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HW 7

4.43 For $x \in [0,1)$, let $\sum_1^\infty a_n(x)2^{-n}$ be the binary expansion of x. (If x is a dyadic rational, choose the expansion such that $a_n(x) = 0$ for n large.) Then the sequence $(a_n) \in \{0,1\}^{[0,1)}$ has no pointwise convergent sequence.

Proof. Suppose not. Then there exists a convergent subsequence (a_{n_k}). Let $x \in [0,1)$ be defined by letting $x_{n_k} = -1 + (-1)^k$ and $x_n = 0$ otherwise. Then $a_{n_k}(x)$ does not converge. □

52 The one-point compactification of $\mathbb{R}^n$ is homeomorphic to the n-sphere $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$.

Proof. The homeomorphism is the usual stereographic projection sending the north pole to ∞. It is easy to see that this map is a bijection and a local homeomorphism away from the north pole.

Note that if any closed subset of S^n does not contain the north pole then it has a positive distance to the north pole. Hence the subset maps to a bounded and closed, hence compact, set. Thus, by taking complements, the inverse map is continuous at the north pole. Moreover, the preimage of a compact set in $\mathbb{R}^n$ under the map is compact and does not contain the north pole, so by taking complements the map is continuous at ∞. Hence the map is a homeomorphism. □

60 The product of countably many sequentially compact spaces is sequentially compact.

Proof. Let $(X_n)_{n=1}^\infty$ be sequentially compact. Let $(x_n) \subset \prod_n X_n$. Pick $N_1 \subset \mathbb{N}$ such that $(\pi_1(x_n))_{n \in N_1}$ converges. Pick $N_2 \subset N_1$ such that $(\pi_2(x_n))_{n \in N_2}$ converges, and so on. Let (n_k) be defined by picking $n_1 \in N_1$, $n_2 > n_1$ with $n_2 \in N_2$, and so on. Then $\pi_i(x_{n_k})$ converges for all i. Hence (x_{n_k}) converges. □

69 Let A be a nonempty set, and let $X = [0,1]^A$. The algebra generated by the coordinate maps $\pi_\alpha : X \to [0,1]$ for $\alpha \in A$ and the constant function 1 is dense in $C(X)$.

Proof. By Stone-Weierstrauss it suffices to show that this algebra separates points. This is obvious. □

74 Consider $\mathbb{N}$ as a subset of its Stone-Cech compactification $\beta\mathbb{N}$.

a. If A and B are disjoint subsets of $\mathbb{N}$, their closures in $\beta\mathbb{N}$ are disjoint. (Hint: $\chi_A \in C(\mathbb{N}, I)$.)

b. No sequence in $\mathbb{N}$ converges in $\beta\mathbb{N}$ unless it is eventually constant.

Proof. For part (a), since $\mathbb{N}$ is discrete, $\chi_A : \mathbb{N} \to I$ is continuous, and so is χ_B. Therefore they have unique continuous extensions to $\beta\mathbb{N}$, f and g respectively. Suppose $x \in \overline{A} \cap \overline{B} \subset \beta\mathbb{N}$. Then $f(x) = 1 = g(x)$.

$\square$