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HW 1

1 Let $f : X \to Y$. Prove that

a) if $A, B \subset Y$, then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ and $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

b) For a family $(A_\lambda)_{\lambda\in\Lambda} \subset P(X)$, show that $f^{-1}(\bigcup_{\lambda\in\Lambda} A_\lambda) = \bigcup_{\lambda\in\Lambda} f^{-1}(A_\lambda)$ and $f^{-1}(\bigcap_{\lambda\in\Lambda} A_\lambda) = \bigcap_{\lambda\in\Lambda} f^{-1}(A_\lambda)$

and give examples for the following situations

c) $f^{-1}(f(A)) \neq A$, for some $A \subset X$,

d) $f(f^{-1}(B)) \neq B$ for some $B \subset Y$,

e) $f(\cap_{\lambda\in\Lambda}) \neq \cap_{\lambda\in\Lambda} f(A_\lambda)$, for some family $(A_\lambda)_{\lambda\in\Lambda} \subset P(X)$.

Proof. (a) is a subcase of (b). To prove the first part of (b),

$$\begin{aligned} x \in f^{-1}(\bigcup_{\lambda\in\Lambda} A_\lambda) &\iff f(x) \in \bigcup_{\lambda\in\Lambda} A_\lambda \\ &\iff f(x) \in A_\lambda \text{ for some } \lambda \\ &\iff x \in f^{-1}(A_\lambda) \text{ for some } \lambda \\ &\iff x \in \bigcup_{\lambda} f^{-1}(A_\lambda). \end{aligned}$$

For the second part,

$$\begin{aligned} x \in f^{-1}(\bigcap_{\lambda\in\Lambda} A_\lambda) &\iff f(x) \in \bigcap_{\lambda\in\Lambda} A_\lambda \\ &\iff f(x) \in A_\lambda \text{ for all } \lambda \\ &\iff x \in f^{-1}(A_\lambda) \text{ for all } \lambda \\ &\iff x \in \bigcap_{\lambda} f^{-1}(A_\lambda) \end{aligned}$$

For (c), let $X = \{0, 1\}$ and $Y = \{0\}$. Let $A = \{0\} \subset X$. Let $f : X \to Y$ be the constant function. Then $f^{-1}(f(A)) = f^{-1}(Y) = X \neq A$.

For (d), let $X = \{0\}$ and $B = Y = \{0, 1\}$. Let $f : X \to Y$ be the constant function at 1. Then $f(f^{-1}(B)) = f(X) = \{1\} \neq B$.

For (e), let $X = \{0, 1\}$ and $Y = \{0\}$. Let $A_1 = \{0\}$ and $A_2 = \{1\}$. Let $f : X \to Y$ be the constant function. Then $f(A_1 \cap A_2) = f(\emptyset) = \emptyset$, but $f(A_1) \cap f(A_2) = \{0\}$. $\square$

2 Show that the following two statements are equivalent for two nonempty sets A and B.

a) There is an injection $\phi : A \to B$.

b) There is a surjection $\psi : B \to A$.

Proof. Suppose (a) holds. Let $(U_b)_{b\in B}$ be defined by $U_b = \phi^{-1}(b)$ if $b \in \phi(A)$ and $U_b = A$ otherwise. By the axiom of choice, there exists $f \in \prod_{b\in B} U_b$. Since each $U_b \subset A$, there exist identity injections $i_b : U_b \to A$ for each $b \in B$. Define $\psi : B \to A$ by $\psi(b) = i_b(f(b))$.

To see that ψ is surjective, let $a \in A$. Since ϕ is injective, $\phi^{-1}(\phi(a))$ contains only a. Hence, $f(\phi(a)) \in U_{\phi(a)} = \phi^{-1}(\phi(a))$ implies that $f(\phi(a)) = a$. Thus, $\psi(\phi(a)) = i_{\phi(a)} f(\phi(a)) = i_{\phi(a)}(a) = a$.

Now suppose (b) holds. Since ψ is surjective, AC implies there exists $f \in \prod_{a\in A} \psi^{-1}(a)$. Since each $\psi^{-1}(a) \subset B$, there exist identity injections $i_a : \psi^{-1}(a) \to B$. Define $\phi : A \to B$ by $\phi(a) = i_a(f(a))$.

To see that ϕ is injective, let $b \in B$ and suppose $x, y \in \phi^{-1}(b)$. Then $f(x) \in i_x^{-1}(b) = \{b\}$, and similarly for y. Hence, $f(x) = b = f(y)$. Hence, $b \in \psi^{-1}(x) \cap \psi^{-1}(y) = \psi^{-1}(\{x\} \cap \{y\})$. Thus, $\{x\} \cap \{y\}$ is nonempty, so $x = y$. □

3 Find nonhomeomorphic metric spaces M_1 and M_2 such that there exist injective continuous functions $f : M_1 \to M_2$ and $g : M_2 \to M_2$.

Proof. Let $M_1 = (0, 1)$ and $M_2 = (0, 1) \cup (2, 3)$ with distances inherited from $\mathbb{R}$. Since M_1 is connected but M_2 is disconnected, they cannot be homeomorphic. Let $f : M_1 \to M_2$ be defined by $f(x) = x$, and $g : M_2 \to M_1$ be defined by $g(x) = x/3$. □

4 Prove that every real vector space has a basis.

Proof. Let V be a real vector space. Let $\mathcal{I}$ be the collection of linearly independent subsets of V. $\mathcal{I}$ is partially ordered by inclusion. We may assume that $V \neq \{0\}$ is nonempty since the proposition is trivial otherwise. By AC, there exists $v \in V \setminus \{0\}$, so $\{v\} \in \mathcal{I}$. In particular, $\mathcal{I}$ is nonempty.

Let $\mathcal{J} \subset \mathcal{I}$ be a nonempty chain, and $B := \bigcup \mathcal{J}$. I claim that B is linearly independent, hence a bound for $\mathcal{J}$. Let $\sum_{w\in W} \alpha_w w = 0$ for a nonempty finite set $W \subset B$. By the definition of B, each w lies in some $J_w \in \mathcal{J}$. Since $\{J_w\}_{w\in W}$ is a nonempty finite chain, it follows that $\bigcup_w J_w \in \{J_w\} \subset \mathcal{J}$. Hence, $\bigcup_w J_w$ is linearly independent, so $\alpha_w = 0$ for all w. Thus, B is linearly independent.

Hence, every chain in $\mathcal{I}$ is bounded, so Zorn's Lemma implies that $\mathcal{I}$ has a maximal element M. If $\text{span}(M) = V$, we are done. Otherwise, there exists $v \in V \setminus \text{span}(M)$. If $\alpha v + \sum_{m\in M} \beta_m m = 0$ for (β_m) zero except on a finite set, then $\alpha v \in \text{span}(V)$. Thus $\alpha = 0$, so $\beta_m = 0$ for all m. Hence $\{v\} \cup M$ is linearly independent, contradicting the maximality of M. □

5 Prove that any partial order $\leq$ on a set X can be extended to a linear order on the set.

Proof. Let $\mathcal{O} \subset P(X \times X)$ be the collection of partial orders containing $\leq$. $\mathcal{O}$ is partially ordered by inclusion. Let $\mathcal{U} \subset \mathcal{O}$ be a nonempty chain, and $U = \bigcup \mathcal{U}$. Since $\mathcal{U}$ is nonempty, U is a superset of a partial order, hence is reflexive. For transitivity, suppose xUy and yUz. Then xRy and ySz for some $R, S \in \mathcal{U}$. Let $T = R \cup S$. Then xTy and yTz, so xTz which implies xUz. A similar argument shows that U is antisymmetric. Hence, U is a bound for $\mathcal{U}$. Thus, every chain in $\mathcal{O}$ is bounded. Moreover, $\mathcal{O}$ is nonempty since it contains $\leq$. Hence, Zorn's Lemma implies there exists a maximal element $M \in \mathcal{O}$.

I claim that M is linearly ordered. Suppose $a, b \in X$ with neither aMb nor bMa. Define a relation $N \in P(X \times X)$ by $N = M \cup \{(a,b)\}$. Let T be the transitive closure of N. That is, xTy iff there is a finite sequence $(x_i)_{i=1}^n \subset X$ such that $x_1 = x$, $x_n = y$ and $x_i N x_{i+1}$ for all $1 \leq i < n$. Since $T \supset N \supset M$, T is reflexive. T is transitive since we can concatenate the sequences for xTy and yTz.

For anti-symmetry, suppose xTy and yTx. By concatenation, we get a sequence $(x_i)_{i=1}^n$ with $x_1 = x_n = x$, $x_m = y$ for some $1 < m < n$, and $x_i N x_{i+1}$ for all $1 \leq i < n$. If none of the (x_i, x_{i+1}) is equal to (a, b), then every such pair is in M. Hence, by the transitivity of M, $xMx_2M \ldots MyM \ldots Mx$ implies xMy and yMx, so $x = y$.

The other case is that there exists an $(x_i, x_{i+1}) = (a, b)$. If only one such pair exists, then $(x_k, x_{k+1}) \in M$ for $k \neq i$. The transitivity of M implies that xMa and bMx. Hence bMa, a contradiction. If there exists another pair $(x_j, x_{j+1}) = (a, b)$, WLOG assume i is of minimal index and j is the index of the next such pair. Then $x_{i+1}Mx_{i+2}M \ldots Mx_j \implies bMa$, a contradiction. $\square$

6 Find a sequence of Riemann integrable functions (f_n) defined on $[0,1]$, so that for all $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ so that

$$\int_0^1 |f_m(x) - f_n(x)|\, dx < \epsilon \text{ whenever } m, n \geq n_0,$$

but there is no Riemann integrable function f so that

$$\lim_{n\to\infty} \int_0^1 |f(x) - f_n(x)|\, dx = 0.$$

Proof. Pick any $0 < a < 1$ and a strictly decreasing sequence $a_n \to a$ with $a_0 = 1$. Let $E_0 = [0,1]$. Given E_n a disjoint union of 2^n closed intervals of length $a_n 2^{-n}$, define E_{n+1} by removing an open interval from the center of each interval of E_n so that E_{n+1} consists of 2^{n+1} closed intervals of length $a_{n+1}2^{-n-1}$. Let $E = \bigcap_n E_n$.

Let $f_n = \chi_{E_n}$. Each f_n is Riemann integrable since it has only finitely many points of discontinuity. Since (E_n) is a descending sequence of sets of finite measure, $m(E) = m(\bigcap_n E_n) = \lim_{n\to\infty} m(E_n) = \lim_{n\to\infty} a_n = a$. Hence $\int |\chi_E - f_n| = \int \chi_{E \setminus E_n} = a - a_n \to 0$. Thus, $f_n \to \chi_E$ in L_1. In particular, (f_n) is Cauchy in L_1.

Since $f_n \to \chi_E$ in L_1, it suffices to show that there is no Riemann integrable function in the L_1 equivalence class of χ_E. Let g differ from χ_E on a set of measure 0. Pick any $x \in E$ such that $g(x) = 1$. I claim that g is discontinuous at x. Let U be a neighborhood of x. Since E cannot contain any intervals, it follows that $V := U \cap E^c$ is a nonempty open set. Thus $m(V) > 0$, so $g(y) = 0$ for some $y \in V$. Hence g is discontinuous at x. Thus, g is discontinous on E a.e. Since E has positive measure, g cannot be Riemann integrable. $\square$