University of Pretoria
Department of Statistics
WST322 Actuarial Statistics
Tutorial 6 Memorandum – 1 and 8 September 2011

Question 1

Q&A Part 2: 2.11 - 2.26

Question 2

Consider the following scenario: An insurance company issues a policy to n policyholders and claims are made independently, where the number of claims follow a negative binomial distribution with parameters k and p. Claim sizes have a Pareto (α, λ) distribution. The aggregate claims for a one-year period are denoted by S.

(a) Write down expressions for the mean, variance and moment generating function of S.

$$E(S) = E(N)E(X) = \frac{kq}{p} \times \frac{\lambda}{\alpha - 1}$$

$$Var(S) = E[N]Var(X) + Var(N)[E(X)]^2$$

$$= \frac{kq}{p} \times \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} + \frac{kq}{p^2} \times \frac{\lambda^2}{(\alpha-1)^2}$$

$$M_S(t) = M_N(\log M_X(t))$$

$$= \left(\frac{p}{1 - qM_X(t)}\right)^k$$

(b) The company considers to using proportional reinsurance with retention level ε. Write down expressions for the mean and variance of the net amount of aggregate claims and then calculate their values if ε =0.8, k = 6, p = 0.01, α = 3 and λ = 750.

$$E(\varepsilon S) = \varepsilon \frac{kq}{p} \times \frac{\lambda}{\alpha - 1} = 178200$$

$$Var(\varepsilon S) = \varepsilon^2 \{\frac{kq}{p} \times \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} + \frac{kq}{p^2} \times \frac{\lambda^2}{(\alpha-1)^2}\} = 5506380000 = (74205)^2$$

(c) If the direct insurer receives a net premium income of c per policyholder (c = gross premium – reinsurance premium) and has an initial surplus (from the preceding years) of U = 5000 available, determine n so that the probability that the claims exceed the combined policy income and surplus will be at most 0.25, given that c = 500 and if a normal approximation is used for the distribution of S .

$$total = U + nc - S$$

$P(Total < 0) < 0.25$

$i.e \;\; P(S > U + nc) < 0.25$

$i.e. \;\; P(Z < \dfrac{U + nc - 178000}{74205}) > 0.75$

$i.e \;\; \dfrac{U + nc - 178000}{74205} > 0.67$

$i.e. \; n > \dfrac{0.67 \times 74205 + 178200 - 5000}{500} = 446$

Question 3

Consider the collective risk model $S = \sum_{i=1}^{N} X_i$, where X_i represents the claim size, with continuous distribution function $F(x)$ and raw moments m_1, m_2 and m_3, and N, the number of claims, having some discrete distribution.

(a) Derive a general expression for the moment generating function of *S*.

$$
\begin{aligned}
M_S(t) &= E_N\left[E\left[\exp\left\{ t\sum_{i=1}^{n} X_i \right\} \middle| N = n \right] \right] \\
&= E_N\left[\prod_{i=1}^{n} E\left[\exp\{tX_i\} \middle| N = n\right] \right] \quad \text{since the } X_i\text{'s are independent} \\
&= E_N\left[\prod_{i=1}^{n} M_{X_i}(t) \middle| N = n \right] \\
&= E_N\left[\prod_{i=1}^{n} M_X(t) \middle| N = n \right] \qquad \text{since the } X_i\text{'s are indentically distributed} \\
&= E_N\left[(M_X(t))^N \right] \\
&= E_N\left[e^{N \ln M_X(t)} \right] = M_N(\ln M_X(t))
\end{aligned}
$$

(b) Define the cumulant generating function, say $C_X(t)$, of a random variable X and explain (without proof) the relationship between $C_X(t)$ and the first three moments of X .

$$
\begin{aligned}
&C_X(t) = \ln M_X(t) \\
&C_X'(0) = \mu_X \\
&C_X''(0) = \sigma_X^2 = E\left[(X - \mu_X)^2\right] \\
&C_X'''(0) = skewness(X) = E\left[(X - \mu_X)^3\right]
\end{aligned}
$$

(c) Derive the coefficient of skewness of *S* if *S* has a compound Poisson distribution with parameters *F(x)* and λ. Explain how this result can be used to support the assumption that *S* has an approximate normal distribution.

$$\begin{aligned} M_S(t) &= M_N(\ln M_X(t)) \textit{ from } (a) \\ &= e^{\lambda\left(e^{\ln M_X(t)}-1\right)} \\ &= e^{\lambda(M_X(t)-1)} \\ C_S(t) &= \ln M_S(t) \\ &= \lambda(M_X(t)) \end{aligned}$$

$$C_S^{'}(t) = \lambda(M_X'(t)) \Rightarrow C_S^{'}(0) = \lambda(M_X'(0)) = \lambda m_1$$

$$C_S^{''}(t) = \lambda(M_X''(t)) \Rightarrow C_S''(0) = \lambda(M_X''(0)) = \lambda m_2$$

$$C_S^{'''}(t) = \lambda(M_X'''(t)) \Rightarrow C_S^{'''}(0) = \lambda(M_X'''(0)) = \lambda m_3$$

Coefficient of Skewness $= \dfrac{\lambda m_3}{(\lambda m_2)^{3/2}} = \dfrac{m_3}{(m_2)^{3/2}\sqrt{\lambda}} \xrightarrow[\infty]{\lambda} 0$ *thus S is approximately symmetric for large λ i.e. for large expected number of claims. Since the normal distribution is symmetric, we have an indication that S may be approximately normal for large λ.*

Question 4

Consider a portfolio of policies where the annual individual claim numbers are Poisson(μ) distributed and the number of claiming policy holders, *N*, has a Poisson(λ) distribution.

(a) Find an expression for the cumulant generating function of *S*, the aggregate claim numbers.

$$S = \sum_{i=1}^{N} X_i \sim CompPoisson(\lambda, Poisson(\mu))$$

$$M_S(t) = M_N(\ln M_X(t)) = e^{\lambda(M_X(t)-1)} = e^{\lambda\left(e^{\mu(e^t-1)}-1\right)}$$

$$K_S(t) = \lambda\left(e^{\mu(e^t-1)}-1\right)$$

(b) Use (a) to derive expressions for *E*(*S*), *Var*(*S*) and skew(*S*).

$$K_S'(t) = \lambda\left(e^{\mu(e^t-1)}-1\right)\mu e^t \;\therefore\; K_S'(0) = \lambda\mu = E[S]$$

$$K_S''(t) = \lambda\left(e^{\mu(e^t-1)}-1\right)\left(\mu e^t\right)^2 + \lambda\left(e^{\mu(e^t-1)}-1\right)\mu e^t \;\therefore\; K_S''(0) = \lambda\mu^2 + \lambda\mu = \lambda(\mu^2+\mu) = \mathrm{var}(S)$$

$$K_S'''(t) = \lambda\left(e^{\mu(e^t-1)}-1\right)\left(\mu e^t\right)^3 + \lambda\left(e^{\mu(e^t-1)}-1\right)2\left(\mu e^t\right)^2 + \lambda\left(e^{\mu(e^t-1)}-1\right)\left(\mu e^t\right)^2 + \lambda\left(e^{\mu(e^t-1)}-1\right)\mu e^t$$

$$\therefore K_S'''(0) = \lambda\mu^3 + 2\lambda\mu^2 + \lambda\mu^2 + \lambda\mu = \lambda(\mu^3 + 3\mu^2 + \mu) = skewness(S)$$

(c) Suppose each policyholder claims only a few times per year, under what conditions can the normal approximation for the distribution of S be used? Explain your answer.

$CoS(S) = \dfrac{skewness(S)}{\mathrm{var}(S)^{3/2}} = \dfrac{\lambda(\mu^3+3\mu^2+\mu)}{\left(\lambda(\mu^2+\mu)\right)^{3/2}}$ *which is a function of μ and $1/\sqrt{\lambda}$. For large λ, $Cos(S) \to 0$ which implies that the normal approximation will be good for large λ i.e. a large number of policyholders claiming.*

Question 5

A medical insurance company has three large corporate clients. The annual claim numbers of each corporate client has a Poisson(μ) distribution (same for all clients), where μ comes from a Gamma(α, β) distribution. Claim sizes are Pareto(δ, λ) distributed. A summary of the relevant information for each corporate client is given in the table below, which shows the parameter values and numbers of employees per client.

Client i	δ_i	λ_i	n_i
A	3	2000	2500
B	2.5	1800	1500
C	3.5	7500	1000

(a) Derive a general expression (in terms of the parameters α, β, δ_i etc) for the expected value and variance of the aggregate claims for all three corporate clients (do not use the numerical values given in the table here). (Hint: $E(S) = E(N)E(X)$ and $Var(S) = E[N]Var(X) + Var(N)[E(X)]^2$ as well as $Var(S) = E[Var(S \mid \mu)] + Var[E(S \mid \mu)]$).

$$E[S_i \mid \mu] = \mu m_1 = \mu \frac{\lambda_i}{\delta_i - 1}$$

$$E[\sum_{i=1}^{n} S_i \mid \mu] = \mu \sum_{i=1}^{n} \frac{\lambda_i}{\delta_i - 1}$$

$$\text{var}[S_i \mid \mu] = \mu m_2 = 2\mu \frac{\lambda^2{}_i}{(\delta_i - 1)(\delta_i - 2)}$$

$$\text{var}[\sum_{i=1}^{n} S_i \mid \mu] = \sum_{i=1}^{n} \text{var}[S_i \mid \mu] = 2\mu \sum_{i=1}^{n} \frac{\lambda^2{}_i}{(\delta_i - 1)(\delta_i - 2)} \quad \textit{by independence}$$

$$E[S = \sum_{i=1}^{n} S_i] = E_\mu[E[S \mid \mu]] = E_\mu[\mu \sum_{i=1}^{n} \frac{\lambda_i}{\delta_i - 1}] = \frac{\alpha}{\beta} \sum_{i=1}^{n} \frac{\lambda_i}{\delta_i - 1}$$

$$\begin{aligned}
\text{var}(S) &= E_\mu[\text{var}(S \mid \mu)] + \text{var}_\mu(E[S \mid \mu]) \\
&= E_\mu[2\mu \sum_{i=1}^{n} \frac{\lambda^2{}_i}{(\delta_i - 1)(\delta_i - 2)}] + \text{var}_\mu(\mu \sum_{i=1}^{n} \frac{\lambda_i}{\delta_i - 1}) \\
&= 2\frac{\alpha}{\beta} \sum_{i=1}^{n} \frac{\lambda^2{}_i}{(\delta_i - 1)(\delta_i - 2)} + \frac{\alpha}{\beta^2} \left(\sum_{i=1}^{n} \frac{\lambda_i}{\delta_i - 1} \right)^2
\end{aligned}$$

(b) Using the numerical values given in the table, together with $\alpha = 200$ and $\beta = 4$, calculate the probability of ruin after 3 years if an initial surplus of $U = 350\,000$ is set aside.

$$E[S(3)] = 3(260000) = 780000$$
$$\text{var}(S(3)) = 3(2470000000) = (86081.3569)^2$$
$$P[ruin] = P\left[S > U + \sum_i n_i(12)(100)(3)\right]$$
$$= P\left[Z > \frac{350000 + 5000(1200)(3) - 780000}{86081.3569}\right]$$
$$= 1 - \Phi(204.11) = 0$$

(c) One of the corporate clients extends the policy arrangements with its employees to also include an annual life insurance cover, where claims under this cover are assumed to vary according to a lognormal distribution, with parameters μ_j and σ_j and the probability for individual *j* to claim is q_j. Write down the model for the aggregate claim sizes over one year for all employees of this client, together with the expressions for the expected value and variance of the aggregate.

$$S = \sum_{j=1}^{n_i} Y_j \text{ where } Y_j = X_j N_j \text{ and } N_j \sim Bin(1, q_j) \text{ (individual risk model) and } X_j \sim LogN(\mu_j, \sigma_j^2)$$
$$E[Y_j] = q_j E[X_j]$$
$$\text{var}[Y_j] = E[N_j]\text{var}[X_j] + \text{var}(N_j)E[X_j]^2 = q_j \text{ var}[X_j] + q_j(1-q_j)E[X_j]^2 \text{ with}$$
$$E[X_j] = \exp(\mu_j + \frac{1}{2}\sigma_j^2) \text{ and } \text{var}(X_j) = \exp(2\mu_j + \sigma_j^2)\left(\exp(\sigma_j^2) - 1\right)$$
$$\text{Thus } E[S] = \sum_{j=1}^{n_i} E[Y_j] = \sum_{j=1}^{n_i} q_j E[X_j] \text{ and } \text{var}(S) = \sum_{j=1}^{n_i} \text{var}(Y_j) = \sum_{j=1}^{n_i} q_j \text{ var}(X_j) + q_j(1-q_j)E[X_j]^2$$

(d) Suppose the life cover claim sizes for all employees of a client come from the same lognormal distribution and the probability to claim for all employees is the same, say *q*. Write down expressions for the expected value and variance of the aggregate claim sizes. Also, if for client A: μ_A = 12, σ_A = 1 and q_A = 0.001, and the aggregate premium income is equal to the pure premium, what is the value of the loading factor that the employer should use so that the probability to exceed the premium income will be less than 10%.

$$E[S] = nq\exp\left\{\mu + \frac{1}{2}\sigma^2\right\} \text{ and } \text{var}(S) = nq\exp\{2\mu + \sigma^2\}\left(\exp\{\sigma^2\} - 1\right) + nq(1-q)\left(\exp\left\{\mu + \frac{1}{2}\sigma^2\right\}\right)^2$$
$$E[S_A] = 670843.2163$$
$$\text{var}(S_A) = 4.89144011 \times 10^{11} = (699388.312)^2$$
$$\textit{Premium Income} = (1+\theta)E[S_A]$$
$$P[S > (1+\theta)E[S_A]] = P\left[Z > \frac{0.3(670843.2163)}{699388.312}\right] = 1 - \Phi(0.29) = 0.3859$$

<u>Question 6</u>

A risk consists of independent policies, where the claim numbers follow a negative binomial distribution with parameters *k* and *p*, while the claim sizes are distributed according to a gamma distribution with parameters α and λ.

(a) Find an expression for the cumulant generating function of $S=\sum_{i=1}^{N} X_i$, the aggregate claims, where $C_S(t)=\log(M_S(t))$.

$X \sim \text{gamma}(\alpha, \lambda)$ $\quad M_X(t) = \left(1-\frac{t}{\lambda}\right)^{-\alpha}$

$N \sim \text{negbin}(k, p)$ $\quad M_N(t) = \dfrac{p^k}{(1-qe^t)^k}$ ✓

$$M_S(t) = M_N(\log M_X(t)) = \frac{p^k}{\left(1-qe^{\log M_X(t)}\right)^k} = \frac{p^k}{\left[1-qM_X(t)\right]^k} \checkmark$$

$$C_S(t) = \log(M_S(t))$$
$$= k\log p - k\log(1-qM_X(t)) \checkmark$$

(b) Use the cumulant generating function to derive expressions for the mean and variance of S and compare the expressions with expressions that you derived from the following two formulas:

$E(S)=E(N)E(X)$ and $Var(S)=Var(X)E(N)+Var(N)(E(X))^2$

$$C_S'(t) = \frac{-k}{1-qM_X(t)}\left(-qM_X'(t)\right) = \frac{kq}{1-qM_X(t)} M_X'(t) \quad ✓$$

$$\therefore C_S'(0) = \frac{kq}{1-q(1)} m_1 = \frac{kq}{1-q}\,\frac{\alpha}{\lambda} \quad ✓ = E[S]$$

$$C_S''(t) = \frac{kq}{1-qM_X(t)} M_X''(t) + kq\left(1-qM_X(t)\right)^{-2}(-1)\,M_X'(t)\left(-q\,M_X'(t)\right)$$

$$= \frac{kq}{1-qM_X(t)} M_X''(t) + \frac{kq^2\left(M_X'(t)\right)^2}{\left(1-qM_X(t)\right)^2} \quad ✓$$

$$\therefore C_S''(0) = \frac{kq}{1-q(1)} m_2 + \frac{kq^2 (m_1)^2}{(1-q(1))^2} \quad ✓$$

$$= \frac{kq}{p}\left(\frac{\alpha}{\lambda^2} + \left(\frac{\alpha}{\lambda}\right)^2\right) + \frac{kq^2}{p^2}\left(\frac{\alpha}{\lambda}\right)^2$$

$$= \frac{kq}{p}\left(\frac{\alpha+\alpha^2}{\lambda^2}\right) + \frac{kq^2}{p^2}\,\frac{\alpha^2}{\lambda^2} = \frac{kq\alpha}{\lambda^2 p}\left[\frac{1+\alpha}{1} + \frac{q\alpha}{p}\right]$$

$$= \frac{\alpha kq}{\lambda^2 p}\left[\frac{p+\alpha p+q\alpha}{p}\right] = \frac{\alpha kq}{\lambda^2 p^2}(p+\alpha) \quad ✓$$

And: $E[S] = E[N]E[X] = \frac{kq}{p}\,\frac{\alpha}{\lambda}$ ✓

$$\left.\begin{aligned} var(S) &= var(X)E[N] + E[X]^2 var(N) = \frac{\alpha}{\lambda^2}\,\frac{kq}{p} + \frac{\alpha^2}{\lambda^2}\left(\frac{kq}{p^2}\right)_2 \\ &= \frac{\alpha kq}{\lambda^2 p}\left(1+\frac{\alpha}{p}\right) = \frac{\alpha kq\,(p+\alpha)}{\lambda^2 p} \quad ✓ \end{aligned}\right\} \text{check}$$

Question 7

An insurance portfolio consists of 3 groups of policyholders, differing in claim numbers and claim sizes. Specifically, it is known that for group i the following information is known:

- Claim size X_{ij} is lognormally distributed with parameters μ_i and σ_i^2, $j = 1, 2, \ldots, N_i$, $i = 1, 2, 3$;
- N_i is Po(λ_i) distributed ;
- λ_i is gamma distributed with parameters α_i and δ_i;
- $S_i = \sum_{i=1}^{N_i} X_{ij}$

while the aggregate claim size over all groups is defined by $S = \sum_{i=1}^{3} S_i$.

(a) Prove that N_i is negative binomially distributed with parameters α_i and $\frac{\delta_i}{1+\delta_i}$.

... 1 year.

(a) $P(N=n) = \int_0^\infty e^{-\lambda} \frac{\lambda^n}{n!} * \frac{\delta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\delta} d\lambda$ ✓

$$= \frac{\delta^\alpha}{\Gamma(\alpha)\, n!} \int_0^\infty \lambda^{\alpha+n-1} e^{-\lambda(1+\delta)} d\lambda$$

3)

$$= \frac{\delta^\alpha}{\Gamma(\alpha)\, n!} \frac{\Gamma(\alpha+n)}{(1+\delta)^{\alpha+n}} = \frac{(\alpha+n-1)!}{(\alpha-1)!\, n!} \left(\frac{1}{1+\delta}\right)^n \left(\frac{\delta}{1+\delta}\right)^\alpha$$ ✓ ✓

the density function of the neg. binomial $(\overset{q}{\alpha}; \overset{p}{\tfrac{\delta}{1+\delta}})$ distribution

(b) Prove that $E(S) = \sum_{i=1}^{3} \frac{\alpha_i}{\delta_i} e^{\mu_i + \sigma_i^2/2}$ and $Var(S) = \sum_{i=1}^{3} \frac{\alpha_i}{\delta_i^2} \left(\delta_i e^{\sigma_i^2} + 1 \right) e^{2\mu_i + \sigma_i^2}$ (Hint: $V(S) = E(N)V(X) + V(N)[E(X)]^2$)

(b) $E(X_{ij}) = e^{\mu_i + \sigma_i^2/2}$; $Var(X_{ij}) = e^{2\mu_i + \sigma_i^2}(e^{\sigma_i^2} - 1)$

$$E(N_i) = \frac{\alpha_i/(1+\delta_i)}{\delta_i/(1+\delta_i)} = \frac{\alpha_i}{\delta_i}$$ ✓

$$V(N_i) = \frac{\alpha_i/(1+\delta_i)}{\delta_i^2/(1+\delta_i)^2} = \frac{\alpha_i(1+\delta_i)}{\delta_i^2}$$ ✓

$$\therefore E(S_i) = E(N_i)E(X_{ij}) = \frac{\alpha_i}{\delta_i} e^{\mu_i + \sigma_i^2/2}$$ ✓

$$V(S_i) = \frac{\alpha_i}{\delta_i} e^{2\mu_i + \sigma_i^2}(e^{\sigma_i^2} - 1) + \frac{\alpha_i(1+\delta_i)}{\delta_i^2}(e^{2\mu_i + \sigma_i^2})$$ ✓

$$= \frac{\alpha_i}{\delta_i}(e^{2\mu_i + \sigma_i^2})\left\{(e^{\sigma_i^2} - 1) + \frac{(1+\delta_i)}{\delta_i}\right\}$$

$$= \frac{\alpha_i}{\delta_i}(e^{2\mu_i + \sigma_i^2})\left\{e^{\sigma_i^2} + \frac{1}{\delta_i}\right\}$$

$$= \frac{\alpha_i}{\delta_i^2}(e^{2\mu_i + \sigma_i^2})(\delta_i e^{\sigma_i^2} + 1)$$ ✓

From these, $E(S)$ and $V(S)$ are found as required.

(c) The following data was made available:

i	μ_i	σ_i^2	α_i	δ_i	n_i
1	5	2	15	1	7000
2	7	3	24	1.5	3000
3	10	5	32	2	2000

(i) Find the values of E(S) and Var(S) for the given data.

(ii) Now, suppose S approximately has a normal distribution and the insurer set an initial surplus of R7.2 million aside. Premiums are collected at a constant rate of R200 per year per policyholder. Determine the probability of ruin after 1 year.

(c) (i) $E(S_1) = 15e^{5+1} = 6051.432$

$E(S_2) = 16e^{7+1.5} = 78636.301$

$E(S_3) = 16e^{10+2.5} = 4293396.584$

$\Sigma ES_i = 4378084.32$

$V(S_1) = 15e^{10+2}(1e^{2}+1) = 20480386.13$

(8) $= 10\frac{2}{3}e^{17}(1.5e^{3}+1) = 8020295955$

$= 8e^{25}(2e^{5}+1) = 1.715596324 \times 10^{14}$

$\therefore V(S) = 1.71567673 \times 10^{14}$ $\sqrt{V(S)} = 13098384.37$

(ii) $P\left(S > \frac{2.400.000 + 7.200.000 - 4.378.084.32}{13098384.37}\right)$

(2) $= 1 - \Phi(0.40)$

$= 0.3446.$

Question 8

Individual claim numbers X_i in a portfolio of policies follow a Poisson(μ) distribution. Let $S = \sum_{i=1}^{N} X_i$ denote the aggregate claim numbers in the portfolio, where N has a Poisson(λ) distribution.

a) Derive the moment generating function $M_S(t)$ of S.

4(a) $M_X(t) = e^{\mu(e^t-1)}$ and $M_N(t) = e^{\lambda(e^t-1)}$

$M_S(t) = M_N(\log M_X(t))$

$= e^{\lambda(e^{\log M_X(t)}-1)}$

3) $= e^{\lambda(M_X(t)-1)} = e^{\lambda\left(e^{\mu(e^t-1)}-1\right)}$

b) Find an expression for the cumulant generating function of S.

1) (b) $K_S(t) = \ln M_S(t) = \lambda\left\{e^{\mu(e^t-1)} - 1\right\}$

c) Derive the mean, variance and coefficient of skewness of S using (b).
d) Show that S will still have a positively skewed distribution for small λ even is $\mu \to \infty$.

(c) $K_S'(t) = \lambda e^{\mu(e^t-1)}\mu e^t$

$K_S'(0) = \lambda\mu$ = mean ✓

$K_S''(t) = \lambda e^{\mu(e^t-1)}\left(\mu e^t\right)^2 + \lambda e^{\mu(e^t-1)}\mu e^t$

$K_S''(0) = \lambda\mu^2 + \lambda\mu$ = variance ✓

$K_S'''(t) = \lambda e^{\mu(e^t-1)}\left(\mu e^t\right)^3 + \lambda e^{\mu(e^t-1)}2\mu^2 e^t + \lambda e^{\mu(e^t-1)}\left(\mu e^t\right)^2 + \lambda e^{\mu(e^t-1)}\mu e^t$ ✓

$K_S'''(0) = \lambda\mu^3 + 2\lambda\mu^2 + 2\mu^2 + \lambda\mu$

$= \lambda\mu\left(\mu^2 + 3\mu + 1\right)$ = skewness ✓

Coefficient of skewness $= \dfrac{\lambda\mu\left(\mu^2+3\mu+1\right)}{\left\{\lambda\mu(\mu+1)\right\}^{3/2}}$ (✓)

$= \dfrac{\mu^2+3\mu+1}{\sqrt{\lambda}\sqrt{\mu}\,(\mu+1)^{3/2}}$

$= \left(1+\dfrac{3}{\mu}+\dfrac{1}{\mu^2}\right) \Big/ \sqrt{\lambda}\left(1+\dfrac{1}{\mu}\right)^{3/2} \xrightarrow[\infty]{\mu} \dfrac{1}{\sqrt{\lambda}} > 0$ ✓

Question 9

a) Consider a collective risk model $S = \sum_{i=1}^{N} X_i$ where $S = 0$ if $N = 0$ and N is a discrete random variable. Derive general expressions for $E[S]$ and $\text{var}(S)$ in the collective risk model. Also prove that the moment generating function of S is given in terms of the moment generating function X as follows: $M_S(t) = M_N(\ln M_X(t))$

(i) $E(S) = E_N[E(S \mid N)]$

$E(S \mid N = n) = \sum_{i=1}^{n} E(X_i) = nm_1$ √

$\therefore E(S \mid N) = Nm_1$ and $E(S) = E(Nm_1) = E(N)m_1$

(ii) $V(S) = E[V(S \mid N)] + V[E(S \mid N)]$

$V(S \mid N = n) = V\left(\sum_{i=1}^{n} X_i\right) = nV(X_i) = n\left(m_2 - m_1^2\right)$

$\therefore V(S \mid N) = N\left(m_2 - m_1^2\right)$ √

$V[E(S \mid N)] = V(Nm_1) = m_1^2 V(N)$ √

$\therefore V(S) = E(N)\left(m_2 - m_1^2\right) + m_1^2 V(N)$ √

(iii) $M_S(t) = E[e^{St}] = E\left[E\left(e^{t\sum_{i=1}^{N} X_i}\right) \mid N = n\right]$

$$E\left(e^{t\sum_{i=1}^{n} X_i}\right) = \prod_{1}^{n} E(e^{tX_i}) = (M_X(t))^n \quad \surd$$

$$\therefore E\left(e^{t\sum_{i=1}^{N} X_i} \mid N\right) = \{M_X(t)\}^N = e^{N\ln(M_X(t))} \quad \surd$$

and $M_S(t) = E\left[e^{N\ln M_X(t)}\right] = M_N(\ln M_X(t)) \quad \surd$

b) An insurance company insures 5 risks, each of which produces aggregate claims. For each risk, the claim sizes follow an exponential distribution and the claims arise according to a Poisson process. The parameters of the Poisson processes and exponential distributions are given in the second and third columns of the table below. It is assumed that the risks are independent.

Risk	λ	α	b	q	n
1	20	0.01	50000	0.002	1200
2	15	0.04	40000	0.003	750
3	25	0.02	30000	0.005	1500
4	10	0.02	20000	0.008	900
5	30	0.025	10000	0.01	1600

(i) Find the mean and variance of S, the overall claim amount.

(ii) Determine the probability that S will exceed an amount of 7500, assuming S is approximately normally distributed.

(i) For risk i:

$$E[S_i] = \frac{\lambda_i}{\alpha_i}$$

$$var(S_i) = \frac{\lambda_i}{\alpha_i^2}$$

$$\therefore E[S] = \sum_{i=1}^{5} E[S_i] = \sum_{i=1}^{5} \frac{\lambda_i}{\alpha_i} = 5325$$

$(20\times100 + 15\times25 + 25\times50 + 10\times50 + 30\times40)$

$$var(S) = \sum_{i=1}^{5} var(S_i) \quad \text{(risks are independent)}$$

$$= 20\times100^2 + 15\times25^2 + 25\times50^2 + 10\times50^2 + 30\times40^2$$

$$= 344875$$

(ii) $P[S > 7500] = 1 - \Phi\left(\frac{7500 - 5325}{\sqrt{344875}}\right)$

$= 1 - \Phi(3.70)$

$= 0.00011$

c) Suppose it is now also known that the 5 risks in (b) are for 5 groups of policyholders for which the insurance company supplies group life coverage with a fixed benefit amount b. The probability to die in the next year for an individual in each group is given by q. Columns 4 and 5 in the table give the respective benefit amounts and probabilities to die, while column 6 gives the number of policyholders in each group.

(i) State the model and relevant assumptions for modeling the aggregate claims through the individual risk model, carefully defining all symbols used. Find the mean and variance of the tital benefit amount paid out during the year.

(ii) What should the premium per individual policyholder be if the probability for the total benefit claims to exceed the total premium income must not exceed 0.01? (Assume the total benefit claims to be normally distributed.)

(c) (i) Let $T_i = \sum_{j=1}^{n_i} X_j$ and $T = \sum_{i=1}^{5} T_i$ √

Assumptions

- X_j independent, $j = 1,2,\ldots n_i$
- T_i independent, $i = 1,2,\ldots 5$

In general $X_j = bI_j,\ I_j = \begin{cases} 1, \text{ probability } q \\ 0, \text{ probability } 1-q \end{cases}$ √

$E(X_j) = bq$

$V(X_j) = b^2 q(1-q)$

$\therefore E(T_i) = \sum_{j=1}^{n_i} b_i q_i = n_i b_i q_i$ √

$\therefore V(T_i) = \sum_{j=1}^{n_i} b_i^2 q_i (1-q_i) = n_i b_i^2 q_i (1-q_i)$ √

$\therefore E(T) = \sum_{i=1}^{5} E(T_i) = \sum_{i=1}^{5} n_i b_i q_i$ √

$V(T) = \sum_{i=1}^{5} V(T_i) = \sum_{i=1}^{5} n_i b_i^2 q_i (1-q_i)$ √

For the given values,

$$E(T) = 1200 \times 50000 \times 0.002 + 750 \times 40000 \times 0.003 + 1500 \times 30000 \times 0.005 + 900 \times 20000 \times 0.008 + 1600 \times 10000 \times 0.010 = 739000$$ √ √

$$V(T) = 1200 \times (50000)^2 \times 0.002 \times 0.998 + 750 \times (40000)^2 \times 0.003 \times 0.997 + 1500 \times (30000)^2 \times 0.005 \times 0.995 + 900 \times (20000)^2 \times 0.008 \times 0.992 + 1600 \times (10000)^2 \times 0.010 \times 0.990 = 2.073441 \times 10^{10}$$

$\sqrt{V(T)} = 143994.479$ √ √

$n = \sum n_i = 5950$

(ii) $P(T < p) = \Phi\left(\dfrac{p - 739000}{143994.479}\right) = 0.99$, where p = total premium

$\therefore \dfrac{p - 739000}{143994.479} = 2.326$ √

$\therefore p = 1073931.158$ √

$\therefore$ Individual premium $1073931.158 / 5950 = 180.49$ √

Question 10

An insurance portfolio contains 100 policies which are categorized into three independent categories of policyholders, namely A, B and C. The probability of a claim on an individual policy is p, and at most one claim per year is possible. Individual claim sizes are assumed to be exponentially distributed with mean 4 and are independent from claim to claim. The value of p, depending on the category of the policyholder, is

Category	*Value of p*	*Proportion of policyholders*
A	0.1	20%
B	0.2	60%
C	0.15	20%

Denote by S the total amount claimed in one year.

a) Calculate $E[S]$ and $\text{var}(S)$.

$$E[S] = E_p\left[E[S \mid p]\right]$$

and since $E[S|p] = E[N|p]E[X]$ where $N \sim bin(100, p)(1/2\ mark)$ and $X \sim \exp(1/4)$

$$= (100p)(4) = 400p$$

$$E[S] = E_p[400p] = 400((0.1)(0.2) + (0.2)(0.6) + (0.15)(0.2))$$

$$= 68$$

$$\text{var}(S) = E_p\left[\text{var}(S \mid p)\right] + \text{var}_p\left(E[S \mid p]\right)$$

and since $\text{var}(S|p) = E[N|p]\,\text{var}(X) + \text{var}(N \mid p)E[X]^2$

$$= (100p)(4^2) + 100p(1-p)4^2$$

$$= 3200p - 1600p^2$$

$$\text{var}(S) = E_p\left[3200p - 1600p^2\right] + \text{var}_p(400p) = 3200E_p[p] - 1600E_p[p^2] + 400^2\,\text{var}_p(p)$$

$$= 3200(0.17) - 1600(0.0305) + 400^2(0.0305 - 0.17^2)$$

$$= 751.2 = (27.41)^2$$

b) Show that skewness of a compound binomial distribution with parameters q and n and distribution function $F_X(x)$ is given by $nqm_3 - 3nq^2m_2m_1 + 2nq^3m_1^3$.

$$skewness(S) = \left.\frac{d^3}{dt^3}\log M_S(t)\right|_{t=0}$$

$$= \left.\frac{d^3}{dt^3} n\log(qM_X(t) + (1-q))\right|_{t=0}$$

$$= \left.\frac{nq\left(\frac{d^3}{dt^3}M_X(t)\right)}{qM_X(t)+1-q} - \frac{3nq^2\left(\frac{d^2}{dt^2}M_X(t)\right)\left(\frac{d}{dt}M_X(t)\right)}{(qM_X(t)+1-q)^2} + \frac{2n\left(q\frac{d}{dt}M_X(t)\right)^3}{(qM_X(t)+1-q)^3}\right|_{t=0}$$

$$= nqm_3 - 3nq^2m_2m_1 + 2nq^3m_1^3$$

c) *Explain* how you would go about finding a value for which the probability is 0.3 that S will exceed that value. Justify all assumptions needed.

Since the skewness of S *is* $E_p\left[skewness(S \mid p)\right] = E_p\left[100pm_3 - 2(100)p^2m_2m_1 + 2(100)p^3m_1^3\right]$.

$E[p^2] = 0.0305$
$E[p^3] = 0.005675$
$m_1 = E[X] = 4$
$m_2 = E[X^2] = 16 + 4^2 = 32$
$m_3 = E[X^3] = \dfrac{3!}{(1/4)^3} = 384$

So

$skewness(S) = E_p[skewness(S \mid p)]$

$= 100(4)(0.17) - 2(100)(32)(4)(0.0305) + 2(100)(4)^3(0.005675) = -640.16$

Thus the skewness is far from 0 so a normal assumption for the distribution of S would not be justified. If it was we would solve $P[S > v] = 0.3$ by normalizing S with the expected value and variance calculated in (a).

d) The insurer thinks that a better way to model their claim numbers may be with Poisson distributions with parameters as given in the table below.

Category	*Value of λ*	*Proportion of policyholders*
A	2	20%
B	12	60%
C	3	20%

(i) Explain how the values of λ were decided on.

Since the expected value for the Poisson is the same as it's parameter and when the claim numbers were distributed binomial the expected claim numbers were $100(0.2)(0.1) = 2$, $100(0.6)(0.2) = 12$ ***and*** $100(0.2)(0.15) = 3$ ***for each category.***

(ii) What will the distribution for S be now? Give all necessary parameters.

Each category will have a compound Poisson distribution with parameters 2, 12 and 3 respectively. Then S is the sum of three compound Poisson's so is also a compound Poisson with parameter 2+12+3=17 and underlying distribution exp(1/4).

Question 11

Give a full explanation of the collective risk model and individual risk model, including all assumptions. State clearly the differences between the two insurance models and for what type of insurance policies each would be applicable.

<u>Collective Risk Model:</u> $S = \sum_{i=1}^{N} X_i$ ***and*** $S = 0$ ***if*** $N = 0$***, where*** N ***is the claim numbers and*** X ***are the claim amounts,*** N ***and*** X ***independent.***

- ***number of claims from each risk not restricted to 1***
- ***e.g. motor insurance i.e. short term insurance***
- X_i ***'s i.i.d***

<u>Individual Risk Model:</u> $S = \sum_{i=1}^{n} Y_i$

- ***fixed number of risks*** n

- ***independent risks*** $Y_i \in \{0, X_i\}$
- ***number of claims from each risk is 0 or 1 i.e. restricted***
- ***e.g. life insurance***

Question 12

A bicycle wheel manufacturer claims that its products are virtually indestructible in accidents and therefore offers a guarantee to purchasers of pairs of its wheels. There are 250 bicycles covered, each of which has a probability *p* of being involved in an accident (independently). Despite the manufacturer's publicity, if a bicycle is involved in an accident, there is in fact a probability of 0.1 for each wheel (independently) that the wheel will need to be replaced at a cost of R100. Let *S* denote the total cost of replacement wheels in a year.

a) Show that the moment generating function of *S* is given by $M_S(t) = \left[\dfrac{pe^{200t} + 18pe^{100t} + 81p}{100} + 1 - p\right]^{250}$.

$S = \sum_{i=1}^{N} Y_i$ where $Y_i = \begin{cases} \sum_{j=1}^{N} (100) & \text{if } N = 1,2 \\ 0 & \text{if } N = 0 \end{cases}$

Claim Size Distribution $\longrightarrow$ So $Y_i = \begin{cases} 200 & \\ 100 & \text{with prob.} \\ 0 & \end{cases} \begin{matrix} (0.1)^2 \\ 2(0.1)(0.9) \\ 0.81 \end{matrix}$

Number of accidents $N \sim bin(250, p)$

So $M_N(t) = (1-p+pe^t)^{250}$

and $M_Y(t) = E[e^{tY}]$

$= e^{t(0)}(0.81) + e^{t(100)}(0.18) + e^{t(200)}(0.01)$

So $M_S(t) = M_N(\log M_Y(t))$

$= (1-p+pe^{\ln M_Y(t)})^{250}$

$= (1-p+pM_Y(t))^{250}$

$= \left[\dfrac{81p + 18pe^{100t} + pe^{200t}}{100} + 1-p\right]^{250}$

b) Using (a), show that $E[S] = 5000p$ and $Var(S) = 550{,}000\,p - 100000\,p^2$.

$E[S] = M_S'(0)$ ✓

$$= 250\left[\frac{81p + 18pe^{100t} + pe^{200t}}{100} + 1 - p\right]^{249}\left(\frac{18pe^{100t}(100) + pe^{200t}(200)}{100}\right)\Bigg|_{t=0}$$

$$= 250\left[\frac{81p + 18p + p}{100} + 1 - p\right]^{249}\left[\frac{1800p + 200p}{100}\right]$$

$$= \frac{250(2000p)}{100}$$

$$= \underline{5000p}$$

$E[S^2] = M_S''(0)$ ✓

$$= 250(249)\left[\frac{81p + 18pe^{100t} + pe^{200t}}{100} + 1 - p\right]^{248}\left(\frac{18pe^{100t}(100) + pe^{200t}(200)}{100}\right)^2$$

$$+ 250\left[\frac{81p + 18pe^{100t} + pe^{200t}}{100} + 1 - p\right]^{249}\left(\frac{18pe^{100t}100^2 + pe^{200t}200^2}{100}\right)\Bigg|_{t=0}$$

$$= (250)(249)(1) * \left(\frac{2000p}{100}\right)^2 + (250)(1)\left(\frac{100^2(18)p + (200^2)p}{100}\right)$$

$$= 24\,900\,000p^2 + 550\,000p$$ ✓

So $\operatorname{var}(S) = E[S^2] - E[S]^2 = \underline{-100\,000p^2 + 550\,000p}$

Suppose instead that the manufacturer models the cost of replacement wheels as a random variable *T* based on a portfolio of 500 wheels, each of which (independently) has a probability of 0.1p of requiring replacement.

b) Derive expressions for $E[T]$ and $\operatorname{Var}(T)$ in terms of *p*.

Number of wheels needing replacement :

$W \sim Bin(500, 0.1p)$ ✓

And the $T = 100W$ ✓

$E[T] = 100E[W] = 100(500)(0.1p) = 5000p$

$\operatorname{var}(T) = 100^2\operatorname{var}(W) = 100^2(500)(0.1p)(1-0.1p)$ ✓

$= 500\,000p - 50\,000p^2$

d) Suppose $p = 0.05$.

(i) Calculate the mean and variance of *S* and *T*.

$E[S] = 250 = E[T]$

$var(S) = 27\ 250$

$= (165.08)^2$

$var(T) = 24\ 875 = (157.72)^2$

(ii) Calculate the probabilities that *S* and *T* exceed R500, assuming a normal approximation.

$P[S > 500] = P\left[Z > \frac{500 - 250}{165.08}\right]$

$= P[Z > 1.514]$

$= 1 - 0.93448$

$= 0.06552$

$P[T > 500] = P[Z > 1.59] = 1 - 0.94408 = 0.05592$

(iii) Comment on the differences.

iii) Variance of S is larger ∴ higher probability in (ii)

Fewer accidents under S ⇒ high loss since 2 wheels.

But T ⇒ only 1 wheel

Question 13

The total claims arising from a certain portfolio of insurance policies over a given month is represented by

$$S = \begin{cases} \sum_{i=1}^{N} X_i & \text{if } N > 0 \\ 0 & \text{if } N = 0 \end{cases}.$$

where N has a Poisson distribution with mean 2 and $X_1, X_2, ..., X_N$ is a sequence of independent and identically distributed random variables that are also independent of *N*. Their distribution is such that $P[X_i = 1] = \frac{1}{3}$ and $P[X_i = 2] = \frac{2}{3}$. An aggregate reinsurance contract has been arranged such that the amount paid by the reinsurer is *S*–3 (if *S* > 3) and zero otherwise. The aggregate claims paid by the direct insurer and the reinsurer are denoted by S_I and S_R, respectively. Calculate $E[S_I]$ and $E[S_R]$.

$S = \sum_{i=1}^{N} X_i$ with $N \sim \text{Poisson}(2)$

$P[X_i = 1] = \frac{1}{3}$ $\quad P[X_i = 2] = \frac{2}{3}$

$E[S] = E[N]E[X] = (2)\left(1\left(\frac{1}{3}\right) + 2\left(\frac{2}{3}\right)\right) = \frac{10}{3}$

$S_I = S$ if $S \leq 3$

$S = S_I + S_R$

$S_R = S - 3$ if $S > 3$

$P[S_I = 0] = P[N=0] = e^{-2} = 0.13534$

$P[S_I = 1] = P[N=1]P[X_1 = 1] = \frac{e^{-2}(2^1)}{1!}\left(\frac{1}{3}\right) = 0.09022$

$P[S_I = 2] = P[N=1]P[X_1 = 2] + P[N=2]P[X_1=1]P[X_2=1]$

$= \frac{e^{-2}(2^1)}{1!}\left(\frac{2}{3}\right) + \frac{e^{-2}2^2}{2^2}\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)$

$= 0.21052$

$P[S_I = 3] = 1 - 0.13534 - 0.09022 - 0.21052 = 0.56392$

So $E[S_I] = 2.20303 \rightarrow \left(0(0.13534) + (1)(0.09022) + (2)(0.21052) + (3)(0.56392)\right)$

$E[S_R] = E[S - S_I] = \frac{10}{3} - 2.20303 = 1.1303$

Question 14

Describe the collective and individual risk models respectively, clearly indicating the differences between the two models. Also, give an example of an insurance product where each would be applicable.

collective risk model

(claims for a group of policyholders)

$S = \begin{cases} \sum_{i=1}^{N} X_i & \text{if } N > 0 \\ 0 & \text{if } N = 0 \end{cases}$

no. of claims over the term (discrete)

aggregate claims paid over the term (is a compound random variable)

individual claim size (amount of the i^{th} claim) (discrete / continuous)

e.g. car insurance

aggregate claims

$S = Y_1 + Y_2 + \dots + Y_n \rightarrow$ fixed no. of risks

* Y_j : jth risk claim amount
  - ↳ no. of claims from jth risk : $N_j \in \{0, 1\}$
  - ↳ $P[\text{claim from } j\text{th risk}] = q_j$
  - ↳ $X_j = Y_j \mid N_j = 1$: If a claim occurs the claim amount is X_j, with
    - distribution $F_j(x)$
    - mean μ_j
    - variance σ_j^2

e.g. life insurance

Individual Risk Model	vs. Collective Risk Model
■ n is fixed	■ N is a random variable
■ Number of claims from each risk is restricted to 0 or 1	■ No restriction on the number of claims
■ Individual risks are independent	■ Individual claim amounts are independent

Question 15

Consider the collective risk model $S = \sum_{i=1}^{N} X_i$, where X_i represents the claim size, with continuous distribution function $F_X(x)$ and raw moments m_1, m_2 and m_3, and N, the number of claims, having some discrete distribution.

(a) Derive a general expression for the variance of the random variable S.

$$\operatorname{var}(S) = E_N[\operatorname{var}(S \mid N)] + \operatorname{var}_N(E[S \mid N])$$

$$\operatorname{var}(S \mid N = n) = \operatorname{var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{var}(X_i) (independence)$$

$$= \sum_{i=1}^{n} (m_2 - m_1^2) = n(m_2 - m_1^2)$$

$$E[S \mid N = n] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = nm_1$$

Thus

$$\begin{aligned}\operatorname{var}(S) &= E_N[N(m_2 - m_1^2)] + \operatorname{var}_N(Nm_1) \\ &= (m_2 - m_1^2)E_N[N] + m_1^2 \operatorname{var}_N(N) \\ &= \operatorname{var}(X)E_N[N] + E[X]^2 \operatorname{var}_N(N)\end{aligned}$$

(b) Define the cumulant generating function, $C_X(t)$, of a general random variable X and explain (without proof) the relationship between $C_X(t)$ and the first three moments of X.

$$C_X(t) = \ln M_X(t)$$
$$C_X^{'}(0) = \mu_X$$
$$C_X^{''}(0) = \sigma_X^2$$
$$C_X^{'''}(0) = skewness(X)$$

(c) Derive formulas for the expected value and variance of S using its cumulant generating function if S has a compound binomial distribution with parameters $F_X(x)$, k and p. Make use of the formula for the skewness of S, given by $kpm_3 - 3kp^2 m_2 m_1 + 2kp^3 m_1^3$ to explain if it is justified or not to assume that S has an approximate normal distribution. How do the formulas for the expected value and variance affect the normal assumption if it can be assumed?

$$\begin{aligned} M_S(t) &= M_N(\ln M_X(t)) \\ &= (1-p+pe^{\ln M_X(t)})^k \\ &= (1-p+pM_X(t))^k \end{aligned}$$

$$\begin{aligned} C_S(t) &= \ln M_S(t) \\ &= k\ln(1-p+pM_X(t)) \end{aligned}$$

$$C_S^{'}(t) = \frac{k}{1-p+pM_X(t)} p.M_X'(t) \Rightarrow C_S^{'}(0) = \frac{k}{1-p+p(1)} p.M_X'(0)$$

$$= \frac{k}{1-p+p} pm_1 = \frac{k}{1-p+p} pm_1 = kpm_1 = E[S]$$

$$C_S^{''}(t) = k(-1)(1-p+pM_X(t))^{-2}(pM_X'(t))^2 + \frac{k}{1-p+pM_X(t)} pM_X''(t)$$

$$\Rightarrow C_S''(0) = k(-1)(1-p+p(1))^{-2}(pm_1)^2 + \frac{k}{1-p+p(1)} pm_2 = -kp^2 m_1^2 + kpm_2 = kp(m_2 - pm_1^2)$$

$$= \mathrm{var}(S)$$

Coefficient of Skewness

$$= \frac{kpm_3 - 3kp^2 m_2 m_1 + 2kp^3 m_1^3}{(kp(m_2 - pm_1^2))^{3/2}} = \frac{pm_3 - 3p^2 m_2 m_1 + 2p^3 m}{(p(m_2 - pm_1^2))^{3/2}\sqrt{k}} \xrightarrow[\infty]{k} 0$$

thus S is approximately symmetric for large k i.e. for a large upper limit on the number of claims. Since the normal distribution is symmetric, we have an indication that S may be approximately normal for large k. Var(S) and E[S] both also increase as k increases thus the normal distribution assumed will become more spread out and the peak will shift upwards.

Question 16

Consider two group life insurance policies belonging to two independent companies. Company A has 3000 independent lives insured for R100 000 each and they pay a premium of R80 per life. Company B has 500 independent lives insured for R80 000 each and they pay a premium of R75 per life. Let $\mu_i, i = A, B$ be the probability of a claim for Company i.

a) Provide two loss distributions, from the tables, which would be suitable to model $\mu_i, i = A, B$ for each company.

U(0,1)(continuous)

Beta (α_i, β_i)

b) Suppose $\mu_i \sim Beta(2, \beta_i)$ for $i = A, B$. What is probability of a single claim that each company can expect if the parameters in the table below have been calculated for each company by the insurer.

	β_i

Company A	198
Company B	98

$$Company\ A = \frac{2}{2+198} = \frac{2}{200} = 1\%$$

$$Company\ B = \frac{2}{2+98} = \frac{2}{100} = 2\%$$

c) Derive formulas, and then calculate values, for the expected total claim amount for each company and the two portfolios together. Comment on the premiums charges by the insurance company.

$$E[S_A] = E\left[\sum_{k=1}^{n_A} X_{Ak}\right] = \sum_{k=1}^{n_A} E[X_{Ak}]$$

But $X_{Ak} = \sum_{j=1}^{N} Y_{Aj}$ ***where*** $N \sim bin(1, \mu_A)$ ***and*** $Y_{Aj} = 100000$

$$\begin{aligned} E[X_{Ak}] &= E[N](100000) \\ &= E_{\mu_A}[N \mid \mu_A](100000) \\ &= E_{\mu_A}[\mu_A(1)](100000) \\ &= \frac{2}{2+\beta_A}(100000) \end{aligned}$$

So

$$E[S_A] = \sum_{k=1}^{n_A} \frac{2}{2+\beta_A}(100000) = n_A \frac{2}{2+\beta_A}(100000)$$

$$= (3000)\frac{2}{2+198}(100000)$$

$$= R3000000$$

Similarly

$$E[S_B] = n_B \frac{2}{2+\beta_B}(80000)$$

$$= (500)\frac{2}{2+98}(80000)$$

$$= R800000$$

So $E[S] = E[S_A] + E[S_B] = R3800\,000$

Expected income = R80*3000+R75*500=R277500. Outgo is much larger then income thus the premiums charged are not high enough!

(d) Derive formulas, and then calculate values, for the variance of the total claim amount for each company and the two portfolios together.

$$\mathrm{var}[S_A] = \mathrm{var}\left[\sum_{k=1}^{n_A} X_{Ak}\right] = \sum_{k=1}^{n_A} \mathrm{var}[X_{Ak}](independence)$$

But $X_{Ak} = \sum_{j=1}^{N} Y_{Aj}$ ***where*** $N \sim bin(1, \mu_A)$ ***and*** $Y_{Aj} = 100000$

$$\begin{aligned}
\text{var}[X_{Ak}] &= E[N]\text{var}(Y_{Aj}) + \text{var}(N)E[Y_{Aj}]^2 \\
&= 0 + \text{var}(N)E[Y_{Aj}]^2 \\
&= \left(E_{\mu_A}[\text{var}(N \mid \mu_A)] + \text{var}_{\mu_A}[E(N \mid \mu_A)]\right)(100000)^2 \\
&= \left(E_{\mu_A}[(1)\mu_A(1-\mu_A)] + \text{var}_{\mu_A}[\mu_A(1)]\right)(100000)^2 \\
&= \left(E_{\mu_A}[\mu_A - \mu_A^2] + \text{var}_{\mu_A}[\mu_A]\right)(100000)^2 \\
&= \left(E_{\mu_A}[\mu_A] - E_{\mu_A}[\mu_A^2] + E_{\mu_A}[\mu_A^2] - E_{\mu_A}[\mu_A]^2\right)(100000)^2 \\
&= \left(E_{\mu_A}[\mu_A] - E_{\mu_A}[\mu_A]^2\right)(100000)^2 \\
&= \left[\frac{2}{2+\beta_A} - \left(\frac{2}{2+\beta_A}\right)^2\right](100000)^2
\end{aligned}$$

So

$$\text{var}[S_A] = \sum_{k=1}^{n_A}\left[\frac{2}{2+\beta_A} - \left(\frac{2}{2+\beta_A}\right)^2\right](100000)^2 = n_A\left[\frac{2}{2+\beta_A} - \left(\frac{2}{2+\beta_A}\right)^2\right](100000)^2$$

$$= (3000)\left(\frac{2}{2+198} - \left(\frac{2}{2+198}\right)^2\right)(100000)^2$$

$$= R(544977.06)^2$$

Similarly

$$\text{var}[S_B] = n_B\left[\frac{2}{2+\beta_B} - \left(\frac{2}{2+\beta_B}\right)^2\right](80000)^2$$

$$= (500)\left(\frac{2}{2+98} - \left(\frac{2}{2+98}\right)^2\right)(80000)^2$$

$$= R(250439.61)^2$$

So

$$\text{var}[S] = \text{var}[S_A] + \text{var}[S_B] (\textit{independence})$$

$$= R(599766.62)^2$$

Question 17

Consider three independent policyholders who can each claim from their car insurance a maximum of 10 times over a 3 year period. The underlying claims X arise from a $Beta(\alpha,\delta)$ and the probability of a claim for each policyholder is μ which has the following distribution: $\mu = \begin{cases} 0.2 & \textit{with probability } 0.4 \\ 0.3 & \textit{with probability } 0.2 \\ 0.4 & \textit{with probability } 0.4 \end{cases}$.

a) Decide whether a collective or individual risk model would be best suited to model the total claims over the three year period from this risk and describe your model fully (formulas, distributions, assumptions etc.)

$X \sim \text{Beta}(\alpha, \delta)$

• 1 time unit = 3 years'

• restriction on the maximum no. of claims, thus $N \sim bin(\cdot,\cdot)$

$\mu = \begin{cases} 0.2 & \text{with prob } 0.4 \\ 0.3 & \text{with prob. } 0.2 \\ 0.4 & \text{with prob. } 0.4 \end{cases}$

collective risk model:

$S = S_1 + S_2 + S_3$, S_i's are independent

where $S_i = \sum_{j=1}^{N} X_j$ where $X_j \sim \text{Beta}(\alpha, \delta)$

$N|\mu \sim bin(10, \mu)$ with μ's distribution as given

• N & X_j's are independent

b) Calculate the variance of the total claim amount over the three year period from this risk, correct to 4 decimal places, if $\alpha = 3$ and $\delta = 5$.

$var(S) = \sum_{i=1}^{3} var(S_i)$ (S_i's are independent)

and $var(S_i) = E[N]var(X_j) + var(N)E[X_j]^2$

$= E_\mu\left[E[N|\mu]\right]var(X_j) + \left[var_\mu(E[N|\mu]) + E_\mu[var(N|\mu)]\right] \cdot E[X_j]^2$

$= E_\mu\left[10\mu\right]var(X) + \left[var_\mu(10\mu) + E_\mu(10\mu(1-\mu))\right]E[X_j]^2$

$= 10E[\mu]var(X) + \left[10^2 var(\mu) + 10E[\mu] - 10E[\mu^2]\right]E[X_j]^2$

$= 10E[\mu]var(X) + \left[(10^2-10)E[\mu^2] - 10^2E[\mu]^2 + 10E[\mu]\right]E[X]^2$

BUT $E[\mu] = 0.2 \times 0.4 + 0.3 \times 0.2 + 0.4 \times 0.4 = 0.3$

and $E[\mu^2] = 0.2^2 \times 0.4 + 0.3^2 \times 0.2 + 0.4^2 \times 0.4 = 0.098$

$(E(N) = 3,\ var(N) = 2.82)$ $(var(X) = 0.02604,\ E[X] = 0.375)$

So $var(S_i) = 10(0.3)\left(\frac{\alpha\delta}{(\alpha+\delta)^2(\alpha+\delta+1)}\right) + \left[90(0.098) - 100(0.3)^2 + 10(0.3)\right]\left[\frac{\alpha}{\alpha+\delta}\right]^2$

$= 3\left(\frac{3(5)}{8^2(9)}\right) + \left[8.82 - 9 + 3\right]\left(\frac{3}{8}\right)^2$ [1

$= 0.4747$ So $var(S) = 3(0.4747) = 1.4241$

c) How would your answer for (b) change if the policyholders each had the same probability of claiming, μ, and were thus not independent? You only need indicate where in your calculations the changes will occur and how they change, and do not need to give the final numerical answer.

c) Firstly $var(S) = var\left(\sum_{i=1}^{3} S_i\right) \neq \sum_{i=1}^{3} var(S_i)$ ✓

S_i's are not independent but $S_i|\mu$'s are ✓ [2]

$\therefore var\left(\sum_{i=1}^{3} S_i\right) = var_{\mu}\left(E\left[\sum_{i=1}^{3} S_i \,|\, \mu\right]\right) + E_{\mu}\left[\underbrace{var\left(\sum_{i=1}^{3} S_i / \mu\right)}\right]$ ✓

$= \sum_{i=1}^{3} var(S_i|\mu)$

d) Decide whether a collective or individual risk model would be best suited to model the total claims over the three year period from this risk if each policyholder was only allowed to claim once for a set claim size of 100 000 and describe your model fully (formulas, distributions, assumptions etc.)

individual risk model: ✓

$S = S_1 + S_2 + S_3$ ✓ S_i's independent

where $S_i = \sum_{j=1}^{N} Y_j$ ✓ where $N \sim bin(1, \mu)$ ✓

$Y_j = \begin{cases} 100\,000 & \text{if } N=1 \\ 0 & \text{if } N=0 \end{cases}$