Paul Gustafson
Math 663

Problems

1 Give a proof of the mean ergodic theorem using the spectral theorem for unitary operators.

Proof. Let U be a unitary operator on a Hilbert space H. By the spectral theorem, there exists a unitary map $T : H \to L^2(X,\mu)$ for some finite measure space (X,μ) with $U = T^{-1}ST$ where S is multiplication by a function f taking values on the unit circle.

Note that we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N} f^n = \chi_{f^{-1}(1)}$. Thus, $P := \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N} U^n$ exists and is a orthogonal projection.

I claim that P is the orthogonal projection onto $\ker(I-U)$. If $v \in \ker(I-U)$, then $T^{-1}STv = Uv = v$. Hence $S(Tv) = Tv$. Therefore $f(x) = 1$ for all x where $Tv(x) \neq 0$. This implies that $Pv = T^{-1}\chi_{f^{-1}(1)}Tv = v$. All these steps are reversible, so the range of P is precisely $\ker(I-U)$. □

2 Prove Khintchine's recurrence theorem: If G is a countable amenable group and G acts on (X,μ) via a p.m.p. action then for every measurable set $A \subset X$ and $\epsilon > 0$ the set $S := \{s \in G : \mu(sA \cap A) \geq \mu(A)^2 - \epsilon\}$ is syndetic.

Proof. Let $t \in G$, $\{F_n\}_{n=1}^{\infty}$ be a tempered Folner sequence in G, and P be the orthogonal projection onto the subspace of G-invariant functions in $L^2(X)$. From the mean ergodic theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{|F_n|}\sum_{s\in tF_n} \mu(sA\cap A) &= \lim_{n\to\infty} \frac{1}{|F_n|}\sum_{s\in tF_n} \langle \chi_{sA}, \chi_A\rangle \\
&= \langle \lim_{n\to\infty} \frac{1}{|F_n|}\sum_{s\in F_n} s\chi_A, t^{-1}\chi_A\rangle \\
&= \langle P\chi_A, t^{-1}\chi_A\rangle \\
&= \langle tP\chi_A, \chi_A\rangle \\
&= \langle P\chi_A, \chi_A\rangle \\
&= \|P\chi_A\|_2^2 \\
&\geq \langle P\chi_A, 1\rangle^2 \\
&= \langle \chi_A, 1\rangle^2 \\
&= \mu(A)^2.
\end{aligned}$$

Note that in the limiting step the error is

$$|\langle P\chi_A - \frac{1}{|F_n|}\sum_{s\in F_n} s\chi_A, t^{-1}\chi_A\rangle| \leq \|P\chi_A - \frac{1}{|F_n|}\sum_{s\in F_n} s\chi_A\|_2\|\chi_A\|_2,$$

and the last bound is independent of t.

It follows that by choosing n sufficiently large we can ensure that

$$\frac{1}{|F_n|}\sum_{s\in tF_n}\mu(sA\cap A)\geq\mu(A)^2-\epsilon$$

for all $t\in G$. This implies that for any $t\in G$ there exists $s\in F_n$ such that $ts\in S$. Thus S is syndetic. $\square$

5 Give examples of unitary representations π and ρ of $\mathbb{Z}$ such that $\pi\otimes\rho$ is ergodic but neither π nor ρ is weakly mixing.

Proof. Let $\pi=\rho$ be the one-dimensional representation taking 1 to multiplication by i. Since this representation is finite-dimensional, it is not weakly mixing. Moreover, $(\pi\otimes\rho)(1)$ acts by multiplication by -1, so $\pi\otimes\rho$ is ergodic. $\square$

7 Show that a countable discrete group G is amenable iff every continuous action of G on a compact Hausdorff space has an invariant Borel probability measure.

Proof. Suppose G is amenable. The canonical map $\beta : l^\infty(G)\to C(\beta G)$ (extending a bounded function to the Stone-Cech compactification) is a G-equivariant C^*-algebra isomorphism. Thus, the left-invariant mean on $l^\infty(G)$ induces a G-invariant state on βG. By the Riesz Representation theorem, this gives us an invariant Borel probability measure on βG.

Now suppose G acts continuously on a compact Hausdorff space K. Fix any point $x_0\in K$. Define $f:G\to K$ by $f(s)=sx_0$. Clearly f is G-equivariant, so $\beta f:\beta G\to K$ is equivariant also. The pushforward of the measure on βG by the function βf is the desired measure.

Now suppose the converse holds. Then action of G on βG gives us an invariant Borel probability measure on βG. Integrating against this measure and making use of the properties of the map β mentioned above, we get an invariant mean on $l^\infty(G)$. $\square$

8 Show that a subgroup H of an amenable countable discrete group G is amenable.

Proof. Suppose H were not amenable. Then it admits a paradoxical decomposition $C\sim D\sim H$. Let R be a complete set of representatives for the right cosets of H in G. Then $\{C_iR\}_{C_i\in C}$, $\{D_iR\}_{D_i\in D}$ forms a paradoxical decomposition for G, a contradiction. $\square$