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HW 2

1 Let E_{ij} denote the standard basis of $M_4(\mathbb{R})$, and $F_{ij} = E_{ij} - E_{ji}$. Let $\alpha \in \mathbb{R} \setminus \{0\}$ and $A = F_{21} + \alpha F_{43}$. Let $B = F_{32}$. We consider the following system on $M := SO(4, \mathbb{R})$:

$$\dot{E} = E(A + uB), \quad E \in M, u \in \{-1, 1\}.$$

a Show that the system is controllable iff $\alpha \neq \pm 1$.

Proof. Since $SO(4)$ is a compact Lie group, the system is controllable iff it is bracket generating.

Fix any $E \in M$. To begin calculating the brackets, we have

$$\begin{aligned}[F_{ij}, F_{kl}] &= [E_{ij} - E_{ji}, E_{kl} - E_{lk}] \\ &= \delta_{jk}E_{il} - \delta_{jl}E_{ik} - \delta_{ik}E_{jl} + \delta_{il}E_{jk} - (\delta_{li}E_{kj} - \delta_{lj}E_{ki} - \delta_{ki}E_{lj} + \delta_{kj}E_{ki}) \\ &= \delta_{jk}F_{il} - \delta_{jl}F_{ik} - \delta_{ik}F_{jl} + \delta_{il}F_{jk}\end{aligned}$$

Hence

$$\begin{aligned}[A, B] &= [F_{21} + \alpha F_{43}, F_{32}] \\ &= F_{13} + \alpha F_{42}\end{aligned}$$

$$\begin{aligned}[A, [A, B]] &= [F_{21} + \alpha F_{43}, F_{13} + \alpha F_{42}] \\ &= F_{23} + \alpha F_{14} - \alpha F_{41} - \alpha^2 F_{32} \\ &= 2\alpha F_{14} + (1 + \alpha^2)F_{23}\end{aligned}$$

$$\begin{aligned}[B, [A, B]] &= [F_{32}, F_{13} + \alpha F_{42}] \\ &= F_{21} - \alpha F_{34} \\ &= A\end{aligned}$$

Let $C = (2\alpha)^{-1}([A, [A, B]] + (1+\alpha^2)B) = F_{14}$. Then $Lie^3_E = \operatorname{span}(A, B, [A, B], C)$.

Then we have

$$\begin{aligned}[A, C] &= [F_{21} + \alpha F_{43}, F_{14}] \\ &= \alpha(F_{24} + F_{31}) \\ &= -\alpha F_{13} - F_{42}\end{aligned}$$

$$\begin{aligned}[B, C] &= [F_{32}, \alpha F_{14}] \\ &= 0\end{aligned}$$

$$\begin{aligned}[C, [A, B]] &= [F_{14}, F_{13} + \alpha F_{42}] \\ &= -F_{43} + \alpha F_{12}\end{aligned}$$

Case $\alpha^2 = 1$. We have $\alpha[A, C] = -F_{13} - \alpha F_{42} = -[A, B]$, and $\alpha[C, [A, B]] = F_{12} - \alpha F_{43} = -A$. Thus $[Lie^3_E, Lie^3_E] = Lie^3_E$, and it is easy to see that $\dim(Lie^3_E) = 4$. Since this holds for all points of M, Lie^3 defines an involutive distribution of M. By the Frobenius theorem there exists an immersed submanifold N containing E such that $T_q(N) = Lie^3_q(M)$ for all $q \in N$. In particular, the orbit of E and the admissible set lie in N. Since $\dim(N) = 4 < 6 = \dim(M)$, the inclusion map $N \to M$ is everywhere singular. Hence, Sard's theorem implies that N has measure 0 as a subset of M, so $N \neq M$.

Case $\alpha^2 \neq 1$. Let $D = (-\alpha^2 + 1)^{-1}([A, C] + [A, B]) = F_{13}$. Since $[B, D] = [F_{32}, F_{13}] = F_{21}$, we have

$$(F_{12}, F_{13}, F_{14}, F_{23}, F_{24}, F_{34}) = (-[B, D], D, C, -B, [A, C]+\alpha D, [C, [A, B]]+\alpha[B, D]).$$

Hence $\mathcal{F} := \{A + uB : u \in \{-1, 1\}\}$ is bracket-generating. Since $SO(4)$ is connected, this implies the system is controllable. □

b Will the answer of the previous item change if $u \in \{2, 3\}$ instead of $\{-1, 1\}$?

Proof. No, $\text{span}(A+2B, A+3B) = \text{span}(A, B) = \text{span}(A+B, A-B)$, so Lie^1, hence every Lie^n, will be the same as in (a). □

c Assume that $\alpha = \pm 1$. Prove that for any point $E \in M$ the attainable set from E coincides with the orbit, and find the dimension of every orbit.

Proof. Since $SO(4)$ is a compact Lie group, the family of vector fields $\mathcal{F}$ corresponding to the control system is Poisson stable (as shown in class). Hence, for every $f \in \mathcal{F}$, $-f$ is compatible with $\mathcal{F}$. Thus the attainable set is dense in the orbit of E.

By part (a), the orbit of E lies in an immersed submanifold N with $T_q(N) = Lie^3(q)$ for all $q \in N$. By passing to the connected component of N containing E, WLOG N is connected. Then by the Rachevskii-Chow theorem, $\mathcal{F}$ is controllable on N. Hence by Krener's theorem, the attainable set of E is equal to N, which coincides with the orbit of E. For every point $q \in M$, the orbit of q has dimension $\dim(Lie^3(q)) = 4$. □

d Assume that $\alpha = 1$. Define complex scalar multiplication on $\mathbb{R}^4$ by $ie_1 = -e_4$, $ie_4 = e_1$, $ie_2 = e_3$, and $ie_3 = -e_2$. Show that a matrix D belongs to the tangent space at the identity I to the orbit iff the corresponding linear operator $\widehat{D}$ is also linear over $\mathbb{C}$ and the 2x2 matrix D_1 corresponding to this operator in the complex basis (e_1, e_2) is satisfies $\overline{D_1^T} = D_1$.

Proof. Suppose $v = \sum_{j=1}^4 v_j e_j \in \mathbb{R}^4$ and $U \in M_4(\mathbb{R})$, then $U(iv) = U(v_1(-e_4) + v_2(e_3) + v_3(-e_2) + v_4(e_1)) = v_4 U e_1 - v_3 U e_2 + v_2 U e_3 - v_1 U e_4$. We also have $F_{jk} e_l = E_{jk} e_l - E_{kj} e_l = \delta_{kl} e_j - \delta_{jl} e_k$.

Hence,

$$\begin{aligned} iF_{jk}(v) &= i \sum_l v_l(\delta_{kl} e_j - \delta_{jl} e_k) \\ &= v_k(ie_j) + v_j(-ie_k) \end{aligned}$$

and

$$\begin{aligned} F_{jk}(iv) &= v_4 F_{jk} e_1 - v_3 F_{jk} e_2 + v_2 F_{jk} e_3 - v_1 F_{jk} e_4 \\ &= v_4(\delta_{k1} e_j - \delta_{j1} e_k) - v_3(\delta_{k2} e_j - \delta_{j2} e_k) + v_2(\delta_{k3} e_j - \delta_{j3} e_k) - v_1(\delta_{k4} e_j - \delta_{j4} e_k) \\ &= (\delta_{k1} v_4 - \delta_{k2} v_3 + \delta_{k3} v_2 - \delta_{k4} v_1) e_j + (-\delta_{j1} v_4 + \delta_{j2} v_3 - \delta_{j3} v_2 + \delta_{j4} v_1) e_k \end{aligned}$$

Hence,

$$\begin{aligned} F_{12}(iv) &= -v_3 e_1 - v_4 e_2 = -v_3(ie_4) - v_4(-ie_3) = iF_{34} v \\ F_{13}(iv) &= v_2 e_1 - v_4 e_3 = v_2(ie_4) - v_4(ie_2) = iF_{42} v \\ F_{14}(iv) &= -v_1 e_1 - v_4 e_4 = -v_1(ie_4) - v_4(-ie_1) = iF_{14} v \\ F_{23}(iv) &= v_2 e_2 + v_3 e_3 = v_2(-ie_3) + v_3(ie_2) = iF_{23} v \\ F_{24}(iv) &= -v_1 e_2 + v_3 e_4 = -v_1(-ie_3) + v_3(-ie_1) = iF_{31} v \\ F_{34}(iv) &= -v_1 e_3 - v_2 e_4 = -v_1(ie_2) - v_2(-ie_1) = iF_{12} v \end{aligned}$$

Thus, $\mathcal{B} := \{F_{14}, F_{23}, F_{34} + F_{12}, F_{13} + F_{42}\}$ is an $\mathbb{R}$-basis for the the subspace of $T_I(M)$ consisting of all $\mathbb{C}$-linear operators. From (a), we know that $\mathcal{B}$ is also an $\mathbb{R}$-basis for $Lie_I^3(M) = T_I(O_I(\mathcal{F}))$. Hence, $T_I(O_I(\mathcal{F}))$ coincides with the $\mathbb{C}$-linear subspace of $T_I(M)$. In the $\mathbb{C}$-basis $\{e_1, e_2\}$ for $\mathbb{R}^4$ with the given complex structure, we have

$$F_{14} = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \quad F_{23} = \begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}$$

and

$$F_{34} + F_{12} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad F_{13} + F_{42} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

All of these matrices are self adjoint. Hence each element of span($\mathcal{B}$) is also self-adjoint. $\square$