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HW 1

1 Let $f : X \to Y$. Prove that

a) if $A, B \subset Y$, then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

b) For a family $(A_\lambda)_{\lambda \in \Lambda} \subset P(X)$, show that $f^{-1}(\bigcup_{\lambda \in \Lambda} A_\lambda) = \bigcup_{\lambda \in \Lambda} f^{-1}(A_\lambda)$ and $f^{-1}(\bigcap_{\lambda \in \Lambda} A_\lambda) = \bigcap_{\lambda \in \Lambda} f^{-1}(A_\lambda)$

and give examples for the following situations

c) $f^{-1}(f(A)) \neq A$, for some $A \subset X$,

d) $f(f^{-1}(B)) \neq B$ for some $B \subset Y$,

e) $f(\cap_{\lambda \in \Lambda}) \neq \cap_{\lambda \in \Lambda} f(A_\lambda)$, for some family $(A_\lambda)_{\lambda \in \Lambda} \subset P(X)$.

Proof. (a) is a subcase of (b). To prove the first part of (b),

$$\begin{aligned} x \in f^{-1}(\bigcup_{\lambda \in \Lambda} A_\lambda) \iff & \quad f(x) \in \bigcup_{\lambda \in \Lambda} A_\lambda \\ \iff & \quad f(x) \in A_\lambda \text{ for some } \lambda \\ \iff & \quad x \in f^{-1}(A_\lambda) \text{ for some } \lambda \\ \iff & \quad x \in \bigcup_\lambda f^{-1}(A_\lambda). \end{aligned}$$

For the second part,

$$\begin{aligned} x \in f^{-1}(\bigcap_{\lambda \in \Lambda} A_\lambda) \iff & \quad f(x) \in \bigcap_{\lambda \in \Lambda} A_\lambda \\ \iff & \quad f(x) \in A_\lambda \text{ for all } \lambda \\ \iff & \quad x \in f^{-1}(A_\lambda) \text{ for all } \lambda \\ \iff & \quad x \in \bigcap_\lambda f^{-1}(A_\lambda) \end{aligned}$$

For (c), let $X = \{0, 1\}$ and $Y = \{0\}$. Let $A = \{0\} \subset X$. Let $f : X \to Y$ be the constant function. Then $f^{-1}(f(A)) = f^{-1}(Y) = X \neq A$.

For (d), let $X = \{0\}$ and $B = Y = \{0, 1\}$. Let $f : X \to Y$ be the constant function at 1. Then $f(f^{-1}(B)) = f(X) = \{1\} \neq B$.

For (e), let $X = \{0, 1\}$ and $Y = \{0\}$. Let $A_1 = \{0\}$ and $A_2 = \{1\}$. Let $f : X \to Y$ be the constant function. Then $f(A_1 \cap A_2) = f(\emptyset) = \emptyset$, but $f(A_1) \cap f(A_2) = \{0\}$.

□

2 Show that the following two statements are equivalent for two nonempty sets A and B.

a) There is an injection $\phi : A \to B$.

b) There is a surjection $\psi : B \to A$.

Proof. Suppose (a) holds. Let $(U_b)_{b\in B}$ be defined by $U_b = \phi^{-1}(\{b\})$ if $b \in \phi(A)$ and $U_b = A$ otherwise. By the axiom of choice, there exists $f \in \prod_{b\in B} U_b$. Since each $U_b \subset A$, there exist identity injections $i_b : U_b \to A$ for each $b \in B$. Define $\psi : B \to A$ by $\psi(b) = i_b(f(b))$.

To see that ψ is surjective, let $a \in A$. Since ϕ is injective, $\phi^{-1}(\phi(\{a\}))$ contains only a. Hence, $f(\phi(a)) \in (U_{\phi(a)} = \phi^{-1}(\phi(\{a\})))$ implies that $f(\phi(a)) = a$. Thus, $\psi(\phi(a)) = i_{\phi(a)} f(\phi(a)) = i_{\phi(a)}(a) = a$.

Now suppose (b) holds. Let $(U_a)_{a\in A}$ be defined by $U_a = \psi^{-1}(\{a\})$, which are non-empty since ψ is surjective. By AC, there exists $f \in \prod_{a\in A} U_a$. Since each $U_a \subset B$, there exist identity injections $i_a : U_a \to B$. Define $\phi : A \to B$ by $\phi(a) = i_a(f(a))$.

To see that ϕ is injective, let $b \in B$ and suppose $x, y \in \phi^{-1}(\{b\})$. Then $x \in f^{-1}(i_x^{-1}(\{b\})$, so $f(x) \in i_x^{-1}(\{b\}) = \{b\}$ and similarly for y. Hence, $f(x) = b = f(y)$. Hence, $b \in (U_x \cap U_y)$. But $U_x \cap U_y = \psi^{-1}(\{x\}) \cap \psi^{-1}(\{y\}) = \psi^{-1}(\{x\} \cap \{y\})$. Thus, $\{x\} \cap \{y\}$ is nonempty, so $x = y$.

$\square$