Multivariable Calculus Exam II Corrections

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In-Class Portion

1. Consider the function $f(x, y) = \dfrac{x^4 - 4y^2}{x^2 + 2y^2}$

   (b) (2 points each) We will investigate $\lim_{(x,y)\to(0,0)} f(x, y)$.

   ii. Find the limit, along the path $y = 0$:

   Solution.

$$\begin{aligned} \lim_{x\to 0} f(x, 0) &= \lim_{x\to 0} \frac{x^4}{x^2} \\ &= \lim_{x\to 0} x^2 \\ &= \boxed{0}. \end{aligned}$$

   iii. Find the limit, along the path $y = x$.

   Solution.

$$\begin{aligned} \lim_{y\to 0} f(y, y) &= \lim_{y\to 0} \frac{y^4 - 4y^2}{y^2 + 2y^2} \\ &= \lim_{y\to 0} \frac{y^4 - 4y^2}{3y^2} \\ &= \lim_{y\to 0} \frac{y^2(y^2 - 4)}{3y^2} \\ &= \lim_{y\to 0} \frac{y^2 - 4}{3} \\ &= \boxed{-\frac{4}{3}}. \end{aligned}$$

   iv. What do your answers indicate about this limit?

   Solution. The limit does not exist, since the limits for each path give different values.

2. (10 points) Find an equation of the tangent plane to $g(x, y) = x^2 e^{x+2y}$ at point $(2, -1)$.

Solution. First, we need to find the partial derivatives of g:

$$\begin{aligned} g_x(x, y) &= \frac{\partial}{\partial x}\left[x^2 e^{x+2y}\right] \\ &= \frac{\partial}{\partial x}\left[x^2\right] e^{x+2y} + x^2 \frac{\partial}{\partial x}\left[e^{x+2y}\right] \\ &= 2xe^{x+2y} + x^2 e^{x+2y} \\ &= e^{x+2y}\left(2x + x^2\right), \text{ and} \\ g_y(x, y) &= \frac{\partial}{\partial y}\left[x^2 e^{x+2y}\right] \\ &= x^2 \frac{\partial}{\partial y}\left[e^{x+2y}\right] \\ &= 2x^2 e^{x+2y}. \end{aligned}$$

Now that we have our partials, we can evaluate them at the point $(2, -1)$:

$$\begin{aligned} g_x(2, -1) &= e^{2-2}\left(2(2) + (2)^2\right) \\ &= e^0\,(4 + 4) \\ &= 8, \text{ and} \\ g_y(2, -1) &= 2(2^2)e^{2-2} \\ &= 8e^0 \\ &= 8. \end{aligned}$$

We also need to find z_0, which is given by solving $g(2, -1)$:

$$z_0 = g(2, -1) = 2^2 e^{2-2} = 4e^0 = 4.$$

Finally, we can write the equation of the tangent plane:

$$\boxed{z = 4 + 8(x - 2) + 8(y + 1).}$$

3. (10 points) Find the directional derivative of $h(x) = \sqrt{x+y} - x^2 + \frac{1}{\pi}\sin(\pi y)$, at the point $(3, 1)$ in the direction $\langle 5, -2\rangle$.

Solution. Similar to the previous problem, we need to find the partial derivatives of h:

$$\begin{aligned} h_x(x,y) &= \frac{\partial}{\partial x}\left[\sqrt{x+y} - x^2 + \frac{1}{\pi}\sin(\pi y)\right] \\ &= \frac{1}{2\sqrt{x+y}} - 2x, \text{ and} \\ h_y(x,y) &= \frac{\partial}{\partial y}\left[\sqrt{x+y} - x^2 + \frac{1}{\pi}\sin(\pi y)\right] \\ &= \frac{1}{2\sqrt{x+y}} + \cos(\pi y). \end{aligned}$$

Then, we evaluate them at the point $(3, 1)$:

$$\begin{aligned} h_x(3,1) &= \frac{1}{2\sqrt{3+1}} - 2(3) \\ &= \frac{1}{4} - 6 \\ &= -\frac{23}{4}, \text{ and} \\ h_y(3,1) &= \frac{1}{2\sqrt{3+1}} + \cos(\pi) \\ &= \frac{1}{4} - 1 \\ &= -\frac{3}{4}. \end{aligned}$$

Before we can build the directional derivative, we need to find the magnitude of $\langle 5, -2\rangle$:

$$\|\mathbf{v}\| = \|\langle 5, -2\rangle\| = \sqrt{5^2 + (-2)^2} = \sqrt{25+4} = \sqrt{29}.$$

Now we can build the directional derivative:

$$\begin{aligned} D_{\mathbf{v}} &= \frac{(h_x(3,1) + h_y(3,1)) \cdot \mathbf{v}}{\sqrt{29}} \\ &= \frac{-\frac{23}{4}(5) - \frac{3}{4}(-2)}{\sqrt{29}} \\ &= \frac{-\frac{115}{4} + \frac{6}{4}}{\sqrt{29}} \\ &= \frac{-\frac{109}{4}}{\sqrt{29}} \\ &= \boxed{-\frac{109}{4\sqrt{29}}}. \end{aligned}$$

4. (12 points) For the function $k(x, y) = x^3 - 3x + 3xy^2$, find each critical point, and identify each as a local minimum, local maximum, or saddle point. [I guarantee there will be no "inconclusive."]

Solution. We start by finding the partial derivatives of k:

$$\begin{aligned} k_x(x, y) &= 3x^2 - 3 + 3y^2, \\ k_y(x, y) &= 6xy. \end{aligned}$$

From the second equation, we see that either $y = 0$ or $x = 0$. If $y = 0$, then we have:

$$\begin{aligned} k_x(x, 0) &= 3x^2 - 3 \\ &= 0 \\ x^2 &= 1 \\ x &= \pm 1. \end{aligned}$$

If $x = 0$, then we have:

$$\begin{aligned} k_x(0, y) &= -3 + 3y^2 \\ &= 0 \\ y^2 &= 1 \\ y &= \pm 1. \end{aligned}$$

So, we have the following critical points:

$$(1, 0), (-1, 0), (0, 1), (0, -1).$$

Now we need to find the second partial derivatives:

$$\begin{aligned} k_{xx}(x, y) &= 6x, \\ k_{yy}(x, y) &= 6x, \\ k_{xy}(x, y) &= 6y. \end{aligned}$$

We will use the following equation for D in order to classify our critical points:

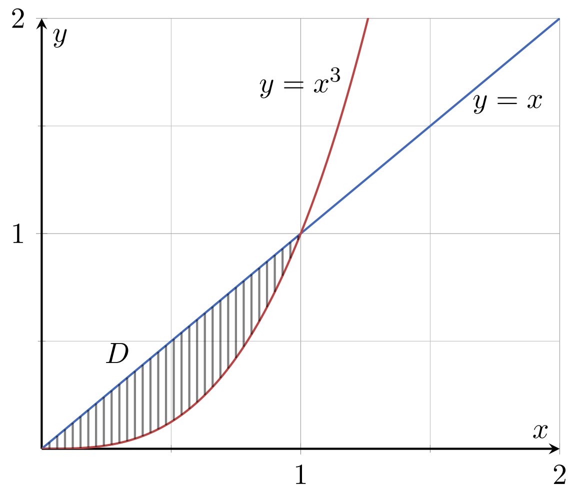
$$D = k_{xx}(x_0, y_0)k_{yy}(x_0, y_0) - \left(k_{xy}(x_0, y_0)\right)^2.$$

Thus, we get the following table:

Critical Point	**D**	**Conclusion**
$(1, 0)$	$6(1)(6(1)) - (6(0))^2 = 36$	$(D > 0) \wedge (f_{xx} > 0) \Rightarrow$ Local minimum
$(-1, 0)$	$6(-1)(6(-1)) - (6(0))^2 = 36$	$(D > 0) \wedge (f_{xx} < 0) \Rightarrow$ Local minimum
$(0, 1)$	$6(0)(6(0)) - (6(1))^2 = -36$	$(D < 0) \Rightarrow$ Saddle point
$(0, -1)$	$6(0)(6(0)) - (6(-1))^2 = -36$	$(D < 0) \Rightarrow$ Saddle point

5. (10 points) Find the value of $\iint_D 12xy^2\,dA$ where D is the region in the first quadrant between $y = x$ and $y = x^3$.

Solution. Our region D is bounded by the first quadrant and the curves $y = x$ and $y = x^3$:



Thus, this gives us the region $D = \{(x, y) \,|\, 0 \le x \le 1, x^3 \le y \le x\}$.

Allowing us to write the double integral as:

$$\begin{aligned}
\iint_D 12xy^2\,dA &= \int_0^1 \int_{x^3}^{x} 12xy^2\,dy\,dx \\
&= \int_0^1 4x\left[y^3\right]_{x^3}^{x}\,dx \\
&= 4\int_0^1 x\left[x^3 - (x^3)^3\right]\,dx \\
&= 4\int_0^1 x^4(1 - x^6)\,dx \\
&= 4\left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1 \\
&= 4\left(\frac{1}{5} - \frac{1}{7}\right) \\
&= 4\left(\frac{2}{35}\right) \\
&= \boxed{\frac{8}{35}}.
\end{aligned}$$

6. (10 points) The solid E is the region in the cylinder $x^2+y^2=1$ which lives below the plane $z=4$ and above $z=1-x^2-y^2$. [See picture]. Determine $\iiint_E (x^2+y^2)\,dV$.

Solution. First, we can change our integral to cylindrical coordinates:

$$\begin{aligned} x &= r\cos(\theta), \\ y &= r\sin(\theta), \\ z &= z, \\ dV &= r\,dr\,d\theta\,dz. \end{aligned}$$

Thus, we can rewrite and solve our integral:

$$\begin{aligned} \iiint_E (x^2+y^2)\,dV &= \int_0^{2\pi}\int_0^1\int_{1-r^2}^4 (r^2)\cdot(r)\,dz\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 r^3\,[z]_{1-r^2}^4\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^1 r^3\big(4-(1-r^2)\big)\,dr\,d\theta \\ &= 2\pi\int_0^1 r^3(3+r^2)\,dr \\ &= 2\pi\left[\frac{3r^4}{4}+\frac{r^6}{6}\right]_0^1 \\ &= 2\pi\left(\frac{3}{4}+\frac{1}{6}\right) \\ &= 2\pi\left(\frac{9}{12}+\frac{2}{12}\right) \\ &= 2\pi\left(\frac{11}{12}\right) \\ &= \boxed{\frac{11\pi}{6}}. \end{aligned}$$

(Note the change in order of integration from $dr\,d\theta\,dz$ to $d\theta\,dr\,dz$. This is because the limits of integration for z are dependent on r, and the limits of integration for r are dependent on θ.)