# Principal component analysis as a game with Nash Equilibrium


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# PLAN

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# PROBLEM STATEMENT

The **principle components (PC)** of data are the vectors that align with the directions of maximum variance.

Principal components analysis (PCA) has **two main purposes**:

- Interpretable features
- Data compression

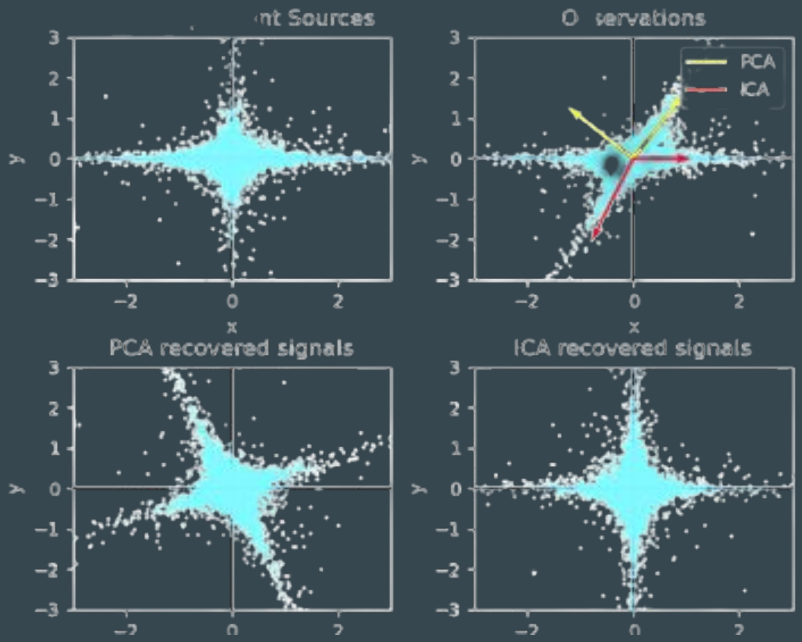
**PCA solution** of dataset X:

- Eigenvectors of $X^TX$ or right singular vectors of X.

# RECENT METHODS

$X - n\, x\, d$

- Full SVD
- Power method
- Oja’s algorithm



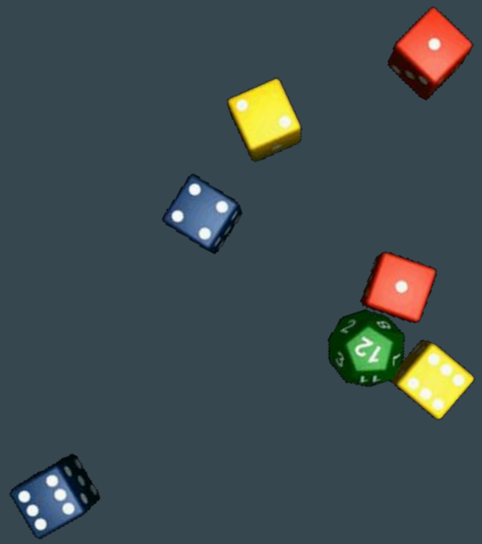
# PCA AS EIGENGAME

Considering **PCA as a game**, where

- **Player** is an eigenvector

- **Player's objective** is to maximize their own local utility function

- **Our purpose** is to reach Nash equilibrium



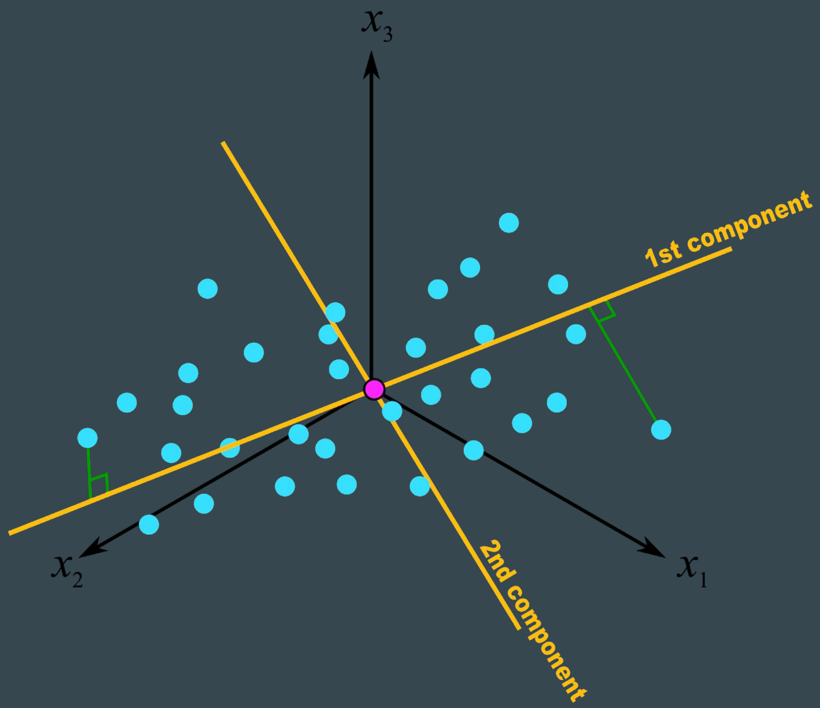
# PCA AS EIGENGAME

$$X^T X = M$$

$$V^T M V = V^T V \Lambda = \Lambda$$

$$R(\hat{V}) = \hat{V}^T M \hat{V}$$

$$\max_{\hat{V}^T \hat{V} = I} \sum_i R_{ii} - \sum_{i \neq j} R_{ij}^2$$



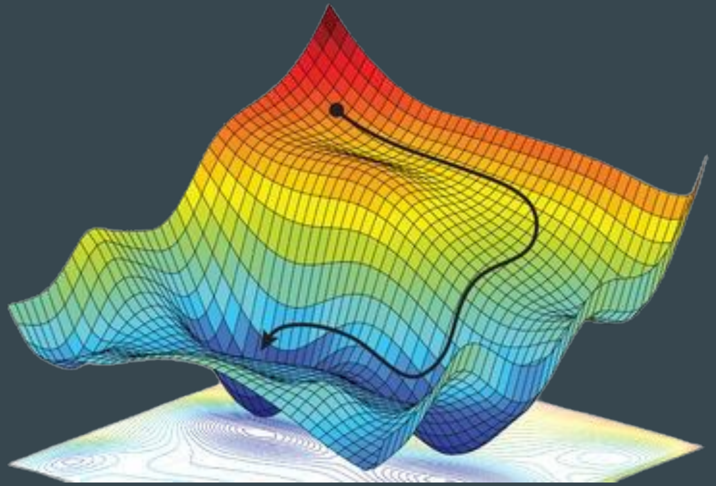
# PCA AS EIGENGAME

**Utility functions:**

$$\max_{\hat{v}_1^T \hat{v}_1 = 1} \langle \hat{v}_1, M \hat{v}_1 \rangle$$

$$\max_{\hat{v}_2^T \hat{v}_2 = 1, \hat{v}_1^T \hat{v}_2 = 0} \langle \hat{v}_2, M \hat{v}_2 \rangle - \frac{\langle \hat{v}_2, M \hat{v}_1 \rangle^2}{\langle \hat{v}_2, M \hat{v}_2 \rangle}$$

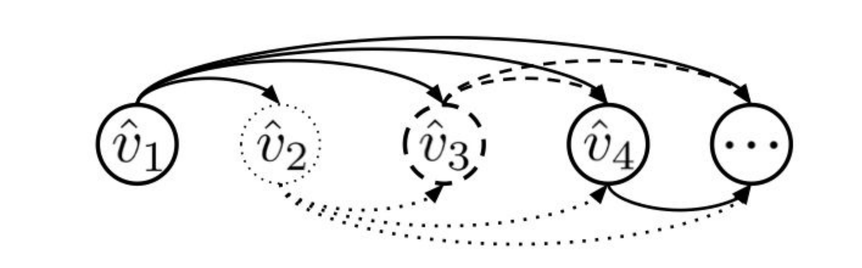
$$u_i(\hat{v}_i | \hat{v}_{j<i}) = \hat{v}_i^T M \hat{v}_i - \sum_{j<i} \frac{\left(\hat{v}_i{}^T M \hat{v}_j\right)^2}{\hat{v}_j{}^T M \hat{v}_j} = ||X \hat{v}_i||^2 - \sum_{j<i} \frac{\langle X \hat{v}_i, X \hat{v}_j \rangle^2}{\langle X \hat{v}_j, X \hat{v}_j \rangle}$$



# PCA AS EIGENGAME

**<u>Nash equilibrium</u>** - strategy profile when no player can do better by unilaterally changing their strategy.

**<u>Theorem:</u>** Assume that the *top-k* eigenvalues of $X^T X$ are distinct. Then the *top-k* eigenvectors form the unique strict-Nash equilibrium of the proposed game.



# ALGORITHM EIGENGAME

**Utility gradient**

$$\nabla_{\hat{v}_i} u_i(\hat{v}_i | \hat{v}_{j<i}) = 2M \left[ \hat{v}_i - \sum_{j<i} \frac{\hat{v}_i^T M \hat{v}_j}{\hat{v}_j^T M \hat{v}_j} \hat{v}_j \right] =$$

$$2X^T \left[ X\hat{v}_i - \sum_{j<i} \frac{\langle X\hat{v}_i, X\hat{v}_j \rangle}{\langle X\hat{v}_j, X\hat{v}_j \rangle} X\hat{v}_j \right]$$

**Algorithm 1** EigenGame$^R$-Sequential

Given: matrix $X \in \mathbb{R}^{n \times d}$, maximum error tolerance $\rho_i$, initial vector $\hat{v}_i^0 \in \mathcal{S}^{d-1}$, learned approximate parents $\hat{v}_{j<i}$, and step size $\alpha$.

$\hat{v}_i \leftarrow \hat{v}_i^0$

$t_i = \lceil \frac{5}{4} \min(||\nabla_{\hat{v}_i^0} u_i||/2, \rho_i)^{-2} \rceil$

**for** $t = 1 : t_i$ **do**

  `rewards` $\leftarrow X\hat{v}_i$

  `penalties` $\leftarrow \sum_{j<i} \frac{\langle X\hat{v}_i, X\hat{v}_j \rangle}{\langle X\hat{v}_j, X\hat{v}_j \rangle} X\hat{v}_j$

  $\nabla_{\hat{v}_i} \leftarrow 2X^\top \left[ \texttt{rewards} - \texttt{penalties} \right]$

  $\nabla^R_{\hat{v}_i} \leftarrow \nabla_{\hat{v}_i} - \langle \nabla_{\hat{v}_i}, \hat{v}_i \rangle \hat{v}_i$

  $\hat{v}_i' \leftarrow \hat{v}_i + \alpha \nabla^R_{\hat{v}_i}$

  $\hat{v}_i \leftarrow \frac{\hat{v}_i'}{||\hat{v}_i'||}$

**end for**

return $\hat{v}_i$

# DATA

**Synthetic data:**

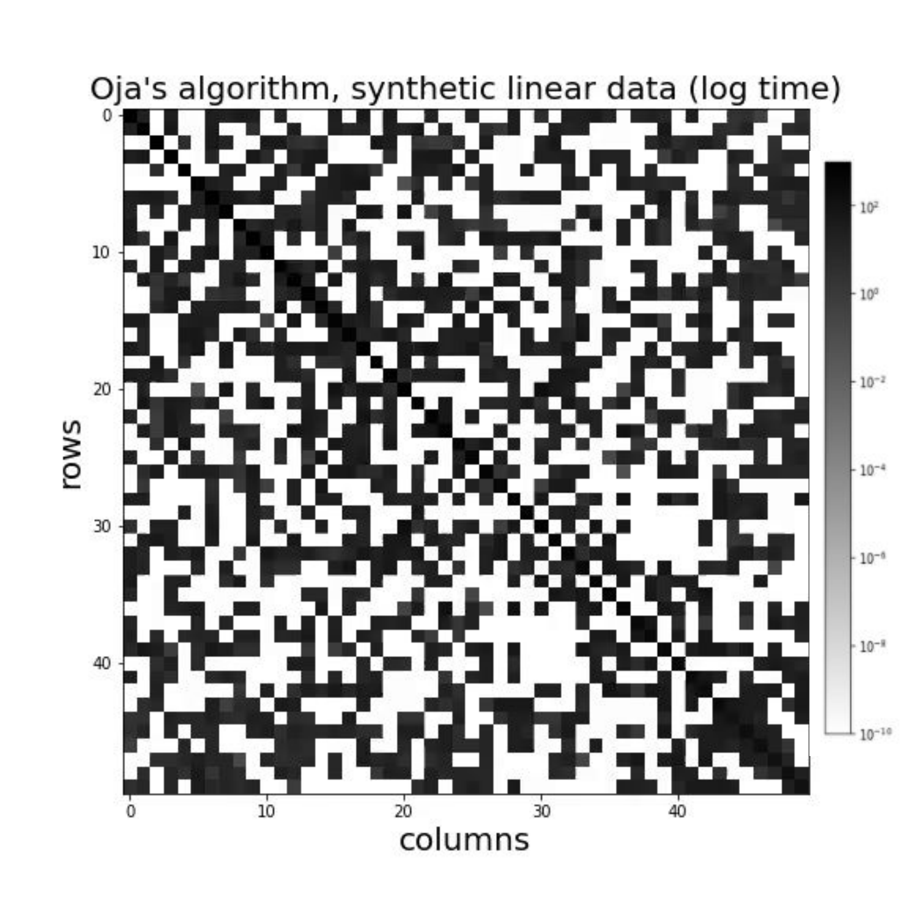
- *V* is initialized randomly so $M \in \mathbb{R}^{50\times 50}$ is constructed as a diagonal matrix without loss of generality
- Linear spectrum ranges from 1 to 1000 with equal spacing
- Exponential spectrum ranges from $10^3$ to $10^0$ with equal spacing on the exponents

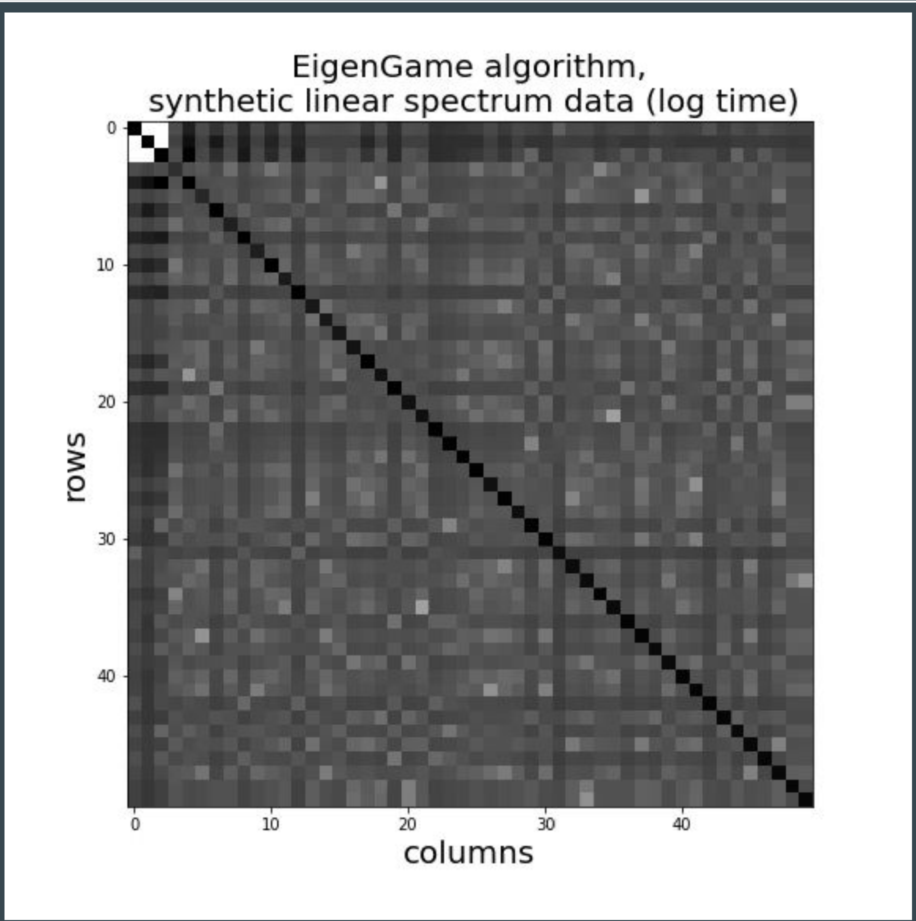
**MNIST dataset**

**IRIS dataset**

**Digits**

# EXPERIMENTS



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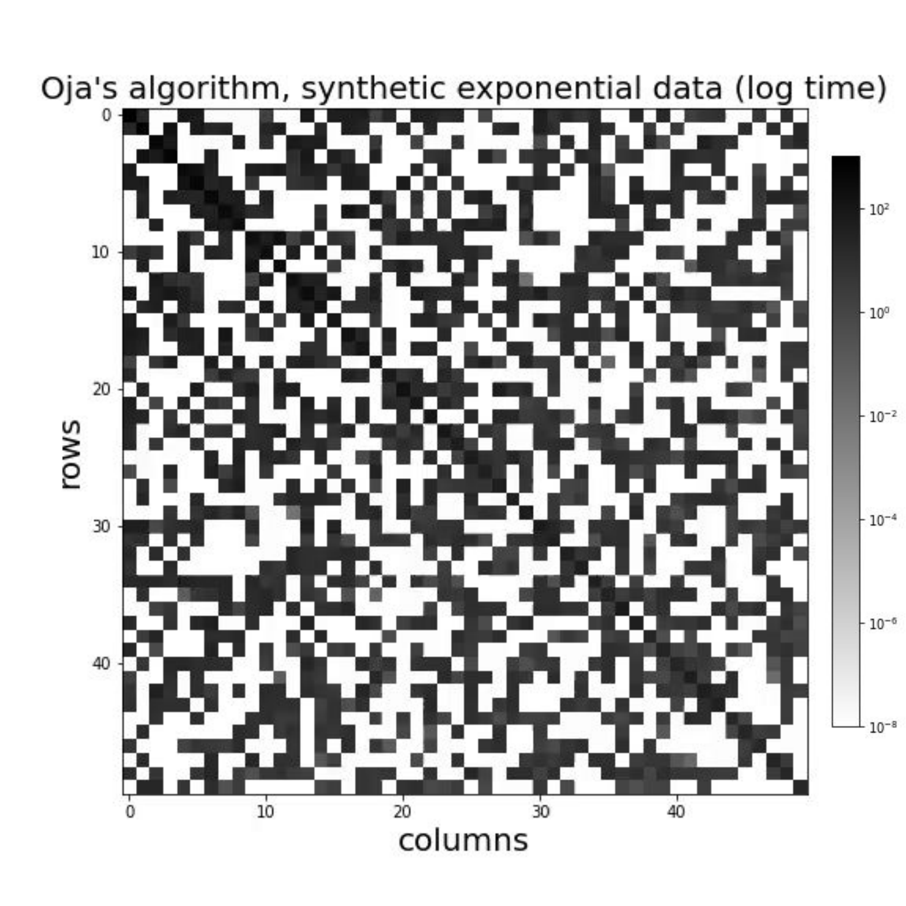


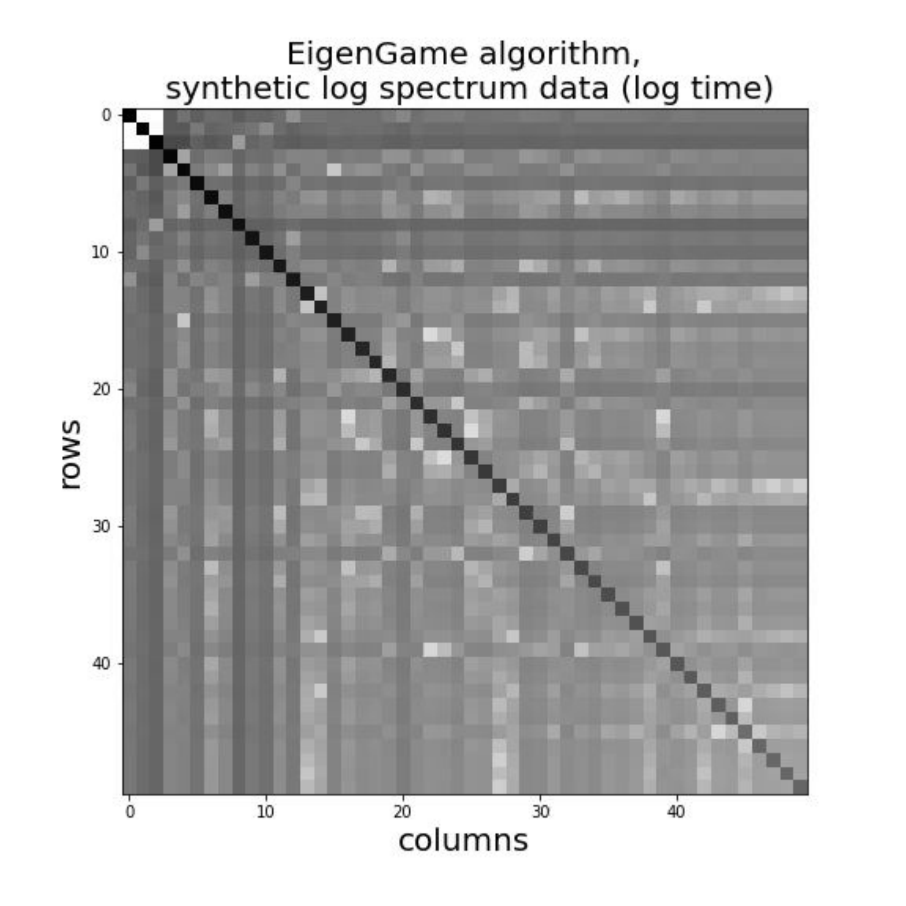

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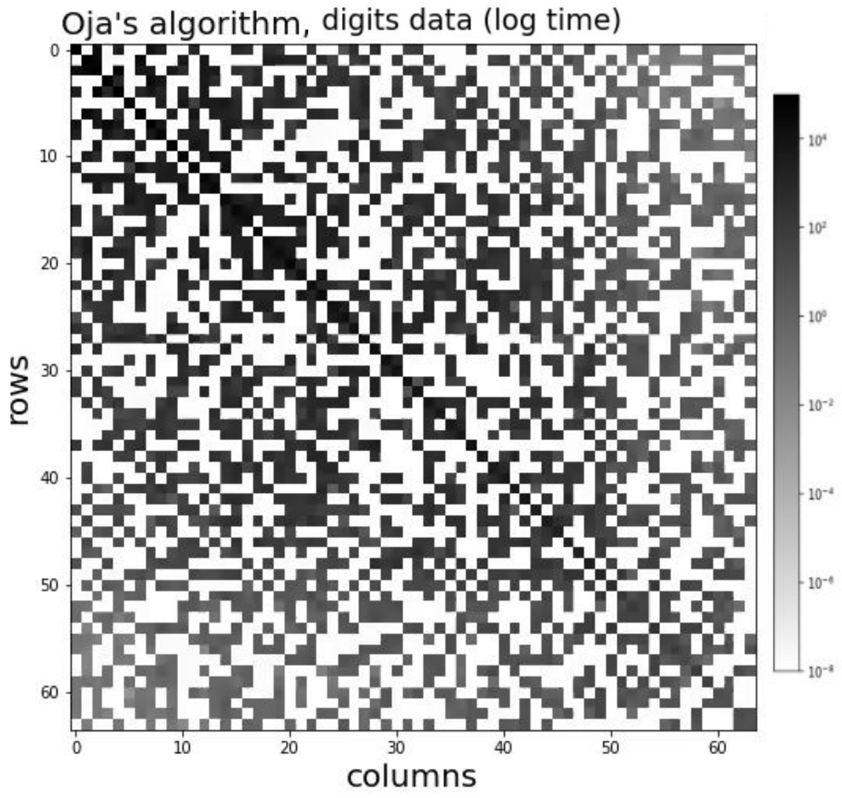


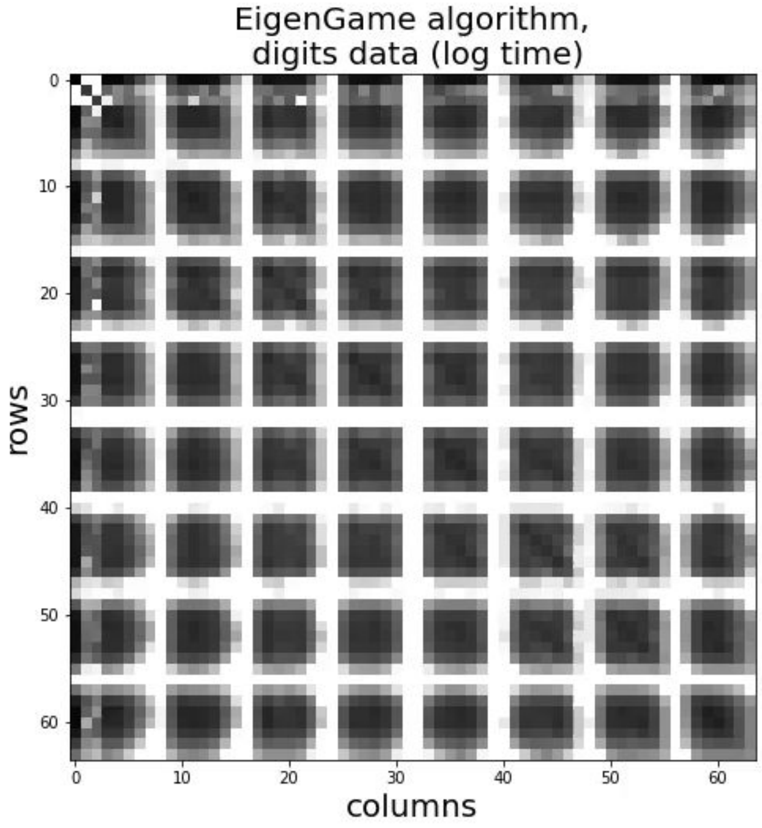

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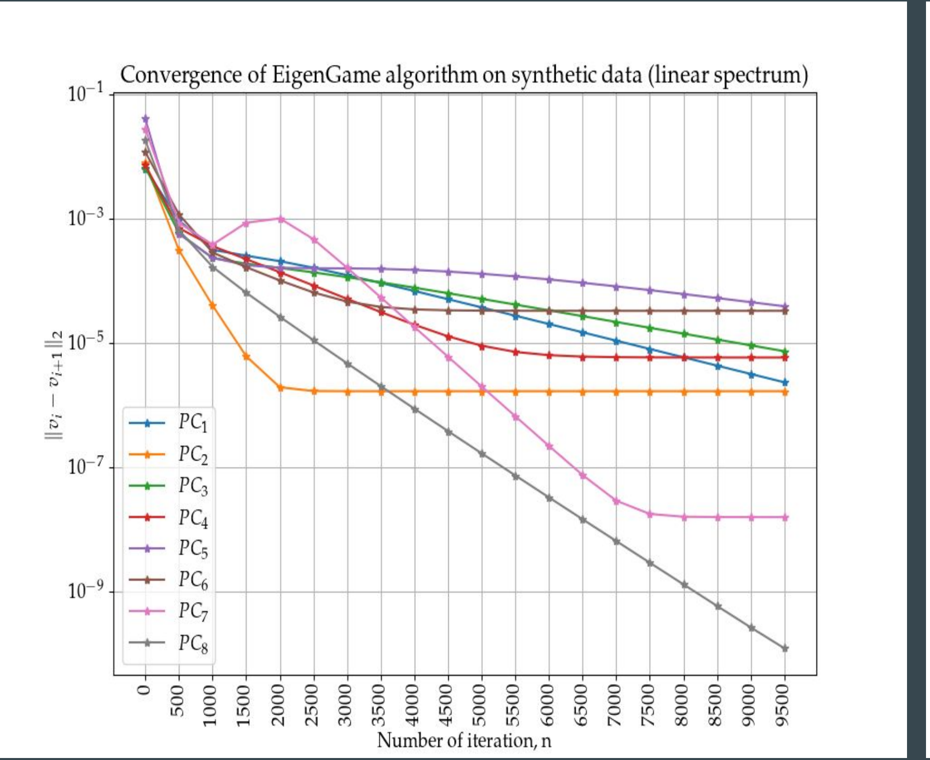


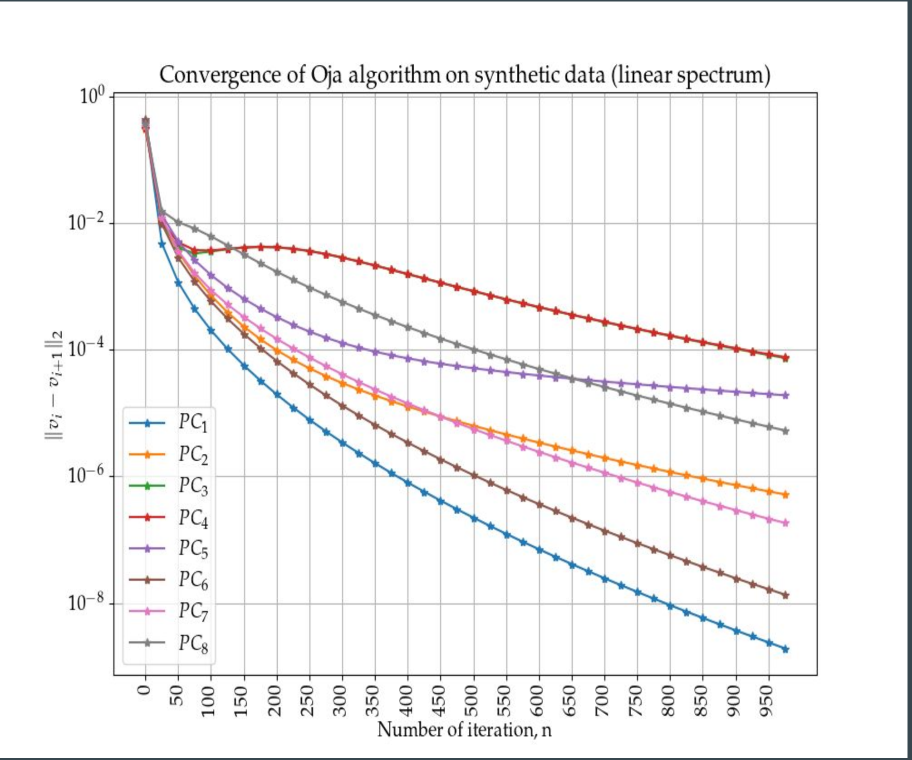

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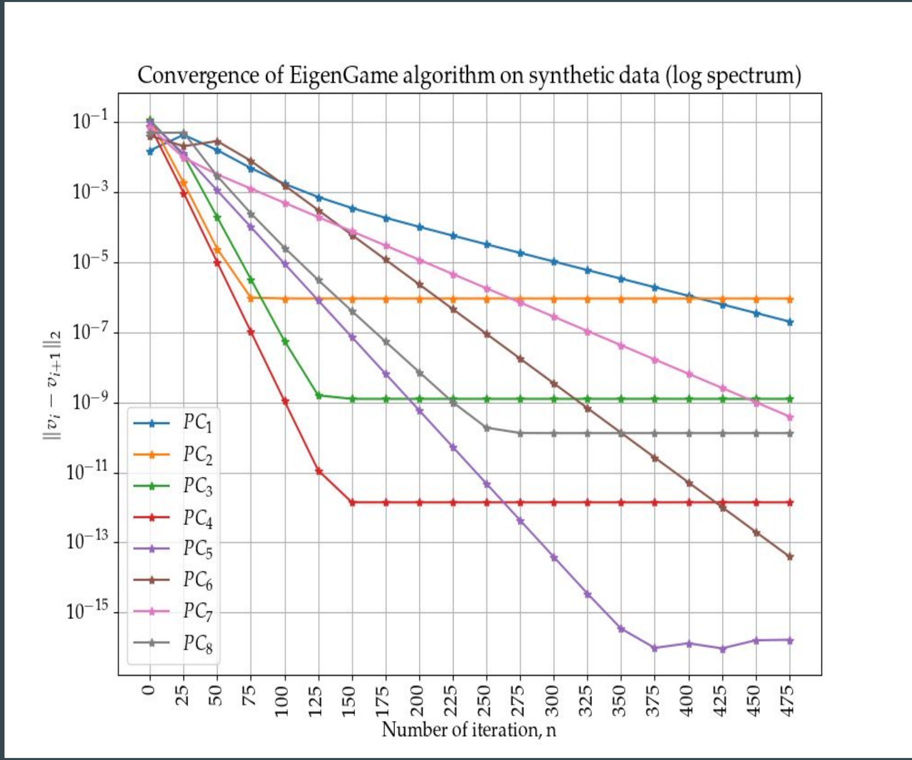


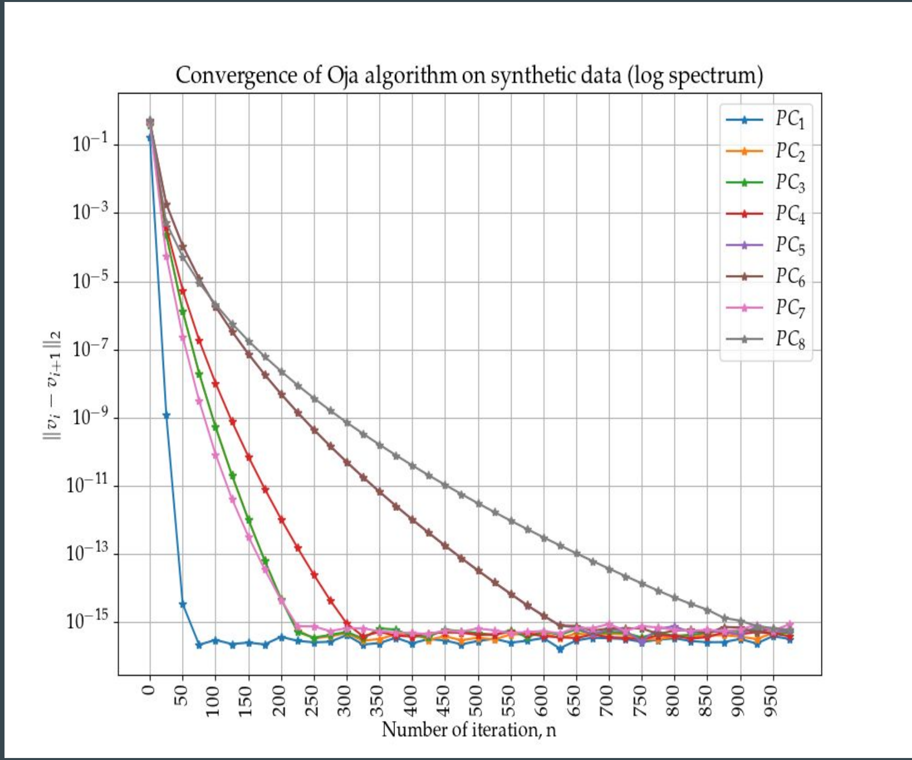

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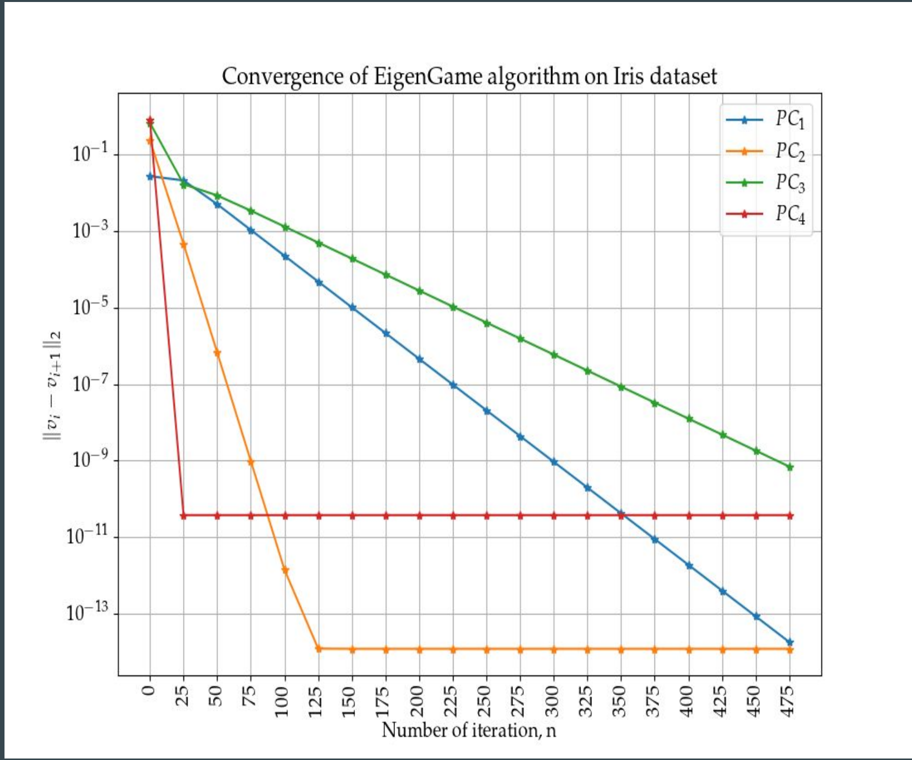


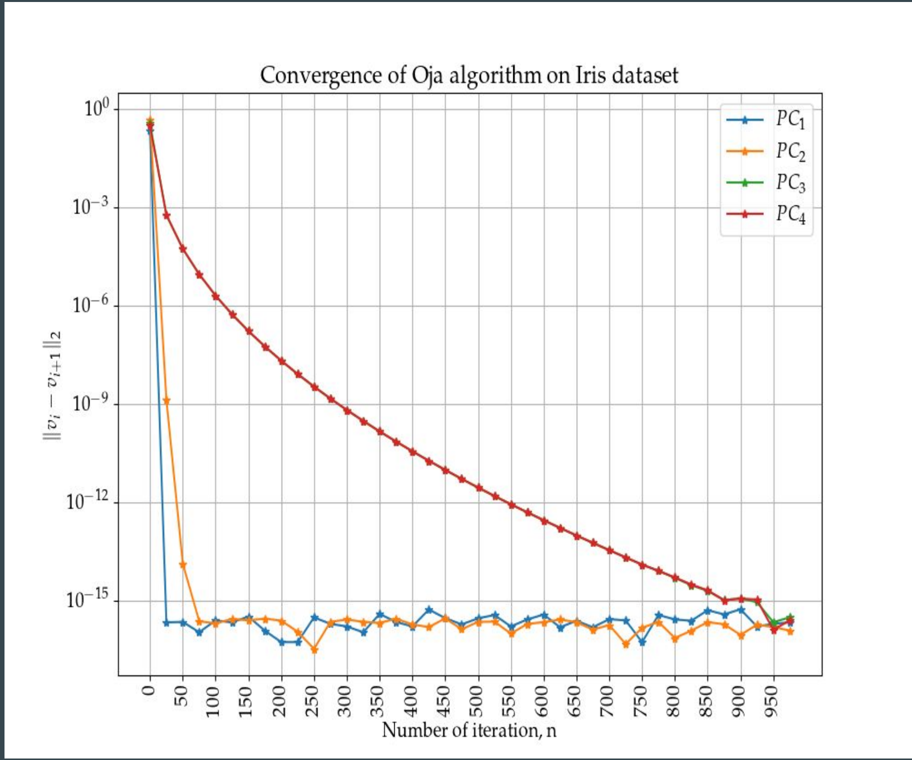

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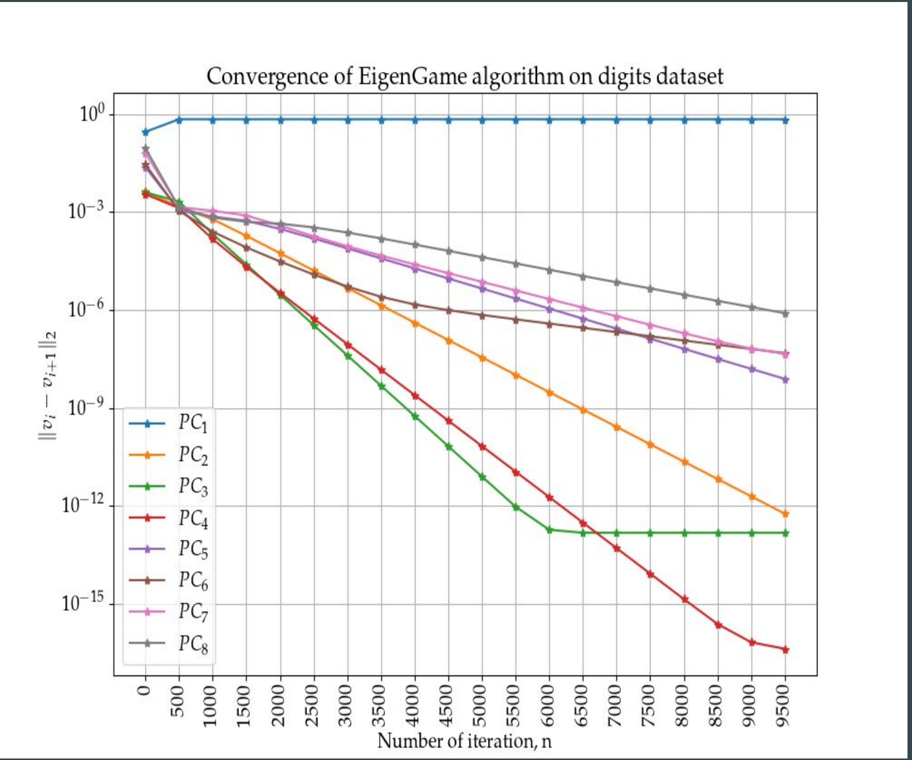


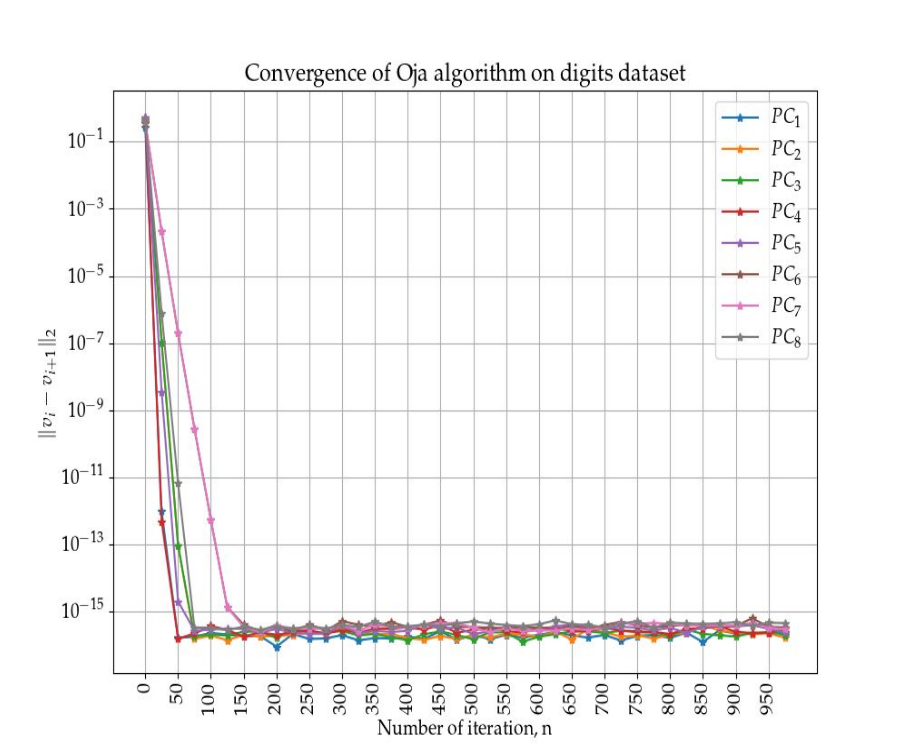

# RESULTS

- Realization of EigenGame algorithm
- Realization of Oja’s algorithm
- Diagonal convergence
- Checking algorithm on different datasets (synthetic data, MNIST, IRIS, digits)
- Visualizations of algorithms

# FURTHER DEVELOPMENT

1. **Parallelization:** faster computation

2. **Deep Variants:** $X\hat{v}_i \longrightarrow f_i(X|weights)$

3. **Scale:** improve efficiency

4. **Core ML:** accelerating training

# Thank you for your attention!

# REFERENCES


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