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HW 9

1 If $f \in L_1(0, \infty)$, define

$$g(s) = \int_0^\infty e^{-st} f(t) dt, \quad 0 < s < \infty.$$

Prove that $g(s)$ is differentiable on $(0, \infty)$ and that

$$g'(s) = -\int_0^\infty t e^{-st} f(t) dt, \quad 0 < s < \infty.$$

Proof.

□

2 Let (Ω, μ, Σ) be a finite measure space and (f_n) be a sequence of measurable functions on Ω . Suppose that for each $\omega \in \Omega$ there is an $M_\omega \in \mathbb{R}$ so that for all $k \in \mathbb{N}$, $|f_k(\omega)| \leq M_\omega$. Let $\epsilon > 0$. Show that there is a measurable $A \subset \Omega$ and an $M \in \mathbb{R}$ so that $\mu(\Omega \setminus A) < \epsilon$ and $f_k(\omega) < M$ for all $k \in \mathbb{N}$ and all $\omega \in A$.

3 57/page 77. Show that $\int_0^\infty e^{-sx} x^{-1} \sin x dx = \arctan(s^{-1})$ for $s > 0$ by integrating $e^{-sxy} \sin x$ with respect to x and y . (Hints: $\tan(\frac{\pi}{2}) = (\tan \theta)^{-1}$ and Exercise 31d.)

4 60/page 77. $\Gamma(x)\Gamma(y)/\Gamma(x+y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt$. (Recall that Γ was defined in Section 2.3. Write $\Gamma(x)\Gamma(y)$ as a double integral and use the argument of the exponential as a new variable of integration.)

5 Given a bounded function $f : [a, b] \rightarrow \mathbb{R}$, define

$$H(x) = \lim_{\delta \rightarrow 0} \sup_{|x-y| \leq \delta} f(y), \text{ and } h(x) = \lim_{\delta \rightarrow 0} \inf_{|x-y| \leq \delta} f(y)$$

a) For $x \in [a, b]$, f continuous at $x \iff H(x) = h(x)$.

b) Assume now that (P_k) is an increasing sequence of partitions of $[a, b]$ for which the mesh converges to zero. Write $P_k = (t_0^{(k)}, t_1^{(k)}, \dots, t_{n_k}^{(k)})$. Define for $x \in [a, b]$,

$$G(x) = \lim_{k \rightarrow \infty} G_{P_k}(x) \text{ and } g(x) = \lim_{k \rightarrow \infty} g_{P_k}(x),$$

where for a partition $P = (t_0, t_1, \dots, t_n)$

$$G_P = \sum_{i=1}^n \chi_{(t_{i-1}, t_i]} \sup_{t \in (t_{i-1}, t_i]} f(t) \text{ and } g_P = \sum_{i=1}^n \chi_{(t_{i-1}, t_i]} \inf_{t \in (t_{i-1}, t_i]} f(t).$$

Prove that $H = G$ and $h = g$ m -a.e.

c) Show that f is Riemann integrable \iff the set of discontinuities of f has Lebesgue measure zero.

6 Problem 30/page 60. Hint: AM-GM. Show that $\lim_{k \rightarrow \infty} \int_0^k x^n (1 - k^{-1}x)^k dx = n!$.

7 Problem 1/88. Let ν be a signed measure on (X, \mathcal{M}) . If (E_j) is an increasing sequence in \mathcal{M} , the $\nu(\bigcup_1^\infty E_j) = \lim_{j \rightarrow \infty} \nu(E_j)$. If (E_j) is a decreasing sequence in \mathcal{M} and $\nu(E_1)$ is finite, then $\nu(\bigcap_1^\infty E_j) = \lim_{j \rightarrow \infty} \nu(E_j)$.

8 Problem 4/88. If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

9 Problem 7/88. Suppose that ν is a signed measure on (X, \mathcal{M}) and $E \in \mathcal{M}$.

a. $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$.

b. $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \dots, E_n \text{ are disjoint, and } \bigcup_1^n E_j = E\}$.