MAP551 - PC6 - Dynamics around critical points, hyperbolicity, continuation and bifurcations.

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0.1 Introduction

0.2 Some toy dynamical systems and course application

0.2.1 A vector with a singular germ non C^0-conjugated with the one related to its linearization

0.2.1.1

We linearize on $X(0,0) = 0$ which yields : $DX(0,0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus, $X_1(x_1, x_2) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$.

The solutions are consequently circles, for example : $x_1'' = -x_2' = x_1$.

0.2.1.2

After calculation :

$$\frac{d\Phi}{dt}(t) = -2\|\Phi(t)\|^4$$

Which yields with $u = \|\Phi(t)\|^2$:

$$d_t u = -2u^2 \Leftrightarrow d_t(\frac{1}{u}) = -\frac{-d_t u}{u^2} = 2$$

Which in turn yields : $u(t) = \frac{u_0}{1+2u_0 t} \to 0$ when $t \to +\infty$.

That means that $\|\Phi(t)\|^2 \to 0$. Consequently, the solutions all converge towards 0 even though we do not know how they converge.

0.2.1.3

The two germs can not be C^0-conjugated because in the linear-system, the solutions do not converge towards 0 (they are circles) whereas the solutions of the non-linear one do. Consequently, there are plenty of neighbourhoods for which both vector field are different.

This is not in contradiction with Hartman-Grobman's theorem (HG) since the Jacobian matrix of the linear system admit for eagen-values $\{-i, i\}$, both having a real part equal to 0. Thus, HG does not apply.

0.2.2 Local / global, stable / unstable manifold

0.2.2.1

We have : $d_t f(x_1, x_2) = d_t x_2 x_2 - x_1 + d_t x_1 {x_1}^2 = (1 - {x_1}^2)x_2 - x_1 + x_2 {x_1}^2) = 0$ which proves that f is a first integral of X.

0.2.2.2

The figure 1 shows the phase portrait for the current system. For plotting reasons, the two parts of the blue curve do not cross in $(-1, 0)$ but should.

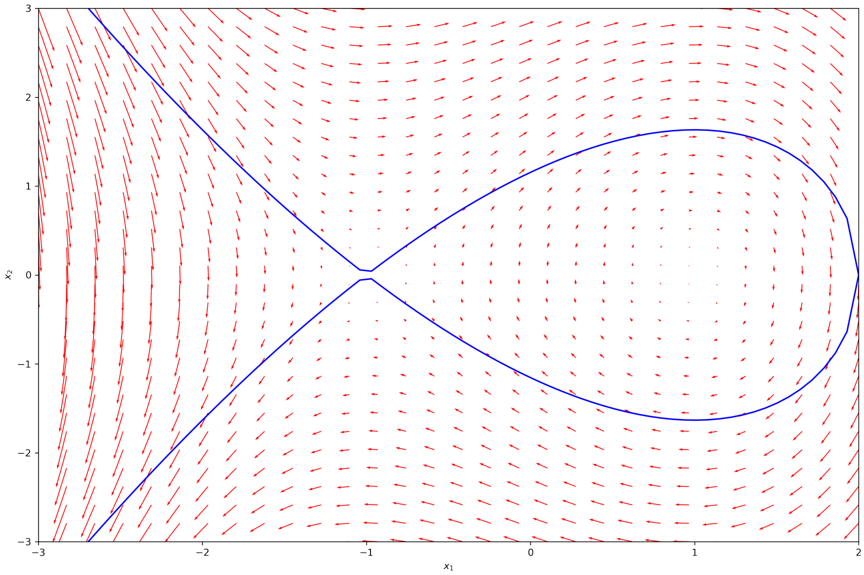

Figure 1: The figure shows the phase portrait for $a = (-1, 0)$. It also displays the gradient for some (x_1, x_2).

Having access to the first integral gives a lot of information on the trajectories.

0.2.2.3

The jacobian is : $DX(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ -2x_1 & 0 \end{pmatrix}$ which yields $DX(-1, 0) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$.

The eigen-values for $DX(-1, 0)$ are : $\{\pm\sqrt{2}\}$.

Consequently, the HG's theorem applies and we can study the eigenspaces:

$\lambda = -\sqrt{2}$

$$A \begin{pmatrix} x1 \\ x2 \end{pmatrix} = -\sqrt{2} \begin{pmatrix} x1 \\ x2 \end{pmatrix}$$

Which yields : $\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix}$.

This is the stable eigenspace which is consistant with the phase portrait previously plotted as it gives for stable manifold (localy, around a) the initial conditions which are close to the straight line defined by : $x_2 = -\frac{x_1}{\sqrt{2}}$

$\lambda = -\sqrt{2}$ We do the same for $\lambda = \sqrt{2}$ and get : $\begin{pmatrix} x1 \\ x2 \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$.

This is the stable eigenspace which is consistant with the phase portrait previously plotted as it gives for stable manifold (locally, around a) the initial conditions which are close to the straight line defined by : $x_2 = \frac{x_1}{\sqrt{2}}$

Once again, this is only valid close to a since we are using HG's theorem which is valid localy where we know that the linear system is close to the real one.

Globally, we see on figure 1 that the stability domain (where we converge toward a) far from a are more complicated that the previous conditions show.

0.2.3 Stable/unstable manifold

0.2.3.1

Solving this system yields :

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x_1(0)e^{-t} \\ \frac{x_2(0)}{\sqrt{1-2\alpha {x_2}^2(0)t}} \end{pmatrix}$$

where x_2 was obtained by the classical method for solving Bernoulli's differential equations.

0.2.3.2

$\alpha < 0$: In this case, the denominator of x_2 can never be equal to zero and for $t \to +\infty$, $x_2(t) \to 0$. This is the stable manifold.

$\alpha > 0$: In this case, the denominator of x_2 reaches 0 for $t = \frac{1}{2\alpha {x_2}^2(0)}$. This is the unstable manifold with a polynomial divergence

We see that it also depends on $x_2(0)$, and in the case of $x_2(0) = 0$, the solution simply remains null.

0.2.3.3

The eigen vector associated with the 0 eigenvalue is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. That means that the dynamic along this eigenspace is the one of x_2 described previously. This dynamic is roughly plotted in the figure 2.

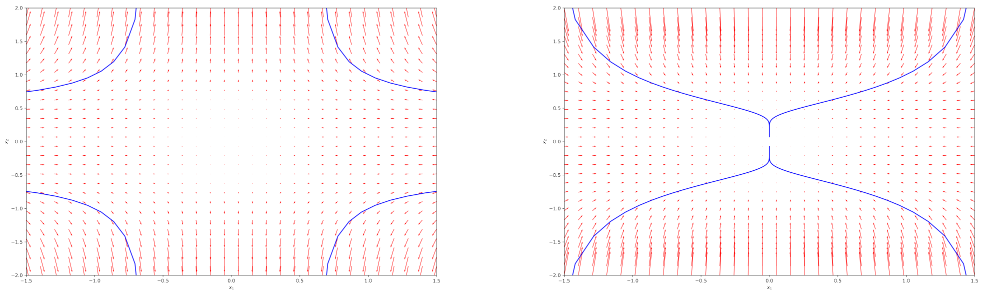

Figure 2: The figure shows the phase portrait for $(x_1(0), x_2(0)) = (1, 1)$. It also displays the gradient for some (x_1, x_2). Left is the case with $\alpha = 1$, right is the case with $\alpha = -1$. This is consistant with the analysis of the previous question.

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0.3 Study of the ω-limit sets of the Brusselator model

0.3.1 Study of equilibria

We have :

$$\begin{pmatrix} d_t y_1 \\ d_t y_2 \end{pmatrix} = \begin{pmatrix} a - (b+1)y_1 + {y_1}^2 y_2 \\ b y_1 - {y_1}^2 y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Which yields :

$$\begin{pmatrix} (b+1)y_1 + {y_1}^2 y_2 \\ y_1(b - y_1 y_2) \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix}$$

However, $y_1 \neq 0$ which implies the necessary condition that $b = y_1 y_2$ which in turn yields $y_1 = a$.

Consequently, the equilibrium is reached for : $y_1 = a$ and $y_2 = \frac{b}{a}$ with $a \neq 0$.

0.3.1.1

The jacobian at point (a, b) equals : $DX(a, \frac{b}{a}) = \begin{pmatrix} b-1 & a^2 \\ -b & -a^2 \end{pmatrix} = A$.

Which as for characteristic polynomial : $\chi_A(x) = x^2 + tr(A)x + det(A)$ where :

$$trA = b - 1 - a^2 detA \qquad = -(b-1)a^2 + ba^2 = a^2 > 0$$

In addition, $tr(A) = 0 \Leftrightarrow a = \sqrt{b-1}$ which yields :

$$\begin{cases} a < \sqrt{b-1} \Leftrightarrow tr(A) > 0 \\ a > \sqrt{b-1} \Leftrightarrow tr(A) < 0 \end{cases} \tag{1}$$

The first case is unstable (the solution gets further from the equilibrium) while the second is stable.

The non hyperbolic case happens when $4det(A) - tr(A)^2 = 0$. This function is plotted in figure 3.

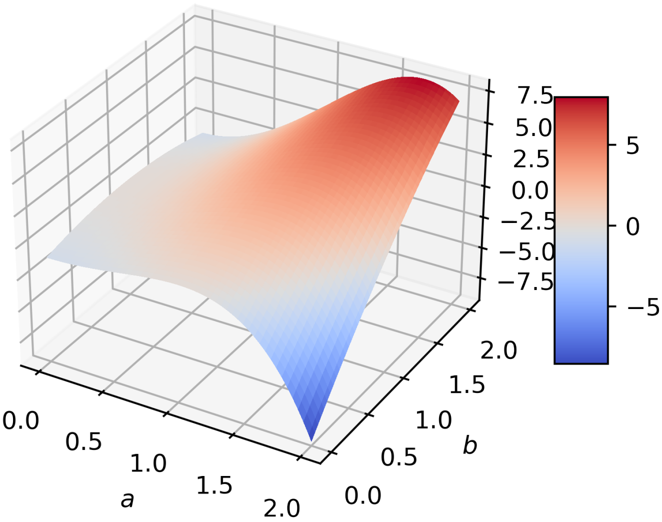

Figure 3: Plot of : $a, b \to 4a^2 - (b - 1 - a^2)^2$.

0.3.1.2

0.3.1.3

For $b = 3$, the solution is not hyperbolic if $Tr(A) = 0$ which yields $a = \sqrt{2}$ which is the value of a for which there is a change in the system. For this set of parameters $(\sqrt{2}, 3)$, at least of the real part of the eigenvalues changes sign, in this case both as it is illustrated in 4. Changing signs means there is a limit cycle as displayed in figure 5.

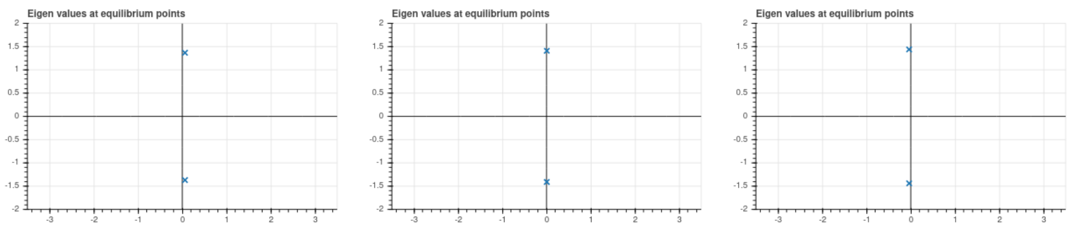

Figure 4: Eigen values for several values of a and $b = 3$. From left to right : $a = 1.37$, $a = 1.41$, $a = 1.44$. The eigen values real part go from being positive to negative.

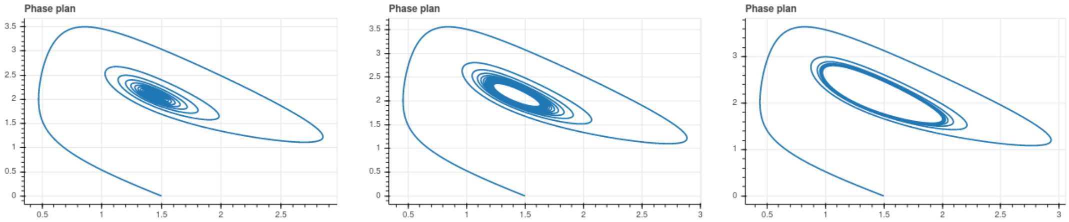

Figure 5: Phase plan for several values of a and $b = 3$. From left to right : $a = 1.37$, $a = 1.41$, $a = 1.44$.

0.3.2 Proof of the existence of a limit Cycle? - Poincaré-Bendixon Theorem

0.3.2.1

We are considering $a \in [0,\sqrt{2}[$ since a is given positive.

0.3.2.2

We could use the Poincaré-Bendixon Theorem, unfortunately we need an invariant compact space which is a hypothesis difficult to verify.

0.3.2.3

Based on the picture given in figure 1, we can say :

- If we start with strictly positive initial conditions below the magenta line, then based on the gradient, we can not cross the axes : $u_1 = (1, 0)$ and $u_2 = (0, 1)$. If we did, then by unicity of the solution, we would have the null solution which by hypothesis is not the case.

- We then need to verify that the solution stay below the magenta line. For this we can use the gradient and prove that it is, on the line, orthogonal to $\vec{n} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

In this case, the space defined by : $u_1 > 0, u_2 > 0, u_1 + u_2 < 4$ is a invariant compact space. We could then use the Poincaré Bendixon Theorem to prove the existence of a limit cycle.

0.4 Continuation of equilibria - limit points / Hopf / pitch-fork

0.4.1 Brusselator model - Hopf bifurcation - change of stability

0.4.1.1

For a autonomous differential equation $x' = f(x, \lambda)$, where λ is a parameter in $\mathbb{R}$.

The idea for continuation of the equilibria branch is to use another equilibrium already none λ_0, x_0 such that $f(x_0, \lambda_0) = 0$.

If we remain close enough of the equilibrium, then we can write $\lambda_1 = \lambda_0 + \delta\lambda$ and try to find x_1 such that $f(x_1, \lambda_1) = 0$.

To solve this equation, the Newton's algorithm is used. This algorithm works really well when starting close to the solution (here at x_0) and for function from $\mathbb{R}^n \to \mathbb{R}^n$. The algorithm is initialized at (x_0, λ_1).

This is the first algorithm.

The second, which is better, requires $D_x f(x, \lambda)$ to have a inverse. In that case, we can compute $\frac{dx}{d\lambda}$ which allows to initialize at $(x_0 + x'(\lambda_0)\delta\lambda, \lambda_1)$ which is closer tot the solution.

From the previous questions, we can say that for $b = 3$, the Hopf bifurcation happens for $a = \sqrt{2}$.

0.4.1.2

The figure 6 shows the ouput for the algorithm (order 0) which seems consistant with $a = \sqrt{2}$.

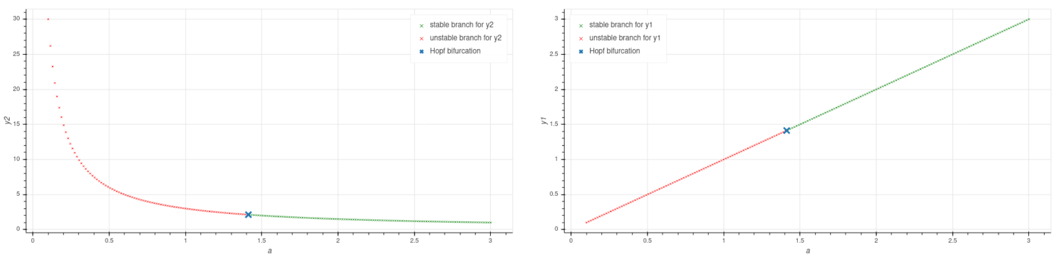

Figure 6: Output of the algorithm of continuation of the equilibria usiung a natural continuation

0.4.1.3

The accuracy of the algorithm is based on the accuracy of Newton's algorithm : has it converged ? Are the stopping criteria adapted ? etc.

0.4.2 Thermal explosion - limit point - unstable branch

0.4.2.1

Using $g(\theta) = F_k e^\theta - \theta$ which is obviously C^2 we have :

$$\begin{cases} g'(\theta) &= F_k e^\theta - 1 \\ g''(\theta) &= F_k e^\theta \end{cases} \tag{2}$$

If we plot g for several F_k, we have :

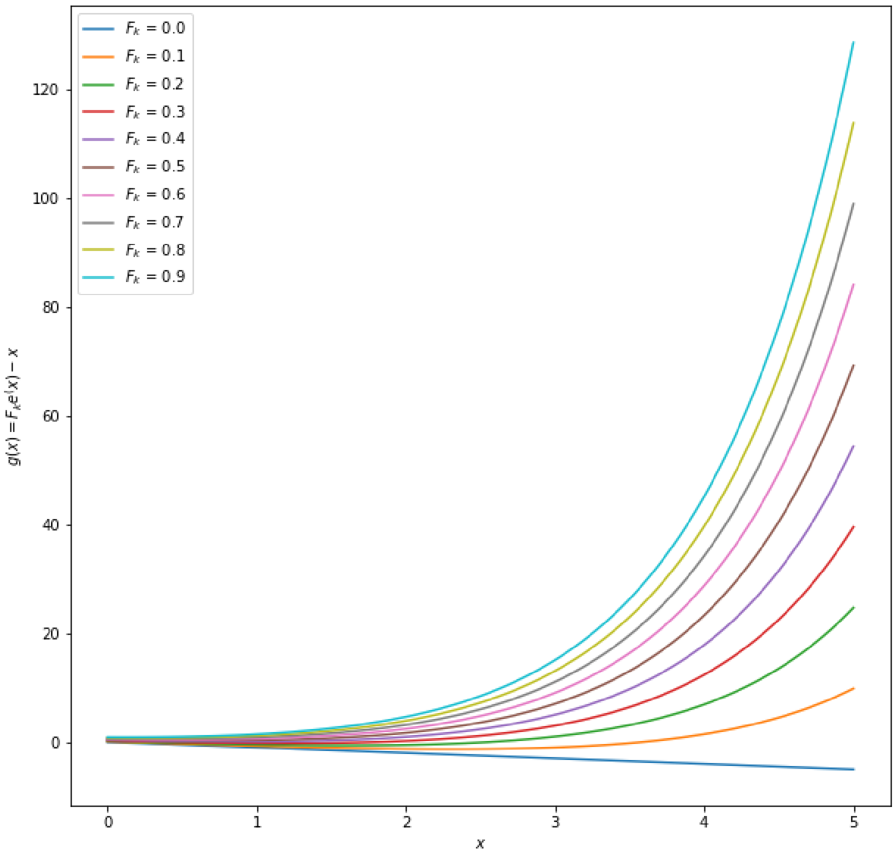

Figure 7: Graph showing the evolution of $g(x)$ for several F_k value.

Note that we are interested in the equilibria (meaning such that $g(x_{eq}) = 0$) that are positives.

A necessary condition is to have the minimum of the function below 0. To get this minimum, we compute $g'(\theta_{min}) = 0$ which yields $F_k e^{\theta_{min}} = 1$ which means : $\theta_{min} = -ln(F_k)$. First, θ has to be positive, thus : $F_k < 1$. Then, $g(\theta_{min}) = 1 - \theta_{min} = 1 + ln(F_k) < 0$ yields : $F_k < e^{-1}$.

For $F_k = e^{-1}$, we are on the limit of the equilibrium. Increasing only a little bit F_k yields to no more equilibrium.

0.4.2.2

We suppose that $F_k < e^{-1}$, such that there are two equilibria.

To study the stability, computing the derivative of g (since it is equal to $d_{t^2}\theta$) in the given point is enough. If it is negative, then it a stable equilibrium. If not, then the equilibrium is unstable.

The figure 7 shows that the first position is stable while the second is unstable. In theory, we know that g strictly decreases before θ_{min} and thus the first time it crosses the x-axis is a stable position.

The figure 8 shows the graph with the equilibria of the system in the y-axis as a function of F_k.

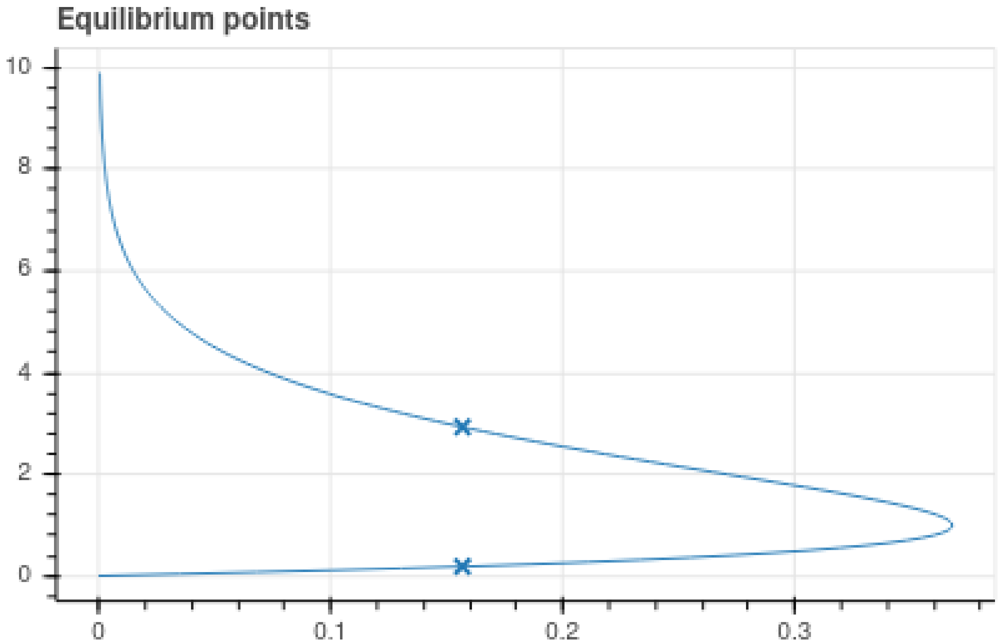

Figure 8: Graph showing the equilibra for various value of F_k (x-axis).

At the limit point for $F_k = e^{-1}$, we only have one equilibrium.

0.4.2.3

The figure 9 shows the result of the algorithm used to computed and go through the bifurcation point ($F_k = e^{-1}$, where there is a abrupt change in the dynamic of the system).

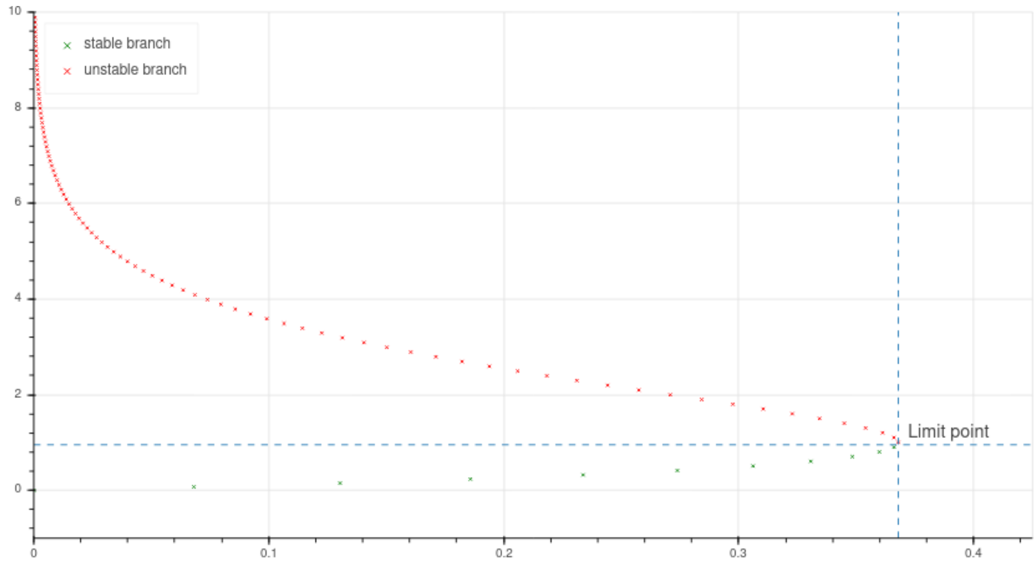

Figure 9: Graph showing the result of the continuation algorithm using pseudo arclength starting from $\theta = 0$ and $F_k = 0$.

The way this algorithm works and allow to go through the bifurcation point is solving for both λ (here F_k) and x (here θ). To do that we need another equation (two unknowns). We use :

$$\begin{cases} f(x, \lambda) = 0 \\ < (x, \lambda) - (\hat{x}_0, \lambda_0), \hat{x}_0 - x_0 >= 0 \end{cases} \tag{3}$$