Exercice 1 :

0) For a given $c \in \mathbb{R}^d$, $b \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times d}$,

$$\min_x c^T x \quad \text{st} \quad \begin{array}{l} Ax = b \\ x \geq 0 \end{array} \qquad (P)$$

and

$$\max_y b^T y \quad \text{st} \quad A^T y \leq c \qquad (D)$$

1) For (P), we have :

$$g(\lambda, \nu) = \inf_{x \in D} \left(L(x, \lambda, \nu) \right)$$

st : $L(x, \lambda, \nu) = c^T x + \nu^T (Ax - b) - \lambda^T x$
$= -b^T \nu + (c + A^T \nu - \lambda)^T x$

L is linear in x, hence :

$$g(\lambda, \nu) = \begin{cases} -b^T \nu & A^T \nu - \lambda + c = 0 \\ -\infty & \text{otherwise} \end{cases}$$

g is linear on affine domain $\{(\lambda, \nu) / A^T \nu - \lambda + c = 0\}$
hence it is concave.

lower bound property : $p^* \geq -b^T \nu$ if $A^T \nu + c \geq 0$

Dual problem : maximize $-b^T \nu$
subject to $A^T \nu + c \geq 0$ $\qquad (P')$

2). Le problème (D) est équivalent à résoudre le problème suivant :

$$-\min\left(-b^T y\right)$$
$$\text{tq} \quad A^T y - c \leq 0$$

On peut définir la fonction de Lagrange associée (sans prendre en compte pour le moment le signe "-" devant le $\min(-b^T y)$).

$$\mathcal{L}(y, \lambda) = -b^T y + \lambda^T (A^T y - c)$$
$$= (A\lambda - b)^T y - \lambda^T c$$

$$g(\lambda) = \inf_y \mathcal{L}(y, \lambda) = \begin{cases} -\lambda^T c & \text{if } A\lambda - b = 0 \\ -\infty & \text{otherwise} \end{cases}$$

lower bond property : $p^* \geq -\lambda^T c$ if $A\lambda - b \geq 0$

In our problem here we need to have an upper bond property (due to the "-").

Langrage duale problem :

$$\begin{aligned} &\text{minimize} && -\lambda^T c \\ &\text{subject to} && A\lambda - b = 0 \\ &\text{and} && \lambda \geq 0 \end{aligned} \qquad (D')$$

Paul CHAUVIN

Ex1: 3) We have :

(P) : $\min_x c^T x$ st $Ax - b$, $x \geq 0$

(D) $\max(-b^T \lambda)$ st $A^T \lambda + c \geq 0$

dual of P (P') : $\max -b^T \lambda$ st $A^T \lambda + c \geq 0$

dual of D (D') : $\min -\lambda^T c$ st $A\lambda - b = 0$ and $\lambda \geq 0$

Hence, here, the problem we have is actually (P) − (D) because

$$\min_{x,y} c^T x - b^T y = \min_x c^T x - \max_y (-b^T y)$$

st $x \geq 0$

$A^T y \leq c$

Hence, the dual of our problem is actually $P' - D' = \max(-b^T \lambda) - \min(-\lambda^T c)$

$= P - D$

<u>Conclusion</u>: The problem is self-dual.

Ex 1.4: * We have $\min_{x,y} c^T x - b^T y$

$$= \min_{x} c^T x - \max_{y} (-b^T y)$$

We can minimize the 1st term with x, and maximize the 2nd terme with y,
Hence we can find x^* and y^* independantly from each other.

We obtain x^* by solving P and y^* by solving D.

* x^* : solution of P
y^* : solution of D

(P) is the dual of D, and we have strong duality constraint, hence $x^* = y^*$ and $c^T x^* = b^T y^*$

because $c^T \neq b^T \Rightarrow x^* = y^* = 0$

<u>conclusion</u>: The optimal value for the self-dual problem is exactly 0.

Paul CHAUVIN

Exercice 2

1). Let's compute the conjugate of $\|x\|_1$

If $\|y\|_* > 1$, by definition of the dual norm, there is a $z \in \mathbb{R}^n$ with $\|z\| \leq 1$ and $y^T z > 1$. Taking $x = tz$ and letting $t \to +\infty$, we have

$$y^T x - \|x\| = t(y^T z - \|z\|) \to +\infty$$

Hence, $f^*(y) = +\infty$.

Conversely, if $\|y\|_* \leq 1$, we have $y^T x \leq \|x\|$ for all x. Then, for all x, $y^T x - \|x\| \leq 0$. Therefore, $x = 0$ is the value that maximizes $y^T x - \|x\|$, with maximum value 0.

Conclusion: $f^*(y) = \begin{cases} 0 & \|y\|_* \leq 1 \\ \infty & \text{otherwise} \end{cases}$

Ex 2.2: We have

$$\nabla \left(\|Ax - b\|_2^2 + \|x\|_1 \right)$$

$$= \nabla \|Ax - b\|_2^2 + \nabla \|x\|_1$$

$$= 2\left(A^T A x - A^T b\right) + 2z$$

Hence, we need to have $A^T A x = A^T b - z$

$$x = (A^T A)^{-1} \left(A^T b - z\right)$$

Our dual problem hence become:

$$\max_{\text{st } \lambda \geq 0} g(\lambda) = \max \Big(\| A(A^T A)^{-1}(A^T b - z) - b\|_2^2 + z^T (A^T A)^{-1} (A^T b - z) \Big)$$

$$= \max_z z^T B z - z^T x_{LS}$$

$$\text{st } -\lambda \leq z \leq \lambda$$

with $BA^T b = x_{LS}$

Ex 3.1:
$$\min_{w} \frac{1}{n} \sum_{i=1}^{n} \mathcal{L}(w, x_i, y_i) + \frac{\tau}{2} \|w\|_2^2$$
is equivalent to
$$\min_{w} \frac{1}{n\tau} \sum_{i=1}^{n} \max(0; 1-y_i(w^T x_i)) + \frac{1}{2} \|w\|_2^2$$
is equivalent to
$$\min_{w,z} \frac{1}{n\tau} 1^T z + \frac{1}{2} \|w\|_2^2$$
$$\text{st} \quad z_i \geqslant 1 - y_i(w^T x_i) \quad \forall i \in 1, \dots n$$
$$z \geqslant 0.$$
Conclusion: (Sep1) $\Leftrightarrow$ (Sep2)

Ex 3.2: The dual of Sep (2) is

$$\max \ \mathbb{1}^T \lambda + \frac{1}{2} \| \lambda^T \mathrm{diag}(y)\, x \|_2^2 - (\lambda^T \mathrm{diag}(y)\, x)^2$$

$$\text{st} \ \frac{1}{m\tau} \mathbb{1}^T - \lambda \geqslant 0$$

$$\lambda \geqslant 0$$