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## Midterm

**1** Use the Courant-Fischer mini-max theorem to show that  $\lambda_2 < 0$  for the matrix

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

*Proof.* The characteristic polynomial for  $A$  is  $f(x) := x^3 + 6 + 6 - 9x - 4x - x = x^3 - 14x + 12$ . We have  $\lim_{x \rightarrow -\infty} f(x) < 0$ ,  $f(0) > 0$ ,  $f(1) < 0$ , and  $\lim_{x \rightarrow \infty} f(x) > 0$ . Thus  $\lambda_2 < 0$ .  $\square$

**2** Let  $A$  be an  $n \times n$  complex matrix that satisfies  $A^*A = AA^*$ . Show that  $A$  is diagonalizable and that there is a unitary matrix  $U$  for which  $U^*AU = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ .

*Proof. Step 1:  $A$  and  $A^*$  are simultaneously diagonalizable.* Let  $J \in M_n(\mathbb{C})$  be the Jordan Normal Form for  $A$ . I claim that  $J$  is diagonal. Suppose not. Then  $J$  contains an  $m \times m$  Jordan block  $B$  for  $1 < m \leq n$ . If  $\lambda$  is the generalized eigenvalue corresponding for  $B$ , then we have  $[B, B^*]_{11} = (BB^*)_{11} - (B^*B)_{11} = (|\lambda|^2 + 1) - |\lambda|^2 \neq 0$ . Hence  $[J, J^*] \neq 0$ , so  $[A, A^*] \neq 0$ , a contradiction. Thus,  $J$  is diagonal. The matrix  $J^* = \overline{J}^T$  is clearly diagonal also.

*Step 2:  $A$  is unitarily diagonalizable.* The proof is by induction on  $n$ . The base case is trivial. For the inductive step, recall that  $A$  must have an eigenvector. Let  $v$  be an normalized eigenvector of  $A$ . Let  $w \in v^\perp$ . Then  $\langle v, Aw \rangle = \langle A^*v, w \rangle = 0$  since  $v$  is an eigenvector of both  $A$  and  $A^*$  by Step 1. Hence  $v^\perp$  is an invariant subspace of  $A$ , and we can apply the inductive hypothesis to  $A|_{v^\perp}$ .  $\square$

**3** Let  $f$  be continuous on  $[0, 1]$ , with  $f(0) = f(1) = 0$  and let  $s \in S^{1/n}(1, 0)$  be the linear spline interpolant to  $f$ , with knots at  $x_j = \frac{j}{n}$ .

(a) Let  $\lambda \in \mathbb{R}$ . Show that  $\left| \int_0^1 s(x) e^{i\lambda x} dx \right| \leq \frac{2n^2}{\lambda^2} \omega(f, 1/n)$ .

*Proof.* We have

$$\begin{aligned}
\left| \int_0^1 s(x) e^{i\lambda x} dx \right| &= \left| \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} s(x) e^{i\lambda x} dx \right| \\
&= \left| \sum_{k=0}^{n-1} \left[ \frac{1}{i\lambda} s(x) e^{i\lambda x} \right]_{x=k/n}^{(k+1)/n} - \frac{1}{i\lambda} \int_{k/n}^{(k+1)/n} s'(x) e^{i\lambda x} dx \right| \\
&= \left| \sum_{k=0}^{n-1} \frac{1}{i\lambda} \int_{k/n}^{(k+1)/n} s'(x) e^{i\lambda x} dx \right| \\
&= \left| -\frac{1}{\lambda^2} \sum_{k=0}^{n-1} [s'(x) e^{i\lambda x}]_{x=k/n}^{(k+1)/n} \right| \\
&\leq \frac{1}{\lambda^2} \sum_{k=0}^{n-1} \left| s' \left( \frac{k+1}{n} - \right) \right| + \left| s' \left( \frac{k}{n} + \right) \right| \\
&\leq \frac{1}{\lambda^2} \sum_{k=0}^{n-1} 2n\omega(f, 1/n) \\
&= \frac{2n^2}{\lambda^2} \omega(f, 1/n).
\end{aligned}$$

□

(b) Use the previous part to show that  $\left| \int_0^1 f(x) e^{i\lambda x} dx \right| \leq \omega(f, 1/n) + \frac{2n^2}{\lambda^2} \omega(f, 1/n)$ .

*Proof.* We have

$$\begin{aligned}
\left| \int_0^1 f(x) e^{i\lambda x} dx \right| &\leq \left| \int_0^1 f(x) - s(x) e^{i\lambda x} dx \right| + \left| \int_0^1 s(x) e^{i\lambda x} dx \right| \\
&\leq \int_0^1 |f(x) - s(x)| dx + \frac{2n^2}{\lambda^2} \omega(f, 1/n) \\
&\leq \int_0^1 \omega(f, 1/n) dx + \frac{2n^2}{\lambda^2} \omega(f, 1/n) \\
&\leq \omega(f, 1/n) + \frac{2n^2}{\lambda^2} \omega(f, 1/n)
\end{aligned}$$

□

**4** Let  $\{\phi_n(x)\}_{n=0}^\infty$  be a set of polynomials orthogonal with respect to a weight function  $w(x)$  on a domain  $[a, b]$ . Assume that the degree of  $\phi_n$  is  $n$ , and that the coefficient of  $x^n$  in  $\phi_n(x)$  is  $k_n > 0$ . In addition, suppose that the continuous functions are dense in  $L_w^2[a, b] = \{f : [a, b] \rightarrow \mathbb{C} : \int_a^b |f(x)|^2 w(x) dx < \infty\}$ .

(a) Show that  $\phi_n$  is orthogonal to all polynomials of degree  $n-1$  or less.

*Proof.* The set  $\{\phi_k\}_{1 \leq k < n}$  spans the polynomials of degree less than  $n - 1$ .  $\square$

- (b) Show that  $\{\phi_n\}_{n=0}^\infty$  is complete in  $L_w^2[a, b]$ .

*Proof.*  $\square$

- (c) Show that the polynomials satisfy the recurrence relation  $\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) + C_n\phi_{n-1}(x)$ . Find  $A_n$  in terms of the  $k_n$ 's.

*Proof.*  $\square$

**5** Suppose that  $f(\theta)$  is a  $2\pi$ -periodic function in  $C^m(\mathbb{R})$ , and that  $f^{(m+1)}$  is piecewise continuous and  $2\pi$ -periodic. Here  $m > 0$  is a fixed integer. Let  $c_k$  denote the  $k$ -th (complex) Fourier coefficient for  $f$  and let  $c_k^{(j)}$  denote the  $k$ -th Fourier coefficient for  $f^{(j)}$ .

- (a) Show that  $c_k^{(j)} = (ik)^j c_k$  for  $1 \leq j \leq m + 1$ .

*Proof.* Integrate by parts  $j$  times.  $\square$

- (b) For  $k \neq 0$ , show that the Fourier coefficient  $c_k$  satisfies the bound

$$|c_k| \leq \frac{1}{2\pi|k|^{m+1}} \|f^{(m+1)}\|_{L_1[0, 2\pi]}$$

*Proof.* Integrate by parts.  $\square$

- (c) Let  $S_n(\theta) = \sum_{k=-n}^n c_k e^{ik\theta}$  be the  $n$ -th partial sum of the Fourier series for  $f$ ,  $n \geq 1$ . Show that both of the following hold for  $f$ :

$$\|f - S_n\|_{L_2} \leq C \frac{\|f^{(m+1)}\|_{L_1}}{n^{m+\frac{1}{2}}} \text{ and } \|f - S_n\|_{C[0, 2\pi]} \leq C' \frac{\|f^{(m+1)}\|_{L_1}}{n^m}.$$

*Proof.*  $\square$

- (d) Let  $f(x)$  be the  $2\pi$ -periodic function that equals  $x^2(2\pi - x)^2$  when  $x \in [0, 2\pi]$ . Verify that  $f$  satisfies the conditions above with  $m = 1$ . With the help of (a), calculate the Fourier coefficients for  $f$ . (Hint: look at  $f''$ .)