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HW 1

J1.1.1 Show that no Banach space has a countable Hamel basis.

Proof. Let X be a Banach space. Suppose $(x_i)_{i=1}^{\infty} \subset X$ is a countable Hamel basis. Let $X_n = \text{span}((x_i)_{i=1}^n)$ for each n. I claim each X_n is nowhere dense. Suppose X_n contained some ball $B_\epsilon(x)$. Then it contains $B_\epsilon(0)$, which contains a multiple of x_{n+1}, a contradiction. Hence, each X_n is nowhere dense. But $U = \bigcup_n X_n$, which contradicts the Baire Category Theorem. □

J1.2 Show that l_∞ is not separable.

Proof. Let $U \subset l_\infty$ be the set of all sequences consisting of only 0 and 1. Note that $|U| = |P(\mathbb{N})| > |\mathbb{N}|$. Let $x, y \in U$ be distinct. Then $\|x - y\| = 1$, so $B_{1/2}(x) \cap B_{1/2}(y) = \emptyset$. Thus, $(B_{1/2}(x))_{x \in U}$ is an uncountable set of disjoint balls, so l_∞ cannot be separable. □

J1.3 Show that every Banach space with a basis is separable.

Proof. Let (x_n) be a basis for the Banach space X. By normalizing, we can assume $\|x_n\| = 1$ for all n. Let Q be a countable dense set in the scalar field. Let $U = \bigcup_n \bigoplus_{k=1}^n Qx_k$. Then U is countable since it is the countable union of countable sets. To see that U is dense, let $x \in B$ and $\epsilon > 0$. Then $x = \sum_n a_n x_n$ for some scalars a_n. Pick N such that $\|x - \sum_{n=1}^N a_n x_n\| < \epsilon/2$. For $n \leq N$, pick q_n within $\frac{\epsilon}{2N}$ of a_n. Then $\|x - \sum_{n=1}^N q_n x_n\| \leq \|x - \sum_{n=1}^N a_n x_n| + \| \sum_{n=1}^N (a_n - q_n) x_n\| \leq \epsilon/2 + \sum_{n=1}^N |a_n - q_n| < \epsilon$. □

J1.4 Find a basis (x_n) for a normed space X for which some $x_1^\#$ is not continuous. (Hint: consider the algebraic span of $(e_i) \subset l_2$).

Let $X = \text{span}\,(e_i)_{i=1}^\infty \subset l_2(\mathbb{N})$. Define $x_1 = e_1$ and $x_n = ne_1 + e_n$ for $n \geq 2$. Note that (x_n) span X since they span $(e_i)_{i=1}^n$. To see that (x_n) is a basis, suppose $\sum_{n=1}^\infty a_n x_n = 0$. If $a_k \neq 0$ for some $k \geq 2$, then $e_k^*(\sum_{n=1}^\infty a_n x_n) = \sum_n a_n e_k^*(x_n) = a_k$, a contradiction. Hence $a_k = 0$ for $k \geq 2$, so $a_1 = 0$ also.

To see that $x_1^\#$ is not continuous, note that for $n \geq 2$ we have $|x_1^\#(e_n)| = |x_1^\#(-nx_1 + x_n)| = n \to \infty$.