Problem Set 3
CSCE 440/640

Due dates: Electronic submission of the pdf file of this homework is due on **9/21/2016 before 2:50pm** on ecampus.tamu.edu, a signed paper copy of the pdf file is due on **9/21/2014** at the beginning of class.

Name: (put your name here)

Resources. I used the program "julia" to do some matrix multiplication.

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

Signature: ____________________________

Read chapter 4 in the lecture notes and make five insightful comments on perusall. Read chapter 6 in the textbook.

Problem 1. (10 points) Exercise 2.24 in the lecture notes.

Solution.

$$P(0) = \left(\frac{1}{3}\right)^2 + \left(\frac{\sqrt{5}}{3}\right)^2 = \frac{1}{9} + \frac{5}{9} = \frac{2}{3},$$

and

$$P(1) = \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{1}{3}.$$

The resulting states are

$$v_0 = \frac{1}{\sqrt{6}} |00\rangle + \sqrt{\frac{5}{6}} |10\rangle ,$$

and

$$v_1 = |01\rangle .$$

Problem 2. (20 points) Exercise 2.26 in the lecture notes.

Solution. The quantum circuit begins by inserting a $|0\rangle$-valued qubit at the least significant position. The resulting state is $|0000\rangle$.

Now apply (c) of problem 2.27. At the end of this circuit, first three qubits are in the desired state.

Problem 3. (20 points) Exercise 2.27 in the lecture notes. (a) Design the circuit, (b) prove the correctness of the circuit and (c) show how to create the state.

Solution. (a) Insert a qubit in the $|0\rangle$ state in the least significant position. Let's denote the position as -1. Then execute $\Lambda_{0,-1} \circ \Lambda_{1,-1} \circ \Lambda_{2,-1}$.

(b) Each controlled-not $\Lambda_{x,-1}$ flips the -1-bit iff the x-qubit has value $|1\rangle$. In other words, if v_i denotes the value of the i-th bit, then $\Lambda_{x,-1}$'s only effect is to replaces v_{-1} with $v_{-1} \oplus v_x$. Hence, the final value of the -1-bit is $0 \oplus x_2 \oplus x_1 \oplus x_0 = x_2 \oplus x_1 \oplus x_0$.

(c) Starting with $|0000\rangle$, apply the map $H \otimes H \otimes H \otimes 1$. After this application, the state is a uniformly distributed superposition between $|0\rangle$ and $|1\rangle$ in the three most significant bits, and the state of the least significant bit is $|0\rangle$.

Apply the circuit in (a) to the three most significant bits, inserting the new bit at position -1. The state of the other four qubits remains unchanged, but the new qubit is in state $|0\rangle$ if the first three qubits have even parity, and $|1\rangle$ otherwise.

Finally apply a $\Lambda_{-1,1}$. This turns the odd parity states into even parity states. By inspection, this gives the desired state in the most significant four qubits.

Problem 4. (20 points)
(a) Exercise 6.1.1 (a) in the textbook KLM (should read Figure 6.1)
(b) Exercise 6.1.1 (b) in the textbook KLM

Solution. (a) Since U is unitary, $\overline{U}^T U = I$. This means $\delta_{jk} = \sum_i \overline{u}_{ij} u_{ik}$, i.e. that the columns of U are orthonormal. Setting $k = j$, we get $1 = \sum_i |u_{ij}|^2$.
(b) Any example given from (a) will have the rows summing to zero as well because the rows of a unitary matrix are orthonormal, too. But this doesn't have to be the case for a stochastic matrix of this type. A counterexample is

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

Problem 5. (10 points) Exercise 3.4 in the lecture notes.

Solution. Just write out where each basis vector goes and eyeball it.
(a) If $f(x) = 0$, then the corresponding circuit is just the identity.
(b) If $f(x) = 1$, then the corresponding circuit is $1 \otimes X$.
(c) If $f(x) = x$, then the corresponding circuit is $\Lambda_{1,0}$.
(d) If $f(x) = 1 + x$, then the corresponding circuit is $(X \otimes 1)\Lambda_{1,0}(X \otimes 1)$.

Problem 6. (20 points) Consider a system of two quantum bits and a controlled-not gate $\lambda_{0,1}(X)$ that has the least significant bit as a control bit and acts on the most significant quantum bit. Dispel the myth that the control bit of the controlled-not gate remains unaffected. Specifically, describe the action of the controlled-not gate on the following four states:

$$|0_H\rangle \otimes |0_H\rangle, \quad |0_H\rangle \otimes |1_H\rangle, \quad |1_H\rangle \otimes |0_H\rangle, \quad |1_H\rangle \otimes |1_H\rangle,$$

where

$$|0_H\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \quad \text{and} \quad |1_H\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle.$$

Express the result in terms of the $|0_H\rangle$ and $|1_H\rangle$ basis.

Solution. The transition matrices from the H-basis to the standard basis and vice versa are both

$$S := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

In the standard basis,

$$\Lambda_{0,1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Conjugating by the transition matrix gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus,

$$\begin{aligned}
\Lambda_{0,1}(|0_H\rangle \otimes |0_H\rangle) &= |0_H\rangle \otimes |0_H\rangle \\
\Lambda_{0,1}(|0_H\rangle \otimes |1_H\rangle) &= |0_H\rangle \otimes |1_H\rangle \\
\Lambda_{0,1}(|1_H\rangle \otimes |0_H\rangle) &= |1_H\rangle \otimes |1_H\rangle \\
\Lambda_{0,1}(|1_H\rangle \otimes |1_H\rangle) &= |1_H\rangle \otimes |0_H\rangle
\end{aligned}$$

Checklist:

☐ Did you add your name?
☐ Did you disclose all resources that you have used?
(This includes all people, books, websites, etc. that you have consulted)
☐ Did you sign that you followed the Aggie honor code?
☐ Did you solve all problems?
☐ Did you submit the pdf file resulting from your latex source file on ecampus?
☐ Did you submit a hardcopy of the pdf file in class?