Error Analysis

The Mean

The mean (or average) is the statistical method of obtaining a figure, which is closer to the actual true value from repeated measurements showing variations in the data.

In the table below are the times measured for a pendulum to swing back to its point of release using a stopwatch. Each time the pendulum is released from the same point.

i	1	2	3	4	5	6	7	8	9	10
Time (s)	5.14	5.18	5.23	5.13	5.11	5.19	5.23	5.29	5.06	5.25

The mean ($\overline{x}$) is calculated by summing all the readings and dividing by the number of readings (n). This is represented algebraically as follows

$$\overline{x} = \frac{\sum_{i=1}^{i=n} x_i}{n}$$

So using the data in the table we find that the mean is

$$\overline{x} = \frac{(5.14 + 5.18 + 5.23 + 5.13 + 5.11 + 5.19 + 5.23 + 5.29 + 5.06 + 5.25)}{n}$$

$$\overline{x} = \frac{51.81}{10} = 5.181 = 5.18 \text{ s}$$

So what is the uncertainty in the calculated mean?
A quick and simple method is to divide the range by the number of readings (n).

$$\text{Uncertainty in mean } = \frac{\text{range}}{n}$$

i.e.

$$\text{Uncertainty in mean } = \frac{(5.29 - 5.06)}{10}$$

$$\text{Uncertainty in mean } = 0.023 \text{ s}$$

So from the experiments we can say the time taken for the pendulum to swing back to its release point is 5.18±0.02 s.
However, a more elegant method to obtain an uncertainty is to resort to statistics again!

The Uncertainty in the Mean

We could examine the spread or dispersion of the data in relation to the calculated mean ($\overline{x}$) of 5.18 s (5.181). The difference between the i th measurement and the mean is referred to as the **deviation of the mean (d_i)** and is shown in the table below.

i	x_i (s)	$d_i = x_i - \overline{x}$ (s)	$d_i^2 = (x_i - \overline{x})^2$ (s^2)
1	5.14	-0.041	0.001681
2	5.18	-0.001	0.000001
3	5.23	0.049	0.002401
4	5.13	-0.051	0.002601
5	5.11	-0.071	0.005041
6	5.19	0.009	0.000081
7	5.23	0.049	0.002401
8	5.29	0.109	0.011881
9	5.06	-0.121	0.014641
10	5.25	0.069	0.004761
		$\sum(x_i - \overline{x}) = 0$	$\sum(x_i - \overline{x})^2 = 0.04549$

However, if we take the mean of the deviation (d_i) we find that it comes out to be zero, since the sum of the deviations is equal to zero.

Instead we consider the **square of the mean** (d_i^2) to get a handle on the variation of the data as shown in the table.

Now if we take the mean of the **square of the mean** (d_i^2) we obtain a quantity referred to as the **variance** (σ^2). This is represented algebraically as follows

$$\sigma^2 = \frac{\sum (x_i - \overline{x})^2}{n}$$

The value of the variance for our pendulum data, which gives you an indication of the variation in the measurements, therefore is

$$\sigma^2 = \frac{0.04549}{10} = 0.004549 \text{ s}^2$$

In general, the practice is to obtain the **standard deviation** (σ) of the data set to indicate the variation in the measurement. I.e.

$$\sigma = \left(\frac{\sum (x_i - \overline{x})^2}{n} \right)^{\frac{1}{2}}$$

$$\sigma = \left(\frac{0.04549}{10} \right)^{\frac{1}{2}}$$

$$\sigma = 0.06745 \text{ s}$$

It is important to realise that the **standard deviation** (σ) is not the uncertainty in the mean, just an indicator of the dispersion in the collected data.
We define the **uncertainty in the mean** (the standard error of the mean) ($\sigma_{\overline{x}}$) as

$$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$$

We can now finally determine standard error of the mean as

$$\sigma_{\overline{x}} = \frac{0.06745}{\sqrt{10}} = 0.02$$

Therefore we can say the time taken for the pendulum to return to its release point is **5.18 ± 0.02 s.**

The key equations are summarised in the table below.

Population	Sample
Mean $$\overline{x} = \frac{\sum_{i=1}^{i=n} x_i}{n}$$	Mean $$\overline{x} = \frac{\sum_{i=1}^{i=n} x_i}{n}$$
Variance $$\sigma^2 = \frac{\sum (x_i - \overline{x})^2}{n}$$	Variance $$\sigma^2 = \frac{\sum (x_i - \overline{x})^2}{n-1}$$
Standard Deviation $$\sigma = \left(\frac{\sum (x_i - \overline{x})^2}{n} \right)^{\frac{1}{2}}$$	Standard Deviation $$\sigma = \left(\frac{\sum (x_i - \overline{x})^2}{n-1} \right)^{\frac{1}{2}}$$
Uncertainty in the mean (Standard error in the mean) $$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$$	Uncertainty in the mean (Standard error in the mean) $$\sigma_{\overline{x}} = \frac{\sigma}{\sqrt{n}}$$
Best estimate $$\overline{x} \pm \sigma_{\overline{x}}$$	Best estimate $$\overline{x} \pm \sigma_{\overline{x}}$$

What are these other expressions containing this (n-1) factor?
These expressions (Sample) allow us to estimate the standard deviation from a very large group or population of data. A data set may consist of several thousand measurements; we can take a small sample set (n=25 say) to estimate the mean and the standard deviation of the population. To obtain a value as close as possible to the actual value one would need to take a large number of measurements. If we use the expressions labelled as “population” they would tend to underestimate the **standard deviation** (σ), particularly for small n. Other reasons also exist such as degrees of freedom etc... So we **generally** utilise the expressions referred to as the “sample” containing **(n-1)** in the denominator. If n is very large, then the (–1) becomes negligible.

Propagation of errors

Generally we measure a number of quantities experimentally to calculate a specific value. For example, we may want to determine the centripetal acceleration (a_c) of a golf ball of mass (m), travelling at a constant speed (v) in a circular orbit of radius (r). The centripetal acceleration is given by

$$a_c = \frac{mv^2}{r}$$

All the experimentally measured quantities (m, v, r) will have an associated error as calculated in the previous section ($\Delta m,\ \Delta v,\ \Delta r$). Note $\sigma_{\bar{m}} = \Delta m$, etc... just a different notation people use.
It is important that we understand how we propagate or carry over these uncertainties into the final result Δa_c .

How do we combine these errors to obtain an estimate of the error (Δa_c) in (a_c)?

The first thing we need to determine if these errors associated with the quantities are correlated or uncorrelated.

Uncorrelated errors are **random** in nature and are **independent** of each other. Any fluctuation in Δm should not influence or be related to Δv , Δr in any fashion. They should display a Gaussin like trend.

Correlated errors are **dependent** or **influenced** by each other. For example, you are given 100 steel ball bearings of mass (m). You are then asked to determine the total mass of the 100 ball bearings using a device, which can only measure 1 ball bearing at a time. If you just decide to measure the mass m of one of the bearings and determine its associated error Δm , and then assume that all the bearings have the same error Δm and mass m , then you have just correlated the errors. You are making the assumption that each bearing is identical. They are no longer independent. However, if you measured each single ball bearing and its error, then the errors are uncorrelated. Even if the errors of each measurement may be the same.

To combine errors we use the expression below known as the "error propagation equation" which provides a calculus approach to dealing with the propagation of errors. One should refer to the literature to obtain further details about its origin.

Error propagation equation
$\sigma_{\bar{x}}^2 = \left(\frac{\partial x}{\partial a}\right)^2 \sigma_{\bar{a}}^2 + \left(\frac{\partial x}{\partial b}\right)^2 \sigma_{\bar{b}}^2 + \left(\frac{\partial x}{\partial c}\right)^2 \sigma_{\bar{c}}^2 + \ldots\ldots + 2\sigma_{\bar{a}\bar{b}\bar{c}}^2 \left(\frac{\partial x}{\partial a}\right)\left(\frac{\partial x}{\partial b}\right)\left(\frac{\partial x}{\partial c}\right) + \ldots$

If the errors are correlated then we need to take account of the cross terms (shown in red - $2\sigma_{\bar{a}\bar{b}\bar{c}}^2 ..$) involving the respective variables and multiply them by their derivatives. Again refer to literature for a more detailed approach regarding correlated errors.

For uncorrelated errors this term averages to zero. But generally this term is neglected for uncorrelated errors.

The table below shows the expressions for uncorrelated errors in the different notations and forms you may come across in the literature. These are the general expressions one can use to deal with any dependable variable that is a function of many different measured/given variables. I.e. $x = f(a,b,c..)$.

Uncorrelated	Uncorrelated Non-Statistical
$\left(\delta x\right)^2 = \left(\frac{\partial x}{\partial a}\delta a\right)^2 + \left(\frac{\partial x}{\partial b}\delta b\right)^2 + \left(\frac{\partial x}{\partial c}\delta c\right)^2 + ..$	
$\left(\Delta x\right)^2 = \left(\frac{\partial x}{\partial a}\Delta a\right)^2 + \left(\frac{\partial x}{\partial b}\Delta b\right)^2 + \left(\frac{\partial x}{\partial c}\Delta c\right)^2 + ..$	
$\left(\Delta x\right)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$	$\delta x = \left\lvert\frac{\partial x}{\partial a}\right\rvert \delta a + \left\lvert\frac{\partial x}{\partial b}\right\rvert \delta b + \left\lvert\frac{\partial x}{\partial c}\right\rvert \delta c + ..$
$\left(\Delta x\right)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$	$\Delta x = \left\lvert\frac{\partial x}{\partial a}\right\rvert \Delta a + \left\lvert\frac{\partial x}{\partial b}\right\rvert \Delta b + \left\lvert\frac{\partial x}{\partial c}\right\rvert \Delta c + ..$
$\sigma_x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \sigma_a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \sigma_b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \sigma_c^2 + ..$	
$\sigma_{\bar{x}}^2 = \left(\frac{\partial x}{\partial a}\right)^2 \sigma_{\bar{a}}^2 + \left(\frac{\partial x}{\partial b}\right)^2 \sigma_{\bar{b}}^2 + \left(\frac{\partial x}{\partial c}\right)^2 \sigma_{\bar{c}}^2 + ..$	

If the variables are correlated then you could use the Uncorrelated Non- Statistical expressions since this will provide you with the largest possible error you could have!

The expressions may initially look daunting, but they simply involve partial differentiating the function with respect to each variable.

What is partial differentiation? – this involves treating all the other variables as constants and taking the normal derivative of the function!

Lets begin with determining some general rules for the propagation of errors using the above. This will give us a set of expressions to deal with typical expressions we may encounter.

We will consider that a dependable variable x depends on variables a and b, which have associated errors of Δa and Δb. We want to find Δx in each case. Where m, n, p are constants. We will assume that all errors are uncorrelated.

Addition and subtraction

[1] If $x = a + b$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 \quad \text{where} \quad \frac{\partial x}{\partial a} = 1, \quad \frac{\partial x}{\partial b} = 1$$

$$\Delta x^2 = (1)^2 \Delta a^2 + (1)^2 \Delta b^2$$

So

$$\boxed{\Delta x^2 = \Delta a^2 + \Delta b^2}$$

[2] If $x = 7a + 3b$, what is the error in Δx.

We can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 \quad \text{where} \quad \frac{\partial x}{\partial a} = 7, \quad \frac{\partial x}{\partial b} = 3$$

$$\Delta x^2 = (7)^2 \Delta a^2 + (3)^2 \Delta b^2$$

So

$$\Delta x^2 = 49\Delta a^2 + 9\Delta b^2$$

Now if $x = ma + nb$ where m and n are just constants then

$$\boxed{\Delta x^2 = m^2\Delta a^2 + n^2\Delta b^2}$$

[2] If $x = ma - nb$, what is the error in Δx.

We can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 \quad \text{where} \quad \frac{\partial x}{\partial a} = m, \quad \frac{\partial x}{\partial b} = -n$$

$$\Delta x^2 = (m)^2 \Delta a^2 + (-n)^2 \Delta b^2$$

So

$$\boxed{\Delta x^2 = m^2\Delta a^2 + n^2\Delta b^2}$$

Multiplication and Division

[1] If $x = ab$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 \quad \text{where} \quad \frac{\partial x}{\partial a} = b\ , \quad \frac{\partial x}{\partial b} = a$$

$$\Delta x^2 = (b)^2 \Delta a^2 + (a)^2 \Delta b^2$$

Dividing though by $(ab)^2$

$$\frac{\Delta x^2}{(ab)^2} = \frac{(b)^2 \Delta a^2}{(ab)^2} + \frac{(a)^2 \Delta b^2}{(ab)^2} \quad \text{Note } x^2 = (ab)^2$$

so

$$\boxed{\frac{\Delta x^2}{x^2} = \frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2}}$$

[2] If $x = \frac{a}{b}$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 \quad \text{where} \quad \frac{\partial x}{\partial a} = \frac{1}{b}\ , \quad \frac{\partial x}{\partial b} = \frac{-a}{b^2}$$

$$\Delta x^2 = \left(\frac{1}{b}\right)^2 \Delta a^2 + \left(\frac{-a}{b^2}\right)^2 \Delta b^2$$

Dividing though by $\left(a/b\right)^2$

$$\frac{\Delta x^2}{\left(a/b\right)^2} = \left(\frac{1}{b}\right)^2 \Delta a^2 \left(\frac{b}{a}\right)^2 + \left(\frac{a}{bb}\right)^2 \Delta b^2 \left(\frac{b}{a}\right)^2$$

Note $x^2 = (a/b)^2$, so

$$\boxed{\frac{\Delta x^2}{x^2} = \frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2}}$$

[3] If $x = 7ab$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 \quad \text{where} \quad \frac{\partial x}{\partial a} = 7b\ , \quad \frac{\partial x}{\partial b} = 7a$$

$$\Delta x^2 = (7b)^2 \Delta a^2 + (7a)^2 \Delta b^2$$

Dividing though by $(7ab)^2$

$$\frac{\Delta x^2}{(7ab)^2} = \frac{(7b)^2 \Delta a^2}{(7ab)^2} + \frac{(7a)^2 \Delta b^2}{(7ab)^2} \quad \text{Note } x^2 = (7ab)^2$$

$$\boxed{\frac{\Delta x^2}{x^2} = \frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2}}$$

[3] If $x = m\frac{a}{b}$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 \quad \text{where} \quad \frac{\partial x}{\partial a} = \frac{m}{b} \ , \quad \frac{\partial x}{\partial b} = \frac{-ma}{b^2}$$

$$\Delta x^2 = \left(\frac{m}{b}\right)^2 \Delta a^2 + \left(\frac{-ma}{b^2}\right)^2 \Delta b^2$$

Dividing though by $\left(m\frac{a}{b}\right)^2$

$$\frac{\Delta x^2}{\left(m\frac{a}{b}\right)^2} = \left(\frac{m}{b}\right)^2 \Delta a^2 \left(\frac{b}{ma}\right)^2 + \left(\frac{ma}{bb}\right)^2 \Delta b^2 \left(\frac{b}{ma}\right)^2$$

Note $x^2 = \left(m\frac{a}{b}\right)^2$, so

$$\boxed{\frac{\Delta x^2}{x^2} = \frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2}}$$

Powers

[1] If $x = a^3$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a \quad \text{where} \quad \frac{\partial x}{\partial a} = 3a^2$$

$$\Delta x = \left(3a^2\right)\Delta a$$

Dividing though by a^3

$$\frac{\Delta x}{a^3} = \frac{\left(3a^2\right)\Delta a}{a^2 a} \quad \text{Note } x = a^3$$

so

$$\frac{\Delta x}{x} = 3\frac{\Delta a}{a}$$

[2] If $x = ma^n$, what is the error in Δx where m and n are just constants.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a \quad \text{where} \quad \frac{\partial x}{\partial a} = mna^{n-1}$$

$$\Delta x = \left(mna^{n-1}\right)\Delta a$$

Dividing though by ma^n

$$\frac{\Delta x}{ma^n} = \frac{\left(mna^{n-1}\right)\Delta a}{ma^n}$$

$$\frac{\Delta x}{ma^n} = \frac{\left(mna^n a^{-1}\right)\Delta a}{ma^n}$$

$$\frac{\Delta x}{ma^n} = \left(na^{-1}\right)\Delta a$$

$$\frac{\Delta x}{ma^n} = \frac{n\Delta a}{a} \quad \text{where x} = ma^n$$

so

$$\boxed{\frac{\Delta x}{x} = n\frac{\Delta a}{a}}$$

Exponentials

[1] If $x = e^{3a}$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \quad \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a \quad \text{where} \quad \frac{\partial x}{\partial a} = 3e^{3a}$$

$$\Delta x = \left(3e^{3a}\right)\Delta a$$

Dividing though by e^{3a}

$$\frac{\Delta x}{e^{3a}} = \frac{\left(3e^{3a}\right)\Delta a}{e^{3a}} \quad \text{Note } x = e^{3a}$$

so

$$\frac{\Delta x}{x} = 3\Delta a$$

[2] If $x = me^{na}$, what is the error in Δx where m and n are just constants.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \quad \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a \quad \text{where} \quad \frac{\partial x}{\partial a} = nme^{na}$$

$$\Delta x = \left(nme^{na}\right)\Delta a$$

Dividing though by me^{na}

$$\frac{\Delta x}{me^{na}} = \frac{\left(nme^{na}\right)\Delta a}{me^{na}} \quad \text{Note } x = me^{na}$$

so

$$\boxed{\frac{\Delta x}{x} = n\Delta a}$$

[3] If $x = 6m^{3a}$, what is the error in Δx where m is a constant.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \quad \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a$$

$$x = 6m^{3a} \quad \text{using the relation } \boxed{e^{\ln z} = z}$$

$$x = 6\left(e^{\ln m}\right)^{3a} = 6\left(e^{3\ln m}\right)^{a} \quad \text{Let } c = 3\ln m$$

$$x = 6\left(e^{c}\right)^{a} = 6e^{ca} \quad \text{where} \quad \frac{\partial x}{\partial a} = 6ce^{ca}$$

$$\Delta x = \left(6ce^{ca}\right)\Delta a$$

Dividing though by $6e^{ca}$

$$\frac{\Delta x}{6e^{ca}} = \frac{\left(6ce^{ca}\right)\Delta a}{6e^{ca}} \quad \text{Note } x = 6e^{ca}$$

so

$$\frac{\Delta x}{x} = c\Delta a \qquad \text{Note } c = 3\ln m$$

$$\frac{\Delta x}{x} = (3\ln m)\Delta a$$

[4] If $x = pm^{na}$, what is the error in Δx where m, n, p are constants.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a$$

$$x = pm^{na} \quad \text{using the relation } \boxed{e^{\ln z} = z}$$

$$x = p\left(e^{\ln m}\right)^{na} = p\left(e^{n\ln m}\right)^{a} \qquad \text{Let } c = n\ln m$$

$$x = p\left(e^{c}\right)^{a} = pe^{ca} \qquad \text{where} \quad \frac{\partial x}{\partial a} = pce^{ca}$$

$$\Delta x = \left(pce^{ca}\right)\Delta a$$

Dividing though by pe^{ca}

$$\frac{\Delta x}{pe^{ca}} = \frac{\left(pce^{ca}\right)\Delta a}{pe^{ca}} \quad \text{Note } x = pe^{ca}$$

so

$$\frac{\Delta x}{x} = c\Delta a \qquad \text{Note } c = n\ln m$$

$$\boxed{\frac{\Delta x}{x} = (n\ln m)\Delta a}$$

Logarithms

[1] If $x = \ln(3a)$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a \quad \text{where} \quad \frac{\partial x}{\partial a} = \left(\frac{1}{3a}\right).(3) = \frac{1}{a}$$

$$\Delta x = \left(\frac{1}{a}\right)\Delta a$$

so

$$\boxed{\Delta x = \frac{\Delta a}{a}}$$

[2] If $x = 4\ln(6a)$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a \quad \text{where} \quad \frac{\partial x}{\partial a} = 4.\left(\frac{1}{6a}\right).(6) = \frac{4}{a}$$

$$\Delta x = 4\left(\frac{1}{a}\right)\Delta a$$

so

$$\Delta x = 4\frac{\Delta a}{a}$$

[3] If $x = m\ln(na)$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a \quad \text{where} \quad \frac{\partial x}{\partial a} = m.\left(\frac{1}{na}\right).(n) = \frac{m}{a}$$

$$\Delta x = m\left(\frac{1}{a}\right)\Delta a$$

so

$$\boxed{\Delta x = m\frac{\Delta a}{a}}$$

Trigonometric Functions

[1] If $x = 3\sin(4a)$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \quad \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a \quad \text{where} \quad \frac{\partial x}{\partial a} = 3\cos(4a).4$$

$$\Delta x = (12\cos(4a))\Delta a$$

Therefore

$$\Delta x = 12\cos(4a)\Delta a$$

[2] If $x = m\cos(na)$, what is the error in Δx.

So we can use

$$\Delta x^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 \quad \Rightarrow \quad \Delta x = \left(\frac{\partial x}{\partial a}\right)\Delta a \quad \text{where} \quad \frac{\partial x}{\partial a} = -m\sin(na).n$$

$$\Delta x = (mn\sin(na)).\Delta a$$

Therefore

$$\boxed{\Delta x = mn\sin(na)\Delta a}$$

Remember when working with angles always use radians!

Summary of general expressions

	Expression $x = f(a,b)$	Error in Δx
1	$x = a \pm b$ $x = ma \pm nb$	$\Delta x^2 = \Delta a^2 + \Delta b^2$ $\Delta x^2 = m^2 \Delta a^2 + n^2 \Delta b^2$
2	$x = m\frac{a}{b}$; $x = mab$; $x = \frac{a}{b}$; $x = ab$	$\frac{\Delta x^2}{x^2} = \frac{\Delta a^2}{a^2} + \frac{\Delta b^2}{b^2}$
3	$x = ma^n$	$\frac{\Delta x}{x} = n\frac{\Delta a}{a}$
4	$x = me^{na}$	$\frac{\Delta x}{x} = n\Delta a$
5	$x = pm^{na}$	$\frac{\Delta x}{x} = (n \ln m)\Delta a$
6	$x = m\ln(na)$	$\Delta x = m\frac{\Delta a}{a}$
7	$x = m\cos(na)$	$\Delta x = mn\sin(na)\Delta a$
8	$x = m\sin(na)$	$\Delta x = mn\cos(na)\Delta a$
9	$x = m\tan(na)$	$\Delta x = mn\frac{1}{\cos^2(na)}\Delta a$

General Examples

[1] Now coming back to our equation for centripetal acceleration,

$$a_c = \frac{mv^2}{r}$$

where the experimentally measured quantities (m, v, r) have errors ($\Delta m,\ \Delta v,\ \Delta r$). Find Δa_c .

$$a_c = \frac{mv^2}{r}$$

$$a_c = \frac{mB}{r} \qquad \text{where } B = v^2$$

We can use rule [2]

$$\left(\frac{\Delta a_c}{a_c}\right)^2 = \left(\frac{\Delta m}{m}\right)^2 + \left(\frac{\Delta r}{r}\right)^2 + \left(\frac{\Delta B}{B}\right)^2$$

Now >>

$$B = v^2$$

we can use rule [3]

$$\frac{\Delta B}{B} = 2\frac{\Delta v}{v}$$

Therefore

$$\boxed{\left(\frac{\Delta a_c}{a_c}\right)^2 = \left(\frac{\Delta m}{m}\right)^2 + \left(\frac{\Delta r}{r}\right)^2 + \left(2\frac{\Delta v}{v}\right)^2}$$

If we use the general expression for error propagation we should get the same result.

$$(\Delta x)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $a_c = \frac{mv^2}{r}$ we have

$$(\Delta a_c)^2 = \left(\frac{\partial a_c}{\partial m}\right)^2 \Delta m^2 + \left(\frac{\partial a_c}{\partial r}\right)^2 \Delta r^2 + \left(\frac{\partial a_c}{\partial v}\right)^2 \Delta v^2$$

where

$$\frac{\partial a_c}{\partial m} = \frac{v^2}{r} \quad ; \quad \frac{\partial a_c}{\partial r} = -\frac{mv^2}{r^2} \quad ; \quad \frac{\partial a_c}{\partial v} = \frac{2mv}{r}$$

Substituting

$$(\Delta a_c)^2 = \left(\frac{v^2}{r}\right)^2 \Delta m^2 + \left(-\frac{mv^2}{r^2}\right)^2 \Delta r^2 + \left(\frac{2mv}{r}\right)^2 \Delta v^2$$

To get a relative error as above, we divide through by $(a_c)^2$

$$\frac{(\Delta a_c)^2}{(a_c)^2} = \left(\frac{v^2}{r}\right)^2 \Delta m^2 \left(\frac{r}{mv^2}\right)^2 + \left(\frac{mv^2}{r^2}\right)^2 \Delta r^2 \left(\frac{r}{mv^2}\right)^2 + \left(\frac{2mv}{r}\right)^2 \Delta v^2 \left(\frac{r}{mv^2}\right)^2$$

$$\boxed{\left(\frac{\Delta a_c}{a_c}\right)^2 = \left(\frac{\Delta m}{m}\right)^2 + \left(\frac{\Delta r}{r}\right)^2 + \left(2\frac{\Delta v}{v}\right)^2}$$

[2] If $v = v_0 + at$, what is the error in Δv assuming errors in Δv_0 ; Δa ; Δt .

$$v = v_0 + at$$
$$v = v_0 + B \qquad \text{where } B = at$$

We can use rule [1]

$$(\Delta v)^2 = (\Delta v_0)^2 + (\Delta B)^2$$

Now >>

$$B = at$$

we can use rule [2]

$$\left(\frac{\Delta B}{B}\right)^2 = \left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta t}{t}\right)^2$$

where

$$(\Delta B)^2 = \left[\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta t}{t}\right)^2\right](at)^2$$

Note $B = at$

Therefore

$$\boxed{(\Delta v)^2 = (\Delta v_0)^2 + \left[\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta t}{t}\right)^2\right](at)^2}$$

Now using the general expression for error propagation instead we get.

$$(\Delta x)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $v = v_0 + at$ we have

$$(\Delta v)^2 = \left(\frac{\partial v}{\partial v_0}\right)^2 \Delta {v_0}^2 + \left(\frac{\partial v}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial v}{\partial t}\right)^2 \Delta t^2$$

where

$$\frac{\partial v}{\partial v_0} = 1 \;;\; \frac{\partial v}{\partial a} = t \;;\; \frac{\partial v}{\partial t} = a$$

Substituting

$$(\Delta v)^2 = (1)^2 \Delta {v_0}^2 + (t)^2 \Delta a^2 + (a)^2 \Delta t^2$$
$$(\Delta v)^2 = (\Delta v_0)^2 + (t)^2 \Delta a^2 + (a)^2 \Delta t^2$$

We need to rearrange the algebra if you want it in the same form as the above.

If we factorise out $(at)^2$

$$(\Delta v)^2 = (\Delta v_0)^2 + \left[\frac{(t)^2 \Delta a^2}{(at)^2} + \frac{(a)^2 \Delta t^2}{(at)^2}\right](at)^2$$

we get

$$\boxed{(\Delta v)^2 = (\Delta v_0)^2 + \left[\left(\frac{\Delta a}{a}\right)^2 + \left(\frac{\Delta t}{t}\right)^2\right](at)^2}$$

[3] if $P = VI$, what is the error in ΔP assuming errors in ΔV ; ΔI .

$$P = VI$$

We can use rule [1]

$$\boxed{\left(\frac{\Delta P}{P}\right)^2 = \left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2}$$

Now using the general expression for error propagation instead we get.

$$\left(\Delta x\right)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $P = VI$ we have

$$\left(\Delta P\right)^2 = \left(\frac{\partial P}{\partial V}\right)^2 \Delta V^2 + \left(\frac{\partial P}{\partial I}\right)^2 \Delta I^2$$

where

$$\frac{\partial P}{\partial V} = I \quad ; \quad \frac{\partial P}{\partial I} = V$$

So now substituting

$$\left(\Delta P\right)^2 = \left(I\right)^2 \left(\Delta V\right)^2 + \left(V\right)^2 \left(\Delta I\right)^2$$

$$\left(\Delta P\right)^2 = \left(VI\right)^2 \left[\left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2\right]$$

$$\left(\frac{\Delta P}{P}\right)^2 = \left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2$$

$$\boxed{\left(\frac{\Delta P}{P}\right)^2 = \left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2}$$

[4] if $P = VI + B$, what is the error in ΔP assuming errors in ΔV ; ΔI ; ΔB .

$$P = VI + B$$

Let $P = A + B$ where $A = VI$

We can use rule [1]

$$\left(\Delta P\right)^2 = \left(\Delta A\right)^2 + \left(\Delta B\right)^2$$

now the error in A can be found using rule [2]

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2$$

$$(\Delta A)^2 = (VI)^2\left[\left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2\right]$$

Substituting we obtain

$$\boxed{(\Delta P)^2 = (VI)^2\left[\left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2\right] + (\Delta B)^2}$$

Now using the general expression for error propagation instead we get.

$$(\Delta x)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $P = VI + B$ we have

$$(\Delta P)^2 = \left(\frac{\partial P}{\partial V}\right)^2 \Delta V^2 + \left(\frac{\partial P}{\partial I}\right)^2 \Delta I^2 + \left(\frac{\partial P}{\partial B}\right)^2 \Delta B^2$$

where

$$\frac{\partial P}{\partial V} = I \quad ; \quad \frac{\partial P}{\partial I} = V \quad ; \quad \frac{\partial P}{\partial B} = 1$$

So now substituting

$$(\Delta P)^2 = (I)^2 (\Delta V)^2 + (V)^2 (\Delta I)^2 + (1)^2 (\Delta B)^2$$

$$\boxed{(\Delta P)^2 = (VI)^2\left[\left(\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta I}{I}\right)^2\right] + (\Delta B)^2}$$

[5] if $z = V^2 T$, what is the error in Δz assuming errors in ΔV ; ΔT .

$$z = V^2 T$$

Let $z = AT$ where $A = V^2$

We can use rule [2]

$$\left(\frac{\Delta z}{z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta T}{T}\right)^2$$

now the error in A can be found using rule [3]

$$\Delta A = 2V.\Delta V$$

$$\frac{\Delta A}{A} = \frac{2V.\Delta V}{V^2} = \frac{2\Delta V}{V}$$

$$\left(\frac{\Delta A}{A}\right)^2 = \left(2\frac{\Delta V}{V}\right)^2$$

Substituting we obtain

$$\boxed{\left(\frac{\Delta z}{z}\right)^2 = \left(2\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta T}{T}\right)^2}$$

Now using the general expression for error propagation instead we get.

$$(\Delta x)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $z = V^2T$ we have

$$(\Delta z)^2 = \left(\frac{\partial z}{\partial V}\right)^2 \Delta V^2 + \left(\frac{\partial z}{\partial T}\right)^2 \Delta T^2$$

where

$$\frac{\partial z}{\partial V} = 2VT \quad ; \quad \frac{\partial z}{\partial T} = V^2$$

So now substituting

$$(\Delta z)^2 = (2VT)^2 (\Delta V)^2 + (V^2)^2 (\Delta T)^2$$

$$(\Delta z)^2 = (V^2T)^2 \left[\left(\frac{2\Delta V}{V}\right)^2 + \left(\frac{\Delta T}{T}\right)^2\right]$$

$$\boxed{\left(\frac{\Delta z}{z}\right)^2 = \left(2\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta T}{T}\right)^2}$$

[6] if $z = V^2T + D$, what is the error in Δz assuming errors in ΔV ; ΔT ; ΔD.

$$z = V^2T + D$$

Let $z = A + D$ where $A = V^2T$

We can use rule [1]

$$(\Delta z)^2 = (\Delta A)^2 + (\Delta D)^2$$

Now let $A = BT$ where $B = v^2$.

So the error in $A = BT$ can be found using rule [2]

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta T}{T}\right)^2$$

The error in $B = v^2$ can be found using rule [3]

$$\Delta B = 2V.\Delta V$$

$$\frac{\Delta B}{B} = \frac{2V.\Delta V}{V^2} = \frac{2\Delta V}{V}$$

$$\left(\frac{\Delta B}{B}\right)^2 = \left(2\frac{\Delta V}{V}\right)^2$$

This gives

$$\left(\frac{\Delta A}{A}\right)^2 = \left(2\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta T}{T}\right)^2$$

$$\left(\Delta A\right)^2 = \left(V^2 T\right)\left[\left(2\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta T}{T}\right)^2\right]$$

Substituting we obtain

$$\boxed{\left(\Delta z\right)^2 = \left(V^2 T\right)\left[\left(2\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta T}{T}\right)^2\right] + \left(\Delta D\right)^2}$$

Now using the general expression for error propagation instead we get.

$$\left(\Delta x\right)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $z = V^2 T + D$ we have

$$\left(\Delta z\right)^2 = \left(\frac{\partial z}{\partial V}\right)^2 \Delta V^2 + \left(\frac{\partial z}{\partial T}\right)^2 \Delta T^2 + \left(\frac{\partial z}{\partial D}\right)^2 \Delta D^2$$

where

$$\frac{\partial z}{\partial V} = 2VT \quad ; \quad \frac{\partial z}{\partial T} = V^2 \quad ; \quad \frac{\partial z}{\partial D} = 1$$

So now substituting

$$\left(\Delta z\right)^2 = \left(2VT\right)^2\left(\Delta V\right)^2 + \left(V^2\right)^2\left(\Delta T\right)^2 + \left(1\right)^2\left(\Delta D\right)^2$$

$$\left(\Delta z\right)^2 = \left(V^2 T\right)^2\left[\left(\frac{2\Delta V}{V}\right)^2 + \left(\frac{\Delta T}{T}\right)^2\right] + \left(\Delta D\right)^2$$

$$\boxed{\left(\Delta z\right)^2 = \left(V^2 T\right)^2\left[\left(2\frac{\Delta V}{V}\right)^2 + \left(\frac{\Delta T}{T}\right)^2\right] + \left(\Delta D\right)^2}$$

[7] if $z = \dfrac{I\lambda}{\mu^3} + KL^2$, what is the error in Δz assuming errors in ΔI ; $\Delta\lambda$; $\Delta\mu$; ΔK ; ΔL.

$$z = \frac{I\lambda}{\mu^3} + KL^2$$

Let $z = A + B$ where $A = \dfrac{I\lambda}{\mu^3}$; $B = KL^2$

We can use rule [1]

$$(\Delta z)^2 = (\Delta A)^2 + (\Delta B)^2$$

$$A = \frac{I\lambda}{\mu^3}$$

Let $A = \dfrac{I\lambda}{D}$ where $D = \mu^3$

We can use rule [2]

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{\Delta I}{I}\right)^2 + \left(\frac{\Delta\lambda}{\lambda}\right)^2 + \left(\frac{\Delta D}{D}\right)^2$$

Now the error in $D = \mu^3$ using [3]

$$\Delta D = 3\mu^2 \Delta\mu$$

$$\frac{\Delta D}{D} = \frac{3\mu^2.\Delta\mu}{\mu^3}$$

$$\left(\frac{\Delta D}{D}\right)^2 = \left(3\frac{\Delta\mu}{\mu}\right)^2$$

Giving

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{\Delta I}{I}\right)^2 + \left(\frac{\Delta\lambda}{\lambda}\right)^2 + \left(3\frac{\Delta\mu}{\mu}\right)^2$$

$$(\Delta A)^2 = \left(\frac{I\lambda}{\mu^3}\right)^2 \left[\left(\frac{\Delta I}{I}\right)^2 + \left(\frac{\Delta\lambda}{\lambda}\right)^2 + \left(3\frac{\Delta\mu}{\mu}\right)^2\right]$$

$$B = KL^2$$

Let $B = KC$ where $C = L^2$

We can use rule [2]

$$\left(\frac{\Delta B}{B}\right)^2 = \left(\frac{\Delta C}{C}\right)^2 + \left(\frac{\Delta K}{K}\right)^2$$

Now the error in $C = L^2$ using [3]

$$\Delta C = 2L.\Delta L$$

$$\frac{\Delta C}{C} = \frac{2L.\Delta L}{L^2}$$

$$\left(\frac{\Delta C}{C}\right)^2 = \left(2\frac{\Delta L}{L}\right)^2$$

Giving

$$\left(\frac{\Delta B}{B}\right)^2 = \left(2\frac{\Delta L}{L}\right)^2 + \left(\frac{\Delta K}{K}\right)^2$$

$$(\Delta B)^2 = \left(KL^2\right)^2 \left[\left(2\frac{\Delta L}{L}\right)^2 + \left(\frac{\Delta K}{K}\right)^2\right]$$

So now substituting

$$\boxed{(\Delta z)^2 = \left(\frac{I\lambda}{\mu^3}\right)^2 \left[\left(\frac{\Delta I}{I}\right)^2 + \left(\frac{\Delta\lambda}{\lambda}\right)^2 + \left(3\frac{\Delta\mu}{\mu}\right)^2\right] + \left(KL^2\right)^2 \left[\left(2\frac{\Delta L}{L}\right)^2 + \left(\frac{\Delta K}{K}\right)^2\right]}$$

Now using the general expression for error propagation instead we get.

$$\left(\Delta x\right)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $z = \frac{I\lambda}{\mu^3} + KL^2$ we have

$$\left(\Delta z\right)^2 = \left(\frac{\partial z}{\partial I}\right)^2 \Delta I^2 + \left(\frac{\partial z}{\partial \lambda}\right)^2 \Delta \lambda^2 + \left(\frac{\partial z}{\partial \mu}\right)^2 \Delta \mu^2 + \left(\frac{\partial z}{\partial K}\right)^2 \Delta K^2 + \left(\frac{\partial z}{\partial L}\right)^2 \Delta L^2$$

where

$$\frac{\partial z}{\partial I} = \frac{\lambda}{\mu^3} \;\; ; \;\; \frac{\partial z}{\partial \lambda} = \frac{I}{\mu^3} \;\; ; \;\; \frac{\partial z}{\partial \mu} = -3\frac{I\lambda}{\mu^4} \;\; ; \;\; \frac{\partial z}{\partial K} = L^2 \;\; ; \;\; \frac{\partial z}{\partial L} = 2LK$$

So now substituting

$$\left(\Delta z\right)^2 = \left(\frac{\lambda}{\mu^3}\right)^2 \left(\Delta I\right)^2 + \left(\frac{I}{\mu^3}\right)^2 \left(\Delta \lambda\right)^2 + \left(3\frac{I\lambda}{\mu^4}\right)^2 \left(\Delta \mu\right)^2 + \left(L^2\right)^2 \left(\Delta K\right)^2 + \left(2LK\right)^2 \left(\Delta L\right)^2$$

Note we can ignore all minus signs which will be squared

$$\boxed{\left(\Delta z\right)^2 = \left(\frac{I\lambda}{\mu^3}\right)^2 \left[\left(\frac{\Delta I}{I}\right)^2 + \left(\frac{\Delta \lambda}{\lambda}\right)^2 + \left(3\frac{\Delta \mu}{\mu}\right)^2\right] + \left(KL^2\right)^2 \left[\left(\frac{\Delta K}{K}\right)^2 + \left(2\frac{\Delta L}{L}\right)^2\right]}$$

[8] if $z = \frac{x}{L^2} + \frac{J^2}{\left(t-q\right)^2}$, what is the error in Δz assuming errors in Δx ; ΔL ; ΔJ ; Δt ; Δq.

$$z = \frac{x}{L^2} + \frac{J^2}{\left(t-q\right)^2}$$

Let $z = A + B$ where $A = \frac{x}{L^2}$; $B = \frac{J^2}{\left(t-q\right)^2}$

We can use rule [1]

$$\left(\Delta z\right)^2 = \left(\Delta A\right)^2 + \left(\Delta B\right)^2$$

$$A = \frac{x}{L^2}$$

Let $A = \dfrac{x}{C}$ where $C = L^2$

We can use rule [2]

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta C}{C}\right)^2$$

Now the error in $C = L^2$ using [3]

$$\Delta C = 2L\Delta L$$

$$\frac{\Delta C}{C} = \frac{2L.\Delta L}{L^2}$$

$$\left(\frac{\Delta C}{C}\right)^2 = \left(2\frac{\Delta L}{L}\right)^2$$

Giving

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{\Delta x}{x}\right)^2 + \left(2\frac{\Delta L}{L}\right)^2$$

$$(\Delta A)^2 = (A)^2\left[\left(\frac{\Delta x}{x}\right)^2 + \left(2\frac{\Delta L}{L}\right)^2\right]$$

$$(\Delta A)^2 = \left(\frac{x}{L^2}\right)^2\left[\left(\frac{\Delta x}{x}\right)^2 + \left(2\frac{\Delta L}{L}\right)^2\right]$$

$$B = \frac{J^2}{(t-q)^2}$$

Let $B = \dfrac{D}{E}$

where $D = J^2$; $E = (t-q)^2$

We can use rule [2]

$$\left(\frac{\Delta B}{B}\right)^2 = \left(\frac{\Delta D}{D}\right)^2 + \left(\frac{\Delta E}{E}\right)^2$$

Now the error in $D = J^2$ using [3]

$$\Delta D = 2J\Delta J$$

$$\frac{\Delta D}{D} = \frac{2J\Delta J}{J^2}$$

$$\left(\frac{\Delta D}{D}\right)^2 = \left(2\frac{\Delta J}{J}\right)^2$$

So

$$\left(\frac{\Delta B}{B}\right)^2 = \left(2\frac{\Delta J}{J}\right)^2 + \left(\frac{\Delta E}{E}\right)^2$$

Now the error in $E = (t-q)^2$

Let $E = H^2$ where $H = (t-q)$

Now the error in $E = H^2$ using [3]

$$\Delta E = 2H\Delta H$$

$$\frac{\Delta E}{E} = \frac{2H.\Delta H}{H^2}$$

$$\left(\frac{\Delta E}{E}\right)^2 = \left(\frac{2\Delta H}{H}\right)^2$$

$$\left(\frac{\Delta E}{E}\right)^2 = \left(\frac{2}{H}\right)^2 (\Delta H)^2$$

The error in $H = (t-q)$ using [1]

$$\left(\Delta H\right)^2 = \left(\Delta t\right)^2 + \left(\Delta q\right)^2$$

Therefore

$$\left(\frac{\Delta E}{E}\right)^2 = \left(\frac{2}{H}\right)^2 \left[\left(\Delta t\right)^2 + \left(\Delta q\right)^2\right]$$

$$\left(\frac{\Delta E}{E}\right)^2 = \left(2\frac{\Delta t}{H}\right)^2 + \left(2\frac{\Delta q}{H}\right)^2$$

$$\left(\frac{\Delta E}{E}\right)^2 = \left(2\frac{\Delta t}{\left(t-q\right)}\right)^2 + \left(2\frac{\Delta q}{\left(t-q\right)}\right)^2$$

So the error in B

$$\left(\frac{\Delta B}{B}\right)^2 = \left(2\frac{\Delta J}{J}\right)^2 + \left(2\frac{\Delta t}{\left(t-q\right)}\right)^2 + \left(2\frac{\Delta q}{\left(t-q\right)}\right)^2$$

$$\left(\Delta B\right)^2 = \left(\frac{J^2}{\left(t-q\right)^2}\right)^2 \left[\left(2\frac{\Delta J}{J}\right)^2 + \left(2\frac{\Delta t}{\left(t-q\right)}\right)^2 + \left(2\frac{\Delta q}{\left(t-q\right)}\right)^2\right]$$

So now substituting

$$\boxed{\left(\Delta z\right)^2 = \left(\frac{x}{L^2}\right)^2 \left[\left(\frac{\Delta x}{x}\right)^2 + \left(2\frac{\Delta L}{L}\right)^2\right] + \left(\frac{J^2}{\left(t-q\right)^2}\right)^2 \left[\left(2\frac{\Delta J}{J}\right)^2 + \left(\frac{2\Delta t}{\left(t-q\right)}\right)^2 + \left(\frac{2\Delta q}{\left(t-q\right)}\right)^2\right]}$$

Now using the general expression for error propagation instead we get.

$$\left(\Delta x\right)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $z = \frac{x}{L^2} + \frac{J^2}{\left(t-q\right)^2}$ we have

$$\left(\Delta z\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 \Delta x^2 + \left(\frac{\partial z}{\partial L}\right)^2 \Delta L^2 + \left(\frac{\partial z}{\partial J}\right)^2 \Delta J^2 + \left(\frac{\partial z}{\partial t}\right)^2 \Delta t^2 + \left(\frac{\partial z}{\partial q}\right)^2 \Delta q^2$$

where

$$\frac{\partial z}{\partial x} = \frac{1}{L^2} \;\; ; \;\; \frac{\partial z}{\partial L} = -2\frac{x}{L^3} \;\; ; \;\; \frac{\partial z}{\partial J} = \frac{2J}{(t-q)^2} \;\; ; \;\; \frac{\partial z}{\partial t} = -2\frac{J^2}{\left(t-q\right)^3} \;\; ; \;\; \frac{\partial z}{\partial q} = 2\frac{J^2}{\left(t-q\right)^3}$$

So now substituting

$$\left(\Delta z\right)^2 = \left(\frac{1}{L^2}\right)^2 \left(\Delta x\right)^2 + \left(-2\frac{x}{L^3}\right)^2 \left(\Delta L\right)^2 + \left(2\frac{J}{(t-q)^2}\right)^2 \left(\Delta J\right)^2$$

$$+\left(-2\frac{J^2}{(t-q)^3}\right)^2(\Delta t)^2+\left(2\frac{J^2}{(t-q)^3}\right)^2(\Delta q)^2$$

Note we can ignore all minus signs which will be squared and re arranging

$$\boxed{(\Delta z)^2=\left(\frac{x}{L^2}\right)^2\left[\left(\frac{\Delta x}{x}\right)^2+\left(2\frac{\Delta L}{L}\right)^2\right]+\left(\frac{J^2}{(t-q)^2}\right)^2\left[\left(\frac{2\Delta J}{J}\right)^2+\left(\frac{2\Delta t}{(t-q)}\right)^2+\left(\frac{2\Delta q}{(t-q)}\right)^2\right]}$$

This example indicates how partial differentiation is much quicker!

[9] if $z=6A\sin 4\theta$, what is the error in Δz assuming errors in ΔA; $\Delta\theta$.

$$z=6A\sin 4\theta$$

Let $z=6AB$ where $B=\sin 4\theta$

We can use rule [1]

$$\left(\frac{\Delta z}{z}\right)^2=\left(\frac{\Delta A}{A}\right)^2+\left(\frac{\Delta B}{B}\right)^2$$

Now the error in $B=\sin 4\theta$ using rule [8]

$$\Delta B=\cos 4\theta.(4).\Delta\theta$$

$$\frac{\Delta B}{B}=\frac{4\cos 4\theta}{\sin 4\theta}.\Delta\theta$$

$$\left(\frac{\Delta B}{B}\right)^2=\left(4\cos 4\theta\frac{\Delta\theta}{\sin 4\theta}\right)^2$$

So now substituting

$$\boxed{\left(\frac{\Delta z}{z}\right)^2=\left(\frac{\Delta A}{A}\right)^2+\left(4\cos 4\theta\frac{\Delta\theta}{\sin 4\theta}\right)^2}$$

Now using the general expression for error propagation instead we get.

$$(\Delta x)^2=\left(\frac{\partial x}{\partial a}\right)^2\Delta a^2+\left(\frac{\partial x}{\partial b}\right)^2\Delta b^2+\left(\frac{\partial x}{\partial c}\right)^2\Delta c^2+..$$

So for $z=6A\sin 4\theta$ we have

$$(\Delta z)^2=\left(\frac{\partial z}{\partial A}\right)^2\Delta A^2+\left(\frac{\partial z}{\partial\theta}\right)^2\Delta\theta^2$$

where

$$\frac{\partial z}{\partial A} = 6\sin 4\theta \quad ; \quad \frac{\partial z}{\partial \theta} = 6A\cos 4\theta.(4)$$

So now substituting

$$\left(\frac{\Delta z}{z}\right)^2 = \left(\frac{6\sin 4\theta}{6A\sin 4\theta}\right)^2 (\Delta A)^2 + \left(\frac{6A\cos 4\theta.(4)}{6A\sin 4\theta}\right)^2 (\Delta\theta)^2$$

$$\boxed{\left(\frac{\Delta z}{z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2 + \left(4\cos 4\theta\frac{\Delta\theta}{\sin 4\theta}\right)^2}$$

[10] if $z = \frac{q\sin^2 3\theta}{T}$, what is the error in Δz assuming errors in Δq; $\Delta\theta$; ΔT .

$$z = \frac{q\sin^2 3\theta}{T}$$

Let $z = \frac{qB}{T}$ where $B = \sin^2 3\theta$

We can use rule [1]

$$\left(\frac{\Delta z}{z}\right)^2 = \left(\frac{\Delta q}{q}\right)^2 + \left(\frac{\Delta B}{B}\right)^2 + \left(\frac{\Delta T}{T}\right)^2$$

Now the error in $B = \sin^2 3\theta$ using rule [8]

$$\Delta B = 2\sin 3\theta.\cos 3\theta.(3).\Delta\theta$$

$$\frac{\Delta B}{B} = \frac{2\sin 3\theta.\cos 3\theta.(3).\Delta\theta}{\sin^2 3\theta}$$

$$\frac{\Delta B}{B} = \frac{6\cos 3\theta.\Delta\theta}{\sin 3\theta}$$

$$\left(\frac{\Delta B}{B}\right)^2 = \left(6\cos 3\theta.\frac{\Delta\theta}{\sin 3\theta}\right)^2$$

So now substituting

$$\boxed{\left(\frac{\Delta z}{Z}\right)^2 = \left(\frac{\Delta q}{q}\right)^2 + \left(6\cos 3\theta\frac{\Delta\theta}{\sin 3\theta}\right)^2 + \left(\frac{\Delta T}{T}\right)^2}$$

Now using the general expression for error propagation instead we get.

$$(\Delta x)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $z = \dfrac{q\sin^2 3\theta}{T}$ we have

$$(\Delta z)^2 = \left(\frac{\partial z}{\partial q}\right)^2 \Delta q^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \Delta\theta^2 + \left(\frac{\partial z}{\partial T}\right)^2 \Delta T^2$$

where

$$\frac{\partial z}{\partial q} = \frac{\sin^2 3\theta}{T}$$

$$\frac{\partial z}{\partial \theta} = \frac{q.(2).\sin 3\theta.\cos 3\theta.(3)}{T} = \frac{6q\sin 3\theta.\cos 3\theta}{T}$$

$$\frac{\partial z}{\partial T} = (-1).\frac{q\sin^2 3\theta}{T^2}$$

So now substituting

$$(\Delta z)^2 = \left(\frac{\sin^2 3\theta}{T}\right)^2 \Delta q^2 + \left(\frac{6q\sin 3\theta.\cos 3\theta}{T}\right)^2 \Delta\theta^2 + \left((-1).\frac{q\sin^2 3\theta}{T^2}\right)^2 \Delta T^2$$

$$(\Delta z)^2 = \left(\frac{q\sin^2 3\theta}{T}\right)^2 \left[\left(\frac{\Delta q}{q}\right)^2 + \left(6\cos 3\theta \frac{\Delta\theta}{\sin 3\theta}\right)^2 + \left(\frac{\Delta T}{T}\right)^2\right]$$

$$\boxed{\left(\frac{\Delta z}{z}\right)^2 = \left(\frac{\Delta q}{q}\right)^2 + \left(6\cos 3\theta \frac{\Delta\theta}{\sin 3\theta}\right)^2 + \left(\frac{\Delta T}{T}\right)^2}$$

[11] if $z = \dfrac{d\cos^2 5\theta}{T} + k\sin 2x$, what is the error in Δz assuming errors in Δd ; $\Delta\theta$; ΔT ; Δk ; Δx .

$$z = \frac{d\cos^2 5\theta}{T} + k\sin 2x$$

Let $z = A + B$ where $A = \dfrac{d\cos^2 5\theta}{T}$; $B = k\sin 2x$

We can use rule [1]

$$(\Delta z)^2 = (\Delta A)^2 + (\Delta B)^2$$

Now the error in $A = \dfrac{d\cos^2 5\theta}{T}$

Let $A = \dfrac{dS}{T}$ where $S = \cos^2 5\theta$

So the error in $A = \dfrac{dS}{T}$ using [2]

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{\Delta d}{d}\right)^2 + \left(\frac{\Delta S}{S}\right)^2 + \left(\frac{\Delta T}{T}\right)^2$$

The error in $S = \cos^2 5\theta$ using [7]

$$\Delta S = -2\cos 5\theta.\sin 5\theta.(5).\Delta\theta$$
$$\frac{\Delta S}{S} = \frac{-2\cos 5\theta.\sin 5\theta.(5).\Delta\theta}{\cos^2 5\theta}$$
$$\frac{\Delta S}{S} = \frac{-10\sin 5\theta.\Delta\theta}{\cos 5\theta}$$
$$\left(\frac{\Delta S}{S}\right)^2 = \left(-10\sin 5\theta.\frac{\Delta\theta}{\cos 5\theta}\right)^2$$

This gives

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{\Delta d}{d}\right)^2 + \left(10\sin 5\theta.\frac{\Delta\theta}{\cos 5\theta}\right)^2 + \left(\frac{\Delta T}{T}\right)^2$$

Now the error in $B = k\sin 2x$

Let $B = ky$ where $y = \sin 2x$

So the error in $B = ky$ using [2]

$$\left(\frac{\Delta B}{B}\right)^2 = \left(\frac{\Delta k}{k}\right)^2 + \left(\frac{\Delta y}{y}\right)^2$$

So the error in $y = \sin 2x$ using rule [8]

$$\Delta y = \cos 2x.(2).\Delta x$$
$$\frac{\Delta y}{y} = \frac{2\cos 2x.\Delta x}{\sin 2x}$$
$$\frac{\Delta y}{y} = \frac{2\cos 2x.\Delta x}{\sin 2x}$$
$$\left(\frac{\Delta y}{y}\right)^2 = \left(2\cos 2x.\frac{\Delta x}{\sin 2x}\right)^2$$

This gives

$$\left(\frac{\Delta B}{B}\right)^2 = \left(\frac{\Delta k}{k}\right)^2 + \left(2\cos 2x.\frac{\Delta x}{\sin 2x}\right)^2$$

$$(\Delta A)^2 = \left(\frac{d\cos^2 5\theta}{T}\right)^2 \left[\left(\frac{\Delta d}{d}\right)^2 + \left(10\sin 5\theta.\frac{\Delta\theta}{\cos 5\theta}\right)^2 + \left(\frac{\Delta T}{T}\right)^2\right]$$
$$(\Delta B)^2 = (k\sin 2x)^2 \left[\left(\frac{\Delta k}{k}\right)^2 + \left(2\cos 2x.\frac{\Delta x}{\sin 2x}\right)^2\right]$$

Substituting we get

$$\boxed{\begin{aligned}(\Delta z)^2 = {}&\left(\frac{d\cos^2 5\theta}{T}\right)^2 \left[\left(\frac{\Delta d}{d}\right)^2 + \left(10\sin 5\theta.\frac{\Delta\theta}{\cos 5\theta}\right)^2 + \left(\frac{\Delta T}{T}\right)^2\right] \\ &+ (k\sin 2x)^2 \left[\left(\frac{\Delta k}{k}\right)^2 + \left(2\cos 2x.\frac{\Delta x}{\sin 2x}\right)^2\right]\end{aligned}}$$

Now using the general expression for error propagation instead we get.

$$\left(\Delta x\right)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $z = \dfrac{d\cos^2 5\theta}{T} + k\sin 2x$ we have

$$\left(\Delta z\right)^2 = \left(\frac{\partial z}{\partial d}\right)^2 \Delta d^2 + \left(\frac{\partial z}{\partial \theta}\right)^2 \Delta \theta^2 + \left(\frac{\partial z}{\partial T}\right)^2 \Delta T^2 + \left(\frac{\partial z}{\partial k}\right)^2 \Delta k^2 + \left(\frac{\partial z}{\partial x}\right)^2 \Delta x^2$$

where

$$\frac{\partial z}{\partial d} = \frac{\cos^2 5\theta}{T}$$

$$\frac{\partial z}{\partial \theta} = \frac{d(2\cos 5\theta).(-\sin 5\theta).(5)}{T} = \frac{-10d\cos 5\theta.\sin 5\theta}{T}$$

$$\frac{\partial z}{\partial T} = (-1)\frac{d\cos^2 5\theta}{T^2}$$

$$\frac{\partial z}{\partial k} = \sin 2x$$

$$\frac{\partial z}{\partial x} = k\cos 2x.(2) = 2k\cos 2x$$

Substituting and ignoring any minus signs

$$\left(\Delta z\right)^2 = \left(\frac{\cos^2 5\theta}{T}\right)^2 \Delta d^2 + \left(\frac{10d\cos 5\theta.\sin 5\theta}{T}\right)^2 \Delta\theta^2 + \left(\frac{d\cos^2 5\theta}{T^2}\right)^2 \Delta T^2 + \left(\sin 2x\right)^2 \Delta k^2 + \left(2k\cos 2x\right)^2 \Delta x^2$$

$$\boxed{\begin{aligned}\left(\Delta z\right)^2 = \left(\frac{d\cos^2 5\theta}{T}\right)^2 &\left[\left(\frac{\Delta d}{d}\right)^2 + \left(10\sin 5\theta\frac{\Delta\theta}{\cos 5\theta}\right)^2 + \left(\frac{\Delta T}{T}\right)^2\right] \\ + \left(\sin 2x\right)^2 &\left[\left(\frac{\Delta k}{k}\right)^2 + \left(2\cos 2x\frac{\Delta x}{\sin 2x}\right)^2\right]\end{aligned}}$$

[12] if $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, what is the error in Δf assuming errors in Δu ; Δv .

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$$

Let $A = B + C$ where $A = \frac{1}{f}$; $B = \frac{1}{u}$; $C = \frac{1}{v}$

We can use rule [1]

$$(\Delta A)^2 = (\Delta B)^2 + (\Delta C)^2$$

$$A = \frac{1}{f}$$

We can use rule [2] to obtain

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{0}{1}\right)^2 + \left(\frac{\Delta f}{f}\right)^2$$

$$\left(\frac{\Delta A}{A}\right)^2 = \left(\frac{\Delta f}{f}\right)^2$$

$$(\Delta A)^2 = \left(\frac{\Delta f}{f}\right)^2 A^2$$

$$(\Delta A)^2 = \left(\frac{\Delta f}{f}\right)^2 \left(\frac{1}{f}\right)^2$$

$$(\Delta A)^2 = \left(\frac{\Delta f}{f^2}\right)^2$$

$$B = \frac{1}{u}$$

We can use rule [2] to obtain

$$\left(\frac{\Delta B}{B}\right)^2 = \left(\frac{0}{1}\right)^2 + \left(\frac{\Delta u}{u}\right)^2$$

$$\left(\frac{\Delta B}{B}\right)^2 = \left(\frac{\Delta u}{u}\right)^2$$

$$(\Delta B)^2 = \left(\frac{\Delta u}{u}\right)^2 B^2$$

$$(\Delta B)^2 = \left(\frac{\Delta u}{u}\right)^2 \left(\frac{1}{u}\right)^2$$

$$(\Delta B)^2 = \left(\frac{\Delta u}{u^2}\right)^2$$

$$C = \frac{1}{v}$$

We can use rule [2] to obtain

$$\left(\frac{\Delta C}{C}\right)^2 = \left(\frac{0}{1}\right)^2 + \left(\frac{\Delta v}{v}\right)^2 = \left(\frac{\Delta v}{v}\right)^2$$

$$(\Delta C)^2 = \left(\frac{\Delta v}{v}\right)^2 C^2$$

$$(\Delta C)^2 = \left(\frac{\Delta v}{v}\right)^2 \left(\frac{1}{v}\right)^2$$

$$(\Delta C)^2 = \left(\frac{\Delta v}{v^2}\right)^2$$

Substituting for ΔA ; ΔB and ΔC into the expression

$$\left(\frac{\Delta f}{f^2}\right)^2 = \left(\frac{\Delta u}{u^2}\right)^2 + \left(\frac{\Delta v}{v^2}\right)^2$$

Re arranging we obtain

$$\boxed{\left(\frac{\Delta f}{f}\right)^2 = f^2\left[\left(\frac{\Delta u}{u^2}\right)^2 + \left(\frac{\Delta v}{v^2}\right)^2\right]}$$

You may have been tempted to first rearrange the lens formula to obtain

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} = \frac{v}{uv} + \frac{u}{uv} = \frac{v+u}{uv}$$

$$f = \frac{uv}{u+v}$$

However, you will notice that u and v are now both in the denominator and nominator. We cannot use the simple rules any more like previous since the variables will not be independent any more as before. In this form you must use the general expression for error propagation.

Now using the general expression for error propagation instead we get.

$$(\Delta x)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $f = \dfrac{uv}{u+v}$ we have

$$(\Delta f)^2 = \left(\frac{\partial f}{\partial u}\right)^2 \Delta u^2 + \left(\frac{\partial f}{\partial v}\right)^2 \Delta v^2$$

where

$$\frac{\partial f}{\partial u} = \frac{(v)(u+v)-(1)(uv)}{(u+v)^2} = \frac{uv+v^2-uv}{(u+v)^2}$$

$$\frac{\partial f}{\partial u} = \frac{v^2}{(u+v)^2}$$

$$\frac{\partial f}{\partial v} = \frac{(u)(u+v)-(1)(uv)}{(u+v)^2} = \frac{uv+u^2-uv}{(u+v)^2}$$

$$\frac{\partial f}{\partial v} = \frac{u^2}{(u+v)^2}$$

So now substituting

$$\left(\Delta f_x\right)^2 = \left(\frac{v^2}{\left(u+v\right)^2}\right)^2 \Delta u^2 + \left(\frac{u^2}{\left(u+v\right)^2}\right)^2 \Delta v^2$$

$$\left(\Delta f_x\right)^2 = \frac{1}{\left(u+v\right)^4}\left(v^4\Delta u^2 + u^4\Delta v^2\right)$$

If we multiply top and bottom by u^4 and v^4

$$\left(\Delta f_x\right)^2 = \frac{1}{\left(u+v\right)^4}\left(\frac{u^4}{u^4}v^4\Delta u^2 + \frac{v^4}{v^4}u^4\Delta v^2\right)$$

$$\left(\Delta f_x\right)^2 = \frac{\left(uv\right)^4}{\left(u+v\right)^4}\left(\frac{\Delta u^2}{u^4} + \frac{\Delta v^2}{v^4}\right)$$

where $f = \dfrac{uv}{u+v}$

$$\left(\Delta f_x\right)^2 = f^4\left(\frac{\Delta u^2}{u^4} + \frac{\Delta v^2}{v^4}\right)$$

$$\left(\frac{\Delta f_x}{f}\right)^2 = f^2\left(\frac{\Delta u^2}{u^4} + \frac{\Delta v^2}{v^4}\right)$$

Re arranging we obtain

$$\boxed{\left(\frac{\Delta f_x}{f}\right)^2 = f^2\left[\left(\frac{\Delta u}{u^2}\right)^2 + \left(\frac{\Delta v}{v^2}\right)^2\right]}$$

Ok, we shall repeat the calculation with the original form of the formula

So for $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}$ we let $y = \dfrac{1}{u} + \dfrac{1}{v}$ where $y = \dfrac{1}{f}$

So the error in y is

$$\left(\Delta y\right)^2 = \left(\frac{\partial y}{\partial u}\right)^2 \Delta u^2 + \left(\frac{\partial y}{\partial v}\right)^2 \Delta v^2$$

where

$$\frac{\partial y}{\partial u} = -\frac{1}{u^2}$$
$$\frac{\partial y}{\partial v} = -\frac{1}{v^2}$$

substituting we get

$$\left(\Delta y\right)^2 = \left(\frac{1}{u^2}\right)^2 \Delta u^2 + \left(\frac{1}{v^2}\right)^2 \Delta v^2$$

now the error in $y = \dfrac{1}{f}$ is

$$\left(\Delta y\right)^2 = \left(\frac{1}{f^2}\right)^2 \Delta f^2$$

substituting and rearranging we finally get

$$\left(\frac{1}{f^2}\right)^2 \Delta f^2 = \left(\frac{1}{u^2}\right)^2 \Delta u^2 + \left(\frac{1}{v^2}\right)^2 \Delta v^2$$

$$\frac{1}{f^4}\Delta f^2 = \left(\frac{1}{u^2}\right)^2 \Delta u^2 + \left(\frac{1}{v^2}\right)^2 \Delta v^2$$

$$\frac{1}{f^2}\left(\frac{\Delta f}{f}\right)^2 = \left(\frac{\Delta u}{u^2}\right)^2 + \left(\frac{\Delta v}{v^2}\right)^2$$

$$\boxed{\left(\frac{\Delta f}{f}\right)^2 = f^2\left[\left(\frac{\Delta u}{u^2}\right)^2 + \left(\frac{\Delta v}{v^2}\right)^2\right]}$$

[13] if $B_x = \dfrac{\mu_0 I R^2}{2\left(x^2+R^2\right)^{\frac{3}{2}}}$, what is the error in ΔB_x assuming errors in ΔI ; ΔR ; Δx .

Examining this expression it is clear that we cannot use the simple rules since R^2 is present in both denominator and nominator and unless we can re arrange the expression.
So we must use the general expression for error propagation instead we get.

$$\left(\Delta x\right)^2 = \left(\frac{\partial x}{\partial a}\right)^2 \Delta a^2 + \left(\frac{\partial x}{\partial b}\right)^2 \Delta b^2 + \left(\frac{\partial x}{\partial c}\right)^2 \Delta c^2 + ..$$

So for $B_x = \dfrac{\mu_0 I R^2}{2\left(x^2+R^2\right)^{\frac{3}{2}}}$ we have

$$\left(\Delta B_x\right)^2 = \left(\frac{\partial B_x}{\partial I}\right)^2 \Delta I^2 + \left(\frac{\partial B_x}{\partial x}\right)^2 \Delta x^2 + \left(\frac{\partial B_x}{\partial R}\right)^2 \Delta R^2$$

To simplify the differentiation we can express ΔB_x as

$$\frac{\partial B_x}{\partial I} = \frac{\mu_0 R^2}{2\left(x^2+R^2\right)^{\frac{3}{2}}}$$

$$\frac{\partial B_x}{\partial x} = \frac{(0)\left(2\left(x^2+R^2\right)^{\frac{3}{2}}\right) - \left(2\frac{3}{2}\left(x^2+R^2\right)^{\frac{1}{2}}2x\right)\left(\mu_0 I R^2\right)}{\left(2\left(x^2+R^2\right)^{\frac{3}{2}}\right)^2} = \frac{6x\mu_0 I R^2\left(x^2+R^2\right)^{\frac{1}{2}}}{4\left(x^2+R^2\right)^3}$$

$$\frac{\partial B_x}{\partial R} = \frac{(2\mu_0 IR)\left(2\left(x^2+R^2\right)^{\frac{3}{2}}\right) - \left(2\frac{3}{2}\left(x^2+R^2\right)^{\frac{1}{2}}2R\right)\left(\mu_0 IR^2\right)}{\left(2\left(x^2+R^2\right)^{\frac{3}{2}}\right)^2}$$

$$\frac{\partial B_x}{\partial R} = \frac{4\mu_0 IR\left(x^2+R^2\right)^{\frac{3}{2}} - 6\mu_0 IR^3\left(x^2+R^2\right)^{\frac{1}{2}}}{4\left(x^2+R^2\right)^3}$$

So substituting

$$\left(\Delta B_x\right)^2 = \left(\frac{\mu_0 R^2}{2\left(x^2+R^2\right)^{\frac{3}{2}}}\right)^2 \Delta I^2 + \left(\frac{6x\mu_0 IR^2\left(x^2+R^2\right)^{\frac{1}{2}}}{4\left(x^2+R^2\right)^3}\right)^2 \Delta x^2$$

$$+\left(\frac{4\mu_0 IR\left(x^2+R^2\right)^{\frac{3}{2}} - 6\mu_0 IR^3\left(x^2+R^2\right)^{\frac{1}{2}}}{4\left(x^2+R^2\right)^3}\right)^2 \Delta R^2$$

Dividing through $\Delta B_x{}^2$

$$\left(\frac{\Delta B_x}{B}\right)^2 = \left(\frac{\mu_0 R^2}{2\left(x^2+R^2\right)^{\frac{3}{2}}}\right)^2 \left(\frac{2\left(x^2+R^2\right)^{\frac{3}{2}}}{\mu_0 IR^2}\right)^2 \Delta I^2 + \left(\frac{6x\mu_0 IR^2\left(x^2+R^2\right)^{\frac{1}{2}}}{4\left(x^2+R^2\right)^3}\right)^2 \left(\frac{2\left(x^2+R^2\right)}{\mu_0 IR^2}\right.$$

$$+\left(\frac{4\mu_0 IR\left(x^2+R^2\right)^{\frac{3}{2}} - 6\mu_0 IR^3\left(x^2+R^2\right)^{\frac{1}{2}}}{4\left(x^2+R^2\right)^3}\right)^2 \left(\frac{2\left(x^2+R^2\right)^{\frac{3}{2}}}{\mu_0 IR^2}\right)^2 \Delta R^2$$

$$\left(\frac{\Delta B_x}{B}\right)^2 = \left(\frac{\Delta I}{I}\right)^2 + \left(\frac{3x.\Delta x}{\left(x^2+R^2\right)}\right)^2 + \left(\frac{4\mu_0 IR\left(x^2+R^2\right)^{\frac{3}{2}}}{4\left(x^2+R^2\right)^3} - \frac{6\mu_0 IR^3\left(x^2+R^2\right)^{\frac{1}{2}}}{4\left(x^2+R^2\right)^3}\right)^2 \left(\frac{2\left(x^2+R^2\right)^{\frac{3}{2}}}{\mu_0 IR^2}\right)^2 \Delta R$$

$$\left(\frac{\Delta B_x}{B}\right)^2 = \left(\frac{\Delta I}{I}\right)^2 + \left(\frac{3x.\Delta x}{\left(x^2+R^2\right)}\right)^2 + \left(\frac{2\left(x^2+R^2\right)^3}{R\left(x^2+R^2\right)^3} - \frac{3R\left(x^2+R^2\right)^2}{\left(x^2+R^2\right)^3}\right)^2 \Delta R^2$$

$$\left(\frac{\Delta B_x}{B}\right)^2 = \left(\frac{\Delta I}{I}\right)^2 + \left(\frac{3x.\Delta x}{\left(x^2+R^2\right)}\right)^2 + \left(\frac{2}{R} - \frac{3R}{\left(x^2+R^2\right)}\right)^2 \Delta R^2$$

Finally we obtain the following expression

$$\boxed{\left(\frac{\Delta B_x}{B}\right)^2 = \left(\frac{\Delta I}{I}\right)^2 + \left(\frac{3x.\Delta x}{\left(x^2+R^2\right)}\right)^2 + \left(\frac{2\Delta R}{R} - \frac{3R\Delta R}{\left(x^2+R^2\right)}\right)^2}$$