



HENDRIX

COLLEGE

Multivariable Calculus Notes

MATH 230

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1.1 Parametric Equations

1.1.1 Introduction

Most of your calculus experience has been single variable, so that the functions under consideration were typically $f : \mathbb{R} \rightarrow \mathbb{R}$. Our course is divided into roughly 3 sections:

- Parametric Equations/Functions: Functions of the form $f : \mathbb{R} \rightarrow \mathbb{R}^n$ (Chapters 1 - 3)
- Scalar Functions: Functions of the form $f : \mathbb{R}^n \rightarrow \mathbb{R}$ (Chapters 4 - 5)
- Vector Fields: Functions of the form $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (Chapter 6)

1.1.2 Parametric Equations

A *parametric equation* (or, *sometimes parametric function* or *vector-valued function*) is a function of the form $f : \mathbb{R} \rightarrow \mathbb{R}^n$. We will typically consider $n = 2$ or $n = 3$ and call the input variable the parameter, usually denoted by t . We write them as

$$f(t) = \begin{cases} x(t) \\ y(t) \end{cases} \quad \text{or} \quad f(t) = \begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}.$$

A *parametric curve* is the set of points $(x(t), y(t))$ in \mathbb{R}^2 or $(x(t), y(t), z(t))$ in \mathbb{R}^3 traced out. Note that in general, the curve may not be a function for y in terms of x , but is a function of the parameter t .

1.1.3 Graphing Parametric Curves in the Second Dimension

Elimination of the Parameter

In some cases, we can explicitly solve for t in terms of one of x or y . When this is possible, you can write $y(x)$ or $x(y)$ and use your “regular” algebraic knowledge. We call this process *eliminating the parameter*.

Using Technology

- Your TI-84 can graph this if you switch to **par** mode.
- Likewise, GeoGebra can do this, using the **curve** function.
 - In general, the syntax is: `curve(x(t), y(t), t, min, max)`



1.1.4 The Cycloid

A wheel of radius a is rolling along a flat road at a constant velocity. The curve generated by a point along the edge of the wheel traces out a shape called a *cycloid*. Let t represent the angle - in radians!!!! - rotated through, and that the point of interest starts at the origin. Before we find the equations for the point, let's find the location of the center of the circle:

$$f_{\text{center}}(t) = \begin{cases} x(t) = at \\ y(t) = a \end{cases}$$

Then, relative to the center, our point along the edge has equations

$$f(t) = \begin{cases} x(t) = -a \sin(t) \\ y(t) = -a \cos(t) \end{cases}$$

Thus, our point has parametric equations

$$f(t) = \begin{cases} x(t) = a(t - \sin(t)) \\ y(t) = a(1 - \cos(t)) \end{cases}$$

1.1.5 Final Notes

Next time, we'll start asking Calculus-y questions: What are the velocities in the x , y , and total directions? What total distance does it travel? What is the area of the region under one period of the cycloid?

- The syllabus has a number of practice problems to work on. These are not required, and not to be turned in, but are for you to work before class next time.
- We will talk about them at the start of the next class. You should try them beforehand.
- The most common reason for a lack of success in this class is not spending time working problems on your own.

1.2 Calculus of Parametric Curves

For this section, we will have a parametric curve in \mathbb{R}^2 , defined by $f(t) = \begin{cases} x(t) \\ y(t) \end{cases}$. In many cases, the curve does not describe y as a function of x . However, we can still carry over many ideas from single variable calculus.



1.2.1 Slope for a Parametric Curve

Given a point t_0 , the *slope of the curve* in the xy -plane is given by

$$\left. \frac{dy}{dx} \right|_{t=t_0} = \left. \frac{dy/dt}{dx/dt} \right|_{t=t_0}.$$

Note that this is undefined when $x'(t_0) = 0$.

The *tangent line* at t_0 is given by

$$y = \left(\left. \frac{dy}{dx} \right|_{t=t_0} \right) (x - x(t_0)) + y(t_0).$$

1.2.2 Second Derivative

The value of the second derivative for the curve at t_0 is given by

$$\left. \frac{d^2y}{dx^2} \right|_{t=t_0} = \frac{d}{dt} \left(\left. \frac{dy}{dx} \right|_{t=t_0} \right) = \frac{d}{dt} \left(\frac{dy/dt}{dx/dt} \right) \Big|_{t=t_0}.$$

Note the benefit of Leibnitz notation for each of these two derivatives!

1.2.3 Area Under a Curve

Suppose that a parametric curve is non-self intersecting. Then, the signed area of the region between the curve and the x -axis on the t interval $[t_a, t_b]$ is given by

$$A = \int_{t_a}^{t_b} y(t) \frac{dx}{dt} dt = \int_{t_a}^{t_b} y(t) \frac{dx}{dt} dt.$$

1.2.4 Arc Length

The *arc length* of a parametric curve over the t interval $[t_a, t_b]$ is given by

$$s = \int_{t_a}^{t_b} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt.$$

1.2.5 Surface Area

The *surface area* of the region obtained by rotating a non-self intersecting parametric curve is given by

$$S = \int_{t_a}^{t_b} 2\pi y(t) \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt.$$



1.2.6 The Cycloid

We can apply each of the above to the cycloid:

- **Derivative:** $\frac{dy}{dx} = \frac{dy}{dt} = \frac{\sin(t)}{1-\cos(t)}$. Note that the slope is then independent of the radius of the wheel and that the slope is undefined at each of $t = \dots, -4\pi, -2\pi, 0, 2\pi, 4\pi, \dots$
- **Concavity:** $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{\sin(t)}{1-\cos(t)} \right)$. After some work, we find that $\frac{d^2y}{dx^2} = -\frac{a}{y^2}$, which shows that the cycloid is always concave down.
- **Area:** The area of one period of the cycloid $A = 3\pi a^2$, after some work.

$$\begin{aligned}
 A &= \int_0^{2\pi} (a - a \cos t)(a - a \cos t) dt \\
 &= a^2 \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t) dt \\
 &= a^2 \left(2\pi + \int_0^{2\pi} 1 - 2 \cos t + \cos^2 t \right) dt \\
 &= a^2 \left(2\pi + \left(\frac{t}{2} + \frac{1}{4} \sin(2t) \right) \right) \Big|_0^{2\pi} \\
 &= 3\pi a^2.
 \end{aligned}$$

- **Arc Length:** The arc length of one period of the cycloid is $s = 8a$, again after some work.

$$\begin{aligned}
 S &= \int_0^{2\pi} \sqrt{(a - a \cos t)^2 + (a \sin t)^2} dt \\
 &= a \int_0^{2\pi} \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} dt \\
 &= a \int_0^{2\pi} \sqrt{2 - 2 \cos t} dt \\
 &= \sqrt{2}a \int_0^{2\pi} \sqrt{1 - \cos t} dt \\
 &= \sqrt{2}a \int_0^{2\pi} \sqrt{2 \sin^2 \left(\frac{t}{2} \right)} dt \\
 &= \sqrt{2}a \cdot \sqrt{2} \int_0^{2\pi} \sin \left(\frac{t}{2} \right) dt \\
 &= 2a \left(-2 \cos \left(\frac{t}{2} \right) \right) \Big|_0^{2\pi} \\
 &= 8a.
 \end{aligned}$$



- *Surface Area*: The surface area of the solid obtained by rotating one period of the cycloid around the x -axis is $S = \frac{64\pi a^2}{3}$, after a lot of tedious work.