

Towards finiteness for mapping class group representations from group-theoretical categories

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Preliminary Exam
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Definition of the mapping class group

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- Examples
 - $MCG(\Sigma_{0,1}^m) = B_m$
 - $MCG(\Sigma_{1,0}^0) = SL(2, \mathbb{Z})$

Introduction to the problem

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- They also asked if, more generally, all mapping class group representations associated to $\text{Mod}(D^\omega(G))$ have finite image.
- In this talk, I'll work through the genus 2 case.

Theorem (Ng–Schauenberg [Ng2010])

Every modular representation associated to a modular category has finite image.

Other Related Work

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Theorem (Fjelstad–Fuchs [?fjfu])

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Theorem (Fjelstad–Fuchs [?fjfu])

Every mapping class group representation of a closed surface with at most one marked point associated to $\text{Mod}(D(G))$ has finite image.

- Fjelstad and Fuchs use Lyubashenko's method of constructing projective representations of mapping class groups from factorizable ribbon Hopf algebras (in this case $D(G)$).
- We will use a different construction due to Kirillov. In our case, this construction corresponds to the twisted Dijkgraaf-Witten theory.

Outline

- 1 Input data
- 2 Generators for the mapping class group
- 3 The representation space
- 4 A spanning set for the representation space
- 5 Action of the generators on the spanning set
- 6 Sketch of a proof of finiteness
- 7 Future Directions

Input data

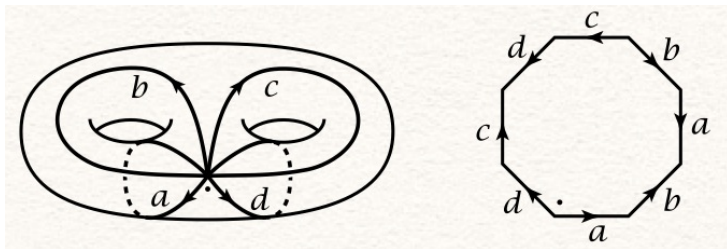


Figure: A genus 2 surface Σ as a quotient of its fundamental polygon. Image source: Hatcher's *Algebraic Topology*.

- Oriented closed surface Σ of genus 2

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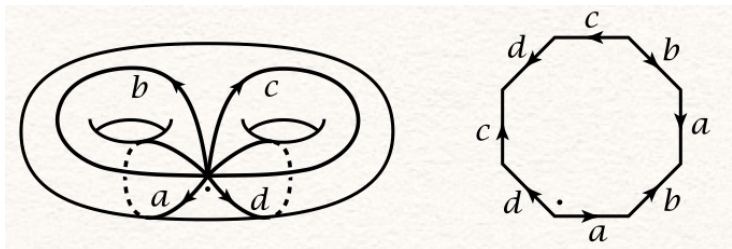


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- Oriented closed surface Σ of genus 2
- Finite group G

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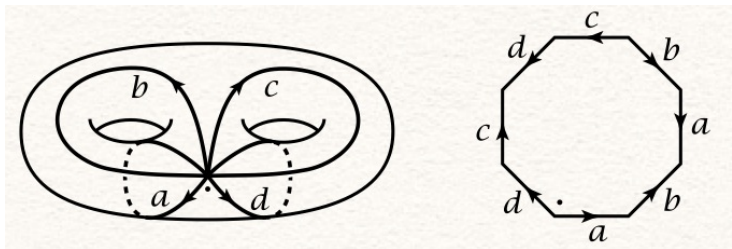


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- Oriented closed surface Σ of genus 2
- Finite group G
- Normalized 3-cocycle $\omega : G \times G \times G \rightarrow U(1)$.

Generators for the mapping class group

- A theorem of Lickorish [?lickorish1964finite] implies that $\text{MCG}(\Sigma)$ is generated by the Dehn twists $T_a, T_b, T_c, T_d, T_{a^{-1}d}$.

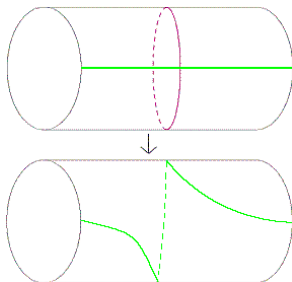


Figure: A Dehn twist with respect to the red curve. Image source: Wikipedia article on Dehn twists.

The representation space

- Using Kirillov's definitions [?kirillovStringNets], the representation space is

$$H := \frac{\text{Vect}_G^\omega\text{-colored graphs in } \Sigma}{\text{local relations}}$$

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- The Drinfel'd center $\mathcal{Z}(\text{Vect}_G^\omega)$ is braided monoidally equivalent to $\text{Mod}(D^\omega(G))$ (well-known according to [?0704.0195]).
- Hence, the mapping class group representation on H should be equivalent to the mapping class group representation associated to $\text{Mod}(D^\omega(G))$ by the Reshitikhin-Turaev construction [?1012.0560, preprint].

The spherical category Vect_G^ω

- The spherical fusion category Vect_G^ω is the category of G -graded finite-dimensional vector spaces with the following modified structural morphisms from [?math/0601012], where V_g is the simple object:

- The associator $a_{g,h,k} : (V_g \otimes V_h) \otimes V_k \rightarrow V_g \otimes (V_h \otimes V_k)$

$$a_{g,h,k} = \omega(g, h, k)$$

- The evaluator $ev_g : V_g^* \otimes V_g \rightarrow 1$

$$ev_g = \omega(g^{-1}, g, g^{-1})$$

- The pivotal structure $j_g : V_g^{**} \rightarrow V_g$

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Colored graphs in Σ

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- A *coloring* of Γ is the following data:
 - Choice of an object $V(\mathbf{e}) \in \text{Obj } \mathcal{A}$ for every oriented edge $\mathbf{e} \in E^{or}$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.

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 - Choice of an object $V(\mathbf{e}) \in \text{Obj } \mathcal{A}$ for every oriented edge $\mathbf{e} \in E^{or}$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.
 - Choice of a vector $\varphi(v) \in \text{Hom}_{\mathcal{A}}(1, V_1 \otimes \cdots \otimes V_n)$ for every interior vertex v , where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are edges incident to v , taken in counterclockwise order and with outward orientation.

Local relations

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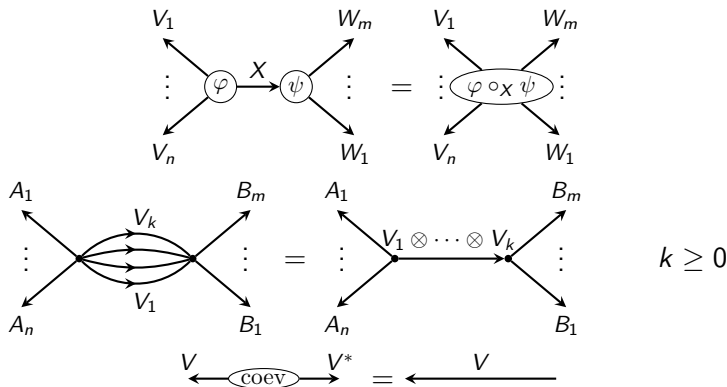


Figure: The remaining local relations. Image source: [?kirillovStringNets].

Consequences of the local relations

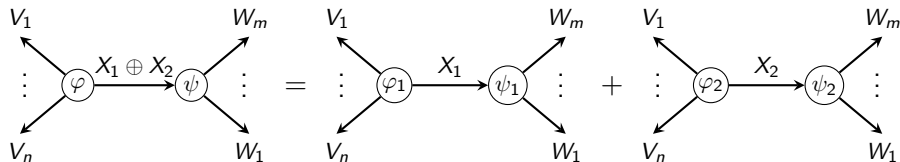


Figure: Additivity in edge colorings. Here φ_1, φ_2 are compositions of φ with projector $X_1 \oplus X_2 \rightarrow X_1$ (respectively, $X_1 \oplus X_2 \rightarrow X_2$), and similarly for ψ_1, ψ_2 . Image source: [?kirillovStringNets].

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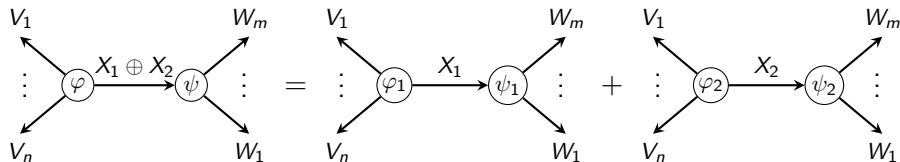


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- Additivity in edge colorings
- A colored graph may be evaluated on any disk $D \subset S$, giving an equivalent colored graph Γ' such that Γ' is identical to Γ outside of D , has the same colored edges crossing ∂D , and contains at most one colored vertex within D .

A spanning set for the representation space

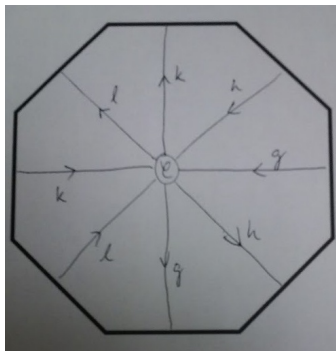


Figure: The spanning set S consists of all such colored graphs, where the edge labels vary over all 4-tuples $g, h, k, l \in G$ satisfying $[g, h][k, l] = 1$ and $\varphi := \varphi_{g,h,k,l}$ is the canonical basis element of the one-dimensional space $\text{Hom}(1, ((\cdots ((V_g \otimes V_h) \otimes V_g^{-1}) \otimes \cdots \otimes V_l^{-1}))$.

Action of the Dehn twist T_a on the spanning set I

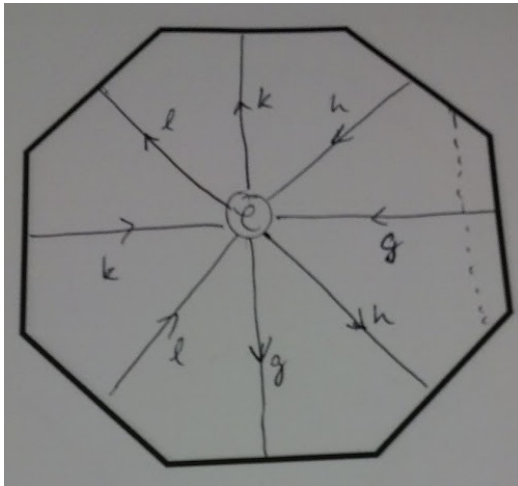


Figure: The dashed line is a simple closed curve isotopic to a .

Action of the Dehn twist T_a on the spanning set II

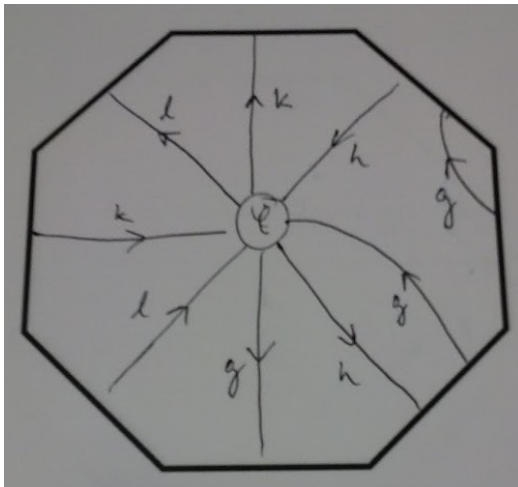


Figure: The result of the twist T_a .

Action of the Dehn twist T_a on the spanning set III

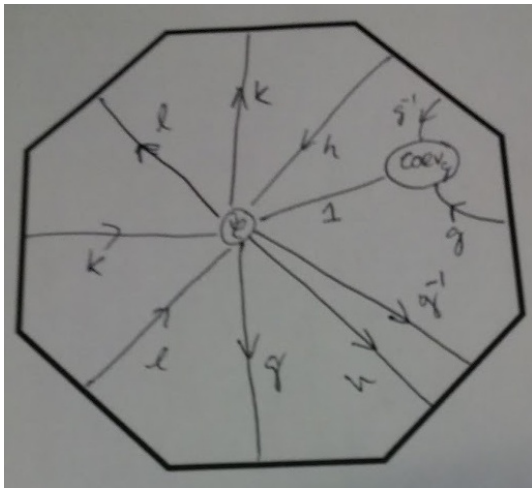


Figure: Using the local relations.

Action of the Dehn twist T_a on the spanning set IV

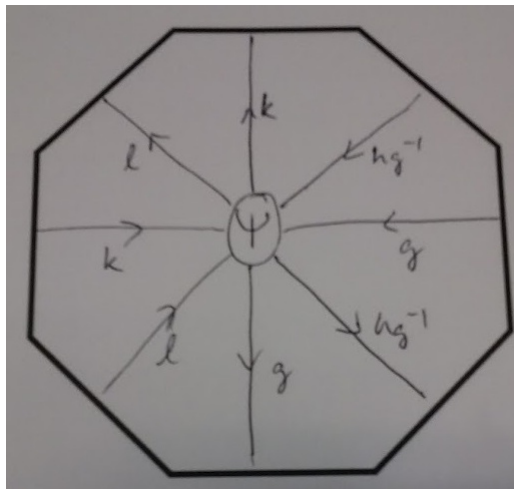


Figure: The result. The map ψ differs from $\phi_{g, hg^{-1}, k, l}$ by a product of factors in $\text{Im}(\omega)$.

Action of the Dehn twist $T_{a^{-1}d}$ on the spanning set I

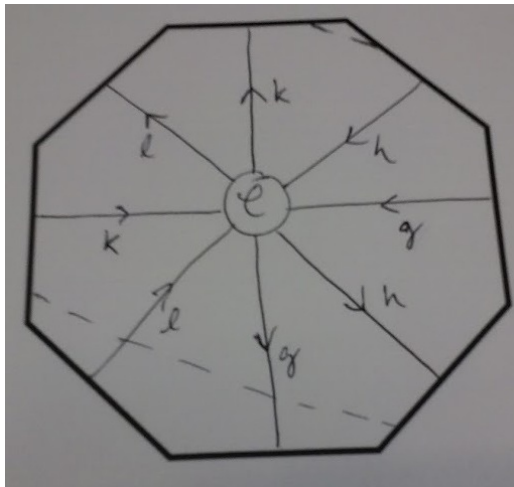


Figure: The dashed line is a simple closed curve isotopic to $a^{-1}d$.

Action of the Dehn twist $T_{a^{-1}d}$ on the spanning set II

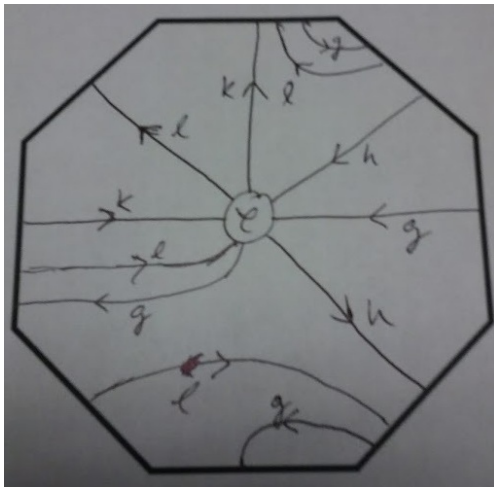


Figure: The result of the twist $T_{a^{-1}d}$.

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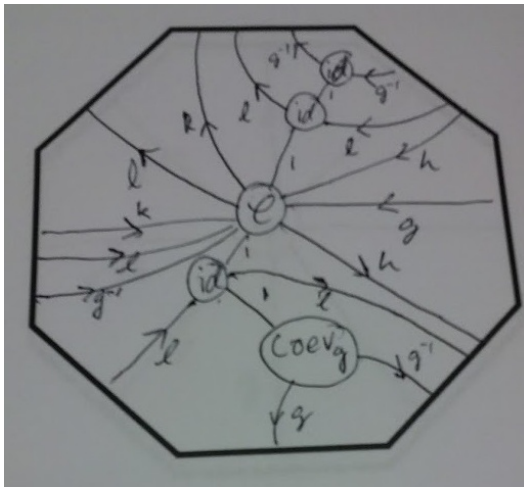


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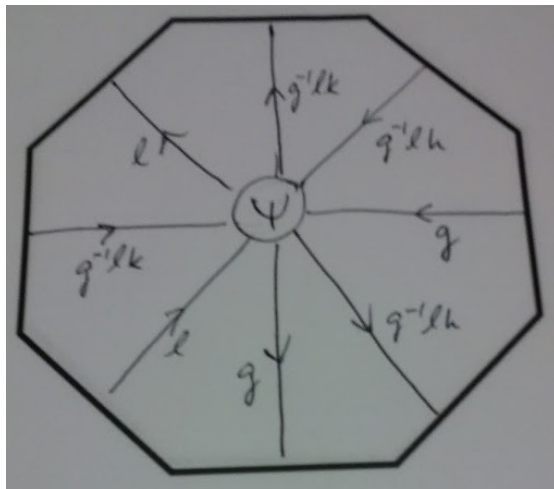


Figure: The result. Again, ψ differs from $\phi_{g, g^{-1}lh, g^{-1}lk, l}$ by a product of factors in $\text{Im}(\omega)$.

Sketch of a proof of finiteness

Proposition

Let $\rho : \text{MCG}(\Sigma) \rightarrow \text{PGL}(H)$ be the representation defined above. Then $|\text{Im}(\rho)| < \infty$.

Sketch of proof.

- For any k , let R denote the set of $|G|$ -th roots of unity. Then ω is cohomologous to a cocycle taking values in R (follows from [weibel1995introduction, Theorem 6.58]). Hence, WLOG ω takes values in R .

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- Add marked points using the Birman exact sequence.

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- Add marked points using the Birman exact sequence.
- Look at the simplest undetermined cases of weakly integral modular categories.

Acknowledgements

- Thanks to my advisor Eric Rowell, my father Robert Gustafson, and Zhengnan Wang for enlightening discussions.

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- Thanks for listening!

References I