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Bonus exercises

1 Let $\lambda : [0,T] \to [0,1]$ be measurable. Let

$$A_n = \bigcup_{k=0}^{n-1} [kT/n, kT/n + \int_{kT/n}^{(k+1)T/n} \lambda(t)\,dt).$$

Show that for any $\phi \in L_1[0,T]$,

$$\int_0^t \chi_{A_n}(\tau)\phi(\tau)\,d\tau \to \int_0^t \lambda(\tau)\phi(\tau)\,d\tau,$$

uniformly on $[0,T]$.

Proof. We first consider the case $\phi = \chi_{(a,b)}$. Pick nonnegative integers $k_a, k_b \leq n$ such that $|k_aT/n - a| < T/n$ and $|b - k_bT/n| < T/n$. Then

$$\begin{aligned}
\left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))\phi(\tau)\,d\tau \right| &= \left| \int_a^b (\lambda(\tau) - \chi_{A_n}(\tau))\,d\tau \right| \\
&= \left| \left(\int_a^{k_aT/n} + \int_{k_bT/n}^b + \int_{k_aT/n}^{k_bT/n} \right) (\lambda(\tau) - \chi_{A_n}(\tau))\,d\tau \right| \\
&\leq |k_aT/n - a|(2) + |b - k_bT/n|(2) + 0 \\
&\leq 4T/n,
\end{aligned}$$

which goes to 0 uniformly in t.

By linearity, we get the same result for step functions.

Let $\epsilon > 0$ and $\phi \in L_1[0,T]$ be arbitrary. We can pick a step function h such that $\|\phi - h\|_{L_1[0,T]} < \epsilon/(2T)$. Then

$$\begin{aligned}
\left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))\phi(\tau)\,d\tau \right| &\leq \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))h(\tau)\,d\tau \right| + \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))(h - \phi)(\tau)\,d\tau \right| \\
&\leq \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))h(\tau)\,d\tau \right| + 2t\epsilon/(2T) \\
&\leq \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))h(\tau)\,d\tau \right| + \epsilon. \\
&\leq 2\epsilon,
\end{aligned}$$

uniformly in t for n sufficiently large by the step function case. $\square$

2 Let Y_1 and Y_2 be two complete vector fields on $\mathbb{R}^n$, which are also Lipshitzian, i.e. there exist $L_i > 0$ such that $\|Y_i(x_1) - Y_i(x_2)\| \leq L_1\|x_1 x_2\|$ for any $x_1, x_2 \in \mathbb{R}^n$, $i \in \{1, 2\}$. Let T, λ, A_n be as in the previous problem. Further, assume that $q(t)$ is the trajectory of the vector field $\lambda(t)Y_1 + (1-\lambda(t))Y_2$ with $q(0) = q_0$ and $(q_n(t))$ are the trajectories of the vector fields $\chi_{A_n}Y_1 + (1-\chi_{A_n}(t))Y_2$ with $q_n(0) = q_n$ and such that $q_n \to q_0$. Prove that $q_n(t) \to q(t)$ uniformly on $[0, T]$. (Hint: Use the Gronwall inequality: if $y(\tau), \beta(\tau)$ are continuous and take non-negative values on $[0, T]$ and $\alpha \geq 0$ with $y(t) \leq \alpha + \int_0^t \beta(\tau)y(\tau)\,d\tau$ for $t \in [0, T]$, then $y(t) \leq \alpha e^{\int_0^t \beta(\tau)\,d\tau}$.)

Proof. □

3 Let V be a finite dimensional complex vector space and $T : V \to V$ be a linear transformation. Let $p(x)$ denote the characteristic polynomial of T, and $m(x)$ denote the minimal polynomial of T. Find a necessary and sufficient condition on the Jordan Normal Form of T for $p(x) = m(x)$.

Proof. I claim that $p = m$ if and only if the geometric multiplicity of each eigenvalue of T is 1. This means that each Jordan block has a distinct eigenvalue.

Let (λ_j) be an enumeration of the eigenvalues of T without multiplicity. Let s_j denote the size of the largest Jordan block corresponding to each eigenvalue λ_j. Then $(T - \lambda_j I)^{s_j}$ kills the generalized eigenspace V_j for λ_j. Thus $m(x)$ divides $\prod_j (x - \lambda_j)^{s_j}$. Moreover, each $(x - \lambda_j)^{s_j}$ generates the T-annihilator for the basis vector acted on by the last column of the largest Jordan block for λ_j. Thus, each $(x-\lambda_j)^{s_j}$ divides $m(x)$. Hence $m(x) = \prod_j (x-\lambda_j)^{s_j}$. It follows that $p = m$ iff each λ_j corresponds to exactly one Jordan block. □