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HW 6

1 Assume $(X, \mathcal{M}, \mu)$ is a complete measure space.

a) If $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable and $f = g$ μ-a.e., then g is also measurable.

b) If $f_n : X \to \mathbb{R}$ is $\mathcal{M}$-measurable and $\lim_{n\to\infty} f_n = f$ μ-a.e., then f is measurable.

Proof. For (a), let $N = \{x : f(x) \neq g(x)\}$. Since $\mathcal{M}$ is complete, $N \in \mathcal{M}$. Let $a \in \mathbb{R}$. We need to show that $A := g^{-1}((a, \infty))$ is in $\mathcal{M}$. To see this, note that $A \cap N^c = f^{-1}((a, \infty)) \cap N^c$, which is in $\mathcal{M}$. Hence $A = (A \cap N) \cup (A \cap N^c)$ is in $\mathcal{M}$ since $A \cap N$ is a null set.

For (b), let $N = \{x : f(x) \neq \lim_n f_n\} \in \mathcal{M}$. Then $f_n \chi_{N^c} \to f \chi_{N^c}$ pointwise. Hence $f\chi_{N^c}$ is measurable by the first exercise of the last homework set. Thus, by (a), f is measurable. $\square$

2 If $f \in \mathcal{L}^+$ and $\int f \, d\mu < \infty$, then for any $\epsilon > 0$, there is an $E \in \mathcal{M}$, so that $\mu(E) < \infty$ and $\int_E f \, d\mu > \int f \, d\mu - \epsilon$.

Proof. There exists a simple function $0 \leq \phi \leq f$ with $\int f - \int \phi \leq \epsilon$. Write ϕ in standard form as $\phi = \sum_{n=1}^N a_n \chi_{A_n}$, where $a_n \neq 0$. Since ϕ is integrable, $\mu(A_n) < \infty$ for all n. Thus if $E := \bigcup_n A_n$, then $\mu(E) = \sum_n \mu(A_n) < \infty$. Also, we have $\int f - \int_E f \leq \int f - \int_E \phi = \int f - \int \phi \leq \epsilon$. $\square$

3 Let $f \in \mathcal{L}^+(\mu)$. If $\int f \, d\mu < \infty$, then $\mu(\{x \in X : f(x) = \infty\}) = 0$.

Proof. Let $E := \{x \in X : f(x) = \infty\}$. Suppose $\mu(E) \neq 0$. Let $\phi_n = n\chi_E$. Then $\int f \geq \int \phi_n \to \infty$, a contradiction. $\square$

4 Prove Fatou's Lemma without using the Monotone Convergence Theorem, and deduce the MCT from Fatou's Lemma. Fatou's lemma states that if $(f_n) \subset \mathcal{L}^+$, then $\int \liminf f_n \leq \liminf \int f_n$.

Proof. Step 1. Suppose ρ is a measure on $\mathcal{M}$, and $(E_n) \subset \mathcal{M}$. Then $\rho(\liminf E_n) = \rho(\bigcup_n \bigcap_{k\geq n} E_k) = \lim_n \rho(\bigcap_{k\geq n} E_k) \leq \lim_n \inf_{k\geq n} \rho(E_k) = \liminf_n \rho(E_n)$.

Step 2. Let ϕ be a simple function such that $0 \leq \phi \leq \liminf f_n$. Let $E_n = \{x : \phi(x) \leq f_n(x)\}$. Then

$$\begin{aligned}
\liminf E_n &= \bigcup_n \bigcap_{k\geq n} E_k \\
&= \{x : \exists n \forall k \geq n \quad \phi(x) \leq f_k(x)\} \\
&= \{x : \exists n \quad \phi(x) \leq \inf_{k\geq n} f_k(x)\} \\
&\supset \{x : \phi(x) \leq \sup_n \inf_{k\geq n} f_k(x)\} \\
&= \mathbb{R}.
\end{aligned}$$

By Step 1 applied to the measure $A \mapsto \int_A \phi$, we have $\int \phi = \int_{\liminf E_n} \phi \leq \liminf \int_{E_n} \phi \leq \liminf \int_{E_n} f_n \leq \liminf \int f_n$. Hence $\int \liminf f_n \leq \liminf \int f_n$.

For the proof of the MCT, let $(f_n) \subset \mathcal{L}^+$ with $f_n \uparrow f$. First note that $\lim \int f_n$ must exist since $(\int f_n)$ is an increasing sequence. Hence $\int f = \int \lim f_n \leq \liminf \int f_n = \lim \int f_n$. For the reverse inequality, we have $f_n \leq f$ so $\int f_n \leq \int f$. Thus, $\lim \int f_n \leq \int f$. $\square$

5 Let $f : [0,1] \to [0,1]$ be the Cantor function, and C be the Cantor set. Define $g(x) = f(x) + x$ for $x \in [0,1]$.

a) g is a bijection from $[0,1]$ to $[0,2]$ and g^{-1} is continuous.

b) $m(g(C)) = 1$.

c) Using Exercise 29/Chapter 1 show that for some nonmeasurable $A \subset g(C)$, $B = g^{-1}(A)$ is Lebesgue measurable but not Borel measurable.

d) There is a Lebesgue measurable function on $\mathcal{R}$ which is not Borel measurable.

Proof. The Cantor function f is clearly increasing. To see that it is continous, fix $\epsilon > 0$. There exists n such that $2^{-n} < \epsilon$. Let $a, b \in C$ with expansions $a = \sum_j a_j 3^{-j}$ and $b = \sum_j b_j 3^{-j}$. If $|a - b| < 3^{-n}$, then I claim $a_j = b_j$ for $j \leq n$. Suppose not. Let $J < n$ denote the first index such that $a_J \neq b_J$. WLOG $2 = a_J > b_J = 0$. Then $a - b = \sum_{j=J}^{\infty} (a_j - b_j) 3^{-j} \geq (2)3^{-J} - \sum_{j=J+1}^{\infty} (2)3^{-j} = (2)3^{-J} - (2)3^{-J-1}\frac{1}{1-1/3} = (2)3^{-J} - (2)3^{-J-1}\frac{3}{2} = 3^{-J} > 3^{-n}$, a contradiction. Hence, $a_j = b_j$ for $j \leq n$, so $|f(a) - f(b)| \leq \sum_{j=n+1}^{\infty} 2^{-j} = 2^{-n} < \epsilon$. Thus f is continuous.

Since g is the sum of an increasing function and a strictly increasing function, g is strictly increasing. In particular, g is injective. Since f is continuous, so is g. Since $g(0) = 0$ and $g(1) = 2$, the intermediate value theorem implies that g surjects onto $[0,2]$.

To see that g^{-1} is continuous, suppose $F \subset [0,1]$ is closed. Then F is compact. Let $(U_\alpha)_{\alpha \in A}$ be an open cover of $g(F)$. Then $(g^{-1}(U_\alpha))$ is an open cover of F. Let $(g^{-1}(U_\alpha))_{\alpha \in G}$ be a finite subcover of F. Then $(U_\alpha)_{\alpha \in G}$ is a subcover of $g(F)$, since g is a bijection. Thus $g(F)$ is compact, hence closed. Thus g^{-1} is continuous.

For (b), from the definition of the Cantor set we have $C^c = \bigcup_n I_n$ for some countable collection of disjoint intervals I_n with $\sum_n m(I_n) = 1$. The construction of f implies that, for each n, $f(I_n) = \{x_n\}$ for some singleton $\{x_n\}$. Hence $m(g(I_n)) = m(x_n + I_n) = m(I_n)$. Since g is a bijection, $(g(I_n))_n$ remain pairwise disjoint. Thus, $m(g(C)) = 2 - m(g(C^c)) = 2 - \sum_n m(g(I_n)) = 2 - \sum_n m(I_n) = 1$.

For (c), Exercise 29/Chapter 1 implies that there exists a nonmeasurable set $A \subset g(C)$. The proof of this exercise is simple (pick an interval $[i, i+1]$ for which $E \cap [i, i+1]$ has positive measure and then do the Vitali construction using $E \cap N_r$ instead of N_r). Since $m(C) = 0$ and the Lebesgue measure is complete, $B := g^{-1}(A)$ is Lebesgue measurable. However, if B were Borel measurable, then $A = g(B)$ would be Borel since g^{-1} is continuous hence Borel measurable. Thus, B is not Borel.

For (d), such a function is χ_B. $\square$