PC6 : Dynamics around critical points, hyperbolicity, continuation and bifurcations

1 Introduction

The Petite Classe is divided into three parts. The first part is devoted to the study of invariant manifolds, which we will conduct on toy examples to illustrate the notions seen in the course. The second part is devoted to a full study of the Brusselator system of equations [6], but from a different perspective compared to what we have done previously in terms of integrating the system in time. We will focus in this PC on the study of branches of equilibria. The third part of the PC is devoted to the introduction of the notion of numerical continuation of equilibria along branches parametrized by one of several parameters. Without resorting to a numerical integration in time, which could be difficult near bifurcation points, or even impossible or misleading near limit points or unstable equilibria, we want to follow branches of equilibria and detect bifurcations, even unstable branches of solutions. We apply the simplest methods (natural continuation and pseudo-arclength continuation) to a series of system we have already investigated and follow several branches of critical points.

2 Some toy dynamical systems and course application

2.1 A vector field with a singular germ non $\mathcal{C}^0$-conjugated with the one related to its linerarization

We consider the following vector field:

$$X(x_1, x_2) = \left(-x_2 - x_1(x_1^2 + x_2^2), x_1 - x_2(x_1^2 + x_2^2)\right), \tag{1}$$

which admits a singular germ $(X, 0)$.

2.1.1 Show that the orbits of the linearized vector field X_1 are circles.

2.1.2 Going back to the nonlinear vector field X, and denoting by $\phi(t) = (x(t), y(t))$ a trajectory, study the evolution of $||\phi(t)||^2$.

2.1.3 Explain why the two germs $(X, 0)$ and $(X_1, 0)$ can not be $\mathcal{C}^0$-conjugated. Is this in contradiction with Hartman-Grobman's theorem?

2.2 Local / global stable/unstable manifold

We consider the following vector field:

$$X(x_1, x_2) = (x_2, 1 - x_1^2), \tag{2}$$

which admits a singular hyperbolic germ at $a = (-1, 0)$.

2.2.1 Show that $f(x_1, x_2) = \frac{x_2^2}{2} - x_1 + \frac{x_1^3}{3}$ is a first integral of X.

2.2.2 Represent the phase portrait of X.

2.2.3 Identify the unstable manifold at a and comment on the local and global structures.

2.3 Stable/unstable manifold

We consider the following vector field:

$$X(x_1, x_2) = (-x_1, \alpha x_2^3), \tag{3}$$

which admits a singular germ at $O = (0, 0)$.

2.3.1 Solve for the exact solution of the system.

2.3.2 Identify the stable and unstable manifold depending on the sign of α.

2.3.3 Depending on the sign of α, describes the dynamics on the eigenspace associated with the eigenvector of the zero eigenvalue of the Jacobian matrix.

3 Study of the ω-limit sets of the Brusselator model

The dynamics of the oscillating reaction discovered by Belousov and Zhabotinsky [2, 1], can be modeled through the so-called Brusselator model [6], which we have already integrated through many numerical methods:

$$\begin{cases} \mathrm{d}_t y_1 = a - (b+1) y_1 + y_1^2 y_2, \\ \mathrm{d}_t y_2 = b\, y_1 - y_1^2 y_2, \\ y_1(0) = y_1^0, \\ y_2(0) = y_2^0, \end{cases} \tag{4}$$

where a and b are two positive parameters. The purpose here is to focus the study on ω-limit sets, among which equilibria, and on their stability. The reader is encouraged to make the link with the previous system studied in the previous PCs.

3.1 Study of equilibria

3.1.1 Identify the critical points or equilibria of the dynamics (in terms of a and b).

3.1.2 Study the stability of such points. Identify the set of parameters for which the equilibria are hyperbolic.

3.1.3 What is the value of a for which the equilibrium point in not hyperbolic for $b = 3$. What happens in terms of the eigenvalues of the system at this point? Identify the set of parameters for which there is a change in stability of the system. What happens to the eigenvalues? Illustrate a typical behavior of the system on each side of the Hopf bifurcation point.

3.2 Proof of the existence of a limit Cycle? - Poincaré-Bendixon Theorem

We focus in this part on the case $b = 3$ and we consider the branch of points for which the equilibrium point is unstable

3.2.1 What range of a parameters are we considering here?

3.2.2 What theorem from the course could you maybe use here to prove the existence of a limit cycle? What would be the main difficulty?

3.2.3 We encountered what seems to be limit cycles in many other examples in this course, including the Rosenzweig-MacArthur system from population dynamics (see PC 2)

$$\begin{cases} u_1' = u_1 \left(1 - \dfrac{u_1}{\gamma}\right) - \dfrac{u_1 u_2}{1 + u_1} \\ u_2' = u_2 \Big(\dfrac{u_1}{1 + u_1} - \alpha\Big) \end{cases} \tag{5}$$

with $\beta = 1$, $\gamma = 2.5$ and $\alpha < \alpha_{cr} \approx 0.43$. A phase portrait of this system for $\alpha = 0.4$, with one trajectory, is given in Figure 1. Based on this picture, how could you prove the existence of a limit cycle?

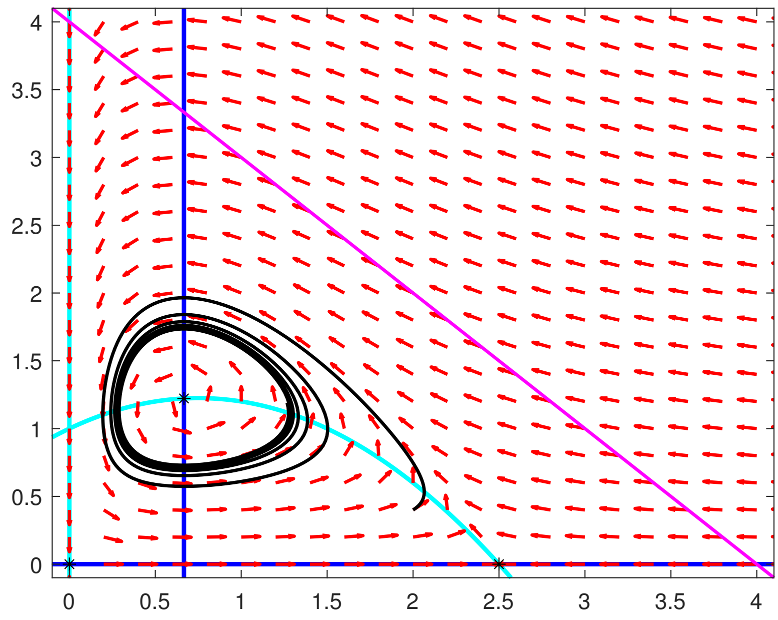

Figure 1: Phase portrait of the Rosenzweig-MacArthur system (5), for $\alpha = 0.4$, $\beta = 1$ and $\gamma = 2.5$. The cyan and blue curves represent the nullclines $u_1' = 0$ and $u_2' = 0$ respectively. The magenta line is the line $u_1 + u_2 = 4$, and the red arrows represent the direction of the vector field on a grid in phase space.

4 Continuation of equilibria - limit points / Hopf / pitchfork

In this section, we envision a completely different approach in order to study the equilibria branches of a system. Starting from an equilibrium point of the system, we will investigate a numerical method in order to follow a branch of such equilibria, without resorting to the simulation of the time dynamics of the system. The reason for that is two-fold. First we want to focus on the behavior of such branches without having to conduct several costly simulations; second, we want to be able to visit some branches, which are unstable and for which the time simulation would not lead to any piece of information. We also want to follow such branches and identify bifurcation point, as well as potentially follow new branches issued from the bifurcation point as in the pitchfork bifurcation.

The description of the continuation method used in the notebook is provided in the appendix and is inspired from [5, 4] as well as used at SANDIA at the beginning of the years 2000[1]. The idea of this section is to illustrate the principles of pseudo-arclength continuation in order to identify critical point branches on some very simple systems, knowing perfectly that there are some more involved algorithms in order to treat more complex cases, which are generalization of the proposed algorithms in this PC.

4.1 Brusselator model - Hopf bifurcation - change of stability

For the Brusselator model, we will propose a continuation of the equilibria branch using a natural continuation.

4.1.1 Describe the two algorithms based on the appendix for the continuation of equilibria of the system we can use. Considering we have fixed $b = 3$, identify the branch of equilibria and recall the stability analysis of such points, and the location of the Hopf bifurcation.

4.1.2 Use the natural algorithm in order to reproduce the branch of equilibria, and check that the output corresponds to the theoretical values.

4.1.3 Comment on the notion of accuracy for such an algorithm.

[1]LOCA: Library Of Continuation Algorithms – http://www.cs.sandia.gov/loca/

4.2 Thermal explosion - limit point - unstable branch

We focus in this subsection on the thermal explosion equation with heat loses. Compared to the study we have conducted in PC1, we rather use another form of the system:

$$\mathrm{d}_t\theta = \mathrm{F_k}\,\exp(\theta) - \theta, \tag{6}$$

where the Frank-Kamenetskii parameter, $\mathrm{F_k} \geq 0$, is the inverse of the α_0 parameter we used in PC 1.

4.2.1 Describe the positive equilibria of the system, depending on the value of $\mathrm{F_k}$. What happens for $\mathrm{F_k} = 1/e$? We call such a point a limit point.

4.2.2 Study the stability of the equilibria on the two branches. Propose a figure with the graph of equilibria in θ as a function of $\mathrm{F_k}$. What happens at that limit point in terms of branches of equilibria?

4.2.3 Use the continuation algorithm starting from the point $(\theta = 0, \mathrm{F_k} = 0)$ in order to represent the continuation of the equilibria and explain how we can go "through" the limit point using pseudo arclength. Is such a point a bifurcation point?

4.3 Bead on a hoop configuration - pitchfork bifurcation - new branches of equilibria

A circular wire hoop rotates with constant angular velocity ω about a vertical diameter. A small bead moves, with or without friction, along the hoop, as presented in Figure 2, where the dynamics of the bead is described through the θ angle. The equation of motion, using the standard notation in classical mechanics, can be shown to be [7]:

$$\ddot{\theta} = -\omega_c^2 \sin\theta + \omega^2 \sin\theta\, \cos\theta - \alpha\theta \tag{7}$$

with $\omega_c = \sqrt{g/R}$, where the gravity acceleration is denoted by g and the radius of the hoop is denoted R. The coefficient α is related to the friction in the system and can be idealized to be zero in the frictionless configuration.

Let $y_1 = \theta$ and $y_2 = \dot{\theta}$, its time derivative. Then, we can switch to a first order system of differential equations:

$$\begin{cases} \mathrm{d}_t y_1 = y_2 \\ \mathrm{d}_t y_2 = (\omega^2 \cos y_1 - \omega_c^2)\sin y_1 - \alpha\, y_2 \end{cases} \tag{8}$$

4.3.1 Depending on the rotation velocity of the hoop, identify the number of the equilibria in the system and give their analytic expression in the frictionless configuration. Do we have to work in the (y_1, y_2) plane, or is it sufficient to work with y_1 alone? Explain.

4.3.2 Explain what is the influence of friction on the equilibria.

4.3.3 Conduct a stability analysis on the various branches of equilibria in both the frictionless and with friction configurations. Are the equilibria hyperbolic ones?

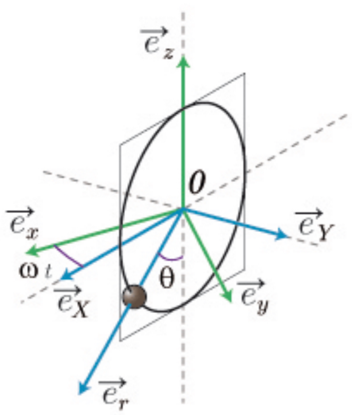

Figure 2: Various referentials for the study of the bead on a rotating hoop.

4.3.4 Explain how the case with friction is the right framework in order to study the pitchfork bifurcation. When does this bifurcation take place? Is the equilibrium point hyperbolic at that location?

4.3.5 Use the continuation tool provided in the notebook in order to get the various branches of equilibria. Explain how one could detect the location of the bifurcation point, and switch from one branch to another.

References

[1] I.R. Epstein and J.A. Pojman. *An Introduction to Nonlinear Chemical Dynamics.* Oxford University Press, 1998. Oscillations, Waves, Patterns and Chaos.

[2] P. Gray and S.K. Scott. *Chemical Oscillations and Instabilities.* Oxford Univ. Press, 1994.

[3] E. Hairer and G. Wanner. *Solving ordinary differential equations. II - Stiff and differential-algebraic problems.* Springer-Verlag, Berlin, 1991.

[4] S. Karkar. *Numerical Methods for nonlinear dynamical systems. Application to self-sustained oscillations in musical instruments.* PhD thesis, Aix-Marseille Université, 2012.

[5] H. B. Keller. Numerical solution of bifurcation and nonlinear eigenvalue problems. In *Applications of bifurcation theory (Proc. Advanced Sem., Univ. Wisconsin, Madison, Wis., 1976)*, pages 359–384. Publ. Math. Res. Center, No. 38. Academic Press, New York, 1977.

[6] R. Lefever and G. Nicolis. Chemical instabilities and sustained oscillations. *J. Theoretical Biology*, 30(2):267–284, 1971.

[7] L. A. Raviola, M. E. Véliz, H.D. Salomone, N. A. Olivieri, and E.E. Rodríguez. The bead on a rotating hoop revisited: an unexpected resonance. *European Journal of Physics*, 38:015005, 2016.

[8] B. van der Pol. On “relaxation-oscillations”. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 7:978–992, 1926.

[9] B. van der Pol. *Selected scientific papers.* 2 vols. Edited by H. Bremmer and C. J. Bouwkamp, with an introduction bt H. B. G. Gasimir. North-Holland Publishing Co., Amsterdam, 1960.