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HW 6, due April 4

35.18 Consider the subnormal series $0 \to A_3 \times 0 \to S_3 \times 0 \to S_3 \times A_3 \to S_3 \times S_3$. All the factor groups have prime order so are simple, abelian. Thus, $S_3 \times S_3$ is solvable.
19 Yes, let σ be a 90-degree rotation and τ a reflection. Note that $\langle \sigma \rangle$ is cyclic of order 4 and normal in D_4. Hence, we have the subnormal series $0 \to C_2 \to C_4 \to D_4$, which is a composition series since the orders of all the factor groups are prime (2, actually).
36.5 Each Sylow 3-subgroup of S_4 are generated by one of the following 3-cycles: $(1,2,3), (1,2,4), (1,3,4), (2,3,4)$. The fact that they are conjugate is a consequence of the Sylow theorems, but you could just conjugate by transpositions if you want to be explicit. For example, $(3,4)(1,2,3)(3,4) = (1,2,4)$, so the corresponding 3-Sylow subgroups are conjugate.
13 The only divisor of 45 that is congruent to 1 mod 3 is 1. Thus, the 3-Sylow subgroup (of order 9) is normal in the whole group.
15 P is obviously a p-Sylow subgroup of $N[N[P]]$. Suppose Q is a p-Sylow subgroup of $N[N[P]]$. Then $Q = gPg^{-1}$ for some $g \in N[N[P]]$. Since $N[P]$ is normal in $N[N[P]]$, this implies $Q \subset N[P]$. Hence, Q and P are p-Sylow subgroups of $N[P]$, so $Q = P$ since P is normal in $N[P]$. Thus, P is the unique p-Sylow subgroup of $N[N[P]]$, so is normal in $N[N[P]]$.
18 Note that 3, 5, and 15 are not congruent to 1 mod 17. Hence, the only divisor of 255 that is congruent to 1 mod 17 is 1. Thus, the 17-Sylow subgroup is normal in the whole group.
19 Presumably $m \neq 1$ or else we have the counterexample C_p. Since $n_p \equiv 1 \pmod{p}$, $n_p \mid m$. This implies $n_p = 1$ since $1 < m < p$. Thus, the p-Sylow subgroup is normal in the whole group.
37.4 Call the group G. By the Sylow theorems, $n_5 = 1$, $n_7 = 1$, and $n_{47} = 1$. Hence, the corresponding Sylow subgroups are normal in G. Since they have prime order, they are cyclic and have trivial intersection. Hence, using the trick from class (proved below), each pair of Sylow subgroups commutes pointwise.

Trick from class: If $H, K \lhd G$ with $H \cap K = \{e\}$ and $h \in H$, $k \in K$; then $hk = kh$. Proof of trick: $hkh^{-1}k^{-1} = k'k^{-1} \in K$ and $hkh^{-1}k^{-1} = hh' \in H$, so $hkh^{-1}k^{-1} = e$.

Let $x, y, z \in G$ have orders 5,7, and 47, respectively. Since x, y commute, xy has order 35 (x^k and y^k only have the same order for $35 \mid k$). Similarly, xyz has order (5)(7)(47).
5 Call the group G. $96 = (32)(3)$, so the possibilities are $n_2 = 1$ or $n_2 = 3$. WLOG $n_2 = 3$ since G is not simple if $n_2 = 1$. But $(n_2)! = 6 < 96 = |G|$. Hence, by a theorem proved in class, G is not simple (consider the transitive action of G on the set of 3-Sylow subgroups by conjugation).

6 $160 = (32)(5)$, so $n_2 = 1$ or $n_2 = 5$. WLOG, $n_2 = 5$. But $5! = 120 < 160$, so G is not simple.

8

a. Note that $\tau\sigma\tau^{-1}(\tau a_i) = \tau\sigma a_i = \tau a_{i'}$ where $i' = i + 1 \pmod m$. If $x \notin (\tau a_i)_i$ for any i, then $\tau\sigma\tau^{-1}(x) = \tau\tau^{-1}(x) = x$ since $\tau^{-1}x \notin (a_i)_i$.

b. It suffices to show that $(1, 2, \dots, m)$ is conjugate to each $(a_1, a_2, \dots, a_m)$. By part(a), this is obvious: just define τ by $i \mapsto a_i$ for $1 \leq i \leq m$ and extend this to a bijection of $[n]$ however you like.

c. Let $\sigma = \prod_i \sigma_i$ and $\eta = \prod_i \eta_i$ denote two such products of disjoint cycles with each $\sigma_i = (\sigma_{i1}, \dots, \sigma_{i,r_i})$ and $eta_i = (\eta_{i1}, \dots, \eta_{i,r_i})$. Since the σ_{ij} are distinct and the η_{ij} are distinct, there exists $\tau \in S_n$ such that $\tau(\sigma_{ij}) = \eta_{ij}$ for all i, j.

   By the fact that conjugation by τ is an homomorphism and by part (a), $\tau\sigma\tau^{-1} = \prod_i \tau\sigma_i\tau_{-1} = \prod_i \eta_i = \eta$.

d. Let $P(n)$ denote the set of partitions of n. For any $\sigma \in S_n$, let $(O_{\sigma,i})_{1\leq i\leq s}$ denote the orbits of $[n]$ under σ. Define the map $\phi : S_n \to P(n)$ by $\phi(\sigma) = (|O_{\sigma,i}|)_i$.

   This map is surjective: a partition $Q = (t_i)_{i=1}^s$ of n is the image of $\eta = (1, 2, \dots, t_1)(t_1 + 1, \dots, t_2) \dots (n - t_s + 1, \dots, n)$.

   Suppose $\sigma \in \phi^{-1}(Q)$. For each i, pick $x_i \in O_{\sigma,i}$. Then

$$\sigma = \prod_{i=1}^{s} (x_i, \sigma(x_i), \dots, \sigma^{t_i} x_i) \tag{1}$$

   Hence, by part (c), the preimage of any partition is a subset of a conjugacy class. Moreover, by part (a) and equation (1), if η is conjugate to σ, then $\phi(\eta) = Q$. Hence, the preimage of any partition is equal to a conjugacy class. Thus, $p(n)$ is the number of conjugacy classes of S_n.

e. $1, 2, 3, 5, 7, 11, 15$