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MATH 663 - Subfactors, Knots, and Planar Algebras (Fall 2017)

HW 1

1 Let $\phi : \mathcal{B}(H) \to \mathbb{C}$ be a linear functional. Show that the following statements are equivalent:

(a) There are $n \in \mathbb{N}$ and $(\xi_i)_{i=1}^n, (\eta_i)_{i=1}^n \subset H$ such that

$$\phi(x) = \sum_{i=1}^n \langle x\xi_i | \eta_i \rangle \qquad (x \in \mathcal{B}(H))$$

(b) ϕ is continuous with respect to the weak operator topology.

(c) ϕ is continuous with respect to the strong operator topology.

Proof. (a) $\implies$ (b): Let $(x_\lambda)_\lambda \subset \mathcal{B}(H)$ be a net such that $x_\lambda \overset{WOT}{\to} x$. Then

$$\begin{aligned}
\lim_\lambda \phi(x_\lambda) &= \sum_{i=1}^n \lim_\lambda \langle x_\lambda \xi_i | \eta_i \rangle \\
&= \sum_{i=1}^n \langle \lim_\lambda x_\lambda \xi_i | \eta_i \rangle \\
&= \phi(x),
\end{aligned}$$

where the second equality follows from the definition of the WOT.

(b) $\implies$ (c): Suppose ϕ is continuous wrt the WOT. Further suppose $x_\lambda \overset{SOT}{\to} x \in \mathcal{B}(H)$. Then $x_\lambda \overset{WOT}{\to} x \in \mathcal{B}(H)$, so $\phi(x_\lambda) \to \phi(x)$.

(c) $\implies$ (a): Suppose ϕ is continuous with respect to the SOT. By the definition of the SOT, there exists an $r > 0$ and $\xi_1, \ldots, \xi_n$ such that $\|x\xi_i\| < r$ for all i implies that $|\phi(x)| < 1$. This implies that there exists δ such that $\sum_i \|x\xi_i\|^2 < \delta$ implies $|\phi(x)| < 1$.

Define $j : \mathcal{B}(H) \to H^{\oplus n}$ by $j(x) = \bigoplus_i x\xi_i$. Let $K = \operatorname{im}(j)$. Let $\psi : K \to \mathbb{C}$ be the linear functional defined by

$$\psi(\bigoplus_i x\xi_i) = \phi(x).$$

By the Hahn-Banach theorem, we can extend ψ to $H^{\oplus n}$. Hence, by the Riesz Representation Theorem, we can write

$$\phi(x) = \psi j(x) = \sum_{i=1}^n \langle x\xi_i, \eta_i \rangle$$

for some $(\eta_i) \subset H$. $\square$

2 Let H be an infinite dimensional Hilbert space. Show by means of explicit examples that the norm topology, the strong operator topology, and the weak operator topology are all inequivalent on $\mathcal{B}(H)$.

Proof. Define $x_n \in \mathcal{B}(\ell^2(\mathbb{N}))$ by $x_n(e_i) = 0$ if $i \leq n$ and $x_n(e_i) = e_i$ if $i > n$. Then $x_n \to 0$ in the SOT. On the other hand, $\|x_n\| = 1$ for all n.

Define $y_n \in \mathcal{B}(\ell^2(\mathbb{N}))$ by $y_n(e_i) = e_{i+n}$. Then $x_n \to 0$ in the WOT. On the other hand, $x_n(e_1) = e_n$ for all n, which doesn't converge. $\square$

3 Show that $\mathcal{B}(H)$ is a factor.

Proof. The set of bounded operators $\mathcal{B}(H)$ is obviously a von Neumann algebra (it's the commutant of the identity). To see that it is a factor, we need to show that $\mathcal{B}(H) \cap Z(\mathcal{B}(H)) = \mathbb{C}$. In other words, we need to show that $Z(\mathcal{B}(H)) = \mathbb{C}$.

Suppose $x \in Z(\mathcal{B}(H))$ and $\xi \in H$ is a nonzero vector. Let p be the projection onto line generated by ξ. Then $(1-p)x\xi = (1-p)xp\xi = x(1-p)p\xi = 0$. Thus, $px\xi = x\xi$, so ξ is an eigenvector of x. Thus, every nonzero vector in H is an eigenvector of x. It suffices to show that they all have the same eigenvalue.

Suppose $x\xi = \alpha\xi$ and $x\eta = \beta\eta$ for some $\alpha, \beta \in \mathbb{C}$ and linearly independent $\xi, \eta \in H$. Then $\alpha\xi + \beta\eta = x(\xi+\eta) = \lambda(\xi+\eta)$ for some $\lambda \in \mathbb{C}$. Thus, $\alpha = \lambda = \beta$. Thus, x is a scalar matrix. $\square$

4 Let S be a self-adjoint subset of $\mathcal{B}(H)$. Show that S' is a von Neumann algebra.

Proof. First, I claim that S' is a *-subalgebra of $\mathcal{B}(H)$. Suppose $x, y \in S'$ and $u \in S$. Then $xyu = uxy$, and $\alpha x + \beta y)u = u(\alpha x + \beta u$ for all $\alpha, \beta \in \mathbb{C}$. Moreover, $x^*u = (u^*x)^* = (xu^*)^* = ux^*$. Hence, S' is a *-algebra.

Since S' obiously contains $1_{\mathcal{B}(H)}$, it suffices to show that S' is weakly closed. Let $(x_\lambda) \subset S'$ be a net such that $x_\lambda \to x \in \mathcal{B}(H)$ in the weak operator topology. Let $u \in M$ be arbitrary.

$$\begin{aligned} 0 &= \langle (x_\lambda u - u x_\lambda)\xi, \eta \rangle \\ &= \langle x_\lambda u \xi, \eta \rangle - \langle x_\lambda \xi, u^* \eta \rangle \\ &\to \langle x u \xi, \eta \rangle - \langle x \xi, u^* \eta \rangle \\ &= \langle (xu - ux)\xi, \eta \rangle \end{aligned}$$

Thus, $x \in S'$. Hence, S' is weakly closed. $\square$

5 Let e be a finite projection in a von Neumann algebra M. Let $f \leq e$ be another projection. Show that f is also finite.

Proof. Let $g \in P(M)$ be a projection such that $f \sim g \leq f$. We have $e - f \geq 0$ and $e - f + g \leq e$. Moreover, $(e-f) \perp g$ and $(e-f) \perp f$. Hence, $(e-f) + f \sim (e-f) + g \leq (e-f) + f$. Thus, since e is finite, we have $e - f + g = e$. Thus, $f = g$. Thus, f is finite. $\square$

6 It is know that if M is a factor, and $p, q \in P(M)$, then either $p \preceq q$ or $q \preceq p$. Using this fact, show that if M is a II_1-factor then $p \sim q$ if and only if $\tau(p) = \tau(q)$, where τ is the unique normal faithful tracial state on M.

Proof. If $p \sim q$, then there exists $u \in M$ such that $p = u^*u$ and $q = uu^*$. Thus $\tau(p) = \tau(u^*u) = \tau(uu^*) = \tau(q)$ since τ is a trace.

Conversely, suppose $\tau(p) = \tau(q)$. WLOG $p \preceq q$. Then there exists a projection $r \in P(M)$ such that $r \leq q$ and $r \sim p$. Since $r \leq q$, we can write $r - q = x^*x$ for some $x \in M$. Since $r \sim p$, the first part of this problem implies $\tau(r) = \tau(p) = \tau(q)$. Hence, $\tau(x^*x) = \tau(q - r) = 0$. Hence, since τ is faithful, $x = 0$. Thus, $q = r$, so $q \sim p$. □

7 Let $M \subset \mathcal{B}(H)$ be a von Neumann algebra. A vector $\xi \in H$ is called cyclic for M if $H = \overline{M\xi}^{\|\cdot\|}$. We call ξ separating for M if for each $x \in M$, $x\xi = 0 \implies x = 0$. Show that ξ is cyclic for M if and only if ξ is separating for M'.

Proof. Suppose ξ is separating for M'. Let p be the projection onto $\overline{M\xi}$. Since M is unital, $(p-1)\xi = 0$. Since ξ is separating for M', it is enough to show that $p - 1 \in M'$. Or, equivalently, show that $p \in M'$.

Suppose $x \in M$ and $v \in M\xi$. Then $xpv = xv = pxv$. Thus $xpv = pxv$ for all $v \in M\xi$. Since $xp - px$ is a bounded operator, the same identity holds for all $v \in \overline{M\xi}$. If $v \in (M\xi)^\perp$, then $xpv = 0$. On the other hand, for all $w \in M\xi$, we have $\langle xv, w\rangle = \langle v, xw\rangle = 0$. Thus, $pxv = 0$. Thus, since $H = \overline{M\xi} \oplus (M\xi)^\perp$, we have $px = xp$. Thus, $p \in M'$.

Conversely suppose ξ is cyclic for M. Further suppose that $x\xi = 0$ for some $x \in M'$. Then $xy\xi = yx\xi = 0$ for all $y \in M$. Thus, $xM\xi = 0$. Since x is bounded, this implies $0 = x\overline{M\xi} = xH$. Thus, $x = 0$.

□

8 Let Γ be a group. Recall from class the definition of the (left) group von Neumann algebra $L\Gamma = \lambda(\mathbb{C}\Gamma)'' \subset \mathcal{B}(\ell^2\Gamma)$ and the normal tracial state $\tau : L\Gamma \to \mathbb{C}$; $\tau(x) = \langle x\delta_e | \delta_e\rangle$.

(a) Consider the right regular representation $\rho : \mathbb{C}\gamma \to \mathcal{B}(\ell^2\Gamma)$; $\rho(g)\delta_h = \delta_{hg^{-1}}$, $g, h \in \Gamma$. Show that $\rho(\mathbb{C}\Gamma) \subset L\Gamma'$.

Proof. Let $g, h, k \in G$. Then $\rho(g)\lambda(h)\delta_k = \delta_{hkg^{-1}} = \lambda(h)\rho(g)\delta_k$. Linearizing, we have $\rho(\mathbb{C}\Gamma) \subset \lambda(\mathbb{C}\Gamma)'$.

Let $x \in L\Gamma'$ and $y \in \rho(\mathbb{C}\Gamma)$. Then there exists a net $(x_i) \subset \lambda(\mathbb{C}\Gamma)$ such that $x_i \to x$ in the WOT. Thus, for all $\xi, \eta \in \ell^2\Gamma$, we have

$$\begin{aligned}
0 &= \langle (x_i y - y x_i)\xi, \eta\rangle \\
&= \langle x_i y\xi, \eta\rangle - \langle x_i\xi, y^*\eta\rangle \\
&\to \langle xy\xi, \eta\rangle - \langle x\xi, y^*\eta\rangle \\
&= \langle (xy - yx)\xi, \eta\rangle
\end{aligned}$$

Hence, x and y commute. Since x and y were arbitrary, this implies $\rho(\mathbb{C}\Gamma) \subset L\Gamma'$. □

(b) Define a linear map $\Lambda_\tau : L\Gamma \to \ell^2\Gamma$ by $\Lambda(x) = \hat{x} = x\delta_e$. Use part (a) above to show that Λ_τ is injective. Hence any $x \in L\Gamma$ is uniquely represented by a "Fourier series $\hat{x} = \sum_{g\in\Gamma} \hat{x}(g)\delta_g \in \ell^2\Gamma$.

Proof. Suppose $\Lambda_\tau(x) = 0$. Then for all $g \in \Gamma$, we have $0 = \rho(g)\Lambda_\tau(x) = \rho(g)x\delta_e = x\delta_g$, where the last equality follows from part (a). Thus, $x = 0$. Thus, Λ_τ is injective. □

(c) Use the above to conclude that τ is a faithful state on $L\Gamma$.

Proof. Suppose $\tau(x^*x) = 0$. Then $0 = \langle x^*x\delta_e, \delta_e\rangle = \langle x\delta_e, x\delta_e\rangle$. Thus $x\delta_e = 0$, so part (b) implies that $x = 0$. □

(d) A group is said to have infinite conjugacy classes (icc) if for every $h \neq e$, the conjugacy class C_h of h is infinite. Show that if $x \in L\Gamma \cap L\Gamma'$, then $\hat{x}$ is constant on conjugacy classes. Conclude that if Γ is icc, then $L\Gamma$ is a II_1-factor.

Proof. Suppose $x \in L\Gamma \cap L\Gamma'$, and $g, h \in \Gamma$. Then

$$\begin{aligned}
\hat{x}(g) &= \langle x\delta_e, \delta_g\rangle \\
&= \langle \lambda(h)x\delta_e, \lambda(h)\delta_g\rangle \\
&= \langle x\delta_h, \delta_{hg}\rangle \\
&= \langle x\rho(h)\delta_e, \delta_{hg}\rangle \\
&= \langle \rho(h)x\delta_e, \delta_{hg}\rangle \\
&= \langle x\delta_e, \rho(h^{-1})\delta_{hg}\rangle \\
&= \langle x\delta_e, \delta_{hgh^{-1}}\rangle \\
&= \hat{x}(hgh^{-1})
\end{aligned}$$

Now suppose $L\Gamma$ is icc, and $x \in L\Gamma \cap L\Gamma'$. Since $\hat{x}$ is constant on conjugacy classes, it must be zero for all non-trivial conjugacy classes (otherwise, its ℓ^2-norm would be infinite). Hence $L\Gamma \cap L\Gamma' = \mathbb{C}$, so $L\Gamma$ is a factor. Since τ is a normal, faithful, tracial state, $L\Gamma$ is finite. Hence, since $L\Gamma$ is infinite dimensional, it is a II_1-factor. □

(e) Conversely, show that if Γ is not icc, then $L\Gamma \cap L\Gamma' \neq \mathbb{C}1$.

Proof. Let $C \subset \Gamma$ be a nontrivial, finite conjugacy class. Then $\lambda(\delta_C) \in L\Gamma$. Moreover, if $g \in \Gamma$, then $\lambda(g)\lambda(\delta_C)\lambda(g^{-1}) = \lambda(\delta_C)$. Hence, by linearity, $\lambda(\delta_C) \in \mathbb{C}\Gamma'$. Moreover, if we have a net $(x_i) \subset \lambda(\mathbb{C}\Gamma)$ with $x_i \to x$ in the WOT, we have

$$\begin{aligned}
0 &= \langle (x_i\lambda(\delta_C) - \lambda(\delta_C)x_i)\xi, \eta\rangle \\
&\to \langle (x\lambda(\delta_C) - \lambda(\delta_C)x)\xi, \eta\rangle,
\end{aligned}$$

for all $\xi, \eta \in H$. Thus $\lambda(\delta_C) \in L\Gamma'$. □

9 Consider the group S_∞ given by all finite permutations of $\mathbb{N}$ and the non-commutative free group $\mathbb{F}_2$ on two generators. Show that both of these groups are icc.

Proof. Let $\sigma \in S_\infty$ be a nontrivial permutation. Then there exist $x \neq y \in \mathbb{N}$ such that $\sigma(x) = y$. For $n \in \mathbb{N}$, let $\tau_n \in S_\infty$ be the transposition interchange y and n. Then for all n greater than x and y, we have $\tau_n \sigma \tau_n^{-1}(x) = \tau_n \sigma(x) = \tau_n y = n$. Thus, $\tau_n \sigma \tau_n^{-1}$ are distinct for infinitely many n.

Let $a, b \in \mathbb{F}_2$ be the standard generators. Let $g \in \mathbb{F}_2$ be a nontrivial element. WLOG the first letter of the reduced word for g is a. I claim that the conjugates $g_n := b^n g b^{-n}$ are distinct for all $n \geq 0$. This is because the reduced word for g_n must start with $b^n a$ since the b^{-n} can only cancel b's on the right side of the this a. $\square$