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HW 1

1 Write the control system on $M = \mathbb{R}^2 \times \mathbb{T}^3$ corresponding to the car with two off-hook trailers system.

Proof. Let $n_i = (\cos\theta_i, \sin\theta_i)$ and $n_i' = (-\sin\theta_i, \cos\theta_i)$ for $0 \le i \le 2$. Then $n_i \cdot n_j = \cos(\theta_i - \theta_j) = n_i' \cdot n_j'$ and $n_i \cdot n_j' = \sin(\theta_i - \theta_j)$.

Let v_2 denote the velocity of the car, and v_i denote the velocity of the $(n-i)$-th trailer. Let $v_{1.5}$ denote the velocity of the first hook, and $v_{0.5}$ denote the velocity of the second hook. Let $\omega_i = \frac{\partial\theta_i}{\partial t}$.

In the case of linear motion of the car, we have $v_2 = vn_2$ and $\omega_2 = 0$. Hence,

$$v_{1.5} = vn_2$$

$$\begin{aligned} v_1 &= (v_{1.5} \cdot n_1)n_1 \\ &= (vn_2 \cdot n_1)n_1 \\ &= v\cos(\theta_2 - \theta_1)n_1 \end{aligned}$$

$$\begin{aligned} \omega_1 &= v_{1.5} \cdot n_1' \\ &= vn_2 \cdot n_1' \\ &= v\sin(\theta_2 - \theta_1) \end{aligned}$$

$$\begin{aligned} v_{0.5} &= v_1 - \omega_1 n_1' \\ &= v\cos(\theta_2 - \theta_1)n_1 - v\sin(\theta_2 - \theta_1)n_1' \end{aligned}$$

$$\begin{aligned} \omega_0 &= v_{0.5} \cdot n_0' \\ &= v\cos(\theta_2 - \theta_1)n_1 \cdot n_0' - v\sin(\theta_2 - \theta_1)n_1' \cdot n_0' \\ &= v\cos(\theta_2 - \theta_1)\sin(\theta_1 - \theta_0) - v\sin(\theta_2 - \theta_1)\cos(\theta_1 - \theta_0) \\ &= v\sin((\theta_1 - \theta_0) - (\theta_2 - \theta_1)) \\ &= v\sin(2\theta_1 - \theta_0 - \theta_2). \end{aligned}$$

For the case of the car turning, we have $v_2 = 0$ and $\omega_2 = \omega$. Hence,

$$v_{1.5} = -\omega n_2'$$

$$\begin{aligned} v_1 &= (v_{1.5} \cdot n_1) n_1 \\ &= (-\omega n_2' \cdot n_1) n_1 \\ &= \omega \sin(\theta_2 - \theta_1) n_1 \end{aligned}$$

$$\begin{aligned} \omega_1 &= v_{1.5} \cdot n_1' \\ &= -\omega n_2' \cdot n_1' \\ &= -\omega \cos(\theta_2 - \theta_1) \end{aligned}$$

$$\begin{aligned} v_{0.5} &= v_1 - \omega_1 n_1' \\ &= \omega \sin(\theta_2 - \theta_1) n_1 + \omega \cos(\theta_2 - \theta_1) n_1' \end{aligned}$$

$$\begin{aligned} \omega_0 &= v_{0.5} \cdot n_0' \\ &= \omega \sin(\theta_2 - \theta_1) n_1 \cdot n_0' + \omega \cos(\theta_2 - \theta_1) n_1' \cdot n_0' \\ &= \omega \sin(\theta_2 - \theta_1) \sin(\theta_1 - \theta_0) + \omega \cos(\theta_2 - \theta_1) \cos(\theta_1 - \theta_0) \\ &= \omega \cos(2\theta_1 - \theta_0 - \theta_2) \end{aligned}$$

Hence the control system for M is given by the family of vector fields $\mathcal{F} = \{\pm X_1, \pm X_2\}$, where

$$X_1 = \cos(\theta_2) \frac{\partial}{\partial x} + \sin(\theta_2) \frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + A \frac{\partial}{\partial \theta_0}$$

with $A = \sin(2\theta_1 - \theta_0 - \theta_2)$, and

$$X_2 = \frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + B \frac{\partial}{\partial \theta_0}$$

with $B = \cos(2\theta_1 - \theta_0 - \theta_2)$. $\square$

2 Find all points $q \in M$ such that $\mathcal{F}$ is bracket-generating. At these points, calculate the degree of nonholonomy of $\mathcal{F}$.

Proof. Hence,

$$\begin{aligned}
[X_1, X_2] = \Bigg[&\cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + A\frac{\partial}{\partial \theta_0}, \\
&\frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + B\frac{\partial}{\partial \theta_0}\Bigg] \\
= {}&\sin(\theta_2 - \theta_1)\left(-\sin(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial B}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + A\frac{\partial B}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
&- \left(-\sin(\theta_2)\frac{\partial}{\partial x} + \cos(\theta_2)\frac{\partial}{\partial y} + \cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_2}\frac{\partial}{\partial \theta_0}\right) \\
&+ \cos(\theta_2 - \theta_1)\left(-\cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) - B\frac{\partial A}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
= {}&\sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} \\
&+ \left(-\sin^2(\theta_2 - \theta_1) - \cos(\theta_2 - \theta_1) - \cos^2(\theta_2 - \theta_1)\right)\frac{\partial}{\partial \theta_1} \\
&+ \left(\sin(\theta_2 - \theta_1)\frac{\partial B}{\partial \theta_1} + A\frac{\partial B}{\partial \theta_0} - \frac{\partial A}{\partial \theta_2} + \cos(\theta_2 - \theta_1)\frac{\partial A}{\partial \theta_1} - B\frac{\partial A}{\partial \theta_0}\right)\frac{\partial}{\partial \theta_0} \\
= {}&\sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + (-1 - \cos(\theta_2 - \theta_1))\frac{\partial}{\partial \theta_1} \\
&+ \left(\sin(\theta_2 - \theta_1)(-2A) + A^2 + B + \cos(\theta_2 - \theta_1)(2B) - B(-B)\right)\frac{\partial}{\partial \theta_0} \\
= {}&\sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + (-1 - \cos(\theta_2 - \theta_1))\frac{\partial}{\partial \theta_1} \\
&+ \left(2\cos((\theta_2 - \theta_1) + (2\theta_1 - \theta_0 - \theta_2)) + B + 1\right)\frac{\partial}{\partial \theta_0} \\
= {}&\sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + C\frac{\partial}{\partial \theta_1} + D\frac{\partial}{\partial \theta_0},
\end{aligned}$$

where $C = -1 - \cos(\theta_2 - \theta_1)$ and $D = 2\cos(\theta_1 - \theta_0) + \cos(2\theta_1 - \theta_0 - \theta_2) + 1$.

Hence

$$\begin{aligned}
\frac{\partial D}{\partial \theta_2} &= \sin(2\theta_1 - \theta_0 - \theta_2) \\
\frac{\partial D}{\partial \theta_1} &= -2\sin(\theta_1 - \theta_0) - 2\sin(2\theta_1 - \theta_0 - \theta_2) \\
\frac{\partial D}{\partial \theta_0} &= 2\sin(\theta_1 - \theta_0) + \sin(2\theta_1 - \theta_0 - \theta_2)
\end{aligned}$$

Then

$$
\begin{aligned}
[X_1,[X_1,X_2]] &= \left[\cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \sin(\theta_2-\theta_1)\frac{\partial}{\partial \theta_1} + A\frac{\partial}{\partial \theta_0},\right. \\
&\qquad \left.\sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + C\frac{\partial}{\partial \theta_1} + D\frac{\partial}{\partial \theta_0}\right] \\
&= \sin(\theta_2-\theta_1)\left(\frac{\partial C}{\partial \theta_1}\frac{\partial}{\partial \theta_1} + \frac{\partial D}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + A\frac{\partial D}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
&\qquad - C\left(-\cos(\theta_2-\theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + D\frac{\partial A}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
&= \left(\sin(\theta_2-\theta_1)\frac{\partial C}{\partial \theta_1} + C\cos(\theta_2-\theta_1)\right)\frac{\partial}{\partial \theta_1} \\
&\qquad \left(\sin(\theta_2-\theta_1)\frac{\partial D}{\partial \theta_1} + A\frac{\partial D}{\partial \theta_0} - C\frac{\partial A}{\partial \theta_1} + D\frac{\partial A}{\partial \theta_0}\right)\frac{\partial}{\partial \theta_0} \\
&= \left(\sin(\theta_2-\theta_1)\frac{\partial C}{\partial \theta_1} + C\cos(\theta_2-\theta_1)\right)\frac{\partial}{\partial \theta_1} \\
&\qquad \left(\sin(\theta_2-\theta_1)\frac{\partial D}{\partial \theta_1} + A\frac{\partial D}{\partial \theta_0} - C(2B) + D(-B)\right)\frac{\partial}{\partial \theta_0}
\end{aligned}
$$

and

$$
\begin{aligned}
[X_2,[X_1,X_2]] = \Bigg[&\frac{\partial}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial}{\partial\theta_1} + B\frac{\partial}{\partial\theta_0},\\
&\sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + C\frac{\partial}{\partial\theta_1} + D\frac{\partial}{\partial\theta_0}\Bigg]\\
={}& \cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \frac{\partial C}{\partial\theta_2}\frac{\partial}{\partial\theta_1} + \frac{\partial D}{\partial\theta_2}\frac{\partial}{\partial\theta_0}\\
&- \cos(\theta_2-\theta_1)\left(\frac{\partial C}{\partial\theta_1}\frac{\partial}{\partial\theta_1} + \frac{\partial D}{\partial\theta_1}\frac{\partial}{\partial\theta_0}\right) + B\frac{\partial D}{\partial\theta_0}\frac{\partial}{\partial\theta_0}\\
&- C\left(-\sin(\theta_2-\theta_1)\frac{\partial}{\partial\theta_1} + \frac{\partial B}{\partial\theta_1}\frac{\partial}{\partial\theta_0}\right) - D\frac{\partial B}{\partial\theta_0}\frac{\partial}{\partial\theta_0}\\
={}& \cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y}\\
&+ \left(\frac{\partial C}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial C}{\partial\theta_1} + C\sin(\theta_2-\theta_1)\right)\frac{\partial}{\partial\theta_1}\\
&+ \left(\frac{\partial D}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial D}{\partial\theta_1} + B\frac{\partial D}{\partial\theta_0} - C\frac{\partial B}{\partial\theta_1} - D\frac{\partial B}{\partial\theta_0}\right)\frac{\partial}{\partial\theta_0}\\
={}& \cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y}\\
&+ \left(\frac{\partial C}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial C}{\partial\theta_1} + C\sin(\theta_2-\theta_1)\right)\frac{\partial}{\partial\theta_1}\\
&+ \left(\frac{\partial D}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial D}{\partial\theta_1} + B\frac{\partial D}{\partial\theta_0} - C(-2A) - DA\right)\frac{\partial}{\partial\theta_0}.
\end{aligned}
$$

Letting T be the matrix with rows $X_1, X_2, [X_1,X_2], [X_1,[X_1,X_2]], [X_2,[X_1,X_2]]$, using MATLAB we find that $\det(T) = \sin(\theta_2-\theta_1) - \sin(\theta_1-\theta_0) + \sin(\theta_2 - 2\theta_1 + \theta_0)$.

If $\det(T) \neq 0$, then $Lie^3_q = T_qM$, and the degree of nonholonomy at q is 3.

On the other hand, if $\det(T) = 0$ then let $\alpha = \theta_2 - \theta_1$ and $\beta = \theta_1 - \theta_0$. Then we have $0 = \det(T) = \sin(\alpha) - \sin(\beta) + \sin(\alpha - \beta) = \sin(\alpha) - \sin(\beta) + \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) = \sin(\alpha)(1+\cos(\beta)) - \sin(\beta)(1+\cos(\alpha)$. If either $\sin(\alpha) = 0$ or $\sin(\beta) = 0$, then $(\alpha,\beta) \in \{(0,0)\} \cup \{\pi\} \times S^1\} \cup \{S^1 \times \pi\}$.

Otherwise, we have $\frac{1+\cos(\beta)}{\sin(\beta)} = \frac{1+\cos(\alpha)}{\sin(\alpha)}$. Let $f : (0, 2\pi) \to \mathbb{R}$ be defined by $f(\pi) = 0$ and $f(x) = \frac{1+\cos(x)}{\sin(x)}$ otherwise. If we identify S^1 with $[0, 2\pi)$, we have $f(\alpha) = f(\beta)$. Note that f is continuous, and $f'(x) = -1 - \frac{(1+\cos(x))\cos(x)}{\sin^2(x)} = -1 - \frac{\cos(x)}{1-\cos(x)} < 0$ for all x. Hence f is monotone decreasing. Thus, $\alpha = \beta$.

Thus, the points q such that $Lie^3_q \neq T_qM$ are those points such that $\alpha = \pi$, or $\beta = \pi$, or $\beta - \alpha = 0$. In the original variables, this means $\theta_2 - \theta_1 = \pi$. or $\theta_1 - \theta_0 = \pi$, or $2\theta_1 - \theta_0 - \theta_2 = 0$.

Suppose $q \in M$ such that $Lie^4_q \neq T_qM$. Using MATLAB, I found that $X_1, X_2, [X_1,X_2], [X_1,[X_1[X_1,X_2]], [X_2,[X_1,X_2]]$ have determinant $\sin(\alpha) + \sin(\beta) +$

$\sin(\alpha+\beta)$, which must be 0 at q. Hence if $\alpha = \beta$, then $0 = 2\sin(\alpha) + \sin(2\alpha) = 2\sin(\alpha)(1 + \cos(\alpha)$. Hence $\alpha \in \{0, \pi\}$ if $\alpha = \beta$.

From MATLAB, we also have $\det(X_1, X_2, [X_1, X_2], [X_1[X_1, X_2]], [X_2, [X_2, [X_1, X_2]]) = 2\cos(\beta) + \cos(\alpha+\beta) + 2\cos(\alpha) + \cos(\alpha-\beta) + 2$. If this determinant is zero, we cannot have $\alpha = \beta = 0$. The only remaining case is either $\alpha = \pi$ or $\beta = \pi$. Suppose $\alpha = \pi$. Then we have $0 = 2\cos(\beta) + \cos(\pi+\beta) - 2 + \cos(\pi-\beta) + 2 = 2\cos(\beta) - \cos(\beta) - \cos(\beta) = 0$.

$\square$

3 Let $\widetilde{M}$ denote the set of bracket-generating points of $\mathcal{F}$. Prove that the system is controllable on $\widetilde{M}$.