University of Pretoria
Department of Statistics
WST322 Actuarial Statistics
Tutorial Chapter 9

Question 1

Q&A Part 3 – 3.1 - 3.12

Question 2

Consider a portfolio of n policies where the i^{th} policy produces an aggregate claim amount of S_i which has a compound Poisson distribution with parameters λ and the lognormal distribution with parameters μ and σ^2. It is also known that λ has an exponential distribution with mean M.

a) Determine the mean and variance of S_i and of $\sum_{i=1}^{n} S_i$.

b) If μ =5, σ^2 =3 and M = 1.5 and the initial surplus is R4000, calculate the probability of ruin in 3 years of time if there are 300 policyholders, each paying a premium of R500 per year if A normal approximation is used for the distribution of $\sum_{i=1}^{n} S_i$.

Question 3

A group of policy holders experience losses for which the individual loss distribution has an annual mean and variance R2000 and R100 000 respectively. The number of claims in a year from this group of policy holders follows a Poisson distribution with parameter λ and it seems realistic to assume that λ has an exponential distribution with mean 50.

(a) Give a detailed description of the distribution of $S(t)$ = aggregate claim size after t years. Write down expressions for the mean and variance of $S(t)$.

(b) The insurance company charges a premium of $(1+\theta)E(S(t))$, with θ = premium loading = 0.2 and wishes to determine the initial surplus, U, of this portfolio so that the probability of ruin after at most 5 years is only 10%. Assuming $S(t)$ to be approximately normally distributed, find U.

Question 4

An insurer has issued two five-year term assurance policies to two individuals involved in a dangerous sport. Premiums are payable annually in advance, and claims are paid at the end of the year of death.

Individual	*Annual Premium*	*Sum Assured*	*Annual P[death]*
A	100	1700	0.05
B	50	400	0.1

Assume that the probability of death is constant over each of the five years of the policy. Suppose that the insurer has an initial surplus of U. Assuming U = 1000,

(a) Determine the distribution of $S(1)$, the surplus at the end of the first year, and hence calculate $\psi(U,1)$.
(b) Determine the possible values of $S(2)$ and hence calculate $\psi(U,2)$.

Question 5
a) If X and Y are independent Poisson random variables with mean λ, derive the moment generating function of X, and hence show that $X + Y$ also has a Poisson distribution.
b) An insurer has a portfolio of 100 policies. Annual premiums of 0.2 units per policy are payable annually in advance. Claims, which are paid at the end of the year, are for a fixed sum of 1 unit per claim. Annual claims numbers on each policy are Poisson distributed with mean 0.18. Calculate, assuming a normal approximation how much initial capital is needed in order to ensure that the probability of ruin at the end of the year is less than 1%.

Question 6
Consider a Poisson process $\{N(t)\}$ with parameter λ with underlying claims arising from an exponential distribution with expected value 100.
a) Consider the inequality $\Psi(U) \le e^{-RU}$. What is the name of this inequality and why is it useful? Discuss the relationship between $\Psi(U)$ and U , and $\Psi(U)$ and R.
b) Provide an equation to solve for the adjustment coefficient if the insurer uses a loading of 10% on their premiums and solve for the adjustment coefficient.
c) If $X \sim U(50,150)$ instead, derive an equation for the adjustment coefficient. You do not need to simplify the equation.
d) If an excess-of-loss reinsurance policy is now put into place with excess level M , derive the expected value of the reinsurer's claims and the moment generating function of the insurer's claims. You can assume the reinsurer knows about all claims. What value of M would be a good choice between the insurer and reinsurer? Then provide an equation in terms of the formulas derived, as well as M and the reinsurer's loading, for which the adjustment coefficient can be solved for. Assume the claims are distributed exponentially as before.

Question 7
Consider the aggregate claims from a risk where the claims arise according to a Poisson process with parameter λ and the claim sizes are represented by X .
a) Define the adjustment coefficient.
b) Derive the upper bound $\frac{2\theta m_1}{m_2}$ for the adjustment coefficient, where θ is the premium loading used by the insurer, and $m_i = E[X^i]$.
c) Under what condition will a lower bound for the adjustment coefficient exist?

Question 8 Consider a Poisson process $\{N(t)\}_{t\ge 0}$ with parameter λ which models the number of claims for an aggregate claims process $\{S(t)\}_{t\ge 0}$ with underlying claims X . The number of claims and actual claim sizes are all assumed independent.
a) Derive formulas for the expected value and variance of the aggregate claims at time 1.
Assuming a normal distribution approximation for $S(t)$, calculate the probability of ruin at time 1, if the average claim size is 2500 with variance 70^2, the initial surplus set aside is 100 000, the premium rate is 1000 and $\lambda = 2$.

Question 9
Consider the compound Poisson process.

a) If the underlying claim distribution is $\exp(2)$ and the loading factor is 0.4, determine the adjustment coefficient for this process.
b) If proportional reinsurance is put in place with retention level $\alpha = 0.7$, determine and describe the effect of this reinsurance on the adjustment coefficient, if the reinsurer uses a loading of 0.45.
c) Determine the optimum value of the retention level α in order to minimize risk. Also determine the implied adjustment coefficient.

Question 10
The annual aggregate claim numbers from a group of general insurance policies have a compound Poisson distribution with parameter 20. Individual claim amounts have a Pareto distribution with parameters $\alpha = 5$ and $\lambda = 6$.
a) Calculate the expected value and variance of the annual aggregate claim amount, S, that arises from this group of policies.

Suppose that the values in (a) are given in terms of millions of Rands. Also suppose that there are 10 000 policies in the portfolio. The insurer that underwrites these policies sets aside R25 million as initial surplus for this group of policies, and insurer charges an annual premium of R3 000 for each policy. Interest and expenses can be ignored.
b) Assuming that S follows a normal distribution, estimate the probability of ruin for this portfolio at the end of one year.

Question 11 The annual number of claims from a portfolio of policies has a Poisson distribution with parameter λ. The individual claim amount distribution is uniform with parameters $a = 0$ and $b = 100$. The insurer of the portfolio uses a premium loading of $\theta = 10\%$. Excess of loss reinsurance, with retention level M, and a reinsurance premium loading of $\xi = 20\%$ is considered for the portfolio.

a) Show that the expected premium income, net of reinsurance, is given by $(-0.006M^2 + 1.2M - 5)\lambda$.

b) Find the moment generating function (in terms of M) for the insurer's individual claim amount random variable, net of reinsurance.

c) Set up an equation that gives the adjustment coefficient, R, implicitly in terms of M.

d) Find the minimum retention level that will ensure that the insurer's net premium exceeds the insurer's expected claims, net of reinsurance.

Question 12 a) Sort the following ruin probabilities from smallest to largest: $\psi(200,200)$, $\psi_{12}(200,10)$, $\psi(200)$, $\psi(100)$, $\psi(200,10)$, $\psi(0)$.

b) Suppose that individual claim amounts of a compound Poisson process are bounded above by a constant M. Let R be the adjustment coefficient of the process. Assume that for $0 \le x \le M$ the following inequality holds: $\exp(Rx) \le \frac{x}{M}\exp(RM) + 1 - \frac{x}{M}$. Show that R is bounded below by $\frac{1}{M}\log\left(\frac{c}{\lambda m_1}\right)$.

Question 13 Consider a portfolio of policies for which the annual number of claims has a Poisson distribution with parameter λ. The individual claim amounts are modeled uniformly with a maximum claim amount of 250. A premium loading of $\theta = 11\%$ is used. In order to reduce risk for the insurer an excess of loss reinsurance arrangement, with retention level M, and a reinsurance premium loading of $\xi = 20\%$ is considered for the portfolio.
a) What range of values can M be? Why?
b) Derive a formula for the expected annual premium income for the insurer in terms of M and λ.

c) Derive the moment generating function, in terms of M, for the insurer's individual claim amount.
d) Set up an equation to solve for the adjustment coefficient, R, implicitly in terms of M.
e) Find the optimum retention level that will ensure that the insurer's net premium exceeds the insurer's expected claims, net of reinsurance.

Question 14 An insurer has two policyholders, A and B, who are insured for income protection in the event that they can no longer work. A successful claim pays out R150 000 and R200 000 annually for 20 years to each policyholder respectively. The probability of each policyholder claiming this year is 0.05 and 0.01 respectively, with this probability increasing by 1% each subsequent year, and the policyholders each pay an annual premium of R1500 and R2000 respectively this year. Suppose the insurer has an initial surplus of $U =$ R100 000. Let $S(t)$ be the aggregate surplus for these two policyholders for the period of t years.

Provide a distribution for $S(1)$ and calculate $\Psi(U,1)$. You may use the given table.

Claims	$S(1)$	**Probability**

Question 15
a) Define the concepts of ruin, ruin in finite time and ultimate ruin, including the necessary formulas to explain fully.
b) State Lundberg's inequality, explaining all terms involved as well as how the inequality is useful.
c) If the process under consideration is a compound Poisson process with parameter λ and the insurer uses a loading of θ, how is the adjustment coefficient determined?
d) Derive an upper bound for the adjustment coefficient of this process.
e) The insurer is considering introducing proportional reinsurance with level $\alpha = 0.25$. The reinsurer using a loading of δ. How is the adjustment coefficient now determined?
f) Determine the new upper bound for the adjustment coefficient as derived in (d) as a result of the proportional reinsurance in (e).

Question 16
Consider a random variable S with a compound Poisson distribution with parameter λ and underlying claims distribution $f_X(x)$.

a) Derive the coefficient of skewness for S and indicate when a normal assumption for S is valid.
b) An insurer currently using the model S for the aggregate claims from a portfolio decides to take out excess-of-loss reinsurance with retention level M.
    (i) If the reinsurer is only aware of claims in which they make part payment, provide a distribution for the number of claims for the reinsurer indicating all parameters and their distributions.

(ii) How would the reinsurer solve for their adjustment coefficient under this model if they use a loading of δ?

c) In each row indicate which ruin probability in the columns is larger or less than each row probability. Hint: you only need to fill in the upper or lower triangle of the table.

	$\Psi(500)$	$\Psi_1(500)$	$\Psi(100,10)$	$\Psi_1(100,10)$	$\Psi(500,10)$	$\Psi_1(500,10)$	$\Psi(100)$	$\Psi_1(100)$
$\Psi(500)$								
$\Psi_1(500)$								
$\Psi(100,10)$								
$\Psi_1(100,10)$								
$\Psi(500,10)$								
$\Psi_1(500,10)$								
$\Psi(100)$								
$\Psi_1(100)$								

Question 17 Consider the equation $\lambda M_X(R) = cR + \lambda$ for calculating the adjustment coefficient for Lundberg's inequality under a compound Poisson surplus process.

a) Derive an upper bound for R.

b) If a premium loading of θ is used and $X \sim gamma(1,2)$, derive a formula for R in terms of θ.

Question 18 An institution holds a portfolio of independent loans on which a series of payments need to be made. There can be more than one default on a loan and the total number of defaults over a year is N. The average number of defaults on the portfolio over a year is λ. Each time a default occurs, the institution suffers a loss of R X_i (X_i is independent of N). Let S denote the total loss on the portfolio.

a) Write down an expression for S, and determine a suitable distribution for N. The state which type of model it represents.

b) Suppose the total income on interest is $c = R15$ per year. Write down an equation for the institution's surplus at the end of the year.

c) Let $\lambda = 2$, $E[X_i] = R5$ and $\mathrm{var}(X_i) = 2$, and assume that S is approximately normally distributed. What level of capital should the institution hold to be at least 97.5% certain that losses from defaults will not exceed interest income?

d) Suppose the institution holds the amount of initial capital calculated in (c). What is the probability of ruin at the end of year 1?

Question 19 a) Explain why $\Psi(U) > \Psi(U,t)$.

b) State Lundberg's inequality and explain its implication. Explain all components of the inequality.

c) Derive an upper bound for the adjustment coefficient R under a compound Poisson model with parameter λ.

d) Derive an equation to determine the adjustment co-efficient R under a compound Poisson model with parameter λ with excess-of-loss reinsurance in place with a retention level K. The insurer uses a loading of δ and the reinsurer uses a loading of σ. You may assume the reinsurer knows about all claims that occur.

Question 20 a) State Lundberg's inequality defining all terms and explaining what it is used for.
b) An insurer models hail storm claim as follows:
- The number of storms over a time period t is distributed Poisson with a rate λ.
- The number of claims for the i^{th} storm is distributed Poisson with a rate μ_i and where $u_1, u_2, u_3, \ldots$ are i.i.d. realisations from a U(2,10) distribution.
- The individual claim values are distributed gamma(α, β) and are independent.
- The storms are independent.

(i) Provide a formula for $S(t)$, the aggregate claims from hail storms over time period t, describing all random variables.
(ii) Derive $E[S(t)]$.
(iii) Derive an expression for the variance of the aggregate claim amount per storm.
(iv) Derive an expression for the variance of $S(t)$.
(v) Provide an equation to solve for the required initial surplus for a probability of ruin at $t = 2$ to be 0.05, if a premium loading of θ is used.

Question 21 Derive an equation to solve for the adjustment coefficient if proportional reinsurance with proportion paid by the insurer of δ. It is known that the underlying claims follow an $\exp(\alpha)$ distribution, and the insurer uses a premium loading of θ and the reinsurer uses a premium loading of β. The number of claims is assumed to follow a Poisson distribution with parameter σ.

Question 22 Consider the aggregate claims process $\{S(t)\}_{t \geq 0}$ with $N(t)$ modelled as a Poisson process with parameter λ so that $S(t)$ is then a compound Poisson process.
a) Find $E[S(t)]$ and $\text{var}(S(t))$ in terms of the moments of the underlying claim size distribution.
b) The adjustment coefficient R is the unique positive root of the equation $\lambda M_X(R) = \lambda + cR$. Provide a formula for the premium income per unit time c if the insurer uses a loading of θ.
c) Derive an upper bound for the adjustment coefficient R.
d) Derive an equation to solve for the adjustment coefficient if excess-of-loss reinsurance is put in place with retention level M and the reinsurer is aware of all claims. The reinsurer uses a loading of α. You do not need to simplify any integrals.

Question 23 a) Explain why the following two statements are true
(i) $\Psi_h(U_2, t) \leq \Psi_h(U_1, t)$ for $U_1 \leq U_2$
(ii) $\Psi_h(U, t) \leq \Psi(U, t)$
b) Consider an aggregate claims process $S(t)$ for a portfolio with the number of claims $N(t)$ a Poisson process with parameter λ. If the underlying claims X follow an exp(0.01) distribution, derive the adjustment coefficient for this portfolio in terms of the loading factor θ.

Question 24 a) For an aggregate claims process $S(t)$ of a portfolio with the number of claims $N(t)$ a Poisson process with parameter λ, if proportional reinsurance with percentage retained δ is put in place, derive an equation to solve for the adjustment coefficient if the underlying claims X follow a N(100,20) distribution. You can assume that the insurer's loading is α and the reinsurer's loading β. Show clearly all steps of your workings. The equation should be in terms of δ, α and β.
b) Discuss the applicability of using the normal distribution to model the claim size in (a).
c) Describe how you would, in practice, solve the equation obtained in (a).