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HW 4, due 2/21

18.3 Prove that $\int_1^\infty (1/x)\,dx = \infty$.

Proof. Let $f(x) = (\chi_{(1,\infty)}(x))(1/x)$. Define $\phi_m = \sum_{n=1}^m \frac{1}{n+1}\chi_{(n,n+1)}$. Then for all $m \geq 1$, we have $\phi_m \leq f$. Hence, by the monotonicity of the integral, $\int f\,dm \geq \int \phi_m\,dm = \sum_{n=1}^m \frac{1}{n+1} \to \infty$ as $m \to \infty$. $\square$

4 Find (f_n) nonnegative measurable functions that converge uniformly to 0, but $\lim_{n\to\infty} \int f_n = 1$.

Proof. Let $f_n = (1/n)\chi_{(0,n)}$. $\square$

6 Suppose (f_n) nonnegative, measurable decrease pointwise to f, and that $\int f_k < \infty$ for some k. Prove that $\int f = \lim_{n\to\infty} f_n$. Also, give an example showing that the condition $\int f_k < \infty$ is necessary.

Proof. For the counterexample, let $f_n = \chi_{(n,\infty)}$ for $n \geq 1$.

For the other part of the problem, for all $n \geq k$, let $g_n = f_k - f_n$. Since (f_n) is nonnegative and decreasing, $(g_n)_{n\geq k}$ is increasing and nonnegative. Since $g_n \leq f_k$, we have $\int g_n < \infty$ for all $n \geq k$. Hence, using the linearity of the integral on integrable functions and the MCT,

$$\begin{aligned}
\int f\,dm &= -\int (f - f_k)\,dm + \int f_k\,dm \\
&= -\int \lim_{n\to\infty} f_k - f_n\,dm + \int f_k\,dm \\
&= -\lim_{n\to\infty}\left(\int f_k - f_n\,dm\right) + \int f_k\,dm \\
&= -\lim_{n\to\infty}\int -f_n\,dm \\
&= \lim_{n\to\infty}\int f_n\,dm
\end{aligned}$$

$\square$

7 Let $\mu : \mathcal{A} \to [0,\infty]$ be a nonnegative, finitely additive, set function defined on a σ-algebra $\mathcal{A}$. Prove that:

1. $\mu(E) \leq \mu(F)$ whenever $E, F \in \mathcal{A}$ satisfy $E \subset F$.
2. if $\mu(\emptyset) \neq 0$, then $\mu(E) = \infty$ for all $E \in \mathcal{A}$.

Proof. For (1), we have $\mu(F) = \mu(E) + \mu(E \setminus F) \geq \mu(E)$. For (2), if $\mu(\emptyset) \neq 0$, we have $\mu(E) = \mu(E \cup \bigcup_{i=1}^n \emptyset) = \mu(E) + n\mu(\emptyset) \to \infty$ as $n \to \infty$. $\square$

8 Define μ and $\mathcal{A}$ as in (7). Prove that TFAE:

1. $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ for every pairwise disjoint $(E_n) \subset \mathcal{A}$.
2. $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$ for every increasing $(E_n) \subset \mathcal{A}$.

Proof. To prove (2) implies (1), let $F_k = \bigcup_{n=1}^{k} E_n$. Then (F_k) is an increasing sequences of sets in $\mathcal{A}$, so, by (2), $\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcup_{k=1}^{\infty} F_k) = \lim_{k\to\infty} \mu(F_k) = \lim_{k\to\infty} \sum_{n=1}^{k} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.

For (1) implies (2), let (F_n) be the disjointification of (E_n). That is, $F_n := E_n \backslash (\bigcup_{k<n} E_k)$, so for all N, we have $\bigcup_{n=1}^{N} F_n = \bigcup_{n=1}^{N} E_n$. Then, applying (1) to F_n, we have $\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(F_n) = \lim_{N\to\infty} \mu(E_N)$. □

15 Let f be nonnegative and measurable. Prove that $\int f < \infty$ if and only if $L := \sum_{i=-\infty}^{\infty} 2^k\, m\{f > 2^k\} < \infty$.

Proof. Note that $f \leq \sum_{k=-\infty}^{\infty} 2^{k+1} \chi_{\{2^k < f \leq 2^{k+1}} \leq 2 \sum_{k=-\infty}^{\infty} 2^k \chi_{\{f>2^k\}}$. Hence, $\int f \leq \int 2 \sum_{k=-\infty}^{\infty} 2^k \chi_{\{f>2^k\}} = 2 \sum_{k=-\infty}^{\infty} \int 2^k \chi_{\{f>2^k\}} = 2L$.

For the opposite inequality, we have

$$\begin{aligned}
\sum_{k=\infty}^{\infty} 2^k \chi_{\{f>2^k\}} &= \sum_{k=-\infty}^{\infty} \sum_{j=k}^{\infty} 2^k \chi_{\{2^j < f \leq 2^{j+1}\}} \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{j} 2^k \chi_{\{2^j < f \leq 2^{j+1}\}} \\
&= \sum_{j=-\infty}^{\infty} 2^{j+1} \chi_{\{2^j < f \leq 2^{j+1}\}} \\
&\leq 2f.
\end{aligned}$$

□

16 Let $f \geq 0$ be integrable. Given $\epsilon > 0$, show that there is a measurable set E with $m(E) < \infty$ such that $\int_E f > \int f - \epsilon$. Moreover, show that E can be chosen so that f is bounded on E.

Proof. For $k \geq 1$, define $E_k := f^{-1}([k-1, k))$. By Corollary 18.12, we have $\int f = \sum_k \int_{E_k} f$. Since $\int f < \infty$, we may pick N such that $\sum_{k>N} \int_{E_k} < \epsilon/2$. Hence, if $F := \bigcup_{k\leq N} E_k$, then $f < N$ on F and $\int_F f = \sum_{k\leq N} \int_{E_k} f > \int f - \epsilon/2$.

Next, pick an an integrable, nonnegative, simple function $\phi \leq \chi_F f$ such that $\int_F f - \int \phi \leq \epsilon/2$. Write ϕ in standard form as $\phi = \sum_{i=0}^{n} a_i \chi_{A_i}$ where $a_0 = 0$. Note that since ϕ is integrable, we have $m(A_0^c) = \sum_{i=1}^{n} m(A_i) \leq (\min_{i\geq 1} a_i)^{-1} \sum_{i=1}^{n} a_i m(A_i) = (\min_{i\geq 1} a_i)^{-1} \int \phi \leq \infty$. Hence, if $E := A_0^c$, we have $\int_E f \geq \int_E \phi = \int_{\mathbb{R}} \phi \geq \int_F f - \epsilon/2 > \int f - \epsilon$. Note that f is bounded on E since $\phi \leq \chi_F f$ implies $E = A_0^c \subset F$.

□

17 If f is nonnegative and integrable, prove that the function $F(x) = \int_{-\infty}^{x} f$ is continuous. In fact, even more is true: Given $\epsilon > 0$, show that there is a $\delta > 0$ such that $\int_E f < \epsilon$ whenever $m(E) < \delta$. [Hint: This is easy when f is bounded; see (16)]

Proof. By (16), pick a measurable set $F \in \mathbb{R}$ such that $f|_F \leq M$ for some bound $M > 0$, and $\int f - \int_F f < \epsilon/2$. If $m(E) < \epsilon/(2M) =: \delta$, then $\int_E f = \int_{E\cap F} f + \int_{E\cap F^c} f \leq \int_{E\cap F} M + \int_{F^c} f \leq M\, m(E) + (\int f - \int_F f) < \epsilon/2 + \epsilon/2$.

This implies uniform continuity because for any $x < y$ with $y - x < \delta$, we have, by the linearity of the integral on integrable functions, $F(y) - F(x) = \int_{(x,y)} f < \epsilon$.

□

14 Define $f : [0,1] \to [0,\infty)$ by $f(x) = 0$ if x is rational and $f(x) = 2^n$ if x is irrational with exactly $n = 0, 1, 2, \ldots$ leading zeros in its decimal expansion. Show that f is measurable, and find $\int_0^1 f$.

Proof. Note that if $a \leq 0$, then $\{f \geq a\} = [0,1]$. If $0 < a < 1$, then $\{f \geq a\} = [0,1] \setminus \mathbb{Q}$. If $a \geq 1$, let 2^n be the minimal power of 2 such that $2^n \geq a$. Then $\{f \geq a\} = \{f \geq 2^n\} = (0, 10^{-n}) \setminus \mathbb{Q}$ where I use the convention that zeros in front of the decimal do not count, so, for example, $0.34\ldots$ has zero leading zeros. Since the inverse image of each ray is measurable, f is measurable.

Define $\phi_m = \sum_{n=0}^{m} 2^n \chi_{(10^{-n-1}, 10^{-n})}$. Then $\phi_m \to f$ a.e., so $\int f = \lim_{m\to\infty} \int \phi_m = \sum_{n=0}^{\infty} 2^n (10^{-n} - 10^{-n-1}) = (1 - 1/10) \sum_{n=0}^{\infty} (2/10)^n = (9/10)(5/4) = 9/8$.

□

J18.1 Suppose that f is a nonnegative integrable function and A is a measurable set. Define F on $\mathbb{R}$ by $F(t) = m_f(A + t)$. Show that F is a continuous function. Recall that $m_f(E) := \int \chi_E f\, dm$. (Hint: First treat the case where A is a bounded interval.)

Proof. To show that F is continuous at t, it suffices to show that every sequence $t_n \to t$ has $F(t_n) \to F(t)$. WLOG, by replacing A with $A - t$ and t_n with $t_n - t$, we can assume $t = 0$.

Case A an interval: If A is an interval and $x \in A^\circ$, pick δ so small that $A_\delta(x) \subset A$. Then if $|t_n| < \delta$, we have $x + t_n \in A$. Hence, $\chi_{A+t_n}(x) = \chi_A(x)$ for all large n. Similarly, if $x \in (A^c)^\circ$, we have, for all large n, that $x + t_n \in A^c$ so $\chi_{A+t_n}(x) = \chi_A(x)$. Since an interval has at most two boundary points, we have $\chi_{A+t_n} \to \chi_A$ a.e. Hence, $\chi_{A+t_n} f \to \chi_A f$ a.e., so since these functions are dominated by f, we have $\int \chi_{A+t_n} f \to \int \chi_A f$. Thus, $F(t_n) \to F(t)$ for every $t_n \to t$, so F is continuous at t.

General case

□