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HW 7

19.43 No to both; let $f(x) = x\chi_{[0,1)}$. Then $||f||_\infty = 1$, but $\{|f| = 1\} = \emptyset$.
50 Note that if $f \in L_\infty$ then f is a.e. equal to a bounded function, so the simple functions of the Basic Construction converge uniformly to f a.e. Hence, the simple functions are dense in L_∞. If E is of finite measure, all simple functions defined on E are integrable. If $m(E) = \infty$, then the integrable simple functions are not dense: take $f = \chi_E$. If $||\phi - f||_\infty < 1/2$, then $||\phi||_\infty > 1/2$, so $\int |\phi| = \infty$.
51 There's a typo in the problem statement: the exponent of $m(E)$ should be $1/p$, not $(1 - 1/p)$. To see why the latter can't be right, let $f = \chi_E$, then $||f||_p = m(E)^{1/p} = m(E)^{1/p}||f||_\infty$. Take $m(E) = 2$ and $p = 1$ to see that the problem can't be correct as stated.

If $f \in L_\infty(E)$ with $m(E) < \infty$ and $1 \le p < \infty$, we have $||f||_p \le ||||f||_\infty||_p = (m(E))^{1/p}||f||_\infty$. This implies $L_\infty(E) \subset L_p(\mathbb{R})$ with the convention that a function f defined on E is set to 0 outside of E.

If $E = [0, 1]$, this inequality reduces to $||f||_p \le ||f||_\infty$. To see $||f||_1 \le ||f||_p$, use Hölder's inequality: $||f||_1 = ||(1)f||_1 \le ||1||_q||f||_p = ||f||_p$
52 Let $f \in L_\infty[0, 1]$. To see that $||f||_p$ is increasing as a function of p, let $1 \le r < s \le \infty$. By Hölder,

$$||f||_r = (\int |f|^r(1))^{1/r} \le ((\int |f|^{r(s/r)})^{r/s})^{1/r} = ||f||_s.$$

Since by (51) every $||f||_p$ is bounded above by $||f||_\infty$, we have $\lim_{p\to\infty} ||f||_p$ exists.

To show that $||f||_\infty \le \lim_{p\to\infty} ||f||_p$, let $\epsilon > 0$. We have

$$\begin{aligned}
||f||_p &\ge (\int_{\{|f|>||f||_\infty-\epsilon\}} |f|^p)^{1/p} \\
&\ge ((||f||_\infty - \epsilon)^p\, m\{|f| > ||f||_\infty - \epsilon\})^{1/p} \\
&= (||f||_\infty - \epsilon)(m\{|f| > ||f||_\infty - \epsilon\})^{1/p} \\
&\to ||f||_\infty - \epsilon,
\end{aligned}$$

as $p \to \infty$, since $m\{|f| > ||f||_\infty - \epsilon\} > 0$. Thus, $||f||_\infty \le \lim_{p\to\infty} ||f||_p \le ||f||_\infty$, so $\lim_{p\to\infty} = ||f||_\infty$.
62 Pick a step function h such that $||f - h||_p < \epsilon/2$. If $m(A) < \delta := (\frac{\epsilon}{2||h||_\infty})^p$,

then

$$\begin{aligned}
||f\chi_A||_p &\le ||h\chi_A||_p + ||(f-h)\chi_A||_p \\
&\le ||||h||_\infty \chi_A||_p + (\epsilon/2) \\
&\le ||h||_\infty m(A)^{1/p} + (\epsilon/2) \\
&< \epsilon.
\end{aligned}$$

If $p = \infty$, this will not work. Take $f(x) := 1$. If $m(A) > 0$, then $||f\chi_A||_\infty = ||\chi_A||_\infty = 1$.

64(a) *Case* $p > 1$. For the boundedness, since $1 < p < \infty$, we can use Hölder:

$$|h(x)| = |\int fT_x(g)| \le ||f||_p ||T_x(g)||_q = ||f||_p ||g||_q,$$

where the last equality follows from (63), which was proved in class.

For continuity,

$$|h(x) - h(y)| = |\int f \cdot (T_x - T_y)g)| \le ||f||_p ||(T_x - T_y)g||_q \to 0$$

as $y \to x$ by (63)(c).

Case $p = 1$. For the boundedness, note

$$|h(x)| \le \int |fT_x(g)| \le \int |f| \cdot ||g||_\infty = ||f||_1 ||g||_\infty.$$

For continuity, first note the previous estimate shows that $fT_x(g) \in L_1$, so by (63) we have

$$\begin{aligned}
h(x) &= \int fT_x(g) \\
&= \int (fT_x(g))^+ - \int (fT_x(g))^- \\
&= \int T_{-x}((fT_x(g))^+) - \int T_{-x}((fT_x(g))^-) \\
&= \int (T_{-x}(f)g)^+ - \int (T_{-x}(f)g)^- \\
&= \int T_{-x}(f)g,
\end{aligned}$$

where the penultimate equality is justified by the fact that for any function F, we have $T_x(F^+) = T_x(1/2(|F| + F)) = 1/2(|T_xF| + T_xF) = (T_xF)^+$ and similarly for F^-.

Thus, $|h(x) - h(y)| = |\int fT_{x-y}(g)| = |\int (T_{y-x}f)g| \le ||g||_\infty \int |T_{y-x}(f)| \to 0$ as $y \to x$ by (63)(c).

64(b) Example: Let $f := 1/(1+x^2)$ and $g = \sin(e^x)$. Then $f \in L_1$ and $g \in L_\infty$. The difference quotient at 0 is

$$(h(y) - h(0))/y = \int f(t)(g(t+y) - g(t))/y\,dt = \int (1/(1+t^2))(\sin(e^{t+y}) - \sin(e^t)/y\,dt$$

From Fatou's lemma,

$$\liminf_{y\to 0} \int (1/(1+t^2))|(\sin(e^{t+y}) - \sin(e^t)/y|dt \geq \int (1/(1+t^2))|e^t \cos(e^t)|,$$

and that last integral has to diverge (just look at the parts where $|cos(e^t)| > 1/2$ and get a divergent series, I think).