WST 311

Assignment A: 5-9 February 2018

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1. Calculate the eigenvalues and corresponding normalized eigenvectors for the matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & -5 \\ -5 & 1 \end{pmatrix}.$$

2. Let $\boldsymbol{X}_1 : p \times 1$, $\boldsymbol{X}_2 : p \times 1$ and $\boldsymbol{Y} : q \times 1$ be random vectors with $\boldsymbol{A} : n \times p$ a matrix of constants and $\boldsymbol{b} : q \times 1$ a constant vector. Use the definition of a covariance and show that

$$cov\left(\boldsymbol{X}_1 + \boldsymbol{A}\boldsymbol{X}_2, (\boldsymbol{Y} + \boldsymbol{b})'\right) = cov\left(\boldsymbol{X}_1, \boldsymbol{Y}'\right) + \boldsymbol{A}cov\left(\boldsymbol{X}_2, \boldsymbol{Y}'\right).$$

3. If $cov(\boldsymbol{X}, \boldsymbol{X}') = \boldsymbol{\Sigma} : p \times p = (\sigma_{ij})$ and $\boldsymbol{a} : p \times 1$ constant then

$$var(\boldsymbol{a}'\boldsymbol{X}) = \sum_{i=1}^{p}\sum_{j=1}^{p} a_i a_j \sigma_{ij} = \sum_{i=1}^{p} a_i^2 \sigma_{ii} + 2\sum_{i<j}^{p} a_i a_j \sigma_{ij}$$

Show this.

4. Suppose $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ is a 3×1 vector with jointly distributed random variables such that

$$E\left(\boldsymbol{X}\right) = \boldsymbol{\mu} \text{ and } \boldsymbol{\Sigma} = \boldsymbol{cov}\left(\boldsymbol{X}, \boldsymbol{X}'\right) = \begin{pmatrix} 50 & 36 & 18 \\ 36 & 36 & 0 \\ 18 & 0 & 72 \end{pmatrix}.$$

Consider the linear combinations

$$\boldsymbol{b}'\boldsymbol{X} = 2X_1 + 2X_2 - X_3,$$

$$\boldsymbol{c}'\boldsymbol{X} = X_1 - X_2 + 3X_3$$

and

$$\boldsymbol{d}'\boldsymbol{X} = X_1 + X_3.$$

Use PROC IML to answer the following:

(a) Calculate $var\left(\boldsymbol{b}'\boldsymbol{X}\right)$.

(b) Calculate $var\left(\boldsymbol{c}'\boldsymbol{X}\right)$.

(c) Calculate $cov\left(\boldsymbol{b}'\boldsymbol{X}, \boldsymbol{c}'\boldsymbol{X}\right)$.

(d) Calculate $cor\left(\boldsymbol{b}'\boldsymbol{X}, \boldsymbol{c}'\boldsymbol{X}\right)$.

(e) Let $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{b}' \\ \boldsymbol{c}' \end{pmatrix}$. Calculate $cov\left(\boldsymbol{A}\boldsymbol{X}, (\boldsymbol{A}\boldsymbol{X})'\right)$.

(f) Let $\boldsymbol{B} = \begin{pmatrix} \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{d} \end{pmatrix}$. Calculate $cov\left(\boldsymbol{B}'\boldsymbol{X}, \left(\boldsymbol{B}'\boldsymbol{X}\right)'\right) = \begin{pmatrix} \varsigma_{11} & \varsigma_{12} & \varsigma_{13} \\ \varsigma_{21} & \varsigma_{22} & \varsigma_{23} \\ \varsigma_{31} & \varsigma_{32} & \varsigma_{33} \end{pmatrix}$ and use this to give:

i. ς_{23}
ii. $cov\left(X_1 + X_3, 2X_1 + 2X_2 - X_3\right)$.

(g) Calculate the following for $\boldsymbol{\Sigma}$:

i. The eigenvalues and normalized eigenvectors of $\boldsymbol{\Sigma}$.
ii. Calculate $\boldsymbol{\Sigma}^{\frac{1}{2}}$, the symmetric square root of $\boldsymbol{\Sigma}$.
iii. Use the eigenvalues of $\boldsymbol{\Sigma}$ to calculate $|\boldsymbol{\Sigma}|$ and $\mathrm{tr}(\boldsymbol{\Sigma})$.

5. Consider the data for *Iris Setosa* in the iris flower data discussed in class (see example Iris.pdf and dataset Iris.xls given on ClickUP under Assignment A). Let $\boldsymbol{x}_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ x_{i3} \\ x_{i4} \end{pmatrix}$, $i = 1, 2, \ldots, 50$ indicate the vectors of observations for petal width (PW), petal length (PL), sepal width (SW), and sepal length (SL) respectively and let

$$\boldsymbol{X} : 50 \times 4 = \begin{pmatrix} \boldsymbol{x}_1' \\ \boldsymbol{x}_2' \\ \vdots \\ \boldsymbol{x}_{50}' \end{pmatrix}$$

be the observed data matrix. Use SAS IML to calculate the sample mean ($\overline{\boldsymbol{x}}$), sample covariance matrix $\left(\boldsymbol{S} = \frac{1}{49}\mathbf{X}'\left(\boldsymbol{I}_{50} - \frac{1}{50}\mathbf{1}_{50}\mathbf{1}_{50}'\right)\mathbf{X}\right)$ and sample correlation matrix ($\boldsymbol{R}$) for *Iris Setosa*.