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Bonus exercises

1 Let $\lambda : [0,T] \to [0,1]$ be measurable. Let

$$A_n = \bigcup_{k=0}^{n-1} [kT/n, kT/n + \int_{kT/n}^{(k+1)T/n} \lambda(t)\, dt).$$

Show that for any $\phi \in L_1[0,T]$,

$$\int_0^t \chi_{A_n}(\tau)\phi(\tau)\, d\tau \to \int_0^t \lambda(\tau)\phi(\tau)\, d\tau,$$

uniformly on $[0,T]$.

Proof. We first consider the case $\phi = \chi_{(a,b)}$. Pick nonnegative integers $k_a, k_b \le n$ such that $|k_aT/n - a| < T/n$ and $|b - k_bT/n| < T/n$. Then

$$\begin{aligned}
\left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))\phi(\tau)\, d\tau \right| &= \left| \int_a^b (\lambda(\tau) - \chi_{A_n}(\tau))\, d\tau \right| \\
&= \left| \left(\int_a^{k_aT/n} + \int_{k_bT/n}^b + \int_{k_aT/n}^{k_bT/n} \right) (\lambda(\tau) - \chi_{A_n}(\tau))\, d\tau \right| \\
&\le |k_aT/n - a|(2) + |b - k_bT/n|(2) + 0 \\
&\le 4T/n,
\end{aligned}$$

which goes to 0 uniformly in t.

By linearity, we get the same result for step functions.

Let $\epsilon > 0$ and $\phi \in L_1[0,T]$ be arbitrary. We can pick a step function h such that $\|\phi - h\|_{L_1[0,T]} < \epsilon/(2T)$. Then

$$\begin{aligned}
\left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))\phi(\tau)\, d\tau \right| &\le \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))h(\tau)\, d\tau \right| + \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))(h - \phi)(\tau)\, d\tau \right| \\
&\le \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))h(\tau)\, d\tau \right| + 2t\epsilon/(2T) \\
&\le \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))h(\tau)\, d\tau \right| + \epsilon. \\
&\le 2\epsilon,
\end{aligned}$$

uniformly in t for n sufficiently large by the step function case. $\square$