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HW 7

19.43 No to both; let $f(x) = x\chi_{[0,1)}$. Then $||f||_\infty = 1$, but $\{|f| = 1\} = \emptyset$.
50 Note that if $f \in L_\infty$ then f is a.e. equal to a bounded function, so the simple functions of the Basic Construction converge uniformly to f a.e. Hence, the simple functions are dense in L_∞. If E is of finite measure, all simple functions defined on E are integrable. If $m(E) = \infty$, then the integrable simple functions are not dense: take $f = \chi_E$. If $||\phi - f||_\infty < 1/2$, then $||\phi||_\infty > 1/2$, so $\int |\phi| = \infty$.
51 There's a typo in the problem statement: the exponent of $m(E)$ should be $1/p$, not $(1 - 1/p)$. To see why the latter can't be right, let $f = \chi_E$, then $||f||_p = m(E)^{1/p} = m(E)^{1/p}||f||_\infty$. Take $m(E) = 2$ and $p = 1$ to see that the problem can't be correct as stated.

If $f \in L_\infty(E)$ with $m(E) < \infty$ and $1 \le p < \infty$, we have $||f||_p \le ||(||f||_\infty)||_p = (m(E))^{1/p}||f||_\infty$. This implies $L_\infty(E) \subset L_p(\mathbb{R})$ with the convention that a function f defined on E is set to 0 outside of E.

If $E = [0, 1]$, this inequality reduces to $||f||_p \le ||f||_\infty$. To see $||f||_1 \le ||f||_p$, use Hölder's inequality: $||f||_1 = ||(1)f||_1 \le ||1||_q||f||_p = ||f||_p$
52 Let $f \in L_\infty[0, 1]$. To see that $||f||_p$ is increasing as a function of p, let $1 \le r < s \le \infty$. By Hölder,

$$||f||_r = (\int |f|^r(1))^{1/r} \le ((\int |f|^{r(s/r)})^{r/s})^{1/r} = ||f||_s.$$

Since by (51) every $||f||_p$ is bounded above by $||f||_\infty$, we have $\lim_{p\to\infty} ||f||_p$ exists.

To show that $||f||_\infty \le \lim_{p\to\infty} ||f||_p$, let $\epsilon > 0$. We have

$$\begin{aligned} ||f||_p &\ge \left(\int_{\{|f|>||f||_\infty-\epsilon\}} |f|^p \right)^{1/p} \\ &\ge ((||f||_\infty - \epsilon)^p \, m\{|f| > ||f||_\infty - \epsilon\})^{1/p} \\ &= (||f||_\infty - \epsilon)(m\{|f| > ||f||_\infty - \epsilon\})^{1/p} \\ &\to ||f||_\infty - \epsilon, \end{aligned}$$

as $p \to \infty$, since $m\{|f| > ||f||_\infty - \epsilon\} > 0$. Thus, $||f||_\infty \le \lim_{p\to\infty} ||f||_p \le ||f||_\infty$, so $\lim_{p\to\infty} = ||f||_\infty$.
62 Pick a step function h such that $||f - h||_p < \epsilon/2$. If $m(A) < \delta := (\frac{\epsilon}{2||h||_\infty})^p$,

then

$$\begin{aligned}
||f\chi_A||_p &\le ||h\chi_A||_p + ||(f-h)\chi_A||_p \\
&\le ||||h||_\infty \chi_A||_p + (\epsilon/2) \\
&\le ||h||_\infty m(A)^{1/p} + (\epsilon/2) \\
&< \epsilon.
\end{aligned}$$

If $p = \infty$, this will not work. Take $f(x) := 1$. If $m(A) > 0$, then $||f\chi_A||_\infty = ||\chi_A||_\infty = 1$.

64(a) *Case* $p > 1$. For the boundedness, since $1 < p < \infty$, we can use Hölder:

$$|h(x)| = |\int fT_x(g)| \le ||f||_p ||T_x(g)||_q = ||f||_p ||g||_q,$$

where the last equality follows from (63), which was proved in class.

For continuity,

$$|h(x) - h(y)| = |\int f \cdot (T_x - T_y)g)| \le ||f||_p ||(T_x - T_y)g||_q \to 0$$

as $y \to x$ by (63)(c).

Case $p = 1$. For the boundedness, note

$$|h(x)| \le \int |fT_x(g)| \le \int |f| \cdot ||g||_\infty = ||f||_1 ||g||_\infty.$$

For continuity, first note the previous estimate shows that $fT_x(g) \in L_1$, so by (63) we have

$$\begin{aligned}
h(x) &= \int fT_x(g) \\
&= \int (fT_x(g))^+ - \int (fT_x(g))^- \\
&= \int T_{-x}((fT_x(g))^+) - \int T_{-x}((fT_x(g))^-) \\
&= \int (T_{-x}(f)g)^+ - \int (T_{-x}(f)g)^- \\
&= \int T_{-x}(f)g,
\end{aligned}$$

where the penultimate equality is justified by the fact that for any function F, we have $T_x(F^+) = T_x(1/2(|F| + F)) = 1/2(|T_xF| + T_xF) = (T_xF)^+$ and similarly for F^-.

Thus, $|h(x) - h(y)| = |\int (T_{-x} - T_{-y})(f)g| \le ||g||_\infty \int |(T_{-x} - T_{-y})(f)| \to 0$ as $y \to x$ by (63)(c).

64(b) Suppose g is continuously differentiable.

Case $f = \chi_{[a,b]}$. We have $h(x) = \int_a^b g(x+t)\,dt$, so $(h(x+k)-h(x))/k = \int_a^b (g(x+k+t)-g(x+t))/k\,dt \to \int_a^b g'(x+t)\,dt\,(k \to 0)$ by the DCT, since $|(g(x+k+t)-g(x+t))/k| \leq sup_{t\in[a-1,b+1]}|g'(x+t)|$ for $t \in [a,b]$ and $k < 1/2$ by the MVT.

The step function case follows by linearity of the integral.

Pick a step functions s_n such that $||f - s_n||_p \to 0$ and $s_n \to f$ a.e. Then

$$(h(x+k)-h(x))/k = \int f(t)(g(x+k+t)-g(x+t))/kdt$$