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## 1 $\mathbb{F}_2$ and the Fano plane

### 1.1 Introduction

The purpose of this paper is to answer Exercise 2.5 (p. 96) of Greenberg [1]:

Let  $\mathbb{F}_2$  be the field of two elements  $\{0, 1\}$ , whose multiplication and addition have the usual tables except that  $1 + 1 = 0$ . Show that  $\mathbb{F}_2^2$  is isomorphic to the smallest affine plane. Show that  $P^2(\mathbb{F}_2)$  is isomorphic to the Fano plane.

We will need a few preliminary definitions from Greenberg.

**Definition 1.** An *incidence geometry*  $(\mathcal{P}, \mathcal{L}, \mathcal{I})$  consists of a set of points  $\mathcal{P}$ , a set of lines  $\mathcal{L}$ , and an incidence relation  $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$  such that:

1. Every pair of distinct points is incident to a unique line.
2. Every line is incident to at least two distinct points.
3. There exist three distinct noncollinear points.

**Definition 2.** Two lines are *parallel* if there is no point incident to both lines.

**Definition 3.** A *projective plane* is an incidence geometry in which:

1. No two lines are parallel.
2. Every line is incident to at least three distinct points.

**Definition 4.** An *affine plane* is an incidence geometry in which, for every line  $l$  and point  $P$  not incident to  $l$ , there exists a unique line  $m$  incident to  $P$  and parallel to  $l$ .

### 1.2 The affine plane $\mathbb{F}_2^2$

As in  $\mathbb{R}^2$ , the points in  $\mathbb{F}_2^2$  are simply the elements of the vector space  $\mathbb{F}_2^2$ , i.e. ordered pairs of elements of  $\mathbb{F}_2$ .

Also analogously to  $\mathbb{R}^2$ , the lines in  $\mathbb{F}_2^2$  are cosets of 1-dimensional subspaces of  $\mathbb{F}_2^2$ . That is, every line in  $\mathbb{F}_2^2$  can be written as  $V + h$  for some 1-dimensional subspace  $V \subset \mathbb{F}_2^2$  and  $h \in \mathbb{F}_2^2$ .

Incidence in  $\mathbb{F}_2^2$  corresponds to inclusion. For example, the point  $(1, 1) \in \mathbb{F}_2^2$  is incident to the line  $\{(1, 0)t + (0, 1) : t \in \mathbb{F}_2\}$ , since  $(1, 1) = (1, 0)(1) + (0, 1)$ .

As Greenberg notes, the smallest affine plane, call it  $\mathcal{A}$ , consists of a set of four points  $\{A, B, C, D\}$  and a set of six lines  $\{\{A, B\}, \{B, C\}, \{C, D\}, \{D, A\}, \{A, C\}, \{B, D\}\}$ ,

where incidence corresponds to inclusion. For example, the point  $B$  is incident to the line  $\{A, B\}$ .

To see that  $\mathcal{A}$  and  $\mathbb{F}_2^2$  are isomorphic, first note that each 1-dimensional subspace over  $\mathbb{F}_2$  has exactly 2 elements, so each line in  $\mathbb{F}_2^2$  has 2 elements. Conversely, given two elements  $a, b \in \mathbb{F}_2^2$ , the line  $L((b-a)t, a)$  passes through  $a$  and  $b$ . Thus, the lines in  $\mathbb{F}_2^2$  are precisely the two-element subsets of  $\mathbb{F}_2^2$ .

Therefore, an arbitrary bijection  $f$  from the points of  $\mathbb{F}_2^2$  to the points of  $\mathcal{A}$  induces a bijection of lines (two-element subsets), and since inclusion is preserved under bijections, incidence is also preserved.

### 1.3 $P^2(K)$

For an arbitrary field  $K$ , the points of the projective space  $P^2(K)$  are the 1-dimensional subspaces of  $K^3$ . The lines are the 2-dimensional subspaces of  $K^3$ . Incidence corresponds to containment.

Projective points in  $P^2(K)$  are usually denoted  $(a:b:c)$  for some generator  $(a, b, c) \in K^3 \setminus \{0\}$ . Then  $(a:b:c) = (d:e:f)$  iff  $(a, b, c)$  is a nonzero multiple of  $(d, e, f)$ .

To label the projective lines, given an element  $a \in K^3 \setminus \{0\}$ , consider the rank-1 linear transformation  $T(a) : K^3 \rightarrow K^3$  defined by  $(T(a))(x) := \sum_{i=1}^3 a_i x_i$ . By the rank-nullity theorem, the nullity of  $T(a)$  is 2.

Moreover, if  $V \subset K^3$  is a 2-dimensional subspace, then I claim  $V = \ker(T(a))$  for some  $a \in K^3 \setminus \{0\}$ . To see this, pick a basis  $\{v, w\}$  for  $V$ . Since

$$\begin{aligned} 3 &\geq \dim(\ker(T(v)) + \ker(T(w))) \\ &= \dim(\ker(T(v)) + \ker(T(w)) - \dim(\ker(T(v)) \cap \ker(T(w)))) \\ &= 4 - \dim(\ker(T(v)) \cap \ker(T(w))), \end{aligned}$$

we have  $\dim(\ker(T(v)) \cap \ker(T(w))) \geq 1$ .

Hence, we may pick  $a \in \ker(T(v)) \cap \ker(T(w)) \setminus \{0\}$ . Thus, if  $x \in V$ , then  $x = \alpha v + \beta w$  for some  $\alpha, \beta \in K$ , so

$$\begin{aligned} (T(a))(x) &= \sum_{i=1}^3 a_i(\alpha v_i + \beta w_i) \\ &= \alpha \sum_{i=1}^3 v_i a_i + \beta \sum_{i=1}^3 w_i a_i \\ &= \alpha(T(v))(a) + \beta(T(w))(a) \\ &= 0. \end{aligned}$$

Therefore,  $V \subset \ker(T(a))$ , so by dimension counting,  $V = \ker(T(a))$ . Hence, the lines in  $P^2(K)$ , as 2-dimensional subspaces of  $K^3$ , are precisely the elements of the set  $\{\ker(T(a)) : a \in K^3 \setminus \{0\}\}$ .

**Example 1.** The projective line  $\{x + y + z = 0 : (x:y:z) \in P^2(\mathbb{R})\}$  is incident to the point  $(1:0:-1) \in P^2(\mathbb{R})$  since  $t + 0 - t = 0$  for all  $t \in \mathbb{R}$ .

#### 1.4 $P^2(\mathbb{F}_2)$ as the Fano plane

A simplification occurs in  $P^2(\mathbb{F}_2)$ : there is a correspondence between each point in  $P^2(\mathbb{F}_2)$ , as a 1-dimensional subspace of  $\mathbb{F}_2^3$ , and its unique nonzero element in  $\mathbb{F}_2^3$ . Since, on the other hand, each non-zero element in  $\mathbb{F}_2^3$  generates a 1-dimensional subspace of  $\mathbb{F}_2^3$ , this correspondence defines a bijection from  $P^2(\mathbb{F}_2)$  to  $\mathbb{F}_2^3 \setminus \{0\}$ . Hence, there are  $2^3 - 1 = 7$  points in  $P^2(\mathbb{F}_2)$ , and there is only one  $(x:y:z)$ -representation for each point.

Since every subspace of  $\mathbb{F}_2^3$  contains 0, a 1-dimensional subspace  $V \subset \mathbb{F}_2^3$  lies within a 2-dimensional subspace  $W \subset \mathbb{F}_2^3$  iff the unique nonzero element in  $V$  lies within  $W$ . Hence, since each 2-dimensional subspace of  $\mathbb{F}_2^3$  contains exactly  $2^2 - 1 = 3$  nonzero  $\mathbb{F}_2^3$ -elements, each line in  $P^2(\mathbb{F}_2)$  is incident to precisely 3 projective points.

Since we have seen that no two nonzero elements of an  $\mathbb{F}_2$ -vector space are linearly dependent, each pair of distinct nonzero elements in  $\mathbb{F}_2^3$  determines a 2-dimensional subspace. Hence, since each 2-dimensional subspace of  $\mathbb{F}_2^3$  contains exactly  $\binom{3}{2} = 3$  distinct pairs of nonzero points, there are  $(1/3)\binom{7}{2} = 7$  dimension-2 subspaces of  $\mathbb{F}_2^3$ , i.e. projective lines in  $P^2(\mathbb{F}_2)$ .

From the previous section, each projective line can be written as  $\ker(T(a))$  for some  $a \in \mathbb{F}_2^3 \setminus \{0\}$ . Since there are 7 lines, the elements of  $(\ker(T(a)))_{a \in \mathbb{F}_2^3 \setminus \{0\}}$  must be distinct. Thus, Figure 1 defines an explicit isomorphism between  $P^2(\mathbb{F}_2)$  and the Fano plane.

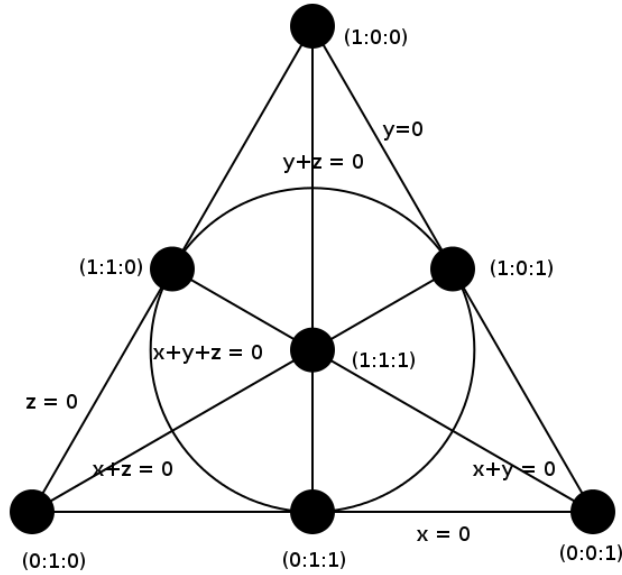


Figure 1: An isomorphism between  $P^2(\mathbb{F}_2)$  and the Fano plane

However, this isomorphism is far from unique: the group of automorphisms of the Fano plane has order 168. Indeed, to see that the order of this group is at most 168, first pick any 3 noncollinear points. Under any automorphism, these 3 points must map to 3 noncollinear points, so there are  $(7)(6)(4) = 168$  choices for the images of these three points. However, each pair of these points determines a distinct line, and the sole other point on that line must remain collinear with the pair. Hence, since there are 3 such pairs, the images of 3 more points are fixed. But there are only 7 points, so the last point's image is also determined.

For the opposite inequality, first note that the standard action of  $GL(3, \mathbb{F}_2)$  on  $\mathbb{F}_2^3$  preserves subspaces, subspace dimension, and inclusion. Hence, this group action induces an incidence-preserving action on  $P^2(\mathbb{F}_2)$ . Moreover, if  $g \in GL(3, \mathbb{F}_2)$  fixes  $P^2(\mathbb{F}_2)$ , then, since every 1-dimensional subspace of  $\mathbb{F}_2^3$  has only one nonzero point,  $g$  must fix  $\mathbb{F}_2^3$ . Thus,  $GL(3, \mathbb{F}_2)$  acts faithfully on  $P^2(\mathbb{F}_2)$ , so is isomorphic to a subgroup of the automorphism group of the Fano plane. Finally, by counting row choices,  $|GL(3, \mathbb{F}_2)| = (2^3 - 1)(2^3 - 2)(2^3 - 4) = 168$ , so  $GL(3, \mathbb{F}_2)$  must be the whole automorphism group.

## References

- [1] Marvin J Greenberg. *Euclidean and non-Euclidean geometries: Development and history*. WH Freeman, 2007.