2. Suppose that $f\colon X \to Y$ is a one-to-one and onto function.

   (a) Briefly, explain how you know $f^{-1}\colon Y \to X$ is a function which indeed uses all of Y as its domain.

   Solution. $\ldots Y \to X$, then by definition, it too, must map each x to y by definition. Hence, each y in the domain is mapped to a unique x is in the co-domain.

   (b) Show that f^{-1} is one-to-one.

   Solution. Let $f\colon X \to Y$ be a one-to-one and onto function. Also let f^{-1} be the inverse function of f, defined as $f^{-1}\colon Y \to X$.

   To prove that f^{-1} is one-to-one, we must show that if $f^{-1}(y_1) = f^{-1}(y_2)$ for any $y_1, y_2 \in Y$, then $y_1 = y_2$.

   Assume $y_1, y_2 \in Y$ and that $f^{-1}(y_1) = f^{-1}(y_2)$. Because of the definition of the inverse function, since $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$, then $f(x_1) = y_1$ and $f(x_2) = y_2$ which implies $x_1 = x_2$ by substitution. Since, $f(x_1) = y_1$ and $f(x_2) = y_2$, and given that f is one-to-one, it follows that $y_1 = y_2$.

   (c) Show that f^{-1} is onto.

   Solution. Let $f\colon X \to Y$ be a one-to-one and onto function. Also let f^{-1} be the inverse function of f, defined as $f^{-1}\colon Y \to X$.

   To prove that f^{-1} is onto, we must show that for every element $x \in X$, there exists an element $y \in Y$ such that $f^{-1}(y) = x$.

   Let any $x \in X$. Since f is onto, there exists a $y \in Y$ such that $f(x) = y$. Then, by the definition of the inverse function, if $f(x) = y$ then $f^{-1}(y) = x$. Thus, f^{-1} is onto.

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3. Construct a function $f\colon \mathbb{Z} \to \mathbb{N}$ which is onto, but is not one-to-one. Justify your answer.

Solution. Consider the piece-wise function (where $z \in \mathbb{Z}$), $f(z) = \begin{cases} -z & \text{if } z < 0, \\ z+1 & \text{if } z \geq 0 \end{cases}$

This function is onto because for every natural number $n \in \mathbb{N}$ in the co-domain, there exists at least one element in the domain, $z \in \mathbb{Z}$, that maps to it. In other words, if $n > 0$, then $f(-n) = n$ (for negative numbers), and $f(n-1) = n$ (for non-negative inputs). Thus, at least one integer maps to every natural number.

However, this function is not one-to-one because for any $n > 0$, there are two $z \in \mathbb{Z}$ such that $z_1 = -(n)$ and $z_2 = n - 1$ such that $f(z_1) = n$ and $f(z_2) = n$

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4. Construct a function $g\colon \mathbb{N} \to \mathbb{N}$ which is one-to-one, but not onto. Justify your answer.

Solution. Consider $g(n) = n + 1$ (where $n \in \mathbb{N}$).

This function is one-to-one because for every $n \in N$, $g(n)$ maps to $n + 1$. By definition, this mapping is one-to-one because for every $n_1, n_2 \in N$, if $g(n_1) = g(n_2)$, then $n_1 + 1 = n_2 + 1$, which simplifies to $n_1 = n_2$.

It is not onto because there does not exist an $n \in N$ for which $g(n) = 1$. In other words, 1 (in the co-domain) is never mapped to by an element in the domain.

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5. Find a function $h\colon \mathbb{N} \to \mathbb{Z}$ which is *both* one-to-one and onto. Justify your answer.

Solution. Consider the piece-wise function (where $n \in \mathbb{N}$), $h(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ -\frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$

For even $n \in \mathbb{N}$, $h(n)$ produces a non-negative integer[1], and for odd n, it produces a negative integer. The really cool thing about this, is that for each corresponding mapping, if n is even, $h(n)$ produces a non-negative integer, and if n is odd, $h(n)$ produces a negative integer. This ensures that each n maps to a unique $z \in \mathbb{Z}$. Thus, $h(n)$ is one-to-one.

For every integer $z \in \mathbb{Z}$, there exists a natural number $n \in \mathbb{N}$ such that $h(n) = z$. Furthermore, if $z \geq 0$, we can take $n = 2z$, which is an even natural number that maps to z (by the definition of even numbers). Conversely, if $z < 0$, we can take $n = -2z - 1$, which is an odd natural number that maps to z (by the definition of odd numbers). Thus, this ensures that every integer is the image of some natural number. Thus, $h(n)$ is onto.

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6. Suppose that R is an equivalence relation on X and let $a, b \in X$ so that $a \not R\, b$, that is, a is not related to b. Show that

$$[a] \cap [b] = \emptyset$$

Goal: Prove that if a and b are not related by R, then their equivalence classes have no elements in common. In other words, the intersection of the two equivalence classes has no elements.

Solution. For the sake of contradiction, suppose that $[a] \cap [b] \neq \emptyset$. By the denial of the empty set, this means there must exist an element $c \in X$ such that $c \in [a]$ and $c \in [b]$. Then, by the definition of equivalence classes, this means that $c\ R\ a$, and $c\ R\ b$. Given these relations, by respectively using symmetry and transitivity, we can show that since $c\ R\ a$, then $a\ R\ c$, and because $a\ R\ c$ and $c\ R\ b$, then $a\ R\ b$.

However, $a\ R\ b$ contradicts the given information, so our assumption must be false. Hence, it must be the case that $[a] \cap [b] = \emptyset$.

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[1] Why do we always say 'non-negative' and not 'positive?'

For the following problem, define $R\colon V \to V$ by $a\ R\ b \iff b$ is a path connected to a:

7. Suppose $G = (V, E)$ is a simple graph, which might not be connected. Let $v \in V$. For each vertex $u \in V$, we say that v is a *path connected* to u provided that there exists a path from v to u. Is R an equivalence relation? [Hint: R is reflexive, since any vertex v is path connected to itself by the simple path: v.]

Solution. Let $u, v \in V$ with $R\colon V \to V$ by $a\ R\ b$ if, and only if, b is a path connected to a. To show that R is an equivalence relation, we need to show three things:

(a) **Reflexivity**:
For R to be reflexive, every vertex $v \in V$ must be a path connected to itself. This is proven true by the hint provided.

(b) **Symmetry**:
For R to be symmetric, if $v\ R\ u$ (indicative of a path connected from v to u), then $u\ R\ v$ (path connected from u to v by definition). Because the given information does not specify that G or the relation $a\ R\ b$ is a *directed graph*, one can assume that this *path connected* is non-directed, and hence, is intrinsically symmetric because the of the non-directional nature of the graph or path. In other words, if $v\ R\ u$, then $u\ R\ v$ and vice versa because it does not matter at which vertex you begin at (in this context).

(c) **Transitivity**:
For R to be transitive, if $v\ R\ u$, and $u\ R\ w$, then $v\ R\ w$. Because the vertices v, u and u, w are path connected, there exists edges that connect these vertices together to form a path. Hence, because there is a path that stretches from v to w, then $v\ R\ w$.

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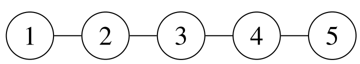
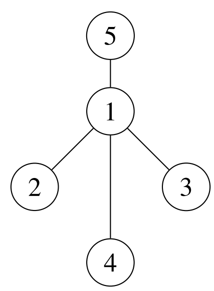
8. Draw two trees with 5 vertices which are not isomorphic. Explain how you know they are not isomorphic.



Graph G



Graph H

Solution. Let $G = (V, E)$ and $H = (W, F)$ (with V, W being the vertex sets, and E, F being the edge sets). Then,

$G = \{\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}\}\}$ and
$F = \{\{1, 2, 3, 4, 5\}, \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}\}\}$.

For G to be *isomorphic* to H, the process must preserve the degrees of corresponding vertices. Since G has vertices degrees of 1, 2, 2, 2, and 1 for vertices 1-5 and H has vertices 2-5 with $\deg(1)$, and vertex 1 with $\deg(4)$ these graphs are not bijective.

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9. Suppose that T is a tree. Show that T must contain at least two vertices of odd degree.

   Solution. Suppose T is a tree. By the Handshake Theorem, we know that the sum of the degrees of T's vertices is always twice the sum of the edges. So, $\Sigma \deg(v) = 2E$ (where E is the set of edges). Then, we know that by Theorem 1, a tree with n vertices, has $n - 1$ edges. So, we can rewrite the equation to be $\Sigma \deg(v) = 2(n - 1)$.

   Now, assume for the sake of contradiction that T has fewer than two vertices of odd degree. Then, there are two possible cases for T's vertices' degree. Either:

   (a) No vertices of odd degree, or
   (b) exactly one vertex of odd degree.

   We know that the first case must be false because the lemma to Theorem 1 says that, "If G is a tree, G has at least one vertex of degree 1."
   And well, we know that it cannot be the second case because Corollary 2 to the Handshake Theorem says that no graph can have exactly one vertex of odd degree.

   Therefore, it must be the case that our assumption was false, and T must have an even number of vertices with an odd degree. And as shown, since T cannot have no vertices of odd degree (0), and 1 is odd, then the minimum number of even vertices with odd degree T can have is 2.