University of Pretoria
Department of Statistics
WST322 Actuarial Statistics
Tutorial Chapters 7 and 8

Question 1

Q&A Part 2 2.11 - 2.26

Question 2

Consider the collective risk model $S = \sum_{i=1}^{N} X_i$, where X_i represents the claim size, with continuous distribution function $F(x)$ and raw moments m_1, m_2 and m_3, and N, the number of claims, having some discrete distribution.

(a) Derive a general expression for the moment generating function of S.

(b) Define the cumulant generating function, say $C_X(t)$, of a random variable X and explain (without proof) the relationship between $C_X(t)$ and the first three moments of X.

(c) Derive the coefficient of skewness of S if S has a compound Poisson distribution with parameters $F(x)$ and λ. Explain how this result can be used to support the assumption that S has an approximate normal distribution.

Question 3

Consider a portfolio of policies where the annual individual claim numbers are Poisson(μ) distributed and the number of claiming policy holders, N, has a Poisson(λ) distribution.

(a) Find an expression for the cumulant generating function of S, the aggregate claim numbers.

(b) Use (a) to derive expressions for $E(S)$, $Var(S)$ and skew(S).

(c) Suppose each policyholder claims only a few times per year, under what conditions can the normal approximation for the distribution of S be used? Explain your answer.

Question 4

A risk consists of independent policies, where the claim numbers follow a negative binomial distribution with parameters k and p, while the claim sizes are distributed according to a gamma distribution with parameters α and λ.

(a) Find an expression for the cumulant generating function of $S = \sum_{i=1}^{N} X_i$, the aggregate claims, where $C_S(t) = \log(M_S(t))$.

(b) Use the cumulant generating function to derive expressions for the mean and variance of S and compare the expressions with expressions that you derived from the following two formulas:

$E(S)=E(N)E(X)$ and $Var(S)=Var(X)E(N)+Var(N)(E(X))^2$

Question 5

Individual claim numbers X_i in a portfolio of policies follow a Poisson(μ) distribution. Let $S=\sum_{i=1}^{N} X_i$ denote the aggregate claim numbers in the portfolio, where N has a Poisson(λ) distribution.

a) Derive the moment generating function $M_S(t)$ of S.

b) Find an expression for the cumulant generating function of S.

c) Derive the mean, variance and coefficient of skewness of S using (b).

d) Show that S will still have a positively skewed distribution for small λ even is $\mu \to \infty$.

Question 6

a) Consider a collective risk model $S=\sum_{i=1}^{N} X_i$ where $S=0$ if $N=0$ and N is a discrete random variable. Derive general expressions for $E[S]$ and $\text{var}(S)$ in the collective risk model. Also prove that the moment generating function of S is given in terms of the moment generating function X as follows: $M_S(t)=M_N(\ln M_X(t))$

b) An insurance company insures 5 risks, each of which produces aggregate claims. For each risk, the claim sizes follow an exponential distribution and the claims arise according to a Poisson process. The parameters of the Poisson processes and exponential distributions are given in the second and third columns of the table below. It is assumed that the risks are independent.

Risk	λ	α	b	q	n
1	20	0.01	50000	0.002	1200
2	15	0.04	40000	0.003	750
3	25	0.02	30000	0.005	1500
4	10	0.02	20000	0.008	900
5	30	0.025	10000	0.01	1600

(i) Find the mean and variance of S, the overall claim amount.

(ii) Determine the probability that S will exceed an amount of 7500, assuming S is approximately normally distributed.

c) Suppose it is now also known that the 5 risks in (b) are for 5 groups of policyholders for which the insurance company supplies group life coverage with a fixed benefit amount b. The probability to die in the next year for an individual in each group is given by q. Columns 4 and 5 in the table give the respective benefit amounts and probabilities to die, while column 6 gives the number of policyholders in each group.

(i) State the model and relevant assumptions for modeling the aggregate claims through the individual risk model, carefully defining all symbols used. Find the mean and variance of the tital benefit amount paid out during the year.

(ii) What should the premium per individual policyholder be if the probability for the total benefit claims to exceed the total premium income must not exceed 0.01? (Assume the total benefit claims to be normally distributed.)

Question 7

An insurance portfolio contains 100 policies which are categorized into three independent categories of policyholders, namely A, B and C. The probability of a claim on an individual policy is p, and at most one claim per year is possible. Individual claim sizes are assumed to be exponentially distributed with mean 4 and are independent from claim to claim. The value of p, depending on the category of the policyholder, is

Category	*Value of* p	*Proportion of policyholders*
A	0.1	20%
B	0.2	60%
C	0.15	20%

Denote by S the total amount claimed in one year.

a) Calculate $E[S]$ and $\text{var}(S)$.

b) Show that skewness of a compound binomial distribution with parameters q and n and distribution function $F_X(x)$ is given by $nqm_3 - 3nq^2m_2m_1 + 2nq^3m_1^3$.

c) *Explain* how you would go about finding a value for which the probability is 0.3 that S will exceed that value. Justify all assumptions needed.

d) The insurer thinks that a better way to model their claim numbers may be with Poisson distributions with parameters as given in the table below.

Category	*Value of* λ	*Proportion of policyholders*
A	2	20%
B	12	60%
C	3	20%

(i) Explain how the values of λ were decided on.

(ii) What will the distribution for S be now? Give all necessary parameters.

Question 8

Give a full explanation of the collective risk model and individual risk model, including all assumptions. State clearly the differences between the two insurance models and for what type of insurance policies each would be applicable.

Question 9

A bicycle wheel manufacturer claims that its products are virtually indestructible in accidents and therefore offers a guarantee to purchasers of pairs of its wheels. There are 250 bicycles covered, each of which has a probability p of being involved in an accident (independently). Despite the manufacturer's publicity, if a bicycle is involved in an accident, there is in fact a probability of 0.1 for each wheel (independently) that the wheel will need to be replaced at a cost of R100. Let S denote the total cost of replacement wheels in a year.

a) Show that the moment generating function of S is given by $M_S(t) = \left[\dfrac{pe^{200t} + 18pe^{100t} + 81p}{100} + 1 - p\right]^{250}$.

b) Using (a), show that $E[S] = 5000p$ and $Var(S) = 550{,}000\,p - 100000\,p^2$.

Suppose instead that the manufacturer models the cost of replacement wheels as a random variable T based on a portfolio of 500 wheels, each of which (independently) has a probability of 0.1p of requiring replacement.

c) Derive expressions for $E[T]$ and Var(T) in terms of p.

d) Suppose $p = 0.05$.

(i) Calculate the mean and variance of S and T.

(ii) Calculate the probabilities that S and T exceed R500, assuming a normal approximation.

(iii) Comment on the differences.

Question 10

The total claims arising from a certain portfolio of insurance policies over a given month is represented by

$$S = \begin{cases} \sum_{i=1}^{N} X_i & if \ \ N > 0 \\ 0 & if \ N = 0 \end{cases}.$$

where N has a Poisson distribution with mean 2 and $X_1, X_2, ..., X_N$ is a sequence of independent and identically distributed random variables that are also independent of N. Their distribution is such that $P[X_i = 1] = 1/3$ and $P[X_i = 2] = 2/3$. An aggregate reinsurance contract has been arranged such that the amount paid by the reinsurer is $S-3$ (if $S > 3$) and zero otherwise. The aggregate claims paid by the direct insurer and the reinsurer are denoted by S_I and S_R, respectively. Calculate $E[S_I]$ and $E[S_R]$.

Question 11

Describe the collective and individual risk models respectively, clearly indicating the differences between the two models. Also, give an example of an insurance product where each would be applicable.

Question 12

Consider the collective risk model $S = \sum_{i=1}^{N} X_i$, where X_i represents the claim size, with continuous distribution function $F_X(x)$ and raw moments m_1, m_2 and m_3, and N, the number of claims, having some discrete distribution.

(a) Derive a general expression for the variance of the random variable S.

(b) Define the cumulant generating function, $C_X(t)$, of a general random variable X and explain (without proof) the relationship between $C_X(t)$ and the first three moments of X.

(c) Derive formulas for the expected value and variance of S using its cumulant generating function if S has a compound binomial distribution with parameters $F_X(x)$, k and p. Make use of the formula for the skewness of S, given by $kpm_3 - 3kp^2m_2m_1 + 2kp^3m_1^3$ to explain if it is justified or not to assume that S has an approximate normal distribution. How do the formulas for the expected value and variance affect the normal assumption if it can be assumed?

Question 13

Consider two group life insurance policies belonging to two independent companies. Company A has 3000 independent lives insured for R100 000 each and they pay a premium of R80 per life. Company B has 500 independent lives insured for R80 000 each and they pay a premium of R75 per life. Let $\mu_i, i = A, B$ be the probability of a claim for Company i.

a) Provide two loss distributions, from the tables, which would be suitable to model $\mu_i, i = A, B$ for each company.

b) Suppose $\mu_i \sim Beta(2, \beta_i)$ for $i = A, B$. What is probability of a single claim that each company can expect if the parameters in the table below have been calculated for each company by the insurer.

	β_i
Company A	198
Company B	98

c) Derive formulas, and then calculate values, for the expected total claim amount for each company and the two portfolios together. Comment on the premiums charges by the insurance company.

d) Derive formulas, and then calculate values, for the variance of the total claim amount for each company and the two portfolios together.

Question 14

Consider three independent policyholders who can each claim from their car insurance a maximum of 10 times over a 3 year period. The underlying claims X arise from a $Beta(\alpha,\delta)$ and the probability of a claim for each policyholder is μ which has the following distribution: $\mu = \begin{cases} 0.2 & with\ probability\ 0.4 \\ 0.3 & with\ probability\ 0.2 \\ 0.4 & with\ probability\ 0.4 \end{cases}$.

a) Decide whether a collective or individual risk model would be best suited to model the total claims over the three year period from this risk and describe your model fully (formulas, distributions, assumptions etc.)

b) Calculate the variance of the total claim amount over the three year period from this risk, correct to 4 decimal places, if $\alpha = 3$ and $\delta = 5$.

c) How would your answer for (b) change if the policyholders each had the same probability of claiming, μ, and were thus not independent? You only need indicate where in your calculations the changes will occur and how they change, and do not need to give the final numerical answer.

d) Decide whether a collective or individual risk model would be best suited to model the total claims over the three year period from this risk if each policyholder was only allowed to claim once for a set claim size of 100 000 and describe your model fully (formulas, distributions, assumptions etc.)

Question 15

a) Show that $S = X_1 + X_2$ has a $\text{gamma}(2,2)$ distribution where X_1 and X_2 are i.i.d. and follow an $\exp(2)$ distribution.

b) Suppose that $\{X_i\}_{i=1}^{\infty}$ are discrete random variables distributed on the positive integers. For a compound random variable $S = \sum_{i=1}^{N} X_i$ we have that $S = 0$ when $N = 0$. Derive a general formula for the distribution and density function of S, namely $G_S(s)$ and $g_S(s)$. Clearly define any notation used.

Question 16 Let S_1 and S_2 be independent compound Poisson random variables with parameters λ_i and $F_i(x)$, $i = 1,2$ respectively, where $F_i(x)$ denotes the distribution function of the individual claim amounts underlying S_i. Define $A = S_1 + S_2$.

Prove that A has a compound Poisson distribution with parameters Λ and $F(x)$, where $\Lambda = \lambda_1 + \lambda_2$ and $F(x) = \frac{1}{\Lambda}(\lambda_1 F_1(x) + \lambda_2 F_2(x))$.

The following notation should be used: The moment generating functions for A, $F_i(x)$ and $F(x)$ should be denoted by $M_A(t)$, $M_i(t)$ and $M(t)$ respectively.

Question 17 Suppose that the aggregate claims, denoted by S, has a compound distribution, with individual claim sizes that are independent and identically distributed. Also, the number of claims is independent of the respective claim sizes. Derive the following formulas:

(a) $E(S) = E(N)E(X)$

(b) $\text{var}(S) = E(N)\,\text{var}(X) + \text{var}(N)[E(X)]^2$

(c) $M_S(t) = M_N(\log M_X(t))$

Question 18

Consider the following two scenarios:

Scenario 1: A group of policyholders for car insurance are grouped together by an insurance company as they are considered homogeneous with regard to their predicted claim patterns.

Scenario 2: A group of n policyholders for life insurance are grouped together by an insurance company as they are considered homogeneous with regard to their predicted claim patterns.

In each scenario the insurer is interested in the aggregate claim amount from the portfolio of policyholders.

Use the two scenarios to clearly explain the collective and individual risk models respectively, making sure you

i) explain the terms ‘collective’ and ‘individual’ in each case,

ii) mention claim numbers per policyholder, and claim numbers for the group of policyholders,

iii) provide formulas, explain all concepts, distributions and assumptions.

Question 19

Consider a portfolio of n policies in which each policyholder is insured for two risks, namely car insurance and car scratch repair. The number of claims per policyholder per year on the car insurance is unlimited but each policyholder is limited to 2 claims per year on car scratch repair. The number of car insurance claims from the portfolio are known to follow a Poisson distribution with parameter α and the underlying claims are from a LogNormal(μ,σ^2). The probability of a policyholder claiming for scratch repair is p and the claim amounts for scratch repair are standard at R500. The parameters μ,σ^2 and p are known, however α is assumed to have a Pareto(β,δ,κ) distribution.

a) Provide a model for the aggregate claims from this portfolio for a single year.

b) Suppose the insurer has second similar portfolio with m policyholders, the only difference being the parameters $\mu,\sigma^2,\beta,\delta,\kappa$ and p, say $\mu_{(2)},\sigma_{(2)}{}^2,\beta_{(2)},\delta_{(2)},\kappa_{(2)}$ and $p_{(2)}$, Determine a formula for the variance of the aggregate yearly claims from these two portfolios combined of *only* the car insurance claims.

Question 20

Complete the following proof for a compound distribution $T=\sum_{i=1}^{M} Y_i$ where M and $\{Y_i\}$ are independent and $\{Y_i\}$ are i.i.d.

(i) $E[T]=E[M]E[Y]$

Proof

$$E[T] \quad = E_M\left[E[T\mid M]\right]=E_M\left[E\left[\sum_{i=1}^{M} Y_i \mid M\right]\right]$$

Now $$E\left[\sum_{i=1}^{M} Y_i \mid M=m\right] \qquad = E[\_\_\_\_\_\_\_\_\_\_\_\_]$$

$$=\sum_{i=1}^{m} E[\_\_\_\_\_\_\_\_\_\_\_\_\_\_] \text{ (since \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)}$$

So

$E[T] \quad = E_M\left[ME[Y]\right]$(since____) $= E[Y]E[M]$

(ii) $\quad \mathrm{var}(T) = \mathrm{var}(M)E[Y]^2 + E[M]\mathrm{var}(Y)$

Proof

$\mathrm{var}(T) \ =$ ____

$\quad = \mathrm{var}_M\left(ME[Y]\right) + E_M\left[\mathrm{var}(T \mid M)\right]$

Now

$\mathrm{var}(T \mid M = m) = \mathrm{var}\left(\sum_{i=1}^{m} Y_i\right) = \sum_{i=1}^{m} \mathrm{var}(Y_i)$ (since____

$\quad = m\,\mathrm{var}(Y)$ (since____

So

$\mathrm{var}(T) \ = \mathrm{var}_M\left(ME[Y]\right) + E_M\left[M\ \mathrm{var}(Y)\right] =$ ____

(iii) $\quad M_T(w) = M_M(\ln M_Y(w))$

Proof

$M_T(w) = E[\_\_\_\_\_\_\_\_\_\_]$ (1/2 *mark*) $= E_M\left[E\left[e^{w\sum_{i=1}^{M} Y_i} \mid M\right]\right]$

Now

$E\left[e^{w\sum_{i=1}^{M} Y_i} \mid M = m\right] \quad = E\left[e^{\_\_\_\_}.e^{\_\_\_\_}\ldots e^{\_\_\_\_}\right]$

$\quad = \prod_{i=1}^{m} E[e^{wY_i}]$ (since ________________________________)

$\quad = \left(E[e^{wY}]\right)^m$ (since _________________________________)

So

$M_T(w) = E_M\left[M_Y(w)^M\right] = E_M\left[e^{\_\_\_\_\_\_\_\_}\right] = M_M(\ln M_Y(w))$.

Question 21

Complete the table below describing the collective and individual risk models respectively.

	Collective Risk Model	Individual Risk Model
Explain the reason for the name of the model i.e. ‘collective’ and ‘individual’.		
Provide formulas for the model.		
Provide assumption(s) of the model		
Claims numbers per policyholder.		
Claim numbers for the group.		
Example of insurance for this model		

Question 22

Consider a portfolio of k policyholders in which each policyholder is insured for **two** risks. The first risk provides car insurance for each policyholder with the number of yearly claims per policyholder distributed $Bin(5, p)$ and the underlying claims from a LogNormal(μ, σ^2). The second risk provides life insurance to the policyholder with a sum of R150 000 on death and the probability of a policyholder claiming is q. The parameters μ, σ^2 are known, however p is assumed to have a Beta(β, δ) distribution.

a) Provide a model for the aggregate claim amounts for car insurance risk of the portfolio, with all applicable distributions.

b) Provide a model for the aggregate claim amounts for the life insurance risk of the portfolio, with all applicable distributions.

c) Derive a formula for the expected aggregate claims on the portfolio, in terms of β, δ, μ and σ.

d) Suppose the insurer has a second similar portfolio with m policyholders, and is only interested in the car insurance part of the two portfolios. The claim amounts for this second portfolio are distributed $LogN(\eta, \lambda^2)$ and the number of yearly claims per policyholder distributed $Bin(4, p)$.

(i) Describe (fully) the dependency (if any) between the two car insurance risks in the two portfolios.

(ii) Derive a formula for the variance of the combined portfolios described in (d) in terms of $\mu, \sigma, \beta, \delta, \eta$ and λ.

Question 23

Consider a collection of n portfolios in which each portfolio is modelled as an individual risk model. Each portfolio $i = 1, 2, ..., n$ has the following parameters:

- Probability of an individual claiming: q_i
- Underlying claims distribution: gamma(α_i, λ_i)

a) Set up a model specification for this collection of portfolios, namely equations and distributions.

b) Are the individual portfolios independent? Explain.

c) Calculate $E[S]$ and $\mathrm{var}(S)$.

d) The underlying claims are gamma(α, λ) instead of gamma(α_i, λ_i) how do you answers to (b) and (c) change, with $\alpha = 12$ and $\lambda = 3$?

Question 24

Consider a compound random variable $S = \sum_{i=1}^{N} X_i$. Derive $E[S], \mathrm{var}(S)$ and $M_S(t)$ stating all assumptions.

Question 25

Consider a portfolio of m policyholders in which each policyholder is insured for **two** risks. The first risk provides lump sum disability insurance for each policyholder i with the number of claims restricted to 1 per policyholder over a lifetime, with a probability of a policyholder claiming of p and a lump sum payout of R500 000. The second risk provides life insurance to the policyholder i with a sum of R150 000 on death and the probability of a policyholder claiming of q_i. The parameters p and q are unknown with $P[p = 0.4] = 0.5$ and $P[p = 0.3] = 0.5$ and $q_i \sim Beta(\alpha, \beta)$.

a) Provide a model for the aggregate claim amounts for the disability insurance risk of the portfolio, with all applicable distributions.

b) Provide a model for the aggregate claim amounts for the life insurance risk of the portfolio, with all applicable distributions.

c) Are the individual policyholders independent? Explain.
d) Derive a formula for the expected aggregate claims on the portfolio.
e) Derive a formula for the variance of the only the disability insurance part of the portfolio.

Question 26 Describe in full the collective risk model with all its assumptions and give a practical example where it can be applied. Also provide formulas for the expected aggregate claim amount and the variance of the aggregate claim amount, and comment on the claim numbers per policyholder and claims numbers for the whole portfolio.

Question 27 Consider the collective risk model $S = \sum_{i=1}^{N} X_i$, and let $M_Y(t)$ denote the moment generating function of a random variable Y.

a) What assumptions are required for the identity $M_S(t) = M_N(\log M_X(t))$ to hold?
b) Prove the identity $M_S(t) = M_N(\log M_X(t))$.

Question 28 An insurer has d groups of independent life insurance policies. For the j^{th} group, $j = 1,2,..,d$, we have the following:

- n_j : number of lives in the group
- p_j : probability of a death for a single life
- B_j : the amount paid upon the death of a life (this is a fixed monetary amount).
- Y_{ij} : the random variable representing the amount paid for the i^{th} life in group j

Let S be the total claim amount from all the groups.

a) Derive expressions for $E[S]$ and $\text{var}(S)$ in terms of n_j, p_j and B_j.
b) How do your answers to (a) change if only the groups are independent but the individual policies within each group are not? Explain why.

Question 29 Claims on a group of insurance policies follow a compound binomial distribution with parameters n and p, where $P[p = 0.1] = P[p = 0.3] = 0.5$. Derive $E[S]$ and $\text{var}(S)$, where S is the total claim amount, and $m_1 = E[X]$ and $m_2 = E[X^2]$.

Question 30 Give a full explanation of the collective risk model and individual risk model, including all assumptions. State clearly the differences between the two insurance models and for what type of insurance policies each would be applicable.

Question 31 Consider a compound random variable $S = \sum_{i=1}^{N} X_i$.

a) Derive the $\text{var}(S)$ under the assumption that N and $\{X_i\}$ are independent and X_i's are i.i.d.
b) Derive the $\text{var}(S)$ for a compound Poisson process with parameter μ.
c) Derive the coefficient of skewness for a compound Poisson process with parameter μ. You may use without proof that in this case $E[S] = m_1\mu$ and $skewness(S) = m_3\mu$. Also discuss the symmetry of the distribution of S for large claim numbers.

Question 32 An inexperienced actuary has been assigned to investigate a proportional reinsurance policy for a certain portfolio of the company he works for. He recalls from his undergraduate studies that reinsurance affects the variance and expected value of the total claim amount in some way. By deriving the $E[S]$ and $\text{var}(S)$ before reinsurance and $E[S_I]$ and $\text{var}(S_I)$ after proportional reinsurance with proportion α is applied, discuss the implication. The number of claims has a Poisson(λ) distribution and the claim amounts are distributed Gamma(σ, β).

Question 33 Consider a portfolio of m polices in which each policy has a maximum of one claim. A claim occurs with probability p and if a claim occurs it follows a $\text{N}(\mu,\sigma^2)$ distribution. The variance σ^2 is however unknown but σ is presumed to follow a $\text{Beta}(a,b)$ distribution.

a) Explain the term *conditionally independent random variables.*

b) Derive the variance of the total claim amount on this portfolio.

Question 34 Consider the collective risk model.

a) Provide a real life example of such a model and explain why it fits such a model.

b) Derive a formula for the variance of the aggregate claims S for the collective risk model.

c) Use your answer in (b) to determine the variance of the compound Poisson distribution.

Question 35 Consider a portfolio with n policies where the aggregate claims from the i^{th} policy has a compound Poisson distribution with parameters λ_i and $F_X(x)$. The λ_i's are unknown and are assumed to come from an exponential distribution with parameter 0.3.

a) Provide equations for the aggregate claim amount from the portfolio, providing the distribution of each random variable.

b) Derive the expected value and variance of the claim numbers per policy.

c) Explain why the policies are independent.

d) Determine the expected value and variance of the whole portfolio in terms of the moments of X .

Question 36 Consider the individual risk model.

a) Define the model fully and show that the compound binomial model can be used in the model.

b) Derive the expected value and variance of the aggregate claims for the individual risk model.

Question 37 Consider a portfolio in which each policy can have a maximum of 3 claims per year, with the probability of a single claim being p. This probability is, however, unknown and assumed to follow a $\text{Beta}(2,4)$ distribution.

a) Explain why the Beta is an appropriate choice for the distribution of p .

b) Write down the random variable and distribution for the number of claims per policy.

c) Provide a formula for the aggregate claims from the portfolio if the underlying claims are represented with the random variable X .

d) If the moments of X are given by $m_i, i = 1,2,3,...$ derive a formula for the expected value of the aggregate claim amounts.

e) If the moments of X are given by $m_i, i = 1,2,3,...$ derive a formula for the variance of the aggregate claim amounts.