Practice Set VI

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1. (3 points) Suppose you have an unlimited number of 3-cent and 5-cent stamps. You can then make a total of 3 cents, 5 cents, 6 cents, and each value from 8-cents and above. Use either regular or strong induction to show that each of 8-cents, 9-cents, 10-cents, ... can be made.

   Proof.

   (a) **Base Cases**:
   $n = 8¢$: $3¢ + 5¢$
   $n = 9¢$: $3 \cdot 3¢$
   $n = 10¢$: $2 \cdot 5¢$

   (b) **Inductive Hypothesis**:
   Suppose $n > 10¢$, and $8¢ \leq k < n$, we can make k cents.

   (c) **Inductive Step**:
   Our goal is to make n cent stamps. To do that, consider $n - 3$ cent stamps. Since n is greater than 10, $n - 3$ is at least 8 cent stamps.

   By the inductive hypothesis, we can make $n - 3$ cent stamps because this amount is within the range of our assumption. To make n cent stamps, simply add one 3 cent stamp to $n - 3$. We have made n stamps.

   $\square$

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2. (3 points) Write a recursive definition for the set of positive numbers which are multiples of either 3 or 5: $\{3, 5, 6, 9, 10, 12, 15, 18, 20, 21, \ldots\}$.

   Skip.

3. (3 points) Write a recursive definition for the set of positive powers of 3 : $\{3, 9, 27, 81, \dots\}$.

Solution. Let X be recursively defined as:

- **B**:
  $3 \in X$
- **R**:
  If $3^n \in X$, then $3^{n+1} \in X$.

4. (3 points) Use structural induction and the definition that you wrote in Problem 3 above to show that each element in this set is odd.

Proof.

- **Base Case**:
  For $n = 1$: $3^1 = 3$, which is clearly odd (i.e., has the form of $2m + 1$ where m is just 1).

- **Inductive Hypothesis**:
  Suppose that 3^n is odd for some $n \in \mathbb{N}$. This means there must exist an integer k such that $3^n = 2k + 1$.

- **Inductive Step**:
  We need to show that 3^{n+1} is also odd under the assumption that 3^n is odd.

  Rewriting 3^{n+1} as $3 \cdot 3^n$, we can use the inductive hypothesis to substitute and expand the expression:

$$\begin{aligned} 3^{n+1} &= 3 \cdot (2k + 1) \\ &= 6k + 3 \\ &= 6k + 2 + 1 \\ &= 2 \cdot (3k + 1) + 1 \end{aligned}$$

  Because $(3k+1)$ must be an integer by definition, 3^{n+1} must be odd by definition of odd numbers. In other words, 3^{n-1} is equivalent to some form $2m + 1$ (or in this case, $3m + 1$) for some $m \in \mathbb{Z}$.

$\square$

5. (3 points) Use induction to prove that $5^n - 1$ is divisible by 4 for each integer $n \geq 0$.

Proof.

- **Base Case**:
  For $n = 0$: $5^0 - 1 = 0$; which is divisible by 4 because $0 = 0 \times 4$.

- **Inductive Hypothesis**:
  Suppose for some integer $n \geq 0$, $5^n - 1$ is divisible by 4.

- **Inductive Step**:
  We need to show that $5^{n+1} - 1$ is also divisible by 4 under the assumption that $5^n - 1$ is divisible by 4.

  Rewriting $5^{n+1} - 1$ as $5 \cdot 5^n - 1$, we can use the inductive hypothesis to substitute and expand the expression:

$$
\begin{aligned}
5 \cdot 5^n - 1 &= (4 + 1) \cdot 5^n - 1 \\
&= (4 \cdot 5^n) + (1 \cdot 5^n) - 1 \\
&= (4 \cdot 5^n) + (5^n - 1)
\end{aligned}
$$

  Now, we know that $4 \cdot 5^n$ is divisible by 4, because itself is a multiple of 4. Additionally, we know that $5^n - 1$ is also divisible by 4 by the inductive hypothesis. Hence, when we add these two expressions together, we will get a number that is also a multiple of 4 by definition. Therefore, $5^{n+1} - 1$ is divisible by 4.

$\square$

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6. (3 points) Suppose that $f\colon \mathbb{N} \to \mathbb{N}$ is a function with two properties:

- $f(1) = 2$
- $f(a+b) = f(a) \cdot f(b)$ for all $a, b \in \mathbb{N}$.

Show, by inducting on n, that $f(n) = 2^n$.

Proof.

- **Base Case**:
  For $n = 1$: $2^1 = 2$, so our base case for $f(n) = 2^n$ is satisfied.

- **Inductive Hypothesis**:
  Suppose for some $n \in \mathbb{N}$, the statement, $f(n) = 2^n$.

- **Inductive Step**:
  We need to show that the properties of $f(n)$ hold for $f(n+1) = 2^{n+1}$.

  Given that $f(a+b) = f(a) \cdot f(b)$, for all $a, b \in \mathbb{N}$, we can rewrite $f(n+1)$ as $f(n) \cdot f(1)$. Then, by the inductive hypothesis, we know that:

$$\begin{aligned} f(n) + f(1) &= 2^n + 2 \\ &= 2^{n+1} \end{aligned}$$

  Hence, this means that $f(n+1)$ upholds the specified characteristics of f.

$\square$

7. We have previously used that, given a set of numbers $s(n)$ for integers $n \geq 0$ if the k^{th} sequence of differences is constant (and not 0) then $s(n)$ is generated by a polynomial of degree k. We will justify that here. Suppose that $s(n)$ can be written as a polynomial $a_k n^k + a_{k-1} n^{k-1} + \cdots + a_1 n + a_0$. We wish to show that the k^{th} sequence of differences from s is constant. If we write that constant as c, then $a_k = c/k!$. We will show this by induction on k.

   (a) **Base Case**: Our base case is when $k = 1$ Suppose that $s(n) = a_1 n + a_0$. We need to show two related facts:

      i. (2 points) $s(n) - s(n-1)$ is constant.

      Solution.

$$\begin{aligned} s(n) - s(n-1) &= (a_1 n + a_0) - (a_1(n-1) + a_0) \\ &= a_1 n + a_0 - a_1 n + a_1 - a_0 \\ &= (a_1 n - a_1 n) + (a_0 - a_0) + a_1 \\ &= a_1 \end{aligned}$$

      Thus, $s(n) - s(n-1)$ yields a constant answer, a_1.

      ii. (2 points) Let us call that constant number c. Show that $s(n) = (c/1!) \cdot n$ plus another smaller powered term.

      Solution. Start with $s(n) = c \cdot n + a_0$ from part (i). Considering the factorial of 1 is 1, the substituted expression fits the expression $(c/1!) \cdot n + a_0$. This is the case because we can divide any number by 1, as it is the multiplication and division identity.

   (b) **Inductive Hypothesis**: Let $k > 1$ and suppose that for any $f(n) = b_{k-1} n^{k-1} + b_{k-2} + \cdots + b_1 n + b_0$ the following are true:

      i. the $(k-1)^{\text{st}}$ sequence of differences for f is constant.

      ii. if we denote this constant c, then $f(n) = \dfrac{c}{(k-1)!} \cdot n^{k-1} +$ other smaller terms.

   (c) **Inductive Step**: Suppose that $s(n) = a_k n^k +$ other smaller terms.

      i. (2 points) Show that s has a constant k^{th} sequence of differences. [Hint: Define $g(n) = s(n) - s(n-1)$ and explain how we know that g has a constant $(k-1)^{\text{st}}$ sequence of differences]

      Solution. Define $g(n) = s(n) - s(n-1)$. To show that $g(n)$ has a constant $(k-1)^{\text{st}}$ sequence of differences, consider:

$$s(n) = a_k n^k + \text{lower terms}$$

$$s(n-1) = a_k(n-1)^k + \text{lower terms}$$

Now, we must calculate $g(n)$. First, we will substitute: $g(n) = [a_k n^k + \text{lower terms}] - [a_k(n-1)^k + \text{lower terms}]$. We will focus on the highest-order term.

To determine the value of $a_k(n-1)^k$, we can use the Binomial Theorem[1]. Thus,

$$a_k(n-1)^k = a_k \left[\sum_{i=0}^{k} \binom{k}{i} n^{k-i}(-1)^i \right]$$

Now, when we subtract $a_k n^k$ from the summation expression when $i = 0$, we will end up with:

$$g(n) = a_k n^k - a_k n^k - a_k \left[\sum_{i=1}^{k} \binom{k}{i} n^{k-i}(-1)^i \right]$$

Which simplifies to:

$$g(n) = -a_k \left[\sum_{i=1}^{k} \binom{k}{i} n^{k-i}(-1)^i \right]$$

Finally, we can conclude that since $g(n)$ is a polynomial of degree $k-1$, taking the $(k-1)^{\text{st}}$ difference of $g(n)$ will yield a constant by the inductive hypothesis. This is the case because a polynomial of degree d has a d^{th} sequence of differences that is constant.

ii. (2 points) Show that if this constant is c, then $a_k = c/k!$.

We know that the k^{th} difference of n^k, multiplied by the coefficient a_k simplifies to $a_k \cdot k!$. Therefore, if that difference is the constant c, we have $c = a_k \cdot k!$, and by solving for a_k, we get $a_k = c/k!$.

[1] Huge thanks to Tanvi Kiran for this hint she provided. If it wasn't for her, I would have never figured this problem out.

The Last Part of 7 (in class):

$g(n) = s(n) - s(n-1) + \cdots \Rightarrow s(n) = a_k\not{n}^k + a_{k-1}\not{n}^{k-1} + \cdots \Rightarrow s(n-1) = a_k(\not{n}^k - kn^n k - 1 + \cdots) + a_{k-1}(n\not{y}m \Rightarrow a_k n^{k-1} +$ smaller terms. By the inductive hypothesis and since this is a $(k-1^{\text{st}})$ degree polynomial, it has a constant $(k-1^{\text{st}})$ sequence of differences.

This implies s has a constant k^{th} sequence of difference.

Let c be that constant. By the inductive hypothesis, $g(n) = \frac{c}{(k-1)!} \cdot n^{k-1} + \cdots$. But we proved just earlier, that this expression is equal to $a_k k n^{k-1} + \cdots$. Hence, because they are the same polynomial, it must be the case that $\frac{c}{(k-1)!}$ is equal to $a_k k n^{k-1}$.

Solving for a_k, we get $\frac{c}{(k-1)!} = a_k k \Rightarrow \frac{c}{k(k-1)!} = a_k \Rightarrow \frac{c}{k!} = a_k$.