University of Pretoria
Department of Statistics
WST322 Actuarial Statistics
Tutorial 1 - Chapter 1

Question 1

Part 1 questions in the core notes for Chapter 1.

Question 2

Consider a driver who drives to work every morning via three possible routes (1, 2 and 3). Each morning he decides which route to take without any knowledge of accidents on each route. The following matrix represents losses in time (minutes) to the driver with respect to the route chosen,

Driver's Chosen Route

	Route 1	Route 2	Route 3
Accident on Route 1	40	18	15
Accident on Route 2	20	20	25
Accident on Route 3	28	10	30

(a) Define what is meant by a two-player zero-sum game and explain whether or not this concept applies to the game given above.

(b) Determine the minimax route for the driver, first eliminating any dominating strategies.

(c) Suppose the driver listens to the news before he leaves home to go to work and becomes aware of an accident on Route 2 (but no accidents on either of the other routes).

(i) Discuss if it would still be feasible for him to make use of Route 2.

(ii) Find a randomized strategy he could make use of to decide between Route 1 and 3 if he decides not to use Route 2 and the accident on Route 2 can be ignored but an accident on Route 1 or 3 may have occurred after he listened to the news.

(iii) Using your result in (ii), explain how he could make practical use of the randomized strategy derived.

Question 3

A financial institution will make a decision on one of three possible investment options, namely property market, money market and stock exchange market. The decision will depend on international oil price movements and political stability in third world countries. International analyzers maintain that the probability of the oil price staying stable to be 80%, while the probability for third world stability when oil prices rise is 40%. They also believe that third world political stability has a 75% probability when oil prices are stable. The following matrix represents the profit to be gained (in millions) from each of these options, depending on the different combinations of these two factors.

Options for financial institution

Oil price and political stability	Property market	Money market	Stock exchange
Price stable, politics stable	8	6	14
Price up, politics stable	10	7	9
Price stable, politics unstable	7	5	6
Price up, politics unstable	12	8	4

Clearly showing your arguments, state which option should the financial institution choose, when it uses a
(a) minimax strategy

(b) Bayes strategy

Also give the relevant maximum profits in both cases.

Question 4

The payoff matrix (in terms of losses of A) in a zero-sum two player game is given by:

Player A

Player B	**I**	**II**	**III**
1	1	5	6
2	3	-1	2
3	-2	1	4

(a) Find the saddlepoint, if one exists (Hint: First eliminate dominated strategies).
(b) Describe how a randomized strategy where the objective is to minimize the maximum expected loss can be implemented by Player A, and determine the corresponding value of the game.

Question 5

Mary and John are looking to buy a house. They have been approved by three banks for a home loan, say bank A, B and C, and must decide which loan to take. The three loans are for the same amount but differ with respect to the deposit amount, fees and early payment options. The interest rate at present is 12% (θ_2) but there is a probability of 0.05 that it drops by 0.5% (θ_1) and a probability of 0.4 that it rises by 1% (θ_3) by the time they take out the loan. Ignore the effects of future inflation. They can only afford a maximum payment of R5000 per month on their loan. The following shows the number of years they would take to pay off the loan:

Bank

Interest Rate	**A**	**B**	**C**
θ_1	15	18	20
θ_2	20	20	25
θ_3	30	27	26

(a) Would Mary and John disregard any of the loans given these estimates for the loan term? Explain.

(b) Which loan should they choose under a minimax strategy?

(c) Suppose they decide to draw an observation from a $bin(2,\frac{1}{3})$ distribution, say x, and decide on two possible decision functions, namely:

$$d_1(x)=\begin{cases} A & if\ \ x=2 \\ B & if\ \ x=1 \\ C & if\ \ x=0 \end{cases} \qquad \text{or} \qquad d_2(x)=\begin{cases} B & if\ \ x=0 \\ C & if\ \ x=1\ or\ 2 \end{cases}.$$

(i) Calculate the matrix of risks associated with these decision functions.

(ii) Use the Bayes criterion to decide between the 2 decision functions.

(iii) Suppose they observe $x=0$ after generating a random binomial observation. Which loan would they choose?

<u>Question 6</u>

In a certain game a parameter θ can be either 1, 2 or 3. A statistician who needs to decide between these values of θ, knows that the loss incurred for a decision θ^* is $|\theta^*-\theta|$.

(a) Construct a suitable loss matrix.

(b) If it is assumed in a Bayesian context that $P(\theta=i)=ip$, i = 1, 2 , $0<p<\frac{1}{2}$, determine the Bayes risk for each decision.

(c) Give a strategy for each value of p, $0<p<\frac{1}{2}$. Clearly identify any possible dominated strategy.

<u>Question 7</u>

An insurer is considering whether to outsource its advertising. If it decides not to outsource, it expects to spend R14,000,000 next year on its advertising, and believes that this would result in a portfolio of 100000 policies. It has received quotations from two different companies for outsourcing its advertising. Company A would cost R21,000,000 per year, and believes that this would result in the business expanding to 125000 policies. Company B would cost R30,000,000 per year, and believes that this would expand the business to 140000 policies. At present, each policy returns a profit to the company of R300 per year; but this is not guaranteed in the future. The company has assessed that it will stay at this level with probability 0.6, but could reduce to R200 per year with probability 0.25, or increase to R400 per year with probability 0.15.

a) Explain which of the three options can be immediately discarded.

b) Determine the Bayes solution to the problem of maximising the profit to the company over the coming year.

<u>Question 8</u>

a) Explain clearly the difference between a zero-sum two-player game and a statistical game.

A manufacturer of specialist beds must decide which bed to make in the coming year. There are three possible choices: basic, deluxe or supreme each with different once off initial manufacturing costs. The manufacturer has fixed costs of R1,300,000. The revenue and once off costs for each product are as follows:

Once off Costs **Revenue per Bed Sold**

Basic	100000	1.00
Deluxe	400000	1.20
Supreme	1 000000	1.50

Last year the manufacturer sold 2,100,000 items and is preparing forecasts of profitability for the coming year based on three scenarios: Low sales (70% of last year's level), Medium sales (same as last year) and High sales (15% higher than last year).

b) Determine the annual profits under each possible combination.

c) Determine the minimax solution to this problem.

d) Determine the Bayes criterion solution based on the annual profit given that P(Low) = 0.25 and P(Medium) = 0.6.

Question 9

Consider a brother and his younger sister playing a game of 'rock, paper, scissors'. The brother allows his sister to win if they both show the same hand, otherwise 'rock' beats 'scissors', 'scissors' beats 'paper' and 'paper' beats 'rock'. The winner gets given 1 sweet from the other player.

a) Complete the loss matrix below for the brother.

		Big Brother		
		Rock	Paper	Scissors
Little Sister	Rock			
	Paper			
	Scissors			

b) Are there any dominated strategies for the brother? Explain.

c) Determine a minimax strategy for the brother.

d) The brother considers using a randomized strategy to optimize his game. He decides to choose 'rock' and 'paper' with equal probability p. Determine the value of p which will optimize his game.

e) How would the brother implement his strategy in (d)?

Question 10

John has just started his first job and is moving out of his parent's home and must decide between renting or buying a flat for himself. The following loss matrix shows his respective gains depending on whether the property market is good or bad.

	Good Market	**Bad Market**
Rent	-2	1
Buy	4	-1

a) Are there any dominated strategies? Explain.

b) Determine a minimax strategy for John.

c) Suppose John has a sample of data of recent profits/losses on sales of houses in the area, $x_1, x_2, \ldots, x_n$. He decides to make use of the following decision function: $d(\mathbf{x}) = \begin{cases} rent \ if \ \mathbf{x} > 0 \\ buy \ if \ \mathbf{x} < 0 \end{cases}$ where $\mathbf{x} = \sum_{i=1}^{n} x_i$. Explain what may motivate John to choose this decision function.

d) Provide an additional decision function he could alternatively use.

e) Describe two decision criteria with which he could decide between the two decision functions above.

Question 11

The UN is discussing setting pollution limits for a certain country at either *medium* or *low*. The country can also decide whether to set its pollution limits at *medium* or *low*. Setting *medium* pollution limits will cost the country 5000, and setting *low* pollution limits will cost the country 10000. If the country's limit exceeds the UN's decision, they will be fined 6000. All costs to the country can be considered gains for the UN.

a) Set up a zero-sum two-player loss matrix for the country and the UN. Explain why your matrix is zero-sum.
b) Using your loss matrix in (a) determine a minimax strategy for the country as well as the UN.
c) Describe the solution you obtained in (b) theoretically as well as why it is significant.
d) Describe a second method that could be used by the country to decide between *low* and *medium* pollution limits.

Question 12 Two new graduates, John and Jake, are considering a number of job offers each. Job A pays R30 000 per month and job B pays R26 000 per month. Job A is situated 45km away from John and 30 km away from Jake, while Job B is situated 12km away from John and 17.5km away from Jake. For John the distance to travel by car to and from job A each day is 90km, and for job B 24km. For Jake the distance to travel by car to and from job A each day is 60km and for job B 35km. The cost per km for petrol is R1.20 for both John and Jake. For job A both John and Jake have the alternative option of using the Gautrain to travel to and from job A. For John this will cost R84 per return trip and for Jake R70 per return trip. Assume 20 days in a working month. Each job is only available to either John or Jake, not to them both. However, if one graduate takes a job first, the other graduate has the option of a job at the same company but in a more junior position with a R2000 pay cut.

a) Is this a statistical game or a two-player game? Explain.
b) Set up a 3×3 playoff matrix for the game, with values in each cell given by a vector (L_{John}, L_{Jake}) where L_i represents the monthly income for graduate i after travelling expenses are taken into account. Assume John takes the job for below the diagonal and Jake for above the diagonal, and on the diagonal have two entries.

		Jake		
		Job A with car	Job A with train	Job B with car
John	Job A with car	John takes the job Jake takes the job	Jake takes the job	Jake takes the job
	Job A with train	John takes the job	John takes the job Jake takes the job	Jake takes the job
	Job B with car	John takes the job	John takes the job	John takes the job Jake takes the job

c) d) e) f) g) h) i) j) k) l) m) n) o)

c) Is the game zero-sum? Explain.

Consider only the diagonal entries of your payoff matrix in (b).

d) Determine a minimax strategy for John and Jake by considering what can happen in each diagonal entry only.

e) Comment on your answers obtained in (d). Does it provide a useful solution?

Question 13 Two class mates Mary and John have a joint assignment to do. They decide to do the two questions individually instead of as a group and then hand it in. However, they have left it to the last minute and didn't organize with other who would do which of the two questions. They do not have a way to contact each other so they each decide to pick one of the questions and hope the other does the opposite question. The two questions are worth 10 and 12 marks respectively. Mary expects to get 80% for question 1 and 70% for question 2. John expects to get 70% for question 1 and 75% for question 2. If they both do the same question they will hand in the answer they expect to get the most marks for and will get zero for the unanswered question.

a) Complete the matrix below where the cells represent the total expected marks they will get for the assignment as a group. Is your matrix zero-sum?

		John	
		Question 1	Question 2
Mary	Question 1		
	Question 2		

b) Determine a minimax strategy for each player. Do the chosen strategies provide a good solution?
c) Determine a randomised strategy for each player correct to 4 decimal places.
d) A uniform random number of 0.6378 is drawn. Which strategy should each player chose?

Question 14 Mary and John are playing a card guessing game. Mary chooses a card from a 10, 9 or 8 of hearts and cannot change her decision. John then guesses which card she chose. John gets points equivalent to the card value if he guesses the correct card or only the absolute difference if he chooses incorrectly.

a) Set up a loss matrix for this game indicating clearly if the values are losses or gains to the respective player.
b) Is the game between Mary and Johan a zero-sum two-player game or a statistical game? Provide reasons.
c) Describe the following two strategies theoretically in terms of the game described above:
   (i) Minimax strategy
   (ii) Randomised strategy
d) Determine a minimax strategy.

Question 15 A driver who commutes from Pretoria to Johannesburg and back each day is weighing up the advantage of using e-tolled roads. The e-tolled route is shorter but involves the cost of e-tolls. The non-e- tolled route is longer. The petrol price varies from high, to stable, to low. Ifthe petrol price is stable the petrol cost for each trip to the driver on the e-tolled route is p_1, and on the non-e-tolled route it is p_2. The cost increases by a factor $0 < k < I$ if the petrol price is high, and decreases by a factor $0 < c < I$ if the petrol price is low. The e-tolls cost per trip stays constant at a value of e, thus each trip total cost is *e-toll +petrol*. The table below provides the losses for each combination.

Petrol Price	**E-tolled Route**	**Non-e-tolled Route**
High	$e+(l+k)P1$	$(l+k)p,$
Stable	$e+P1$	$P2$
Low	$e+(1-c)p_1$	$(!-c)p,$

a) What should the relationship between p_1 and p_2 be in order for the non-e-tolled route to be the minimax strategy for the player?

b) If the probability of high petrol prices and low petrol prices is identical, determine a randomized strategy for the driver, given $k = 0.4, c = 0.4, p_1 = 50, p_2 = 80$.

Question 16 Alex bets Charl as to the final pass rate of a certain module's group. The pass rate will either be 75%, 70% or 65%. Charl can choose any of these three options. Alex pays Charl R50 if he chooses the correct pass rate, but Charl pays Alex R50 is he chooses a lower or higher pass rate than expected.

a) Set a gains matrix for this game.
b) Describe different methods to find the optimal strategies.
c) Charl decides to use decision functions based on the semester mark pass rate:

$$d_1(s) = \begin{cases} 75\% & if \quad s \geq 72\% \\ 70\% & if \quad s \in [67\%, 71\%] \\ 65\% & if \quad s \leq 66 \end{cases} \text{ and } d_2(s) = \begin{cases} 75\% & if \quad s \geq 70\% \\ 70\% & if \quad s \in [65\%, 69\%] \\ 65\% & if \quad s \leq 64 \end{cases}$$

Discuss which decision function seems more appropriate.

Question 17 John has been given a bag of balls but has not been told how many balls there are in the bag in total. He only knows there may by 5, 6 or 7 balls in total and that the probabilities of selecting a certain number of balls from the bag is governed by a binomial distribution with parameters n and $p = 0.4$. He needs to decide how many balls are in the bag.

a) Explain why this is a statistical game.
b) Set up a loss matrix for the game if he loses points based on the absolute difference on the total number of balls chosen.
c) Consider the following two decisions functions based on an observed number of balls, x, selected from the bag by John's friend:

$$d_1(x) = \begin{cases} 5 \ if \ \ x = 4 \\ 6 \ if \ \ x = 5; \\ 7 \ if \ \ x = 6 \end{cases} \quad d_2(x) = \begin{cases} 5 \ if \ \ x < 5 \\ 6 \ if \ \ x = 5 \\ 7 \ if \ \ x > 5 \end{cases}.$$

Set up the risk function matrix for these decision functions.
d) Which decision function should he use? Why?

Question 18 Two rugby players are running toward each other in a rugby world cup game. Player A has the ball and must shortly decide if he will pass the ball to a team mate or rather keep running with the ball. Player B is on the opposite team and aims to get the ball from Player A. Player B can attempt either a pass intercept or a tackle. Player A can attempt either a pass or run but doesn't know B's decision. Player A aims to maximize his probability of scoring a try. If B attempts a pass intercept, P(Try) = 0.1 if A passes and 0.8 if A runs. If B attempts a tackle, P(Try) = 0.5 regardless of A's decision.

a) Set up a loss matrix to represent Player A's probability of scoring a try.
b) Explain whether this is a two-player or statistical game.
c) Determine a randomized strategy for Player A based on Player A's probability of passing p.

Question 19 Suppose a random variable X is distributed exponentially with unknown parameter λ which is either 2, 3 or 4. The statistician has to decide which choice for λ is the best and decides to use game theory.

a) Set up a loss matrix if the choice of λ is penalized according to $(\lambda_{\text{actual}} - \lambda_{\text{choice}})^2$.

b) Complete the following matrix of χ^2 distribution probabilities where $Y \sim \chi^2(2\lambda)$:

λ	$P[Y \le 6]$	$P[6 < Y \le 18]$	$P[Y > 18]$	$P[Y > 6]$
2				
3				
4				

c) The statistician decides on the following two decision functions based on i.i.d. observed data x_1, x_2, x_3 to choose the value of λ:

$$d_1(x_1,x_2,x_3) = \begin{cases} 2 & if \quad \sum_i x_i < 1 \\ 3 & if \quad 1 \le \sum_i x_i < 3 \\ 4 & if \quad \sum_i x_i > 3 \end{cases} \text{ and } d_2(x_1,x_2,x_3) = \begin{cases} 2 & if \quad \sum_i x_i < 1 \\ 4 & if \quad \sum_i x_i > 1 \end{cases}.$$

Complete the following expected loss matrix based on these decision functions. Hint: $Y = X_1 + ... + X_3 \sim gamma(\lambda, 3)$. Make sure you motivate how the answer in (b) is used.

λ	d_1	d_2
2		
3		
4		

d) Which decision function should the statistician used based on your answer in (c)? Motivate.

Question 20 A statistician has data which he has good reason to believe follows a Poisson distribution with parameter β. The parameter β is however not known but the statistician believes it to be 2, 2.5 or 3. Three different estimation techniques were used to obtain estimates for β. The realized estimates associated with the three estimation techniques are 2, 2.5 and 3, with the techniques having believed certainty 0.3, 0.4 and 0.3 respectively.

a) Is this a zero-sum two-player game or a statistical game? Explain.
b) Set up a loss matrix for this game if the loss for each decision of the statistician is the absolute value of the difference between the expected value of the Poisson random variable and the true value.
c) Explain why there are no dominating strategies.
d) Use Bayes criterion to determine the statistician's optimal choice of the estimate for β.
e) The statistician considers an alternative approach to modelling X which is distributed $Poisson(\beta)$ by with $\beta \sim U(2,3)$. Provide the density function of X, $f_X(x)$. You do not need to simplify the integral in the density function but should explain how it would determined in a practical situation.

Question 21 A statistician is told that one of two balanced dice has been chosen and rolled. The one dice has six sides numbered from 1 to 6 and the second dice has four sides numbered from 1 to 4. If the statistician identifies the dice that has been rolled correctly he wins R10. The payoff matrix is

		Statistician (gain)	
		4-sided (a_1)	6-sided (a_2)
Nature	4-sided (θ_1)	10	0
(True)	6-sided (θ_2)	0	10

The outcome of the roll is given to the statistician and based on this he decides whether the dice rolled is the 4-sided or the 6-sided dice. The two decision functions that he uses are:

$$d_1(x) = \begin{cases} \text{4-sided} & \text{if } x = 1, 2, 3 \\ \text{6-sided} & \text{if } x = 4, 5, 6 \end{cases} \qquad d_2(x) = \begin{cases} \text{4-sided} & \text{if } x = 1, 2, 3, 4 \\ \text{6-sided} & \text{if } x = 5, 6 \end{cases}$$

The expected loss matrix for the two decision functions is given by

		Statistician (gain)	
		d_1	d_2
Nature	4-sided (θ_1)	$1.5b$	$2b$
(True)	6-sided (θ_2)	b	3.333

a) Calculate $E[L(d_2(x),\theta_1)]$ and use this to find the value of b.

b) Assume that the 4-sided dice is twice as likely to be chosen (nature) compared to the 6-sided dice. Use the Bayes criterion to decide which decision function (d_1 or d_2) should be used. If you did not get the answer for b in the previous part, you may assume that $b > 0$.