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HW 3

1 Problem 8/Page 27. If $(X, \mathcal{M}, \mu)$ is a measure space and $(E_j)_{j=1}^{\infty} \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\bigcup_j E_j) < \infty$.

Proof. Let $F_k := \bigcap_{j \geq k} E_j$. Then (F_k) is an ascending sequence, so $\mu(\liminf E_j) = \mu(\bigcup_k F_k) = \lim_k \mu(F_k)$. For all k, we have $F_k \subset E_k$, so $\mu(F_k) \leq \mu(E_k)$. Hence $\mu(\liminf E_j) = \lim_k \mu(F_k) \leq \liminf \mu(E_k)$.

For the other part, suppose $\mu(\bigcup_j E_j) < \infty$. Let $G_k = \bigcup_{j \geq k} E_j$. Then G_k is a descending sequence and $\mu(G_1) < \infty$, so $\mu(\bigcap_k G_k) = \lim_k \mu(G_k)$. Since $E_k \subset G_k$ for all k, we have $\mu(G_k) \geq \mu(E_k)$. Hence $\mu(\limsup E_j) = \mu(\bigcap_k G_k) = \lim_k \mu(G_k) \geq \limsup \mu(E_k)$.

$\square$

2 Assume μ is finitely additive on a sigma algebra $\mathcal{M}$

a) μ is σ-additive $\equiv$ μ is continuous from below.

b) Assume $\mu(X) < \infty$. Then μ is σ-additive $\equiv$ μ is continuous from above.

Proof. Suppose μ is σ-additive. Let $(E_n) \subset \mathcal{M}$ be an ascending sequence of sets. Let $F_1 = E_1$, and for each $n > 1$, let $F_n = E_n \setminus E_{n-1}$. Then F_n are disjoint, and $\bigcup_{n=1}^{N} F_n = E_N$. Hence $\mu(\bigcup_n E_n) = \mu(\bigcup_n F_n) = \sum_n \mu(F_n) = \lim_{N \to \infty} \sum_{n=1}^{N} \mu(F_n) = \lim_{N \to \infty} \mu(\bigcup_{n=1}^{N} F_n) = \lim_{N \to \infty} E_N$.

For the converse, suppose μ is continuous from below. Let $(F_n) \subset \mathcal{M}$ be a sequence of disjoint sets. Let $E_n = \bigcup_{k=1}^{n} F_k$ for each n. Then (E_n) is an ascending sequence, so $\mu(\bigcup_n E_n) = \lim_{n \to \infty} \mu(E_n)$. Thus, $\mu(\bigcup_n F_n) = \mu(\bigcup_n E_n) = \lim_{n \to \infty} \mu(E_n) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(F_n) = \sum_{k=1}^{\infty} \mu(F_n)$.

For (b), assume $\mu(X) < \infty$. Since $\mu(X) < \infty$, for any set $E \in \mathcal{M}$ we have $\mu(E^c) \leq \mu(X) < \infty$, so $\mu(E) = \mu(X) - \mu(E^c)$.

Suppose μ is σ-additive. Let $(E_n) \subset \mathcal{M}$ be a descending sequence of sets. Then (E_n^c) is a ascending sequence, so part (a) implies that $\mu(\bigcup_n E_n^c) = \lim_{n \to \infty} \mu(E_n^c)$. Hence, $\mu(\bigcap_n E_n) = \mu(X) - \mu((\bigcap_n E_n)^c) = \mu(X) - \mu(\bigcup_n E_n^c) = \mu(X) - \lim_{n \to \infty} \mu(E_n^c) = \mu(X) - \lim_{n \to \infty} \mu(X) - \mu(E_n) = \lim_{n \to \infty} \mu(E_n)$.

For the converse, suppose μ is continuous from above. By part (a), it suffices to show that μ is continuous from below. Let $(E_n) \subset \mathcal{M}$ be an ascending sequence of sets. Then (E_n^c) is descending. Hence, $\mu(\bigcup_n E_n) = \mu(X) - \mu(\bigcap_n E_n^c) = \mu(X) - \lim_{n \to \infty} X - \mu(E_n) = \lim_{n \to \infty} \mu(E_n)$. $\square$

3 Suppose $(X, \mathcal{M}, \mu)$ is a measure space. We call

$$\mathcal{N} = \{A \subset X : \exists B \in \mathcal{M}\ A \subset B \text{ and } \mu(B) = 0\}$$

the *nullsets* of $(X\mathcal{M}, \mu)$.

a) Show that

$$\overline{\mathcal{M}} = \{A \cup N : A \in \mathcal{M} \text{ and } N \in \mathcal{N}\}$$

is a σ-algebra.

b) Show that

$$\overline{\mu} : \overline{\mathcal{M}} \to [0, \infty],\ A \cup N \mapsto \mu(A),\ \text{if } A \in \mathcal{M}, N \in \mathcal{N}$$

is well-defined and a measure.

Proof. For (a), note that $\emptyset \in \overline{\mathcal{M}}$ since $\emptyset \in \mathcal{M} \cap \mathcal{N}$. For closure under complements, let $E \in \overline{\mathcal{M}}$. Then $E = F \cup N$ for some $F \in \mathcal{M}$ and $n \in \mathcal{N}$. Then there exists $B \in \mathcal{M}$ with $N \subset B$ and $\mu(B) = 0$. Let $M = B \setminus N$. Hence $E^c = F^c \cap N^c = F^c \cap (B \setminus M)^c = F^c \cap (B^c \cup M) = (F^c \cap B^c) \cup (F^c \cap M)$, which is in $\overline{\mathcal{M}}$ since $F^c \cap B^c \in \mathcal{M}$ and $F^c \cap M \subset B$.

For closure under countable unions, suppose $(E_n) \subset \overline{\mathcal{M}}$. Then each $E_n = F_n \cup N_n$ for some $F_n \in \mathcal{M}$ and $N_n \in \mathcal{N}$. For each n, pick $B_n \in \mathcal{M}$ with $N_n \subset B_n$ and $\mu(B_n) = 0$. We have $\bigcup_n E_n = (\bigcup_n F_n) \cup (\bigcup_n N_n)$. Further, $\bigcup_n F_n \in \mathcal{M}$ and $\bigcup_n N_n \subset \bigcup_n B_n$ and $\mu(\bigcup_n B_n) \leq \sum_n \mu(B_n) = 0$. Hence, $\bigcup_n E_n \in \overline{\mathcal{M}}$.

For (b), suppose $M \in \overline{\mathcal{M}}$ with $M = A \cup N = A' \cup N'$ for $A, A' \in \mathcal{M}$ and $N, N' \in \mathcal{N}$. We need to show that $\mu(A) = \mu(A')$. By the definition of $\mathcal{N}$, we can pick $B \in \mathcal{M}$ with $\mu(B) = 0$ and $N \subset B$. Thus $A' \subset M \subset (A \cup B)$ implies that $\mu(A') \leq \mu(A \cup B) \leq \mu(A) + \mu(B) = \mu(A)$. The same argument will imply $\mu(A) \leq \mu(A')$, so $\mu(A) = \mu(A')$. Hence, $\overline{\mu}$ is well defined.

Since $\mu(\emptyset) = 0$, we have $\overline{\mu}(\emptyset) = 0$. Suppose $(E_n) \subset \overline{\mathcal{M}}$ is a disjoint sequence of sets with $E_n = A_n \cup N_n$ for $A_n \in \mathcal{M}$ and $N_n \in \mathcal{N}$. Then $\overline{\mu}(\bigcup_n E_n) = \overline{\mu}(\bigcup_n A_n \cup \bigcup_n N_n)$. As we mentioned before, $\bigcup_n N_n \in \mathcal{N}$. Hence, $\overline{\mu}(\bigcup_n E_n) = \mu(\bigcup_n A_n) = \sum_n \mu(A_n) = \sum_n \overline{\mu}(E_n)$. $\square$

4 Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

a) If $E, F \in \mathcal{M}$ and $\mu(E \Delta F) = 0$ then $\mu(E) = \mu(F)$.

b) We say that $E \sim F$ if $\mu(E \Delta F) = 0$. Show that $\sim$ is an equivalence relation.

c) For $E, F \in \mathcal{M}$ put $\rho(E, F) = \mu(E \Delta F)$, show that ρ induces a metric on $\mathcal{M}/\sim$.

Proof. For (a), we have $\mu(E) + \mu(F \setminus E) = \mu(E \cup F) = \mu(F) + \mu(E \setminus F)$. Thus $\mu(E \Delta F) = 0$ implies $\mu(E) = \mu(E \cup F) = \mu(F)$ since $(E \setminus F) \cup (F \setminus E) = E \Delta F$.

For (b), we need to show transitivity (reflexivity and symmetry are obvious). Suppose $\mu(E \Delta F) = 0$ and $\mu(F \Delta G) = 0$. Then $\mu(E \Delta G) = \mu(E \cap G^c) + \mu(E^c \cap G) \leq \mu((E \cup F) \cap G^c) + \mu(E^c \cap (F \cup G)) = \mu((E \setminus F) \cap G^c) + \mu(F \cap G^c) + \mu(E^c \cap F) + \mu(E^c \cap (G \setminus F)) = 0$.

To see that ρ defines a pseudometric on $\mathcal{M}$, we need to show that the triangle inequality holds (symmetry is obvious). Suppose $E, F, G \in \mathcal{M}$. Then, as in (b), $\rho(E, G) = \mu(E \Delta G) \leq \mu((E \setminus F) \cap G^c) + \mu(F \cap G^c) + \mu(E^c \cap F) + \mu(E^c \cap (G \setminus F)) \leq \mu(E \setminus F) + \mu(F \setminus G) + \mu(F \setminus E) + \mu(G \setminus F) = \rho(E, F) + \rho(F, G)$.

Define $\overline{\rho}$ on $\mathcal{M}/\sim$ by $\overline{\rho}(\overline{E}) = \rho(E)$ where $E \in \overline{E}$. To see why $\overline{\rho}$ is well-defined, suppose $E \sim E'$ and $F \sim F'$. Then by the triangle inequality,

$\rho(E', F') \le \rho(E, E') + \rho(E, F) + \rho(F, F') = \rho(E, F)$. Hence $\rho(E', F') = \rho(E, F)$. Thus $\overline{\rho}$ is well-defined.

To see that $\overline{\rho}$ is a metric, suppose $\overline{\rho}(\overline{E}, \overline{E'}) = 0$. Then if E is a representative of $\overline{E}$ and E' is a representative for $\overline{E'}$, then $\rho(E, E') = 0$. Hence $E \sim E'$. The other properties of a metric follow by picking representatives similarly. $\square$

5 If $\mu*$ is an outer measure on X and $(A_j)_{j\in\mathbb{N}}$ a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\bigcup_{j=1}^\infty A_j)) = \sum_{j=1}^\infty \mu^*(E \cap A_j)$ for any $E \subset X$.

Proof. By the definition of outer measure, $\mu^*(E \cap \bigcup_j A_j) \le \sum_j \mu^*(E \cap A_j)$. Suppose this inequality is strict. Then there exists an n such that $\mu^*(E \cap \bigcup_j A_j) < \sum_{j=1}^n \mu^*(E \cap A_j)$.

Now note that if A, B are disjoint sets such that A is μ^*-measurable, then $\mu^*(E \cap (A \cup B)) = \mu^*(E \cap (A \cup B) \cap A) + \mu^*(E \cap (A \cup B) \cap A^c) = \mu^*(E \cap A) + \mu^*(E \cap B)$. By Caratheodory's theorem, $\bigcup_{j=1}^J A_j$ is μ^*-measurable for every J. Hence, $\mu^*(E \cap \bigcup_j A_j) = \mu^*(E \cap A_1) + \mu^*(E \cap \bigcup_{j=2}^\infty A_j) = \ldots = \sum_{j=1}^n \mu^*(E \cap A_j) + \mu^*(E \cap \bigcup_{j=n+1}^\infty A_j) \ge \sum_{j=1}^n \mu^*(E \cap A_j) > \mu^*(E \cap \bigcup_j A_j)$, a contradiction. $\square$

6 Assume that the algebra $\mathcal{A}$ generates the σ-algebra $\mathcal{M}$ and assume that μ is a finite measure on $\mathcal{M}$. Show that for any $\epsilon > 0$ and any $A \in \mathcal{M}$, there is an $\tilde{A} \in \mathcal{A}$ so that $\mu(A \Delta \tilde{A}) < \epsilon$.

Proof. Since μ is finite, the restriction of μ to $\mathcal{A}$ is a finite premeasure. Hence Theorem 1.14 implies that if $E \in \mathcal{M}$ then $\mu(E) = \inf\{\mu(\bigcup_j A_j) : A_j \in \mathcal{A} \text{ and } E \subset \bigcup_j A_j\}$. Let $\epsilon > 0$, and pick $(A_j) \subset \mathcal{A}$ with $A \subset \bigcup_j A_j$ such that $\mu(A \Delta \bigcup_j A_j) = \mu(\bigcup_j A_j) - \mu(A) < \epsilon/2$. Let $B_j = (\bigcup_{k=1}^j A_j) \setminus (\bigcup_{k=1}^{j-1} A_j)$. Then $(B_j) \subset \mathcal{A}$ is a disjoint sequence of sets such that $\bigcup_{j=1}^n B_j = \bigcup_{j=1}^n A_j$ for all n. Since $\mu(\bigcup_j B_j)$ is finite, we can pick n such that, if $\tilde{A} = \bigcup_{j=1}^n B_j$ then $\mu(\tilde{A} \Delta \bigcup_{j=1}^\infty B_j) = \sum_{j=n+1}^\infty \mu(B_j) < \epsilon/2$. Thus $\mu(A \Delta \tilde{A}) \le \mu(A \Delta \bigcup_j B_j) + \mu((\bigcup_j B_j) \Delta \tilde{A}) < \epsilon$. $\square$