Paul Gustafson
Texas A&M University - Math 637
Instructor: Zoran Sunik

HW 4

7 The closure of the open disk contains all the (infinitely many) 1-cells, so the (C) condition fails.

For the (W) condition, suppose $A \cap \overline{e}$ is closed in X for every cell e. Taking e to be the 2-cell, we have that A is closed in X. The converse (that $A \subset X$ closed implies $A \cap \overline{e}$ is closed for all cells e) is obvious.

8 The set $S := \{(1, 1/n) | n \text{ a positive integer } \}$ intersects every 1-cell's closure in at most a single point; however, this set has a limit point at $(1, 0)$. Hence the set S is not closed, contradicting the (W) condition.

For the (C) condition, note that the closure of any 1-cell is simply two of the 0-cells.

9 I have attached a triangulation of $\mathbb{P}^2$ with a spanning tree and labelled directed edges. I left edges which immediately generate trivial loops unlabeled. I also gave edges corresponding to homotopic loops the same label. As a result, the only nontrivial relation is given by the triangle marked by a green circle $A^2 = 1$. Thus, $\pi_1(\mathbb{P}^2) = \mathbb{Z}_2$.

10 I did the same thing here as in (9). This time the presentation is

$$\pi_1(\mathbb{K}^2) = \langle A, B, C | ACB, ABC^{-1} \rangle = \langle A, B | A^2 B^2 \rangle.$$

11 (a) WLOG $y = 0$. Let C be the n-ball centered at $z/2$ with radius $z/2$. Pick $\epsilon > 0$ so small that the ball of radius $z/2 + \epsilon$ centered at $z/2$ remains in $\mathbb{B}^n$. Let this slightly larger ball be denoted D. Let $R \in SO(n)$ be any rotation of C swapping y and z. Let f be the extension of R to D by rotating each sphere centered at $z/2$ with radius $z/2 \leq r \leq z/2 + \epsilon$ by $R^{1-(r-z/2)/\epsilon}$. Extend f to B^n by setting it to be the identity outside of D. This map is the desired homeomorphism.

(b) Pick an open neighborhood V_x of x such that there exists a homeomorphism $V_x \to \mathbb{B}^n$. Pick any smaller open ball $U_x \subset V_x$ containing x. For any $y, z \in U_x$, apply (a) to $\overline{U_x}$ to get a homeomorphism $f : \overline{U_x} \to \overline{U_x}$ that restricts to the identity on the boundary. Extend f to M by setting it to the identity everywhere else.

(c) Let $p : I \to M$ be a path from y to z. For every point x in the image of p, let U_x be chosen as in (b). Let δ be the Lebesgue number of the cover $(p^{-1}(U_x))_x$ of I. Pick points $0 = y_1 < y_2 < \ldots < y_n = 1$ with $y_k - y_{k-1} < \delta$ for all k. Pick a finite set F such that $y_{k-1}, y_k \in p^{-1}(U_x)$ for some $x \in F$ for every $2 \leq k \leq n$. Let f_k be the homeomorphism from (b) mapping $p(y_{k-1})$ to $p(y_k)$. Since permutations of the form $(k-1 \quad k)$ generate S_n, and in particular $(1 \quad n)$, some composition of the f_k will give the desired homeomorphism.