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HW 1

1 Show that every faithful and transitive action of a group G on a set X, where G is abelian, is free (and therefore equivalent to a regular one).

Proof. If X is empty, the conclusion is trivial. Otherwise, let $x \in X$. Suppose $gx = x$ for some $g \in G$. To show that the action is free, it suffices to show that $g = 1$.

Let $y \in X$. Since the action is transitive, $y = hx$ for some $h \in G$. Thus $gy = ghx = hgx = hx = y$. Since y was arbitrary, g fixes all of X. Thus $g = 1$ since the action is faithful. $\square$

2 Consider the following three subspaces of $\mathbb{R}^2$ with metric that is inherited from the Euclidean metric

$$R = \{(x, 0)|x \in \mathbb{R}\}$$

$$L = \{(x, 0)|x \in \mathbb{R}, x \geq 0\} \cup \{(0, y)|y \in \mathbb{R}, y \geq 0\}$$

$$H = \{(x, 0)|x \in \mathbb{R}, x > 0\}.$$

Which of these spaces are quasi-isometric?

Proof. I claim that R and L are quasi-isometric to each other, but H is not quasi-isometric to the others.

For the former, define $\phi : L \to R$ by sending $(x, 0)$ to itself and sending $(0, y)$ to $(-y, 0)$.

Suppose $p, q \in L$. Then $d(p, q) = d(\phi(p), \phi(q))$ unless p, q are on different axes. In this case, WLOG assume $p = (x, 0)$ and $q = (0, y)$ for $x, y \geq 0$. Then $d(p, q) = (x^2+y^2)^{1/2} \leq x+y = d(\phi(p), \phi(q)$. On the other hand, $d(\phi(p), \phi(q)) = x + y = (x^2 + 2xy + y^2)^{1/2} \leq (x^2 + 2(x^2 + y^2) + y^2)^{1/2} = \sqrt{3}(x^2 + y^2)^{1/2} = \sqrt{(3)}d(x, y)$. Hence, L and R are quasi-isometric.

To see that R and H are not quasi-isometric, it suffices to show that $\mathbb{Z}$ and $\mathbb{N}$ are not quasi-isometric. Suppose $\phi : \mathbb{N} \to \mathbb{Z}$ is a quasi-isometry. Then $C^{-1}d(p, q) - K \leq d(\phi(p), \phi(q)) \leq Cd(p, q) + K$ for all $p, q \in \mathbb{N}$ for some $C \geq 1$ and $K \geq 0$. In particular, we have $d(\phi(n), \phi(n + 1)) \leq C + K$ for all $n \in \mathbb{N}$. Since ϕ takes on infinitely many positive values and infinitely many negative values, this implies $\phi(n) \in [0, C + K]$ for infinitely many n. Pick any $p \in \mathbb{N}$ with $\phi(p) \in [0, C + K]$. Pick q with $\phi(q) \in [0, C + K]$ and q so large that $C^{-1}d(p, q) - K > C + K \geq d(\phi(p), \phi(q))$, a contradiction. $\square$