On the Property F Conjecture

Paul Gustafson
Texas A&M University

Outline

- The Property F Conjecture
- Similar conjecture for mapping class groups
- Proof of the modified conjecture in Vect_G^ω-case
- Progress on Property F in the metaplectic case

The Property F conjecture

Conjecture (Rowell)

Let $\mathcal{C}$ be a braided fusion category and let X be a simple object in $\mathcal{C}$. The braid group representations $\mathcal{B}_n$ on $\mathrm{End}(X^{\otimes n})$ have finite image for all $n > 0$ if and only if X is weakly integral (i.e. $\mathrm{FPdim}(X)^2 \in \mathbf{Z}$).

- Verified for modular categories from quantum groups (Rowell, Naidu, Freedman, Larsen, Wang, Wenzl, Jones, Goldschmidt)

A similar conjecture for mapping class groups

Conjecture

The Turaev-Viro-Barrett-Westbury (TVBW) mapping class group representation associated to a compact surface Σ and spherical fusion category $\mathcal{A}$ has finite image iff $\mathcal{A}$ is weakly integral.

A similar conjecture for mapping class groups

Conjecture

The Turaev-Viro-Barrett-Westbury (TVBW) mapping class group representation associated to a compact surface Σ and spherical fusion category $\mathcal{A}$ has finite image iff $\mathcal{A}$ is weakly integral.

- In this talk: $\mathcal{A} = \mathrm{Vect}_G^{\omega}$ – the category of G-graded vector spaces with associativity twisted by a 3-cocycle ω
- This is the same as the twisted Dijkgraaf-Witten TQFT

Related Work

Theorem (Etingof–Rowell–Witherspoon)

The braid group representation associated to the modular category $\mathrm{Mod}(D^\omega(G))$ *has finite image.*

Theorem (Fjelstad–Fuchs)

Every mapping class group representation of a closed surface with at most one marked point associated to $\mathrm{Mod}(D(G))$ *has finite image.*

Theorem (Ng–Schauenberg)

Every modular representation associated to a modular category has finite image.

Main result

Theorem (G.)

The image of any Vect_G^ω *TVBW representation* ρ *of a mapping class group of an orientable, compact surface* Σ *with boundary is finite.*

Idea of proof:

- Find a good finite spanning set S for the representation space
- Calculate the action of each Birman generator on S
- Show that the representation of each Birman generator lies in a quotient of a finite group of monomial matrices.

The TVBW space associated to a 2-manifold

- Kirillov: The TVBW representation space is canonically isomorphic to

$$H := \frac{\text{formal linear combinations of } \mathcal{A}\text{-colored graphs in } \Sigma}{\text{local relations}}$$

Colored graphs in Σ

- Let $\Gamma \subset \Sigma$ be an undirected finite graph embedded in Σ.

Colored graphs in Σ

- Let $\Gamma \subset \Sigma$ be an undirected finite graph embedded in Σ.
- Define E^{or} to be the set of orientation edges of Γ, i.e. pairs $\mathbf{e} = (e, \text{orientation of } e)$; for such an oriented edge $\mathbf{e}$, we denote by $\bar{\mathbf{e}}$ the edge with opposite orientation.

Colored graphs in Σ

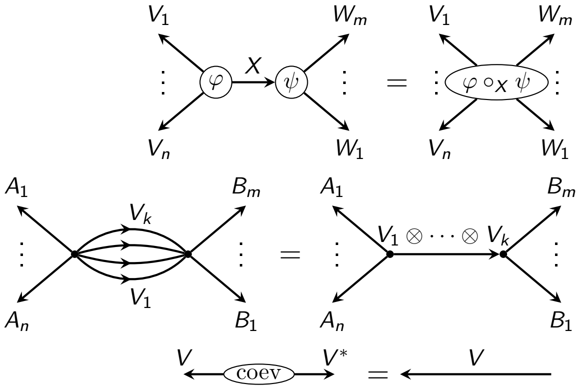
- Let $\Gamma \subset \Sigma$ be an undirected finite graph embedded in Σ.
- Define E^{or} to be the set of orientation edges of Γ, i.e. pairs $\mathbf{e} = (e, \text{orientation of } e)$; for such an oriented edge $\mathbf{e}$, we denote by $\bar{\mathbf{e}}$ the edge with opposite orientation.
- A *coloring* of Γ is the following data:
  - Choice of an object $V(\mathbf{e}) \in \operatorname{Obj} \mathcal{A}$ for every oriented edge $\mathbf{e} \in E^{or}$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.

Colored graphs in Σ

- Let $\Gamma \subset \Sigma$ be an undirected finite graph embedded in Σ.
- Define E^{or} to be the set of orientation edges of Γ, i.e. pairs $\mathbf{e} = (e, \text{orientation of } e)$; for such an oriented edge $\mathbf{e}$, we denote by $\bar{\mathbf{e}}$ the edge with opposite orientation.
- A *coloring* of Γ is the following data:
  - Choice of an object $V(\mathbf{e}) \in \operatorname{Obj} \mathcal{A}$ for every oriented edge $\mathbf{e} \in E^{or}$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.
  - Choice of a vector $\varphi(v) \in \operatorname{Hom}_{\mathcal{A}}(1, V_1 \otimes \cdots \otimes V_n)$ for every interior vertex v, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are edges incident to v, taken in counterclockwise order and with outward orientation.

Local relations

- Isotopy of the graph embedding
- Linearity in the vertex colorings



$k \geq 0$

Figure: The remaining local relations.

Consequences of the local relations

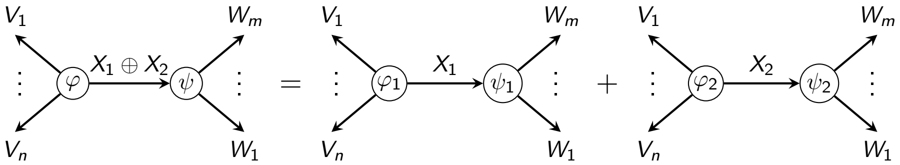


Figure: Additivity in edge colorings. Here φ_1, φ_2 are compositions of φ with projector $X_1 \oplus X_2 \to X_1$ (respectively, $X_1 \oplus X_2 \to X_2$), and similarly for ψ_1, ψ_2.

- Additivity in edge colorings

Theorem (Kirillov, Reshitikhin–Turaev)

A colored graph Γ may be evaluated on any disk $D \subset \Sigma$, giving an equivalent colored graph Γ' such that Γ' is identical to Γ outside of D, has the same colored edges crossing ∂D, and contains at most one colored vertex within D.

Basis for the representation space

By applying the local moves and the preceding theorem, any such representation space has a finite spanning set of "simple" colored graphs with a single vertex, loops for each of the standard generators of $\pi_1(\Sigma)$, and a leg from the vertex to each of the boundary components.

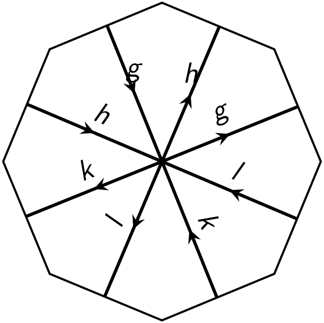


Figure: Element of the spanning set for a genus 2 surface. Here $[g, h][k, l] = 1$, and the vertex is labeled by a "simple" morphism (a $|G|$-th root of unity times a canonical morphism)

Applying the Birman generators to the spanning set

- The next step of the proof is to apply each Birman generator to each element of the spanning set.
- In each case, we relate the resulting colored graph to another element of the spanning set by means of local moves
- The local moves map simple colored graphs to simple colored graphs
- Hence, the Birman generators preserve the finite spanning set.

First Dehn twist

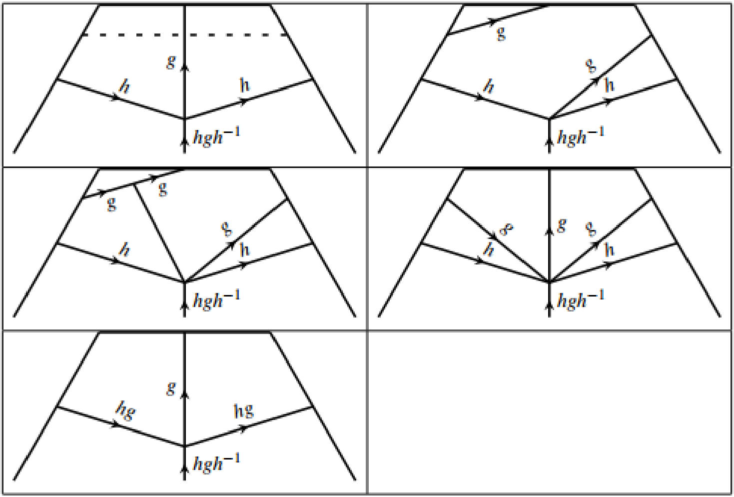


TABLE 1. First type of Dehn twist. Unlabeled interior edges are colored by the group identity element.

Second Dehn twist

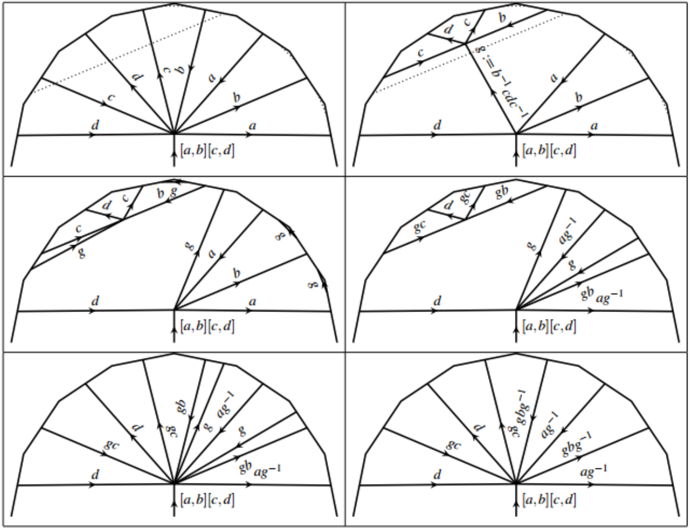


TABLE 2. Second type of Dehn twist.

Braid generator

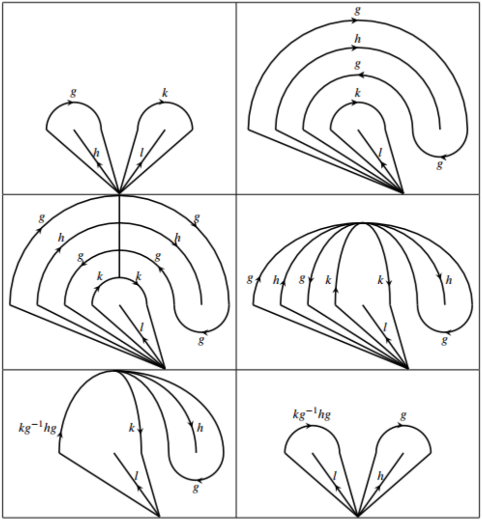


TABLE 3. A braid generator. Unlabeled interior edges are colored by the group identity element.

Dragging a point

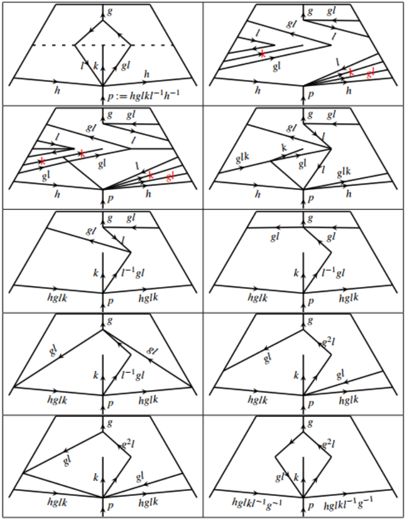


TABLE 4. Pulling a boundary component along a generator for the fundamental group of the corresponding closed surface. Unlabeled interior edges are colored by the group identity element.

Next step: Metaplectic modular categories

A metaplectic modular category is a unitary modular category with the fusion rules of $SO(N)_2$ for odd $N > 1$. It has 2 simple objects X_1, X_2 of dimension $\sqrt{N}$, two simple objects $1, Z$ of dimension 1, and $\frac{N-1}{2}$ objects Y_i, $i = 1, \ldots, \frac{N-1}{2}$ of dimension 2.
The fusion rules are:

1. $Z \otimes Y_i \cong Y_i$, $Z \otimes X_i \cong X_{i+1}$ (modulo 2), $Z^{\otimes 2} \cong 1$,
2. $X_i^{\otimes 2} \cong 1 \oplus \bigoplus_i Y_i$,
3. $X_1 \otimes X_2 \cong Z \oplus \bigoplus_i Y_i$,
4. $Y_i \otimes Y_j \cong Y_{\min\{i+j, N-i-j\}} \oplus Y_{|i-j|}$, for $i \neq j$ and $Y_i^{\otimes 2} = 1 \oplus Z \oplus Y_{\min\{2i, N-2i\}}$.

Related Work

Theorem (Rowell–Wenzl)

The images of the braid group representations on $\mathrm{End}_{SO(N)_2}(S^{\otimes n})$ *for* N *odd are isomorphic to images of braid groups in Gaussian representations; in particular, they are finite groups.*

Theorem (Ardonne–Cheng–Rowell–Wang)

1. *Suppose* $\mathcal{C}$ *is a metaplectic modular category with fusion rules* $SO(N)_2$, *then* $\mathcal{C}$ *is a gauging of the particle-hole symmetry of a* $\mathbb{Z}_N$*-cyclic modular category.*
2. *For* $N = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ *with distinct odd primes* p_i, *there are exactly* 2^{s+1} *many inequivalent metaplectic modular categories.*

Ardonne–Finch–Titsworth classify metaplectic fusion categories up to monoidal equivalence and give modular data for low-rank cases.

Current problem

- Can we modify the standard quantum group construction to construct other metaplectic modular categories?
- In particular, can we flip the signs of the Frobenius-Schur indicators $\nu_2(X_i)$ for the spin objects X_i?
- Conjugating/flipping the sign of $q^{1/2}$ don't work.
- Modify the trace construction?

Thanks

Thanks for listening!