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MATH 663 - Subfactors, Knots, and Planar Algebras (Fall 2017)

HW 2

1 Let M be a factor. Show that M is finite if and only if every isometry $u \in M$ is unitary.

Proof. Suppose M is finite. Let $u \in M$ be an isometry. Then $u^*u = 1$. Let $p = uu^*$. We have $1 \sim p \leq 1$. Thus, $p = 1$, so u is unitary.

Conversely, suppose that every isometry in M is unitary. Let p be a projection such that $1 \sim p \leq 1$. Then there exists an isometry u such that $u^*u = 1$ and $uu^* = p$. Since every isometry in M is unitary, it follows that $p = 1$. $\square$

2 Let Γ be a group. Prove that $L\Gamma' = R\Gamma$, where $R\Gamma \subset \mathcal{B}(\ell^2(\Gamma))$ is the von Neumann algebra generated by the right regular representation $\rho : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$.

Proof. We have $L\Gamma' = J(L\Gamma)J$, where $J(x\delta_e) = x^*\delta_e$. If $g, h \in \Gamma$, we have $J\lambda(g)J\delta_h = J\lambda(g)\lambda(h^{-1})\delta_e = \lambda(hg^{-1})\delta_e = \rho(g)\delta(h)$. By anti-linearity of J, we have $J\lambda(\mathbb{C}\Gamma)J = \rho(\mathbb{C}\Gamma)$. By the continuity of J with respect to the SOT, we have $L\Gamma' = JL\Gamma J = R\Gamma$. $\square$

3 Consider $M = M_n(\mathbb{C})$ equipped with its unique tracial state $\mathrm{Tr} : M_n(\mathbb{C}) \to \mathbb{C}$. Let $e_{ij} \in M$ be the standard matrix units associated to a fixed orthonormal basis $(e_i)_i$ for $\mathbb{C}^n$.

1. Show that the map $e_{ij} \mapsto \frac{1}{\sqrt{n}} e_i \otimes \overline{e_j}$ induces a unitary identification $L^2(M) \cong \mathbb{C}^n \otimes \overline{\mathbb{C}^n}$.

   Proof. We have

$$\begin{aligned}
\langle \sum_{i,j} a_{ij} e_{ij}, \sum_{kl} b_{kl} e_{kl} \rangle_{L^2(M)} &= \mathrm{Tr}(b^* a) \\
&= \mathrm{Tr}(\sum_j \overline{b_{ji}} a_{jk}) \\
&= \frac{1}{n} \sum_{i,j} \overline{b_{ji}} a_{ji} \\
&= \sum_{i,j} \langle a_{ij}(e_i \otimes \overline{e_j}), b_{ij}(e_i \otimes \overline{e_j} \rangle_{\mathbb{C}^n \otimes \overline{\mathbb{C}^n}}
\end{aligned}$$

   $\square$

2. Describe how M acts via the GNS representation on $\mathbb{C}^n \otimes \overline{\mathbb{C}^n}$.

   The image of the action of $e_{ij} \in M$ on $e_{kl} \in L^2(M)$ is $\delta_{jk} e_{il}$. Thus, the image of the action of e_{ij} on $e_k \otimes \overline{e_l}$ is $\delta_{jk} e_i \otimes \overline{e_l}$. This is just the usual matrix multiplication action on the first tensor factor, trivial action on the second factor.

3. Describe how the modular conjugation J acts on $\mathbb{C}^n \otimes \overline{\mathbb{C}^n}$.

   The modular conjugation J acts on $L^2(M)$ by $Jx\xi = x^*\xi$, where $\xi = \sum_i e_{ii}$. Hence, $J \sum_{j,k} a_{jk} e_{jk} = J \sum_{j,k} a_{jk} e_{jk} \xi = \sum_{j,k} \overline{a}_{jk} e_{kj}$. Thus, J acts on $\mathbb{C}^n \otimes \overline{\mathbb{C}^n}$ by $J \sum_{j,k} a_{jk} e_j \otimes \overline{e}_k = \sum_{j,k} \overline{a}_{jk} e_k \otimes \overline{e}_j$.

4. Describe how M' acts on $\mathbb{C}^n \otimes \overline{\mathbb{C}^n}$.

   Since $M' = JMJ$, the action is given by

$$\begin{aligned}
(J \sum_{i,j} a_{ij} e_{ij} J) b_{kl} e_k \otimes \overline{e}_l &= J \sum_{i,j} a_{ij} e_{ij} \overline{b}_{kl} e_l \otimes \overline{e}_k \\
&= J \sum_{i,j} a_{ij} \delta_{jl} \overline{b}_{kl} e_i \otimes \overline{e}_k \\
&= \sum_{i,j} \overline{a}_{ij} \delta_{jl} b_{kl} e_k \otimes \overline{e}_i \\
&= \sum_{i,j} b_{kl} e_k \otimes \overline{a}_{ij} e_{ij} \overline{e}_l.
\end{aligned}$$

   This is just matrix multiplication on the second tensor factor.

4 Give an example of a group Γ and an ergodic probability measure preserving action $\Gamma \curvearrowright (X, \Sigma, \mu)$ so that

$$L^\infty(X) \rtimes_\alpha \Gamma \cong M_n(\mathbb{C}).$$

Proof. Let $\Gamma = \mathbb{Z}_n$ and $X = \mathbb{Z}_n$ with the counting measure and translation action α. This action is free and ergodic. Thus, Γ acts freely and ergodically on $L^\infty(X)$. Thus, a theorem in class implies that $L^\infty(M) \rtimes_\alpha \Gamma$ is a factor. Since this factor has dimension n^2, it is isomorphic to $M_n(\mathbb{C})$. $\square$

5 A II_1-factor (M, τ) is said to have *property Gamma* if there exists a sequence of unitaries $(u_n)_{n\in\mathbb{N}} \subset M$ such that $\tau(u_n) = 0$ and

$$\|u_n x - x u_n\|_2 \to 0 \qquad (x \in M).$$

Prove that $L(S_\infty)$ has property Gamma.

Proof. Let $u_n = (n \quad n+1)$ be the transposition. Since $\tau(x) = \langle x\delta_e, \delta_e \rangle$, we have $\tau(u_n) = 0$. Let $x \in S_\infty$. Then $x \in S_m \subset S_\infty$ for some finite m. By the far commutation relation, we have $u_n x = x u_n$ for $n > m$. This implies that for all $x \in \mathbb{C}S_\infty$, we have $u_n x - x u_n = 0$ for all large n. The normality of $\| \cdot \|_2$ then implies that $L(S_\infty)$ has property Gamma. $\square$

6 (Bonus problem) Show that $L\mathbb{F}_2$ does not have property Gamma. Deduce that $L\mathbb{F}_2$ is not AFD.

Proof. See p. 485 of Effros, E. Property Γ and inner amenability. $\square$

7 Let $M \subset \mathcal{B}(H)$ be a von Neumann algebra and let K be a Hilbert space. Consider the von Neumann algebra $M \otimes 1 \subset \mathcal{B}(H \otimes K)$. Show that $(M \otimes 1)' = M' \bar{\otimes} \mathcal{B}(K)$. (Here, $M' \bar{\otimes} \mathcal{B}(K)$ is defined as the von Neumann algebra generated the algebraic tensor product $M' \otimes \mathcal{B}(K)$ inside $\mathcal{B}(H \otimes K)$.)

Proof. Clearly $M' \bar{\otimes} \mathcal{B}(K) \subset (M \otimes 1)'$. For the reverse inclusion, suppose that $x \in (M \otimes 1)'$. Let $(e_i)_{i \in I}$ be an o.n.b. for H and $(f_j)_{j \in J}$ an o.n.b. for K. We can write

$$x(e_i \otimes f_j) = \sum_{k,l} x_{ij}^{kl} e_k \otimes f_l.$$

After some algebra, the commutation relation $x(y \otimes 1) = (y \otimes 1)x$ for $y \in M$ becomes

$$\sum_k y_i^k x_{kj}^{lm} = \sum_k x_{ij}^{km} y_k^l,$$

for all l, m, i, j. Letting $p_r : H \otimes K \to H \otimes f_r \simeq H$ denote the projection, we have $p_j x p_m \in B(H)$ with matrix coefficients $(x_{bj}^{am})_{ab}$. Fixing m and j in the commutation equation above gives the equation $p_j x p_m (y \otimes 1) = (y \otimes 1) p_j x p_m$ for all j, m. Thus, $p_j x p_m \in (M \otimes 1)'$.

Let $\mathcal{F}$ denote the net of finite subsets of the index set J. For $\lambda \in \mathcal{F}$, let $x_\lambda = \sum_{j,m \in \mathcal{F}} p_j x p_m \in M' \otimes \mathcal{B}(K)$. Let $\xi \in H \otimes K$. Then $\xi = \sum_{i,j} \xi_{ij} e_i \otimes f_j$. We have

$$\begin{aligned}
\|(x - x_\lambda)\xi\| &= \|\sum_{l \notin \mathcal{F}} x_{ij}^{kl} \xi_{kl} e_k \otimes f_l + \sum_{j \notin \mathcal{F}, l \in \mathcal{F}} x_{ij}^{kl} \xi_{kl} e_k \otimes f_l\| \\
&\leq \|x\| \left(\sum_{l \notin \mathcal{F}} |\xi_{kl}|^2 \right)^{1/2} + \|\sum_{j \notin \mathcal{F}} x_{ij}^{kl} \xi_{kl} e_k \otimes f_l\| \\
&\to 0,
\end{aligned}$$

where the first term goes to 0 since ξ is square-summable and the second because $x\xi$ is square-summable. Thus, $x_\lambda \to x$ in the SOT. $\square$