RESEARCH STATEMENT

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Overview

My research applies category theory and low-dimensional topology to questions from condensed matter physics and pure mathematics. My main project is the Property F conjecture [13], which relates universality of topological quantum computation to algebraic properties of fusion categories. My main result is the proof of an extended version of the Property F conjecture for the representation categories of twisted quantum doubles of finite groups [11]. I am also interested in applying functional programming and dependent type theory to mathematics.

My current projects include:

- Investigating the effects of symmetry gauging on braid group representations associated to modular tensor categories
- Proving Property F for metaplectic modular categories
- Classifying even metaplectic modular categories
- Developing a Haskell library for quantum mapping class group representations
- Optimizing cubicaltt, a typechecker for a constructive univalent type theory

Potential future projects include:

- Investigating braiding for fermionic (2+1)-TQFTs
- Constructing (3+1)-TQFTS

Background

Recently, a major program in condensed matter physics has been to classify topological phases of matter giving rise to quasiparticles with exotic braiding statistics. In 2016, the Nobel Prize in Physics was awarded to Thouless, Haldane, and Kosterlitz for related work on "theoretical discoveries of topological phase transitions and topological phases of matter." Category theory (more specifically, fusion category theory) has proven to be the correct framework for this classification program. Briefly, 2-dimensional topological phases of matter correspond to (2+1)-topological quantum field theories (TQFTs) parametrized by unitary modular tensor categories (UMTCs), a highly structured type of fusion category.

The holy grail for this program is to build a universal topological quantum computer. A topological quantum computer is a theoretical quantum computer that performs computations by braiding quasiparticles. The braiding statistics of a quasiparticle, images of a family of braid group representations, form the gate set for a quantum circuit model for the computer. If the braiding statistics are insufficiently expressive, this computer may not hold any advantage over a classical computer. On the other extreme, a universal quantum computer is as expressive as possible. More precisely, a universal quantum computer can efficiently simulate any other quantum computer. One path to universality for topological quantum computation is to create a quasiparticle whose braiding statistics are expressive enough to simulate any quantum circuit. Translated into mathematics, this corresponds to finding UMTCs whose associated braid group representations' images are dense in (projective) unitary groups [7].

The Property F conjecture [13] pushes this translation one step further, from analysis into algebra. It states that the braid group representations associated to a simple object in a braided fusion category have finite image if and only if the object is weakly integral, a purely algebraic condition. In practice, most infinite UMTC-associated braid group images turn out to be dense in the appropriate unitary groups. Thus, proving the Property F conjecture would be a major milestone in classifying topological phases of matter capable of universal quantum computation by quasiparticle braiding.

Although verifying the Property F conjecture in full generality has proven elusive, it has been verified for many classes of UMTCs. In particular, it has been verified for all UMTCs arising from the classical quantum group construction and various cases related to Hecke- and BMW-algebras [6, 7, 8, 9, 10, 15, 16]. It has also been verified for representation categories of twisted quantum doubles of finite groups [5].

Results

Since a $(2+1)$-TQFT gives (projective) representations of all mapping class groups of oriented, compact surfaces, we can generalize the Property F conjecture to consider all such mapping class groups, not just the braid group. My thesis proves this modified Property F conjecture for the representation categories of twisted quantum doubles of finite groups. In particular, my result answers a question of Etingof, Rowell and Witherspoon [5] regarding finiteness of images of arbitrary mapping class group representations in the affirmative.

My approach is to translate the problem into manipulation of colored graphs embedded in the given surface. To do this translation, I use the fact that any mapping class group representation associated to $\mathrm{Mod}(D^{\omega}(G))$ is isomorphic to a Turaev-Viro-Barrett-Westbury (TVBW) representation [3] associated to the spherical fusion category Vec_G^{ω} of twisted G-graded vector spaces. As shown by Kirillov [12] , the representation space for this TVBW representation is canonically isomorphic to a vector space spanned by Vec_G^{ω}-colored graphs embedded in the surface. By analyzing the action of the Birman generators [4] on a finite spanning set of colored graphs, I found that the mapping class group acts by permutations on a slightly larger finite spanning set. This implies that the representation has finite image.

Current Research

I am currently further investigating Property F and related conjectures. The most promising path towards proving the Property F conjecture for all UMTCs appears to be break it into two sub-conjectures based on the notion of gauging [2], a procedure for building a modular tensor category using a smaller modular tensor category as input. Physically, gauging corresponds to promoting a global topological symmetry to a local symmetry. Mathematically, gauging corresponds to first extending a UMTC by a finite group G and then equivariantizing. Conjecturally, any weakly integral category can be obtained by gauging a small class of very well-understood categories (a pointed modular category or a pointed modular category tensor an Ising category). It is also conjectured that gauging preserves Property F, i.e. the property that all simple objects associated braid group representations have finite image. Proving both of these two conjecture would imply the Property F conjecture.

Progress on the first conjecture is underway. A recent result of Natale [14] shows that all weakly group theoretical categories are gaugings of pointed modular categories or pointed tensor Ising modular categories. Moreover, all known weakly integral modular categories are weakly group-theoretical, and the weak integrality is conjecturally equivalent being weakly group-theoretical. Thus, Natale's result is a large step towards proving that all weakly integral categories are obtained by gauging very simple ones, reducing this conjecture to another, potentially more tenable, conjecture.

Property F for Metaplectic Modular Categories

My main current focus is the conjecture that gauging preserves Property F. A testbed for this conjecture is the class of metaplectic modular categories. These categories, UMTCs with the same fusion rules of the quantum group UMTC $SO(N)_2$ for N odd, form some of the simplest non-trivial examples of strictly weakly integral categories. Moreover, every metaplectic modular category is the gauging of a $\mathbb{Z}_n$-cyclic pointed modular category [1]. Therefore, metaplectic modular categories serve as a perfect test case: if we can prove Property F for all metaplectic categories, we will have taken the first step towards understanding how Property F relates to gauging.

Recently, Rowell and Wenzl showed that the classical quantum group UMTC $SO(N)_2$ has Property F [16]. However, their proof uses the quantum group machinery heavily, making the task of generalizing their result to all metaplectic modular categories nontrivial. I am currently pursuing three approaches to the more general metaplectic case: (i) comparing the metaplectic R-matrices to the $SO(N)_2$ ones, (ii) modifying the quantum group construction to change the Frobenius-Schur indicator of the fundamental spinor object, and (iii) directly analyzing the effects of gauging on R-matrices.

Classification of Even Metaplectic Modular Categories

Together with Paul Bruillard, Julia Plavnik, and Eric Rowell, I am also working on classification of even metaplectic modular categories, UMTCs with the fusion rules of $SO(N)_2$ for N even. This classification is of interest because it fits directly into the (2+1)-topological phase of matter classification program. Additionally, this classification is a test case for the conjecture that all weakly integral modular categories are gaugings of pointed modular categories or pointed tensor Isings.

As in the (odd) metaplectic case, all even metaplectic modular categories are expected to be gaugings of $\mathbb{Z}_N$-cyclic modular categories, but the situation is more complicated due to the power of 2 in the prime factorization of N. In fact, the somewhat awkward nomenclature (even metaplectic vs. metaplectic) reflects the fact that powers of 2 in the prime factorization of N introduce a complication in the fusion rules. Gauging is also more complicated: the 2-torsion elements of $\mathbb{Z}_N$ introduce possible cohomological obstructions and increase the number of possible fusion rules for the extension. Thus, understanding and classifying these categories will be a significant step towards understanding gauging.

A Haskell library for quantum mapping class group representations

I am also working on a Haskell library for quantum mapping class group represenations. The executable currently calculates the image of a braid group generator in the Turaev-Viro-Barett-Westbury (TVBW) representation associated to an arbitrary spherical fusion category in terms of 300 structural morphisms (associators, coevaluators, and evaluators – each tensored with identity morphisms).

The next goal is to compute matrix coefficients of mapping class group generators in particular TVBW representations. To investigate braiding, I attempted to do this for Tambara Yamagami categories, but it turned out to be computationally intractable even for the simplest nontrivial cases (possibly for "deep" reasons: we may need to take advantage of deeper structure). A simpler first step is to calculate matrix coefficients of braid group representations from pointed spherical fusion categories. After that, it may be possible to compute matrix coefficients for more complicated categories using gauging.

Optimizing cubicaltt

I am also interested in univalent type theories. My long-term goal is to facilitate machine-checked mathematical proofs. Univalent type theories allow a homotopical version of logic that greatly simplifies the process of formalizing mathematics. However, one major drawback of early univalent type theories was a lack of canonicity – it was possible to construct terms of the type of natural numbers that would not reduce to a canonical natural number $(0, 1, 2, \ldots)$.

Cubical type theory solves this problem, in theory, since the type theory has the canonicity property. Moreover, a type checker named "cubicaltt" for the type theory has been implemented. However, the normal forms of relatively simple terms can be enormous. The normal form of the univalence axiom, for example, is on the order of tens of megabytes. This makes serious computation with the univalence axiom quickly intractable.

Last summer, I experimented with cubicaltt at a Mathematical Research Community on Homotopy Type Theory. We wanted to calculate $\pi_4(S^3)$ by using the long exact sequence associated to the Hopf fibration, based on some type-theoretical work in Guillaume Brunerie's thesis. We were able to get further than he did before, but again we hit a computational roadblock due to huge normalized terms.

Since then, I have profiled the cubicaltt interpreter (which is also written in Haskell) to narrow down the source of the computational overhead. The next step in the project is to look for a more computationally-efficient representation of certain terms that currently have huge overhead.

References

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