Theorem 1. *In $\mathcal{N}(3)$ every almost disjoint family can be extended to a maximal almost disjoint family.*

Proof. Assume that $\mathcal{A}$ is an almost disjoint family of subsets of X in $\mathcal{N}(3)$ with (finite) support E. We may assume without loss of generality that $\mathcal{A}$ is maximal among almost disjoint families in $\mathcal{N}(3)$ with support E.

Claim. $\mathcal{A}$ is a maximal almost disjoint family in $\mathcal{N}(2)$.

Proof. Assume $\mathcal{A}$ is not maximal and that B is a subset of X which is in $\mathcal{N}(3)$ but not in $\mathcal{A}$ for which $\mathcal{A} \cup \{B\}$ is an almost disjoint family. Assume that the (least) support of B is E'. Since $\mathcal{A}$ is maximal among almost disjoint families with support E, there is an atom $a_0 \in E' \setminus E$.

We first note that

$$\forall \psi \in \mathrm{fix}_G(E), \forall A \in \mathcal{A}, \psi(B) \cap A \text{ is finite.} \tag{1}$$

(If $\psi(B) \cap A$ is infinite for some $A \in \mathcal{A}$ then $\psi^{-1}(\psi(B) \cap A) = B \cap \psi^{-1}(A)$ is infinite. This contradicts the almost disjointness of $\mathcal{A} \cup \{B\}$ since $\psi^{-1}(A) \in \mathcal{A}$.)

□