WST 311
Assignment B: 12-16 February 2018

1. Let $\mathbf{X} : p \times 1 = \begin{pmatrix} \boldsymbol{X}_1 : p_1 \times 1 \\ \boldsymbol{X}_2 : p_2 \times 1 \end{pmatrix}$. Then $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are independent if and only if $M_{\boldsymbol{X}}(\boldsymbol{t}) = M_{\boldsymbol{X}_1}(\boldsymbol{t}_1)M_{\boldsymbol{X}_2}(\boldsymbol{t}_2)$. Prove this.

2. Let $\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ where $X_1,\ X_2$ and X_3 are independent and exponential, $X_i \sim EXP(1)$.

   (a) Derive the pdf of $\mathbf{X}$ as well as the moment generating function of $\mathbf{X}$

   (b) Let $Y_1 = X_1$, $Y_2 = X_1 + X_2$ and $Y_3 = X_1 + X_2 + X_3$. Calculate the pdf of $\boldsymbol{Y}$ and the moment generating function of $\boldsymbol{Y}$.

   (c) Use moment generating functions and show that $Y_1,\ Y_2$ and Y_3 are dependent.

   (d) Calculate $E(\boldsymbol{Y})$.

3. Let $\mathbf{X} : 2 \times 1 = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ have a multivariate normal distribution with mean and covariance given by

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}.$$

   Show that the density function is given by

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 \right] \right\}.$$

4. Let $\mathbf{X}'$ be the random vector $\begin{pmatrix} X_1, & X_2, & X_3, & X_4 \end{pmatrix}$ with mean vector $\boldsymbol{\mu}_X' = \begin{pmatrix} 4, & 3, & 2, & 1 \end{pmatrix}$ and

$$\boldsymbol{\Sigma}_{\boldsymbol{X}} = \begin{pmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 7 & -2 \\ 2 & 0 & -2 & 4 \end{pmatrix}.$$

   Partition $\boldsymbol{X}$ as

$$\left(\begin{array}{c} X_1 \\ X_2 \\ \hline X_3 \\ X_4 \end{array}\right) = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{pmatrix}.$$

   Let

$$\boldsymbol{A} = \begin{pmatrix} 1 & 2 \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

   Use PROC IML to calculate the following.

   (a) $E(\boldsymbol{X}_2)$

   (b) $E(\boldsymbol{A}\boldsymbol{X}_2)$

   (c) $cov(\boldsymbol{X}_2)$

   (d) $cov(\boldsymbol{A}\boldsymbol{X}_2)$

   (e) $cov(\boldsymbol{X}_1, \boldsymbol{X}_2')$

   (f) $cov(\boldsymbol{A}\boldsymbol{X}_1, (\boldsymbol{B}\boldsymbol{X}_2)')$

5. Write a SAS/IML program simulating the following theoretical fact:

   Let $X_1,\ X_2,\ \ldots,\ X_n$ be a random sample from $N(0,1)$ distribution. It is then known that $\overline{X}$ will follow a normal distribution, with mean 0 and standard deviation $\frac{1}{\sqrt{n}}$.

   Accept that $\overline{X}$ follows a normal distribution. Demonstrate through simulation that the mean is 0 and the standard deviation is $\frac{1}{\sqrt{n}}$. Simulate 1000 samples, each of size 300. Do the simulation without using a do loop! Give the theoretical and empirical values for the mean and standard deviation.

6. Work through Example A, Questions 1, 2 and 3.