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HW 4

1 Let A and B be self-adjoint matrices, which may be real or complex. We say that $A \le B$ if and only if $\langle A\mathbf{x}, \mathbf{x}\rangle \le \langle B\mathbf{x}, \mathbf{x}\rangle$ for all $\mathbf{x}$.

a. If $\lambda_1 \ge \lambda_2, \dots, \lambda_n$ are the eigenvalues of A and $\tilde{\lambda}_1 \ge \tilde{\lambda}_2, \dots, \tilde{\lambda}_n$ are the eigenvalues of B, then show that $\lambda_k \le \tilde{\lambda}_k$.

b. Show that $\mathrm{Trace}(A) \le \mathrm{Trace}(B)$ if $A \le B$.

c. Show that if we increase a diagonal entry of A, then the resulting matrix B satisfies $A \le B$.

d. (Keener, problem 1.3(b)). Use the previous part to estimate the lowest eigenvalue of the matrix below. Keener gets $-\frac{1}{3}$. Using matlab you get less than about -2. Can you beat $-\frac{1}{3}$?

$$A = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & -4 \\ 4 & -4 & 3 \end{pmatrix}$$

Proof. For (a),

Since the trace of a matrix is the sum of its eigenvalues, (b) follows directly from (a).

□

2 Let A be a self-adjoint matrix with eigenvalues $\lambda_1 \ge \lambda_2, \dots, \ge \lambda_n$. Show that for $2 \le k < n$ we have

$$\max_U \sum_{j=1}^{k} \langle Au_j, u_j\rangle = \sum_{j=1}^{k} \lambda_j,$$

where $U = \{u_1, \dots, u_k\}$ is any o.n. set. (Hint: Put A in diagonal form and use a judicious choice of B.)

Proof. □

3 Show that ℓ^∞ is a Banach space under the norm $\|\{x_j\}\| = \sup_j |x_j|$

Proof. To see that $\|\cdot\|$ is a norm, we need to show that it is positive definite, homogenous, and satisfies the triangle inequality. The norm is clearly nonnegative since the absolute value function is nonnegative. Moreover if $x = (x_j) \in \ell^\infty$ and $\|x\| = 0$, then $\sup_j |x_j| = 0$. Hence $\|x_j| \le 0$ for all j, so $x_j = 0$ for all j.

For homogeneity, let $c \in \mathbb{R}$. Then $\|cx\| = \sup_j |cx_j| = |c| \sup_j |x_j| = |c|\|x\|$. For the triangle inequality, let $y = (y_j)$. Then $\|x + y\| = \sup_j |x_j + y_j| \le \sup_j |x_j| + |y_j| \le \sup_j |x_j| + \sup_j |y_j| \le \|x\| + \|y\|$.

To see that ℓ^∞ is complete, suppose $(x_n) \subset \ell^\infty$ is Cauchy. Write each x_n as $(x_{nj})_j$.

Fix j. Since (x_n) is Cauchy, given $\epsilon > 0$ there exists N such that $\|x_n - x_m\| < \epsilon$ for all $n, m \geq N$. Thus $|x_{nj} - x_{mj}| \leq \sup_k \|x_{nk} - x_{mk}\| < \epsilon$ for all $n, m \geq N$. Hence $(x_{nj})_n$ is Cauchy in $\mathbb{R}$, so has a limit y_j.

Let $y = (y_j)_j \in \ell^\infty$. I need to show that $y \in \ell^\infty$ and $x_n \to y$. For the former, note that since (x_n) is Cauchy, there exists M such that $\|x_n\| \leq M$ for all n. Hence $|x_{nj}| \leq M$ for all n, j. Thus for each j, we have $|y_j| = |\lim_n x_{nj}| = \lim_n |x_{nj}| \leq M$. Thus, $y \in \ell^\infty$.

To see that $x_n \to y$, pick $\epsilon > 0$. Since (x_n) is Cauchy, we can pick N such that $\|x_n - x_m\| < \epsilon/2$ for all $n, m \geq N_1$. Since each $x_{nj} \to y_j$ for each $1 \leq j \leq N$, we can pick N_j such that $\|x_{nj} - y_j\| < \epsilon/2$ for all $n \geq N_j$. Let $K = \max(N, \max_j N_j))$. Then for $n \geq K$, we have $\|x_n - y\| \leq \|x_n - x_K\| + \|x_K - y\| < \epsilon/2 + \sup_j |x_{Kj} - y_j| < \epsilon/2 + \sup_j(\epsilon/2) = \epsilon/2$. □

4 Show that ℓ^2 is a Hilbert space under the inner product

$$\langle \{x_j\}, \{y_j\} \rangle := \sum_{j=1}^{\infty} \bar{y}_j x_j.$$

Proof. To see that $\langle \cdot, \cdot \rangle$ maps into $\mathbb{R}$, let $x = (x_j) \in \ell^2$ and $y = (y_j) \in \ell^2$. Then for every N, we have $\sum_{j=1}^N \bar{y}_j x_j \leq \left(\sum_{j=1}^N |y_j|^2)^{1/2} (\sum_{j=1}^N |x_j|^2\right)^{1/2} \leq \|x\| \|y\|$ by Cauchy-Schwartz on $\mathbb{C}^N$. Hence, letting $N \to \infty$, we have $\langle x, y \rangle \leq \|x\| \|y\| < \infty$.

To see that $\langle \cdot, \cdot \rangle$ defines an inner product, we need to check that it is positive definite, linear in the first component, and conjugate symmetric. For positive definiteness, note that $\langle x, x \rangle = \sum_j |x_j|^2 \geq 0$, and equality holds iff $x_j = 0$ for all j. Linearity in the first component and conjugate symmetry are both immediate from the definition of $\langle \cdot, \cdot \rangle$.

To see that ℓ^2 is complete, suppose $(x_n) \subset \ell^2$ is Cauchy. For each n, we can write $x_n = (x_{nj})_j$. Fix j, and let $\epsilon > 0$. Since (x_n) is Cauchy, we can pick N such that $\|x_n - x_m\| < \epsilon$ for $n, m \geq N$. Hence $\|x_{nj} - x_{mj}|^2 < \sum_k |x_{nk} - x_{mk}|^2 = \|x_n - x_m\| < \epsilon$. Hence $(x_{nj})_n$ is Cauchy in $\mathbb{R}$, so converges to some y_j.

To see that $y := (y_j)$ is in ℓ^2, we use the fact that (x_n) is bounded in ℓ^2 since it is Cauchy. That is, there exist an M such that $\sum_j |x_{nj}|^2 = \|x_n\| < M$ for all n.

□

5 Let $0 \leq \delta \leq 1$. We define the modulus of continuity for $f \in C[0,1]$ by

$$\omega(f; \delta) := \sup_{|s-t| \leq \delta} |f(s) - f(t)|, \text{ where } \; s, t \in [0,1].$$

a. Explain why $\omega(f; \delta)$ exists for every $f \in C[0,1]$.

b. Fix δ. Let $S_\delta = \{\epsilon > 0 : |f(t) - f(s)| < \epsilon \text{ for all } |s - t| \leq \delta\}$. Show that $\omega(f; \delta) = \inf S_\delta$.

c. Show that $\omega(f; \delta)$ is nondecreasing as a function of δ.

d. Show that $\lim_{\delta \downarrow 0} \omega(f; \delta) = 0$.

Proof. For (a), if $f \in C[0,1]$ then there exists $M > 0$ such that $|f(x)| \leq M$ for all x. This is because the image of a compact set under a continuous function is compact. Hence for all $s, t \in [0,1]$, we have $|f(s) - f(t)| \leq 2M$. Thus $\omega(f;\delta) \leq 2M$.

For (b), if $\epsilon \in S_\delta$, then $\omega(f;\delta) = \sup_{|\,s-t\,| \leq \delta} |f(s) - f(t)| \leq \epsilon$. Hence $\omega(f;\delta) \leq \inf S_\delta$. On the other hand, if $\eta > 0$, then $|f(s) - f(t)| < \omega(f;\delta) + \eta$ for all $|s - t| \leq \delta$. Hence, $\omega(f;\delta) + \eta \in S_\delta$. Thus $inf S_\delta \leq \omega(f;\delta) + \eta$. Letting $\eta \to 0$, we have $\inf S_\delta \leq \omega(f;\delta)$.

For (c), suppose $\delta < \gamma$. If $\epsilon \in S_\gamma$, then $|f(t) - f(s)| < \epsilon$ for all $|s - t| \leq \gamma$, hence for all $|s - t| \leq \delta$. Thus, $S_\gamma \subset S_\delta$. Therefore $\omega(f;\delta) = \inf S_\delta \leq \inf S_\gamma = \omega(f;\gamma)$.

For (d), let $\epsilon > 0$. Since f is continuous on the compact set $[0,1]$, it is uniformly continuous on $[0,1]$. Hence we can pick $\delta > 0$ such that $|f(s) - f(t)| < \epsilon$ for all $|s - t| < \delta$. Thus, $\omega(f;\delta) < \epsilon$. By (c), if $0 < \gamma \leq \delta$, then $\omega(f;\gamma) \leq \omega(f;\delta) < \epsilon$. Hence $\lim_{\delta \downarrow 0} \omega(f;\delta) = 0$. $\square$