Model Order Reduction For Electrical Circuits

With Focus On Balanced Model Reduction

Shakir Sofi, Razan Debo

Skoltech

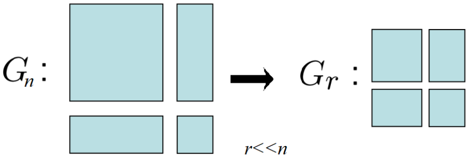
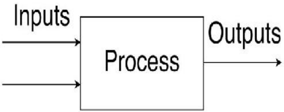


Table of Contents

1. Problem Statement
2. Methods
3. Balanced Truncation
4. Results
5. Conclusion

What is Model Order Reduction ?

Model order reduction (MOR) is the branch of applied maths, and system theory, in which we study the properties of dynamical systems (i.g; any system) in application for reducing the dimensionality, complexity, while preserving the (to the maximum possible extent) input-output (i.g, cause-effect) behavior



Goals

Replicate input-output behavior of large-scale system by smaller system subject to:

- Good approximation (mimic original system as best as possible)
- Preserve system properties, like stability, passivity, etc.
- Small approximation error and/or global error bound.
- Numerically stable and efficient procedures.

Importance and Applications of MOR

- Reduce computational costs (Overcoming curse of dimensionality)
- Requires less storage
- Fast simulations (we can simulate very large systems in little amount of time)
- Reduced models enable rapid prediction, inversion, design, and uncertainty quantification.
- Extracting the essence of complex problems to make them faster and easier to solve.

How To Measure Quality of MOR

- Error should be very small.
- Reduced systems dynamics should converge to dynamics of original system.
- MOR algorithm should preserve system properties like, stability, passivity, etc.

Full model in general: $G : \begin{cases} \dot{x}(t) = f(x(t), u(t)), & x(t) \in \mathbb{R}^n, u \in \mathcal{U} \\ y(t) = g(x(t), u(t)) \end{cases}$

Reduced order model: $G_r : \begin{cases} \dot{z}(t) = f_r(z(t), u(t)), & z(t) \in \mathbb{R}^r, u \in \mathcal{U} \quad r << n \\ y_r(t) = g_r(z(t), u(t)) \end{cases}$

Such that $\|y - y_r\|_2 \leq \text{bound } (r) \cdot \|u\|, \quad \forall u \in \mathcal{U}$

Structure of LTI systems

Full order system:

$$\sum : \begin{cases} \dot{x} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x} \in \mathbb{R}^n \\ \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

Reduced order system:

$$\sum_r : \begin{cases} \dot{\tilde{\mathbf{x}}} = \mathbf{A_r}\tilde{\mathbf{x}} + \mathbf{B_r}\mathbf{u}, \quad \mathbf{z} \in \mathbb{R}^r, \quad r << n \\ \mathbf{y_r} = \mathbf{C_r}\tilde{\mathbf{x}} + \mathbf{D_r}\mathbf{u} \end{cases}$$

where $\dot{\boldsymbol{x}}$ is $\frac{d\mathbf{x}}{dt}$, $\dot{\tilde{\mathbf{x}}}$ is $\frac{d\tilde{\mathbf{x}}}{dt}$, $\mathbf{x}$ being full state vector and $\tilde{\mathbf{x}}$ is reduced state vector $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are full system matrices and $\mathbf{A_r}, \mathbf{B_r}, \mathbf{C_r}, \mathbf{D_r}$ are matrices of reduced system, $\mathbf{u}$ is input vector and $\mathbf{y}$ the output vector.

Methods of MOR: The Big Picture

Three broad categories of MOR Methods:
(a) SVD-based methods
(b) Krylov-based methods
(c) Combined SVD and Krylov methods

Table 1. Overview of approximation methods.

Approximation methods for dynamical systems		
SVD		**Krylov**
Nonlinear Systems	**Linear Systems**	
• POD methods • Empirical grammians	• $\boxed{\text{Balanced truncation}}$ • Hankel approximation OUR FOCUS HERE	• Realization • Interpolation • Lanczos • Arnoldi
SVD-Krylov		

Introduction to MOR methods cont...

- *SVD-based approximation methods* have their roots in the Singular Value Decomposition and the resulting solution of the approximation of matrices by means of matrices of lower rank, which are optimal in the 2-norm (or more generally, in unitarily invariant norms)
- *Krylov-subspace based approximation methods* have roots in Moment Matching (of Impulse responses/ transfer function) i.e, $\mathbf{H}(\mathbf{s})$) of the considered system. For LTI systems $\mathbf{H}(\mathbf{s}) = \mathbf{C}(\mathbf{sI} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ then approximates this transfer function by $\hat{\mathbf{H}}(\mathbf{s})$ by using Krylov-subspace.

For details refer to : Approximation of Large-Scale Dynamical Systems, SIAM, Athanasios C. Antoulas, https://doi.org/10.1137/1.9780898718713

Some Prerequisite's

Controllability and Observability :
A system is said to be ***controllable*** at time t_o if it is possible by means of an unconstrained control/input vector to transfer the system from any initial state $\mathbf{x}(\mathbf{t_o})$ to any other state in a finite interval of time.
A system is said to be ***observable*** at time t_o if,with the system in state $\mathbf{x}(\mathbf{t_o})$,it is possible to determine this state from the observation of the output over a finite time interval.

Controllable LTI system if :
The $n \times n$

$$\mathbf{W}_c(t) = \int_0^t e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^* e^{\mathbf{A}^*\tau} d\tau, \quad \text{is not singular for any } t \geq 0$$

Observable LTI system if :
The $n \times n$

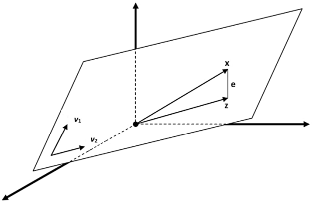
$$\mathbf{W}_o(t) = \int_0^t e^{\mathbf{A}^*\tau}\mathbf{C}^*\mathbf{C} e^{\mathbf{A}\tau} d\tau, \quad \text{is not singular for any } t \geq 0$$

They are the solutions of **Lyapnov equations** for LTI system defined above

Why MOR Works ?

- Fact is, large scale dynamical systems are poorly controllable and observable, that means the cancelation of some variables is possible.
- Important dynamics of underlying system is often restricted to a smaller subspace.
- MOR methods are designed to find the dominant subspaces, so we Project the original system onto it.

$x \approx \Psi z$



These MOR methods will give different strategies to find matrix Ψ

Balanced Model Truncation

Can only singular values or eigen value decide the truncation? lets us check the example, consider the following simplified model:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 10^{-10} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 10^{-10} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

So, the choice the state $z = x_2$, is intuitive (as it is more lightly damped) but this is bad. But choice $z = x_1$ is better, since it is better controlable and observable.

So, we cannot decide simply by looking at matrix A, we must take account of matrix B and C.

Hint: What if we balance the controllability and observability Gramians of a state(*aka Balancing*) then simply truncate the system at **r**.

Balancing Transformation

Let $\mathbf{x} = \mathbf{T}\mathbf{z}$, such that this transformation that makes the controllability and observability Gramians equal and diagonal:

$$\hat{\mathbf{W}}_c = \hat{\mathbf{W}}_o = \mathbf{\Sigma}$$

The transformed (in z coordinates) product of Gramian's will be given by:

$$\hat{\mathbf{W}}_c \hat{\mathbf{W}}_o = \mathbf{T}^{-1} \mathbf{W}_c \mathbf{W}_o \mathbf{T}$$

Plugging in the desired $\hat{\mathbf{W}}_c = \hat{\mathbf{W}}_o = \mathbf{\Sigma}$ yields

$$\mathbf{T}^{-1} \mathbf{W}_c \mathbf{W}_o \mathbf{T} = \mathbf{\Sigma}^2 \quad \Longrightarrow \quad \mathbf{W}_c \mathbf{W}_o \mathbf{T} = \mathbf{T}\mathbf{\Sigma}^2 \quad \text{(spectral decomposition)}$$

But, there can be many such transformation that makes $\hat{\mathbf{W}}_c \hat{\mathbf{W}}_o = \mathbf{\Sigma^2}$, so we need to scale them, let $\mathbf{T}_u$ have unscaled eigenvectors, and the scaled version $\mathbf{T}$ have unit normed columns

$$T_u^{-1} \mathbf{W}_c \mathbf{T}_u^{-*} = \mathbf{\Sigma}_c$$
$$\mathbf{T}_u^{*} \mathbf{W}_o \mathbf{T}_u = \mathbf{\Sigma}_o$$

The scaling that exactly balances these Gramians is then given by $\mathbf{\Sigma}_s = \mathbf{\Sigma}_c^{1/4} \mathbf{\Sigma}_o^{-1/4}$ the exact balancing transformation is given by $\mathbf{T} = \mathbf{T}_u \mathbf{\Sigma}_s$

Now Truncating the Balanced system

Given the new coordinates $\mathbf{z} = \mathbf{T}^{-1}\mathbf{x} \in \mathbb{R}^n$, it is possible to define a reduced-order state

$\tilde{\mathbf{x}} \in \mathbb{R}^r$, as

$$\mathbf{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_r \\ z_{r+1} \\ \vdots \\ z_n \end{bmatrix}, \quad \tilde{\mathbf{x}}$$

in terms of the first r most controllable and observable directions. If we partition the balancing transformation $\mathbf{T}$ and inverse transformation $\mathbf{S} = \mathbf{T}^{-1}$ into the first r modes to be retained and the last $n - r$ modes to be truncated,

$$\mathbf{T} = \begin{bmatrix} \Psi & \mathbf{T}_t \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} \Phi^* \\ \mathbf{S}_t \end{bmatrix}$$

Cont...

Then it is possible to rewrite the transformed dynamics in transformed system as:

$$\frac{d}{dt}\left[\begin{array}{c}\tilde{\mathbf{x}}\\ \hline \mathbf{z}_t\end{array}\right] = \left[\begin{array}{c|c}\mathbf{\Phi}^*\mathbf{A}\mathbf{\Psi} & \mathbf{\Phi}^*\mathbf{A}\mathbf{T}_t\\ \hline \mathbf{S}_t\mathbf{A}\mathbf{\Psi} & \mathbf{S}_t\mathbf{A}\mathbf{T}_t\end{array}\right]\left[\begin{array}{c}\tilde{\mathbf{x}}\\ \hline \mathbf{z}_t\end{array}\right] + \left[\begin{array}{c}\mathbf{\Phi}^*\mathbf{B}\\ \hline \mathbf{S}_t\mathbf{B}\end{array}\right]\mathbf{u}$$

$$\mathbf{y} = \left[\begin{array}{c|c}\mathbf{C}\mathbf{\Psi} & \mathbf{C}\mathbf{T}_t\end{array}\right]\left[\begin{array}{c}\tilde{\mathbf{x}}\\ \hline \mathbf{z}_t\end{array}\right] + \mathbf{D}\mathbf{u}.$$

In balanced truncation, the state z_t is simply truncated (i.e., discarded and set equal to zero), and only the $\tilde{\mathbf{x}}$ equations remain:

$$\frac{d}{dt}\tilde{\mathbf{x}} = \Phi^*\mathbf{A}\Psi\tilde{\mathbf{x}} + \Phi^*\mathbf{B}\mathbf{u}$$
$$\mathbf{y} = \mathbf{C}\Psi\tilde{\mathbf{x}} + \mathbf{D}\mathbf{u}$$

Computational cost is reduced from $\mathbf{O}(n^3)$ to $\mathbf{O}(n.r^2)$

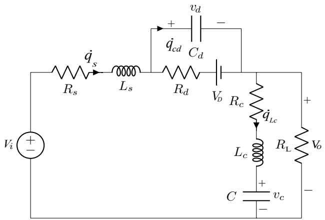
The upper and lower bounds on the error of a given order truncation:

Upper bound: $\|\mathbf{G} - \mathbf{G}_r\|_\infty \leq 2\sum_{j=r+1}^{n} \sigma_j$
Lower bound: $\|\mathbf{G} - \mathbf{G}_r\|_\infty > \sigma_{r+1}$

where σ_j is the j th diagonal entry of the balanced Gramians.

Illustrative Example

Let us consider a circuit as shown below



Using Euler-Lagrangian equations to model this circuit, that is:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}(z,\dot{z},t)}{\partial \dot{z}}\right) = \frac{\partial \mathcal{L}(z,\dot{z},t)}{\partial z} - \frac{\partial \mathcal{D}(\dot{z})}{\partial \dot{z}}$$

$$\mathcal{L}(z,\dot{z},t) = \mathcal{T}(z,\dot{z},t) - \mathcal{V}(z,\dot{z},t)$$

Lagrangian ($\mathcal{L}$) is the difference between the kinetic energy ($\mathcal{T}$) and potential energy ($\mathcal{V}$) of the system. Rayleigh dissipation ($\mathcal{D}$)=$\frac{1}{2}$ power dissipation.

Cont...

$$\mathcal{T} = \frac{1}{2}L_s\dot{q}_s^{\,2} + \frac{1}{2}L_c\dot{q}_{Lc}^2$$

$$\mathcal{V} = \frac{1}{2C_d}q_{cd}^2 + \frac{1}{2C}q_{Lc}^2 - Vi \times q_s + V_D(q_s - q_{cd})$$

$$\mathcal{L} = \frac{1}{2}L_s\dot{q}_s^{\,2} + \frac{1}{2}L_c\dot{q}_{Lc}^2 - \frac{1}{2C_d}q_{cd}^2 - \frac{1}{2C}q_{Lc}^2 + Vi \times q_s - V_D(q_s - q_{cd})$$

$$\mathcal{D} = \frac{1}{2}R_s\left(\dot{q}_s\right)^2 + \frac{1}{2}R_c\left(\dot{q}_{Lc}\right)^2 + \frac{1}{2}R_d\left(\dot{q}_s - \dot{q}_{cd}\right)^2 + \frac{1}{2}R_L\left(\dot{q}_s - \dot{q}_{Lc}\right)^2$$

Putting these equations in Euler-lagrangian equations, and let:

$$\mathbf{x} = (i, v_d, i_{Lc}, v_c)^T = \left(\dot{q}_s, \frac{q_{cd}}{C_d}, \dot{q}_{Lc}, \frac{q_{Lc}}{C}\right)^T$$

Cont...

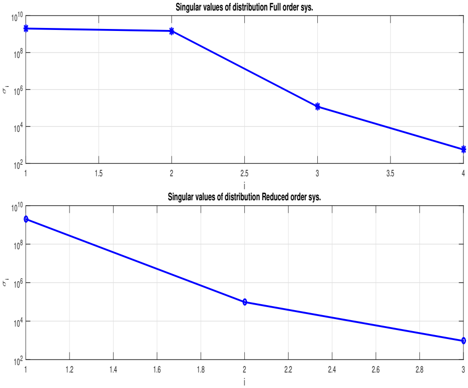
The final state-space LTI model is

$$\mathbf{A} = \begin{bmatrix} \frac{(-R_s - R_L)}{L_s} & \frac{-1}{L_s} & \frac{R_c}{L_s} & 0 \\ \frac{1}{C_d} & \frac{1}{R_d C_d} & 0 & 0 \\ \frac{R_L}{L_c} & 0 & \frac{(-R_L - R_c)}{L_c} & \frac{-1}{L_c} \\ 0 & 0 & \frac{1}{C} & 0 \end{bmatrix} ; \mathbf{x} = \begin{bmatrix} i \\ v_d \\ i_{Lc} \\ v_c \end{bmatrix}$$

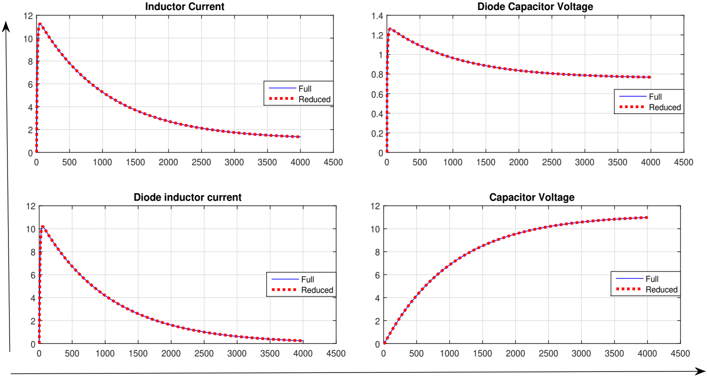
$$\mathbf{B} = \begin{bmatrix} \frac{1}{L_s} & 0 \\ 0 & \frac{1}{R_d C_d} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} ; u = \begin{bmatrix} V_i \\ V_D \end{bmatrix} ; \mathbf{C} = [I]_{4\times 4} ; \mathbf{D} = [0]_{4\times 2}$$

Model truncation to above circuit

With the parameters $R_s = 0.01\Omega$, $C = 1e^{-3}F$, $L_c = 10e^{-6}H$, $R_L = 10\Omega$; $L_s = 10e^{-6}H$, $R_d = 0.05\Omega$, $C_d = 10e^{-9}F$, $R_c = 1\Omega$, $R_L = 10\Omega$, $L_c = 10e^{-9}H$
After doing the *Balanced Model Truncation* this system reduced to third order $(r = 3)$. The singular value of full and reduced order is shown below



Comapring the dynamics



And for Full order system simulation took **4.47165 sec**, while Reduced order took only **3.931263 sec**

Norm Error of approximation is: $\|y - y_r\|_2 \approx 0.067$ **(..seems good..)**

Conclusive Remarks

- MOR for stiff electrical systems suits best to achieve computationally efficient simulation and perform analysis.
- MOR for electrical circuits allow us to design accurate controllers as most of them are based on average behaviour of underlying system.
- Further directions are to use MOR for Non linear electrical circuits, also to reduce circuitry in MEMS systems.
- And many more...

Nowdays, we are sucessfully applying MOR for switched-electrical systems, in which the dynamics are completely dependent on the switches in the electrical systems like; switched power electronic converters, etc,

References

- APPROXIMATION OF LARGE-SCALE DYNAMICAL SYSTEMS: AN OVERVIEW, Athanasios C. ANTOULAS, Dan C. SORENSEN
- Model Reduction for Circuit Simulation, Peter Benner Michael Hinze.
- Numerical Linear Algebra and Applications, Biswa Nath Datta
- Model Reduction Based on Spectral Projection Methods, Peter Benner and Enrique S. Quintana-Ort
- Model Order Reduction in Power Electronics: Issues and Perspectives, S.A Nahvi, Mohammad Abid Bazaz, Hadhiq Khan
- Model Reduction and Approximation Theory and Algorithms, PETER BENNER, KAREN WILLCOX
- MORLAB-Model Order Reduction LABoratory, Max-Planck Institute
- Organization of the SLICOT Library, Benchmarks and examples Model Reduction.

Thank You