Paul Gustafson
Math 643 - Algebraic Topology I

HW 5

3.2

(i) If $f : I \to X$ is a path with $f(0) = f(1) = x_0 \in X$, then there is a continuous $f' : S^1 \to X$ given by $f'(e^{2\pi it}) = f(t)$. If $f, g : I \to X$ are paths with $f(0) = f(1) = x_0 = g(0) = g(1)$ and if $f \simeq g$ rel $\dot{I}$, then $f' \simeq g'$ rel $\{1\}$.

Proof. Let $g : S^1 \to 2\pi I$ be the principle branch of the argument function. Let $h : S^1 \to \dot{I}$ be the composition of $\frac{1}{2\pi}g$ with the natural quotient map. Since $f(0) = f(1)$, the map f induces a map $\bar{f} : \dot{I} \to X$. Then $f' := \bar{f} \circ h$ is the desired map.

For the second part, let $F : f \simeq g$ rel $\dot{I}$. Define F' by applying the first part of this exercise to the function $F(\cdot, t)$. Then $F' : f' \simeq g'$ rel $\{1\}$. □

(ii) If f and g are as above, then $f \simeq f_1$ rel $\dot{I}$ and $g \simeq g_1$ rel $\dot{I}$ implies that $f' * g' \simeq f_1' * g_1'$ rel $\{1\}$.

Proof. I will assume $f' * g'$ is a map from S^1 given by following f' on the arc $[0, \pi]$ and g' on the arc $[\pi, 2\pi]$. Then it is clear that $f' * g' = (f * g)'$. Since $f * g \simeq f_1 * g_1$ rel $\dot{I}$ by Theorem 3.1, we have $f' * g' = (f * g)' \simeq (f_1 * g_1)' = f_1' * g_1'$ rel $\{1\}$ by (i). □

3.4 Let $\sigma : \Delta^2 \to X$ be continuous where $\Delta^2 = [e_0, e_1, e_2]$. Define $\epsilon_0 : I \to \Delta^2$ as the affine map with $\epsilon_0(0) = e_1$ and $\epsilon_0(1) = e_2$; similarly, define ϵ_1 by $\epsilon_1(0) = e_0$ and $\epsilon_1(1) = e_2$, and define ϵ_2 by $\epsilon_2(0) = e_0$ and $\epsilon_2(1) = e_1$. Finally, define $\sigma_i = \sigma \circ \epsilon_i$.

(i) Prove that $(\sigma_0 * \sigma_1^{-1}) * \sigma_2$ is nullhomotopic rel $\dot{I}$. (Hint: Theorem 1.6.)

Proof. Let $p := (\epsilon_0 * \epsilon_1^{-1}) * \epsilon_2$. We have $\sigma p = (\sigma_0 * \sigma_1^{-1}) * \sigma_2$. Let $F : \Delta^2 \times I \to \Delta^2$ be defined by $F(x, t) = te_1 + (1-t)x$. Define $G : I \times I \to X$ by $G(s, t) = \sigma F(p(s), t)$. Then $G(s, 0) = \sigma(F(p(s), 0)) = \sigma p$, and $G(s, 1) = \sigma(F(p(s), 1)) = \sigma(e_1)$. Lastly, $G(0, t) = \sigma F(p(0), t) = \sigma F(e_1, t) = \sigma(e_1)$ and similarly $G(1, t) = \sigma(e_1)$. Thus $G : (\sigma_0 * \sigma_1^{-1}) * \sigma_2 \simeq i_{\sigma(e_1)}$ rel $\dot{I}$. □

(ii) Prove that $(\sigma_1 * \sigma_0^{-1}) * \sigma_2^{-1}$ is nullhomotopic rel $\dot{I}$.

Proof. The proof is completely analogous to that of (i). □

(iii) Let $F : I \times I \to X$ be continuous, and define paths $\alpha, \beta, \gamma, \delta$ in X as indicated in the figure (in the book). Thus $\alpha(t) = F(t, 0)$, $\beta(t) = F(t, 1)$, $\gamma(t) = F(0, t)$, and $\delta(t) = F(1, t)$. Prove that $\alpha \simeq \gamma * \beta * \delta^{-1}$ rel $\dot{I}$.

Proof. It suffices to show that $\gamma * \beta * \delta^{-1} * \alpha^{-1}$ is nullhomotopic rel $\dot{I}$. Let $\epsilon(t) = F(t,t)$. Then by parts (i) and (ii), we have

$$\begin{aligned} \gamma * \beta * \delta^{-1} * \alpha^{-1} &\simeq (\gamma * \beta * \epsilon^{-1}) * (\epsilon * \delta^{-1} * \alpha^{-1}) \text{ rel } \dot{I} \\ &\simeq i_{F(0,0)} \text{ rel } \dot{I}. \end{aligned}$$

□

3.6

(i) If $f \simeq g$ rel $\dot{I}$, then $f^{-1} \simeq g^{-1}$ rel $\dot{I}$, where f, g are paths in X.

Proof. Let $F : f \simeq g$ rel $\dot{I}$. Then $F(1-s,t) : f^{-1} \simeq g^{-1}$ rel $\dot{I}$. □

(ii) If f and g are paths in X with $\omega(f) = \alpha(g)$, then

$$(f * g)^{-1} = g^{-1} * f^{-1}.$$

Proof. We have $(f * g)^{-1}(t) = (f * g)(1-t) = (g^{-1} * f^{-1})(t)$. □

(iii) Give an example of a closed path f with $f * f^{-1} \neq f^{-1} * f$.

Proof. The map $\exp : I \to S^1$ qualifies. □

(iv) Show that if $\alpha(f) = p$ and f is not constant, then $i_p * f \neq f$.

Proof. This is false without the assumption that X is Hausdorff (consider a path f to a two point set with the trivial topology with $f(t) = p$ for $t < 1$ and $f(1) = q$).

So let's assume X is Hausdorff. I claim there exists $t < 1$ with $f(t) \neq p$. Suppose not. Then $f(1) \neq p$. Let U, V be disjoint open neighborhoods of p and $f(1)$. Then since the range of f is connected, there must exist a point q in the range of f but not in $U \cup V$, a contradiction. Therefore, there exists $t < 1$ with $q := f(t) \neq q$.

Since $f^{-1}(q)$ is compact, there exists a minimal such t with $f(t) = q$, say t_0. If $t_0 \leq 1/2$, then $(i_p * f)(t_0) = p \neq q = f(t_0)$. Otherwise, $(i_p * f)(t_0) = f(2t_0 - 1)$. Note that $2t_0 - 1 < t_0$ since $t_0 < 1$. Since t_0 was the minimal t with $f(t) = q$, this implies $(i_p * f)(t_0) \neq q = f(t_0)$. □

3.14 If f is a closed path in S^1 at 1 and if $m \in \mathbb{Z}$, then $t \mapsto f(t)^m$ is a closed path in S^1 at 1 and

$$\deg(f^m) = m \deg f.$$

Proof. Since the function $x \mapsto x^m$ on S^1 is continuous and fixes 1, we have f^m is a closed path in S^1.

Moreover, using the notation of Corollary 3.15, we have $\exp m\tilde{f} = (\exp \tilde{f})^m = f^m$ and $(m\tilde{f})(0) = 0$. Hence by the uniqueness of the lifting, $\widetilde{f^m} = m\tilde{f}$. Hence $\deg f^m = m \deg f$. $\square$

3.23 Let G be a topological group and let H be a normal subgroup. Prove that G/H is a topological group, where G/H is regarded as the quotient space of G by the kernel of the natural map.

Proof. The set of cosets G/H is a group since H is normal. In particular, both multiplication and inversion respect the identification of the elements of a coset. Thus, multiplication and inversion on G/H are continuous. $\square$