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HW 7

1 Assume that $(f_n) \subset L_1(\mu)$ and $f_n \to f$ uniformly.
a) If $\mu(X) < \infty$, then $f \in L_1$ and $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.
b) If $\mu(X) = \infty$, then the conclusion of (a) may fail.

Proof. For (a), we have

$$\begin{aligned} \left| \int f \, d\mu - \int f_n \, d\mu \right| &\leq \int |f - f_n| \, d\mu \\ &\leq \mu(X) \sup_{x \in X} |f(x) - f_n(x)| \\ &\to 0. \end{aligned}$$

For (b), let $(f_n) = 1/n\chi_{[0,n]}$. Then $f_n \to 0$ uniformly on $\mathbb{R}$, but $\int f_n \, dx = 1$ for all n. □

2 Let $f_n, g_n, g \in L_1$, $n \in \mathbb{N}$, and assume that $f_n \to f$, f measurable, and $g_n \to g$ μ-a.e., and that $|f_n| \leq g_n$ and $\int g_n \, d\mu \to \int g \, d\mu$.
Then $\int f_n \, d\mu \to \int f \, d\mu$.

Proof. Following the proof of the DCT, since $|f_n| \leq g_n$ we have $|f| \leq g$. Hence $f \in L_1$. We also have $g_n + f_n \geq 0$ a.e. and $g_n - f_n \geq 0$ a.e. Hence by Fatou's lemma and linearity of the integral on L_1,

$$\int g + \int f \leq \liminf \int (g_n + f_n) = \liminf \int g_n + \int f_n = \int g + \liminf \int f_n$$

The last equality follows from the fact that if $(a_n) \to a$ and $(b_n) \subset \mathbb{R}$, then $\liminf a_n + b_n = a + \liminf b_n$. To see this, pick $\epsilon > 0$ and N such that $|a - a_n| < \epsilon$ for all $n \geq N$. Hence $\liminf a_n + b_n = \liminf(a_n - a) + a + b_n \leq \liminf \epsilon + a + b_n = \epsilon + a + \liminf b_n$, and similarly $\liminf a_n + b_n \geq -\epsilon + a + \liminf b_n$. Hence $\liminf a_n + b_n = a + \liminf b_n$.

Similarly,

$$\int g - \int f \leq \liminf \int (g_n - f_n) = \liminf \int g_n - \int f_n = \int g - \limsup \int f_n$$

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Hence $\limsup \int f_n \leq \int f \leq \liminf \int f_n$, so $\int f = \lim \int f_n$. □

3 Suppose that for $n \in \mathbb{N}$, $f_n = \chi_{E_n}$ for some $E_n \subset \mathbb{R}$, and assume that $f(x) = \lim_{n\to\infty} f_n(x)$ exists a.e.
a) Show that $f = \chi_E$ a.e. for some measurable set $E \subset \mathbb{R}$.

b) Show that for any $g \in L_1$:

$$\int_E g\,dx = \lim_{n\to\infty} \int_{E_n} g\,dx.$$

c) Establish a necessary and sufficient condition for $f_n \to f$ in L_1.

Proof. For (a), we have $\chi_{E_n} \to f$ on N^c for some null set N. Let $x \in N^c$. Since $(\chi_{E_n}(x))_n$ is a convergent discrete-valued sequence, it must be eventually constant. Thus, $f(x) \in \{0,1\}$. Let $E = f^{-1}(1) \cap N^c$. Hence $f = \chi_E$ on N^c, so $f = \chi_E$ a.e. on $\mathbb{R}$. By a previous homework problem, f is measurable since it is the limit of measurable functions. Hence E is measurable.

For (b), we have $\chi_{E_n} g \to \chi_E g$ pointwise a.e. by part (a). Moreover, $\chi_{E_n} g \leq |g| \in L_1$. Hence, by the DCT, we have the desired conclusion.

For (c), one such condition is that $m(E_n) \to m(E)$ with $m(E_n), m(E) < \infty$. Clearly, the latter condition is necessary for f_n, f to be in L_1. For the necessity of the former condition, suppose $f_n \to f$ in L_1. Then $|m(E_n) - m(E)| = |\int f_n - \int f| \leq \int |f_n - f| \to 0$.

For sufficiency, suppose $m(E_n) \to m(E)$ with $m(E_n), m(E) < \infty$. Then $|f_n - f| \leq |f_n| + |f| = \chi_{E_n} + \chi_E$ a.e. Moreover, $\chi_{E_n} + \chi_E \to 2\chi E$ and $\int(\chi_{E_n} + \chi_E) = m(E_n) + m(E) \to 2m(E) = \int(2chi_E)$. Hence, by the generalized DCT (Exercise 2), we have $\int |f_n - f| \to \int \lim_n |f_n - f| = 0$. □

4 Let $L_0([0,1])$ be the space of all measurable functions $f : [0,1] \to \mathbb{R}$.

a) for $f, g \in L_0([0,1])$ put

$$d(f,g) = \int_0^1 \min\{1, |f-g|\}\,dx.$$

Show that $(L_0([0,1]), d)$ is a metric space and that for $f, f_n \in L_0([0,1])$:

$$f_n \to f \text{ in } (L_0([0,1]), d) \iff f_n \to f \text{ in measure.}$$

b) Is there a metric d' on $L_0([0,1])$ for which

$$f_n \to f \text{ in } (L_0([0,1]), d') \iff f_n \to f a.e.$$

Proof. For (a), to see that d is a metric we need to show that d is positive definite, symmetric, and satisfies the triangle inequality. The function d is clearly nonnegative and $0 = d(f,g) = \int_0^1 \min\{1, |f-g|\}\,dx$ implies that $f = g$ a.e. The function d is obviously symmetric. For the triangle inequality, I first claim that for $x, y, z \in \mathbb{R}$ we have $\min\{1, |x-y|\} \leq \min\{1, |x-z|\} + \min\{1, |y-z|\}$.

We have four cases from the RHS of the inequality.

Case $|x-z| \leq 1$ *and* $|y-z| \leq 1$. We have $\min\{1, |x-y|\} \leq |x-y| \leq |x-y| + |y-z| = \min\{1, |x-z|\} + \min\{1, |z-y|\}$.

Case $|x-z| \leq 1$ *and* $|y-z| > 1$. We have $\min\{1, |x-y|\} \leq \min\{1, |x-z| + |z-y|\} \leq \min\{1, 1 + |z-y|\} = 1 + |z-y| = \min\{1, |x-z|\} + \min\{1, |z-y|\}$.

Case $|x-z|>1$ *and* $|y-z|\le 1$. Analogous to previous case.

Case $|x-z|>1$ *and* $|y-z|>1$. We have $\min\{1,|x-y|\}\le\min\{1,|x-z|+|z-y|\}=1\le\min\{1,|x-z|\}+\min\{1,|z-y|\}$.

Hence $\min\{1,|x-y|\}\le\min\{1,|x-z|\}+\min\{1,|y-z|\}$ for all $x,y,z\in\mathbb{R}$. Thus, if $f,g,h\in L_0([0,1])$ then $d(f,g)=\int_0^1\min\{1,|f-g|\}\,dx\le\int_0^1\min\{1,|f-h|\}+\min\{1,|h-g|\}\,dx=\le\int_0^1\min\{1,|f-h|\}\,dx+\int_0^1\min\{1,|h-g|\}\,dx=d(f,h)+d(g,h)$. Thus, d is a metric.

Suppose $f_n\to f$ in $(L_0([0,1]),d)$. Let $0<\epsilon<1$. Pick N such that $d(f,f_n)<\epsilon^2$ for all $n\ge N$. Then for all $n\ge N$, we have

$$\begin{aligned}
m(\{|f-f_n|\ge\epsilon\}) &= \int_{\{|f-f_n|\ge\epsilon\}} dx\\
&= \int_{\{\epsilon\le|f-f_n|<1\}} dx+\int_{\{|f-f_n|\ge 1\}} dx\\
&\le \int_{\{\epsilon\le|f-f_n|<1\}} \frac{|f-f_n|}{\epsilon}\,dx+\int_{\{|f-f_n|\ge 1\}} \min\{1,|f-f_n|\}\,dx\\
&\le \int_{\{\epsilon\le|f-f_n|<1\}} \epsilon^{-1}\min\{1,|f-f_n|\}\,dx+\int_{\{|f-f_n|\ge 1\}} \min\{1,|f-f_n|\}\,dx\\
&\le \epsilon^{-1}\int\min\{1,|f-f_n|\}\,dx\\
&< \epsilon
\end{aligned}$$

Conversely, suppose $f_n\to f$ in measure. Let $0<\epsilon<1$. Pick N such that $m(\{|f-f_n\ge\epsilon\})\le\epsilon$ for all $n\ge N$. Then for all $n\ge N$ we have

$$\begin{aligned}
\int\min\{1,|f_n-f|\}\,dx &= \int_{\{0\le|f_n-f|<\epsilon\}}|f_n-f|\,dx+\int_{\{\epsilon\le|f_n-f|<1\}}|f_n-f|\,dx+\int_{\{|f_n-f|\ge 1\}}dx\\
&\le \int_{\{0\le|f_n-f|<\epsilon\}}\epsilon\,dx+\int_{\{\epsilon\le|f_n-f|<1\}}dx+\int_{\{|f_n-f|\ge 1\}}dx\\
&\le \epsilon+m(\{|f-f_n\ge\epsilon\})\\
&< 2\epsilon.
\end{aligned}$$

For (b), I use the following fact about convergence in metric spaces. Let (M,d) be a metric space, $(x_n)_{n\in\mathbb{N}}\subset M$, and $x\in M$. I claim that if every subsequence of (x_n) has a further subsequence converging to x, then $x_n\to x$. Suppose $x_n\not\to x$. Then there exists $\epsilon>0$ and a subsequence $(x_n)_{n\in N_1}$ such that $d(x,x_n)\ge\epsilon$ for all $n\in N_1$. This subsequence cannot have a further subsequence converging to x, contradicting the hypothesis.

Thus it suffices to find a sequence that does not converge pointwise a.e., but each subsequence has a subsequence that converges a.e. to 0. For $n\in\mathbb{N}$, write n as $n=2^j+k$ for $j\ge 0$ and $0\le k<2^j$. Let $E_n=[k2^{-j},(k+1)2^{-j}]$ and $f_n=\chi_{E_n}$. Every element of $[0,1]$ is contained in infinitely many E_n and infinitely many E_n^c, so (f_n) does not converge pointwise a.e.

On the other hand, suppose $(f_n)_{n\in N_1}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$. For each n, pick $x_n \in E_n$. Then by the sequential compactness of $[0,1]$, there exists an infinite set $N_2 \subset N_1$ and $x_0 \in [0,1]$ such that $x_n \to x_0$ as $n \to \infty, n \in N_2$. Since $\operatorname{diam}(E_n) \to 0$, we have $f_n(x) \to 0$ as $n \to \infty, n \in N_2$ for $x \neq x_0$. Hence, $f_n \to 0$ as $n \to \infty, n \in N_2$ pointwise a.e. $\square$

5

(a) $\displaystyle \lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\, dx = 0$

(b) $\displaystyle \lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} = 0$

(c) $\displaystyle \lim_{n\to\infty} \int_0^\infty n\sin\left(\frac{x}{n}\right) [x(1+x^2)]^{-1}\, dx = \frac{\pi}{2}$

(d) $\displaystyle \lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2}\, dx = \begin{cases} 0 & \text{if } a > 0 \\ \pi/2 & \text{if } a = 0 \\ \pi & \text{if } a < 0 \end{cases}$

Proof. For (a), we have $\int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right)\, dx = \int_0^\infty n^2(1+nu)^{-n}\sin(nu)\, du$.

For $n \geq 2$, we have $\left|n^2(1+nu)^{-n}\sin(nu)\right| \leq n^2(1+nu)^{-2} \leq u^{-2}$, which is integrable on $(1,\infty)$. Hence, by the DCT,

$$\lim_{n\to\infty} \int_1^\infty n^2(1+nu)^{-n}\sin(nu)\, du = \int_1^\infty \lim_{n\to\infty} n^2(1+nu)^{-n}\sin(nu)\, du = 0$$

For the rest of the integral, we have

$$\begin{aligned}
\left|\int_0^1 n^2(1+nu)^{-n}\sin(nu)\, du\right| &\leq \int_0^1 n^2(1+nu)^{-n} nu\, du \\
&= \left[\frac{n^2}{1-n}u(1+nu)^{1-n}\right]_{u=0}^1 - \int_0^1 \frac{n^2}{1-n}(1+nu)^{1-n}\, du \\
&= \frac{n^2}{1-n}(1+n)^{1-n} - \left[\frac{n}{(1-n)(2-n)}(1+nu)^{2-n}\right]_{u=0}^1 \\
&\to 0
\end{aligned}$$

as $n \to \infty$. This proves (a).

For (b), for $0 \leq x \leq 1$ we have $\frac{1+nx^2}{(1+x^2)^n} \leq 1$. Hence, by the DCT we have

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}\, dx = \int_0^1 \lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n}\, dx = 0.$$

For (c), for $x \in [0,\infty)$ we have $\left|n\sin\left(\frac{x}{n}\right)[x(1+x^2)]^{-1}\right| \leq n\left(\frac{x}{n}\right)[x(1+x^2)]^{-1} = (1+x^2)^{-1}$. We have $(1+x^2)^{-1} \in L_1([0,\infty))$ by applying the MCT to the integrals over $[0,M]$ as $M \to \infty$. Hence, by the DCT we have

$$
\begin{aligned}
\lim_{n\to\infty} \int_0^\infty n \sin\left(\frac{x}{n}\right) [x(1+x^2)]^{-1}\, dx &= \int_0^\infty \lim_{n\to\infty} n \sin\left(\frac{x}{n}\right) [x(1+x^2)]^{-1}\, dx \\
&= \int_0^\infty \lim_{m\to 0^+} \frac{\sin(m)}{m} (1+x^2)^{-1}\, dx \\
&= \int_0^\infty (1+x^2)^{-1}\, dx \\
&= \lim_{M\to\infty} \int_0^\infty (1+x^2)^{-1}\, dx \\
&= \lim_{M\to\infty} \tan^{-1}(M) \\
&= \pi/2,
\end{aligned}
$$

where the fourth equality follows from the MCT.

For (d),

$$
\begin{aligned}
\lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2}\, dx &= \lim_{n\to\infty} \int_{na}^\infty \frac{du}{1+u^2} \\
&= \lim_{n\to\infty} \pi/2 - \tan^{-1}(na) \\
&= \begin{cases} 0 & \text{if } a > 0 \\ \pi/2 & \text{if } a = 0 \\ \pi & \text{if } a < 0 \end{cases},
\end{aligned}
$$

where the second equality follows from the MCT applied to $\int_{na}^M \frac{du}{1+u^2}$ as $M \to \infty$.

$\square$