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HW 2

1 Let E be a k-vector space of dimension $n+1$ and let $\mathbb{P}(E)$ be the associated projective space. If $u \in GL(E)$, u induces a bijection $\overline{u}$ from $\mathbb{P}(E)$ to itself which we call a homography.

a) What can we say about u when $\overline{u} = \mathrm{Id}$?

b) Show that the image of a projective subspace of dimension d under a homography is again a projective subspace of dimension d.

c) Conversely, show that if V and W are two projective subspaces of dimension d, then there is a homography $\overline{u}$ such that $\overline{u}(V) = W$.

d) Assume $E = k^2$ and

$$u = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

such that $ad - bc \neq 0$. Take that point $(1,0)$ in $\mathbb{P}^1(k) = \mathbb{P}(E)$ to be the point at infinity, so points x in k can be identified with points $(x,1)$ in $\mathbb{P}^1(k) - \{\infty\}$. Determine $\overline{u}$ explicitly and explain the origins of the word homography.

Proof. For (a), suppose $\overline{u} = \mathrm{Id}$. Let $(e_i)_{i=1}^k$ be a basis for E. Then $ue_i = \lambda_i e_i$ for some $\lambda_i \in k^\times$ for all i. If $\lambda_i \neq \lambda_j$, then $u(e_i + e_j)$ is not a multiple of $e_i + e_j$, a contradiction. Thus we have $u = \lambda \,\mathrm{Id}$ for some $\lambda \in k^\times$. This condition is obviously sufficient as well.

For (b), suppose $\overline{F}$ is a projective subspace of $\mathbb{P}(E)$ of dimension d. Then $F \leq E$ with $\dim(F) = d+1$, so $\dim(u(F)) = d+1$ by the rank-nullity theorem. Hence $\dim(\overline{u}(\overline{F}))) = \dim(\overline{(u(F)}) = d$.

For (c), there exist $F, G \leq E$ with $V = \overline{F}$ and $W = \overline{G}$. Pick a basis $f_1, \ldots, f_d$ for F and extend it to a basis $(f_i)_{i=1}^d$ for E. Do the same for G to get a basis $(g_i)_{i=1}^{n+1}$ for E with $G = \mathrm{span}(g_1, \ldots, g_d)$. Define a linear transformation $u : E \to E$ by sending $f_i \mapsto g_i$ for all i. Then $\overline{u}(V) = \overline{u(F)} = \overline{G} = W$.

For (d), $\overline{u}(x,1) = (\frac{ax+b}{cx+d}, 1)$ for $x \neq -d/c$, and $\overline{u}(-d/c, 1) = (1,0)$. For the point at infinity, $\overline{u}(1,0) = (a/c, 1)$ unless $c = 0$, in which case $\overline{u}(1,0) = (1,0)$.

Homography means "same graph", the transformation is supposed to be only a change of perspective. □

2 Using the same notation as in 1, we denote the canonical projection from $E - \{0\}$ to $\mathbb{P}(E)$ by p. A marking of $\mathbb{P}(E)$ consists of $n+2$ points $x_0, \ldots, x_{n+1}$ of $\mathbb{P}(E)$ such that there is a basis $e_1, \ldots, e_{n+1}$ of E such that $p(e_i) = x_i$ for all i and $p(e_1 + \ldots + e_{n+1}) = x_0$.

a) Assume $n = 1$. Prove that a marking of $\mathbb{P}(E)$ (i.e., the projective line) is exactly the data of three distinct points. (For example, in $\mathbb{P}^1(k)$ we can take $0 = (0,1)$, $\infty = (1,0)$, and $1 = (1,1)$.)

b) Prove that $n+2$ points $x_0, \ldots, x_{n+1} \in \mathbb{P}(E)$ form a marking if and only if no $n+1$ of them are contained in a hyperplane.

c) Prove that if $x_0, \ldots, x_{n+1}$ and $y_0, \ldots, y_{n+1}$ are two markings of $\mathbb{P}(E)$, then there is a unique homography which sends each x_i to y_i. Study the case $n = 1$ in detail.

Proof. For (a), let x_0, x_1, x_2 be a marking of $\mathbb{P}(E)$. Pick a basis e_1, e_2, e_3 for E corresponding to the x_i as in the definition above. Clearly $x_1 \neq x_2$ since e_2 and e_3 are linearly independent. Also $p^{-1}(x_0) = \text{span}(e_1 + e_2)$ does not coincide with $\text{span}(e_1)$ or $\text{span}(e_2)$. Hence all the x_i are distinct.

Conversely, if $x_0, x_1, x_2 \in \mathbb{P}(E)$ are distinct, then no two of them lie in a proper subspace of $\mathbb{P}(E)$ (which must be 0-dimensional). Hence, by part (b) they form a marking.

For (b), suppose $x_0, \ldots, x_{n+1} \in \mathbb{P}(E)$ form a marking with corresponding $e_0, \ldots, e_{n+1}$ as in the definition. It is easy to see that any $n+1$ elements of $\{e_1, \ldots, e_n, \sum_i e_i\}$ form a basis.

Conversely, suppose $x_0, \ldots, x_{n+1} \in \mathbb{P}(E)$ such that no $n+1$ of them are contained in a hyperplane. For each i, pick $e_i \in E$ such that $p(e_i) = x_i$. Any choice of $n+1$ of the e_i must be a basis since the x_i cannot be contained in a proper subspace of $P(E)$. In particular, $e_1, \ldots, e_n$ forms a basis for E.

Hence $e_0 = \sum_{i \geq 1} a_i e_i$ for some $a_i \in k$. Moreover if $a_j = 0$ for some $j \geq 1$ then $(e_i)_{i \geq 0, i \neq j}$ is not a basis for E, a contradiction. Thus the $a_i \neq 0$ for $i \geq 1$. Hence, by replacing e_i with $a_i e_i$, WLOG $a_i = 1$ for all $i \geq 1$. Then $p(e_1 + \ldots + e_n) = x_0$, so the x_i form a marking.

For (c), for each i there exist e_i, f_i such that $p(e_i) = x_i$ and $p(f_i) = y_i$ and such that $\sum_{i \geq 1} e_i = e_0$ and $\sum_{i \geq 1} f_i = f_0$. Since $(e_i)_{i \geq 1}$ is a basis, we can define $u : E \to E$ by $u(e_i) = f_i$ for $i \geq 1$. Then $\overline{u}$ is a homography sending each x_i to y_i.

For the uniqueness part, suppose $\bar{t}$ is a homography sending x_i to y_i for each i. Then $t(e_i) = \lambda_i f_i$ for some $\lambda_i \in k^\times$ for each $i \geq 1$, and $t(e_0) = \alpha f_0$ for some $\alpha \in k^\times$. Then $\sum_{i \geq 1} \lambda_i f_i = t(e_0) = \alpha f_0 = \alpha \sum_{i \geq 1} f_i$. Since $(f_i)_{i \geq 1}$ are linearly independent, we have $\lambda_i = \alpha$ for all $i \geq 1$. Thus $t = \alpha u$, so $\bar{t} = \overline{u}$. $\square$