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Final

1 Let $f \in C[0, 2\pi]$ with $f(0) = f(2\pi)$. Let $Q_{trap}(f) = \frac{2\pi}{n} \sum_{k=0}^{n-1} f\left(\frac{2\pi k}{n}\right)$. Let $E_n = \left| \int_0^{2\pi} f(x)dx - Q_n(f) \right|$.

(a) Show $Q_{trap}(e^{ikx}) = \begin{cases} 0 & k \not\equiv 0 \pmod{n} \\ 2\pi & k \equiv 0 \pmod{n} \end{cases}$

Proof. If $k \equiv 0 \pmod{n}$, we have $Q_{trap} = \frac{2\pi}{n} \sum_{j=0}^{n-1} e^{\frac{2\pi ijk}{n}} = \frac{2\pi}{n} \sum_{j=0}^{n-1} 1 = 2\pi$.

Otherwise, we have

$$\begin{aligned} Q_{trap}(e^{ikx}) &= \frac{2\pi}{n} \sum_{j=0}^{n-1} e^{\frac{2\pi ijk}{n}} \\ &= \frac{2\pi}{n} \frac{1 - e^{2\pi ik}}{1 - e^{\frac{2\pi ik}{n}}} \\ &= 0 \end{aligned}$$

□

(b) Let $f(x)$ be the 2π-periodic function that equals $x^2(2\pi - x)^2$ when $x \in [0, 2\pi]$. Show that $\int_0^{2\pi} f(x)dx = 16\pi^5/15$. Prove that $E_n \le Cn^{-4}$. (Hint: $f(x) = \frac{8\pi^4}{15} - \frac{24}{\pi} \sum_{k \neq 0} e^{ikx} k^{-4}$.)

Proof. We have

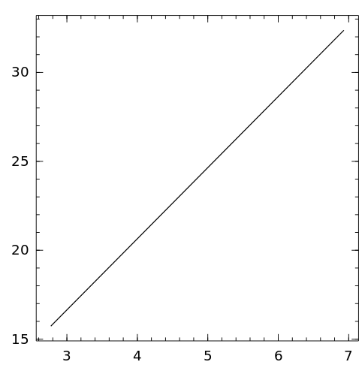
$$\begin{aligned} \int_0^{2\pi} x^2(2\pi - x)^2 dx &= \int_0^{2\pi} 4\pi^2 x^2 - 4\pi x^3 + x^4 \, dx \\ &= \frac{4}{3}\pi^2(2\pi)^3 - \pi(2\pi)^4 + \frac{1}{5}(2\pi)^5 \\ &= \frac{4}{3}\pi^2(2\pi)^3 - \pi(2\pi)^4 + \frac{1}{5}(2\pi)^5 \\ &= (32 \cdot 5 - 16 \cdot 15 + 32 \cdot 3)\pi^5/15 \\ &= 16\pi^5/15 \end{aligned}$$

On the midterm we calculated the Fourier expansion for f, so I will assume the hint without proof. Thus

$$\begin{aligned}
E_n &= |\frac{16\pi^5}{15} - Q_n(\frac{8\pi^4}{15} - \frac{24}{\pi}\sum_{k\geq 0} e^{ikx}k^{-4})| \\
&= |\frac{C}{n}\sum_{j=0}^{n-1}\sum_{k\geq 0} e^{2\pi ikj/n}k^{-4})| \\
&= |\frac{C}{n}\sum_{k\geq 0} k^{-4}\sum_{j=0}^{n-1} e^{2\pi ikj/n}| \\
&= C'\sum_{k\geq 0} k^{-4}Q_n(e^{ikx}) \\
&= C'\sum_{k\geq 0} 2\pi(nk)^{-4} \\
&= C''n^{-4}
\end{aligned}$$

where the interchange of summation is justified by the absolute summability of the series. □

(c) Use Matlab or some other program to plot $\log(E_n)$ vs. $\log n$ for $n = 16, 24, 256, 1024$. This should be a straight line. What is its slope? Does it agree with what you found in (b)?



Proof.

The above plots $\log n$ on the x-axis and $\log E_n$ on the y-axis. The slope according to (b) should be -4. The graph seems to agree. □

2 Let $\mathcal{H}$ be a complex Hilbert space, and let $L \in \mathcal{B}(\mathcal{H})$.

1. Verify that $\langle L(u+e^{i\alpha}v), u+e^{i\alpha}v\rangle - \langle L(u-e^{i\alpha}v), u-e^{i\alpha}v\rangle = 2e^{-i\alpha}\langle Lu, v\rangle + 2e^{-i\alpha}\langle Lv, u\rangle$

Proof. We have

$$\begin{aligned}\langle L(u+e^{i\alpha}v), u+e^{i\alpha}v\rangle - \langle L(u-e^{i\alpha}v), u-e^{i\alpha}v\rangle &= \langle Lu,u\rangle + \langle Lu, e^{i\alpha}v\rangle + \langle e^{i\alpha}v), u\rangle + \langle u,v\rangle \\ &\quad - \langle Lu,u\rangle + \langle Lu, e^{i\alpha}v\rangle + \langle e^{i\alpha}v), u\rangle - \langle u,v\rangle \\ &= 2e^{-i\alpha}\langle Lu,v\rangle + 2e^{-i\alpha}\langle Lv,u\rangle\end{aligned}$$

□

2. Show that if $L = L^*$, then $\|L\| = \sup_{\|u\|=1} |\langle Lu, u\rangle|$.

Proof. Acknowledgement: I looked at http://www.math.washington.edu/ hart/m556/lecture2.pdf for hints.

By Cauchy-Schwarz, we have $\sup_{\|u\|=1} |\langle Lu,u\rangle| \leq \sup_{\|u\|=1} \|Lu\| = \|L\|$.

For the reverse inequality, recall from a previous homework problem that $\|L\| = \sup_{\|u\|=1,\|v\|=1} |\langle Lu, v\rangle\|$. Pick $(u_n),(v_n)$ such that $|\langle Lu_n, v_n\rangle| \to \|L\|$. Pick $\alpha_n \in [0,2\pi]$ such that $e^{i\alpha_n}\langle Lu_n, v_n\rangle = |\langle Lu_n, v_n\rangle|$. Let $w_n = e^{-i\alpha_n}v_n$. Thus, $\langle Lu_n, w_n\rangle \to \|L\|$. In particular, $\|L\| \leq \sup_{\|u\|=1,\|v\|=1} \Re\langle Lu, v\rangle$. Since $\Re\langle Lu,v\rangle \leq |\langle Lu,v\rangle|$, we have $\|L\| = \sup_{\|u\|=1,\|v\|=1} \Re\langle Lu, v\rangle$.

For $\|u\| = \|v\| = 1$, we have

$$\begin{aligned}\Re\langle Lu,v\rangle &= \frac{1}{2}(\langle Lu,v\rangle + \langle v, Lu\rangle) \\ &= \frac{1}{2}(\langle Lu,v\rangle + \langle Lv,u\rangle) \\ &= \frac{1}{4}(\langle L(u+v), u+v\rangle - \langle L(u-v), u-v\rangle) \\ &\leq \frac{1}{4}(\|u+v\|^2 + \|u-v\|^2) \sup_{\|u\|=1} |\langle Lu,u\rangle| \\ &\leq \frac{1}{2}(\|u\|^2 + \|v\|^2) \sup_{\|u\|=1} |\langle Lu,u\rangle| \\ &\leq \frac{1}{2}(\|u\|^2 + \|v\|^2) \sup_{\|u\|=1} |\langle Lu,u\rangle| \\ &= \sup_{\|u\|=1} |\langle Lu,u\rangle|\end{aligned}$$

□

3. Show that if $M = \sup_{\|u\|=1} |\langle Lu, u\rangle|$, then $M \leq \|L\| \leq 2M$, whether or not L is self-adjoint. Give an example that shows that this result is false in a real Hilbert space.

Proof. By Cauchy-Schwarz, we have $M = \sup_{\|u\|=1} |\langle Lu,u\rangle| \leq \sup_{\|u\|=1} \|Lu\| = \|L\|$.

For the other inequality, let $\|u\| = \|v\| = 1$ and $\alpha \in \mathbb{R}$. By part (a),

$$\begin{aligned}
\Re\langle Lu, v\rangle| &= \frac{1}{2}\Re(\langle L(u + e^{i\alpha}v), u + e^{i\alpha}v\rangle - \langle L(u - e^{i\alpha}v), u - e^{i\alpha}v\rangle - 2e^{-i\alpha}\langle Lv, u\rangle) \\
&\le \frac{M}{2}(\|u + e^{i\alpha}v\|^2 + \|u - e^{i\alpha}v\|^2) - \Re(2e^{-i\alpha}\langle Lv, u\rangle) \\
&= M(\|u\|^2 + \|e^{i\alpha}v\|^2) - \Re(2e^{-i\alpha}\langle Lv, u\rangle) \\
&= 2M - \Re(2e^{-i\alpha}\langle Lv, u\rangle)
\end{aligned}$$

By picking α appropriately, we can ensure that $e^{-i\alpha}\langle Lv, u\rangle$ is a nonnegative real number. Thus $\Re\langle Lu, v\rangle| \le 2M$. Thus, $\|L\| \le 2M$.

For the counterexample in the real case, let $\mathcal{H} = \mathbb{R}^2$ with the standard inner product. Let

$$L = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

. Then $\langle Lu, u\rangle = 0$ for all u, but $\|L\| = 1$. □

3 Let $K \in \mathcal{C}(\mathcal{H})$ be self adjoint. Show that the only possible limit point of the set of eigenvalues of K is 0.

Proof. Let $\epsilon > 0$. Let $\mathcal{E}_\lambda$ be the eigenspace corresponding to an eigenvector λ. Let $X = \bigoplus_{|\lambda|>\epsilon} \mathcal{E}_\lambda$. Then X is an invariant subspace for K, and K maps the unit ball B_X of X to a precompact set. Moreover $\epsilon B_X \subset KB_x$. But this implies that B_X is precompact. Hence X must be finite-dimensional, so there can only be finitely many λ with $|\lambda| > \epsilon$. □

4 Let $K \in \mathcal{C}(\mathcal{H})$ be self adjoint. Suppose the range of K contains a dense subset of $\mathcal{H}$, and that an o.n. basis has been chosen for the igenspace of each nonzero eigenvalue of K. Show that the set of all these eigenvectors form a complete orthormal set.

Proof. By a theorem in class, we know that this set (call it S) is orthogonal. Thus we only need to show that the span of S is dense in $\mathcal{H}$. The spectral theorem states that an o.n. basis B of eigenvectors for K exists. We can write each vector in B as a sum of vectors in S since S spans each eigenspace. Thus, $\operatorname{span} S = \operatorname{span} B = \mathcal{H}$. □

5 Let $\mathcal{H} = L^2[0, 1]$. Consider the boundary value problem

$$Lu := \frac{d}{dx}\left((1 + x)\frac{du}{dx}\right) = f(x), u(0) = 0, u'(1) = 0.$$

(a) Find $G(x, y)$, the Green's function for this BVP.

Proof. The fundamental set for this BVP is $\{1, \log(1+x)\}$. The Wronskian of these two functions is $W(x) = \log(1+x)$. Hence, the Green's function is

$$G(x,y) = \begin{cases} \frac{\log(1+x)}{\log(1+y)}, & \text{if } 0 \leq x \leq y \leq 1 \\ \frac{\log(1+y)}{\log(1+x)}, & \text{if } 0 \leq y \leq x \leq 1 \end{cases}$$

□

(b) Let $Gf(x) = \int_0^1 G(x,y)f(y)dy$. Show that the range of G contains a dense set in $\mathcal{H}$.

Proof. Let S be the set of $u \in C^2[0,1]$ with support in $(0,1)$. Note that if $u \in S$ then u satisfies the boundary conditions of the BVP. Hence, we can just plug u into the differential equation to find f such that $Gf = u$. Thus the range of G contains S.

To see that S is dense in $\mathcal{H}$, recall that functions of the form $g(x) = e^{inx}$ form a basis for $L^2[0,1]$. Thus, it suffices to approximate $g(x) = e^{inx}$ by elements of S in the L^2 norm. Letting $\epsilon > 0$, we can cut off the $[0, \epsilon/4)$ and $(1-\epsilon/4, 1]$ ends off g and replace them with a C^2 spline on each end which agree up to second derivatives with g at $\epsilon/4$ and $1-\epsilon/4$. Moreover, we can ensure that the splines are 0 near 0 and 1, respectively, and have sup-norm 2. This new function h is in S and $\|g-h\|_{L^2} < \epsilon$. □

(c) Use it and the previous problem to show that the eigenfunctions for $\frac{d}{dx}\left((1+x)\frac{du}{dx}\right) + \lambda u = 0, u(0) = 0, u'(1) = 0$ form a complete orthogonal set.

Proof. The the function $G(x,y)$ is bounded hence L^2. Thus, the operator G is Hilbert-Schmidt hence compact. Moreover, $G(x,y)$ is symmetric so G is self-adjoint. Thus we can apply part (b) and problem (4) to get that the eigenfunctions form a complete orthogonal set. □

6 Let $\|\cdot\|_{op}$ be the operator norm for $\mathcal{B}(\mathcal{H})$.

1. Show that $(\mathcal{B}(\mathcal{H}), \|\cdot\|_{op})$ is a Banach space.

   Proof. We need to show that $\|\cdot\|_{op}$ is positive definite, homogeneous, and satisfies the triangle inequality.

   It is clearly positive. Moreover, if $\|T\|_{op} = 0$, then $\|Tu\| = 0$ for all $\|u\| = 1$. Hence $Tv = \|v\|T\frac{v}{\|v\|} = 0$ for all $v \in \mathcal{H} \setminus \{0\}$. Thus $Tv = 0$.

   For the homogeneity, we have $\|cT\|_{op} = \sup_{\|u\|=1} |c|\|Tu\| = |c|\|T\|_{op}$ for all $c \in \mathbb{R}$ and $T \in \mathcal{H}$.

   For the triangle inequality, we have $\|S+T\|_{op} = \sup_{\|u\|=1} \|Su + Tu\| \leq \sup_{\|u\|=1} \|Su\| + \|Tu\| \leq \sup_{\|u\|=1} \|Su\| + \sup_{\|v\|=1} \|Tv\| = \|S\|_{op} + \|T\|_{op}$. □

2. Consider the operator $L = I - \lambda M$, with $M \in \mathcal{B}(\mathcal{H})$. Show that if $|\lambda| < \|M\|_{op}^{-1}$, then, in the operator norm, $\sum_{k=0}^{\infty} \lambda^k M^k = (I - \lambda M)^{-1}$.

   Proof. First note that $\|(I - \overline{\lambda} M^*)u\| \geq \|u\| - \|\overline{\lambda} M^* u\| > 0$ for all $u \neq 0$. Thus, $\mathcal{N}(I - \overline{\lambda} M^*) = \{0\}$ so L is invertible by the Fredholm Alternative theorem.

   For $\|u\| = 1$, write $u = (I - \lambda M)v$. Then

$$\begin{aligned}
((I - \lambda M)^{-1} - \sum_{k=0}^{K} \lambda^k M^k)u &= v - \sum_{k=0}^{K} \lambda^k M^k (I - \lambda M)v \\
&= v - \sum_{k=0}^{K} \lambda^k M^k + \sum_{k=0}^{K} \lambda^{k+1} M^{k+1} v \\
&= v - \sum_{k=0}^{K} \lambda^k M^k + \sum_{k=0}^{K} \lambda^{k+1} M^{k+1} v \\
&= \lambda^{k+1} M^{k+1} v \\
&\leq \lambda^{k+1} M^{k+1} \|(I - \lambda M)^{-1}\| \\
&\to 0
\end{aligned}$$

   as $k \to \infty$. Since the last estimate is uniform in u, the converge holds in the operator norm. □

7 Show that if B, B^{-1} are in $\mathcal{B}(\mathcal{H})$, and $K \in \mathcal{C}(\mathcal{H})$, then the range of $L = B + \lambda K$ is closed.

Proof. Suppose $Lx_n \to y$. We need to find x such that $Lx = y$. By passing to a subsequence, since λK is compact, we may assume $\lambda K x_n \to z$. Then $Bx_n \to y - z$. Since B^{-1} is continuous, we have $x_n \to B^{-1}(y - z)$.

Let $x = B^{-1}(y - z)$. Then $Lx = LB^{-1}(y - z) = (y - z) + \lambda K B^{-1}(y - z) = (y - z) + \lambda K \lim_n x_n = y$. □