Probability and Statistics: Practice Set 2

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1. (2 points each) A discrete random variable X has its pmf as given in the table below:

x	$P(X = x)$
1	0.2
3	0.1
4	0.4
7	0.3

(a) Find (as a piecewise-defined function) the CDF for X.

Solution. The CDF for X can be defined as

$$F(x) = \begin{cases} 0, & \text{if } x < 1, \\ 0.2, & \text{if } 1 \leq x < 3, \\ 0.3, & \text{if } 3 \leq x < 4, \\ 0.7, & \text{if } 4 \leq x < 7, \\ 1, & \text{if } x \geq 7. \end{cases}$$

(b) Find the mean $\mu = E(X)$.

Solution. The mean can be calculated with the following:

$$\mu = \sum_{x \in \{1,3,4,7\}} xf(x) = 1(0.2) + 3(0.1) + 4(0.4) + 7(0.3) = 4.2.$$

(c) Find the variance, $\sigma^2 = \text{Var}(X)$.

Solution. Similarly to the previous problem:

$$\sigma^2 = \sum_{x \in \{1,3,4,7\}} (x - 4.2)^2 f(x) = 4.56.$$

(d) Find the Moment Generating Function, $M(t) = E(e^{tX})$ for X.

Solution. The mgf can be found by computing the following:

$$M(t) = \sum_{x \in \{1,3,4,7\}} e^{tx} f(x) = 0.2e^{t1} + 0.1e^{t3} + 0.4e^{t4} + 0.3e^{t7}.$$

2. (2 points each) Suppose that $Y = aX + b$ and let $M_Y(t)$ and $M_X(t)$ be the moment generating functions for Y and X respectively.

   (a) Show that $M_Y(t) = e^{tb}M_X(at)$.

   Solution. We will expand the moment generating function for Y:

$$M_Y(t) = E\left(e^{tY}\right) = E\left(e^{t(aX+b)}\right) = E\left(e^{taX}e^{tb}\right) = E\left(e^{tb}\cdot e^{taX}\right) = e^{tb}\cdot E\left(e^{taX}\right) = e^{tb}M_X(at).$$

   Therefore, $M_Y(t) = e^{tb}M_X(at)$.

   (b) Use the previous result to show that $E(Y) = aE(X) + b$.

   Solution. To use the previous result, we can find the first derivative of $M_Y(t)$. Then, we can evaluate it at 0 since $M'_Y(0) = E(Y)$. Hence, we will find the derivative first:

$$\begin{aligned} M'_Y(t) &= \frac{d}{dt}\left[e^{tb}M_X(at)\right] \\ &= \frac{d}{dt}\left[e^{tb}\right]M_X(at) + e^{tb}\cdot\frac{d}{dt}\left[M_X(at)\right] \\ &= be^{tb}M_X(at) + ae^{tb}M'_X(at). \end{aligned} \tag{1}$$

   Evaluating at $t = 0$:

$$\begin{aligned} E(Y) = M'_Y(0) &= be^{0b}M_X(a0) + ae^{0b}M'_X(a0) \\ &= bM_X(0) + aM'_X(0) \\ &= b\cdot(1) + aE(X) \\ &= aE(X) + b. \end{aligned}$$

   Therefore, $E(Y) = aE(X) + b$.

   (c) Use the previous two results to show that $\mathrm{Var}(Y) = a^2\mathrm{Var}(X)$.

   Solution. We know that $\mathrm{Var}(Y) = E(Y^2) - [E(Y)]^2$. From part (b), we have $E(Y) = aE(X) + b$. We need to find $E(Y^2)$ using $M''_Y(0) = E(Y^2)$.
   Taking the second derivative of equation (1):

$$\begin{aligned} M''_Y(t) &= \frac{d}{dt}\left[be^{tb}M_X(at) + ae^{tb}M'_X(at)\right] \\ &= b^2e^{tb}M_X(at) + abe^{tb}M'_X(at) + abe^{tb}M'_X(at) + a^2e^{tb}M''_X(at) \\ &= b^2e^{tb}M_X(at) + 2abe^{tb}M'_X(at) + a^2e^{tb}M''_X(at). \end{aligned}$$

   Evaluating at $t = 0$:

$$\begin{aligned} E(Y^2) = M''_Y(0) &= b^2M_X(0) + 2abM'_X(0) + a^2M''_X(0) \\ &= b^2\cdot 1 + 2ab\cdot E(X) + a^2\cdot E(X^2) \\ &= b^2 + 2abE(X) + a^2E(X^2). \end{aligned}$$

(Solution continued on the next page)

Therefore:

$$\begin{aligned}
\text{Var}(Y) &= E(Y^2) - [E(Y)]^2 \\
&= [b^2 + 2abE(X) + a^2E(X^2)] - [aE(X) + b]^2 \\
&= b^2 + 2abE(X) + a^2E(X^2) - [a^2E(X)^2 + 2abE(X) + b^2] \\
&= a^2E(X^2) - a^2E(X)^2 \\
&= a^2[E(X^2) - E(X)^2] \\
&= a^2\text{Var}(X).
\end{aligned}$$

3. (2 points each) A basketball player makes 82% of their free-throws. Suppose they take 12 total shots and let X be the number of shots they make. Assume each shot is independent of the others.

(a) Find $P(X = 7)$.

Solution. If X is the number of shots that they make, then $X \sim \texttt{Binomialpdf}(12, .82, 7)$. Thus,

$$P(X = 7) = f(7) = \binom{12}{7}(.82)^7(.18)^5 \approx 3.73\%.$$

(b) Find $P(4 \leq X \leq 10)$.

Solution. This question can be rearranged and solved like the following:

$$P(4 \leq X \leq 10) = P(X \leq 10) - P(X \leq 3) \approx 66.41\%.$$

(c) Find $P(X \geq 7 \mid 4 \leq X \leq 10)$.

Solution. This problem relies upon a conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

where event A is the probability of the basketball player making more than 7 shots, given event B where the total successful shots is between 4 and 10 inclusively. Thus, we need to find the following values with the `Binomialcdf` function:

- $P(A \cap B) = P(7 \leq X \leq 10) = P(X \leq 10) - P(X \leq 6) \approx 0.6525$, and
- $P(B) = P(4 \leq X \leq 10) \approx 0.6641$ (from (b)).

This allows us to solve the conditional probability:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \approx \frac{0.6525}{0.6641} \approx 98.25\%.$$

4. (2 points) A multiple choice exam has 4 choices for each question and 10 total questions. Find the probability that a student guessing at random scores at least a 50% on the exam.

Solution. For this problem, we will be using a binomial distribution. We let X count the number of successes, and set $p = .25$ to be the probability of getting a question correct. Thus, our goal is to find the following:

$$P(X \geq 5) = \sum_{k=5}^{10} \binom{10}{k} (.25)^k (.75)^{10-k} \approx 7.81\%.$$

(Or, we could have equivalently calculated $1 -$ `binomcdf(10,.25,4)`. I just wanted to experiment with this summation route on my calculator.)

5. (2 points) Prof. Seme has a shelf with 25 books. Of these, 8 are Agatha Christie Hercule Poirot novels. If he selects 7, without replacement, what is the probability that exactly 3 are Poirot novels?

Solution. This question requires the use of the hypergeometric distribution. We'll let N be the population of 25 books, $N_1 = 8$ being the Agatha Christie Hercule Poirot novels, and $N_2 = 17$ be the rest of them. We will let X be the number of Poirot novels picked. Hence, our goal is to calculate the following:

$$P(X = 3) = f(3) = \left[\binom{8}{3}\binom{17}{4}\right] \Big/ \binom{25}{7} \approx 27.73\%.$$

6. (3 points) An insurance company offers a product which will payout to a business that needs to close for snow. They will pay nothing for the first snow storm. For each additional storm which causes a closure, they will pay \$10,000, up to a maximum total payment of \$45,000 in a year. The number of snow storms in a given year follows a geometric distribution, with mean $\mu = 2.5$ storms per year. What should their premium (i.e., what should they charge per year) so that they bring in 110% of their expected payout?

Solution. Let X count the number of snow storms that occur in one year. The insurance company's payout, Y, is defined by the piecewise function:

$$Y(X) = \begin{cases} 0, & X \le 1, \\ 10,000 \cdot (X-1), & 2 \le X \le 5, \\ 45,000, & X \ge 6. \end{cases}$$

The formula for the mean of the geometric distribution depends on its definition. Since the random variable X represents the number of storms in a year, it can take the value of zero. This requires the version of the geometric distribution that counts the number of **failures before the first success**, which has a support of $S = \{0, 1, 2, \ldots\}$. The mean for this definition is $\mu = \frac{1-p}{p}$.

$$\mu = \frac{1-p}{p} \implies 2.5 = \frac{1-p}{p} \implies 2.5p = 1-p \implies p = \frac{2}{7}.$$

The expected payout is the sum of each possible payout multiplied by its probability:

$$E[Y] = \sum_{x=2}^{5} Y(x)P(X=x) + Y(6)P(X \ge 6).$$

We can compute this expected value using a table:

Storms (x)	Payout (Y)	Probability $P(X=x) = \left(\frac{5}{7}\right)^x \left(\frac{2}{7}\right)$	Payout · Probability
0 or 1	\$0	-	\$0
2	\$10,000	$(\frac{5}{7})^2(\frac{2}{7}) \approx 0.1458$	\$1,457.73
3	\$20,000	$(\frac{5}{7})^3(\frac{2}{7}) \approx 0.1041$	\$2,082.47
4	\$30,000	$(\frac{5}{7})^4(\frac{2}{7}) \approx 0.0744$	\$2,231.21
5	\$40,000	$(\frac{5}{7})^5(\frac{2}{7}) \approx 0.0531$	\$2,124.99
≥ 6	\$45,000	$P(X \ge 6) = (\frac{5}{7})^6 \approx 0.1328$	\$5,976.48
Total			**\$13,872.88**

Finally, the premium is 110% of the expected payout:

$$\boxed{\text{Premium} = E[Y] \cdot 1.10 = \$13{,}872.88 \cdot 1.10 \approx \$15{,}260.17.}$$

7. (3 points) Suppose that $X \sim b(m, p)$ and $Y \sim b(n, p)$ are two independent random variables. Let $Z = X + Y$. Explain why $Z \sim b(m + n, p)$.

Solution. Since $Z = X + Y$, for Z to equal some value z, it could be that X is 0 and Y could be z, or X is 1, and Y is $z - 1$, and so on. Thus, we can write this as a sum:

$$P(Z = z) = \sum_{k=0}^{z} P((X = k) \cap (Y = z - k)).$$

Because X and Y are independent events, we can rewrite this as the following:

$$P(Z = z) = \sum_{k=0}^{z} P(X = k)P(Y = z - k).$$

Thus, from here, we can show that $Z \sim b(m + n, p)$:

$$\begin{aligned}
P(Z = z) &= \sum_{k=0}^{z} P(X = k)P(Y = z - k) \\
&= \sum_{k=0}^{z} \left[\binom{m}{k} p^k (1-p)^{m-k}\right] \left[\binom{n}{z-k} p^{z-k} (1-p)^{n-(z-k)}\right] \\
&= \sum_{k=0}^{z} \binom{m}{k}\binom{n}{z-k} p^{k+(z-k)} (1-p)^{(m-k)+(n-(z-k))} \\
&= \sum_{k=0}^{z} \binom{m}{k}\binom{n}{z-k} p^{z} (1-p)^{m+n-z}.
\end{aligned}$$

Now, factor the terms that don't depend on k outside the sum:

$$= p^z (1-p)^{m+n-z} \sum_{k=0}^{z} \binom{m}{k}\binom{n}{z-k}.$$

By Vandermonde's Identity, the entire sum simplifies to $\binom{m+n}{z}$:

$$\begin{aligned}
&= p^z (1-p)^{m+n-z} \left[\binom{m+n}{z}\right] \\
&= \binom{m+n}{z} p^z (1-p)^{(m+n)-z}.
\end{aligned}$$

which is corresponds exactly to the pmf formula for the binomial distribution with $m + n$ trials and a success probability of p. Therefore, we have shown that $Z \sim b(m + n, p)$.