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HW 4

1 Define

$$\mathcal{A}^{(\mathbb{Q})} = \left\{ \bigcup_{i=1}^{n} [a_i, b_i) \cap \mathbb{Q} : \begin{array}{c} n \in \mathbb{N}, \{a_i, b_i : 1 \leq i \leq n\} \subset \mathbb{Q} \cup \{\pm\infty\}, \\ \text{and } a_1 < b_1 < a_2 < \ldots < b_n \end{array} \right\}.$$

For $A = \bigcup_{i=1}^{n} [a_i, b_i) \cap \mathbb{Q}$ with $-\infty \leq a_1 < b_1 < a_2 < \ldots < b_n \leq \infty$ put

$$\mu_0(A) = \sum_{i=1}^{n} b_i - a_i.$$

a) $\mathcal{A}^{(\mathbb{Q})}$ is an algebra on $\mathbb{Q}$ and μ_0 is a finitely additive measure on $\mathcal{A}^{(\mathbb{Q})}$.
b) Show that μ_0 is not a premeasure.

Proof. For (a), suppose $E, F \in \mathcal{A}^{(\mathbb{Q})}$ with $E = \bigcup_{i=1}^{n} [a_i, b_i) \subset \mathbb{Q}$ and $F = \bigcup_{i=1}^{m} [c_i, d_i) \subset \mathbb{Q}$ for $a_i, b_i, c_i, d_i \in \mathbb{R}$ for all i. We have $\emptyset \in \mathcal{A}^{(\mathbb{Q})}$, so to show that $\mathcal{A}^{(\mathbb{Q})}$ is an algebra, we only need to show that E^c and $E \cup F$ are in $\mathcal{A}^{(\mathbb{Q})}$. For the former, we have $E^c = [-\infty, a_1) \cup [b_n, \infty) \cup \bigcup_{i=1}^{n-1} [b_i, a_{i+1})$, so $E^c \in \mathcal{A}^{(\mathbb{Q})}$.

For the latter, we have $E \cup F = \bigcup_{i=1}^{n+m} [e_i, f_i)$ for $([e_i, f_i))_i$ a reordering of the concatenation of $([a_i, b_i))$ and $([c_i, d_i))$ such that $e_1 \leq e_2 \leq \ldots \leq e_{n+m}$. Suppose $f_i > e_{i+1}$ for some i. Then $[e_i, f_i) \cup [e_{i+1}, f_{i+1}) = [e_i, f_{i+1})$. Hence, $E \cap F = \bigcup_{i=1}^{n+m-1} [e'_j, f'_j)$ where $[e'_j, f'_j) = ([e_j, f_j))$ for $j < i$, $[e'_i, f'_i) = [e_i, f_{i+1})$, and $[e'_j, f'_j) = [e_{j+1}, f_{j+1}$ for $j > i$. Then $e'_1 \leq e'_2 \leq \ldots \leq e'_{n+m-1}$. We can continue this process until we get $E \cup F = \bigcup_{i=1}^{l} [g_i, h_i)$ for some l, with $g_i \leq g_{i+1}$ for all i and $h_i \leq g_{i+1}$ for all i. Note that $g_i \leq h_i$ by construction. This implies that $E \cup F \in \mathcal{A}^{(\mathbb{Q})}$.

To see that μ_0 is finitely additive on $\mathcal{A}^{(\mathbb{Q})}$, we need to show that $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$ if E, F are disjoint. Using the same notation as above, we have $E \cup F = \bigcup_{i=1}^{n+m} [e_i, f_i)$ for $([e_i, f_i))_i$ a reordering of the concatenation of $([a_i, b_i))$ and $([c_i, d_i))$ such that $e_1 \leq e_2 \leq \ldots \leq e_{n+m}$. If $f_i > e_{i+1}$ for some i, then we contradict $b_j \leq a_{j+1}$, $d_j \leq c_{j+1}$, or the disjointness of E and F. Hence $e_i \leq f_i$ and $f_i \leq e_{i+1}$ for all i, so $\mu_0(E \cup F) = \sum_{i=1}^{n+m} f_i - e_i = \sum_{i=1}^{n} b_i - a_i + \sum_{i=1}^{m} d_i - c_i = \mu_0(E \cup F)$.

For (b), suppose μ_0 is a premeasure. It extends to a measure μ on $\mathcal{M}(\mathcal{A})$. Let $q \in \mathbb{Q}$. Pick any real-valued sequences $a_n \uparrow q$ and $b_n \downarrow q$. Then $q = \bigcap_n (a_n, b_n] \cap \mathbb{Q}$ and $\mu(b_1 - a_1) < \infty$, so $\mu(q) = \lim_{n \to \infty} b_n - a_n = 0$. Since every element in $\mathcal{A}$ is the union of its countably many rational elements, this implies that every element of $\mathcal{A}$ has measure 0, a contradiction. $\square$

2 Let $d \in \mathbb{N}$ and

$$\mathcal{E} = \left\{ \prod_{i=1}^{d} [a_i, b_i) : -\infty \le a_i \le b_i \le \infty \text{ for } i = 1, 2, \dots n \right\}.$$

(if $a_i = \infty$, replace $[a_i, b_i)$ with (a_i, b_i)). Let $\mathcal{A}$ be the algebra generated by $\mathcal{E}$.

a) Show that

$$\mathcal{A} = \left\{ \bigcup_{i=1}^{n} E_i : n \in \mathbb{N}, E_i \in \mathcal{E} \text{ are pairwise disjoint } \right\}.$$

b) Show that there is a measure μ on $\mathcal{M}(\mathcal{A})$ so that

$$\mu(\prod_{i=1}^{d} [a_i, b_i)) = \prod_{j=1}^{d} (b_i - a_i) \text{ whenever } -\infty \le a_i \le b_i \le \infty \text{ for } i = 1, 2, \dots, n.$$

Proof. Let $\mathcal{B} = \{\bigcup_{i=1}^{n} E_i : n \in \mathbb{N}, E_i \in \mathcal{E} \text{ are pairwise disjoint } \}$. Clearly $\mathcal{B} \subset \mathcal{A}$, so it suffices to show that $\mathcal{B}$ is an algebra. Since $\mathcal{E}$ is nonempty, $\mathcal{B}$ must be nonempty.

To see that $\mathcal{B}$ is closed under taking finite intersections, let $B, C \in \mathcal{B}$. Then $B = \bigcup_{i=1}^{m} B_i$ for some $m \in \mathbb{N}$ and disjoint $(B_i) \subset \mathcal{E}$, and $C = \bigcup_{i=1}^{n} C_i$ for some $n \in \mathbb{N}$ and disjoint $(C_i) \subset \mathcal{E}$. Then $B \cap C = \bigcup_{i,j} B_i \cap C_j$. To see that the sets $(B_i \cap C_j)_{i,j}$ are disjoint, suppose $(i, j) \ne (i', j')$. WLOG $i \ne i'$. Then $(B_i \cap C_j) \cap (B_{i'} \cap C_{j'}) = (B_i \cap B_{i'}) \cap (C_j \cap C_{j'}) = \emptyset$ since the (B_i) are disjoint. Hence $(B_i \cap C_j)_{ij}$ are disjoint, so it suffices to break an arbitrary $B_i \cap C_j$ into disjoint elements of $\mathcal{E}$.

Write $B_i = \prod_{i=1}^{d} [a_i, b_i)$ and $C_i = \prod_{i=1}^{d} [c_i, d_i)$. Then $B_i \cap C_i = \prod_{i=1}^{d} [a_i, b_i) \cap [c_i, d_i)$. For each i, we have $[a_i, b_i) \cap [c_i, d_i) = [e_i, f_i)$ for some $-\infty \le e_i \le f_i \le \infty$ by case analysis on the order of a_i, b_i, c_i, d_i. Hence, $B_i \cap C_i \in \mathcal{E}$.

To see that $\mathcal{B}$ is closed under taking complements, let $B \in \mathcal{B}$. Then $B = \bigcup_{i=1}^{n} E_i$ for $E_i \in \mathcal{E}$, and $B^c = \bigcap_i E_i^c$. Since we know that $\mathcal{B}$ is closed under finite intersections, it suffices to show that each $E_i^c \in \mathcal{B}$. Writing E_i as $E_i = \prod_{j=1}^{d} [a_j, b_j)$, let $\mathcal{U} = \{\prod_{j=1}^{d} U_j : \forall j \quad U_j \in \{(-\infty, a_j), [a_j, b_j), [b_j, \infty)\}\} \subset \mathcal{E}$. Then $\mathcal{U}$ is a partition of $\mathbb{R}^d$, and $E_i^c = \bigcup (\mathcal{U} \setminus E)$. Hence $E_i^c \in \mathcal{B}$. $\square$

3 Let μ be a finite measure on $\mathcal{B}_{\mathbb{R}}$. Show that for all $\epsilon > 0$ and all $A \in \mathcal{B}_{\mathbb{R}}$, there is an open set U and a closed set F so that $F \subset A \subset U$ and $\mu(U \setminus F) < \epsilon$. Prove this by showing that

$$\widetilde{\mathcal{M}} := \{A \in \mathcal{B}_{\mathbb{R}} : \forall \epsilon > 0 \exists U \text{ open } \exists C \text{ closed } \quad C \subset A \subset U \text{ and } \mu(U \setminus C < \epsilon\}$$

is a σ-algebra.

Proof. $\square$

4 If $E \in \mathcal{L}$ (the Lebesgue sets) and $m(E) > 0$ then there is for any $\alpha < 1$ and open interval I such that $m(E \cap I) > \alpha m(I)$.