MAP551 - PC 3

Paul Calot

October 17, 2020

0.1 Introduction

0.2 Numerical integration in the framework of application of Peano's theorem.

0.2.1 Study of the ODE

0.2.1.1

The function f has the following properties :

- It is continuous on $\mathbb{R}^+ \times \mathbb{R}$;
- It has a derivative on $\mathbb{R}^+ \times \mathbb{R} \setminus \{(0,0)\}$. Indeed $u \rightarrow \sqrt{|u|}$ is not differentiable at $u = 0$.

From 1), we deduce that Peano's theorem applies and thus there is a solution to (1). From 2), we can say that Cauchy-Lipschitz theorem does not apply.

0.2.1.2

We have first:

$$f(\Delta t, 0) = 4\Delta t cos(\frac{\pi log(\Delta t)}{log 2}) = 2^{-i+2}(-1)^i \tag{1}$$

We also have :

$$\forall |y| > t^2, f(t,y) = 4sign(y)\sqrt{|y|} \tag{2}$$

0.2.1.3

Let us suppose that $u(0) \neq 0$.
First case : $u(0) > 0$.
In this case, $f(t, u(0)) = 4\sqrt{u(0)} > 0$ because $t^2 \leq u(0)$.
Consequently, we have :

$$\forall t \in]0, u(\tilde{t})[, f(t,u) > 0 \tag{3}$$

With $\tilde{t}$, the time such that $\tilde{t}^2 = u$.

The function u is consequently increasing on this interval and remains thus stictly positive. We can write for $t \neq \tilde{t}$:

$$\begin{aligned} d_t u = 4\sqrt{u} &\Leftrightarrow \frac{dt_u}{\sqrt{u}} = 4 \\ &\Leftrightarrow \sqrt{u} = 2t + \sqrt{u(0)} \end{aligned}$$

Then :

$$\begin{aligned} \tilde{t}^2 = u(\tilde{t}) &\Leftrightarrow \tilde{t}^2 = (2\tilde{t} + \sqrt{u(0)})^2 \\ &\Leftrightarrow -\tilde{t} = \sqrt{u(0)} \end{aligned}$$

since everything is positive.

This condition is never met. Consequently, we proved that for an initial condition $u(0) > 0$, $f(\mathbb{R}^+ \times \mathbb{R}^{\times +}) \subset \mathbb{R}^+ \times \mathbb{R}^{\times +}$. Consequently, the Cauchy-Lipchitz theorem applies and there is a unique solution to Cauchy's problem on $\mathbb{R}^{\times +}$.

The same applies for $u(0) < 0$: there is a unique solution to Cauchy's problem on $\mathbb{R}^{\times -}$.

Two examples of such solutions are given in figure 1.

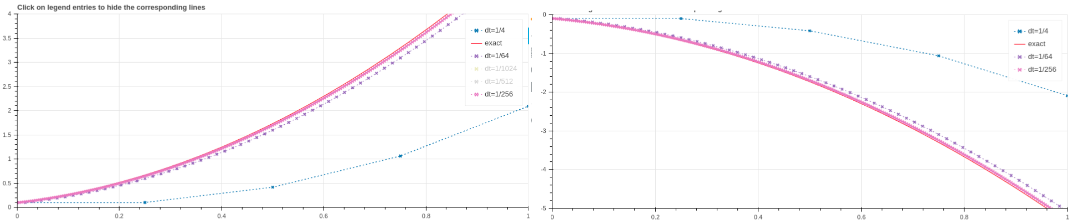

Figure 1: Analytic solutions (and some numerical ones) for $u_0 = 0.1$ (right) and $u_0 = -0.1$ (left).

0.2.2 Numerical integration

0.2.2.1

Recall for forward Euler that $y_{n+1} = y_n + \Delta t f(t, y_n)$ which yields with $\Delta t = 2^i$:

$$y_{n+1} = y_n + 2^i t f(n2^i, y_n)$$

We then have :

- Initial condition : $y_0 = y(0) = 0$
- First step : $y_1 = 0 + 2^i t f(0, 0) = 0$
- Second step : $y_2 = y_1 + \Delta t f(\Delta t, 0) = (-1)^i 4^{-i+1}$ (according to 0.2.1.2)
- Third step : $y_3 = y_2 + \Delta t f(2\Delta t, y_2)$ which yields : $y_3 = 3 \times (-1)^i 4^{-i+1}$.
- ...

By induction, we can prove that $\forall n \in \mathbb{N}, y_n = \alpha_n (-1)^i 4^{-i+1}$ with $\alpha_{n+1} = \alpha_n + 2\sqrt{\alpha_n}$. It's relatively straight forward under the hypothesis that $\forall n \in \mathbb{N}, n > 1, \alpha_n \geq \frac{n^2}{4}$ (which allows to remove the second term in the right hand side of the definition of $f, t > 0$).

Let's now make sure that : $\forall n \in \mathbb{N}, n > 1, \alpha_n \geq \frac{n^2}{4}$.

Once again by induction :

1. For $n = 2$, $\alpha_2 = 1 \geq \frac{1}{2}$

2. Let us suppose that for $k \in \mathbb{N}, k > 1$, $\alpha_k \geq \frac{k^2}{4}$ (inductive hypothesis). Then :

$$\begin{aligned}
\alpha_{k+1} &= \alpha_k + 2\sqrt{\alpha_k} \quad \text{(by recurrence relation)} \\
&\geq \frac{k^2}{4} + 2\sqrt{\frac{k^2}{4}} \quad \text{(by inductive hypothesis)} \\
&= \frac{k^2}{4} + k \\
&= \frac{k^2 + 4k}{4} \\
&\geq \frac{k^2 + 2k + 1}{4} \quad \text{(k>1)} \\
&= \frac{(k+1)^2}{4}
\end{aligned}$$

which proves the case for $k + 1$.

The associated numerical solutions are given in figure 2.

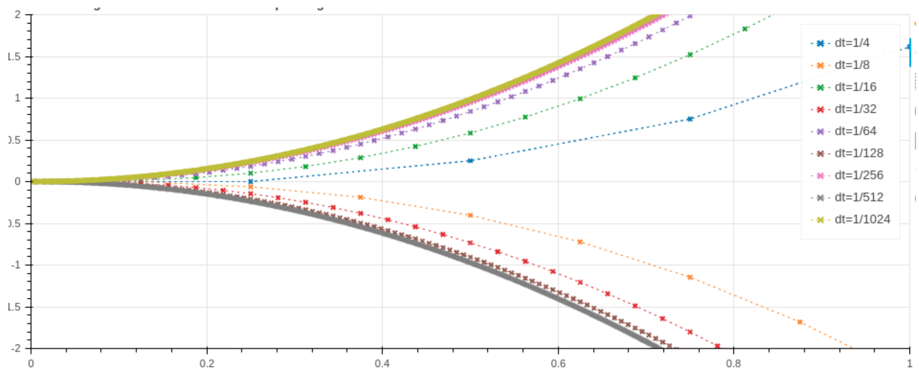

Figure 2: Numerical solutions for various i parities.

0.3 Conservative System and Euler Integration methods

0.3.1

We use system (6) :

$$\begin{aligned}
\frac{\partial E}{\partial t} &= (y_3\frac{\partial y_3}{\partial t} + y_4\frac{\partial y_4}{\partial t}) + (\frac{1-\mu}{{r_1}^2}\frac{\partial r_1}{\partial t} + \frac{\mu}{{r_2}^2}\frac{\partial r_2}{\partial t}) - (y_1\frac{\partial y_1}{\partial t} + y_2\frac{\partial y_2}{\partial t}) \\
&= y_3\Big(y_1 + 2y_4 - \frac{(1-\mu)(y_1+\mu)}{{r_1}^3} - \mu\frac{y_1-1+\mu}{{r_2}^3}\Big) + y_4\Big(y_2 - 2y_3 - \frac{(1-\mu)y_2}{{r_1}^3} - \mu\frac{y_2}{{r_2}^3}\Big) \\
&+ \frac{1-\mu}{{r_1}^2}\frac{y_3(\mu+y_1)+y_2y_4}{\sqrt{(y_1+\mu)^2+{y_2}^2}} + \frac{\mu}{{r_2}^2}\frac{y_3(\mu-1+y_1)+y_2y_4}{\sqrt{(y_1-1+\mu)^2+{y_2}^2}} - y_1y_3 - y_2y_4 \\
&= -\frac{y_3}{r_3}(1-\mu)(y_1+\mu) - \frac{y_3}{{r_2}^3}\mu(y_1-1+\mu) - \frac{y_4y_2}{{r_1}^3}(1-\mu) \\
&- \mu\frac{y_2y_4}{{r_2}^3} + \frac{(1-\mu)}{{r_1}^3}(y_3(y_1+\mu)+y_2y_4) + \frac{\mu}{{r_2}^3}(y_3(y_1-1+\mu)+y_2y_4) \\
&= 0
\end{aligned}$$

This proves that $E(y)$ is an invariant of (6).

In addition : $E(y) = \frac{1}{2}({y_3}^2+{y_4}^2) - (\frac{1-\mu}{r_1} + \frac{\mu}{r_2}) - \frac{1}{2}({y_1}^2+{y_2}^2) = T + V + C$ with :

1. T : kinetic energy ;
2. V : potential energy ;
3. C : This one is harder to interpret. It diminishes as ${y_1}^2+{y_2}^2 = |Y|^2$, where Y is the position in the complex plane of the satellite, increases. It's not a kinetic term (no speed is involved). So the further away from the center of mass of the system Earth + Moon the satellite is, the less energy it has. Consequently, it's a coupling term which acts proportionnaly to the squared distance between the satellite and the origin.

The figure 3 gives the RK45 Scipy solution to this reduced three body problem.
TODO : analysis of the solution ...

0.3.2

0.3.3

0.4 Stability, order and accuracy for non-stiff and stiff equations

0.4.1 Stiffness ?

0.4.1.1

The given Cauchy problem verifies the hypotheses of Cauchy-Lipschitz theorem ($t, u \to k(cos(t) - u(t))$ is of class C^1 on $\mathbb{R}^+ \times \mathbb{R}$ and for all $k > 1$). Consequently, there exists an unique solution and we simply have to verify that this is the one given.

Note that u is indeed of class C^1. We have :

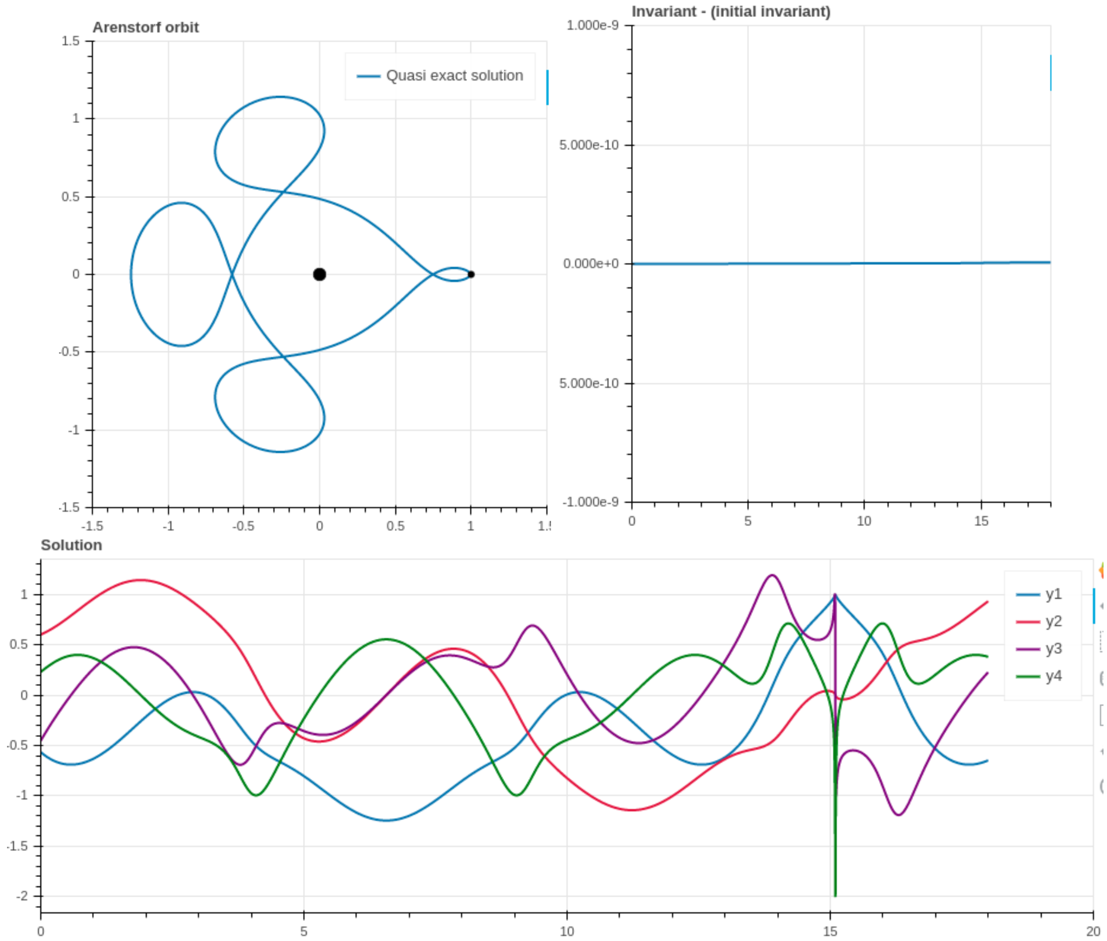

Figure 3: Quasi-exact solution to the reduced three body problem using Scipy solver and for one period only.

1. $d_t u(t) = \frac{k}{k^2+1}(-ksin(t) + cos(t)) - kc_0 e^{-kt}$

2. $k(cos(t) - u(t)) = \left[\frac{k}{k^2+1}(cos(t) - ksin(t))\right] - kc_0 e^{-kt}$

They are equals quantities. Consequently, we found the solution of Cauchy's problem.

0.4.1.2 $k >> 1$

In this case, we have roughly : $u(t) \approx cos(t) + c_0 e^{-kt}$ with $c_0 \approx (u_0 - 1)e^{kt_0}$. That yields to the identification of two regions :

- $\forall kt << 1, u(t) \approx c_0 e^{-kt} + cos(t)$ which decreases very abruptly with t increasing, hence the stiffness.
- $\forall kt >> 1, u(t) \approx cos(t)$.

These two regions are plotted on figure 7.

0.4.1.3 $u_0 = \frac{k^2}{k^2+1}$

This yields to $c_0 \approx 0$ which yields to only one region in this case : $u(t) \approx cos(t)$ (cf. figure 8), hence the lack of stiffness, even for large k (the solution depending only slightly of k).
The stiffness can then come from various origins :

1. Initial condition ;
2. Intrinsically from the system.

In our case, the system seems intrinsically stiff but the specific initial condition $u_0 = \frac{k^2}{k^2+1}$ prevents stiffness from appearing.

0.4.2 Explicit Euler

0.4.2.1

We choose $k = 50$ and $u_0 = 2$.
We have the following regimes, with $T = 2$ sec (cf. 9) :

- $n_t < 50$: divergent oscillations ;
- $n_t \approx 51$: limits of convergence (oscillations still occur) ;
- $n_t > 52$: convergence without (much) oscillations. The more n_t is increased, the less oscillations.

0.4.2.2

For a simpler system defined by

$$\begin{cases} Y' = -kY \\ Y_0 > 0 \end{cases} \tag{4}$$

the forward euler relation yields : $y_{n+1} = (1 + k\Delta t)y_n = z(\Delta t)y_n$.
By studying z, we find that the stability of the scheme requires : $|z(\Delta t)| < 1 \Leftrightarrow dt < \frac{2}{k} \Leftrightarrow n_t > \frac{kT}{2}$.
In our case, we have $T = 2$ sec, $k = 50$, which yields $n_t = 50$. So the scheme is A-stable for $n_t > 50$.

0.4.2.3

From the lesson, we know that the Explicit Euler scheme is consistent of order one. It consequently matches the expected order (cf. 10).

0.4.2.4

In figure 12, we can see that doubling the stiffness requires doubling the discretization accuracy for roughly the same error.

0.4.2.5 $u_0 = \frac{k^2}{k^2+1}$

In this case, it is harder to really observe the same three regimes as previously because the initial error, previously caused by the stiffness which is non-existant here, is much lower. However we roughly have the same behavior as we can see in figure 13.

As for the log-log erorr, we have now (figure 14): As for the global error, we have now (figure 15):

0.4.2.6

Forward euler scheme does not allow for a great stability, however it behaves well even in the presence of stiffness in the system.

0.4.3 Implicit Euler

0.4.3.1

0.4.3.2

0.4.3.3

0.4.3.4

0.4.3.5

0.4.3.6

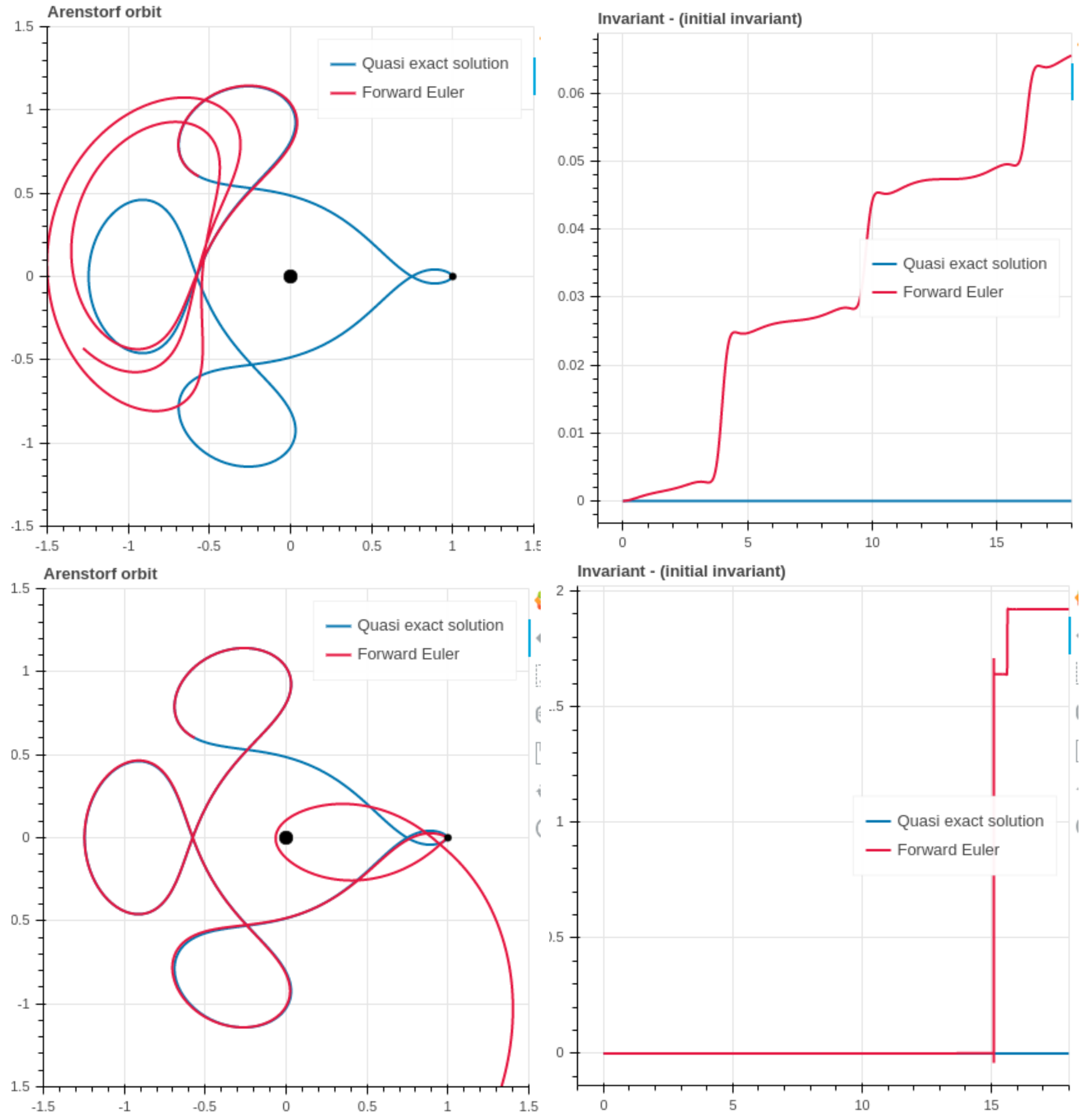

Figure 4: Solution to the reduced three body problem using forward euler scheme. Top is for $n_t = 1800$ points, bottom is for $n_t = 180000$. We see [TODO]

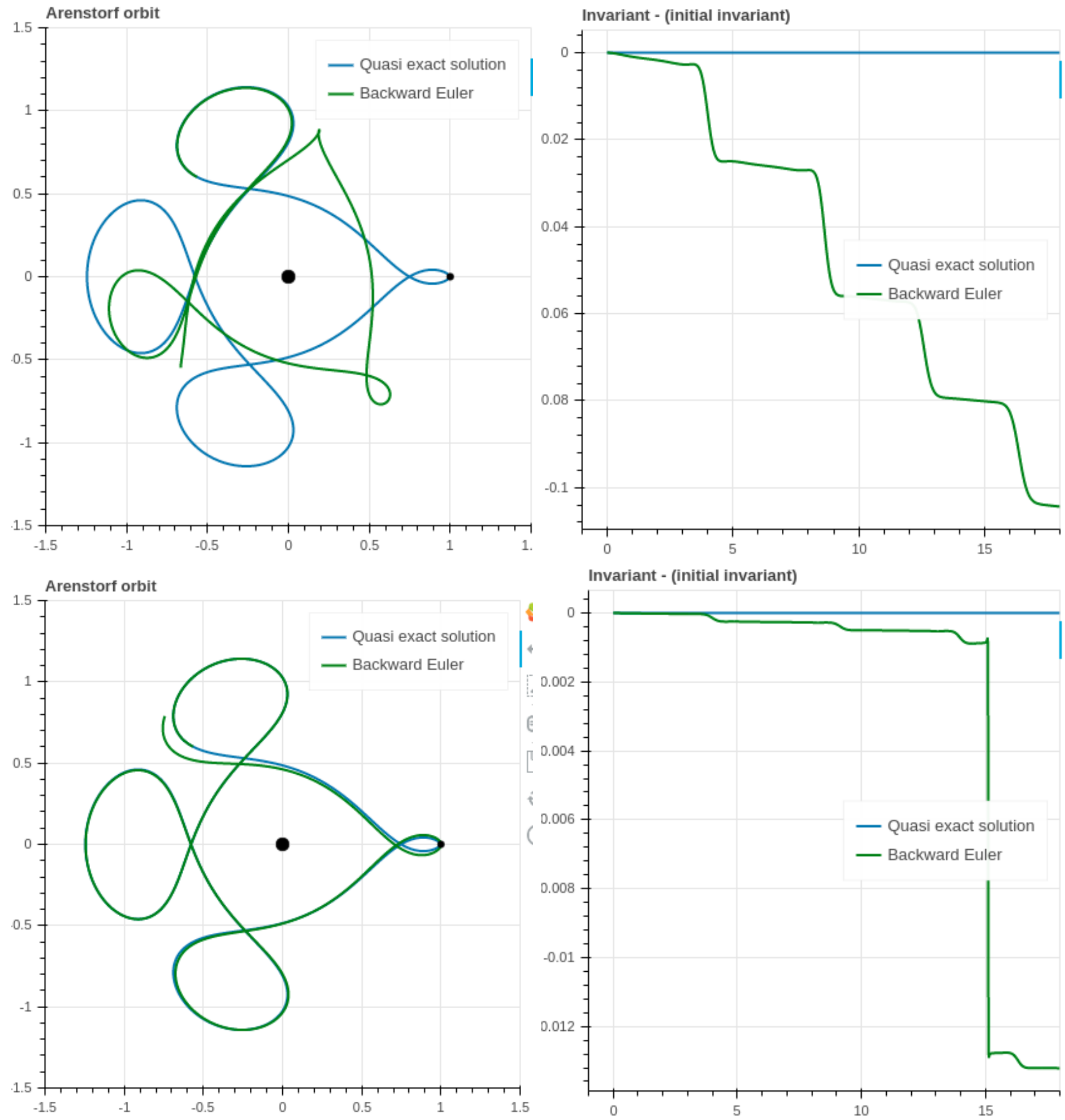

Figure 5: Solution to the reduced three body problem using backward euler scheme. Top is for $n_t = 1800$ points, bottom is for $n_t = 180000$. We see [TODO]

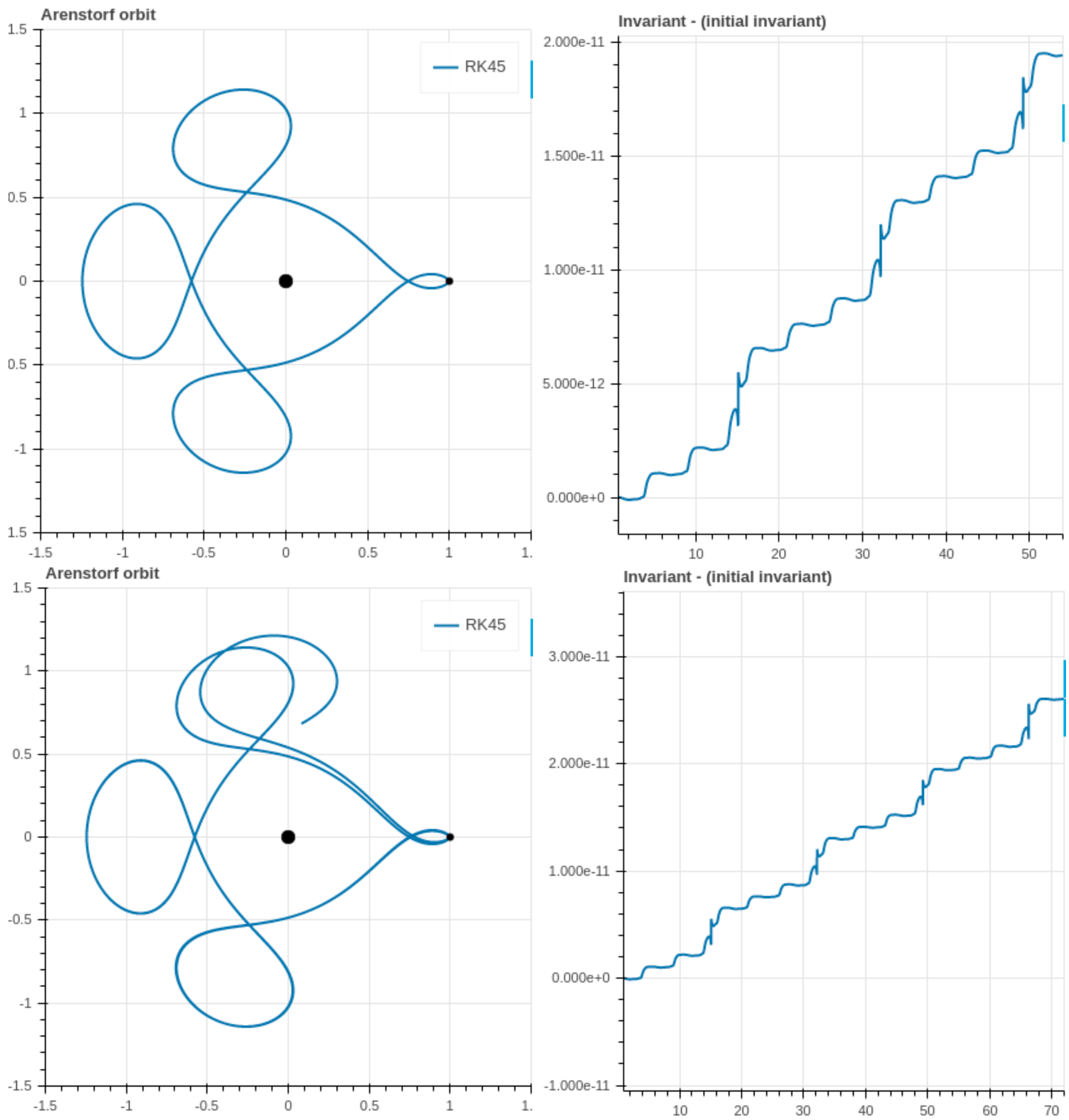

Figure 6: Quasi exact solution using RK45 solver for 3 periods (top images) and 4 periods (bottom images). We see [TODO]

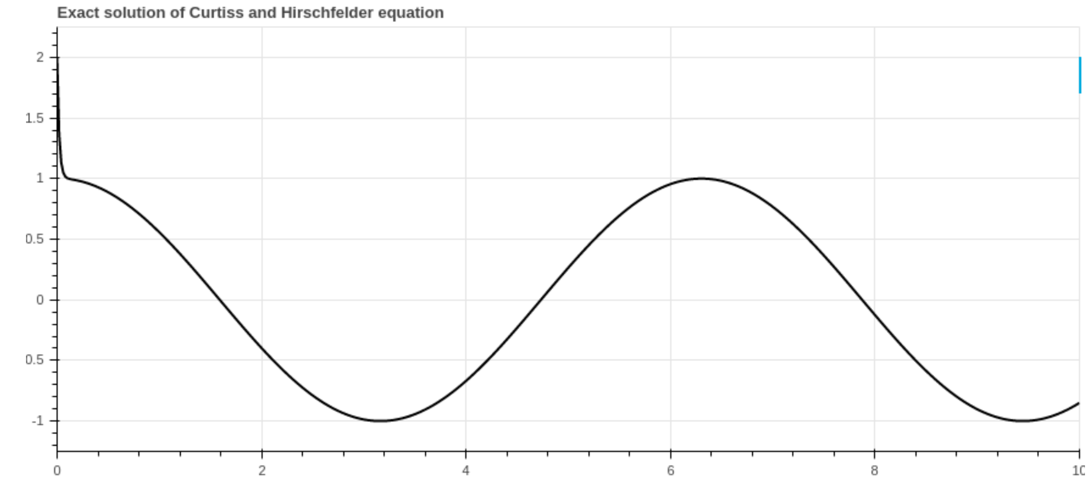

Figure 7: Plotting of the two visible regions. First one is for $t < TODO$, and second therafter. We see [TODO]

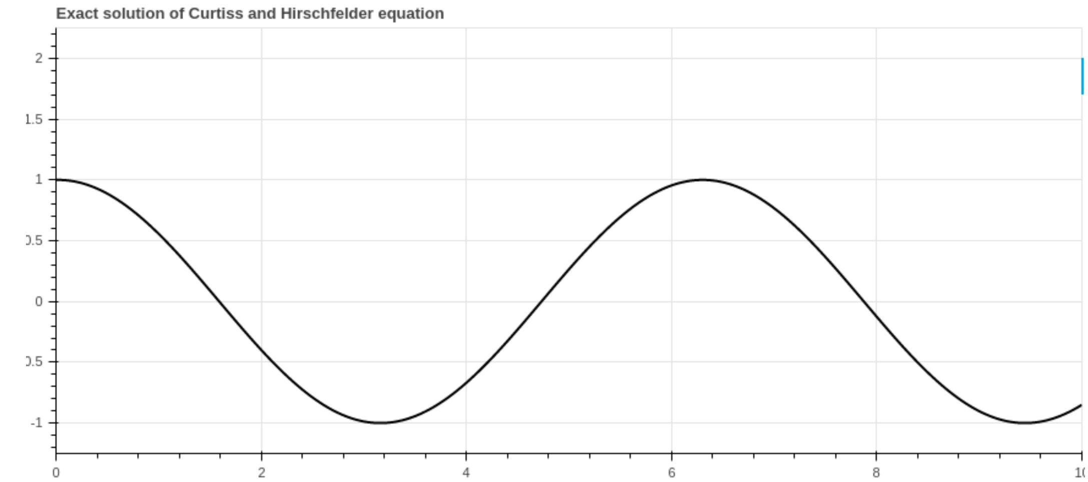

Figure 8: Plotting the numerical solution for $u_0 = \frac{k^2}{k^2+1}$

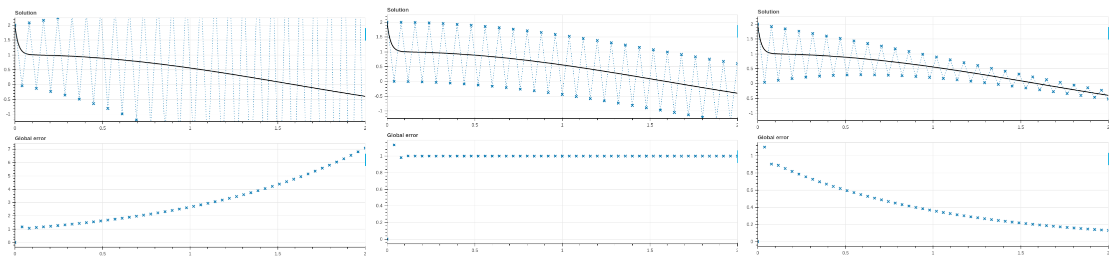

Figure 9: Plotting the three identified regimes for forward euler scheme, $n_t = 50, 51, 52$ (left to right),$k = 50$, $u_0 = 2$ and $T = 2$.

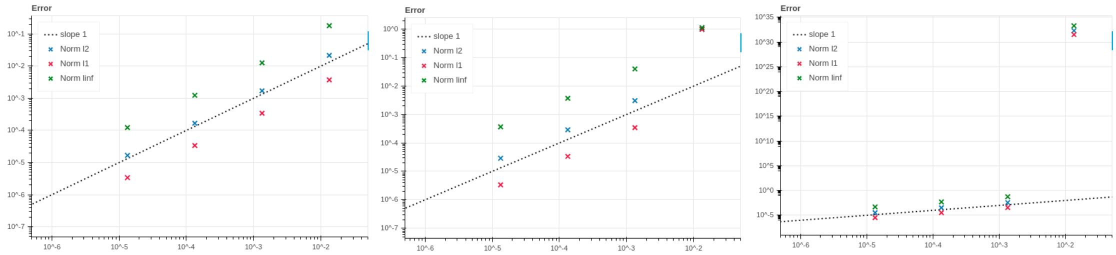

Figure 10: Log-log Plot of the error $\epsilon(\Delta t)$ for $k = 50, 150, 200$, $u_0 = 2$ and $T = 2$.

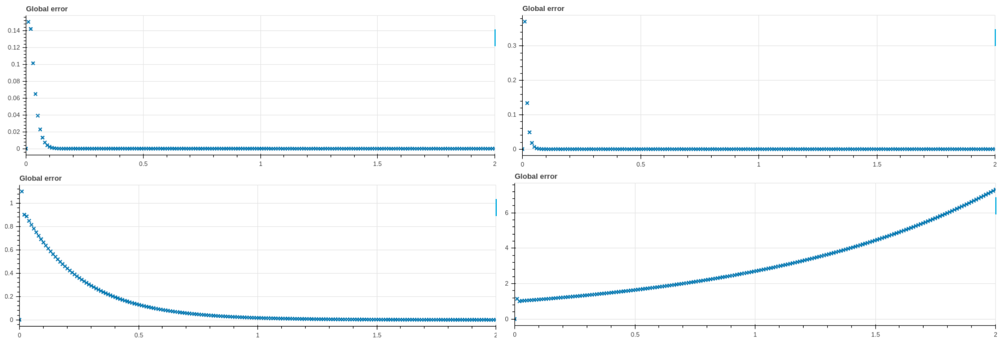

Figure 11: Plot of the global error for various stiffness : $k = 60, 100, 195, 200$ (from left to right, top to bottom), $n_t = 200$, $u_0 = 2$.

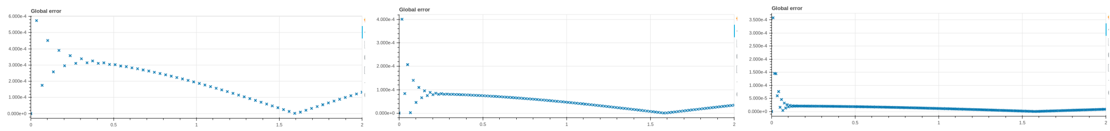

Figure 12: Plot of the global error for (stiffness, discretization step) couples : $(50, 60)$, $(100, 120)$ and $(200, 240)$ (from left to right), $u_0 = 2$.

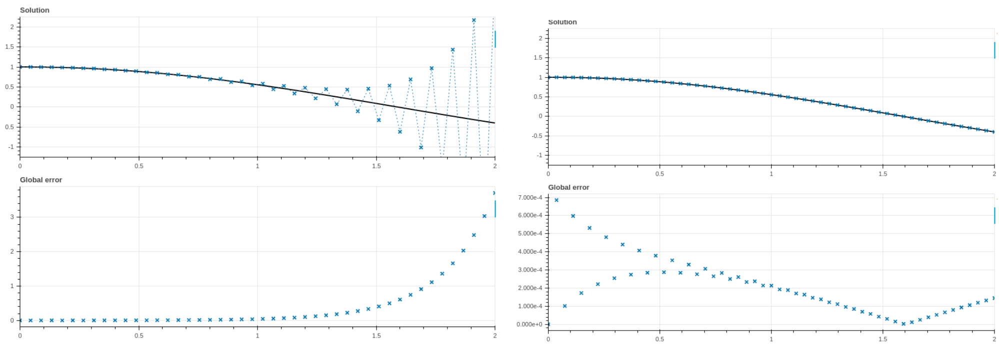

Figure 13: Plotting the identified regimes for forward euler scheme, $n_t = 51, 55$ (left to right),$k = 50$, $u_0 = \frac{k^2}{k^2+1}$ and $T = 2$. The system being not stiff for this initial condition, there is less abrupt changes and the scheme is able to follow the solution better. Knowing what happens between $n_t = 51$ and $n_t = 55$ is harder than previously as regime (divergent, convergent, oscillary) requires long-time view of the numerical integrations which is complicated to have here.

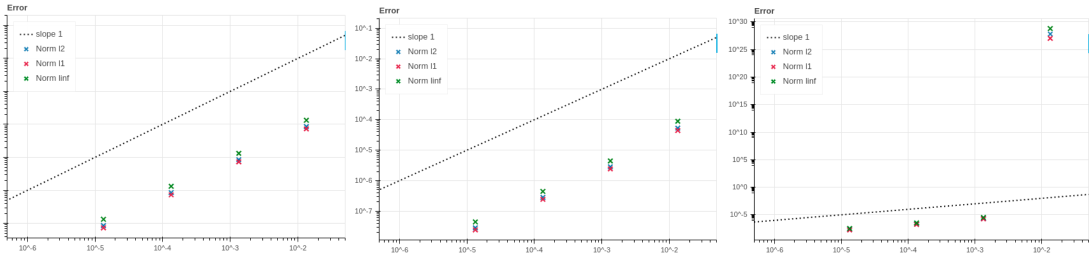

Figure 14: Log-log Plot of the error $\epsilon(\Delta t)$ for $k = 50, 150, 200$, $u_0 = 2$ and $T = 2$.

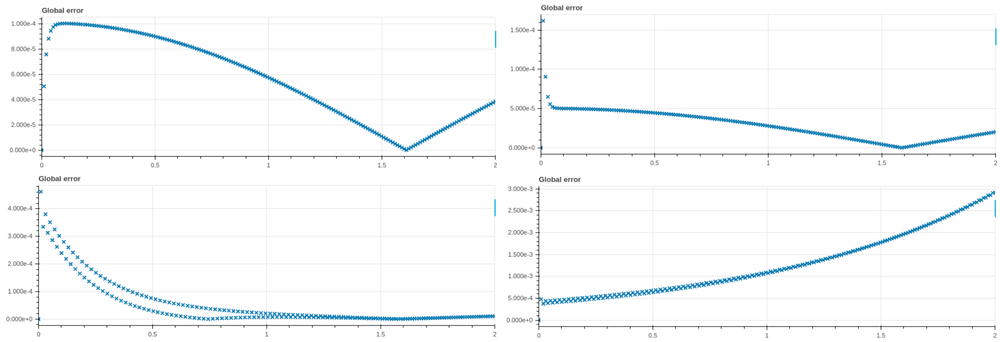

Figure 15: Plot of the global error for various stiffness : $k = 50, 100, 195, 200$ (from left to right, top to bottom), $n_t = 200$, $u_0 = \frac{k^2}{k^2+1}$.

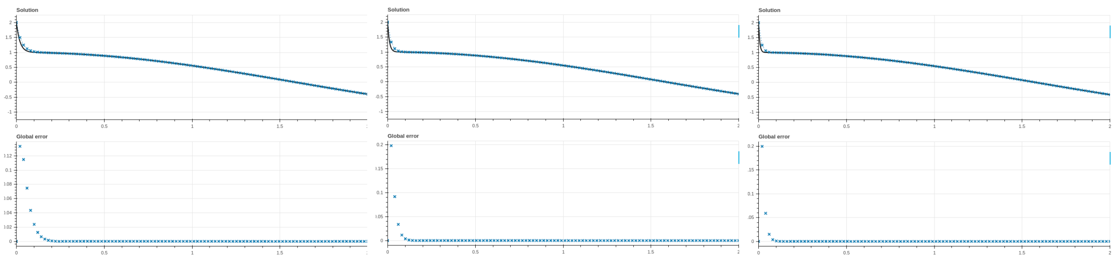

Figure 16: Stiffness influence for backward euler scheme with $n_t = 100$, $u_0 = 2$ and $k = 50, 100, 150$ (left to right).

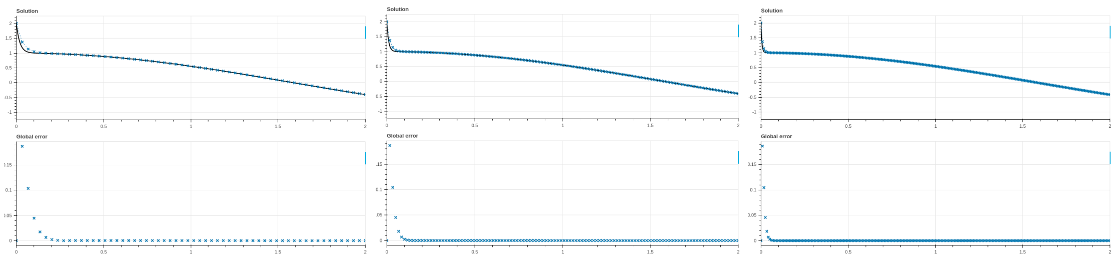

Figure 17: Numerical solutions for three (stiffness, discretization step) couple, from left to right : $(50, 60)$, $(100, 120)$, $(200, 240)$.

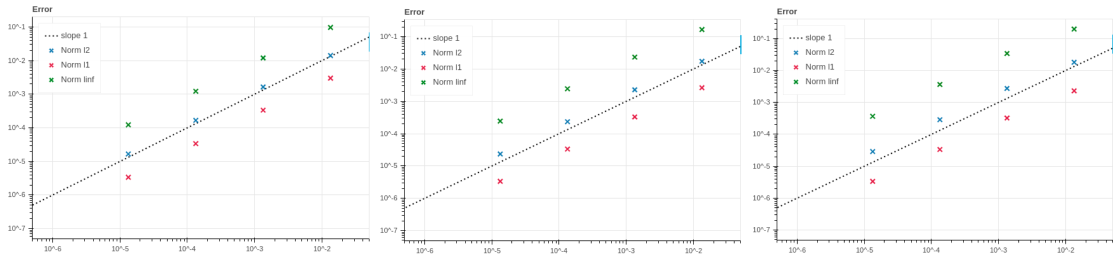

Figure 18: Log-log plot different stiffness value : 50, 100, 150.

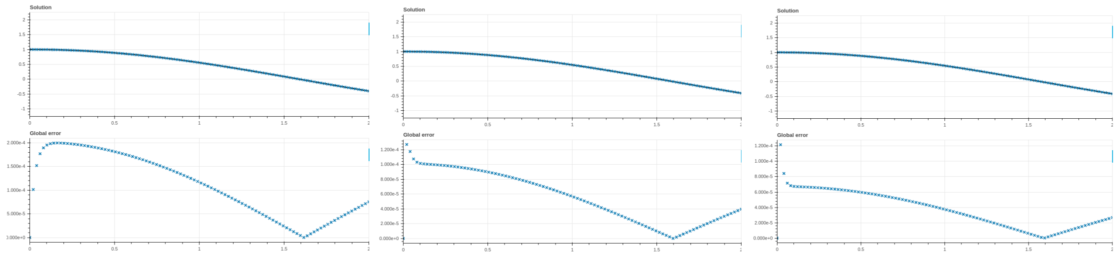

Figure 19: Stiffness influence for backward euler scheme with $n_t = 100$, $u_0 = 2$ and $k = 50, 100, 150$ (left to right).

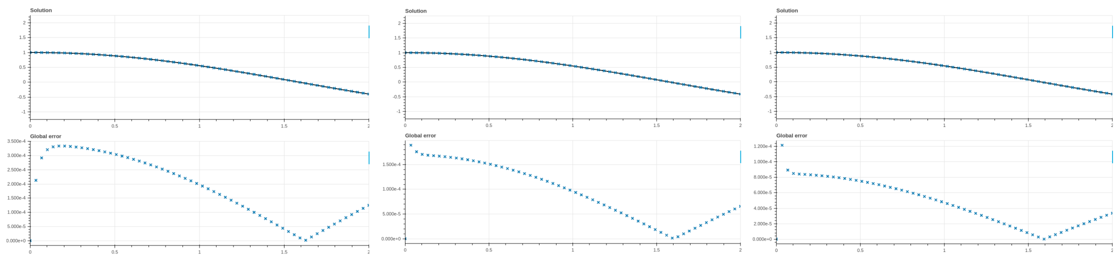

Figure 20: Numerical solutions for three (stiffness, discretization step) couple, from left to right : $(50, 60)$, $(100, 120)$, $(200, 240)$.

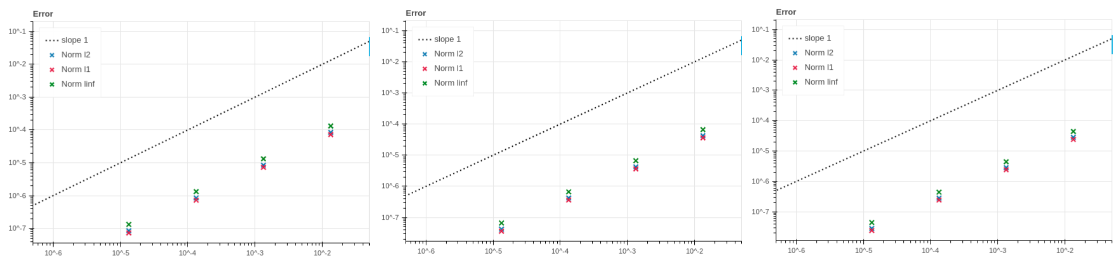

Figure 21: Log-log plot different stiffness value : 50, 100, 150.