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HW 4

B-3 p67 Let $n, d > 0$ and

$$A = \{\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1} | \alpha_i \geq 0 \text{ and} \sum_{i=0}^{n} \alpha_i = d\}.$$

We note that $|A| = \binom{n+d}{d}$. We set $|A| = N + 1$.

The cardinality of the support of $\alpha \in A$ is called the breadth of α.

Consider distinct integers $i, j \in [0, n]$. We denote by (i, j) (resp. (i)) the element $\alpha \in A$ defined by $\alpha_k = 0$ for $k \neq i, j$; $\alpha_i = d - 1$; $\alpha_j = 1$ (resp. $\alpha_k = 0$ for $k \neq i$ and $\alpha_i = d$).

If $X_0, \ldots, X_n$ are variables, then for any $\alpha \in A$ we set $X^\alpha = X_0^{\alpha_0} \cdots X_n^{\alpha_n}$.

We consider the map $\phi : \mathbb{P}^n \to \mathbb{P}^N$ defined by the formula

$$\phi(x_0, \ldots, x_n) = (x^\alpha)_{\alpha \in A}.$$

1) Prove that ϕ is an injective map.

   Proof. Suppose $\phi(x) = \phi(y)$. $\square$

2) Consider the ring homomorphism

$$\theta : k[(Y_\alpha)]_{\alpha \in A} \to k[X_0, \ldots X_n]$$

   defined by $\theta(Y_\alpha) = X^\alpha$. Set $I = \ker \theta$ and consider $V = V_P(I)$ (the Veronese variety). Prove that I is a homogeneous ideal and $\phi(\mathbb{P}^n) \subset V$.

3) Consider $\alpha, \beta, \gamma, \delta \in A$. We assume $\alpha + \beta = \gamma + \delta$. Prove that $Y_\alpha Y_\beta - Y_\gamma Y_\delta$ is in the ideal I.

4) Prove that the open sets $D^+(Y_{(i)})$ cover V. (Hint in book).

5) We define $\psi : D^+(Y_{(i)}) \cap V \to D^+(X_i)$ by the formula

$$\psi((y_\alpha)) = (y_{(i,0)}, y_{(i,1)}, \ldots, y_{(i)}, \ldots, y_{(i,n)}).$$

   Prove that ϕ and ψ are mutually inverse morphisms on the open sets in question.

6) Prove that ϕ gives an isomorphism from $\mathbb{P}^n$ to the Veronese variety V.