# Multivariable Calculus Exam 1

## Basic Derivatives

$$\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}$$

$$\frac{d}{dx}\sin f(x) = \cos f(x) \cdot f'(x)$$

$$\frac{d}{dx}\cos f(x) = -\sin f(x) \cdot f'(x)$$

$$\frac{d}{dx}\tan f(x) = \sec^2 f(x) \cdot f'(x)$$

$$\frac{d}{dx}\cot f(x) = -\csc^2 f(x) \cdot f'(x)$$

$$\frac{d}{dx}\sec f(x) = \sec f(x)\tan f(x) \cdot f'(x)$$

$$\frac{d}{dx}\csc f(x) = -\csc f(x)\cot f(x) \cdot f'(x)$$

$$\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$$

$$\frac{d}{dx}\log_a f(x) = \frac{f'(x)}{f(x)\ln a}$$

$$\frac{d}{dx}(f(x))^n = n(f(x))^{n-1}f'(x)$$

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$$

$$\frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}b^{g(x)} = b^{g(x)}\ln b \cdot g'(x)$$

## Inverse Trigonometric

$$\frac{d}{dx}\arcsin f(x) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

$$\frac{d}{dx}\arccos f(x) = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

$$\frac{d}{dx}\arctan f(x) = \frac{f'(x)}{1+(f(x))^2}$$

$$\frac{d}{dx}\operatorname{arccot} f(x) = -\frac{f'(x)}{1+(f(x))^2}$$

$$\frac{d}{dx}\operatorname{arcsec} f(x) = \frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$$

$$\frac{d}{dx}\operatorname{arccsc} f(x) = -\frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$$

## Chain Rule

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

## Higher-Order Derivatives

$$\frac{d^2}{dx^2}e^x = e^x$$

$$\frac{d^3}{dx^3}\sin x = -\cos x$$

$$\frac{d^4}{dx^4}\cos x = \cos x$$

## Product and Quotient

$$\frac{d}{dx}[u \cdot v] = u' \cdot v + u \cdot v'$$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u' \cdot v - u \cdot v'}{v^2}$$

## Trigonometric Integrals

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin^2 x = \frac{1}{2}(x - \sin x \cos x) + C$$

$$\int \cos^2 x = \frac{1}{2}(x + \sin x \cos x) + C$$

$$\int \tan x \, dx = -\ln|\cos x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

## Reduction Formulas for Sine and Cosine

$$\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$$

$$\int \cos^n x \, dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x \, dx$$

## Inverse Trigonometric Integrals

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2} \, dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x\sqrt{x^2-a^2}} \, dx = \frac{1}{a}\operatorname{arcsec}\left(\frac{x}{a}\right) + C$$

## Regular Integrals and $e$

$$\int x^n \, dx = \frac{1}{n+1}x^{n+1} + C$$

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int e^{ax} \, dx = \frac{1}{a}e^{ax} + C$$

$$\int e^{f(x)}f'(x) \, dx = e^{f(x)} + C$$

## Exponetials

$$\int_0^b e^x \, dx = e^b - 1$$

$$\int_0^b e^{-x} \, dx = 1 - e^{-b}$$

$$\int_0^\infty e^{-x} \, dx = 1$$

$$\int_0^\infty x^n e^{-x} \, dx = n!$$

| $n$ | $\int_0^{\pi/2} \sin^n x \, dx$ | $\int_0^{\pi/2} \cos^n x \, dx$ | $\int_0^{\pi} \sin^n x \, dx$ | $\int_0^{\pi} \cos^n x \, dx$ | $\int_0^{2\pi} \sin^n x \, dx$ | $\int_0^{2\pi} \cos^n x \, dx$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 2 | 0 | 0 | 0 |
| 2 | $\pi/4$ | $\pi/4$ | $\pi/2$ | $\pi/2$ | $\pi$ | $\pi$ |
| 3 | 2/3 | 2/3 | 4/3 | 0 | 0 | 0 |
| 4 | $3\pi/16$ | $3\pi/16$ | $3\pi/8$ | $3\pi/8$ | $3\pi/4$ | $3\pi/4$ |
| 5 | 8/15 | 8/15 | 16/15 | 0 | 0 | 0 |
| 6 | $5\pi/32$ | $5\pi/32$ | $5\pi/16$ | $5\pi/16$ | $5\pi/8$ | $5\pi/8$ |

| Radians | $\sin(\theta)$ | $\cos(\theta)$ | $\tan(\theta)$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |
| $\pi/6$ | $1/2$ | $\sqrt{3}/2$ | $\sqrt{3}/3$ |
| $\pi/4$ | $\sqrt{2}/2$ | $\sqrt{2}/2$ | 1 |
| $\pi/3$ | $\sqrt{3}/2$ | $1/2$ | $\sqrt{3}$ |
| $\pi/2$ | 1 | 0 | $-$ |
| $\pi$ | 0 | $-1$ | 0 |
| $3\pi/2$ | $-1$ | 0 | $-$ |

## Trigonometric Identities

**Pythagorean**

$$\sin^2 \theta + \cos^2 \theta = 1$$
$$\tan^2 \theta + 1 = \sec^2 \theta$$
$$1 + \cot^2 \theta = \csc^2 \theta$$

**Half Angle**

$$\sin^2 \left(\frac{x}{2}\right) = \frac{1 - \cos x}{2}$$
$$\cos^2 \left(\frac{x}{2}\right) = \frac{1 + \cos x}{2}$$
$$\tan^2 \left(\frac{x}{2}\right) = \frac{1 - \cos x}{1 + \cos x}$$

**Double Angle**

$$\sin 2x = 2 \sin x \cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\cos 2x = 2\cos^2 x - 1$$
$$\cos 2x = 1 - 2\sin^2 x$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

**Product to Sum**

$$\sin x \sin y = \frac{1}{2}\left[\cos(x - y) - \cos(x + y)\right]$$
$$\cos x \cos y = \frac{1}{2}\left[\cos(x - y) + \cos(x + y)\right]$$
$$\sin x \cos y = \frac{1}{2}\left[\sin(x + y) + \sin(x - y)\right]$$
$$\cos x \sin y = \frac{1}{2}\left[\sin(x + y) - \sin(x - y)\right]$$

**Sum to Product**

$$\sin x + \sin y = 2 \sin \left(\frac{x + y}{2}\right) \cos \left(\frac{x - y}{2}\right)$$
$$\sin x - \sin y = 2 \cos \left(\frac{x + y}{2}\right) \sin \left(\frac{x - y}{2}\right)$$
$$\cos x + \cos y = 2 \cos \left(\frac{x + y}{2}\right) \cos \left(\frac{x - y}{2}\right)$$
$$\cos x - \cos y = -2 \sin \left(\frac{x + y}{2}\right) \sin \left(\frac{x - y}{2}\right)$$

## Chapter 1: Parametric Equations and Polar Coordinates

- **Slope:** $\left.\dfrac{dy}{dx}\right|_{t=t_0} = \left.\dfrac{dy/dt}{dx/dt}\right|_{t=t_0}$.

  The **tangent line** at $t_0$ is given by

  $$y = \left(\left.\frac{dy}{dx}\right|_{t=t_0}\right)(x - x(t_0)) + y(t_0).$$

- **Concavity:** $\left.\dfrac{d^2y}{dx^2}\right|_{t=t_0} = \left.\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\right|_{t=t_0} = \left.\dfrac{d}{dt}\left(\dfrac{dy/dt}{dx/dt}\right)\right|_{t=t_0}$.

- **Area Under a Curve:** $\displaystyle\int_{t_a}^{t_b} y(t)\frac{dx}{dt}\,dt$.

- **Arc Length:** $\displaystyle\int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

- **Surface Area:** $\displaystyle\int_{t_a}^{t_b} 2\pi y(t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$.

## Chapter 2: Vectors in Space

- ***Direction:*** $P = (x_1, y_1)$ and $Q = (x_2, y_2)$: $\mathbf{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle$.

- ***Vector Sum:*** $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.

- ***Magnitude:*** $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2} = \sqrt{u \cdot u}$.

- ***Dot Product:*** $\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2$.

  - ***Angle:*** $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos(\theta)$, where $0 \leq \theta \leq \pi$ is between $\mathbf{u}$ & $\mathbf{v}$.
  - ***Self-Product:*** $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$.
  - ***Work:*** $W = \mathbf{F} \cdot \mathbf{PQ} = (\|\mathbf{F}\|) \|\mathbf{PQ}\| \cos(\theta)$.

- To ***Normalize*** a vector, divide it by its magnitude $\mathbf{v} = \langle x, y, z \rangle$, then $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \left\langle \frac{x}{\|\mathbf{v}\|}, \frac{y}{\|\mathbf{v}\|}, \frac{z}{\|\mathbf{v}\|} \right\rangle$. $\therefore \mathbf{u} :=$ ***Unit Vector*** in direction of $\mathbf{v}$.

- ***Projection:*** $\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}\right)\mathbf{b}$.

- ***Cross product:*** $\mathbf{u} \times \mathbf{v} = \langle u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3,\ u_1 v_2 - u_2 v_1 \rangle$.

  - ***Angle:*** $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin(\theta)$, where $0 \leq \theta \leq \pi$ is between $\mathbf{u}$ & $\mathbf{v}$.
  - ***Torque:*** $\tau = \mathbf{r} \times \mathbf{F}$    or    $\|\tau\| = \|\mathbf{r}\| \|\mathbf{F}\| \sin(\theta)$

### Parametric Equations Revisted

- ***Vector Equation:*** $\mathbf{r}(t) = \mathbf{r_0} + \mathbf{tV}$.

- ***Parametric Equation:*** $x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$.

- ***Symmetric Equation:*** $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$.

- The ***Line Segment*** from $P$ to $Q$: $\mathbf{r}(t) = (1 - t)\mathbf{p} + t\mathbf{q}$ (where $\mathbf{p}, \mathbf{q}$ are the vector forms of $P, Q$ and $0 \leq t \leq 1$).

- ***Direction Vector:*** $d = \left|\frac{\mathbf{PM} \times \mathbf{v}}{\|\mathbf{v}\|}\right|$.

  - ***Equal***: Same direction vector, share a point.
  - ***Parallel***: Same direction vector, do not share a point.
  - ***Intersecting***: Different direction vectors, share a point.
  - ***Skew***: Different direction vectors, do not share a point.

- If $(x_0, y_0, z_0)$ is a point on a plane, the ***Scalar Equation*** would be: $\langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0 \implies a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

## 2.4 Relations

**Definitions:**

*Relation:*    Defined as $r\colon X \to Y$ if $R \subseteq X \times Y$.
*Reflexive:*    If for each $x \in X$, $x \mathrel{R} x$.
*Symmetric:*    If when $a \mathrel{R} b$, then $a \mathrel{R} b$.
*Transitive:*    If when $a \mathrel{R} b$, and $b \mathrel{R} c$, then $a \mathrel{R} c$.
*Equivalence relation:*    Has all three above traits.
*Equivalence class:*    Because equivalence relations allow us to partition the domain, each partition is then labeled as $[a]$.
*Modulo (%):*    For two integers, $a$ and $b$, $a$ is said to be equivalent to $b$ modulo $n$ if the difference between $a$ and $b$ (that is, $a - b$ is divisible by $n$. This means there exists an integer $k$ such that $a - b = kn$.

**Note:** All functions are relations, but not all relations are functions. Abiding by the definition of a function, if $x$ can be mapped to more than one $y$-value, the statement is not a function. Also note for the relation traits, those items are true on the basis that $R\colon X \to X$ is a relation.

### Prove Modulo Is an Equivalence Class

Let $n \in \mathbb{N}$, and let $a, b, c \in \mathbb{Z}$.

- Is $a \equiv a \bmod n$ reflexive? $a - a = 0$, which is divisible by $n$. So, Yes.

- Is $a \equiv a \bmod n$ symmetric? Well, we must find out if $a \equiv b \bmod n$ is equivalent to $b \equiv a \bmod n$. We know that $(a - b)$ is divisible by $n$, and there are no restrictions to also include that $b - a$ is also divisible by $n$. Yes.

- Is $a \equiv a \bmod n$ transitive? Let $a \equiv b \bmod n$ and $b \equiv c \bmod n$. This means that $a = b + kn$ and $b = c + ln$ for some integers $k$ and $l$. Substituting the second equation into the first, we find that $a = (c + ln) + kn = c + (l + k)n$. So, $a \equiv c \bmod n$, as required for transitivity.

### Integers Modulo $n$

The set of equivalence classes form by this equivalence relation is called the *integers modulo n*, and is denoted as $\mathbb{Z}/n$. We use $[k]$ to denote the equivalence class of $[k]$. For example, elements of $\mathbb{Z}\ 3$ are $[0] = \{\ldots, -9, -8, -6, -3, 0, 3, 6, 9, \ldots\}$, and $[1] = \{\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots\}$.

## 2.6 Graph Theory

**Definitions:**

#### Graphs

*Graph:*    $G = (V, E)$, is a pair of sets $V$, the *vertex set*, and $E$ the *edge set*, so that each element of $E$ has the form $\{v_i, v_j\}, v_1, v_j \in V$.
*Degree:*    The number of edges which include $v$. Granted that $v \in V$.
*Adjacent:*    Vertices $u, v$ belong to $\{u, v\} \in E$.
*Path:*    From vertex $v_0$ to vertex $v_n$ is a sequence $v_0, v_1, v_2 \ldots, v_n$, where each $v_i \in V$ and $\{v_i, v_{i+1}\} \in E$.
*Simple:*    No edge occurs twice in a path.
*Connected:*    If each pair of vertices are adjoined by an edge.

#### Circuits

*Circuit:*    A path with the same starting and ending vertex.
*Complete:*    On $n$ vertices, $K_n$, is the connected graph where each vertex is adjacent to each other.

#### Bipartite Graphs

*Bipartite:*    The vertex set $V = v_1 \cup v_2, v_1 \cap v_2 = \emptyset$, and no vertex in $V_1$ is adjacent to any other in $V_1$, and no vertex in $V_2$ is adjacent to any other in $V_2$.

#### Trees

*Tree:*    A connected graph that has no circuit.
*BiSTree:*    For every node, all elements in the left subtree are less than the node's value, and all elements in the right subtree are greater.
***Lemma:***    If $G$ is a tree, $G$ has at least one vertex of degree 1.

*Proof.* For the sake of contradiction, suppose each vertex has degree $\geq 2$. Pick a vertex, $v_0$. Since, $\deg(v_0) \geq 2$, it is adjacent to some $v_j$. Because $\deg(v_1) \geq 2$, it has an edge distinct from $\{v_0, v_1\}$, follow it to $v_2$. Then, $v_2$ has edge distinct from $\{v_1, v_2\}$, follow it to $v - 3$. If $v_3 = v_0(v_1, v_2)$ we have a circuit. $v_3$ has edge distinct from $\{v_2, v_3\}$. Go to $v_4$. Continue $\ldots$. Either some $v_j$ is visited again, or $v_0, v_1, v_2, \ldots v_n$. Therefore, it must be a circuit.

Hence, $G$ must have a vertex with degree of at least 1 such that $1 \leq$.    $\square$

***Theorem 1:***    A tree with $n$ vertices always has $n - 1$ edges.

*Proof.* By the Lemma, there exists a vertex of degree 1. Remove it and its edge. We still have a tree. This new tree has vertex of degree 1. Remove it and its edge. Continue until you get to $k_2$, then $k_1$. We stop with 1 vertex, 0 edges, we have removed $n - 1$ vertices. Each edge was removed. Thus, we threw out *all $n - 1$* edges.    $\square$

## Euler Circuits

*Euler circuit:* A circuit graph which uses each edge only once.

*Euler path:* A path which uses each edge – start and end vertices are distinct.

**Note:** $G$ has an Euler circuit if, and only if, it is connected and each vertex has an even degree. Intuitively, if each vertex has an even degree, then if you come into the vertex through the entrance (first edge), and you leave through the exit (second edge) you have used up both openings.

$G$ has an Euler path if, and only if, it is connected and has exactly two vertices of odd degree.

## Isomorphisms

Two graphs, $G = (V, E)$ and $H = (W, f)$ are *isomorphic* if there is an $f : V \to W$ which is one-to-one, onto, and $\{v_i, v_j\} \in E \iff \{f(v_i), f(v_j)\} \in F$.

## Vertex Colorings

If $G$ contains a triangle (i.e., if it has a copy of $k_3$, we need at least 3 colors. If $G$ contains a copy of $K_n$, we need at least $n$.

If a graph has no overlapping paths, the graph requires no more than 4 colors.

## Hamilton Graphs

A graph has a *Hamilton Circuit* if there is a circuit that uses each vertex once.

**Note:** This is different from Euler, as Euler uses edges. This is specifically for vertices. If it has a vertex of degree 1, it cannot have a Hamilton circuit.

## Sets

1. Suppose that the universal set $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the particular sets $A = \{2, 3, 5\}$ and $B = \{1, 3, 5, 7, 9\}$. Find each of the following:

(a) $A \cup B = \{1, 2, 3, 5, 7, 9\}$

(b) $A \cap B = \{3, 5\}$

(c) $\{x \in U : x^2 < 10 \wedge x \in A\} = \{2, 3\}$

(d) $B' \cap A = \{2\}$

(e) $\{x \in A : x \geq 7\} = \emptyset$

(f) $\mathcal{P}(A) = \{\emptyset, \{2\}, \{3\}, \{5\}, \{2, 3\}, \{2, 5\}, \{3, 5\}, \{2, 3, 5\}\}$