Computing Quantum Mapping Class Group Representations with Haskell

Paul Gustafson

Motivation

Why should we care about quantum mapping class group representations?

- Topological quantum computation

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- Understand mapping class groups

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- Topological quantum computation
- Understand mapping class groups
- Intrinsic beauty

Mapping class group definition

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- Examples
  - $\mathrm{MCG}(\Sigma_{0,1}^{m}) = B_m$
  - $\mathrm{MCG}(\Sigma_{1,0}^{0}) = SL(2,\mathbb{Z})$
- Birman (1969) found "nice" finite generating set for the mapping class group of any compact surface.

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- History: Witten and Atiyah (1980s)
- Examples of mathematical (2+1)-TQFTs:
  - Reshitikhin-Turaev TQFT (input: modular category)
  - Turaev-Viro-Barret-Westbury TQFT (input: spherical fusion category)

Monoidal categories

A **monoidal category** is a category $\mathcal{C}$ equipped with

- a tensor product – a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$
- an associativity isomorphism – a natural isomorphism $\alpha : (\cdot \otimes \cdot) \otimes \cdot \to \cdot \otimes (\cdot \otimes \cdot)$
- a unit object $1 \in \mathcal{C}$
- a left unitor – a natural isomorphism $\lambda_X : 1 \otimes X \to X$
- a right unitor – a natural isomorphism $\rho_X : X \otimes 1 \to X$,

satisfying certain coherence conditions (the triangle and pentagon axioms).

Rigid monoidal categories

Let X be an object of a monoidal category $\mathcal{C}$. A **left dual** to X is an object X^* equipped with

- an evaluation morphism, $\mathrm{ev}_X : X^* \otimes X \to 1$
- a coevaluation morphism, $\mathrm{coev}_X : 1 \to X \otimes X^*$,

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Right duals are defined similarly. An object X is **rigid** if it has both a left and right dual. A monoidal category is rigid if all of its objects are rigid.

Fusion category

A **fusion category** is a rigid semisimple linear monoidal category (tensor category), with only finitely many isomorphism classes of simple objects, such that the endomorphisms of the unit object form just the ground field k.

- A **pivotal category** is a rigid monoidal category equipped with a pivotal structure, i.e. a tensor natural automorphism $j_X : X \to (X^*)^*$.

Spherical categories

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- A pivotal structure defines categorical left and right traces $\mathrm{End}(X) \to \mathrm{End}(1)$ for every object X. A **spherical category** is a monoidal category such that all left and right traces coincide.

Example: Vect_G^ω

- Let G be a finite group, and $\omega : G \times G \times G \to \mathbb{C}$ be a 3-cocycle. The spherical fusion category Vect_G^ω is the skeletal category of G-graded finite-dimensional vector spaces with the following modified structural morphisms, where V_g is the simple object:
  - The associator $a_{g,h,k} : (V_g \otimes V_h) \otimes V_k \to V_g \otimes (V_h \otimes V_k)$
  $$a_{g,h,k} = \omega(g, h, k)$$
  - The evaluator $ev_g : V_g^* \otimes V_g \to 1$
  $$ev_g = \omega(g^{-1}, g, g^{-1})$$
  - The pivotal structure $j_g : V_g^{**} \to V_g$
  $$j_g = \omega(g^{-1}, g, g^{-1})$$

The TVBW space associated to a 2-manifold

- Let $\mathcal{A}$ be a spherical fusion category, and Σ an oriented compact surface with boundary.
- Using Kirillov's definitions, the representation space we consider is

$$H := \frac{\mathcal{A}\text{-colored graphs in } \Sigma}{\text{local relations}}$$

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- The vector space H is canonically isomorphic to the Turaev-Viro state sum vector space associated to Σ. This isomorphism commutes with the mapping class group action.

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- A *coloring* of Γ is the following data:
    - Choice of an object $V(\mathbf{e}) \in \operatorname{Obj} \mathcal{A}$ for every oriented edge $\mathbf{e} \in E^{or}$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.

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  - Choice of a vector $\varphi(v) \in \operatorname{Hom}_{\mathcal{A}}(1, V_1 \otimes \cdots \otimes V_n)$ for every interior vertex v, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are edges incident to v, taken in counterclockwise order and with outward orientation.

Local relations

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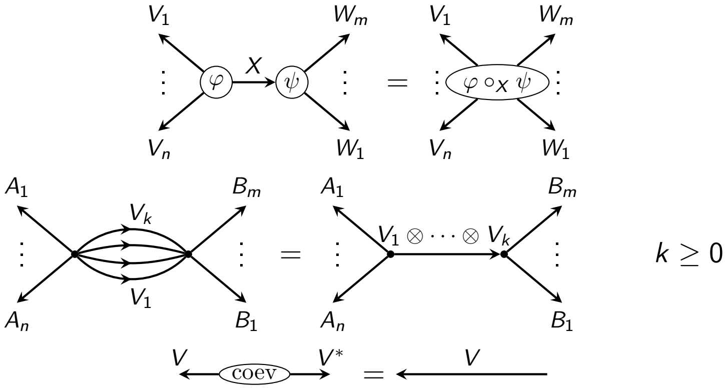

Figure : The remaining local relations.

Consequences of the local relations

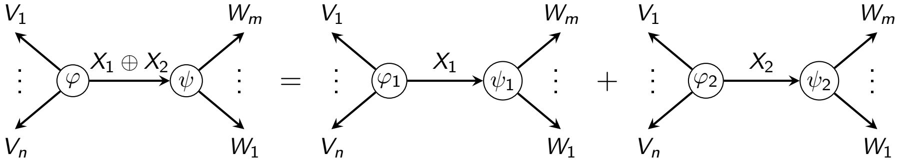

Figure : Additivity in edge colorings. Here φ_1, φ_2 are compositions of φ with projector $X_1 \oplus X_2 \to X_1$ (respectively, $X_1 \oplus X_2 \to X_2$), and similarly for ψ_1, ψ_2.

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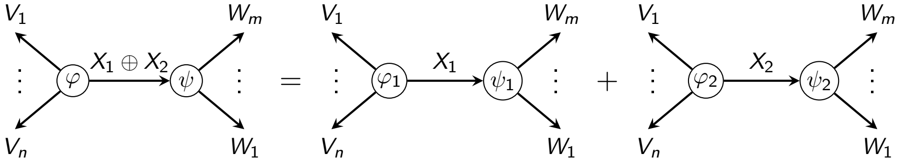

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- Additivity in edge colorings
- A colored graph may be evaluated on any disk $D \subset S$, giving an equivalent colored graph Γ' such that Γ' is identical to Γ outside of D, has the same colored edges crossing ∂D, and contains at most one colored vertex within D.

Overall Strategy

- Find a basis of colored graphs for the representation space for a surface
- "Calculate" the representation of each mapping class group generator with respect to this basis
- Analyze the image of the representation (Is it finite? Can we do universal quantum computation with it (possibly adding extra measurements)?)

Modified Property F conjecture

Conjecture (Rowell)

A TVBW mapping class group representation associated to a spherical fusion category $\mathcal{A}$ has finite image iff $\mathcal{A}$ is weakly integral, i.e. the squared dimension of every simple object is an integer.

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Theorem (Fjelstad–Fuchs)

Every mapping class group representation of a closed surface with at most one marked point associated to $\mathrm{Mod}(D(G))$ *has finite image.*

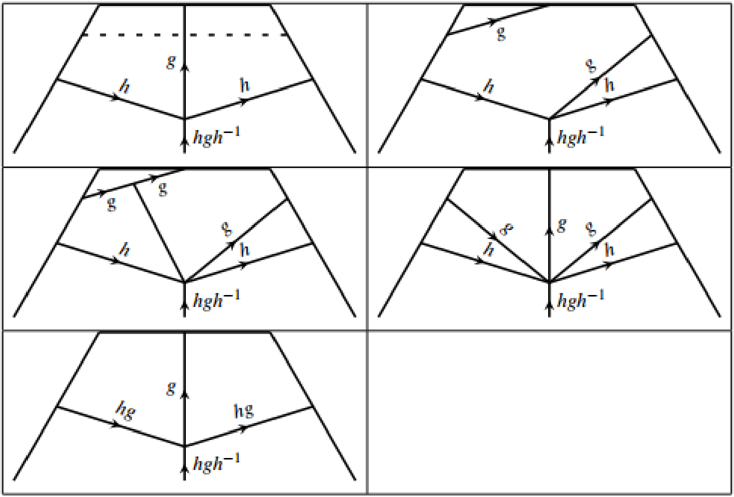

TABLE 1. First type of Dehn twist. Unlabeled interior edges are colored by the group identity element.

Answering ERW's question

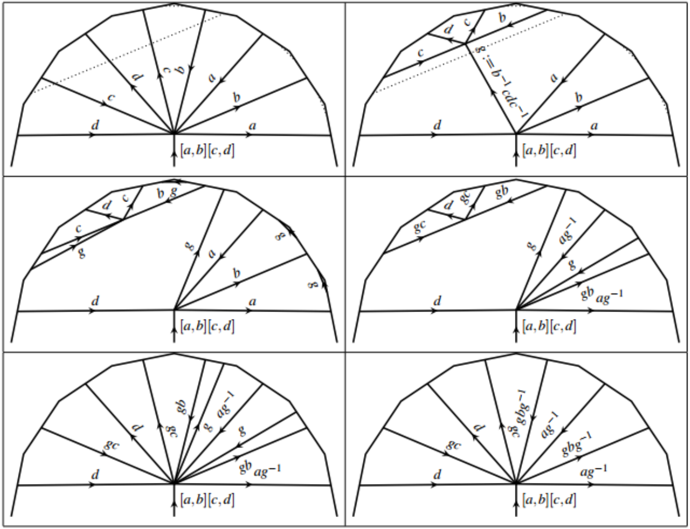

TABLE 2. Second type of Dehn twist.

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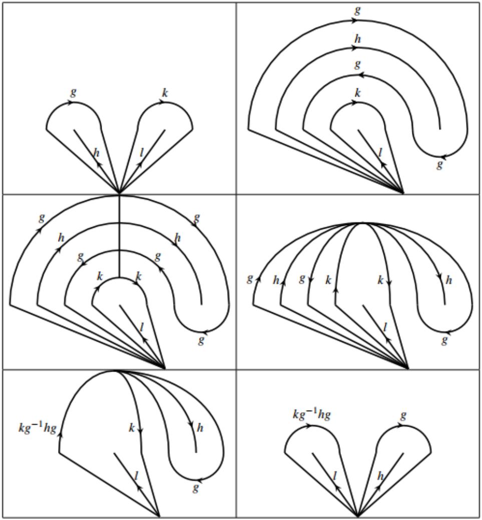

TABLE 3. A braid generator. Unlabeled interior edges are colored by the group identity element.

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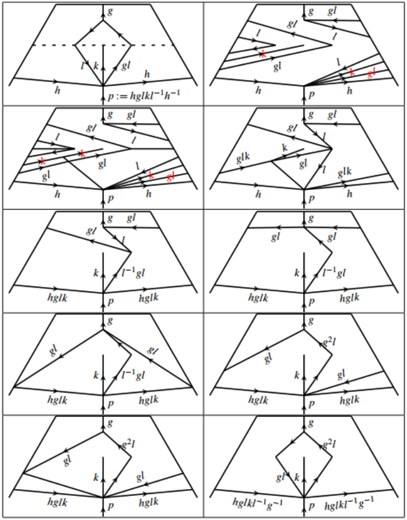

TABLE 4. Pulling a boundary component along a generator for the fundamental group of the corresponding closed surface. Unlabeled interior edges are colored by the group identity element.

First result

Theorem

The image of any Vect_G^ω *TVBW representation* ρ *of a mapping class group of an orientable, compact surface* Σ *with boundary is finite.*

Sketch of proof.

- For any k, let $\mu_{|G|}$ denote the set of $|G|$-th roots of unity. Then ω is cohomologous to a cocycle taking values in $\mu_{|G|}$. Since cohomologous cocycles give rise to equivalent spherical categories Vect_G^ω, WLOG ω takes values in $\mu_{|G|}$.

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- Let $B \subset S$ be a basis for H. Then $\rho(\mathrm{MCG}(\Sigma))B \subset \rho(\mathrm{MCG}(\Sigma))S \subset \mu_{|G|}S$.
- Thus, $|\mathrm{Im}(\rho)| < \infty$.

Next steps

- Tambara-Yamagami categories
- Calculate actual matrices

Calculations = Hard

Easiest one:

$$
\begin{aligned}
&\frac{\omega(h,g,h^{-1})\omega(h,gh^{-1},hg^{-1}h^{-1})\omega(g,h^{-1},hg^{-1}h^{-1})\omega(g,g^{-1}h^{-1},h)}{\omega(g^{-1},g^{-1},g)\omega(g^{-1},g^{-1}h^{-1},h)\omega(g^{-1},h^{-1},hg^{-1}h^{-1})\omega(g,g^{-2}h^{-1},hg)}\cdot\\
&\quad\frac{\omega(g,g^{-1},h^{-1})\omega(g,g^{-1}h^{-1},h)\omega(g^{-1}h^{-1},h,g)\omega(g^{-2}h^{-1},h,g)}{\omega(g,g^{-1}h^{-1},hg^{-1}h^{-1})\omega(hg,h^{-1},hg^{-1}h^{-1})\omega(hg,g,g^{-1}h^{-1})}\\
&=\frac{\omega(h,g,g^{-1}h^{-1})\omega(g,g^{-1}h^{-1},h)}{\omega(g^{-1},g^{-1},g)\omega(g^{-1},g^{-1}h^{-1},h)\omega(g^{-1},h^{-1},hg^{-1}h^{-1})\omega(g,g^{-2}h^{-1},hg)}\cdot\\
&\quad\frac{\omega(g,g^{-1},h^{-1})\omega(g,g^{-1}h^{-1},h)\omega(g^{-1}h^{-1},h,g)\omega(g^{-2}h^{-1},h,g)}{\omega(g,g^{-1}h^{-1},hg^{-1}h^{-1})\omega(hg,g,g^{-1}h^{-1})}\\
&=\frac{\omega(h,g,g^{-1}h^{-1})\omega(g,g^{-1}h^{-1},h)}{\omega(g^{-1},g^{-1},g)\omega(g^{-1},g^{-1}h^{-1},h)\omega(g,g^{-2}h^{-1},hg)}\cdot\\
&\quad\frac{\omega^2(g,g^{-1},h^{-1})\omega(g,g^{-1}h^{-1},h)\omega(g^{-1}h^{-1},h,g)\omega(g^{-2}h^{-1},h,g)}{\omega(g,g^{-1},g^{-1}h^{-1})\omega(hg,g,g^{-1}h^{-1})}\\
&=\frac{\omega(h,g,g^{-1}h^{-1})\omega^2(g,g^{-1}h^{-1},h)\omega^2(g,g^{-1},h^{-1})\omega^2(g^{-1}h^{-1},h,g)}{\omega(g,g^{-2}h^{-1},hg)\omega(g,g^{-1},g^{-1}h^{-1})\omega(hg,g,g^{-1}h^{-1})\omega(g^{-1},g^{-1}h^{-1},hg)}\\
&=\frac{\omega(h,g,g^{-1}h^{-1})\omega^2(g,g^{-1}h^{-1},h)\omega^2(g,g^{-1},h^{-1})\omega^2(g^{-1}h^{-1},h,g)}{\omega(hg,g,g^{-1}h^{-1})}\\
&=\frac{\omega(h,g,g^{-1}h^{-1})\omega^2(g^{-1}h^{-1},h,g)}{\omega^2(g^{-1},h^{-1},h)\omega(hg,g,g^{-1}h^{-1})}
\end{aligned}
$$