

HW 2

1 Using the fact that $\mathcal{B}_{\mathbb{R}}$ is generated by the open intervals, show that:

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\{[a, \infty) : a \text{ rational}\})$$

Proof. It suffices to show both that $\mathcal{B}_{\mathbb{R}}$ contains $[a, \infty)$ for each $a \in \mathbb{Q}$, and that every open interval (x, y) is in $\mathcal{M}(\{[a, \infty) : a \text{ rational}\})$. The former follows from the fact that $[a, \infty) = (-\infty, a)^c$ for each $a \in \mathbb{Q}$.

For the latter, suppose (x, y) is an arbitrary open interval. Pick $(x_n), (y_n) \subset \mathbb{Q}$ with $x_n \uparrow x$ and $y_n \downarrow y$. Then $(x, y) = \bigcup_n [x_n, y_n) = \bigcup_n [x_n, \infty) \cap [y_n, \infty)^c$. \square

2 Problem 1/Page 24. A *ring* is a nonempty family of sets closed under finite unions and differences. A ring that is closed under countable unions is called a σ -ring.

- a. Rings (resp. σ -rings) are closed under finite (resp. countable) intersections.
- b. If \mathcal{R} is a ring (resp. σ -ring), then \mathcal{R} is an algebra (resp. σ -algebra) iff $X \in \mathcal{R}$.
- c. If \mathcal{R} is a σ -ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ -algebra.
- d. If \mathcal{R} is a σ -ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ -algebra.

Proof. For (a), let \mathcal{R} be a ring, and $U, V \in \mathcal{R}$. Let $W = U \cup V$. Then $U \cap V = W \setminus ((W \setminus U) \cup (W \setminus V))$. This is just one of De Morgan's laws in the restricted universe W . A similar argument works for σ -rings with W the countable union of the sets involved.

For (b), let \mathcal{R} be a ring (resp. σ -ring). Suppose $X \in \mathcal{A}$. Since (a) has been verified, we need only check that \mathcal{R} contains complements. This is true since $E^c = X \setminus E$ for any set E . Conversely, suppose \mathcal{R} is an algebra (resp. σ -algebra). Then \mathcal{R} is nonempty, so there exists $E \in \mathcal{R}$. Thus, $X = E \cup E^c \in \mathcal{R}$.

For (c), let $\mathcal{M} = \{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$. Since \mathcal{R} is nonempty, so is \mathcal{M} . It is also clear that \mathcal{M} is closed under complements. For closure under countable unions, let $(E_n) \subset \mathcal{M}$. Then $(E_n) = (A_n) \cup (B_n)$ for sequences $(A_n), (B_n)$ such that each $A_n \in \mathcal{R}$ and each $B_n^c \in \mathcal{R}$. Let $A = \bigcup A_n \in \mathcal{R}$ and $B = \bigcap B_n^c \in \mathcal{R}$. Then $\bigcup E_n = \bigcup A_n \cup \bigcup B_n = A \cup B^c = (A^c \cap B)^c = (B \setminus A)^c \in \mathcal{M}$.

For (d), let $\mathcal{M} = \{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. Since \mathcal{R} is nonempty, there exists $E \in \mathcal{R}$. Hence $\emptyset = E \setminus E \in \mathcal{R}$. Then it follows from the definition of \mathcal{M} that $\emptyset \in \mathcal{M}$. In particular, \mathcal{M} is nonempty. To see that \mathcal{M} is closed under complements, suppose $E \in \mathcal{M}$. Let $F \in \mathcal{R}$. Then $E^c \cap F = F \setminus E \in \mathcal{R}$. Hence, $E^c \in \mathcal{M}$. For closure under countable unions, let $(E_n) \subset \mathcal{M}$. Let $F \in \mathcal{R}$. Then $\bigcup_n (E_n) \cap F = \bigcup_n (E_n \cap F) \in \mathcal{R}$. Hence, $\bigcup_n E_n \in \mathcal{M}$. \square

3 Problem 5/Page 24. $\mathcal{M}(\mathcal{E})$ is the union of the σ -algebras generated by \mathcal{F} as \mathcal{F} ranges over all countable subsets of \mathcal{E} .

Proof. Let

$$\mathcal{H} = \{\mathcal{F} \subset \mathcal{E} : \mathcal{F} \text{ is countable}\},$$

and $\mathcal{U} = \bigcup_{\mathcal{F} \in \mathcal{H}} \mathcal{M}(\mathcal{F})$.

Let $\mathcal{F} \subset \mathcal{E}$ be countable. Then $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E})$. Hence, $\mathcal{U} \subset \mathcal{M}(\mathcal{E})$. For the reverse inclusion, it suffices to show that \mathcal{U} is a σ -algebra, for then \mathcal{U} is a σ -algebra containing (E) , hence containing $\mathcal{M}(\mathcal{E})$.

To see that \mathcal{U} is a σ -algebra, first note that $\emptyset \in \mathcal{H}$, so \mathcal{U} is nonempty. To see that \mathcal{U} is closed under taking complements, let $E \in \mathcal{U}$. Then $E \in \mathcal{M}(\mathcal{F})$ for some countable $\mathcal{F} \subset \mathcal{E}$, so $E^c \in \mathcal{M}(\mathcal{F}) \subset \mathcal{U}$.

For closure under countable union, let $(U_n) \subset \mathcal{U}$. Then each $U_n \in \mathcal{M}(\mathcal{G}_n)$ for some $\mathcal{G}_n \in \mathcal{H}$. Let $\mathcal{G} = \bigcup_n \mathcal{G}_n$. Then \mathcal{G} is countable, and $U_n \in \mathcal{M}(\mathcal{G})$ for all n . Hence, $\bigcup_n U_n \in \mathcal{M}(\mathcal{G}) \subset \mathcal{U}$. \square

4 Show that every σ -algebra has either finite or uncountable many elements.

Proof. Suppose that $\mathcal{M} \subset \mathcal{P}(X)$ is an infinite σ -algebra.

Step 1: \mathcal{M} contains a sequence of disjoint nonempty sets.

Case 1: \mathcal{M} contains an infinite linearly inclusion-ordered subset \mathcal{L} . Let $(E_n)_{n=1}^\infty \subset \mathcal{L}$ be a pairwise distinct sequence of sets. I claim that (E_n) must have a monotone subsequence. Suppose not. Then $(E_n)_{n=1}^\infty$ must be bounded above by some E_{n_1} , for otherwise, given any E_n , there exists $m > n$ with $E_n \subset E_m$. This would define an ascending sequence. Similarly, $(E_n)_{n=n_1}^\infty$ must be bounded below by say E_{m_1} . Then $(E_n)_{n=m_1}^\infty$ must be bounded above by some $E_{n_2} \subset E_{n_1}$. Continuing in this way, we get a subsequence $E_{n_1} \supset E_{n_2} \supset \dots$, a contradiction.

Since the E_n were distinct, this implies that (E_n) contains either a strictly ascending subsequence (A_n) or a strictly descending subsequence (D_n) . In the former case, let $B_n = A_n \setminus (\bigcup_{k=1}^{n-1} A_k)$. In the latter case, let $B_n = D_n \setminus D_{n+1}$. In either case, (B_n) is a sequence of disjoint nonempty sets.

Case 2: every linearly inclusion-ordered subset of \mathcal{M} is finite. Let \mathcal{L}_1 be a maximal linearly ordered subset of $\mathcal{M} \setminus \{\emptyset\}$. Since \mathcal{M} is infinite, \mathcal{L}_1 must be nonempty.

Inductively assume we are given nonempty finite chains $(\mathcal{L}_i)_{i=1}^n \subset \mathcal{P}(\mathcal{M} \setminus \{\emptyset\})$ such that the minimal elements of (\mathcal{L}_i) are pairwise disjoint. Let E_i denote the minimal element of \mathcal{L}_i for each i . Since the \mathcal{L}_i are finite, the set $\mathcal{F} := \{\bigcup_{i=1}^n F_i : F_i \in \mathcal{L}_i \cup \{\emptyset\} \text{ for all } i\}$ is a finite set. Hence there exists a maximal nonempty chain $\mathcal{L}_{n+1} \subset \mathcal{M} \setminus (\mathcal{F} \cup \{\emptyset\})$.

By the Case 2 assumption, \mathcal{L}_{n+1} is finite, so it contains a minimal element E_{n+1} . Suppose $F := E_{n+1} \cap E_k \neq \emptyset$ for some $1 \leq k \leq n$. Then $F \in \mathcal{F}$, for otherwise F contradicts the maximality of \mathcal{L}_{n+1} . But then $F \subsetneq E_{n+1}$, and $G := E_{n+1} \setminus F$ must be in \mathcal{F} since \mathcal{L}_{n+1} is maximal. Hence we can write $G = \bigcup_{i=1}^n G_i$ and $F = \bigcup_{i=1}^n F_i$ with $G_i, F_i \in \mathcal{L}_i \cup \{\emptyset\}$. Then $E_{n+1} = F \cup G \in$

$\mathcal{F} = \bigcup_{i=1}^n (G_i \cup F_i)$, which is in \mathcal{F} since each $\mathcal{L}_i \cup \{\emptyset\}$ is linearly ordered. This contradicts the fact that $E_{n+1} \in \mathcal{L}_{n+1} \subset \mathcal{M} \setminus (\mathcal{F} \cup \emptyset)$.

Hence $(E_i)_{i=1}^{n+1}$ are disjoint, and L_{n+1} is a nonempty finite chain. Thus, by induction, we have the sequence $(E_n)_{n=1}^\infty$ of disjoint nonempty sets.

Step 2: \mathcal{M} is uncountable. From Step 1, there exists a sequence $(M_n) \subset \mathcal{M}$ of disjoint nonempty sets. Define $f : \mathcal{P}(\mathbb{N}) \rightarrow \mathcal{M}$ by $f(U) = \bigcup_{u \in U} M_u$. To see that f is injective, suppose that $U \neq V$. WLOG there exists $t \in U \setminus V$. Since M_t is nonempty, there exists $x \in M_t$, so $x \in \bigcup_{u \in U} M_u = f(U)$. Since the (M_n) are disjoint, $x \notin M_n$ for $n \neq t$. Hence, $x \notin \bigcup_{v \in V} M_v = f(V)$. Thus, $f(U) \neq f(V)$, so f is injective. Thus $\text{card}(\mathcal{M}) \geq \text{card}(\mathcal{P}(\mathbb{N})) > \text{card}(\mathbb{N})$. \square

5 Let $(\Omega_j, \mathcal{M}_j)$ be measure spaces for $j \in [n]$. Show that

$$\mathcal{E} = \left\{ \prod_{j=1}^n E_j : E_j \in \mathcal{M}_j \forall j \right\}$$

is an elementary system.

Proof. Since $\emptyset \in \mathcal{M}_j$ for all j , we have $\emptyset = \prod_{j=1}^n \emptyset \in \mathcal{E}$. Now suppose $E, F \in \mathcal{E}$. Then $E = \prod_j E_j$ and $F = \prod_j F_j$ for $E_j, F_j \in \mathcal{M}_j$ for all j . Hence $E \cap F = \prod_j (E_j \cap F_j) \in \mathcal{E}$. Lastly, we need to check that E^c is the finite union of disjoint elements of \mathcal{E} . Let

$$\mathcal{U} = \left\{ \prod_j U_j : U_j \in \{E_j, E_j^c\} \right\}$$

. Note that \mathcal{U} is a partition of $\prod_j \Omega_j$, and $\mathcal{U} \subset \mathcal{E}$. Hence $E^c = \bigcup (\mathcal{U} \setminus E)$ is a finite union of disjoint elements of \mathcal{E} . \square