University of Pretoria
Department of Statistics
WST322 Actuarial Statistics
Tutorial 7 Memo – 29 September 2011

Question 1

Q&A Part 3: 3.1 – 3.12

Question 2

Consider a portfolio of n policies where the i^{th} policy produces an aggregate claim amount of S_i which has a compound Poisson distribution with parameters λ and the lognormal distribution with parameters μ and σ^2. It is also known that λ has an exponential distribution with mean M.

a) Determine the mean and variance of S_i and of $\sum_{i=1}^{n} S_i$.

$$E(S_i) = E_\lambda[E(S_i \mid \lambda)] = E_\lambda[\lambda m_1] = Me^{\mu+\sigma^2/2}$$

$$E(\sum_{i=1}^{n} S_i) = \sum_{i=1}^{n} E(S_i) = nMe^{\mu+\sigma^2/2}$$

$$Var(S_i \mid \lambda) = \lambda m_2$$

$$\begin{aligned} Var(S) &= E_\lambda[Var(\sum_{i=1}^{n} S_i \mid \lambda)] + Var_\lambda[E(\sum_{i=1}^{n} S_i \mid \lambda)] \\ &= E_\lambda[\sum_{i=1}^{n} Var(S_i \mid \lambda)] + Var_\lambda[\sum_{i=1}^{n} E(S_i \mid \lambda)] \\ &= E_\lambda[\sum_{i=1}^{n} \lambda m_2] + n^2 (m_1)^2 Var_\lambda[\lambda] \\ &= nm_2 M \ + (nm_1 M)^2 \\ &= nMe^{2\mu+2\sigma^2} + (nM)^2 e^{2\mu+\sigma^2} \end{aligned}$$

b) If μ=5, σ^2=3 and M = 1.5 and the initial surplus is R4000, calculate the probability of ruin in 3 years of time if there are 300 policyholders, each paying a premium of R500 per year if A normal approximation is used for the distribution of $\sum_{i=1}^{n} S_i$.

$$\text{Surplus process: } U(t) = U + npt - S(t)$$

$$P(U(t) < 0) = P(S(t) > U + npt)$$

$$= P\left(Z > \frac{U + npt - E(S(t))}{\sqrt{Var(S(t))}}\right)$$

$$= 1 - \Phi\left(\frac{4000 + 300 \times 500 \times 3 - 300 \times 1.5 \times 3 \times e^{6.5}}{\sqrt{300 \times 1.5 \times 3 \times e^{16} + (300 \times 1.5 \times 3)^2 \times e^{13}}}\right)$$

$$= 1 - \Phi(-0.49)$$

$$= 0.68793$$

Question 3

Claims arise in an insurance portfolio according to a Poisson process with parameter λ, while claim sizes are independently distributed according to a lognormal distribution with parameters μ and σ^2. An excess of loss reinsurance treaty is in effect with retention level M.

(a) Write down expressions for S_I, the aggregate claims to be paid by the direct insurer, as well as $E(S_I)$ and $Var(S_I)$. Hint: $\int_L^U x^k f_X(x)dx = e^{\mu k + \frac{1}{2}\sigma^2 k^2}[\Phi(U_k) - \Phi(L_k)]$, $L_k = \frac{\log(L) - \mu}{\sigma} - k\sigma$, $U_k = \frac{\log(U) - \mu}{\sigma} - k\sigma$ for $X \sim LN(\mu, \sigma^2)$.

Let $Y_i = \begin{cases} X_i & \text{if } X_i \leq M \\ M & \text{if } X_i > M \end{cases}$ ✓ i.e. the amount payable by the direct insurer

Then $S_I = \sum_{i=1}^{N} Y_i$ ✓ has a compound Poisson distribution with parameters λ and $F_X(\cdot)$ (a lognormal cdf) ✓

$$E[Y] = \int_0^M x f(x)dx + \int_M^\infty M f(x)dx$$

✓

$$= e^{\mu + \frac{1}{2}\sigma^2}\left[\Phi\left(\frac{\log(M) - \mu}{\sigma} - \sigma\right) - \Phi(-\infty)\right] + M\left[\Phi(\infty) - \Phi\left(\frac{\log(M) - \mu}{\sigma}\right)\right]$$ ✓

and so $E[S_I] = \lambda E[Y]$.

$$E[Y^2] = \int_0^M x^2 f(x)dx + \int_M^\infty M^2 f(x)dx$$

$$= e^{2\mu + 2\sigma^2}\left[\Phi\left(\frac{\log M - \mu}{\sigma} - 2\sigma\right) - \underbrace{\Phi(-\infty)}_{=0}\right] + M^2\left[1 - \Phi\left(\frac{\log M - \mu}{\sigma}\right)\right]$$ ✓ ✓

and so $var(S_I) = \lambda E[Y^2]$ ✓

(b) Premiums are collected from insured policy holders at a rate of c_1 per year and premiums are being paid to the reinsurer at a rate of c_2 per year. An initial surplus of U is set aside to prevent the occurrence of ruin over time. If it is assumed that S_I approximately has a normal distribution, find an expression for U in terms of $E(S_I)$, $Var(S_I)$, c_1 and c_2 if the probability of ruin is not to exceed 0.1 in 3 years.

Surplus Process: $U(t) = U + c_1 t - c_2 t - S_I(t)$

so $U(3) = U + (c_1 - c_2)(3) - S_I(3)$ ✓

$E[S_I(3)] = 3\lambda E[Y]$ and $var(S_I(3)) = 3\lambda E[Y^2]$

$P[ruin] = P[U(3) < 0] = P[S_I(3) > U + 3(c_1 - c_2)]$ ✓

$= P\left[Z > \frac{U + 3(c_1 - c_2) - 3\lambda E[Y]}{\sqrt{3\lambda E[Y^2]}}\right] = 0.1$ ie. $1 - F[z] = 0.1$ OR $F[z] = 0.9$

$\Rightarrow \frac{U + 3(c_1 - c_2) - 3\lambda E[Y]}{\sqrt{3\lambda E[Y^2]}} = 1.29$

$\therefore U = 1.29\sqrt{3\lambda E[Y^2]} - 3(c_1 - c_2) + 3\lambda E[Y]$ ✓

(c) Find the value of U if $\mu = 5, \sigma^2 = 3, \lambda = 20$, $M = 600$, $c_1 = 6000$ and $c_2 = 3500$.

$E[Y] = e^{5 + \frac{1}{2}(3)}\left[\Phi\left(\frac{\log 600 - 5}{\sqrt{3}} - \sqrt{3}\right)\right] + (600)\left[1 - \Phi\left(\frac{\log 600 - 5}{\sqrt{3}}\right)\right]$

$= e^{5 + \frac{1}{2}(3)}\left[\Phi(-0.926)\right] + 600\left[1 - \Phi(0.807)\right]$

$= e^{5 + 3/2}\left(1 - \underset{0.82381}{\Phi(0.926)}\right) + 600\left[1 - 0.79103\right]$

$= \underline{242.57}$ ✓

$E[Y^2] = e^{10+6}\left[\Phi(-2.66) - 0\right] + 600^2\left[1 - \Phi(0.807)\right]$

$= e^{16}\left(1 - \underset{0.99609}{\Phi(2.66)}\right) + 600^2\left[1 - 0.79103\right] = \underline{109\,973,89}$ ✓

$\therefore U = 1.29\sqrt{3(20)(109\,973.89)} - 3(6000 - 3500) + 3(20)(242.57)$

$= \underline{10\,367.88}$ ✓

Question 4 (exam 2005)

A group of policy holders experience losses for which the individual loss distribution has an annual mean and variance R2000 and R100 000 respectively. The number of claims in a year from this group of policy holders follows a Poisson distribution with parameter λ and it seems realistic to assume that λ has an exponential distribution with mean 50.

(a) Give a detailed description of the distribution of $S(t)$ = aggregate claim size after t years. Write down expressions for the mean and variance of $S(t)$.

$S(t)$ has compound Poisson distribution with parameters λt and F_X, where F_X has mean $\overset{2}{\cancel{4}}000$ and λ has density function $f_\lambda(\lambda) = 0.02e^{-0.02\lambda}$ ✓ 3

$E\left(S(t)\mid\lambda\right) = 2000\lambda t$ and $E\left(S(t)\right) = 2000t \times 50 = t \times 10^5$ ✓

$Var\left(S(t)\mid\lambda\right) = \left(100\,000 + 2000^2\right)\lambda t = 41 \times 10^5 \lambda t$ ✓

$\therefore Var\left(S(t)\right) = E_\lambda\left\{41 \times 10^5 \lambda t\right\} + Var_\lambda\left\{2000\lambda t\right\}$ ✓ 5

$= 41 \times 10^5 t \times 50 + 4 \times 10^6 t^2 (50)^2$

$= \cancel{20.5t \times 10^6} + t^2 \times 10^{10}$ ✓

$t\,2050 \times 10^6 + t^2 \times 10^{10}$

$\sqrt{Var(S(t))} = \sqrt{1025 \times 10^6 + 25 \times 10^{10}} = 1000 \times 501.02395$

$= \sqrt{2.51025 \times 10^{11}} = 501023.95$

(b) The insurance company charges a premium of $(1+\theta)E\left(S(t)\right)$, with θ = premium loading = 0.2 and wishes to determine the initial surplus, U, of this portfolio so that the probability of ruin after at most 5 years is only 10%. Assuming $S(t)$ to be approximately normally distributed, find U.

(b) $U(t)$ = Surplus after t years = $U + ct - S(t)$

$= U + 1.2E(S(t)) - S(t)$

$P\left(U(t) < 0\right)$

$= P\left(S(t) > 1.2E(S(t)) + U\right)$ ✓

$E(S(5)) = 5 \times 10^5$

$= P\left(Z > \dfrac{0.2E(S(t)) + U}{\sqrt{Var\left(S(t)\right)}}\right)$

$1.2 \times 5 \times 10^5 = 6 \times 10^5$

(4)

$= P\left(Z > \dfrac{0.2 \times 5 \times 10^5 + U}{\sqrt{102\cancel{,5} \times 10^6 + 25 \times 10^{10}}}\right) = 0.10$ ✓

$\dfrac{10^5 + u}{501023.95}$

i.e. $1.282 = \dfrac{10^5 + U}{500102\cancel{.4894}}$ ✓ or $U = $ R541 131.39 ✓

501023.96 R542 314

Question 5

An insurer has issued two five-year term assurance policies to two individuals involved in a dangerous sport. Premiums are payable annually in advance, and claims are paid at the end of the year of death.

Individual	*Annual Premium*	*Sum Assured*	*Annual P[death]*
A	100	1700	0.05
B	50	400	0.1

Assume that the probability of death is constant over each of the five years of the policy. Suppose that the insurer has an initial surplus of U. Assuming $U = 1000$,

(a) Determine the distribution of $S(1)$, the surplus at the end of the first year, and hence calculate $\psi(U,1)$.

Immediately before the payment of any claims, the insurer has cash reserves of 1000 + 150 =1150.
The distribution of S(1) is given by:

Deaths	***S(1)***	***Prob***
None	***1150***	$0.95 \times 0.9 = 0.855$
A only	$1150 - 1700 = -550$	$0.9 \times 0.05 = 0.045$
B only	$1150 - 400 = 750$	$0.95 \times 0.1 = 0.095$
A and B	$1150 - 2100 = -950$	$0.05 \times 0.1 = 0.005$

And the probability of ruin is given by 0.045 + 0.005 = 0.05.

(b) Determine the possible values of $S(2)$ and hence calculate $\psi(U,2)$.

Assuming the surplus process ends if ruin occurs by time 1, then 2 possible values of S(2) are −550 and −950.

If there are no deaths in year 1, possible values of S(2) are
No deaths: 1150 + 150 = 1300
A only: 1150 + 150 – 1700 = −400
B only: 1150 + 150 – 400 = 900
A and B: 1150 + 150 – 1700 – 400 = −800

If B dies in year 1, the possible values of S(2) are:
A lives: 750 + 100 = 850
A dies: 750 + 100 – 1700 = −850

If A dies in year 1 then ruin occurs so the process stops.

The probability of ruin within 2 years is given by:
$0.05 + 0.855 \times (0.05 \times 0.9 + 0.05 \times 0.1) + 0.095 \times 0.05 = 0.0975$

Alternatively, note that ruin occurs within 2 years if and only if A dies during this time, the probability of which is 0.05 + 0.95 × 0.05 = 0.0975.

Question 6

a) If X and Y are independent Poisson random variables with mean λ, derive the moment generating function of X, and hence show that $X + Y$ also has a Poisson distribution.

$M_X(t) = E[e^{tX}]$ ✓

$= \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \lambda^x / x!$

$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$

$= e^{-\lambda} e^{\lambda e^t}$ ✓

$= e^{\lambda(e^t - 1)}$

$M_{X+Y}(t) = M_X(t) M_Y(t)$ since X and Y are independent

$\left(M_{X+Y}(t) = E[e^{t(X+Y)}] = E[e^{tX} e^{tY}] = E[e^{tX}] E[e^{tY}] \right)$ ✓

$= e^{\lambda(e^t-1)} e^{\lambda(e^t-1)}$ ✓

$= e^{2\lambda(e^t-1)}$

i.e. $X+Y \sim \text{Poisson}(2\lambda)$. ✓

b) An insurer has a portfolio of 100 policies. Annual premiums of 0.2 units per policy are payable annually in advance. Claims, which are paid at the end of the year, are for a fixed sum of 1 unit per claim. Annual claims numbers on each policy are Poisson distributed with mean 0.18. Calculate, assuming a normal approximation how much initial capital is needed in order to ensure that the probability of ruin at the end of the year is less than 1%.

b) 100 policies, each premium = 0.2, each claim = 1, $N \sim \text{Poisson}(0.18)$.

Find $u \ni P[U(1) < 0] < 0.01$: ✓

$U(1) = U + \underbrace{100(0.2)}_{\text{pemiums}} - \underbrace{\sum_{i=1}^{n} Y_i}_{\text{claims}}$ ✓

where $Y_i = \sum_{j=1}^{N} X_j = \sum_{j=1}^{N} (1) = N$ ✓

$E[S] = nE[N] = 100(0.18) = 18$ ✓

$\text{var}(S) = n^2 \text{var}(N) = 100^2(0.18) = 1800$ ✓

So $P[U(1) < 0]$

$= P[U + 20 - S < 0]$

$= P[U + 20 < S]$ ✓

$= P\left[\frac{U + 20 - E[S]}{\sqrt{\text{var}(S)}} < Z\right] < 0.01$

$\Rightarrow \frac{U + 20 - E[S]}{\sqrt{\text{var}(S)}} > 2.32$

$\therefore U > 96.42926394$ ✓

Question 7

Consider a group of 100 life insurance policies. The probability to claim each year on each policy is q where $P[q < 0.5] = 0.6$, and at most one claim is possible on each policy. The sums assured are

$$\begin{cases} 1000000 & \text{if } q < 0.5 \\ 750000 & \text{if } q \geq 0.5 \end{cases}.$$

a) Calculate the expected value of the total claim amount. (Hint: q is a continuous random variable.)

$S = \sum_{i=1}^{100} Y_i$ where $Y_i = \begin{cases} X_i & \text{if } N_i = 1 \\ 0 & \text{if } N_i = 0 \end{cases}$

so that $Y_i \sim$ compound binomial $(1, q)$

and where $X_i = \begin{cases} 1000\,000 & \text{with prob. } 0.6 \\ 750\,000 & \text{with prob. } 0.4 \end{cases}$

So $E[S] = 100 E[Y_i] = 2025000$ ✓

$E[Y_i] = E_q[E[Y_i|q]]$ ✓

$= E_q[(1) q E[X_i]]$

$= [1000000(0.6) + 750000(0.4)] E[q]$

$= 202500$ ✓

$$E[q] = \int_0^{0.5} x(0.6)\,dx + \int_{0.5}^{1} x(0.4)\,dx$$

$$= 0.6 \left.\frac{x^2}{2}\right|_0^{0.5} + 0.4 \left.\frac{x^2}{2}\right|_{0.5}^{1}$$

$$= 0.\cancel{2}25$$

b) If the variance of the total claim amount is 6000000^2, and the total claim amount can be assumed to be normal calculate the probability of ruin at the end of the year. The initial surplus is 10 000000, and the premiums payable on each policy are 2400.

$$U(1) = 10\,000\,000 + 100(2400) - S(1)$$

$$P[\text{ruin}] = P[U(1) < 0]$$

$$= P[S(1) > 10000000 + 100(2400)]$$

$$= P\left[Z > \frac{1000\,0000 + 240000 - 2025000}{\cancel{5000000}\ 6000\,000}\right]$$

$$= 1 - \Phi(1.37)$$

$$= 1 - 0.91466$$

$$= \underline{0.08534}$$

Question 8

Consider a Poisson process $\{N(t)\}$ with parameter λ with underlying claims arising from an exponential distribution with expected value 100.

a) Consider the inequality $\Psi(U) \le e^{-RU}$. What is the name of this inequality and why is it useful? Discuss the relationship between $\Psi(U)$ and U, and $\Psi(U)$ and R.

Lundberg's inequality (1/2 mark)

It provides a relationship between the adjustment coefficient R and the probability of ruin $\Psi(U)$ - a larger R implies a smaller $\Psi(U)$ thus R is maximized in order to provide stability of the portfolio. (1 mark)

The same relationship holds with U. (1/2 mark)

b) Provide an equation to solve for the adjustment coefficient if the insurer uses a loading of 1% on their premiums and solve for the adjustment coefficient.

$$\lambda M_X(R) = \lambda + cR (1/2\ mark)$$

$$\lambda\left(\frac{\frac{1}{100} - R}{\frac{1}{100}}\right)^{-1} = \lambda + 1.1\lambda m_1 R (1/2\ mark)$$

$$\tfrac{1}{100} = \left(\tfrac{1}{100} - R\right)(1 + 1.1(100)R)(1/2\ mark)$$

$$= \tfrac{1}{100} + 1.1R - R - 1.1(100)R^2$$

$$0 = 0.1R - 1.1(100)R^2 = R(0.1 - 1.1(100)R)(1/2\ mark)$$

$$Thus\ R = 0\ or\ R = \frac{0.1}{1.1(100)} = 0.000909091(1/2\ mark)$$

so the unique positive root is 0.000909091.$(1/2\ mark)$

c) If $X \sim U(50,150)$ instead, derive an equation for the adjustment coefficient. You do not need to simplify the equation.

$$\lambda M_X(R) = \lambda + cR$$

$$\lambda\left(\frac{1}{150-50}\frac{1}{R}\right)\left(e^{150R} - e^{50R}\right) = \lambda + 1.1\lambda m_1 R (1/2\ mark)$$

$$\left(\frac{1}{150-50}\frac{1}{R}\right)\left(e^{150R} - e^{50R}\right) = 1 + 1.1(100)R (1/2\ mark)$$

d) If an excess-of-loss reinsurance policy is now put into place with excess level M, derive the expected value of the reinsurer's claims and the moment generating function of the insurer's claims. You can assume the reinsurer knows about all claims. What value of M would be a good choice between the insurer and reinsurer? Then provide an equation in terms of the formulas derived, as well as M and the reinsurer's loading, for which the adjustment coefficient can be solved for. Assume the claims are distributed exponentially as before.

$$E[Z] = \int_M^{\infty} (x - M) f_X(x) dx$$

$$let\ w = x - M$$

$$= \int_0^{\infty} w(0.01)e^{-0.01(w+M)} dw \quad \textbf{(1 ½ marks)}$$

$$= e^{-0.01M}\int_0^{\infty} w(0.01)e^{-0.01w} dw$$

$$= e^{-0.01M}\frac{1}{0.01} = e^{-0.01M}100$$

If $M = 200$ ***then*** $E[Z] = 13.53$

If $M = 150$ ***then*** $E[Z] = 22.31$

If $M = 70$ ***then*** $E[Z] = 49.65853$

So this seems like a good choice since then the reinsurer is expected to pay half of every claim. (1 mark)

$$M_Y(t) = \int_0^M e^{xt} 0.01 e^{-0.01x} dx + e^{tM}\left(1 - F_X(M)\right)$$

$$= \int_0^M 0.01 e^{(t-0.01)x} dx + e^{tM}\left(e^{-0.01M}\right)$$

(1 ½ marks)

$$= \left.\frac{0.01 e^{(t-0.01)x}}{(t-0.01)}\right|_0^M + e^{(t-0.01)M}$$

$$= \frac{0.01 e^{(t-0.01)M} - 0.01}{(t-0.01)} + e^{(t-0.01)M}$$

$$\lambda M_Y(R) = \lambda + c^* R$$

where

$$c^* = 1.1(100)\lambda - (1+\xi)\lambda E[Z]$$

$$= 1.1(100)\lambda - (1+\xi)\lambda e^{-0.01M} 100$$

so we solve $\dfrac{0.01 e^{(R-0.01)M} - 0.01}{(R-0.01)} + e^{(R-0.01)M} = 1 + \left(1.1(100) - (1+\xi)e^{-0.01M} 100\right)R$ ***(1 mark)***

Question 9

Consider the aggregate claims from a risk where the claims arise according to a Poisson process with parameter λ and the claim sizes are represented by X.

a) Define the adjustment coefficient.

R is the unique, positive root of

$\lambda M_X(R) = \lambda + cR$ ✓ [2]

where c = rate of premium income ✓

$M_X(t)$ = m.g.f of the underlying claim sizes ✓

b) Derive the upper bound $\dfrac{2\theta m_1}{m_2}$ for the adjustment coefficient, where θ is the premium loading used by the insurer, and $m_i = E[X^i]$.

$\lambda + cR = \lambda M_X(R)$

$= \lambda \int_0^\infty e^{Rx} f_X(x)dx$ ✓

$> \lambda \int_0^\infty (1 + Rx + \frac{1}{2}R^2x^2) f_X(x)dx$ ✓

$= \lambda(1 + Rm_1 + \frac{1}{2}R^2 m_2)$ ✓

So $(c - \lambda m_1)R > \frac{1}{2}\lambda R^2 m_2$

$\therefore R < \dfrac{2(c - \lambda m_1)}{\lambda m_2} = \dfrac{2((1+\theta)\lambda m_1 - \lambda m_1)}{\lambda m_2}$ ✓

$= \dfrac{2\theta m_1}{m_2}$ ✓

c) Under what condition will a lower bound for the adjustment coefficient exist?

Only if there is an upper bound to an individual claim ✓

e.g. U(a,100). ✓ [1]