CONVEX Optimisation: HW3_LASS0

CHAUVIN Paul

paulchauvin97@gmail.com

PART 1: Derive the dual problem of LASSO and format it as a general Quadratic Problem

CHAUVIN Paul

1) We have $\min_{w} \frac{1}{2}\|Xw - y\|_2^2 + \lambda\|w\|_1$

On introduit $r = Xw - y$:

$$\min_{r,w} \frac{1}{2}\|r\|_2^2 + \lambda\|w\|_1 .$$

Nous avons alors un problème d'optimisation convexe et nous pouvons écrire son Lagrangien

$$\mathcal{L}(w, r, v) = \left(\lambda\|w\|_1 + v^T X w\right) + \left(\frac{1}{2}\|r\|_2^2 - v^T r\right) - \left(v^T y\right)$$

$$g(v) = \inf_{w,r} \mathcal{L}(w,r,v) \qquad \text{(Derive the dual)}$$

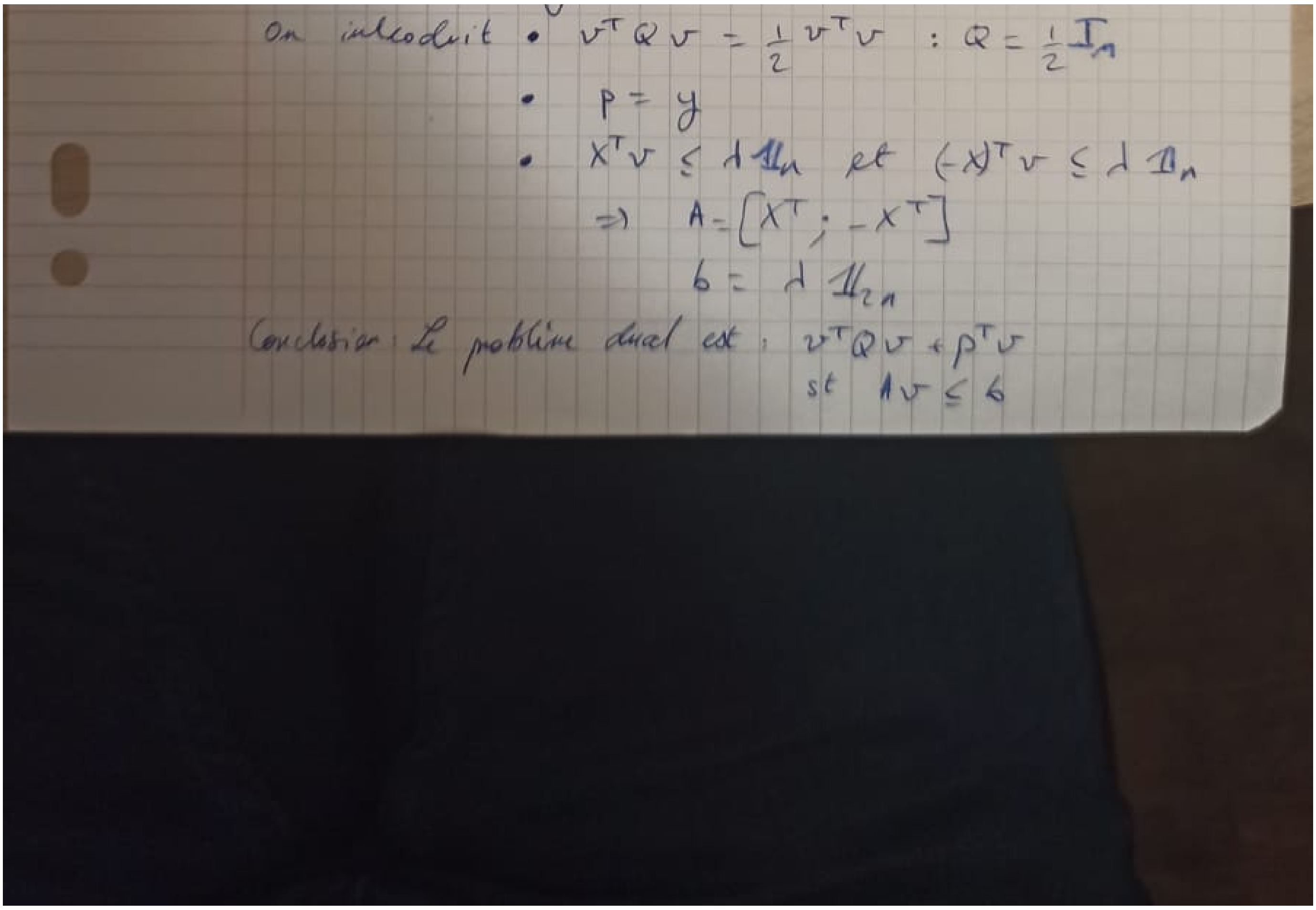
$$\Rightarrow g(v) = \inf_{w}\left(\lambda\|w\|_1 + v^T X w\right) + \inf_{r}\left(\frac{1}{2}\|r\|_2^2 - v^T r\right) - y^T v$$

* $\inf_{w}\left(\lambda\|w\|_1 + v^T X w\right) = -\sup_{w}\left(-\lambda\|w\|_1 - v^T X w\right)$
$$= \begin{cases} 0 & \text{si } 0 \leq \|X^T v\|_\infty \leq \lambda \\ -\infty & \text{sinon} \end{cases}$$

* $\inf_{r}\left(\frac{1}{2}\|r\|_2^2 - v^T r\right) = -\sup_{r}\left(v^T r - \frac{1}{2}\|v\|_2^2\right)$
$$= -\frac{1}{2} v^T v$$

$$\Rightarrow g(v) = -\frac{1}{2} v^T v - y^T v \quad \text{telque } \|X^T v\|_\infty \leq \lambda$$

Or maximiser g revient à minimiser (QP)



On introduit • $v^T Q v = \frac{1}{2} v^T v$: $Q = \frac{1}{2} I_n$

• $p = y$

• $X^T v \leq \lambda \mathbb{1}_n$ et $(-X)^T v \leq \lambda \mathbb{1}_n$

$\Rightarrow A = [X^T ; -X^T]$

$b = \lambda \mathbb{1}_{2n}$

Conclusion : Le problème dual est : $v^T Q v + p^T v$

st $Av \leq b$

Part 2: Implement the barrier method to solve QP

Environment

```
import numpy as np
import matplotlib.pyplot as plt
```

Parameters

```
alpha = 0.1
beta = 0.7
mu = 5
n = 20
d = 15
eps = 10e-6
lambd = 10
```

Useful functions

```
def function(v, Q, p, t0):
    return t0*(np.dot(v.T, np.dot(Q, v)) + np.dot(p.T, v)) - sum([np.log(b[i]-np.do


def function_true(v, Q, p):
    return np.dot(v.T, np.dot(Q, v)) + np.dot(p.T, v)


def line_search(Q, p, v, t, df, dx, t0):
    if function(v + t*dx, Q, p, t0) <= (function(v, Q, p, t0) + alpha*t*np.dot(df.T
        return v + t*dx
    else:
        return line_search(Q, p, v, beta*t, df, dx, t0)
```

Barrier method

```
def centering_step(Q, p, A, b, t, v0, eps, numb_iter=0):
    df = t*(2*np.dot(Q, v0) + p) + \
        sum([A[i, np.newaxis].T/(b[i]-np.dot(A[i], v0)) for i in range(b.shape[0])]
    d2f = 2*t*Q + \
        sum([(np.outer(A[i, np.newaxis].T, A[i, np.newaxis].T)) / ((b[i] - np.dot(A
    dx = -1*np.dot(np.linalg.inv(d2f), df)
    l2 = np.dot(df.T, np.dot(np.linalg.inv(d2f), df))
    if l2/2 <= eps:
        return v0, numb_iter
    v1 = line_search(Q, p, v0, t=1, df=df, dx=dx, t0=t)
    return centering_step(Q, p, A, b, t, v1, eps, numb_iter+1)



def barr_method_inter(Q, p, A, b, v0, eps, t, mu, numb_iter=0, numb_newton = [], v_
    v_center, numb_iter_inter = centering_step(Q, p, A, b, t, v0, eps)
    numb_newton.append(numb_iter)
    v_seq.append(v_center)
    f_seq_true.append(function_true(v_center, Q, p)[0][0])
    if b.shape[0]/t < eps:
        return v_center, numb_newton, v_seq, f_seq_true
    else:
        t = mu*t
        return barr_method_inter(Q, p, A, b, v0, eps, t, mu, numb_iter+numb_iter_in


def barr_method(Q, p, A, b, v0, eps, mu):
    numb_newton = [0]
    v_seq = [v0]
    f_seq_true = [function_true(v0, Q, p)[0][0]]
    return barr_method_inter(Q, p, A, b, v0, eps, 1, mu, 0, numb_newton, v_seq, f_s
```

Part 3: Test our functions

Initialization

```
X = np.random.rand(n,d)
y = np.random.rand(n,1)
Q = 0.5*np.eye(n)
p = -y
A = np.vstack((X.T,-X.T))
b = lambd*np.ones((2*d,1))

v0 = np.zeros((n,1))

mu_list = [2, 5, 10, 15, 20, 25, 30, 40, 50, 70, 85, 100, 130, 170, 200, 250]
mu_list_restricted = [2, 5, 15, 50, 100, 200]
w_center_list = []
f_true_list = []
```

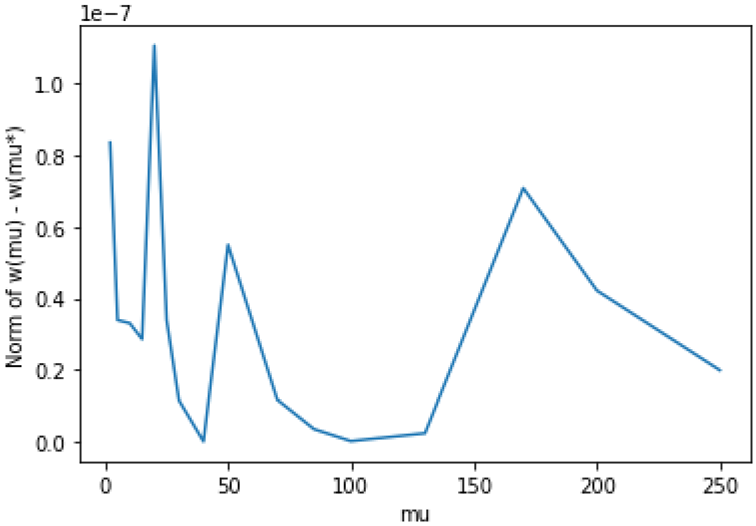
Plot results

```
plt.figure()
for mu in mu_list:
    v_center, numb_newton, v_seq, f_seq_true = barr_method(Q, p, A, b, v0, eps, mu)
    w_center = np.linalg.lstsq(X,-v_center-y)[0]
    w_center_list.append(w_center)
    f_true_list.append(f_seq_true[-1])
    if mu in mu_list_restricted:
        plt.step(numb_newton, f_seq_true - f_seq_true[-1], label='mu = '+str(mu))
plt.legend()
plt.ylabel("Norm of f(mu) - f(mu)*")
plt.xlabel("Centering step number")
plt.semilogy()
plt.show()
```

```
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:4: FutureWarning:
To use the future default and silence this warning we advise to pass `rcond=Nc
  after removing the cwd from sys.path.
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:4: FutureWarning:
To use the future default and silence this warning we advise to pass `rcond=Nc
  after removing the cwd from sys.path.
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:4: FutureWarning:
To use the future default and silence this warning we advise to pass `rcond=Nc
  after removing the cwd from sys.path.
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:4: FutureWarning:
To use the future default and silence this warning we advise to pass `rcond=Nc
  after removing the cwd from sys.path.
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:4: FutureWarning:
To use the future default and silence this warning we advise to pass `rcond=Nc
  after removing the cwd from sys.path.
/usr/local/lib/python3.7/dist-packages/ipykernel_launcher.py:4: FutureWarning:
To use the future default and silence this warning we advise to pass `rcond=Nc
  after removing the cwd from sys.path.
```

Plot W in function of mu


```
mu_min = np.argmin(f_true_list)
plt.figure()
w_diff_norm = [np.linalg.norm(w-w_center_list[mu_min]) for w in w_center_list]
plt.plot(mu_list, w_diff_norm)
plt.ylabel("Norm of w(mu) - w(mu*)")
plt.xlabel("mu")
plt.show()
```




```
mu_min = np.argmin(f_true_list)
plt.figure()
w_diff_norm = [np.linalg.norm(w-w_center_list[mu_min]) for w in w_center_list]
plt.plot(mu_list, w_diff_norm)
plt.ylabel("Norm of w(mu) - w(mu*)")
plt.xlabel("mu")
plt.show()
```

When mu is too small, we need to many iterations to converge. Whereas with smaller mu, it's faster, we need less iteerations, however, each step is going to be more costly. An apporpriate value of mu would be around 40.