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HW 11

1 Let f be increasing on $[0,1]$ and

$$g(x) = \limsup_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}, \quad \text{for } 0 < x < 1.$$

Prove that if $A = \{x \in (0,1) : g(x) > 1\}$ then

$$f(1) - f(0) \geq m^*(A).$$

Hint: Vitali's Lemma.

Proof. To avoid worrying about endpoints, extend f to be constant on $(-\infty, 0]$ and $[1,\infty)$. This does not change A.

For each $x \in A$, pick a sequence $(h_{x,n})$ with $\lim_{n\to\infty} h_{x,n} \to 0$ and

$$\lim_{n\to\infty} \frac{f(x+h_{x,n})-f(x-h_{x,n})}{2h_{x,n}} > 1,$$

for all n.

Then $\mathcal{B} = \{B(h_{x,n}, x) : x \in A\}$ forms a Vitali cover for A. Let $\epsilon > 0$. We can pick a finite set $\mathcal{F} \subset \mathcal{B}$ of disjoint balls with $m(\bigcup \mathcal{F}) > m^*(A) - \epsilon$. Let $(a_i, b_i)_{i=1}^n$ be an enumeration of $\mathcal{F}$ with $a_1 < b_1 < a_2 < \ldots < b_n$. Then $f(1) - f(0) \geq \sum_{i=1}^n f(b_i) - f(a_i) \geq b_i - a_i > m^*(A) - \epsilon$. Letting $\epsilon \to 0$, we have $f(1) - f(0) \geq m^*(A)$. $\square$

2 Let $f : [a,b] \to \mathbb{R}$ be an increasing function. Using Vitali's lemma, show that

$$m(\{D^+ f(x) \neq D^- f(x)\}) = 0.$$

where $D^+(f)$ is the upper derivative from the right, and $D^-(f)$ is the lower derivative from the right.

Proof. Acknowledgement: I looked at http://www.math.ucla.edu/~ralston/245a.1.08f/Vitali.pdf for hints.

It suffices to show that $E_{p,q} = \{x \in [a,b] : D^- f(x) < p < q < D^+ f(x)\}$ has measure 0 for every $p, q \in \mathbb{Q}$ with $p < q$.

Let $\epsilon > 0$. Pick an open set $U \supset E_{p,q}$ with $m(U) < m^*(E_{p,q}) + \epsilon$.

If $x \in E_{p,q}$, then there exist arbitrarily small h for which $\frac{f(x+h)-f(x)}{h} < p$. Thus intervals of the form $[x, x+h) \subset U$ with this property form a Vitali cover for $E_{p,q}$. By the Vitali lemma, we can pick a disjoint finite subset of these intervals $([x_k, x_k + h_k))_{k=1}^n$ such that $\sum_k h_k > m^*(E_{p,q}) - \epsilon$.

Similarly for $y \in E_{p,q} \cap \bigcup_k [x_k, x_k + h_k)$ there exist arbitrarily small l for which $\frac{f(y+l)-f(y)}{l} > q$. Thus, sets of the form $[y, y+l)$ with this property form a Vitali cover for $E_{p,q} \cap \bigcup_k [x_k, x_k + h_k)$. Moreover, by throwing sets out of the cover, we can assume that each interval $[y, y + l)$ lies within an interval $[x_k, x_k + h_k)$. By the Vitali Lemma, we get a disjoint finite subset of these intervals $([y_j, y_j + l_k))_{j=1}^m$ with

$$\begin{aligned}
\sum_k l_k &> m^*(E_{p,q} \cap \bigcup_k [x_k, x_k + h_k)) - \epsilon \\
&= m(\bigcup_k [x_k, x_k + h_k)) - m^*(E_{p,q}^c \cap \bigcup_k [x_k, x_k + h_k)) - \epsilon \\
&> (m^*(E_{p,q}) - \epsilon) - m^*(E_{p,q}^c \cap U) - \epsilon \\
&> m^*(E_{p,q}) - 3\epsilon
\end{aligned}$$

Then we have

$$\begin{aligned}
q(m^*(E_{p,q}) - 3\epsilon) &= q \sum_k l_k \\
&< \sum_j f(y_j + l_j) - f(x_j) \\
&\leq \sum_k f(x_k + h_k) - f(x_k) \\
&< p \sum_k h_k \\
&< p(m^*(E_{p,q}) - \epsilon).
\end{aligned}$$

Letting $\epsilon \to 0$, we have $0 \leq (p - q)m^*(E_{p,q})$, so $m^*(E_{p,q}) = 0$. □

3 Assume that $f : [a, b] \to \mathbb{R}$ is continuous and that $D^+ f(x) > 0$, for all $x \in [a, b]$. Show that f is nondecreasing on $[a, b]$.

Proof. Suppose f is not nondecreasing. Then there exist $a \leq c < d \leq b$ with $f(c) > f(d)$. By the extreme value theorem, f achieves a maximum M on $[c, d]$. Let $u = \sup\{x \in [c, d] : f(x) = M\}$. Since f is continuous, $f(u) = M$. Since $M \geq f(c) > f(d)$, we have $u < d$. Moreover, $f(x) < M$ for all $x \in [u, d]$. Thus $D^+ f(u) \leq 0$, a contradiction. □

4 Determine whether or not the following functions are of bounded variation on $[-1, 1]$.

(a) $f(x) = x^2 \sin(1/x^2), \quad x \neq 0, f(0) = 0$

(b) $f(x) = x^2 \sin(1/x), \quad x \neq 0, f(0) = 0.$

Proof. For (a), we have

$$\begin{aligned} T_{-1}^1(f) &\geq \sum_{n=1}^{N} |f((n\pi)^{-1/2} - f((n\pi + \pi/2)^{-1/2})| \\ &= \sum_{n=1}^{N} |(n\pi + \pi/2)^{-1})| \\ &\to \infty \end{aligned}$$

as $N \to \infty$, so f is not of bounded variation.

For (b), if $(x_n)_{n=0}^N$ is a partition of $[-1, 1]$

$$\begin{aligned} \sum_{n=1}^{N} |f(x_n) - f(x_{n-1})| &\leq C + 2\sum_{n=1}^{\infty} |f((n\pi - \pi/2)^{-1}) - f((n\pi + \pi/2)^{-1})| \\ &= C + 2\sum_{n=1}^{\infty} (n\pi - \pi/2)^{-2} + (n\pi + \pi/2)^{-2}, \end{aligned}$$

which converges. Hence f is of bounded variation. □

5 Let f be of bounded variation on $[a, b]$, then

$$\int_a^b |f'(t)|dt \leq T_a^b(f).$$

Proof. We have

$$\begin{aligned} \int_a^b |f'(t)|dt &= \int_a^b |\frac{1}{2}(T_a^t(f) + f)' - \frac{1}{2}(T_a^t(f) - f)'|dt \\ &\leq \frac{1}{2}\int_a^b |(T_a^t(f) + f)'| + |(T_a^t(f) - f)'|dt \\ &= \frac{1}{2}\int_a^b (T_a^t(f) + f)' + (T_a^t(f) - f)'dt \\ &= \int_a^b (T_a^t(f))'dt \\ &\leq T_a^b(f), \end{aligned}$$

where the last inequality follows from decomposing the function $t \mapsto T_a^t(f)$ into its absolutely continuous and singular parts. □

6 Construct an increasing function on $\mathbb{R}$ whose discontinuities are $\mathbb{Q}$.

Proof. Let δ_x denote the Dirac measure at x. Let (q_n) be an enumeration of $\mathbb{Q}$. Let $\nu = \sum_{n=1}^{\infty} 2^{-n}\delta_{q_n}$. Let $f(x) = \nu((-\infty, x))$. Then f is increasing and has discontinuities at every rational point.

If x is irrational and $\epsilon > 0$, pick N such that $2^{-N} < \epsilon$. Pick $\delta > 0$ such that $d(x, q_n) > \delta$ for all $n \leq N$.

Suppose $d(x, y) < \delta$. WLOG suppose $x < y$. We have $|f(x) - f(y)| = \nu((x, y)) \leq \sum_{n=N+1}^{\infty} 2^{-n} = 2^{-N} < \epsilon$. $\square$