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## HW 7

**1** Assume that $(f_n) \subset L_1(\mu)$ and $f_n \to f$ uniformly.

a) If $\mu(X) < \infty$, then $f \in L_1$ and $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.

b) If $\mu(X) = \infty$, then the conclusion of (a) may fail.

*Proof.* For (a), we have

$$\begin{aligned}\left|\int f \, d\mu - \int f_n \, d\mu\right| &\leq \int |f - f_n| \, d\mu \\ &\leq \mu(X) \sup_{x\in X} |f(x) - f_n(x)| \\ &\to 0.\end{aligned}$$

For (b), let $(f_n) = 1/n\chi_{[0,n]}$. Then $f_n \to 0$ uniformly on $\mathbb{R}$, but $\int f_n \, dx = 1$ for all $n$. □

**2** Let $f_n, g_n, g \in L_1$, $n \in \mathbb{N}$, and assume that $f_n \to f$, $f$ measurable, and $g_n \to g$ $\mu$-a.e., and that $|f_n| \leq g_n$ and $\int g_n \, d\mu \to \int g \, d\mu$.

Then $\int f_n \, d\mu \to \int f \, d\mu$.

*Proof.* Following the proof of the DCT, since $f_n \leq g_n$, we have $f \leq g$, so $f \in L_1$. We also have $g_n + f_n \geq 0$ a.e. and $g_n - f_n \geq 0$ a.e. Hence by Fatou's lemma and linearity of the integral on $L_1$,

$$\int g + \int f \leq \liminf \int (g_n + f_n) = \liminf \int g_n + \int f_n = \int g + \liminf \int f_n$$

The last inequality follows from the fact that if $(a_n) \to a$ and $(b_n) \subset \mathbb{R}$, then $\liminf a_n + b_n = a + \liminf b_n$. To see this, pick $\epsilon > 0$ and $N$ such that $|a - a_n| < \epsilon$ for all $n \geq N$. Hence $\liminf a_n + b_n = \liminf (a_n - a) + a + b_n \leq \liminf \epsilon + a + b_n = \epsilon + a + \liminf b_n$, and similarly $\liminf a_n + b_n \geq -\epsilon + a + \liminf b_n$. Hence $\liminf a_n + b_n = a + \liminf b_n$.

Similarly,

$$\int g - \int f \leq \liminf \int (g_n - f_n) = \liminf \int g_n - \int f_n = \int g - \limsup \int f_n$$

.

Hence $\limsup \int f_n \leq \int f \leq \liminf \int f_n$, so $\int f = \lim \int f_n$. □

**3** Suppose that for $n \in \mathbb{N}$, $f_n = \chi_{E_n}$ for some $E_n \subset \mathbb{R}$, and assume that $f(x) = \lim_{n\to\infty} f_n(x)$ exists a.e.

a) Show that $f = \chi_E$ a.e. for some measurable set $E \subset \mathbb{R}$.

b) Show that for any $g \in L_1$:

$$\int_E g\,dx = \lim_{n\to\infty} \int_{E_n} g\,dx.$$

c) Establish a necessary and sufficient condition for $f_n \to f$ in $L_1$.

*Proof.* For (a), we have $\chi_{E_n} \to f$ on $N^c$ for some null set $N$. Let $x \in N^c$. Since $(\chi_{E_n}(x))_n$ is a convergent discrete-valued sequence, it must be eventually constant. Thus, $f(x) \in \{0, 1\}$. Let $E = f^{-1}(1) \cap N^c$. Hence $f = \chi_E$ on $N^c$, so $f = \chi_E$ a.e. on $\mathbb{R}$. By a previous homework problem, $f$ is measurable since it is the limit of measurable functions. Hence $E$ is measurable.

For (b), we have $\chi_{E_n} g \to \chi_E g$ pointwise a.e. by part (a). Moreover, $\chi_{E_n} g \leq |g| \in L_1$. Hence, by the DCT, we have the desired conclusion.

For (c), one such condition is that $m(E_n) \to m(E)$ with $m(E_n), m(E) < \infty$. Clearly, the latter condition is necessary for $f_n, f$ to be in $L_1$. For the necessity of the former condition, suppose $f_n \to f$ in $L_1$. Then $|m(E_n) - m(E)| = |\int f_n - \int f| \leq \int |f_n - f| \to 0$.

For sufficiency, suppose $m(E_n) \to m(E)$ with $m(E_n), m(E) < \infty$. Then $|f_n - f| \leq |f_n| + |f| = \chi_{E_n} + \chi_E$ a.e. Moreover, $\chi_{E_n} + \chi_E \to 2\chi E$ and $\int(\chi_{E_n} + \chi_E) = m(E_n) + m(E) \to 2m(E) = \int(2chi_E)$. Hence, by the Generalized DCT (Exercise 2), we have $\int |f_n - f| \to \int \lim_n |f_n - f| = 0$. $\square$

**4** Let $L_0([0,1])$ be the space of all measurable functions $f : [0,1] \to \mathbb{R}$.

a) for $f, g \in L_0([0,1])$ put

$$d(f,g) = \int_0^1 \min\{1, |f-g|\}\,dx.$$

Show that $(L_0([0,1]), d)$ is a metric space and that for $f, f_n \in L_0([0,1])$:

$$f_n \to f \text{ in } (L_0([0,1]), d) \iff f_n \to f \text{ in measure.}$$

b) Is there a metric $d'$ on $L_0([0,1])$ for which

$$f_n \to f \text{ in } (L_0([0,1]), d') \iff f_n \to f a.e.$$

*Proof.* For (a), to see that $d$ is a metric, we need to show that $d$ is positive definite, symmetric, and satisfies the triangle inequality. The function $d$ is clearly nonnegative and $0 = d(f,g) = \int_0^1 \min\{1, |f-g|\}\,dx$ implies that $f = g$ a.e. The function $d$ is obviously symmetric. For the triangle inequality, I first claim that for $x, y, z \in \mathbb{R}$ we have $\min\{1, |x-y|\} \leq \min\{1, |x-z|\} + \min\{1, |y-z|\}$.

We have four cases from the RHS of the inequality.

*Case* $|x-z| \leq 1$ *and* $|y-z| \leq 1$. We have $\min\{1, |x-y|\} \leq |x-y| \leq |x-y| + |y-z| = \min\{1, |x-z|\} + \min\{1, |z-y|\}$.

*Case* $|x-z| \leq 1$ *and* $|y-z| > 1$. We have $\min\{1, |x-y|\} \leq \min\{1, |x-z| + |z-y|\} \leq \min\{1, 1 + |z-y|\} = 1 + |z-y| = \min\{1, |x-z|\} + \min\{1, |z-y|\}$.

*Case* $|x-z|>1$ *and* $|y-z|\le 1$. Analogous to previous case.

*Case* $|x-z|>1$ *and* $|y-z|>1$. We have $\min\{1,|x-y|\}\le \min\{1,|x-z|+|z-y|\}=1\le \min\{1,|x-z|\}+\min\{1,|z-y|\}$.

Hence $\min\{1,|x-y|\}\le \min\{1,|x-z|\}+\min\{1,|y-z|\}$ for all $x,y,z\in\mathbb{R}$. Thus, if $f,g,h\in L_0([0,1])$ then $d(f,g)=\int_0^1 \min\{1,|f-g|\}\,dx \le \int_0^1 \min\{1,|f-h|\}+\min\{1,|h-g|\}\,dx \;=\le\; \int_0^1 \min\{1,|f-h|\}\,dx+\int_0^1 \min\{1,|h-g|\}\,dx = d(f,h)+d(g,h)$. Thus, $d$ is a metric.

Suppose $f_n\to f$ in $(L_0([0,1]),d)$. Let $0<\epsilon<1$. Pick $N$ such that $d(f,f_n)<\epsilon^2$ for all $n\ge N$. Then for all $n\ge N$, we have

$$\begin{aligned}
m(\{|f-f_n|\ge\epsilon\}) &= \int_{\{|f-f_n|\ge\epsilon\}} dx \\
&= \int_{\{\epsilon\le|f-f_n|<1\}} dx + \int_{\{|f-f_n|\ge 1\}} dx \\
&\le \int_{\{\epsilon\le|f-f_n|<1\}} \frac{|f-f_n|}{\epsilon}\,dx + \int_{\{|f-f_n|\ge 1\}} \min\{1,|f-f_n|\}\,dx \\
&\le \int_{\{\epsilon\le|f-f_n|<1\}} \epsilon^{-1}\min\{1,|f-f_n|\}\,dx + \int_{\{|f-f_n|\ge 1\}} \min\{1,|f-f_n|\}\,dx \\
&\le \epsilon^{-1}\int \min\{1,|f-f_n|\}\,dx \\
&< \epsilon
\end{aligned}$$

Conversely, suppose $f_n\to f$ in measure. Let $0<\epsilon<1$. Pick $N$ such that $m(\{|f-f_n\ge\epsilon\})\le\epsilon$ for all $n\ge N$. Then for all $n\ge N$ we have

$$\begin{aligned}
\int \min\{1,|f_n-f|\}\,dx &= \int_{\{0\le|f_n-f|<\epsilon\}} |f_n-f|\,dx + \int_{\{\epsilon\le|f_n-f|<1\}} |f_n-f|\,dx + \int_{\{|f_n-f|\ge 1\}} dx \\
&\le \int_{\{0\le|f_n-f|<\epsilon\}} \epsilon\,dx + \int_{\{\epsilon\le|f_n-f|<1\}} dx + \int_{\{|f_n-f|\ge 1\}} dx \\
&\le \epsilon + m(\{|f-f_n\ge\epsilon\}) \\
&< 2\epsilon.
\end{aligned}$$

For (b), I use a fact about convergence in metric spaces. Let $(M,d)$ be a metric space, $(x_n)_{n\in\mathbb{N}}\subset M$ , and $x\in M$. I claim that if every subsequence of $(x_n)$ has a further subsequence converging to $x$, then $x_n\to x$. Suppose $x_n\not\to x$. Then there exists $\epsilon>0$ and a subsequence $(x_n)_{n\in N_1}$ such that $d(x,x_n)\ge\epsilon$ for all $n\in N_1$. This subsequence cannot have a further subsequence converging to $x$, contradicting the hypothesis.

Thus it suffices to find a sequence that does not converge pointwise a.e., but each subsequence has a subsequence that converges to the 0 function. For $n\in\mathbb{N}$, write $n$ as $n=2^j+k$ for $j\ge 0$ and $0\le k<2^j$. Let $E_n=[k2^{-j},(k+1)2^{-j}]$ and $f_n=\chi_{E_n}$. Every element of $[0,1]$ is contained in infinitely many $E_n$ and infinitely many $E_n^c$, so $f_n$ do not converge pointwise a.e.

On the other hand, suppose $(f_n)_{n\in N_1}$ is a subsequence of $(f_n)_{n\in\mathbb{N}}$. For each $n$, pick $x_n \in E_n$. Then by the sequential compactness of $[0,1]$, there exists an infinite set $N_2 \subset N_1$ and $x_0 \in [0,1]$ such that $x_n \to x_0$ as $n \to \infty, n \in N_2$. Since $\operatorname{diam}(E_n) \to 0$, we have $f_n(x) \to 0$ as $n \to \infty, n \in N_2$ for $x \neq x_0$. Hence, $f_n \to 0$ as $n \to \infty, n \in N_2$ pointwise a.e. $\square$

**5**

(a)
$$\lim_{n\to\infty}\int_0^\infty \left(1+\frac{x}{n}\right)^{-n}\sin\left(\frac{x}{n}\right)\,dx = 0$$

(b)
$$\lim_{n\to\infty}\int_0^1 \frac{1+nx^2}{(1+x^2)^2} = 0$$

(c)
$$\lim_{n\to\infty}\int_0^\infty n\sin\left(\frac{x}{n}\right)[x(1+x^2)]^{-1}\,dx = \frac{\pi}{2}$$

(d)
$$\lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}\,dx = \begin{cases} 0 & \text{if } a>0 \\ \pi/2 & \text{if } a=0 \\ \pi & \text{if } a<0 \end{cases}$$

*Proof.* For (a), we have $(1+\frac{x}{n})^n = \sum_{j=0}^n \binom{n}{j}\left(\frac{x}{n}\right)^j$.

For (b), the statement cannot be true since for $n \geq 1$ we have $\int_0^1 \frac{1+nx^2}{(1+x^2)^2}\,dx \geq \int_0^1 \frac{1}{(1+x^2)^2}\,dx > 0$.

For (c),

For (d),

$$\begin{aligned}\lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}\,dx &= \lim_{n\to\infty}\int_{na}^\infty \frac{du}{1+u^2} \\ &= \lim_{n\to\infty}\pi/2 - \tan^{-1}(na) \\ &= \begin{cases} 0 & \text{if } a>0 \\ \pi/2 & \text{if } a=0 \\ \pi & \text{if } a<0 \end{cases}\end{aligned}$$

$\square$