University of Pretoria
Department of Statistics
WST322 Actuarial Statistics
Tutorial 2 – Chapter 2

Question 1

Questions in Q&A Part 1 for Chapter 2.

Question 2 Suppose a bag of balls is used in a competition in which there are a number of red and a number of purple balls. Two balls are picked out of the bag so that the probability that k of these two balls is red follows a binomial distribution with parameters n and p. However, p is not known with certainty and rather follows a $Beta(\varepsilon, \lambda)$ distribution.

(a) Find the distribution of p based on the following sample (in which a number of trial runs are done drawing 2 balls out of the bag): 2 0 1 2 0 0 2 0 2 2 1 1 1 2. Take $\varepsilon = 40$ and $\lambda = 22$.

(b) How would you find an estimator for p under the absolute error loss function? You need not solve for the final estimate.

(c) Suppose now a person enters the competition by giving an initial guess as to how many red balls will be selected in a specific draw. If the player selects the correct number of red balls drawn he is given 3 points and if not he has points taken away equal to the absolute difference between the number guessed and the actual number of red balls drawn. Set up a payoff matrix for the player.

(d) Using a minimax strategy which number of red balls should the player select?

(e) Find the corresponding expected losses for the player in (c) and use it to decide on an optimal strategy for the player. You may assume the estimate for p from (b) is 0.614.

Question 3 (a) Using the usual notation, find an exact expression for $f(\theta \mid \mathbf{X})$ and then prove that $f(\theta \mid \mathbf{X}) \propto f(\mathbf{X} \mid \theta) f(\theta)$ where $\mathbf{X}$ is the sample data and θ the population parameter.

(b) For a sample $x_1, x_2, x_3, \ldots, x_n$ from a LogN(μ, σ^2) distribution where μ follows a U(0,N) distribution, find the posterior distribution of μ.

(c) How would your answer in (b) change if μ was assumed to follow a U(0,1) distribution instead? Explain.

Question 4 An insurer's portfolio consists of three independent policies. Each policy can give rise to at most one claim per month, which occurs with probability θ independently from month to month. The insurer does not know the exact value of θ but instead knows it follows a beta distribution with parameters $\alpha = 2$ and $\beta = 4$. A total of 9 claims are observed from this portfolio over a 12 month period.

a) Find the distribution of θ given this observed data.
b) Based on the distribution you obtained in (a), what do we call the original distribution for θ? Why?
c) Find the Bayesian estimate for θ under the all or nothing loss function.
d) What is the purpose of the loss function in (c) and how does this specific loss function achieve this purpose?
e) What is the Bayesian estimate under the quadratic loss function?
f) Prove the result you make use of in (e).

Question 5 An insurer wishes to estimate the expected number of claims, λ, on a particular type of policy. Prior beliefs about λ are represented by a Gamma (α, β) distribution.

a) For an estimate, *d*, the loss function is defined as $L(\lambda, d) = (\lambda - d)^2 + d^2$. Show that the expected loss is given by $E[L(\lambda, d)] = \frac{\alpha(\alpha+1)}{\beta^2} - \frac{2d\alpha}{\beta} + 2d^2$ and hence determine the Bayesian estimate for λ.

b) If the number of claims are believed to follow a Poisson distribution with parameter λ, and the following claims history is observed,

1,5,6,2,7,2,4,3,4,2,2,5,4,5,1,2,

find the posterior distribution for λ.

c) Using (b), what is the Bayesian estimator for λ under the all-or-nothing loss function?

Question 6 A statistician has obtained a sample of data of size 50 from an experiment and he has calculated the following sample information: $\sum x_i = 250.787$ and $\sum x_i^2 = 1516.705$. He believes the data follows a normal distribution and can with confidence estimate the variance for the distribution of his data as the sample variance reduced by 0.14 (due to noise in the collection of his data). He is unsure of the true mean of the data but from the set-up of his experiment he knows the mean follows a normal distribution with parameters $\mu = 4.7$ and $\sigma^2 = 0.3$.

a) He decides to obtain a Bayesian estimate for the mean.

(i) Give the posterior distribution for the mean based on the sample information.

(ii) Find a Bayesian estimate for the mean under the absolute error loss function. Prove the result you make use of with respect to this loss function.

(iii) Besides the method of Bayesian estimation for an unknown parameter, name three other methods that could also be used for the estimation. Why is the Bayesian method more powerful?

b) He decides to also investigate an alternative method for deciding what the mean for his underlying distribution is. He believes the mean can be one of three values: 4, 5 or 6. If he incorrectly chooses the value the following loss in accuracy in his model results:

		Guess for the Mean		
		$m_1 = 4$	$m_2 = 5$	$m_3 = 6$
True Value for the Mean	$M_1 = 4$	0	1%	2%
	$M_2 = 5$	1.5%	0	1%
	$M_3 = 6$	2.5%	1.5%	0

He decides to make use of one of the following decision functions:

$$d_1(\bar{x}) = \begin{cases} m_1 & \text{if } \bar{x} < 4.5 \\ m_2 & \text{if } 4.5 \le \bar{x} < 5.5 \\ m_3 & \text{if } \bar{x} \ge 5.5 \end{cases}, \quad d_2(\bar{x}) = \begin{cases} m_1 & \text{if } \bar{x} < 4.2 \\ m_2 & \text{if } 4.2 \le \bar{x} < 5.8 \\ m_3 & \text{if } \bar{x} \ge 5.8 \end{cases}, \quad d_3(\bar{x}) = m_2 \text{ for any } \bar{x}$$

(i) Recalling that if $X \sim N(\mu, \sigma^2)$ then $\bar{X} \sim N\left(\mu, \sigma^2/n\right)$, set up a risk matrix for this problem using the decision functions above and the estimate for the mean you obtained in (a).

(ii) Use a minimax criterion to decide which decision function he should use. Show your arguments in full.

(iii) Based on his sample data, which $m_i, i = 1,2,3$ should he choose?

Question 7 Prove/justify the Bayesian posterior distribution formula, namely $f(\theta \mid \underline{X}) \propto (prior)(likelihood)$.

Question 8 Suppose the claims of a risk, denoted with the random variable X, follow a Weibull distribution with parameters λ and σ. The parameter σ is known to be 2 but the parameter λ is unknown and no a priori assumptions are made about λ.

a) Find the posterior distribution for the parameter λ based on a sample $x_1, x_2, ..., x_k$ of claims.

b) If $k = 30$ and $\sum (x_i)^j = 24^j$, find an estimate for λ under the squared error loss function correct to 4 decimal places.

c) How does the squared error loss function measure the error of the estimator? Describe all terms used in the formula.

d) Assume now that λ has a Gamma(a,b) distribution and find the underlying distribution for the claim amounts.

Question 9 a) In a courtroom trial with a male defendant, CCTV evidence from a pub shows a crime being committed by an individual carrying a bottle, but it is unclear whether this is a male or a female. A witness reports seeing the defendant carrying a bottle around the time of the offence. Suppose that at the time of the offence, there are an expected 60% males in the pub. Furthermore, an expected 2% of males and 1% of females carry bottles at any given moment. Use Bayes theorem to determine how strong the evidence is against the defendant?

b) Prove/justify the Bayesian posterior distribution formula, namely $f(\theta \mid \underline{X}) \propto (prior)(likelihood)$

c) A statistician has a sample of data assumed to be from a normal distribution with parameters μ and σ^2. The variance is known to be 2 but the mean is unknown and he has no prior knowledge of the unknown mean. Determine a posterior distribution for the mean.

Question 10 Claim numbers, N, from a certain portfolio follow a Poisson distribution with parameter μ. The parameter μ is however unknown but believed to be either 3,4 or 5.

a) By letting $\mu = Y + 3$ where $Y \sim bin(2, 0.4)$, show that the probability $P[N = n]$ is given by $P[N = n] = \sum_{i=0}^{2} \frac{2e^{-(1+3)} 0.4^i 0.6^{2-i} (i+3)^n}{n! i! (2-i)!}$. What is this type of distribution called?

b) In order to select the correct value of μ game theory is made use of. The following table provides the percentage change from the correct premium value for the respective values of μ.

		Value Chosen for μ		
Correct Value for μ		$\mu = 3$	$\mu = 4$	$\mu = 5$
	$\mu = 3$	0	5%	12%
	$\mu = 4$	-5%	0	5%
	$\mu = 5$	-12%	-5%	0

Determine the expected percentage change from the correct premium for each choice of μ and subsequently determine the optimal choice for μ explaining your choice fully.

Question 11 Claim sizes, X, for a certain portfolio follow a $uniform(a,b)$ distribution. Neither of the parameters a and b are known with certainty but are believed to each independently follow the same distribution.

a) Prove that $f_{a,b}(a,b \mid \mathbf{x}) \propto f_{\mathbf{X}}(\mathbf{x} \mid a,b) f_{a,b}(a,b) = (\text{likelihood}) \times (\text{joint prior})$ gives the joint posterior distribution.

b) If $a \sim \exp(\lambda)$ and $b \sim \exp(\mu)$ derive a formula for the joint posterior distribution based on a sample $x_1, ..., x_{10}$.

c) What can be said about the posterior independence of a and b?

Question 12 A biased coin is tossed 10 times and 3 tails are observed. The probability of observing heads is p however p is unknown and no specific prior beliefs about p exist.

a) Provide two example distributions from your tables which could theoretically be used as possible prior distributions for p.
b) Find the distribution for p based on the observed sample.
c) Comment on your answer in (b) with respect to your answer in (a).
d) For $Y \sim beta(\alpha, \beta)$, determine the mode y_{mode}.
e) Find the Bayesian estimate for p under the all-or-nothing loss function. Comment on your answer.

Question 13 Consider $X \sim bin(7, p)$ where p is unknown and needs to be estimated.

a) Provide two suitable prior distributions for p.
b) For both priors in (a) derive the posterior distribution for p based on observed data $x_1, x_2, ..., x_n$.
c) Provide two Bayes estimates for p based on the posteriors derived in (b) and using the quadratic loss function.
d) Determine a method of moments estimate for p. Compare the answer to those in (c).

Question 14 a) Prove the relationship $f(\theta \mid \underline{X}) \propto f(X \mid \theta) f(\theta)$ describing each density function in the expression clearly.
b) Explain how Bayes estimation using the absolute error loss function is achieved. Specifically mention how Bayes estimation is more powerful than classical parameter estimation.
c) Given observations $x_1, x_2, ..., x_n$ from a Weibull(η, μ) distribution where η is unknown, derive a posterior distribution for η if no prior information can be specified about η.

Question 15 a) Discuss one advantage of Bayes estimation over other estimation techniques.
b) Prove that the Bayes estimator under the quadratic loss function is the mean of the posterior distribution.

Question 16 Suppose a random sample of size 10 is drawn from a $Weibull(c,2)$ distribution where $c > 0$ and the values $x_1, x_2, ..., x_{10}$ are observed. The probability density function of $X \mid c \sim Weibull(c,2)$ is

$$f_X(x \mid c) = 2cxe^{-cx^2}, \quad x > 0.$$

Suppose $c \sim gamma(\alpha, \lambda)$.

a) Determine the probability density function of the mixture distribution, $f_X(x)$. Note: you do not have to identify the distribution.
b) Determine the posterior distribution of $C \mid \underline{x}$. Identify the distribution and give its parameters.
c) When using the $U(0, \infty)$ distribution as uninformative prior for c, the posterior distribution is

$$C \mid \underline{x} \sim gamma(n+1, \textstyle\sum_{i=1}^{10} x_i^2).$$

Suppose $\sum_{i=1}^{10} x_i^2 = 2566$. Calculate the Bayes estimate for c under the all-or-nothing loss function.

Question 17 The number of claims, X, on a given insurance policy over one year has probability distribution given by

$$P(X = x) = f(x|\theta) = \theta^x(1 - \theta), \quad x = 0, 1, 2, \ldots$$

where θ is an unknown parameter with $0 < \theta < 1$. Prior beliefs about θ can be described by a distribution with density function

$$g(\theta) \propto \theta^2(1-\theta)^2 .$$

Independent observations $x_1, \ldots, x_{15}$ are available for the number of claims in the previous 15 years. Suppose $\sum_{i=1}^{15} x_i = 8$.

a) Calculate the maximum likelihood estimate for $P(X = 0)$.

b) Calculate the Bayes estimate for θ under the squared error loss function.

c) Determine the probability density function of the mixture distribution $f_X(x)$ for the number of claims X