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Math 689 Commutative and Homological Algebra

HW 1

All rings are commutative with $1 \neq 0$.

1 Let A be a ring and let $\mathfrak{S}$ be the set of all multiplicative subsets of A that do not contain 0, ordered by inclusion. Prove that an element S of $\mathfrak{S}$ is maximal in $\mathfrak{S}$ if and only if $R - S$ is a minimal prime ideal.

Proof. Suppose S is a maximal element of $\mathfrak{S}$. By a proposition from class, there exists a prime ideal I disjoint from S. Suppose J is a prime ideal contained in I. Then $R - J$ is a multiplicative subset of A not containing 0, i.e, an element of $\mathfrak{S}$. But we also have $S \subset R - I \subset R - J$. Thus, by the maximality of S, we have $S = R - I = R - J$. Hence $I = J$. Since this for all prime ideals $J \subset I$, I is a minimal prime ideal.

For the opposite implication, suppose S is not a maximal element of $\mathfrak{S}$. Pick $T \in \mathfrak{S}$ with S properly contained in T. Pick a prime ideal I disjoint from T. Then $I \subset R - T \subset R - S$, where the last inclusion is proper. Therefore, $R - S$ is not a minimal prime ideal. $\square$

2 Let I and J be ideals of a ring A. Show that

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}.$$

Proof. First we have $IJ \subset I \cap J$, so $\sqrt{IJ} \subset \sqrt{I \cap J}$.

Similarly, $I \cap J \subset I$ and $I \cap J \subset J$ implies that $\sqrt{I \cap J} \subset \sqrt{I} \cap \sqrt{J}$.

Lastly, let P be a prime ideal containing IJ. I claim that P contains I or J. Suppose not. Then there exists $x \in I - P$ and $y \in J - P$. But then $xy \in IJ \subset P$, which contradicts the fact that P is a prime ideal. Thus, P contains I or J. Thus every prime ideal containing IJ contains either I or J. Thus, $\sqrt{I} \cap \sqrt{J} \subset \sqrt{IJ}$. $\square$

3 Let $\phi : A \to B$ be a ring homomorphism, and I an ideal of B. Prove that $\phi^{-1}(\sqrt{I}) = \sqrt{\phi^{-1}(I)}$.

Proof. Let $J \subset B$ be a prime ideal. Let $a \in A$ and $x, y \in \phi^{-1}(J)$. Then we have $\phi(x + y) = \phi(x) + \phi(y) \in J$, and $\phi(ax) = \phi(a)\phi(x) \in J$. Thus $\phi^{-1}(J)$ is an ideal. Now suppose $a, b \in A$ with $ab \in \phi^{-1}(J)$. Then $\phi(a)\phi(b) = \phi(ab) \in J$, hence $\phi(a)$ or $\phi(b)$ is in J since J is prime. Thus, $\phi^{-1}(J)$ is also a prime ideal. Since inverse maps preserve intersections and containment, this implies that $\phi^{-1}(\sqrt{I}) \supset \sqrt{\phi^{-1}(I)}$.

For the reverse inclusion, suppose $x^n \in \phi^{-1}(I)$. Then $\phi(x^n) \in I$. Hence $(\phi(x))^n \in I$, so $\phi(x) \in \sqrt{I}$. Thus, $x \in \phi^{-1}(\sqrt{I})$. $\square$

4 Let n be a positive integer, $n \geq 2$. Find $\text{nil}(\mathbb{Z}/(n))$.

Proof. The maximal ideals of $\mathbb{Z}/(n)$ are the ideals generated by the prime factors of n. The intersection of these ideals, $\mathrm{nil}(\mathbb{Z}/(n))$, is the ideal generated by the product of the distinct prime factors of n taken without multiplicity. $\square$

5 Let $\phi : A \to B$ be a surjective ring homomorphism.

(a) Prove that $\phi(\mathrm{rad}(A)) \subset \mathrm{rad}(B)$.

Proof. Let $a \in \mathrm{rad}(A)$ and $b \in B$. Since ϕ is surjective, there exists $c \in A$ with $\phi(c) = b$. Since $a \in \mathrm{rad}(A)$, the element $1 - ca$ is a unit. Thus, $\phi(1-ca) = 1 - b\phi(a)$ is a unit. Since this holds for all $b \in B$, it follows that $\phi(a) \in \mathrm{rad}(B)$. Since this holds for all $a \in \mathrm{rad}(A)$, we have $\phi(\mathrm{rad}(A)) \subset \mathrm{rad}(B)$. $\square$

(b) Give an example to show that the inclusion need not be an equality.

Proof. Let $\phi : \mathbb{Z} \to \mathbb{Z}/(4)$ be the canonical map. We have $\mathrm{rad}(\mathbb{Z}) = (0)$ but $\mathrm{rad}(\mathbb{Z}/(4)) = (2)$. $\square$

6 Let A be a local ring. Prove that A contains no idempotent elements other than 0 and 1.

Proof. Suppose not. Let $a \in A - \{0, 1\}$ be idempotent. Then $a(1-a) = 0$. Thus a and $1-a$ are both non-units. Thus a lies in the unique maximal ideal of A, hence the Jacobson radical of A. This contradicts the fact that $1-a$ is a non-unit. $\square$

7 Let A be a local ring and M, N finitely generated A-modules. Prove that if $M \otimes_A N = 0$, then $M = 0$ or $N = 0$.

Proof. Let k be the residue field of A, and $\mathfrak{m}$ the maximal ideal. We have $0 = (M \otimes_A N) \otimes_A (k \otimes_A k) = (M \otimes_A k) \otimes_A (N \otimes_A k) = (M \otimes_A k) \otimes_k (N \otimes_A k)$. The last expression is just a tensor product of finite dimensional vector spaces over the field k, so either $M \otimes_A k = 0$ or $N \otimes_A k = 0$. WLOG suppose the former. Then $0 = M \otimes_A k = M/\mathfrak{m}M$, so by Nakayama's lemma $M = 0$. $\square$

8 Let A be a ring. Suppose that $A^m \cong A^n$ as A-modules. Show that $m = n$. (Hint: reduce to the case of a field.)

Proof. Let $\mathfrak{m}$ be a maximal ideal of A. Then $A^m/\mathfrak{m}A^m \cong A^n/\mathfrak{m}A^n$ as $A/\mathfrak{m}A$-vector spaces. Since $A^m/\mathfrak{m}A^m \cong (A/\mathfrak{m}A)^m$ and the same for n, the invariance of dimension for vector spaces implies that $m = n$. $\square$

9 Let $I_1, \ldots, I_n$ be ideals of a ring for which $I_1 \cap \ldots \cap I_n = (0)$. Prove that if A/I_j is Noetherian for each j, then A is Noetherian.

Proof. Let $J \subset A$ be an ideal. Since A/I_1 is Noetherian, $J + I_1$ is a finitely generated ideal in A/I_1. We can lift the generators to representatives in J which generate some ideal $F_1 \subset J$. We then have $J = F_1 + J \cap I_1$, where F_1 is finitely generated.

Similarly using A/I_2, we have $J \cap I_1 = F_2 + J \cap I_1 \cap I_2$, where $F_2 \subset J \cap I_2$ is finitely generated. Continue in this way, we finally end up with $J \cap I_1 \cap \ldots \cap I_{n-1} = F_n + J \cap I_1 \cap \ldots \cap I_n = F_n$.

Substituting back, we get $J = F_1 + \ldots + F_n$, where the F_i are all finitely generated. Hence, J is finitely generated. Thus, A is Noetherian. □

10 Let A, B, C be rings and suppose that $\phi : A \to C, \psi : B \to C$ are surjective ring homomorphisms. Prove that if A and B are Noetherian, then $A \times_C B$ is Noetherian. (Hint: use 9.).

Proof. Let $I, J \subset A \times_C B$ be the ideals defined by $I = A \times 0$ and $J = 0 \times B$. Then $I \cap J = 0$, and $A \times_C B/I = B$ and $A \times_C B/J = A$ which are Noetherian. Hence, exercise 9 implies that $A \times_C B$ is Noetherian. □