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HW 9

1 Finish the proof of the Projection Theorem: If for every $f \in \mathcal{H}$ there is a $p \in V$ such that $\|p - f\| = \min_{v \in V} \|v - f\|$ the V is closed.

Proof. $\square$

2 If $L : \mathcal{H} \to \mathcal{H}$ is a bounded linear transformation, then $\overline{R(L)} = N(L^*)^\perp$.
3 Let $\mathcal{H}$ be a Hilbert space of functions that are defined on [0,1]. In addition, suppose that $\mathcal{H} \subset C[0,1]$, with $\|f\|_{C[0,1]} \leq K\|f\|_H$ for all $f \in \mathcal{H}$. (The Sobolev space H^1 has this property.)

a. Show that the point-evaluation functional $\phi_x(f) = f(x)$ is a bounded linear functional on $\mathcal{H}$.

b. Let x be fixed. Show that there is a kernel $k(x, y) \in \mathcal{H}$ such that

$$\phi_x(f) = f(x) = \langle f, k(x, \cdot)\rangle$$

(The kernel $k(x, y)$ is called a reproducing kernel and $\mathcal{H}$ is called a reproducing kernel Hilbert space.)

c. For x, z fixed, show that $k(z, x) = \langle k(z, \cdot), k(x, \cdot)\rangle$. In addition, let $(x_j)_{j=1}^n$ be any finite set of distinct points in $[0, 1]$. Show that the matrix $G_{jk} = k(x_k, x_j)$ is ositive semidefinite; that is $\sum_{j,k} c_k \overline{c_j} k(x_k, x_j) \geq 0$.

d. Suppose the matrix G is positive definite and therefore invertible. Let $f \in \mathcal{H}$. Show that there are unique coefficients $(c_j)_{j=1}^n$ such that $s(x) = \sum_{j=1}^n k(x_j, x)c_j$ interpolates f at the x_j's.

4 Consider the finite rank (degenerate) kernel $k(x, y) = \phi_1(x)\overline{\psi_1}(y)+\phi_2(x)\overline{\psi_2}(y)$, where $\phi_1 = 2x - 1, \phi_2 = x^2, \psi_1 = 1, \psi_2 = 4x - 3$. Let $Ku = \int_0^1 k(x, y)u(y)dy$. Assume that $L := I - \lambda K$ has closed range.

a. For what values of λ does the integral equation

$$u(x) - \lambda \int_0^1 k(x, y)u(y)dy = f(x)$$

have a solution for all $f \in L^2[0, 1]$.

b. For these values, find the solution $u = (I - \lambda K)^{-1}F$ – i.e., find the resolvent.

c. For the values of λ for which the equation does not have a solution for all f, find a condition on f that guarantees a solution exists. Will the solution be unique?

5 Let $S = \{(a_j) \in \ell^2 : \sum_j (1 + j^2)|a_j|^2 \leq 1\}$. Show that S is a compact subset of ℓ^2.