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Threefold Problem Sheet – Roads

This sheet presents three problems based on a common storyline. The three problems share several characteristics and as a result they appear quite similar in nature. But this does not necessarily mean that these problems can be solved using the same strategies and techniques. In fact, the whole idea of these threefold sheets is to highlight how possibly subtle differences and changes in a problem formulation can have a great impact on how the resulting problem can be solved. We believe that studying these sheets is an excellent practice of how to approach the modeling aspects of solving a problem.

Read through the problems and try to work out by yourself possible lines of modeling and then algorithmically solving them. Give reasons of why or why not certain approaches appear plausible. Ideally, substantiate your ideas by sketching corresponding proof ideas.

This sheet was handed out during the tutorial on December 1, 2021, and solved and discussed there as well.

Main Story

There are n cities in AlgoLand, named very creatively with numbers from 1 to n (no two cities have the same "name"). There are, however, no roads yet between them. Since flying cars are still not available in AlgoLand, the government is forced to construct some roads. In each of the following problems you are given a list of possible roads which can be built, and based on the specific task description you have to decide which of these roads will actually be constructed.

Exercise – *Connecting Cities*

As a first step toward building a reliable infrastructure, the people of AlgoLand built a skeleton of the road network. The skeleton of the road network is just a list of possible roads, some of which will be actually built. Moreover, the list of possible roads is such that between every two cities in AlgoLand there exists exactly one path connecting them. You want to start building as many roads as possible at the same time.

There is a constraint, though: to avoid too much disruption you do not want to build two roads ending in the same city.

Input The first line contains $1 \leqslant T \leqslant 30$, the number of test cases. For each test case, in the first line there are two numbers n m, the number of the cities and possible roads. It is guaranteed that $m = n - 1$. The following m lines contain the descriptions of the roads, each of them consisting of two labels s t $(1 \leqslant s < t \leqslant n)$ denoting the cities which are connected by such road.

Points There are two test sets:

1. For the first set, worth 30 points, you may assume that the total number of cities is at most 500.

2. For the second set, worth 70 points, the total number of cities is at most 10^6.

Output For each test case output a single line with the maximum number of roads that can be built at the same time. Furthermore, you can only build a road if it is in the list of possbile roads.

Sample input	Sample output
1	3
8 7	
1 2	
1 3	
1 4	
2 5	
2 6	
4 7	
4 8	

Exercise – *Consecutive Constructions*

The basic roads in Algoland are already built, but the road network still needs to be extended.

The roads to be constructed now are in a more difficult terrain, so, if possible, you want to build them in consecutive intervals. A single construction team with all the heavy equipment can start building the next road in the same city just after finishing the previous one.

The rules are as follows:

- You can use as many teams as you need for operating the construction.

- Each team will start in some city u, and then build a road to a city with larger number v. After finishing the construction the team will be in v. Then it might build another road to a city with even larger number, and so on.

- To avoid disruptions each city can be visited at most once by at most one team. Starting or finishing in a city counts as a visit.

- You want to build as many roads as possible.

Input   The first line contains $1 \leqslant T \leqslant 30$, the number of test cases. For each test case, in the first line there are two numbers n m ($1 \leqslant n \leqslant 500, 0 \leqslant m \leqslant 2000$), the number of the cities and possible roads. The following m lines contain the descriptions of the roads, each of them consisting of two labels s t ($1 \leqslant s < t \leqslant n$).

Points   There are two test sets:

1. For the first set, worth 50 points, you may assume that $n \leqslant 50$ and $m \leqslant 200$.

2. For the second set, worth 50 points, there are no additional assumptions.

Output   For each test case output a single line with the maximum number of roads that can be built.

Sample input	Sample output
1	3
5 4	
1 3	
1 4	
2 3	
4 5	

Exercise – *Missing Roads*

A lot of roads were built in Algoland, but some of them are still missing. Since many accidents happened last year at construction sites, the government decided to decrease the risk of injury by using a a separate construction team for each road. However, as the road-building equipment makes a lot of noise when it is started, no two construction teams are allowed to start building from the same city. There is no problem though in two construction teams ending the road in the same city. A team assigned for one road can *choose* from which city it will start the construction.

There is not a lot of time and all roads must be built simultaneously, but you need to pay attention to the budget. Thus, given a construction cost for each potential road, estimate the minimum possible cost of building at least k of them, without starting a construction of more than one road at each city. Each given possible road has a cost between 1 and 100.

Input   The first line contains $1 \leqslant T \leqslant 30$, the number of test cases. For each test case, in the first line there are n m k ($1 \leqslant n \leqslant 50, 0 \leqslant m \leqslant \binom{n}{2}$, $k \leqslant n$), the number of the cities, possible roads and the number of roads to be built. The following m lines contain the descriptions of the roads and the costs. Each line consists of three numbers s t c ($1 \leqslant s < t \leqslant n, 1 \leqslant c \leqslant 100$), denoting that the cost of building a road between cities s and t is c.

Points   There are two test sets:

1. For the first set, worth 50 points, you may assume that all possible roads have the same cost.

2. For the second set, worth 50 points, there are no additional constraints.

Output   For each test case output a single line with the minimum possible cost of building at least k roads such that no two roads start the construction in the same city. Output no if it is impossible to achieve the goal.

Sample input

```
1
7 12 5
1 2 2
2 3 2
1 3 1
3 4 1
2 4 2
1 4 4
1 7 10
2 6 6
4 6 8
6 7 5
5 6 15
4 5 9
```

Sample output

```
11
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