Computing the long term evolution of the solar system with geometric numerical integrators

Shaula Fiorelli Vilmart, Gilles Vilmart

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Shaula Fiorelli Vilmart • Gilles Vilmart [1]

In this snapshot, we explain how the evolution of the solar system can be computed over long times by taking advantage of geometric numerical methods. Short sample codes are provided for the Sun-Earth-Moon system.[2]

Let us consider the Sun-Earth-Moon system, where for simplicity we neglect the other bodies and influences in the solar system. Surprisingly, applying a standard numerical method yields a dramatically wrong solution, where the Moon is ejected from its orbit (see left picture in Figure 1). In contrast, a well chosen geometric integrator with the same initial data yields the correct behavior (right picture). We explain the main ideas of geometric integration for the long time evolution of such a system.

1 Computing the trajectories

Let us step back in time and imagine we are on the first of January of the year 1600, when Johannes Kepler just moved to Prague to become the new assistant of the astronomer Tycho Brahe. He has to escape from the persecution in Graz, in particular caused by his adhesion to the controversial Copernican theory boldly saying that the planets revolve around the Sun.

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[2] An earlier version in French of this article first appeared in [11].

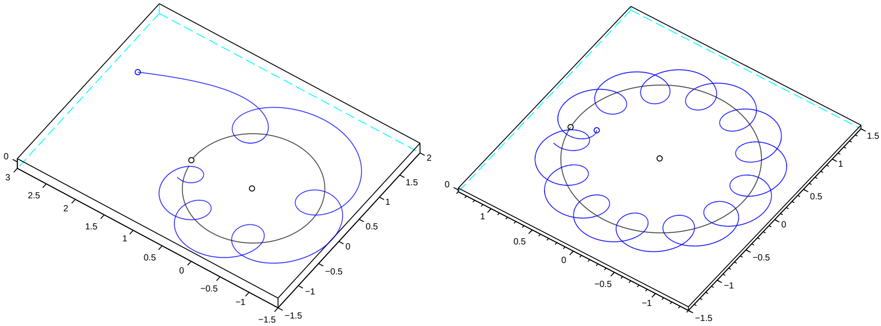

Figure 1: Comparison of two numerical methods for the Sun-Earth-Moon system simulated over one year. The distance between the Moon (blue trajectory) to the Earth (black trajectory) is scaled by a factor 100 in the plots to distinguish better the Earth and the Moon.

How the strange motion of Mars inspired Kepler. Tycho Brahe is very interested in planetary motion and has already calculated very precisely the orbits of known planets. But Mars escapes comprehension: he can not properly predict its trajectory. Without warning of the difficulty, Brahe asks Kepler to calculate the precise orbit of Mars. It will take about six years to Kepler to complete this work. Indeed, while Venus has a nearly circular orbit, the trajectory of Mars is more complex: it turns out to be an ellipse, whose flattening, measured by the eccentricity, is the largest of all planets in the solar system after Mercury.

The three laws of Kepler. This takes Kepler to propose his three basic laws (see Figure 2):

1. The planets describe elliptical orbits and the Sun is a focus;
2. The segment connecting the Sun and the planet sweeps out equal areas during equal times; this *invariant* of the problem is called *the law of equal areas*;
3. The square of the rotational period T of a planet (time between two successive passes in front of a distant star) is proportional to the cube of the semi-major axis a of the elliptical orbit of the planet, that is $T^2 = Constant \cdot a^3$.

Newton's universal law of gravitation. In 1687, Isaac Newton, inspired by the three laws of Kepler, proposes the universal law of gravitation, that all cosmic objects attract each other pairwise with equal forces (but in opposite directions),

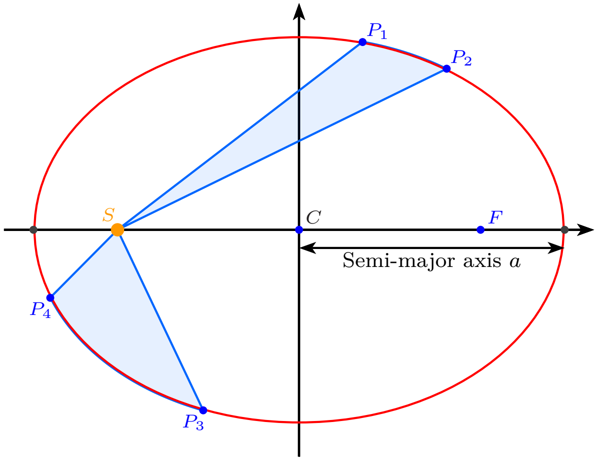

Figure 2: Illustration of Kepler's laws. S et F are the foci of the elliptic trajectory of the planet P. By the law of equal areas, the domains SP_1P_2 and SP_3P_4 have the same area, where the positions P_1, P_2 and P_3, P_4 of the planet are obtain for equal time intervals.

proportional to the product of their masses and inversely proportional to the square of the distance between them. It is this law that we will use to calculate the position of the planets. The gravitational force $\overrightarrow{F}_{S\to P}$ applied by a body S to a body P is given by the following formula:

$$\overrightarrow{F}_{S\to P} = -\overrightarrow{F}_{P\to S} = -\frac{Gm_S m_P}{d^2}\overrightarrow{u},$$

where G is the universal constant of gravitation, m_S, m_P are the masses of the bodies S et P, d is the (Euclidean) distance between S and P, and $\overrightarrow{u}$ is a vector with unit length in the direction from S to P.

Newton's explanation of the law of equal areas. Newton provides in his *Principia Mathematica* (see Figure 3) a justification of the second law of Kepler, by using a method that can be interpreted as the first geometric numerical method, as presented in [3, 9]. The idea is as follows: S is the Sun and a planet is assumed to be located initially at point A. The idea is to apply the Sun's attractive gravitational force not constantly along time, but by impulses. Let us first assume that the Sun applies no gravitational force at all; in this case, the planet moves after some time from A to B in a straight line, with a constant velocity in the direction $\overrightarrow{AB}$. Waiting again, the planet should continue on the same straight line until it reaches the point c, with $\overrightarrow{AB} = \overrightarrow{Bc}$. However, let us apply *an impulse of force* from the Sun to the planet: this force adds a velocity component to the planet motion that Newton represents by the vector $\overrightarrow{BV}$ along the segment SB. The planet's velocity is now the sum of two components: the vector $\overrightarrow{Bc}$ and the vector $\overrightarrow{BV}$, and the resulting vector is $\overrightarrow{BC}$ which defines the point C. The planet thus moves with this new constant velocity until it reaches C. Iterating this process, the planet follows the path $A, B, C, D, E, F, \ldots$. Newton proved that all the triangles SAB, SBC, SCD, SEF *have the same area*: this corresponds to a discrete version of the second law of Kepler (the law of equal areas).

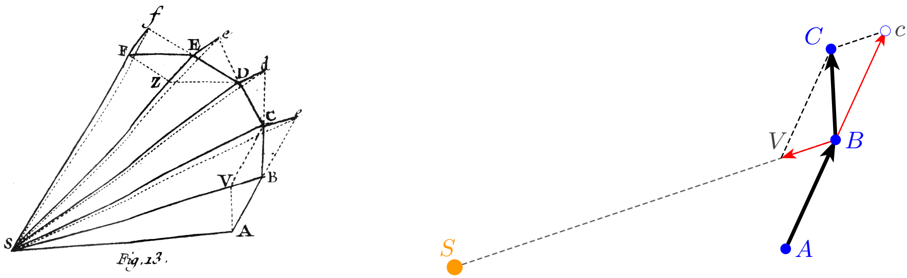

Figure 3: Left: figure from Newton's *Principia Mathematica* (1687) (Book I, Theorem I). Right: the planet's motion subject to a "force impulse".

Newton's geometric proof. We first note that the triangles SAB and SBc have the same area, because they have the same basis $\overrightarrow{AB} = \overrightarrow{Bc}$ and the same height issued from S. Next, observing that $BcCV$ is a parallelogram, we deduce that $\overrightarrow{Cc}$ is parallel to $\overrightarrow{SB}$. The triangles SBc and SBC thus have the same basis $\overrightarrow{SB}$ and the same heights issued from c and C respectively; and hence they have the same area. Thus, the triangles SAB and SBC have the same area. This permits to prove the second law of Kepler for the motion of a body subject to a central force.

In fact, Newton thus proves a discrete version of the law of equal areas (which means here a motion with successive jolts). The process which permits to get from A to B, then B to C, etc., corresponds in fact to a geometric numerical scheme known today as the *symplectic Euler method* that we will present in this snapshot. We will also show that as the time interval between each force impulse tends to zero, the obtained approximation converges towards the solution of the problem.

2 How to solve a differential equation?

Many physical phenomena can be modeled by differential equations that is to say, equations in which the unknown is not a number but a function, and involving one or more derivatives of this function. In practice, it is often difficult or even impossible to find a simple formula for the exact solution of a differential equation. Thus, numerical methods must be used to calculate an approximate solution.

A differential equation example: the problem of de Beaune. Before we address the planetary problem, let us consider first a simpler problem formulated by Florimond de Beaune (1638)[2] and for which the exact solution can be obtained. Find a curve $\mathcal{C}$ in the plane, given by a function $y(t)$, such that the tangent to $\mathcal{C}$ in any point M with abscissa t intersects the horizontal axis at a distance $D = 1$ of the abscissa t (see Figure 4).

We note that the slope of the tangent at point M equals the quotient of the height $y(t)$ over the width $D = 1$, which means that the slope is equal to $y(t)$. We recall in addition that the slope of the tangent to the curve of a function at a given point is by definition the derivative of the function at this point. This yields the differential equation

$$y'(t) = y(t).$$

The general solution of this equation is $y(t) = C \cdot e^t$ (exponential function) where C is a constant that can be determined by adding an initial condition $y(0) = y_0$. Such a differential equation problem with given initial value then possesses a unique solution $y(t) = y_0 e^t$.

There exists a wide range of numerical methods for solving a general differential equation of the form

$$y'(t) = f(y(t)), \qquad y(0) = y_0,$$

where y_0 is a given initial condition, f is a given function and y is an unknown function of time t. Note that for the problem of de Beaune, the function f reduces to the identity, $f(y) = y$. We now present a few very simple examples of such methods, chosen for their importance.

The method of Euler: an approximation of the continuous model. When proving the law of equal areas, Newton used gravitational force impulses, applied to the planet at regular intervals. The numerical methods rely on the same

[2] De Beaune is famous mostly for the problem presented here. It is one of the four problems that he submitted to Decartes who had just published La Géométrie (1637).

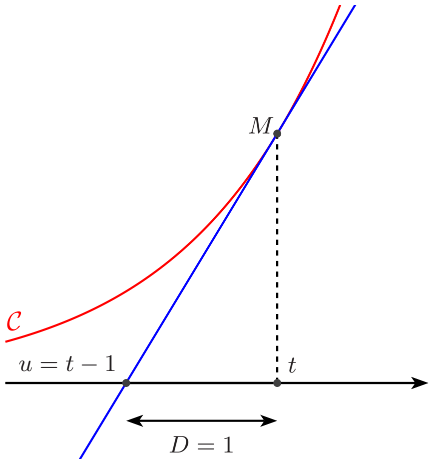

Figure 4: The problem of de Beaune (1638)

idea. We first choose a stepsize h and we compute an approximation $y_n \simeq y(t_n)$ of the continuous solution at times $t_n = nh$, where $n = 0, 1, 2, 3, \ldots$. The first quantity y_0 is known, this is the initial condition. Next, we compute y_1, then y_2, then y_3, and so on. The idea, common to all Euler methods is to approximate the derivative by a difference quotient:

$$y'(t_n) = \lim_{h \to 0} \frac{y(t_n + h) - y(t_n)}{h} \simeq \frac{y_{n+1} - y_n}{h}.$$

Explicit Euler method. This is the simplest numerical method, due to Leonhard Euler (1768).[3] By approximating $f(y(t_n))$ with $f(y_n)$ in the differential equation, we obtain the *explicit Euler method*, $(y_{n+1} - y_n)/h = f(y_n)$, which can be written simply as

$$y_{n+1} = y_n + hf(y_n).$$

Convergence rate of a numerical method. Calculating successively y_1, $y_2, \ldots$, we obtain a polygonal line passing though the points (t_n, y_n) (see Figure 5). One can show that reducing the stepsize h, the obtained polygonal line

[3] This was a great contribution of Euler among the numerous ones in impressively many areas of sciences. In fact, it was published in 1768 (Saint Petersburg) after he left Berlin in 1766 where he wrote his book Institutionum calculi integralis. Thus, Euler's method can already celebrate its 250 year anniversary.

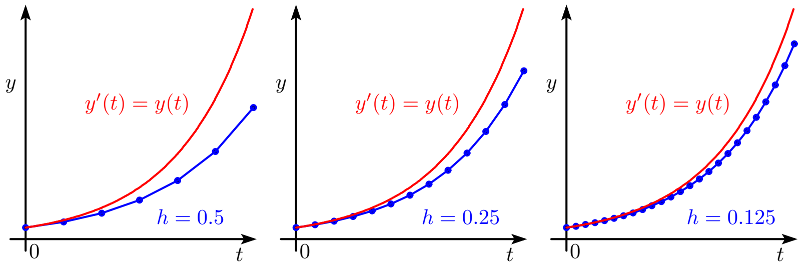

Figure 5: The explicit Euler method for solving the problem of de Beaune $y'(t) = y(t)$, with $y(0) = 1$. The lengths of the stepsize are $h = 0.5, 0.25, 0.125$, respectively.

becomes closer and closer to the exact solution, and converges to the solution as h converges to zero. We observe in addition in Figure 5 that when dividing by a factor two the stepsize h, the numerical solution (blue polygonal line) gets closer to the exact solution (red curve) with an error divided by the same factor two. The convergence rate is thus of order 1. There exist many variants of the Euler method, in particular Runge-Kutta methods, which can be more accurate, with a distance proportional to h^2, h^3, etc., which corresponds to a rate of convergence of order 2, 3, etc.

The implicit Euler method. In contrast to the explicit Euler method, choosing instead the approximation $f(y(t_n)) \simeq f(y_{n+1})$, we get the implicit Euler method

$$y_{n+1} = y_n + hf(y_{n+1}).$$

This method is called implicit because the computation of y_{n+1} requires in general to solve a non-linear system. Indeed, one has to compute y_{n+1} while the value $f(y_{n+1})$ is not a priori known. There exist specific methods for solving such problems.

Many other schemes could be considered. However, in some situations, the rate of convergence is not the only important aspect to be taken into consideration, and having a correct qualitative behavior can be very important, in particular conserving invariants such as the energy of the Sun-Earth-Moon system. This is the aim of the next paragraph.

3 How to conserve the energy of the system ?

We consider the example of a spring oscillating close to its rest position (see Figure 6). In the absence of friction forces (or damping forces), and neglecting

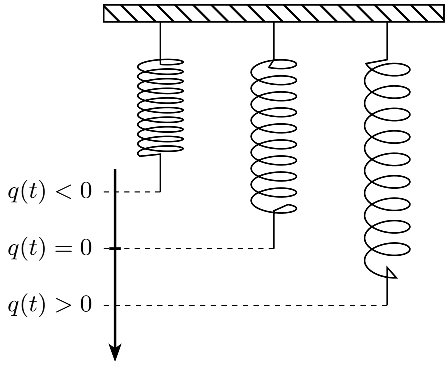

Figure 6: An oscillating spring, where $q(t)$ is the spring elongation at time t.

the gravitational force, we model the motion by a system of two differential equations. We denote by $q(t)$ the elongation at instant t of the spring with respect to its rest position, with $q(t) > 0$ when the spring is stretched, and $q(t) < 0$ when the spring is compressed. In addition, we denote by $p(t)$ the momentum and by m the mass. We can then model the motion using the differential equations

$$q'(t) = \frac{1}{m}p(t), \qquad p'(t) = -kq(t).$$

The first equation above is just the definition of the momentum. The second equation is a consequence of the following two facts: first, if the spring is stretched (or compressed) with $q(t)$ units with respect to its natural rest position, then a restoring force $-kq(t)$ proportional to $q(t)$ but with opposite direction is applied (law of Hooke), where k is a positive constant corresponding to the stiffness of the spring; second, the second law of Newton states that the sum of forces equals the product of mass and acceleration: $\overrightarrow{F}(t) = m\overrightarrow{a}(t)$, where $\overrightarrow{a}(t)$ has coordinates $q''(t)$, and we deduce $-kq(t) = mq''(t) = p'(t)$, which yields the second differential equation.

Energy conservation. This problem possesses an energy:

$$E(p,q) = \text{kinetic energy} + \text{potential energy} = \frac{1}{2m}p^2 + \frac{k}{2}q^2,$$

which is conserved in time by the solution, which means $E(p(t),q(t)) = E(p(0),q(0))$. Indeed, one can compute that the derivative of the energy with respect to time is zero:

$$\frac{d}{dt}E(p(t),q(t)) = \frac{1}{m}p'(t)p(t) + kq'(t)q(t) = \frac{1}{m}(-kq(t))p(t) + \frac{k}{m}p(t)q(t) = 0.$$

This shows that the energy $E(p(t), q(t))$ remains constant in time for the exact solution.

We fix at instant $t = 0$ the initial conditions $p(0) = p_0$ and $q(0) = q_0$. We also fix for simplicity the mass $m = 1$. Applying the explicit Euler method, we obtain the recurrence relation

$$q_{n+1} = q_n + hp_n, \qquad p_{n+1} = p_n - hkq_n.$$

A calculation yields

$$E(p_{n+1}, q_{n+1}) = \frac{1}{2}p_{n+1}^2 + \frac{k}{2}q_{n+1}^2 = (1 + kh^2)\big(\frac{1}{2}p_n^2 + \frac{k}{2}q_n^2\big).$$

Equivalently,

$$E(p_{n+1}, q_{n+1}) = (1 + kh^2)E(p_n, q_n).$$

We see that at each step of the scheme, the energy is *amplified* by the factor $1 + kh^2$ which is strictly larger than 1. Analogously, considering the implicit Euler method,

$$q_{n+1} = q_n + hp_{n+1}, \qquad p_{n+1} = p_n - hkq_{n+1},$$

the same calculation yields

$$E(p_{n+1}, q_{n+1}) = \frac{1}{1 + kh^2}E(p_n, q_n).$$

This time, the energy is not amplified but is is *damped* at each step of the method by the factor $1/(1 + kh^2)$ strictly smaller than 1.

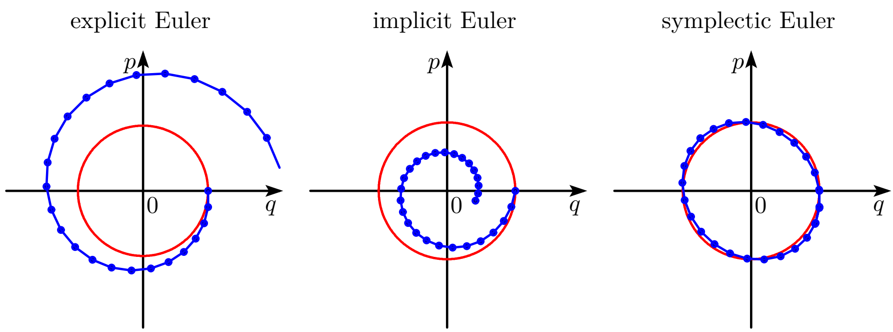

Figure 7: Comparison of the explicit, implicit and symplectic Euler methods for the harmonic oscillator (spring with stiffness parameter $k = 1$).

A state of the spring at a given time t corresponds to the knowledge of both its position q and momentum p. For the exact solution, the conservation of energy implies that these satisfy the identity

$$E(p,q) = \frac{1}{2}p^2 + \frac{k}{2}q^2 = Constant.$$

Setting $k = 1$, the exact solution therefore corresponds to a circle in the (p,q)-plane (see the red curve in Figure 7), passing though the given initial condition, here $q_0 = 1$, $p_0 = 0$ (the spring at initial time is stretch and without velocity). We can observe in Figure 7 the numerical solutions obtained for the spring problem with stepsize $h = 1/4$. We see that the solution obtained with the explicit and implicit Euler methods are spiralling towards the exterior or the interior, respectively. This is due to the amplification or damping factor described previously.

The symplectic method, a method that preserve well the energy over long times. To preserve correctly the energy, the idea is to combine the explicit and implicit Euler methods in the following manner. For the spring problem, the symplectic Euler method is given by

$$q_{n+1} = q_n + hp_n, \qquad p_{n+1} = p_n - hkq_{n+1},$$

We note that this method updates the position and the momentum alternatively, exactly like in the proof by Newton of the law of equal areas presented earlier. We now consider the modified numerical energy, defined as $\tilde{E}_h(p,q) = E(p,q) + hkpq$. A calculation yields

$$\tilde{E}_h(p_{n+1}, q_{n+1}) = \tilde{E}_h(p_n, q_n),$$

This means that the modified energy is exactly conserved by the numerical scheme, without amplification and attenuation factor. In the (p,q)-plane, the curve $\tilde{E}_h(p,q) = \tilde{E}_h(p_0,q_0)$ is in fact an ellipse close to the circle of the exact solution when h is small, because the numerical energy $\tilde{E}_h$ is a small perturbation of size h of the exact energy E. This shows that the numerical error in the energy remains small with size h compared to the exact solution. Indeed, we observe in Figure 7 (right picture) that the numerical trajectory (in blue) remains close to the exact one (in red). The symplectic Euler method is therefore a geometric method (or integrator) well adapted to the problem, because it conserves well the energy of the system (although this energy is not conserved exactly).

4 Try yourself, with the free open source software Scilab

We consider the Sun-Earth-Moon system where we neglect the other planets and influences in the solar system. We represent the positions of these bodies by three functions of time, $\overrightarrow{q}_i(t) \in \mathbb{R}^3$, $i = 0, \dots 2$, where the index $i = 0$ corresponds to the Sun, $i = 1$ corresponds to the Earth, and $i = 2$ corresponds to the Moon. The respective masses of the three bodies are denoted by m_i, $i = 0, 1, 2$, while the universal constant of gravitation is denoted G.[4] We also consider the momenta $\overrightarrow{p}_i(t) = m_i \overrightarrow{q}_i'(t)$. Newton's second law of dynamics then reads

$$\begin{aligned}
\overrightarrow{p}_0' &= \overrightarrow{F}_{E \to S} + \overrightarrow{F}_{M \to S},\\
\overrightarrow{p}_1' &= \overrightarrow{F}_{S \to E} + \overrightarrow{F}_{M \to E},\\
\overrightarrow{p}_2' &= \overrightarrow{F}_{S \to M} + \overrightarrow{F}_{E \to M},
\end{aligned}$$

and we apply our numerical schemes to the above system of differential equations.

body	mass (relative to the Sun)	position ($A.U.$)	velocity ($A.U./day$)
Sun	$m_0 = 1$	0 0 0	0 0 0
Earth	$m_1 = 3.00348959632 \cdot 10^{-6}$	-0.1667743823220 0.9690675883429 -0.0000342671456	-0.0172346557280 -0.0029762680930 -0.0000004154391
Moon	$m_2 = 1.23000383 \cdot 10^{-2}\, m_1$	-0.1694619061456 0.9692330175719 -0.0000266725711	-0.0172817331582 -0.0035325102831 0.0000491191454
Gravitational constant $G = 2.95912208286 \cdot 10^{-4}$.			

Table 1: Initial data from [8] for the Sun, Earth, Moon on 01/01/2016 at 0h00.

We provide in Table 1 the positions and initial velocities (vectors in $\mathbb{R}^3$) for the Sun, the Earth and the Moon at a given date (here 1st of January 2016), expressed in astronomical units, based on the Earth-Sun distance (1 $A.U.$ is about 150 million kilometers), and the time is in earth days.

We give below a short sample code for the free open source software Scilab [10] for computing the evolution of the system with either the explicit Euler method or the symplectic Euler method, as described previously and presented in Figure 1. Notice that these trajectories are almost in a plane, but they evolve in 3D. The Sun itself is slightly moving as well (this is by the way a

[4] The gravitational constant G is among the most difficult physical constants to measure. In precise celestial computations, one uses rather the product of G times the mass as input data (standard gravitational parameter).

common methodology to detect exoplanets), but the software represents the trajectories with respect to the Sun, chosen as a reference, and located at the origin $(0, 0, 0)$. Note that this sample code can be straightforwardly adapted to include additional planets of the solar system, using initial data from [8]. It could also be extended to predict solar (or lunar) eclipses when the Earth moves into the Moon's shadow (or the converse), taking into account the diameters of the bodies.

First, the functions, to be put into a file `euler.sci`.

```
function f=fun_v(q)
  deff('[v]=vecf(v0)','v=v0/norm(v0).^3');
  sun=1:3;earth=4:6;moon=7:9;
  f(sun)  =-G*m0*m1*vecf(q(sun) -q(earth))..
           -G*m0*m2*vecf(q(sun) -q(moon));
  f(earth)=-G*m1*m0*vecf(q(earth)-q(sun))..
           -G*m1*m2*vecf(q(earth)-q(moon));
  f(moon) =-G*m2*m0*vecf(q(moon)-q(sun))..
           -G*m2*m1*vecf(q(moon)-q(earth));
endfunction

function f=fun_u(p)
  f=[p(1:3)/m0;p(4:6)/m1;p(7:9)/m2];
endfunction

function [vp,vq]=euler_symplectic(n,h,p,q)
vp=p;vq=q;
for i=1:n
  q=q+h*fun_u(p);  p=p+h*fun_v(q);
  vq=[vq,q];       vp=[vp,q];
end
endfunction

function [vp,vq]=euler_explicit(n,h,p,q)
vp=p;vq=q;
for i=1:n
  tmp=q; q=q+h*fun_u(p);  p=p+h*fun_v(tmp);
  vq=[vq,q];       vp=[vp,q];
end
endfunction
```

Next, the main script, to be put into a file `calcul.sce`.

```
//Sun-Earth-Moon system integration
m0=1;m1=3.00348959632E-6;m2=m1*1.23000383E-2; //body masses
G=2.95912208286e-4; //gravitational constant

//initial conditions
//source: PORTAIL SYSTEME SOLAIRE
//        OBSERVATOIRE VIRTUEL DE L'IMCCE
//        Observatoire de Paris / CNRS
//http://vo.imcce.fr/webservices/miriade/?forms
//Target: p:Terre, s:Lune
//Epoch: 2016-01-01 00:00:00, 1, 1.0 - day, UTC
//Reference center:
//INPOP Ecliptic Rectangular AstrometricJ2000
```

```
// eclictic coordinates
q0=[0;0;0;-0.1667743823220;0.9690675883429;-0.0000342671456;..
          -0.1694619061456;0.9692330175719;-0.0000266725711];
v0=[0;0;0;-0.0172346557280;-0.0029762680930;-0.0000004154391;..
          -0.0172817331582;-0.0035325102831;0.0000491191454];
// momenta
p0=[v0(1:3)*m0;v0(4:6)*m1;v0(7:9)*m2];

//time integration over 365 days.
[vp,vq]=euler_symplectic(365*10,0.1,p0,q0) //stepsize h=0.1
//[vp,vq]=euler_explicit(365*10,0.1,p0,q0) //stepsize h=0.1

//trajectories with respect to the Sun placed at the origin
vq(4:6,:)=vq(4:6,:)-vq(1:3,:);
vq(7:9,:)=vq(7:9,:)-vq(1:3,:);
comet3d(0,0,0);

//increase by a factor 100 the Earth-Moon distance for visualisation.
vq(7:9,:)=vq(4:6,:)+100*(vq(7:9,:)-vq(4:6,:))
comet3d([vq(4,:)',vq(7,:)'],[vq(5,:)',vq(8,:)'],[vq(6,:)',vq(9,:)']);
```

5 Is the solar system stable?

A question closely related to the topic of this snapshot is the issue of the stability of the solar system. Soon after Newton proposed his universal law of gravitation (1687), many researchers (including Laplace, Lagrange, Poisson, ...) have been studying the question if the regular trajectories of the planets will continue nicely until the end of times, or if collisions or ejections will occur. In 1885, the King Oscar II of Sweden proposed a prize about the stability of Newton's model. This prize was awarded to Poincaré, although he did not really solve the problem. His contribution is however at the origin of the theory of dynamical systems. It also led to important developments in Hamiltonian perturbation theory and gave rise to the Kolmogorov–Arnold–Moser (KAM) theory about the persistence of quasi-periodic motions under small perturbations (see the survey [7]). Unfortunately, this beautiful theory does not apply to realistic solar system models. The initial question "is the solar system stable?" then remained open until the last decades where the final negative answer that the solar system is chaotic was given by the mathematician and astronomer Jacques Laskar and colleagues, based on analytical means but also using numerical methods including geometric integrators.[5] In addition, some of their recent computations show that collisions or ejections could even occur in the next five

[5] The chaotic behaviour of the solar system was shown by Laskar (1989): a small error of a few meters in the initial position of the Earth is amplified by a factor 10 every 10 million years, yielding a huge error of dozens of millions of kilometers after 100 million years. This makes precise numerical predictions based on planetary trajectories in the solar system become infeasible beyond this time horizon of 100 million years.

billion years, that is before the end of the life of the Sun (see the survey [4]).

Notice that the past evolution of the solar system over long times has a surprising application: it serves as a measurement scale for geological dating. Indeed, the position of the planets and their orbital parameters (inclination, etc.) influence how the sediments are deposited on the surface of the Earth, which allows for geological dating by observing these sediment deposits. Such geological calculations are known again through the work of Laskar and colleagues. The calculation named La2010 [5] which uses geometric numerical methods with high order of accuracy, shows that the extinction of dinosaurs (about 65 million years ago) could be slightly older than previously estimated.

6 Conclusion

We have shown, based on the examples of an oscillating spring and the three-body problem Sun-Earth-Moon that for problems with a particular geometric structure, it is essential to use numerical methods that are geometric, preserving the invariants of the system, to get a good qualitative behavior of the numerical solution. We saw that the symplectic Euler method preserves well the energy, a key invariant of the mechanical systems, while the explicit Euler method, and more generally any standard explicit Runge-Kutta methods, do not preserve it and are not suitable for integration over long time intervals.

A mathematical theory, called backward error analysis, permits to demonstrate that symplectic integrators have a good energy conservation for such mechanical systems. This theory of geometric numerical integration [6, 3, 2, 1] reveals to be a powerful tool for the study and design of integrators in many areas of physics (here celestial mechanics), chemistry (molecular dynamics), biology, and it has connections with algebraic tools from other fields of mathematics and physics (renormalization in quantum field theory).

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