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MATH 663 - Subfactors, Knots, and Planar Algebras (Fall 2017)

HW 5

15 Let $n \in \mathbb{N}$ with $n \geq 2$ be fixed. Consider the symmtric matrix $\Lambda \in M_n(\mathbb{C})$ defined by

$$\Lambda_{ij} = \begin{cases} 1, & \text{if } |i-j| = 1 \\ 0, & \text{else} \end{cases}$$

(a) Prove that the eigenvalues of Λ are precisely the zeros of the n-th Chebyshev polynomial S_n of the second kind, i.e.

$$\left\{ 2\cos\left(\frac{k\pi}{n+1}\right) \mid k = 1, \dots, n \right\},$$

where an eigenvector corresponding to the eigenvalue $\lambda_k = 2\cos\left(\frac{k\pi}{n+1}\right)$ is given by

$$t_k = \left(\sin\left(\frac{k\pi}{n+1}\right), \sin\left(\frac{2k\pi}{n+1}\right), \dots, \sin\left(\frac{nk\pi}{n+1}\right) \right)^T$$

(b) Deduce that all values in

$$\left\{ 4\cos^2\left(\frac{\pi}{n+1}\right) \mid n \geq 2 \right\}$$

16 Let a real matrix $P \in M_n(\mathbb{R})$ be a real symmetric matrix with nonnegative entries. Suppose there exists a real eigenvector $y \in \mathbb{R}^n$ of P with positive entries and corresponding eigenvalue $\lambda \geq 0$.

(a) On the set

$$\Gamma_n := \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1, \dots x_n > 0 \}$$

consider the function

$$L : \Gamma_n \to [0, \infty), x \mapsto \max\{ s \geq 0 \mid sx \leq Px \},$$

where $x \leq x'$ means that it holds entry-wise. Prove that

$$\sup_{x \in \Gamma_n} L(x) = \lambda = L(y).$$

(b) Deduce that $\|P\| = \lambda$.

17 Find braids whose closures are the given links, and their associated Jones polynomials.

18 Let $\mathcal{H}$ be a separable complex Hilbert space and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n \xi = \pi(\xi)$$

holds for any $\xi \in \mathcal{H}$, where π denotes the orthogonal projection from $\mathcal{H}$ onto the closed subspace $\mathcal{H}^U$ of all U-invariant vectors in $\mathcal{H}$.