Homework 4: Section 8

Algebra

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Section 8

In Exercises 1, 2, and 5, compute the indicated product involving the following permutations in S_6:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix}, \quad \mu = (\dots)$$

1. $\tau\sigma$

Solution.

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix}$$

2. $\tau^2\sigma$

Solution.

$$\tau\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 5 & 6 & 3 \end{pmatrix}$$

5. $\sigma^{-1}\tau\sigma$

Solution.

$$\sigma^{-1}\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}$$

In Exercises 6 and 8, compute the expressions shown for the permutations σ, τ, and μ defined prior to Exercise 1.

6. $|\langle\sigma\rangle|$

Solution. To find the order of σ, we need to determine the smallest positive integer k such that σ^k is the identity permutation. Thus,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 & 1 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 4 & 6 & 2 & 1 & 3 \end{pmatrix}$$

$$\sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}, \quad \sigma^5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix}, \quad \sigma^6 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Therefore, the order of σ is 6, so $|\langle\sigma\rangle| = 6$.

8. σ^{100}

Solution. From the last problem, we know that σ has order 6. So, $\sigma^{100} = \sigma^4$. We also found that

$$\sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}.$$

Let A be a set and let $\sigma \in S_A$. For a fixed $a \in A$, the set

$$\mathcal{O}_{a,\sigma} = \{\sigma^n(a) : n \in \mathbb{Z}\}$$

is the **orbit** of a **under** σ. In Exercise 12, find the orbit of 1 under the permutation defined prior to Exercise 1.

12. τ

Solution. $\tau^1(1) = 2$, $\tau^2(1) = 4$, $\tau^3(1) = 3$, $\tau^4(1) = 1$. Thus, the orbit of 1 under τ is $\mathcal{O}_{1,\tau} = \{1, 2, 3, 4\}$.

20. Give the multiplication table for the cyclic subgroup of S_5 generated by

$$\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$$

There will be six elements. Let them be ρ, ρ^2, ρ^3, ρ^4, ρ^5, and $\rho^0 = \rho^6$. Is this group isomorphic to S_3?

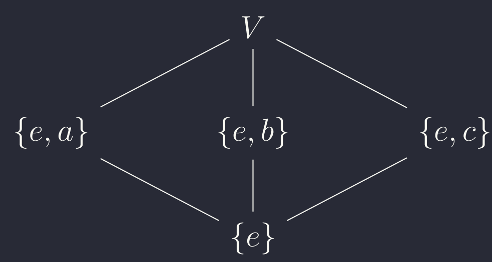
Solution. No. These groups are not isomorphic because S_3 is non-abelian (multiplication table in Example 8.7, and also from class notes), while the cyclic subgroup generated by ρ is abelian, as shown below:

	ρ^0	ρ	ρ^2	ρ^3	ρ^4	ρ^5
ρ^0	ρ^0	ρ	ρ^2	ρ^3	ρ^4	ρ^5
ρ	ρ	ρ^2	ρ^3	ρ^4	ρ^5	ρ^0
ρ^2	ρ^2	ρ^3	ρ^4	ρ^5	ρ^0	ρ
ρ^3	ρ^3	ρ^4	ρ^5	ρ^0	ρ	ρ^2
ρ^4	ρ^4	ρ^5	ρ^0	ρ	ρ^2	ρ^3
ρ^5	ρ^5	ρ^0	ρ	ρ^2	ρ^3	ρ^4

In this section we discussed the group of symmetries of an equilateral triangle and of a square. In Exercises 24 and 25, give a group that we have discussed in the text that is isomorphic to the group of symmetries of the indicated figure. You may want to label some special points on the figure, write some permutations corresponding to symmetries, and compute some products of permutations.

24. The figure in Fig 8.21 (b)

Solution. This figure is isomorphic to the Klien-4 group: $V = \{e, a, b, c\}$.



25. The figure in Fig 8.21 (c)

Solution. This figure is isomorphic to D_4. You can rotate the square 0°, 90°, 180°, or 270° and then flip it across one of its diagonals or across one of its midlines. This gives a total of 8 symmetries, which is the same number of elements in D_4. To show that it is not abelian (like D_4), consider a 90° counter-clockwise rotation followed by a vertical flip. If you track the bottom left corner, $1 \mapsto 2 \mapsto 3$, but if we do the vertical flip first and then rotate, we get $1 \mapsto 4 \mapsto 1$. Thus, the two operations do not commute.

38. Draw the Cayley digraph for D_4. You can sketch the general D_n if you want, but that's optional fun.

Solution. Let r = 1 counterclockwise rotation and s = vertical flip. Then, $D_4 = \{r, s \mid r^4 = s^2 = e,\ rs = sr^{-1}\}$. The Cayley digraph for D_4 is shown below: