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Math 643 - Algebraic Topology I

HW 1

0.6 Let $A = (a_{ij})$ be a real $n \times n$ matrix with $a_{ij} > 0$ for all i, j. Prove that A has a positive eigenvalue λ; moreover there is a corresponding eigenvector $x = (x_i)$ with $x_i > 0$ for all i. (Hint: First define $\sigma : \mathbb{R}^n \to \mathbb{R}$ by $\sigma((x_i)_{i=1}^n) = \sum_i x_i$. Then define $g : \Delta^{n-1} \to \Delta^{n-1}$ by $g(x) = Ax/\sigma(Ax)$. Apply the Brouwer fixed point theorem.)

Proof. First note that A maps the positive orthant into the positive orthant, and $A(\Delta^{n-1})$ does not meet 0. Hence $\sigma(Ax) > 0$ for all x, so g is continuous. Moreover, $\sigma(g(x)) = \sigma(Ax)/\sigma(Ax) = 1$. Hence g maps into Δ^{n-1} since $g(x)$ also maps the positive orthant to itself.

Thus, by the Brouwer fixed point theorem, $g(x) = x$ for some $x = (x_i) \in \Delta^{n-1}$. This means $Ax = \sigma(Ax)x$. As mentioned before, $\sigma(Ax) > 0$. To see that $x_i > 0$ for all i, first pick some j such that $x_j > 0$ (we can do this since $x \in \Delta^{n-1}$). Then for all i, we have $\sigma(Ax)x_i = \langle Ax, e_i \rangle \geq \langle Ax_j, e_i \rangle > 0$. $\square$

0.17 Let $\mathcal{C}$ and $\mathcal{A}$ be categories, and let $\sim$ be a congruence on $\mathcal{C}$. If $T : \mathcal{C} \to \mathcal{A}$ is a functor with $T(f) = T(g)$ whenever $f \sim g$, then T defines a functor $T' : \mathcal{C}' \to \mathcal{A}$ (where $\mathcal{C}'$ is the quotient category) by $T'(X) = T(X)$ for every object X and $T'([f]) = T(f)$ for every morphism f.

Proof. T' is well-defined, and takes identity maps to identity maps. Lastly, $T'([g][f]) = T(gf) = T(g)T(f) = T'([g])T'([f])$. $\square$

0.20(ii) Show that $X \mapsto C(X)$ gives a functor $\mathbf{Top} \to \mathbf{Rings}$.

Proof. Define the functor $F : \mathbf{Top} \to \mathbf{Rings}$ by $F(X) = C(X)$ and if $\phi : X \to Y$ define $F(\phi) : C(Y) \to C(X)$ by $F(\phi)(f) = f(\phi(x))$. Then F is well-defined and takes identities to identities. Suppose $\phi : X \to Y$, $\psi : Y \to Z$, and $f \in C(Z)$. Then $F(\psi\phi)(f) = f(\psi(\phi(x))) = F(\phi)f(\psi(x)) = F(\phi)F(\psi)(f)$. $\square$