WST 311

Assignment C: 19-23 February 2018

1. Let $\boldsymbol{X} : p \times 1$ have a $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution with $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1 : p_1 \times 1 \\ \boldsymbol{X}_2 : p_2 \times 1 \end{pmatrix}$ and

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

   Prove that $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are respectively $N_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$ and $N_{p_2}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_{22})$ distributed and that $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are **independent.**

2. Let

$$\boldsymbol{X}_1 = \begin{pmatrix} X_{11} \\ X_{12} \\ X_{13} \end{pmatrix}, \quad \boldsymbol{X}_2 = \begin{pmatrix} X_{21} \\ X_{22} \\ X_{23} \end{pmatrix} \quad \text{and} \quad \boldsymbol{X}_3 = \begin{pmatrix} X_{31} \\ X_{32} \\ X_{33} \end{pmatrix}$$

   with means

$$\boldsymbol{\mu}_1 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad \boldsymbol{\mu}_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\mu}_3 = \mathbf{0}.$$

   The corresponding covariances are

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} 1 & 0.6 & 0.2 \\ 0.6 & 1 & 0.3 \\ 0.2 & 0.3 & 1 \end{pmatrix}, \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   The random vectors $\boldsymbol{X}_1$, $\boldsymbol{X}_2$ and $\boldsymbol{X}_3$ are independently normally distributed. Let

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \boldsymbol{X}_3 \end{pmatrix} \qquad \boldsymbol{\nu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \boldsymbol{\mu}_3 \end{pmatrix} \qquad \boldsymbol{\Phi} = \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Sigma}_3 \end{pmatrix}$$

   and $\boldsymbol{Y} = \sum_{i=1}^{3} \boldsymbol{X}_i$.

   (a) Derive the distribution for $\boldsymbol{Y}$, and calculate $E(\boldsymbol{Y})$ and $cov(\mathbf{Y}, \mathbf{Y}')$.

   (b) Calculate the moment generating function of $\boldsymbol{Y}$.

3. Suppose that $\boldsymbol{X} : p \times 1$ has a $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ distribution with $Y_1 = \boldsymbol{d}_1'\boldsymbol{X}$ and $Y_2 = \boldsymbol{d}_2'\boldsymbol{X}$. Prove that Y_1 and Y_2 are independent if and only if $\boldsymbol{d}_1'\boldsymbol{\Sigma}\boldsymbol{d}_2 = 0$.

4. Suppose that $\boldsymbol{X} : p \times 1 \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{Y} : p \times 1 \sim N(\boldsymbol{\nu}, \boldsymbol{\Phi})$ and that $\boldsymbol{X}$ and $\boldsymbol{Y}$ are independent. Find the distribution of

   (a) $\boldsymbol{X} + \boldsymbol{Y}$

   (b) $\boldsymbol{X} - \boldsymbol{Y}$.

5.

(a) Use PROC G3D and draw a 3 dimensional graph of a bivariate normal density function, that is for $\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \equiv N_2\left(\begin{pmatrix} 5 \\ 5 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix}\right)$ for each of the $\boldsymbol{\Sigma}'s$ given below.

Start out by defining a grid over the definition region of the density function in PROC IML. Continue by calculating the density function over the grid. Plot the density function using PROC G3D.

Syntax for PROC G3D:

```
proc g3d data=normal;
plot x*y=fxy / side
tilt=45 rotate=35;
```

- 'side' produces a surface graph with a side wall.
- 'tilt' specifies one or more angles ($0° - 90°$) to tilt the graph toward you (default $= 70°$).
- 'rotate' specifies one or more angles at which to rotate the $X - Y$ plane around the perpendicular Z axis (default $= 70°$).

i. $\boldsymbol{\Sigma} = \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}$

ii. $\boldsymbol{\Sigma} = \begin{pmatrix} 4 & 4 \\ 4 & 6 \end{pmatrix}$

iii. $\boldsymbol{\Sigma} = \begin{pmatrix} 4 & -4 \\ -4 & 6 \end{pmatrix}$

(b) Calculate the correlation between X and Y for the three cases in 5(a) and comment on the association between the two random variables.

(c) Refer to 5(a)(ii). Use numerical integration and determine the total volume under the density curve.

(d) Within the context of 5(a)(ii), that is when $\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} 5 \\ 5 \end{pmatrix}, \begin{pmatrix} 4 & 4 \\ 4 & 6 \end{pmatrix}\right)$, draw a graph of a *slice* through the density function where $X = 2$. Use PROC GPLOT to draw the graph. Comment on the shape of the graph. Is it a density function, perhaps a normal density function?

(e) Refer to 5(d). Use numerical integration and determine the area under the curve. How can the curve in 5(d) be transformed to ensure that it is a normal density curve.