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HW 1

1 Let V be a real finite dimensional vector space with inner product $\langle \cdot, \cdot \rangle_V$, and let $B = \{v_1, v_2, \dots, v_n\}$ be an ordered basis for V.

a. If Φ is the associated coordinate map, show that $\langle \cdot, \cdot \rangle_{\mathbb{R}^n} := \langle \Phi^{-1}(\cdot), \Phi^{-1}(\cdot) \rangle$ defines an inner product on $\mathbb{R}^n$.

b. Show that if $x, y \in \mathbb{R}^n$, then $\langle \cdot, \cdot \rangle_{\mathbb{R}^n} = y^T G x$, where $G_{jk} = \langle v_k, v_j \rangle_V$.

Proof. a. Let $x, y, z \in \mathbb{R}^n$ and $\alpha, beta \in \mathbb{R}$.

Symmetry: $\langle x, y \rangle = \langle \Phi^{-1}(x), \Phi^{-1}(y) \rangle = \langle \Phi^{-1}(y), \Phi^{-1}(x) \rangle = \langle y, x \rangle$.

Bilinearity:

$$\begin{aligned} \langle \alpha x + \beta y, z \rangle &= \langle \Phi^{-1}(\alpha x + \beta y), \Phi^{-1}(z) \rangle \\ &= \langle \alpha \Phi^{-1}(x) + \beta \Phi^{-1}(y), \Phi^{-1}(z) \rangle \\ &= \alpha \langle \Phi^{-1}(x), \Phi^{-1}(z) \rangle + \beta \langle \Phi^{-1}(y), \Phi^{-1}(z) \rangle \\ &= \alpha \langle x, z \rangle + \beta \langle y, z \rangle. \end{aligned}$$

Linearity in the second component follows from symmetry.

Positivity: $\langle x, x \rangle = \langle \Phi^{-1} x, \Phi^{-1} x \rangle \geq 0$ with equality iff $\Phi^{-1}(x) = 0 \equiv x = 0$ since Φ is an isomorphism.

b. Since both sides are linear in each variable, it suffices to check the equation for $x = e_j$ and $y = e_k$. $\langle e_j, e_k \rangle = \langle \Phi^{-1}(e_j), \Phi^{-1}(e_k) \rangle = \langle v_j, v_k \rangle = e_k^T G_{kj} e_j$. □

2 In the previous problem, suppose that $B = \{v_1, v_2, \dots, v_n\}$ is simply a subset of vectors in V and $U = \text{span}(B)$. Show that B is a basis for U iff $y^T G x$ is an inner product for $\mathbb{R}^n$.

Proof. The forward implication follows from (1). For the converse, suppose $y^T G x$ is an inner product and $\sum_{i=1}^n a_i v_i = 0$ for $a_i \in \mathbb{R}$. Then $0 = \langle \sum_i a_i v_i, \sum_i a_i v_i \rangle = \sum_{i,j} a_i a_j \langle v_i, v_j \rangle = a^T G a$, where $a = (a_1, a_2, \dots, a_n)$. Since G is positive definite, $a = 0$. Thus B is linearly independent. □

3 Let U be a subspace of an inner product space V.

a. Fix $v \in V$. Show that $p \in U$ satisfies $\min_{u \in U} ||v - u|| = ||v - p||$ iff $v - p$ is orthogonal to the subspace U.

b. Show that p is unique, given that it exists for v.

c. Suppose p exists for every $v \in V$. Define $P : V \to U$ by $Pv := p$. Show that P is linear and $P^2 = P$.

Proof. a. Suppose $\min_{u \in U} ||v - u|| = ||v - p||$. If $v - p$ is not orthogonal to U, then we can pick $u \in U$ such that $\langle v - p, u \rangle \neq 0$. By multiplying u by the appropriate phase, WLOG $\langle v - p, u \rangle > 0$. Let $t \in \mathbb{R}$. Then $||v - p - tu||^2 =$

$||v-p||^2 - 2t\langle v-p,u\rangle + t^2||u||^2$, which is minimized when $t = \frac{\langle v-p,u\rangle}{||u||^2}$. This contradicts the minimality of p.

Conversely suppose $v-p$ is orthogonal to U. Then for any $u \in U$, we have $||v-u||^2 = ||v-p+(p-u)||^2 = ||v-p||^2 + ||p-u||^2$. This is minimized when $u = p$.

b. Suppose both p and q satisfy the conditions in (a). Note that the orthogonal complement to U, $U^\perp$, is a subspace. Moreover if $u \in U \cap U^\perp$, then $\langle u,u\rangle = 0$, so $u = 0$. Hence, since $v-p, v-q \in U^\perp$, we have $(v-p)-(v-q) = q-p \in U^\perp$. Thus, $q-p \in U \cap U^\perp$, so $q = p$.

c. To see that P is linear, let $\alpha, \beta \in \mathbb{C}$ and $v, w \in V$. Then for any $u \in U$, we have $\langle \alpha v + \beta w - (\alpha P(v) + \beta P(w)), u\rangle = \alpha\langle v - P(v), u\rangle + \beta\langle w - P(w), u\rangle = 0$. Hence, $P(\alpha v + \beta w) = \alpha P(v) + \beta P(w)$.

To see that $P^2 = P$, let $v \in V$ and $p = P(v)$. Then $P^2(v) = P(p) = min_{u\in U}||p-u|| = p = P(v)$. □

4 Let V, B, U, G be defined as in problem 1, except B is a basis for U.

a. Let $v \in V$ and $d_k := \langle v, v_k\rangle_V$. Show that $p = \sum_j x_j v_j \in U$ is the orthogonal projection of v onto U iff the x_j's satisfy the *normal equations*, $d_k = \sum_j G_{kj}x_j$.

b. Show that the orthogonal projection $P : V \to U$ exists.

c. Show that if B is orthonormal, then $Pv = \sum_j \langle v, v_j\rangle_V v_j$.

Proof. a. Suppose that p is the orthogonal projection of v onto U. Then $\langle p - v, v_k\rangle = 0$ for every k. Hence $d_k = \langle v, v_k\rangle = \langle p, v_k\rangle = \langle \sum_j x_j v_j, v_k\rangle = \sum_j G_{kj}x_j$.

Conversely, suppose $d_k = \sum_j G_{kj}x_j$ for each k. Then $\langle v, v_k\rangle = \langle \sum_j x_j v_j, v_k\rangle = \langle p, v_k\rangle$ for each k. Hence $\langle p-v, v_k\rangle = 0$ for each k, so $\langle p-v, u\rangle = 0$ for all $u \in U$ since (v_k) is a basis for U.

b. Suppose $z \in \ker(G)$. Then $z^T G z = 0$, so $z = 0$ since G is positive definite. Hence G is invertible, so we can define $x_j = \sum_k (G^{-1})_{jk} d_k = \sum_k (G^{-1})_{jk}\langle v, v_k\rangle_V$.

c. If B is orthonormal, $G_{jk} = \langle v_k, v_j\rangle_V = \delta_{ij}$. Hence from (b), $Pv = \sum_j x_j v_j = \sum_{j,k}(G^{-1})_{jk}\langle v, v_k\rangle_V v_j = \sum_j \langle v, v_j\rangle_V v_j$.

□

5 Equality holds in Schwarz's inequality iff u and v are linearly dependent.

Proof. Suppose u, v are linearly dependent. If either u or v is zero, then equality holds trivially. Otherwise, $u = kv$ for some scalar k. Then $|\langle u, v\rangle| = |\langle u, ku\rangle| = |k|||u||^2 = ||u||||v||$.

Conversely, suppose $|\langle u, v\rangle| = ||u||||v||$. If either u or v is zero, we are done.

Otherwise, let $t = \frac{\langle u,v\rangle}{||v||^2}$. Then

$$\begin{aligned} ||u - tv||^2 &= ||u||^2 - 2\Re(t\langle v, u\rangle) + |t|^2||v||^2 \\ &= ||u||^2 - \frac{2}{||v||^2}|\langle u, v\rangle|^2 + \frac{|\langle u, v\rangle|^2}{||v||^2} \\ &= ||u||^2 - 2||u||^2 + ||u||^2 \\ &= 0. \end{aligned}$$

Hence, $u - tv = 0$, so u, v are linearly dependent. □

6 Suppose that $F \in C[0, 1]$, $F(x) \geq 0$, and $F(x_0) > 0$ for some $x_0 \in [0, 1]$. Show that there is a closed interval $[a, b] \subset [0, 1]$, $a \neq b$, that contains x_0 and on which $F(x) \geq \frac{1}{2}F(x_0)$.

Proof. Since F is continuous, there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|F(x)-F(x_0)| < F(x_0)/2$. Hence, if $x \in [x-\delta/2, x+\delta/2]$, then $F(x)-F(x_0) > -F(x_0)/2$ which implies $F(x) > F(x_0)/2$. □

7 Let $b = \{v_1, \ldots, v_n\}$ be a basis for a vector space V. Define linear functionals ϕ_k for $1 \leq k \leq n$ via $\phi_k(v_j) = \delta_{jk}$.

1. Show that $B^* := \{\phi_1, \ldots, \phi_n\}$ is a basis for V^*.
2. Let $V = \mathbb{R}^n$ and suppose that $B = \{x_1, \ldots, x_n\}$ is a basis of column vectors for $\mathbb{R}^n$, and let $X = [x_1 \cdots x_n]$. Show that R^{n*} may be identified with the set of $1 \times n$ row vectors, and that B^* is then the set of rows of X^{-1}.

Proof. 1) To see that B^* is linearly independent, suppose $\sum_{i=1}^n a_i\phi_i = 0$ for some $a_i \in \mathbb{R}$. For any v_j, applying the left-hand side to v_j yields $a_j = 0$. Hence, $a_i = 0$ for all i.

To see that B^* spans V^*, let $\psi \in V^*$ and $x \in V$. Then $x = \sum_i x_iv_i$ for some $x_i \in \mathbb{R}$. Then $\psi(x) = \sum_i x_i\psi(v_i) = \sum_i x_i\psi(v_i)\phi_i(v_i) = \sum_i \psi(v_i)\phi_i(x_iv_i) = \sum_i \psi(v_i)\phi_i(x) = (\sum_i \psi(v_i)\phi_i)x$. Hence, $\psi = \sum_i \psi(v_i)\phi_i$.

2) For $\psi \in \mathbb{R}^{n*}$ let $T(\psi) = (\psi(e_1), \psi(e_2), \ldots, \psi(e_n))$. Then $T(\psi)$ acting by matrix multiplication on column vectors in $\mathbb{R}^n$ is a linear functional. Moreover, $T(\psi)(e_j) = \psi(e_j)$ for each j, so $T(\psi)$ and ψ must agree everywhere.

For each j, k, $T(\phi_k)(x_j) = \delta_{jk}$. Hence if Y is the matrix with rows $T(\phi_k)$ for $1 \leq k \leq n$, then $YX = I$. Thus, $Y = X^{-1}$. □