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Mapping class groups

- The mapping class group of a compact surface Σ, $\mathrm{MCG}(\Sigma)$, is the group of isotopy classes of orientation-preserving self-homeomorphisms of Σ
  - $\mathrm{MCG}(\mathbf{D}$ with n marked points$) = B_m$
  - $\mathrm{MCG}(\mathbf{T}^2) = SL(2, \mathbb{Z})$

Property F conjecture for mapping class groups (Rowell)

The Turaev-Viro-Barrett-Westbury (TVBW) mapping class group representation associated to a compact surface Σ and spherical fusion category $\mathcal{A}$ has finite image iff $\mathcal{A}$ is weakly integral.

The spherical fusion category Vect_G^ω

Our case: Vect_G^ω, the category G-graded vector spaces with a twist

- The associator $\alpha_{g,h,k} : (V_g \otimes V_h) \otimes V_k \to V_g \otimes (V_h \otimes V_k)$
$$\alpha_{g,h,k} = \omega(g, h, k)$$
- The evaluator $\mathrm{ev}_g : V_g^* \otimes V_g \to 1$
$$ev_g = \omega(g^{-1}, g, g^{-1})$$
- The coevaluator $\mathrm{coev}_g : V_g \otimes V_g^* \to 1$
$$\mathrm{coev}_g = 1$$
- The pivotal structure $j_g : V_g^{**} \to V_g$
$$j_g = \omega(g^{-1}, g, g^{-1})$$

Related Work

- All Vect_G^ω braid group representations have finite images (Etingof–Rowell–Witherspoon)
- If $\omega = 1$, every mapping class group representation of a closed surface with ≤ 1 marked point has finite image (Fjelstad–Fuchs)
- Every $SL(2, \mathbf{Z})$ representation from any modular category has finite image (Ng–Schauenberg)

Main result

The image of any Vect_G^ω TVBW representation ρ of a mapping class group of an orientable, compact surface Σ with boundary is finite.

Proof outline:

- Describe a tractable presentation of the representation space
- Find a good finite spanning set S for the representation space
- Calculate the action of each Birman generator on S
- Show that the representation of each Birman generator lies in a quotient of a finite group of monomial matrices.

The TVBW space associated to a 2-manifold

- Kirillov: The TVBW representation space is canonically isomorphic to
$$H := \frac{\mathbf{C}[\mathcal{A}\text{-colored graphs in } \Sigma]}{\text{local relations}}$$

Local relations

- Isotopy of the graph embedding
- Linearity in the vertex colorings

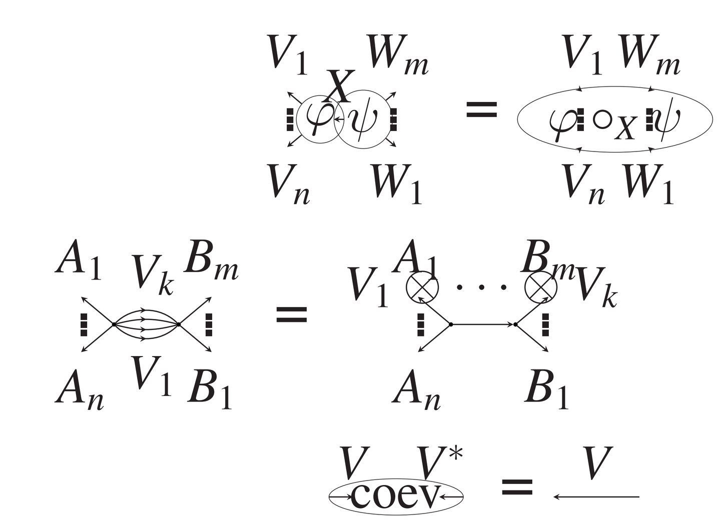


Figure: The remaining local relations.

Spanning set for the representation space

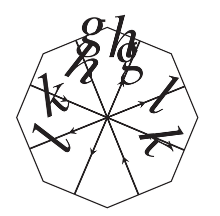


Figure: Element of the spanning set for a genus 2 surface. Here $[g, h][k, l] = 1$, and the vertex is labeled by a "simple" morphism (a $|G|$-th root of unity times a canonical morphism)

Applying the Birman generators to the spanning set

- The next step of the proof is to apply each Birman generator to each element of the spanning set.
- In each case, we relate the resulting colored graph to another element of the spanning set by means of local moves
- The local moves map simple colored graphs to simple colored graphs
- Hence, the Birman generators preserve the finite spanning set.

First Dehn twist

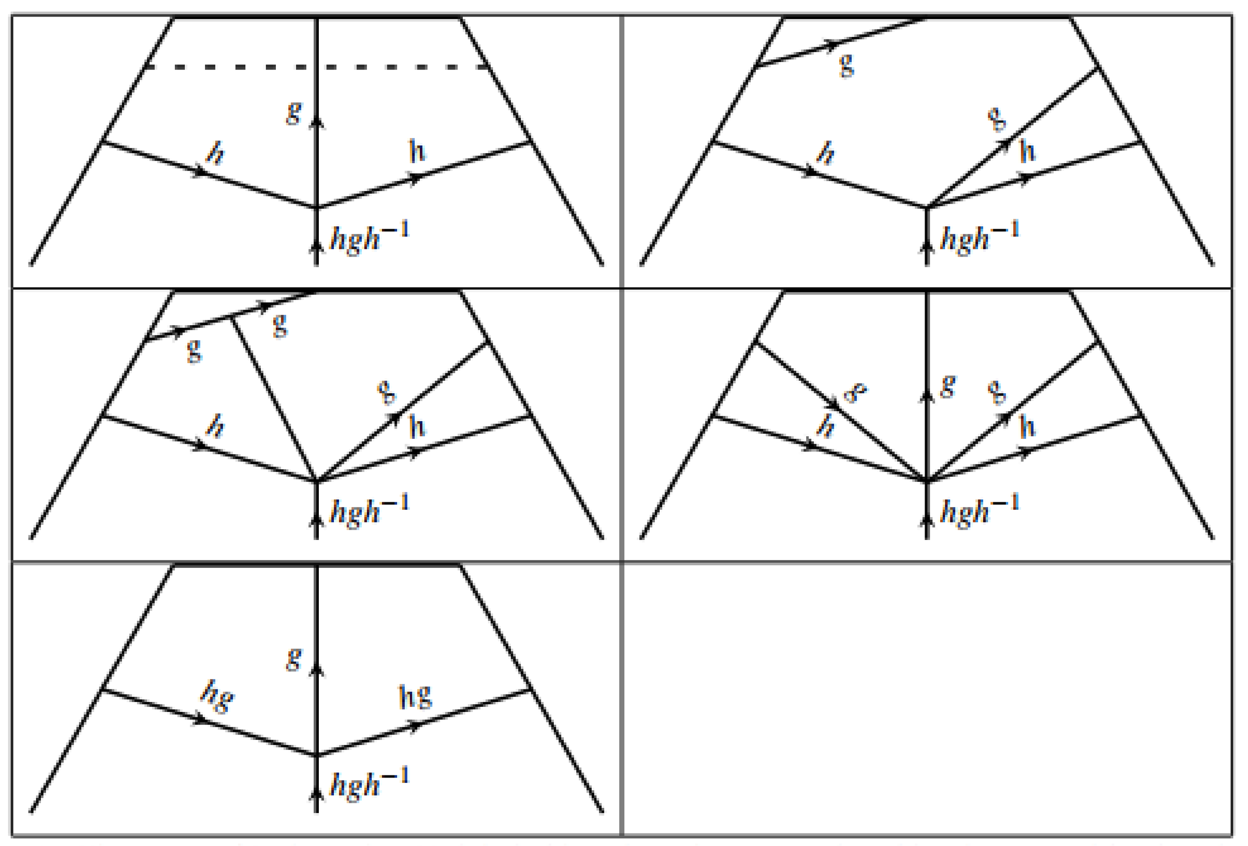
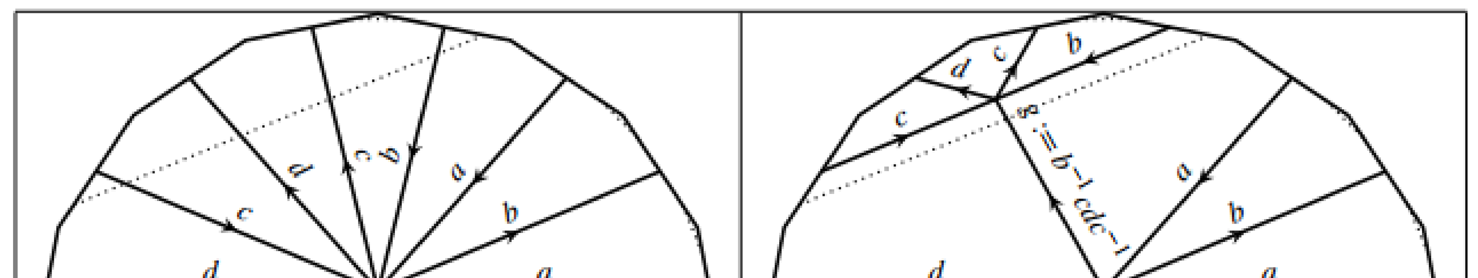


TABLE 1. First type of Dehn twist. Unlabeled interior edges are colored by the group identity element.

Second Dehn twist



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