \frac{x - 4}{\sqrt{x} + 2} \right| \\ &< \frac{\delta}{3} \\ &< \frac{3\epsilon}{3} \\ &= \epsilon. \end{aligned}$$

Notice that we picked  $\delta < 1$  such that  $3 < x < 4$  so  $1 < \sqrt{x} < 2$  and  $3 < \sqrt{x} + 2 < 4$ .

### Theorem 4.2.2: Sequential Criterion for Functional Limits

The following statements are equivalent:

1.  $\lim_{x \rightarrow c} f(x) = L$ .
2. For all sequences  $(x_n)$  where  $x_n \neq c$  and  $(x_n) \rightarrow c$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = L$ .



*Proof.* (1)  $\rightarrow$  (2)

Assume  $\lim_{x \rightarrow c} f(x) = L$ .

Let  $(x_n) \rightarrow c$  with  $x_n \neq c$

Let  $\epsilon > 0$ .

- Since  $\lim_{x \rightarrow c} f(x) = L$ , there exists  $\delta > 0$  such that if  $0 < |x - c| < \delta$ , then  $|f(x) - L| < \epsilon$ .
- Since  $x_n \rightarrow c$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $|x_n - c| < \delta$ .
- Now, for all  $n \geq N$ , it follows that  $x_n - c < \delta$  and thus  $|f(x) - L| < \epsilon$ .

Thus,  $\lim_{n \rightarrow \infty} f(x_n) = L$ .

(2)  $\rightarrow$  (1)

Proof by contrapositive.

Assume (1) is not true. Thus,

$$\lim_{x \rightarrow c} f(x) \neq L.$$

There exists  $\epsilon_0 > 0$  such that for all  $\delta > 0$ , there exists an  $x$  with  $0 < |x - c| < \delta$  and  $|f(x) - L| \geq \epsilon_0$ .

For each  $n \in \mathbb{N}$ , consider  $\delta = \frac{1}{n}$ . There exists  $x_n \in (c - \frac{1}{n}, c + \frac{1}{n})$  with  $x_n \neq c$  such that  $|f(x) - L| \geq \epsilon_0$ .

- Since  $|x_n - c| < \frac{1}{n}$ , we see that  $(x_n) \rightarrow c$ .
- Since for all  $n \in \mathbb{N}$ ,  $|f(x) - L| \geq \epsilon_0$ . Then,  $\lim_{n \rightarrow \infty} f(x) \neq L$ .

Thus,  $\neg(1) \rightarrow \neg(2)$ . So (2)  $\rightarrow$  (1) and (1)  $\rightarrow$  (2). □

If functional limits and sequential limits are the same thing, then everything we know about sequential limits is also true about functional limits.

Recall **Algebraic Limit Theorem**. From this, we can write the functional equivalent:

Assume  $\lim_{x \rightarrow c} f(x) = L$  and  $\lim_{x \rightarrow c} g(x) = M$ . Then,

- $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$
- $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$
- $\lim_{x \rightarrow c} (f(x)g(x)) = LM$
- $\lim_{x \rightarrow c} \left(\frac{f(x)}{g(x)}\right) = \frac{L}{M}$  unless  $M = 0$ .



### Theorem 4.2.3: Divergence Criterion

Let  $f: A \rightarrow \mathbb{R}$  with  $c$  as a limit point of  $A$ . If there exists two sequences  $(x_n)$  and  $(y_n)$  in  $A \setminus \{c\}$  (that both converge to  $c$ ) such that  $\lim_{n \rightarrow \infty} f(x_n) \neq \lim_{n \rightarrow \infty} f(y_n)$ , then  $\lim_{x \rightarrow c} f(x)$  does not exist.

### Example 4.9: Divergence Criterion 1

$f(x) = \frac{x}{|x|} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$  Our goal is to show that  $\lim_{x \rightarrow 0} f(x)$  does not exist.

*Proof.* Let  $(x_n) = (\frac{1}{n})$  and let  $(y_n) = (\frac{-1}{n})$ . We will see that as  $n \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} f(x_n) = 1$  and  $\lim_{n \rightarrow \infty} f(y_n) = -1$ . Thus,  $\lim_{x \rightarrow 0} f(x)$  does not exist.  $\square$

### Example 4.10: Divergence Criterion 2

$g(x) = \sin(\frac{1}{x})$ . Show that  $\lim_{x \rightarrow 0} g(x)$  does not exist.

*Proof.* Let  $(x_n) = (\frac{1}{2\pi n})$  and let  $(y_n) = (\frac{1}{2\pi n + \frac{\pi}{2}})$ . We will see that as  $n \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} g(x_n) = 1$  and  $\lim_{n \rightarrow \infty} g(y_n) = -1$ . Thus,  $\lim_{x \rightarrow 0} g(x)$  does not exist.  $\square$

We say  $\lim_{n \rightarrow \infty} x_n = \infty$  if for all  $M > 0$ , there exists  $N \in \mathbb{N}$  such that for all  $n \geq N$ ,  $x_n > M$ .

We say  $\lim_{x \rightarrow c} f(x) = \infty$  if for all  $M > 0$ , there exists  $\delta > 0$  such that if  $0 < |x - c| < \delta$ , then  $f(x) > M$ . Think of vertical asymptotes.

### Theorem 4.2.4: Infinite Limits Cauchy Criterion

If  $(x_n) \rightarrow \infty$ ,  $(x_n)$  will not be Cauchy. It is possible to have  $x_{n+1} - x_n$  approach 0, but  $(x_n)$  converges to  $\infty$ .





## 4.3 Continuous Functions

### Definition 4.3.1

We say a function  $f$  is *continuous* at  $c$  if

$$\lim_{x \rightarrow c} f(x) = f(c).$$

Equivalent definition:

For all  $\epsilon > 0$ , there exists  $\delta > 0$  such that if  $|x - c| < \delta$ , then

$$|f(x) - f(c)| < \epsilon.$$

Idea: When  $x$  is close to  $c$ ,  $f(x)$  is close to  $f(c)$ . Then, for the topological definition, we can say if  $x \in V_\delta(c)$  then  $f(x) \in V_\epsilon(f(c))$ .

### Definition 4.3.2

We say function  $f$  is *continuous* on a set  $D$  if  $f$  is continuous at every point in  $D$ .

The following are equivalent (TFAE):

1.  $\lim_{x \rightarrow c} f(x) = L$
2. For all sequences  $(x_n)$  such that  $(x_n) \rightarrow c$ , we have  $\lim_{n \rightarrow \infty} f(x_n) = L$ .

Continuous Functions (THM 4.3.2 in book)

Claim: Let  $a \in \mathbb{R}$ . Then  $f(x) = a$  is continuous.

*Proof.* Let  $c \in \mathbb{R}$ . Let  $\epsilon > 0$ . Set  $\delta = \epsilon$ . Now, if  $|x - c| < \delta$ , then  $|f(x) - f(c)| = |a - a| = 0 < \epsilon$ . Thus, constant functions are continuous.  $\square$

Claim:  $f(x) = x$  is continuous.

*Proof.* Let  $c \in \mathbb{R}$ . Let  $\epsilon > 0$ . Set  $\delta = \epsilon$ . If  $|x - c| < \delta$ , then  $|f(x) - f(c)| = |x - c| < \delta = \epsilon$ . Thus, the identity function is continuous.  $\square$

Claim:  $g(x) = \sqrt{x}$  is continuous on  $[0, \infty)$ .

*Proof.* • **Case 1:**  $c \neq 0$

Let  $c \in [0, \infty)$ . Let  $\epsilon > 0$ . Set  $\delta < \epsilon$ . If  $|x - c| < \delta$ , then  $|g(x) - g(c)| = |\sqrt{x} - \sqrt{c}| = \frac{|x - c|}{\sqrt{x} + \sqrt{c}} < \frac{\delta}{\sqrt{c}} < \epsilon$ . Thus,  $g(x) = \sqrt{x}$  is continuous on  $[0, \infty)$ .

• **Case 2:**  $c = 0$

Let  $c \in [0, \infty)$ . Let  $\epsilon > 0$ . Set  $\delta = \epsilon^2$ . If  $|x - 0| < \delta$ , then  $|g(x) - g(0)| = |\sqrt{x} - 0| = \sqrt{x} < \sqrt{\delta} = \epsilon$ . Thus,  $g(x) = \sqrt{x}$  is continuous on  $[0, \infty)$ .



Therefore,  $g(x)$  is continuous for all  $c \in [0, \infty)$ . □

### Theorem 4.3.3: Compositions of Continuous Functions

Let  $f$  be continuous at  $c$ . Let  $g$  be continuous at  $f(c)$ . Then,

$$g \circ f(x) = g(f(x)) \text{ is continuous.}$$

*Proof.* Let  $\epsilon > 0$ . Since  $g$  is continuous at  $f(c)$ , there exists  $\delta_1 > 0$  such that if  $|x - f(c)| < \delta_1$ , then  $|g(x) - g(f(c))| < \epsilon$ . Since  $f$  is continuous at  $c$ , there exists  $\delta_2 > 0$  such that if  $|x - c| < \delta_2$ , then  $|f(x) - f(c)| < \delta_1$ . Thus, if  $|x - c| < \delta_2$ , then  $|g(f(x)) - g(f(c))| < \epsilon$ . Therefore,  $g \circ f(x)$  is continuous. □

### Most Common Applications of Continuity Is with limits.

If  $f$  is continuous at  $c$  and  $(x_n) \rightarrow c$ , then  $\lim_{n \rightarrow \infty} f(x_n) = f(c)$ . Hence,

$$\lim_{n \rightarrow \infty} f(x_n) = f\left(\lim_{n \rightarrow \infty} x_n\right).$$

## 4.4 Continuous Functions on Compact Sets

### Theorem 4.4.1: Extreme Value Theorem

If  $K$  is compact and  $f : K \rightarrow \mathbb{R}$  is continuous, then  $f$  attains a maximum and minimum value on  $K$ .

In other words, there exists  $a \in K$  such that  $f(a) = \sup\{f(x) \mid x \in K\}$ . Also, there exists  $b \in K$  such that  $f(b) = \inf\{f(x) \mid x \in K\}$ .

*Proof.* We know  $f$  is bounded on  $K$  from [Lemma 4.4.2](#) below. Hence,

$$S = \sup\{f(x) \mid x \in K\} \text{ exists.}$$

For every natural number, there exists an  $x_n \in K$  such that  $S - \frac{1}{n} < f(x_n) \leq S$ . It follows that  $\lim_{n \rightarrow \infty} f(x_n) = S$ . So, now we have a sequence,  $(x_n)$  in the compact set  $K$ . Since  $K$  is compact, by the Bolzano-Weierstrass Theorem, there exists a convergent subsequence,

$$(x_{n_j}) \text{ with } a = \lim_{j \rightarrow \infty} x_{n_j} \in K.$$

Since  $f$  is continuous, we have,

$$f(a) = \lim_{j \rightarrow \infty} f(x_{n_j}) = S.$$



By a similar method, there exists  $b \in K$  such that

$$f(b) = \inf\{f(x) \mid x \in K\}.$$

**Note:** This proof hinges on the fact that  $f$  is bounded! We need to show that  $f$  is bounded on  $K$  with a proof with subcovers.  $\square$

### Lemma 4.4.2

How do we know  $f$  is bounded on  $K$ ? That is,

$$f(K) = \{f(x) \mid x \in K\}.$$

Show that  $f(K)$  is bounded.

*Proof.* Let  $c \in K$ . Since  $f$  is continuous at  $c$ , there exists  $\delta_c > 0$  such that if  $|x - c| < \delta_c$ , then

$$|f(x) - f(c)| < 1.$$

Do this over every  $c \in K$ . We get an open cover of  $K$ .

$$\mathcal{O} = \{V_{\delta_c}(c) \mid c \in K\}.$$

Since  $K$  is compact, the Heine-Borel Theorem says there exists a finite subcover. We get  $c_1, c_2, \dots, c_n \in K$  such that  $K \subseteq \bigcup_{i=1}^n V_{\delta_{c_i}}(c_i)$ . Thus,

$$\begin{aligned} f(K) &\subseteq f\left(\bigcup_{i=1}^n V_{\delta_{c_i}}(c_i)\right) \\ &\subseteq \bigcup_{i=1}^n f(V_{\delta_{c_i}}(c_i)) \\ &\subseteq \bigcup_{i=1}^n (f(c_i) - 1, f(c_i) + 1) \\ &\subseteq [\min f(c_i) - 1, \max f(c_i) + 1]. \end{aligned}$$

Therefore,  $f(K)$  is bounded.  $\square$

### Theorem 4.4.3: Preservation of Compact Sets

If  $K$  is compact and  $f: K \rightarrow \mathbb{R}$  is continuous, then  $f(K)$  is compact.

*Proof.* Let  $(y_n)$  be a sequence in  $f(K)$ . We will show  $(y_n)$  has a convergent subsequence with its limit in  $f(K)$ .

For each  $n$  there exists  $x_n \in K$  such that  $f(x_n) = y_n$ . So  $(x_n)$  is a sequence in a



compact set  $K$ . There exists a convergent subsequence  $(x_{n_j})$  with

$$a = \lim_{j \rightarrow \infty} x_{n_j} \in K.$$

Now consider the corresponding subsequence  $(y_{n_j})$  in  $f(K)$ . Since  $f$  is continuous, we have

$$\begin{aligned} \lim_{j \rightarrow \infty} y_{n_j} &= \lim_{j \rightarrow \infty} f(x_{n_j}) \\ &= f(a) \in f(K). \end{aligned}$$

So,  $x_{n_j}$  is a convergent subsequence with limit in  $f(K)$ . Therefore,  $f(K)$  is compact.  $\square$

#### Definition 4.4.4

A function  $f: A \rightarrow \mathbb{R}$  is *uniformly continuous* on  $A$  if for all  $\epsilon > 0$ , there exists  $\delta > 0$  such that for all  $x, c \in A$ , if  $|x - c| < \delta$ , then  $|f(x) - f(c)| < \epsilon$ .

Compare this definition with [Definition 4.3.1](#). The difference is that the  $\delta$  is independent of  $x$ . That is, we have to find one  $\delta$  that needs to work for every point  $x$ .

#### Definition 4.4.5

A function  $f: A \rightarrow \mathbb{R}$  is *not uniformly continuous* on  $A$  if there exists  $\epsilon_0 > 0$  such that for all  $\delta > 0$ , there exists  $x, c \in A$  such that  $|x - c| < \delta$  and  $|f(x) - f(c)| \geq \epsilon_0$ .

#### Theorem 4.4.6

If  $K \subseteq \mathbb{R}$  is compact and  $f: K \rightarrow \mathbb{R}$  is continuous, then  $f$  is uniformly continuous on  $K$ .

*Proof.* Suppose  $f$  is not uniformly continuous on  $K$ . Then, there exists  $\epsilon_0 > 0$  such that for all  $n \in \mathbb{N}$ , there exists  $x_n, y_n \in K$  such that  $|x_n - y_n| < \frac{1}{n}$  and  $|f(x_n) - f(y_n)| \geq \epsilon_0$ .

We now have two sequences  $(x_n)$  and  $(y_n)$  in  $K$ . Since  $K$  is compact, by the Heine-Borel theorem, there exists a convergent subsequence  $(x_{n_i})$  which converges to a point  $x_0 \in K$ .

Since  $K$  is compact,  $(y_n)$  has a convergent subsequence  $(y_{n_{i_j}})$  which converges to a point  $y_0 \in K$ . Notice that since  $(x_{n_{i_j}})$  is a subsequence of  $(x_{n_i})$ , it converges to  $x_0$ . Since  $f$  is continuous:

$$\lim_{j \rightarrow \infty} f(x_{n_{i_j}}) = f(x_0) \quad \text{and} \quad \lim_{j \rightarrow \infty} f(y_{n_{i_j}}) = f(y_0).$$



Because

$$\left| f(x_{n_{i_j}}) - f(y_{n_{i_j}}) \right| < \frac{1}{n_{i_j}},$$

we can see that

$$\lim_{j \rightarrow \infty} |x_{n_{i_j}} - y_{n_{i_j}}| = 0.$$

It follows that  $x_0 = y_0$ . But this is a contradiction because  $|f(x_0) - f(y_0)| \geq \epsilon_0$ . Therefore,  $f$  is uniformly continuous on  $K$ .  $\square$

## 4.5 The Intermediate Value Theorem

### Theorem 4.5.1: Intermediate Value Theorem

Let  $f: [a, b] \rightarrow \mathbb{R}$  be continuous. If  $L$  is a real number satisfying  $f(a) < L < f(b)$  or  $f(b) < L < f(a)$ , then there exists  $c \in (a, b)$  such that  $f(c) = L$ .

**Note:** IVT does not guarantee where, or how many  $c$ 's are in the interval. It only guarantees that at least one  $c$  exists.

*Proof.* (Using **Nested Interval Property**) Without the loss of generality, assume  $f(a) < f(b)$  and let  $y \in (f(a), f(b))$ . Let  $I_1 = [a_1, b_1]$ . Bisect  $I_1$  into two intervals  $[a_1, d]$  and  $d, b_1$  where  $d = \frac{a_1 + b_1}{2}$ .

- If  $f(d) < y$ , set  $a_2 = d$ ,  $b_2 = b_1$ , and  $I_2 = [a_2, b_2]$ . Notice that  $f(a_2) < y < f(b_2)$ .
- If  $f(d) > y$ , then set  $a_2 = a_1$ ,  $b_2 = d$ , and  $I_2 = [a_2, b_2]$ . Notice that  $f(a_2) < y < f(b_2)$ .

Repeat this process indefinitely. We end up with a sequence of nested intervals  $I_n = [a_n, b_n]$ , where

- $I_n \subseteq I_{n-1}$
- $f(a_n) < y < f(b_n)$
- $|a_n - b_n| = \frac{a_n - b_n}{2^{n-1}}$

By the **Nested Interval Property**, there exists a point  $c$  such that  $c \in \bigcap_{n=1}^{\infty} I_n$ . In fact, there is a unique point  $c$  in the intersection. It follows that

$$c = \lim_{n \rightarrow \infty} a_n \quad \text{and} \quad c = \lim_{n \rightarrow \infty} b_n.$$

Since  $f$  is continuous, we have

$$f(c) = \lim_{n \rightarrow \infty} f(a_n) \leq y$$



$$f(c) = \lim_{n \rightarrow \infty} f(b_n) \geq y.$$

Therefore,  $f(c) = y$ .

□

### What Is Important About Continuous Functions?

If  $\lim_{n \rightarrow \infty} (x_n = x)$ , then  $\lim_{n \rightarrow \infty} f(x_n) = f(x)$ .

Think about  $f(x) = 2^x$ . Thus,  $f x$  makes sense if  $x \in \mathbb{Q}$ :

$$2^{\frac{p}{q}} = \sqrt[q]{2^p}.$$

But how do we make sense of something like  $2^\pi$ ?

We can find  $f: \mathbb{Q} \rightarrow \mathbb{R}$  is continuous.

We can define  $f: \mathbb{R} \rightarrow \mathbb{R}$  to be continuous

If  $(q_n)$  is in  $\mathbb{Q}$  and  $(q_n \rightarrow \pi)$ , then we define

$$f(\pi) = \lim_{n \rightarrow \infty} f(q_n).$$

**Exercise: 4.3.11 (Contraction Mapping Theorem)**

Let  $f$  be a function defined on all of  $\mathbb{R}$ , and assume there is a constant  $c$  such that  $0 < c < 1$  and

$$|f(x) - f(y)| \leq c|x - y|$$

for all  $x, y \in \mathbb{R}$ .

- (a) Show that  $f$  is continuous on  $\mathbb{R}$ .
- (b) Pick some point  $y_1 \in \mathbb{R}$  and construct the sequence

$$(y_1, f(y_1), f(f(y_1)), \dots).$$

In general, if  $y_{n+1} = f(y_n)$ , show that the resulting sequence  $(y_n)$  is a Cauchy sequence ([Definition 2.6.1](#)). Hence, we may let  $y = \lim y_n$ .

- (c) Prove that  $y$  is a fixed point of  $f$  (i.e.,  $f(y) = y$ ) and that it is unique in this regard.
- (d) Finally, prove that if  $x$  is *any* arbitrary point in  $\mathbb{R}$ , then the sequence  $(x, f(x), f(f(x)), \dots)$  converges to  $y$  defined in (b).

*Solution.*

- (a) Let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{c}$ . Then, for all  $x, y \in \mathbb{R}$  with  $|x - y| < \delta$ , we have

$$|f(x) - f(y)| \leq c|x - y| < c\left(\frac{\epsilon}{c}\right) = \epsilon.$$

Thus,  $f$  is continuous at every point in  $\mathbb{R}$ .



- (b) *Proof.* Since  $y_{n+1} = f(y_n)$ , we have  $y_2 = f(y_1)$ ,  $y_3 = f(y_2) = f(f(y_1))$ , and so on. Thus, the difference between consequent terms in the sequence is:

$$|y_{n+1} - y_n| = |f(y_n) - f(y_{n-1})| \leq c|y_n - y_{n-1}|.$$

Substituting, we see that:

$$|y_{n+1} - y_n| \leq c|y_n - y_{n-1}|.$$

This inequality shows that our sequence is contracting ([Definition 2.6.3](#)) with  $0 < c < 1$ . Thus, by [Theorem 2.6.4](#), the sequence is Cauchy. □

- (c) Taking the limit as  $n \rightarrow \infty$ , and using the continuity of  $f$ , we have:

$$f(y) = f\left(\lim_{n \rightarrow \infty} y_n\right) = \lim_{n \rightarrow \infty} f(y_n) = \lim_{n \rightarrow \infty} y_{n+1} = y.$$

Suppose there is another fixed point  $z$  such that  $f(z) = z$ . Then

$$|y - z| = |f(y) - f(z)| \leq c|y - z|.$$

Since  $0 < c < 1$ , this implies  $y - z = 0$ , so  $y = z$ . Thus, the fixed point is unique.

- (d) From the work we did in (b), we know the fixed point must be unique. Therefore, for any  $x \in \mathbb{R}$ , it must be the case that the sequence  $(x, f(x), f(f(x)), \dots)$  converges to the fixed point  $y$ .

### Exercise: 4.3.13

Let  $f$  be a function defined on all of  $\mathbb{R}$  that satisfies the additive condition  $f(x + y) = f(x) + f(y)$  for all  $x, y \in \mathbb{R}$ .

- Show that  $f(0) = 0$  and that  $f(-x) = -f(x)$  for all  $x \in \mathbb{R}$ .
- Let  $k = f(1)$ . Show that  $f(n) = kn$  for all  $n \in \mathbb{N}$ , and then prove that  $f(z) = kz$  for all  $z \in \mathbb{Z}$ . Now, prove that  $f(r) = kr$  for any rational number  $r$ .
- Show that if  $f$  is continuous at  $x = 0$ , then  $f$  is continuous at every point in  $\mathbb{R}$  and conclude that  $f(x) = kx$  for all  $x \in \mathbb{R}$ . Thus, any additive function that is continuous at  $x = 0$  must necessarily be a linear function through the origin.

*Solution.*





(a) For  $f(0)$ , we have:

$$\begin{aligned} f(0) &= f(0 + 0) \\ f(0) &= f(0) + f(0) \\ f(0) &= 2f(0) \\ f(0) - f(0) &= 2f(0) - f(0) \\ 0 &= f(0). \end{aligned}$$

For all  $x \in \mathbb{R}$ , we have:

$$\begin{aligned} f(0) &= f(x + (-x)) \\ 0 &= f(x) + f(-x) \\ -f(x) &= f(-x). \end{aligned}$$

(b) Let  $k = f(1)$ .

First, we will show that  $f(n) = kn$  for all  $n \in \mathbb{N}$  by induction.

*Base case:* For  $n = 1$ ,

$$f(1) = k = k \cdot 1.$$

*Inductive step:* Assume  $f(n) = kn$  for some arbitrary  $n \in \mathbb{N}$ . Then,

$$\begin{aligned} f(n+1) &= f(n) + f(1) \\ &= kn + k \\ &= k(n+1). \end{aligned}$$

Thus, by induction,  $f(n) = kn$  for all  $n \in \mathbb{N}$ .

Next, from (a), we have  $f(-x) = -f(x)$ . Hence, for  $z \in \mathbb{Z}$ , if  $z = -n$  where  $n \in \mathbb{N}$ ,

$$f(z) = f(-n) = -f(n) = -kn = k(-n) = kz. \quad (1)$$

Therefore,  $f(z) = kz$  for all integers  $z$ .

Now, observe that for  $p, q \in \mathbb{Z}$  with  $q \neq 0$ , we have:

$$f\left(\underbrace{\frac{p}{q} + \frac{p}{q} + \cdots + \frac{p}{q}}_{q \text{ times}}\right) = f(p).$$

Similarly, by the additive condition ( $q$  times on each side):

$$f\left(\frac{p}{q} + \frac{p}{q} + \cdots + \frac{p}{q}\right) = f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \cdots + f\left(\frac{p}{q}\right).$$



This is equivalent to:

$$q \cdot f\left(\frac{p}{q}\right) = f(p).$$

Therefore,

$$f\left(\frac{p}{q}\right) = \frac{1}{q}f(p)$$

Putting everything together, we let  $r = \frac{p}{q}$ . Then,

$$\begin{aligned} f\left(\frac{p}{q}\right) &= \frac{1}{q}f(p) \\ &= \frac{1}{q}(kp) \quad (\text{from (1)}) \\ &= k\left(\frac{p}{q}\right) \\ &= kr. \end{aligned}$$

Thus,  $f(r) = kr$  for any rational number  $r$ .

(c) Since  $f$  is continuous at  $x = 0$ , we will show  $f$  is continuous everywhere.

Let  $c \in \mathbb{R}$  and let  $\epsilon > 0$ . Since  $f$  is continuous at 0, there exists  $\delta > 0$  such that if  $|h| < \delta$ , then  $|f(h)| < \epsilon$ .

For  $x = c + h$ , we have:

$$\begin{aligned} |f(x) - f(c)| &= |f(c + h) - f(c)| \\ &= |f(c) + f(h) - f(c)| \\ &= |f(h)| \\ &< \epsilon. \end{aligned}$$

Thus,  $f$  is continuous at  $c$ .

Since  $f$  agrees with the continuous function  $kx$  on all rational numbers and  $f$  is continuous on  $\mathbb{R}$ , it follows that  $f(x) = kx$  for all  $x \in \mathbb{R}$ .

Therefore, any additive function that is continuous at  $x = 0$  must be linear,  $f(x) = kx$ .


**Exercise: 4.4.3**

Show that  $f(x) = \frac{1}{x^2}$  is uniformly continuous on the set  $[1, \infty)$  but not on the set  $(0, 1]$ .

*Solution.*

(a) **Uniform Continuity on  $[1, \infty)$ :**

Observe that for all  $x, y \in [1, \infty)$ ,

$$|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{y^2 - x^2}{x^2 y^2} \right| = |y - x| \left( \frac{y + x}{x^2 y^2} \right).$$

Since  $x, y \geq 1$ , we have  $x^2 y^2 \geq 1$ . Therefore,

$$\frac{y + x}{x^2 y^2} = \frac{y}{x^2 y^2} + \frac{x}{x^2 y^2} \leq \frac{1}{x^2 y} + \frac{1}{x y^2} \leq 1 + 1 = 2.$$

Now, let  $\epsilon > 0$ . Choose  $\delta = \frac{\epsilon}{2}$ . Then, when  $|x - y| < \delta$  for  $x, y \in [1, \infty)$ , we have:

$$|f(x) - f(y)| = |y - x| \left( \frac{y + x}{x^2 y^2} \right) < \delta \cdot 2 = \epsilon.$$

(b) **Not Uniformly Continuous on  $(0, 1]$ :**

We will show that  $f$  is not uniformly continuous on  $(0, 1]$  by demonstrating that no matter how small we choose  $\delta > 0$ , there exist  $x, y \in (0, 1]$  such that  $|x - y| < \delta$  but  $|f(x) - f(y)|$  is arbitrarily large.

Let  $\epsilon = 1$ . For the sake of contradiction, assume that  $f$  is uniformly continuous on  $(0, 1]$ ; then, there exists  $\delta > 0$  such that for all  $x, y \in (0, 1]$ , if  $|x - y| < \delta$ , then  $|f(x) - f(y)| < \epsilon$ .

Choose  $x = \frac{\delta}{2}$  and  $y = \frac{\delta}{2} + h$  for some  $h$  with  $0 < h < \frac{\delta}{2}$ . Then  $|x - y| = h < \delta$ .

Compute  $|f(x) - f(y)|$ :

$$|f(x) - f(y)| = \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{y^2 - x^2}{x^2 y^2} \right| = \frac{|y^2 - x^2|}{x^2 y^2}.$$

As  $x \rightarrow 0^+$ ,  $x^2 y^2 \rightarrow 0$ , making the denominator very small and the entire expression very large.



Specifically, take  $x_n = \frac{1}{n}$  and  $y_n = \frac{1}{n+1}$  for  $n \in \mathbb{N}$ . Then,

$$|x_n - y_n| = \left| \frac{1}{n} - \frac{1}{n+1} \right| = \frac{1}{n(n+1)} < \delta \quad \text{for large } n.$$

However,

$$|f(x_n) - f(y_n)| = |n^2 - (n+1)^2| = |n^2 - n^2 - 2n - 1| = 2n + 1.$$

Thus, as  $|x_n - y_n| \rightarrow 0$ ,  $|f(x_n) - f(y_n)| \rightarrow \infty$ . Therefore, by the Sequential Criterion for Absence of Uniform Continuity theorem,  $f$  is not uniformly continuous on  $(0, 1]$ .

### Exercise: 4.4.8

Give an example of each of the following, or provide a short argument for why the request is impossible.

- (a) A continuous function defined on  $[0, 1]$  with range  $(0, 1)$ .
- (b) A continuous function defined on  $(0, 1)$  with range  $[0, 1]$ .
- (c) A continuous function defined on  $(0, 1]$  with range  $(0, 1)$ .

*Solution.*

- (a) **Impossible.** There does not exist a continuous function defined on  $[0, 1]$  with range  $(0, 1)$ . This is because  $[0, 1]$  is a compact set, and the continuous image of a compact set is also compact by the **Preservation of Compact Sets** theorem. However,  $(0, 1)$  is not a compact set since it is not closed in  $\mathbb{R}$ . Therefore, a continuous function cannot map  $[0, 1]$  onto  $(0, 1)$ .

- (b) **Possible.** An example of a continuous function defined on  $(0, 1)$  with range  $[0, 1]$  is:

$$f(x) = \frac{1}{2} \sin(4\pi x) + \frac{1}{2}.$$

This is because  $\sin(4\pi x)$  oscillates between  $-1$  and  $1$ . Therefore,  $\frac{1}{2} \sin(4\pi x)$  oscillates between  $-\frac{1}{2}$  and  $\frac{1}{2}$ . Adding  $\frac{1}{2}$  shifts the range to  $[0, 1]$ .

- (c) **Impossible.** There does not exist a continuous function defined on  $(0, 1]$  with range  $(0, 1)$ . Since  $x = 1$  is in the domain  $(0, 1]$  and  $f$  is continuous at  $x = 1$ , the value  $f(1)$  exists and must be in the range  $(0, 1)$ . However, because  $f$  approaches  $f(1)$  as  $x \rightarrow 1^-$ , the function attains its supremum at  $x = 1$ , meaning  $f(1)$  should be included in the range. But the range  $(0, 1)$  excludes its endpoints, leading to a contradiction. Therefore, such a function cannot exist.