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Math 482 Final Paper: Cantor Spaces in $\mathbb{R}$

This paper describes some basic properties of Cantor subspaces of the real line. It also includes an application of these Cantor subspaces to a characterization of the countability of closed subsets of $\mathbb{R}$ in terms of some simple exterior measures.

Recall that a *perfect* set is a set for which every point is a limit point. A set S is called *totally disconnected* if for every $x, y \in S$, there exist disjoint open sets $U, V \subset S$ such that $x \in U$, $y \in V$, and $U \cup V = S$.

Definition 1. *A* Cantor space *is a non-empty, totally disconnected, perfect, compact metric space.*

Example 1. *Let* $C_0 := [0,1]$, $C_1 := [0,1/3] \cup [2/3,1]$, *and* $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. *Similarly, for* $i > 2$, *let* C_i *be the closed set given by removing the open middle third of each interval of* C_{i-1}. *The* ternary Cantor set

$$\Delta := \bigcap_{i=0}^{\infty} C_i$$

is a Cantor space.

Proof. Since $0 \in C_i$ for all i, Δ is non-empty. Since each interval in C_i is of length 3^{-i}, Δ is totally disconnected. It is closed and bounded, so compact by the Heine-Borel theorem.

To see that Δ is perfect, first note that the endpoints of any interval in any C_i remain endpoints of intervals in C_{i+1}, and $C_{i+1} \subset C_i$. Hence, every point that is an endpoint of an interval in some C_i is in Δ. Now, fix $x \in \Delta$. Given $\epsilon > 0$, there exists a C_i whose intervals are of length less than ϵ. Hence, both endpoints of the interval in C_i containing x are within ϵ of x, and are members of Δ. Thus, x is a limit point, so Δ is perfect. $\square$

Theorem 1. *Let K be a Cantor space. If $A \subset K$ is nonempty and clopen, then A is Cantor.*

Proof. A is compact since it is closed in K, and totally disconnected since it is open. To see that A is perfect, let $x \in A$. Since K is perfect, there exists a sequence $(x_n) \subset K$ such that $x_n \to x$. Since A is open, all but a finite number of x_n lie in A. $\square$

Theorem 2. *If $A \subset \mathbb{R}$ is a Cantor space, then there is a order-preserving homeomorphism $f : A \to \{0,1\}^{\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}}$ is ordered lexicographically and equipped with the product metric $d(x,y) = \sum_{i=1}^{n} |x(i) - y(i)| 2^{-n}$.*

Proof. Step 1. Let $a := \inf(A)$, and $d := \sup(A) - a = \operatorname{diam}(A)$. Since A is totally disconnected, there exists $c \in [a + \frac{d}{4}, a + \frac{3d}{4}] \setminus A$. Then $M_0 := (-\infty, c) \cap A$ and $M_1 := (c, \infty) \cap A$ are clopen relative to A, hence Cantor spaces by Theorem 1. Moreover, $\operatorname{diam}(M_i) \leq \frac{3}{4}\operatorname{diam}(A)$ for $i = 0, 1$.

Step 2. For $n > 1$, apply Step 1 to M_t for each $t \in \{0,1\}^{n-1}$ to get clopen Cantor spaces $M_{t,0}, M_{t,1} \subset M_t$ with $M_{t,0} < M_{t,1}$ and $\operatorname{diam}(M_{t,i}) \leq \frac{3}{4}\operatorname{diam}(M_t)$ for $i = 0, 1$. By recursion on n, for all $r, s \in \{0,1\}^n$ we have $\operatorname{diam}(M_s) \leq \left(\frac{3}{4}\right)^n \operatorname{diam}(A)$, and if $r < s$ in the lexicographical ordering then $M_r < M_s$, i.e. $x \in M_r, y \in M_s$ implies $x < y$. Moreover, for any fixed n, $A = \bigcup_{s \in \{0,1\}^n} M_s$.

Step 3. Fix $x \in A$. The construction in Step 2 generates a descending sequence of sets $(M_{t_n})_{t_n \in \{0,1\}^n}$, each containing x. Since for all n we have $t_{n+1} = t_n, i$ for some $i \in \{0,1\}$, this sequence of sets determines a unique element $f(x) \in \{0,1\}^{\mathbb{N}}$ such that, for any n, the first n entries of $f(x)$ are t_n. To see that f is bijective, note that if $t \in \{0,1\}^{\mathbb{N}}$ and $t_n = (t(1), t(2), ...t(n))$, then $f^{-1}(t) = \bigcap_{n=1}^{\infty} M_{t_n}$ contains exactly one point, since M_{t_n} is a descending chain of compact sets with diameters going to 0.

To see that f is continuous, let $x \in A$. If $x_m \to x$ then, for every M_{t_n} containing x, all but finitely many x_m lie in M_{t_n} since M_{t_n} is open relative to A. Thus, $f(x_m) \to f(x)$ since $\operatorname{diam}(f(M_{t_n})) = 2^{-n} \to 0$ as $n \to \infty$. Since A is compact, the continuity of f implies f^{-1} is also continuous.

To see that f is order-preserving, if $x < y$ there exists n so large that $x \in M_s, y \in M_t$ for s, t of length n with $s \neq t$. By Step 2, this implies $s < t$. Hence, $f(x) < f(t)$. $\square$

Theorem 3. *If $S \subset \mathbb{R}$ is a Cantor space, there exists a nondecreasing, onto, continuous function $g : S \to [0,1]$.*

Proof. Let $h : \{0,1\}^{\mathbb{N}} \to [0,1]$ be defined by $h(x) = \sum_{i=0}^{\infty} x(i)2^{-i}$. Defining f as in Theorem 2, let $g = h \circ f$. Thus, it suffices to show that h is nondecreasing, onto, and continuous.

Let $x, y \in \{0,1\}^{\mathbb{N}}$. Then $|h(x) - h(y)| = |\sum_{i=0}^{\infty}(x(i) - y(i))2^{-i}| \leq \sum_{i=0}^{\infty} |x(i) - y(i)|2^{-i} = d(x,y)$, so h is continuous. If $x < y$, then there exists a minimal n such that $x(n) \neq y(n)$. By the definition of lexicographical ordering, $x(n) = 0$ and $y(n) = 1$. Thus, $h(y) - h(x) = \sum_{i=n}^{\infty}(y(i) - x(i))2^{-i} = 2^{-n} + \sum_{i=n+1}^{\infty}(y(i) - x(i))2^{-i} \geq 2^{-n} + \sum_{i=n+1}^{\infty}(-1)2^{-i} = 0$. Hence, h is nondecreasing. To see that h is onto, let $E_n := \{x \in \{0,1\}^{\mathbb{N}} : x(i) = 0 \text{ for all } i > n\}$. Then each $h(E_n)$ is a 2^{-n+1}-net for $[0,1]$, so the image of h is dense in $[0,1]$. Since S is compact, $h(S)$ is compact, so h is onto. $\square$

Lemma 1. *If $f : [a,b] \to [0,1]$ is nondecreasing and onto, then f is continuous.*

Proof. Let $c \in (a,b]$. Since f is nondecreasing, $\sup_{x<c} f(x) \leq f(c) = \inf_{x \geq c} f(x)$. Hence, since f is onto, $\sup_{x<c} f(x) = f(c)$. To see that $f(c-) = f(c)$, set $\epsilon > 0$. By the definition of supremum, there exists $a < c$ such $f(c) - f(a) < \epsilon$. Then if $a < x < c$, since f is nondecreasing, $f(c) - f(x) < \epsilon$. Hence, $f(c-) = f(c)$. The proof for right continuity is analogous. $\square$

Lemma 2. *Every compact metric space K can be written as $K = A \cup B$, where A is perfect (hence compact), B is countable, and $A \cap B = \emptyset$.*

Proof. Let U be a countable base for K. Let $V := \{S \in U : S \text{ is countable}\}$, and $W := U \setminus V$. Then $B := \bigcup_{S \in V} S$ is countable and open. Let $A := K \setminus B$. Then A is closed, hence compact.

I claim that $\widetilde{W} := \{S \cap A : S \in W\}$ is a base for the topology of A. Suppose $C \subset A$ is open in A, and $x \in C$. Then $C \cup B$ is open in K, so there exists $S \in U$ with $x \in S \subset (C \cup B)$. Since $x \notin B$, S cannot be countable, so $S \in W$. Hence, $x \in S \cap A \subset C$, so $\widetilde{W}$ is a base for A.

Note that every element of W is uncountable, so, since B is countable, every element of $\widetilde{W}$ is also uncountable. Thus, A has no isolated points, so A is perfect. $\square$

Definition 2. *Given an nondecreasing function $\alpha : \mathbb{R} \to \mathbb{R}$, the α-*exterior measure *of a set $E \subset \mathbb{R}$ is defined to be*

$$m_\alpha^*(E) := \inf\{\sum_{i=1}^{\infty} \alpha(b_i) - \alpha(a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i)\}$$

Theorem 4. *Let $E \subset \mathbb{R}$ be a closed set. Then E is countable iff $m_\alpha^*(E) = 0$ for all nondecreasing, continous $\alpha : \mathbb{R} \to \mathbb{R}$.*

Proof. The forward implication is obvious. For the converse, suppose E were uncountable. If E contains a nontrivial interval, then let α be the identity. Since E contains an interval, it contains a compact set of the form $[a, b]$ for $a < b$. Hence, any cover of E by open intervals must contain a finite subcover of $[a, b]$. The sum of the lengths of intervals in this subcover must be at least $b - a$, so $m_\alpha^*(E) \geq b - a > 0$, a contradiction.

Suppose E does not contain any nontrivial intervals. Note that $E \cap [n, n+1]$ must be uncountable for some n, so WLOG, E is compact. Then, by Lemma 2, $E = A \cup B$ where A is a Cantor space and B is countable. Since $A \subset E$, $m_\alpha^*(A) \leq m_\alpha^*(E)$, so it suffices to show that $m_\alpha^*(A) > 0$.

Let $f : A \to [0, 1]$ be the increasing, onto, continuous function defined in Theorem 3. Define

$$\alpha(x) = \begin{cases} 0 & : x \leq \inf(A) \\ \sup\{f(y) : y \in A \cap (-\infty, x)\} & : x > \inf(A) \end{cases}$$

Since A is closed and f is onto $[0, 1]$, α is onto $[0, 1]$. Also, α is clearly nondecreasing. Since α is constant outside $(\inf(A), \sup(A))$, Lemma 1 implies α is continuous.

Let U be a cover of A by open intervals. Since A is compact, there exists a finite subcover $F \subset U$. Denote the elements of F by $((a_i, b_i))_{i=1}^n$, sorted so that $a_i \leq a_{i+1}$ for all $i < n$. If $b_{i+1} < b_i$ for some $i < n$, then $(a_{i+1}, b_{i+1}) \subset (a_i, b_i)$. Since F is finite, we can recursively throw out all such redundant sets. This procedure only reduces the sum of interval lengths of F, so we may assume

$b_i \leq b_{i+1}$ for all $i < n$. For $i < n$, if $b_i \geq a_{i+1}$, then $\alpha(b_i) - \alpha(a_{i+1}) \geq 0$ since α is nondecreasing. On the other hand, if $b_i < a_{i+1}$, then $\alpha(b_i) - \alpha(a_{i+1}) = 0$ since $A \cap [b_i, a_{i+1}] = \emptyset$.

Thus, $\sum_{i=1}^{n} \alpha(b_i) - \alpha(a_i) \geq \alpha(b_n) - \alpha(a_1) = 1$. Hence, $m^*_\alpha(A) \geq 1$. $\square$

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References

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