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HW 11

1 Let f be increasing on $[0, 1]$ and

$$g(x) = \limsup_{h \rightarrow 0} \frac{f(x+h) - f(x-h)}{2h}, \quad \text{for } 0 < x < 1.$$

Prove that if $A = \{x \in (0, 1) : g(x) > 1\}$ then

$$f(1) - f(0) \geq m^*(A).$$

Hint: Vitali's Lemma.

2 Let $f : [a, b] \rightarrow \mathbb{R}$ be an increasing function. Using Vitali's lemma, show that

$$m(\{D^+f(x) \neq D^-f(x)\}) = 0.$$

3 Assume that $f : [a, b] \rightarrow \mathbb{R}$ is continuous and that $D^+f(x) > 0$, for all $x \in [a, b)$. Show that f is nondecreasing on $[a, b]$.

4 Determine whether or not the following functions are of bounded variation on $[-1, 1]$.

(a) $f(x) = x^2 \sin(1/x^2), \quad x \neq 0, f(0) = 0$

(b) $f(x) = x^2 \sin(1/x), \quad x \neq 0, f(0) = 0.$

5 Let f be of bounded variation on $[a, b]$, then

$$\int_a^b |f'(t)| dt \leq T_a^b(f).$$

6 Construct an increasing function on \mathbb{R} whose discontinuities are \mathbb{Q}