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HW 5, due 2/28

18.32 Let $(f_n), f$ be integrable. If $\int |f_n - f| \to 0$, show that $\int_E f_n \to \int_E f$ for all measurable sets E, and that $\int f_n^+ \to \int f^+$.

Proof. Note $\chi_E f_n \in L_1$ and $\chi_E f \in L_1$, so $|\int_E f - \int_E f_n| = |\int_E (f - f_n)| \leq \int_E |f - f_n| \leq \int |f - f_n| \to 0$.

Also, for any real function g, we have $g^+ = (|g| + g)/2$. Hence, $|\int (f_n^+ - f^+)| = (1/2)|\int (|f_n| - |f| + f_n - f)| \leq (1/2)\int (||f_n| - |f|| + |f_n - f|) \leq \int |f_n - f| \to 0$. $\square$

40 Let (f_n), (g_n), and g be integrable, and suppose that $f_n \to f$ a.e., $g_n \to g$ a.e., $|f_n| \leq g_n$ a.e., for all n, and that $\int g_n \to \int g$. Prove that $f \in L_1$ and that $\int f_n \to \int f$.

Proof. Just the proof of the DCT with the obvious substitutions. Since $|f_n| \leq g_n$, we have $|f| \leq g$, so $f \in L_1$. The only other interesting parts are the equality $\liminf_{n\to\infty}(\int g_n + \int f_n) = \int g + \liminf_{n\to\infty} \int f_n$ and the corresponding one for $\limsup$. This follows from the more general fact that if $a_n \to a$ and $(b_n) \subset \mathbb{R}$, then $\liminf_n (a_n + b_n) = a + \liminf_n b_n$.

Indeed, for $\epsilon > 0$, we have $|a_n - a| \leq \epsilon$ for all large n. Hence, $a - \epsilon + \liminf_n b_n \leq a + \liminf_n (a_n - a) + b_n \leq a + \epsilon + \liminf_n b_n$. Letting $\epsilon \to 0$, we have $\liminf_n (a_n + b_n) = a + \liminf_n (a_n - a) + b_n = a + \liminf_n b_n$. $\square$

43(a) Let f be measurable and finite a.e. on $[0,1]$. If $\int_E f = 0$ for all measurable $E \subset [0,1]$ with $m(E) = 1/2$, prove that $f = 0$ a.e. on $[0,1]$.

Proof. *Case* $f \geq 0$. Suppose the conclusion fails. Then $m[f > 0] = m[\cup_n f \geq 1/n] > 0$, so $m[f \geq 1/n] > 0$ for some n. But then either $[0,1/2] \cap [f \geq 1/n]$ or $[1/2,1] \cap [f \geq 1/n]$ has positive measure. WLOG, suppose the former. Then $\int_0^{1/2} f \geq \int_0^{1/2} 1/n \chi_{[f \geq 1/n]} = 1/n\, m([0,1/2] \cap [f \geq 1/n]) > 0$, a contradiction.

General case. Either $m[f \geq 0] \geq 1/2$ or $m[f \leq 0] \leq 1/2$. FIXME

Suppose not. Either $m[f > 0]$ or $m[f < 0]$ is positive; WLOG suppose the former is. Then $m[f > 0] = m[\cup_n f \geq 1/n] > 0$, so $m[f \geq 1/n] > 0$ for some n. If $m[f \geq 1/n] > 1/2$, then pick $E \subset [f \geq 1/n]$ with $m(E) = 1/2$, giving a contradiction. The existence of such an E follows the intermediate value theorem since $g(t) = m([f \geq 1/n] \cap [0,t])$ is continuous.

If $m[f \geq 1/n] < 1/2$, let $F \subset ([0,1] \setminus [f \geq 1/n]$ with $m(F) = 1/2$. Let $G = F \cap [fle0]$. $\square$

43(b) Let f be measurable and finite a.e. on $[0,1]$. If $f > 0$ a.e., show that $\inf \left\{ \int_E f : m(E) \geq 1/2 \right\} > 0$.

Proof. $\square$

44(c) Show that $\lim_n \int_0^1 f_n = 0$ where $f_n(x) = \frac{nx\log x}{1+n^2x^2}$.

Proof. Note that $1 + n^2x^2 \geq 2nx$. Hence, $|f_n| \leq (1/2)|\log x|$. Note that $\int_0^1 |\log x|\,dx = \int_0^1(-\log x)\,dx = [x - x\log x]_0^1 = 1 - \lim_{x\to 0} x\log x = 1 - \lim_{x\to 0}\frac{1/x}{-1/x^2} = 1$. Hence, by the DCT, $\lim_n \int_0^1 f_n = \int_0^1 \lim_{n\to\infty} f_n = 0$. $\square$

44(d) Show that $\lim_n \int_0^1 f_n = 0$ where $f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}$.

Proof. Letting $u = 1+n^2x^2$, we have $\int_0^1 f_n = \int_1^{1+n^2}\frac{n^{-1/2}}{2u}\,du = (n^{-1/2}/2)[\log u]_1^{1+n^2} = (n^{-1/2}/2)\log(1+n^2)$.

Thus, $\lim_n \int_0^1 f_n = \lim_{n\to\infty}\frac{\log(1+n^2)}{2n^{1/2}} = \lim_{n\to\infty}\frac{2n/(1+n^2)}{n^{-1/2}} = 0$. $\square$

47(b) Compute $\lim_{n\to\infty}\int_0^1 \frac{1+nx^2}{(1+x^2)^n}\,dx$.

Proof. By the binomial theorem, $\frac{1+nx^2}{(1+x^2)^n} \leq 1$ for all n. Hence, by the DCT, $\lim_{n\to\infty}\int_0^1 \frac{1+nx^2}{(1+x^2)^n}\,dx = \int_0^1 \lim_{n\to\infty}\frac{1+nx^2}{(1+x^2)^n}\,dx = 0$. $\square$

47(d) Compute $\lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}\,dx$.

Proof. Let $u = nx$. Then $\int_a^\infty \frac{n}{1+n^2x^2}\,dx = \int_{na}^\infty \frac{1}{1+u^2}\,du = [\tan^{-1}(u)]_{na}^\infty = (\pi/2) - \tan^{-1}(na)$. As $n \to \infty$,

$$\int_a^\infty \frac{n}{1+n^2x^2}\,dx = (\pi/2) - \tan^{-1}(na) \to \begin{cases} 0, & a > 0 \\ \pi/2, & a = 0 \\ \pi, & a < 0 \end{cases}.$$

$\square$

49 For which $\alpha \in \mathbb{R}$ is $f(x) := \sum_{n=1}^\infty xn^{-\alpha}e^{-nx}$ continuous on $[0,\infty)$? in $L_1[0,\infty)$?

Proof. First note that, for any α, each term of the series is decreasing in x. Hence, f converges uniformly on every closed interval not containing 0 by the ratio test.

Note that if $\alpha \leq 0$, we have, for $x > 0$, $\sum_{n=1}^\infty xn^{-\alpha}e^{-nx} \geq \sum_{n=1}^\infty xe^{-nx} = x\frac{e^{-x}}{1-e^{-x}} = \frac{x}{e^x-1} \to 1$ as $x \to 0$. But $f(0) = 0$, so f is discontinuous at 0 for $\alpha \leq 0$.

If $\alpha > 0$, then

$$
\begin{aligned}
\lim_{x\to 0} f(x) &= \lim_{x\to 0} \sum_{n=2}^{\infty} x n^{-\alpha} e^{-nx} \\
&\le \lim_{x\to 0} \sum_{n=2}^{\infty} \int_{n-1}^{n} x y^{-\alpha} e^{-yx}\, dy \\
&= \lim_{x\to 0} \int_{1}^{\infty} x y^{-\alpha} e^{-yx}\, dy \\
&= \lim_{x\to 0} \int_{1}^{\infty} (-y^{-\alpha})(-x e^{-yx})\, dy \\
&= \lim_{x\to 0} [-y^{-\alpha} e^{-yx}]_{y=1}^{\infty} - \int_{1}^{\infty} e^{-yx}(\alpha y^{-\alpha-1})\, dy \\
&= \lim_{x\to 0} e^{-x} - \alpha \int_{1}^{\infty} e^{-yx} y^{-\alpha-1})\, dy \\
&= 1 - \alpha \int_{1}^{\infty} \lim_{x\to 0} e^{-yx} y^{-\alpha-1}\, dy \\
&= 1 - \alpha \int_{1}^{\infty} y^{-\alpha-1}\, dy \\
&= 1 + [y^{-\alpha}]_{1}^{\infty} \\
&= 0,
\end{aligned}
$$

where the interchange of limit and integral is justified by the inequality $e^{-yx} y^{-\alpha-1} \le y^{-\alpha-1}$, whose integral converges since $\alpha > 0$. Hence, f is continuous for $\alpha > 0$.

To find out when $f \in L_1[0,\infty)$, note that by the MCT,

$$
\begin{aligned}
\int_0^{\infty} f(x)\, dx &= \sum_{n=1}^{\infty} n^{-\alpha} \int_0^{\infty} x e^{-nx}\, dx \\
&= \sum_{n=1}^{\infty} n^{-\alpha} ([x(-n^{-1}) e^{-nx}]_{x=0}^{\infty} - \int_0^{\infty} (-n^{-1}) e^{-nx}\, dx) \\
&= \sum_{n=1}^{\infty} n^{-\alpha} [-n^{-2} e^{-nx}]_{x=0}^{\infty} \\
&= \sum_{n=1}^{\infty} n^{-2-\alpha},
\end{aligned}
$$

which converges iff $\alpha > -1$. □

55 Prove that if f is integrable on $\mathbb{R}$, then $f(x)\cos(nx)$ is integrable and $\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\cos(nx)\, dx = 0$. The same is true with sin replacing cos.

Proof. To see that $f(x)\cos(nx) \in L_1(\mathbb{R})$, note $|f(x)\cos(nx)| \le |f(x)|$ for all x. The other conclusion follows from (56), replacing t with x and $\sin(xt)$ with $\cos(nx)$ where appropriate. □

60 (a) Show that there is a sequence of polynomials (P_n) such that $P_n \to 0$ pointwise on $[0, 1]$, but with $\int_0^1 P_n(x)\,dx \to 3$.
(b) Find $\int_0^1 \sup_n |P_n(x)|\,dx$.

Proof. $\square$

51 Let (f_n) be a sequence of measurable functions with $|f_n| \le g$ for all n, where $g \in L_1$. If $f_n \to f$ a.e., prove that $f_n \to f$ almost uniformly.

Proof. $\square$

56 Given $f \in L_1(\mathbb{R})$, define $g(x) = \int_{-\infty}^{\infty} f(t)\sin(xt)\,dt$ for $x \in \mathbb{R}$. Show that g is continuous on $\mathbb{R}$ and that $g(x) \to 0$ as $x \to \pm\infty$; hence, g is uniformly continuous on $\mathbb{R}$.

Proof. Case $f = \chi_{(a,b)}$ for $a, b \in \mathbb{R}$. We have $g(x) = \int_a^b \sin(xt)\,dt = (1/x)\int_{xa}^{xb} \sin(t)\,dt = O(1/x)$ since $\int_r^{r+2n\pi} \sin(t) = 0$ for all $r \in \mathbb{R}$, $n \in \mathbb{Z}$. Hence, $g(x) \to 0$ as $|x| \to \infty$.

To see that g is continuous, fix x and suppose $x_n \to x$. Then $\sin(x_n t) \to \sin(xt)$ pointwise on (a, b), hence in measure also. Let $\epsilon > 0$. Convergence in measure says that we can pick N such that, for all $n > N$, we have $m[|\sin(x_n t) - \sin(xt)| \ge \epsilon] < \epsilon$. Then for $n > N$, we have $|g(x) - g(x_n)| \le \int_a^b |\sin(xt) - \sin(x_n t)|\,dt \le 2\,m[|\sin(x_n t) - \sin(xt)| \ge \epsilon] + \epsilon\, m[|\sin(x_n t) - \sin(xt)| < \epsilon] \le 2\epsilon + \epsilon(b - a)$. Hence, g is continuous.

Case f is a step function. We have $f = \sum_{i=1}^m a_i \chi_{A_i}$ a.e., where each A_i is an interval. Both conclusions follow from the linearity of the integral and the previous case.

General case. Let $\epsilon > 0$. Pick a step function h such that $\int |f - h| < \epsilon$. To see that $g(x) \to 0$ as $|x| \to \infty$, note that $|g(x)| = |\int h(t)\sin(xt)\,dt + \int (f(t) - h(t))\sin(xt)\,dt| \le |\int h(t)\sin(xt)\,dt| + \int |f - h| \to \epsilon$ as $|x| \to \infty$ by the step function case.

For the continuity, suppose $x_n \to x$. Then $|g(x_n) - g(x)| = |\int f(t)(\sin(x_n t) - \sin(xt))\,dt| \le |\int h(t)(\sin(x_n t) - \sin(xt))\,dt| + 2\int |f - h| \to 2\epsilon$ as $n \to \infty$. $\square$