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HW 2

1 Using the fact that $\mathcal{B}_{\mathbb{R}}$ is generated by the open intervals, show that:

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\{[a,\infty) : a \text{ rational }\})$$

Proof. It suffices to show both that $\mathcal{B}_{\mathbb{R}}$ contains $[a,\infty)$ for each $a \in \mathbb{Q}$, and that every open interval (x,y) is in $\mathcal{M}(\{[a,\infty) : a \text{ rational }\})$. The former follows from the fact that $[a,\infty) = (-\infty,a)^c$ for each $a \in \mathbb{Q}$.

For the latter, suppose (x,y) is an arbitrary open interval. Pick $(x_n),(y_n) \subset \mathbb{Q}$ with $x_n \uparrow x$ and $y_n \downarrow y$. Then $(x,y) = \bigcup_n [x_n,y_n) = \bigcup_n [x_n,\infty) \cap [y_n,\infty)^c$. $\square$

2 Problem 1/Page 24. A *ring* is a nonempty family of sets closed under finite unions and differences. A ring that is closed under countable unions is called a *σ-ring.*

a. Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.

b. If $\mathcal{R}$ is a ring (resp. σ-ring) , then R is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.

c. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

d. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Proof. For (a), let $\mathcal{R}$ be a ring, and $U, V \in \mathcal{R}$. Let $W = U \cup V$. Then $U \cap V = W \setminus ((W \setminus U) \cup (W \setminus V))$. This is just one of De Morgan's laws in the restricted universe W. A similar argument works for σ-rings with W the countable union of the sets involved.

For (b), let $\mathcal{R}$ be a ring (resp. σ-ring). Suppose $X \in \mathcal{A}$. Since (a) has been verified, we need only check that $\mathcal{R}$ contains complements. This is true since $E^c = X \setminus E$ for any set E. Conversely, suppose $\mathcal{R}$ is an algebra (resp. σ-algebra). Then $\mathcal{R}$ is nonempty, so there exists $E \in \mathcal{R}$. Thus, $X = E \cup E^c \in \mathcal{R}$.

For (c), let $\mathcal{M} = \{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$. Since $\mathcal{R}$ is nonempty, so is $\mathcal{M}$. It is also clear that $\mathcal{M}$ is closed under complements. For closure under countable unions, let $\mathcal{E}$ be a countable subset of $\mathcal{M}$. Then $\mathcal{E} = \mathcal{A} \cup \mathcal{B}$ where $\mathcal{A} := \{E \in \mathcal{E} : E \in \mathcal{R}\}$ and $\mathcal{B} := \{E \in \mathcal{E} : E^{\in}\mathcal{R}\}$. We also have $A := \bigcup \mathcal{A} \in \mathcal{R}$ and $B := \bigcap_{B \in \mathcal{B}} B^c \in \mathcal{R}$. Hence $\bigcup \mathcal{E} = \bigcup \mathcal{A} \cup \bigcup \mathcal{B} = A \cup B^c = (A^c \cap B)^c = (B \setminus A)^c \in \mathcal{M}$.

For (d), let $\mathcal{M} = \{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. Since $\mathcal{R}$ is nonempty, there exists $E \in \mathcal{R}$. Hence $\emptyset = E \setminus E \in \mathcal{R}$. Then it follows from the definition of $\mathcal{M}$ that $\emptyset \in \mathcal{M}$. In particular, $\mathcal{M}$ is nonempty. To see that $\mathcal{M}$ is closed under complements, suppose $E \in \mathcal{M}$. Let $F \in \mathcal{R}$. Then $E^c \cap F = F \setminus E \in \mathcal{R}$. Hence, $E^c \in \mathcal{M}$. For closure under countable unions, let $(E_n) \subset \mathcal{M}$. Let $F \in \mathcal{R}$. Then $\bigcup_n (E_n) \cap F = \bigcup_n (E_n \cap F) \in \mathcal{R}$. Hence, $\bigcup_n E_n \in \mathcal{M}$. $\square$

3 Problem 5/Page 24. $\mathcal{M}(\mathcal{E})$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

Proof. Let

$$\mathcal{H} = \{\mathcal{F} \subset \mathcal{E} : \mathcal{F} \text{ is countable}\},$$

and $\mathcal{U} = \bigcup_{\mathcal{F}\in\mathcal{H}} \mathcal{M}(\mathcal{F})$.

Let $\mathcal{F} \subset \mathcal{E}$ be countable. Then $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E}))$. Hence, $\mathcal{U} \subset \mathcal{M}(\mathcal{E}))$. For the reverse inclusion, it suffices to show that $\mathcal{U}$ is a σ-algebra, for then $\mathcal{U}$ is a σ-algebra containing (E), hence containing $\mathcal{M}(\mathcal{E})$.

To see that $\mathcal{U}$ is a σ-algebra, first note that $\emptyset \in \mathcal{H}$, so $\mathcal{U}$ is nonempty. To see that $\mathcal{U}$ is closed under taking complements, let $E \in \mathcal{U}$. Then $E \in \mathcal{M}(\mathcal{F})$ for some countable $\mathcal{F} \subset \mathcal{E}$, so $E^c \in \mathcal{M}(\mathcal{F}) \subset \mathcal{U}$.

For closure under countable union, let $(U_n) \subset \mathcal{U}$. Then each $U_n \in \mathcal{M}(\mathcal{G}_n)$ for some $\mathcal{G}_n \in \mathcal{H}$. Let $\mathcal{G} = \bigcup_n \mathcal{G}_n$. Then $\mathcal{G}$ is countable, and $U_n \in \mathcal{M}(\mathcal{G})$ for all n. Hence, $\bigcup_n U_n \in \mathcal{M}(\mathcal{G}) \subset \mathcal{U}$. $\square$

4 Show that every σ-algebra has either finite or uncountable many elements.

Proof. Suppose that $\mathcal{M} \subset \mathcal{P}(X)$ is an infinite σ-algebra.

Step 1: Show that $\mathcal{M}$ contains a sequence of disjoint nonempty sets.

Case 1: Assume $\mathcal{M}$ contains an infinite linearly inclusion-ordered subset $\mathcal{L}$. Let $(E_n)_{n=1}^\infty \subset \mathcal{L}$ be a pairwise distinct sequence of sets. I claim that (E_n) must have a monotone subsequence. Suppose not. Then $(E_n)_{n=1}^\infty$ must be bounded above by some E_{n_1}, for otherwise, given any E_n, there exists $m > n$ with $E_n \subset E_m$. This would define an ascending sequence. Similarly, $(E_n)_{n=n_1}^\infty$ must be bounded below by say E_{m_1}. Then $(E_n)_{n=m_1}^\infty$ must be bounded above by some $E_{n_2} \subset E_{n_1}$. Continuing in this way, we get a subsequence $E_{n_1} \supset E_{n_2} \supset \ldots$, a contradiction.

Since the E_n were distinct, this implies that (E_n) contains either a strictly ascending subsequence (A_n) or a strictly descending subsequence (D_n). In the former case, let $B_n = A_n \setminus (\bigcup_{k=1}^{n-1} A_k)$. In the latter case, let $B_n = D_n \setminus D_{n+1}$. In either case, (B_n) is a sequence of disjoint nonempty sets.

Case 2: every linearly inclusion-ordered subset of $\mathcal{M}$ is finite. Let $\mathcal{L}_1$ be a maximal linearly ordered subset of $\mathcal{M} \setminus \{\emptyset\}$. Since $\mathcal{M}$ is infinite, $\mathcal{L}_1$ must be nonempty.

Inductively assume we are given nonempty finite chains $(\mathcal{L}_i)_{i=1}^n \subset \mathcal{P}(\mathcal{M} \setminus \{\emptyset\})$ and sets $\mathcal{F}_i := \{\bigcup_{k=1}^{i-1} F_k : \forall k\, [F_k \in \mathcal{L}_k \cup \{\emptyset\}]\}$ such that each $\mathcal{L}_i$ is maximal in $\mathcal{M} \setminus (\mathcal{F}_i \cup \{\emptyset\})$. Further inductively suppose that the minimal elements of $(\mathcal{L}_i)$ are pairwise disjoint.

Let E_i denote the minimal element of $\mathcal{L}_i$ for each i. Since the $\mathcal{L}_i$ are finite, the set $\mathcal{F}_{n+1} := \{\bigcup_{i=1}^n F_i : \forall i\, [F_i \in \mathcal{L}_i \cup \{\emptyset\}]\}$ is a finite set. Hence there exists a maximal nonempty chain $\mathcal{L}_{n+1} \subset \mathcal{M} \setminus (\mathcal{F}_{n+1} \cup \{\emptyset\})$.

By the Case 2 assumption, $\mathcal{L}_{n+1}$ is finite, so it contains a minimal element E_{n+1}. Suppose $F := E_{n+1} \cap E_k \neq \emptyset$ for some $1 \le k \le n$. First note that $E_{n+1} \neq E_k$ since $E_k \in \mathcal{F}_{n+1}$. Hence $F \neq E_{n+1}$, for otherwise $E_{n+1} \subsetneq E_k$, which contradicts the maximality of $\mathcal{L}_k$. Thus $F \in \mathcal{F}_{n+1}$, for otherwise F contradicts the maximality of $\mathcal{L}_{n+1}$.

Since $F \neq \emptyset$, we have $G := E_{n+1} \setminus F \in \mathcal{F}_{n+1}$ since $\mathcal{L}_{n+1}$ is maximal. Hence we can write $G = \bigcup_{i=1}^n G_i$ and $F = \bigcup_{i=1}^n F_i$ with $G_i, F_i \in \mathcal{L}_i \cup \{\emptyset\}$. Then $E_{n+1} = F \cup G = \bigcup_{i=1}^n (F_i \cup G_i)$, which is in $\mathcal{F}_{n+1}$ since each $F_i \cup G_i \in \mathcal{L}_i \cup \{\emptyset\}$. This contradicts the fact that $E_{n+1} \in \mathcal{L}_{n+1} \subset \mathcal{M} \setminus (\mathcal{F} \cup \{\emptyset\})$.

Hence $(E_i)_{i=1}^{n+1}$ are disjoint, and the other induction hypotheses also hold at $n+1$. Thus, by induction, we have the sequence $(E_n)_{n=1}^\infty$ of disjoint nonempty sets.

Step 2: Show that $\mathcal{M}$ is uncountable. From Step 1, there exists a sequence $(M_n) \subset \mathcal{M}$ of disjoint nonempty sets. Define $f : \mathcal{P}(\mathbb{N}) \to \mathcal{M}$ by $f(U) = \bigcup_{u \in U} M_u$. To see that f is injective, suppose that $U \neq V$. WLOG there exists $t \in U \setminus V$. Since M_t is nonempty, there exists $x \in M_t$, so $x \in \bigcup_{u \in U} M_u = f(U)$. Since the (M_n) are disjoint, $x \notin M_n$ for $n \neq u$. Hence, $x \notin \bigcup_{v \in V} M_v = f(V)$. Thus, $f(U) \neq f(V)$, so f is injective. Thus $\operatorname{card}(\mathcal{M}) \geq \operatorname{card}(\mathcal{P}(\mathbb{N}) > \operatorname{card}(\mathbb{N})$. $\square$

5 Let $(\Omega_j, \mathcal{M}_j)$ be measure spaces for $j \in [n]$. Show that

$$\mathcal{E} = \left\{ \prod_{j=1}^n E_j : E_j \in \mathcal{M}_j \forall j \right\}$$

is an elementary system.

Proof. Since $\emptyset \in \mathcal{M}_j$ for all j, we have $\emptyset = \prod_{j=1}^n \emptyset \in \mathcal{E}$. Now suppose $E, F \in \mathcal{E}$. Then $E = \prod_j E_j$ and $F = \prod_j F_j$ for $E_j, F_j \in \mathcal{M}_j$ for all j. Hence $E \cap F = \prod_j (E_j \cap F_j) \in \mathcal{E}$. Lastly, we need to check that E^c is the finite union of disjoint elements of $\mathcal{E}$. Let

$$\mathcal{U} = \{\prod U_j : U_j \in \{E_j, E_j^c\}\}$$

. Note that $\mathcal{U}$ is a partition of $\prod_j \Omega_j$, and $\mathcal{U} \subset \mathcal{E}$. Hence $E^c = \bigcup(\mathcal{U} \setminus \{E\})$ is a finite union of disjoint elements of $\mathcal{E}$. $\square$