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MATH 663 - Subfactors, Knots, and Planar Algebras (Fall 2017)

HW 4

1 Let (M, τ) be a finite von Neumann algebra, let $1 \in N \subset M$ be a von Neumann subalgebra, and let $E : M \to N$ be the unique τ-preserving conditional expectation. Prove that E is continuous and completely positive.

Proof. The conditional expectation E is a positive linear map between C-* algebras, so E is bounded. Moreover, $E(x) = exe$ where $e : L^2(M) \to L^2(N)$ is the projection. This implies that E is completely positive (by the converse to Stinespring's theorem). $\square$

2 Let $\pi, \sigma \in NC_2(2n)$. Let $\beta \in \mathbb{C} \setminus \{0\}$ and consider the canonical trace $\tau : D_n(\beta) = TL_n(\beta^{-2}) \to \mathbb{C}$. Show that $\tau(D_\sigma^* D_\pi) = \beta^{|\pi \vee \sigma| - n}$.

Proof. Write down the diagram for $D_\sigma^* D_\pi$ as the vertical concatenation of its two factor diagrams. Label points on the boundary of D_π by $1, \ldots, 2n$ in the usual way as if D_σ^* was not there. Label the D_σ^* part according to the usual D_σ labelling, i.e. reflect the usual D_σ^* labelling through its horizontal midline. The σ-labels $n+1$ through $2n$ should agree with the π-labels. Moreover, the "braid closure" of $D_\sigma^* D_\pi$ connects the points labeled 1 to n for π to the correspondingly labeled points for σ. Thus, the connected components of the braid closure correspond to the blocks of $\sigma \vee \pi$. $\square$

3 Prove that the canonical trace τ_n on $TL_n(\lambda)$ is positive semidefinite for all $\lambda \in (0, \frac{1}{4}]$.

Proof. Let

$$\xi_\pi = \sum_{i \in \{1,2\}^{[2n]}} \prod_{\substack{r \sim_\pi s \\ r < s}} F_{i(s)i(r)} e_i,$$

where $F = \beta^{-1/2} \begin{pmatrix} 0 & q^{-1} \\ q & 0 \end{pmatrix}$, and $\beta = q^2 + q^{-2}$ with $q \in \mathbb{R}$. We have

$$\langle \xi_\pi, \xi_\sigma \rangle = \sum_i \prod_{\substack{r \sim_\pi s \\ r < s}} \prod_{\substack{t \sim_\sigma u \\ t < u}} F_{i(s)i(r)} F_{i(t)i(u)} \quad = \sum_i \prod_{b \in \pi \vee \sigma} \prod_{\substack{r < s, t < u \in b \\ r \sim_\pi s \\ t \sim_\sigma u}} F_{i(s)i(r)} F_{i(t)i(u)}$$

Let b be a block in $\pi \vee \sigma$, and let $x_1 \in [2n]$ be the minimal number in the block b. Every element of b is related to two other numbers by π and σ respectively, and the group generated by $\pi, \sigma \subset S_{2n}$ acts transitively on b. Letting $x_{j+1} = \pi x_j$ if $j \geq 1$ is even and $x_{j+1} = \sigma x_j$ if $j \geq 1$ is odd, we have $b = \{x_j\}_{j=1}^{|b|}$. Thus,

$$\langle \xi_\pi, \xi_\sigma \rangle = \sum_i \prod_{b \in \pi \vee \sigma} \prod_{j=1}^{|b|} F_{i(x_j)i(x_{j+1})} \quad = \frac{1}{|\beta|^n} \sum_i \prod_{b \in \pi \vee \sigma} \prod_{j=1}^{|b|} \sqrt{\beta} F_{i(x_j)i(x_{j+1})}$$

For the innermost product to be nonvanishing, since F is 0 on its diagonal, we must have $i(x_{j+1}) \neq i(x_j)$ for all j. Thus, for every block b we get exactly two nonvanishing choices $i|_b$ determined by the values $i(x_1)$ at the block's minimal element x_1. Moreover, the map $j \mapsto x_j$ for $1 \leq j \leq |b|+1$ defines a piecewise linear map $\phi : [1, |b|+1] \to \mathbb{R}_{\geq 0}$ by connecting consecutive points with line segments. The noncrossing condition ensures that there exists at least one, hence at least two, consecutive extrema (i, x_j) and (j, x_j) such that $i - j$ is odd. Moreover, any triple x_{i-1}, x_i, x_{i+1} with x_{i+1} nonextremal contributes a trivial factor to the product. We can contract an "innermost" such triple (i.e. triple at i such that there is no triple x_j, x_{j+1}, x_{j+1} between x_i and one of $x_{i\pm1}$. We get noncrossing pairings $\pi'|_b$ and $\sigma'|_b$, each with one fewer pairing than the unprimed versions. Continuous this process, we end up with a function $\phi' : [1, c] \to \mathbb{R}_{\geq 0}$ with and $\phi'(1) = \phi(1) = \phi(|b|+1) = \phi'(c)$ and with extrema at every integer value. By the non-crossing condition, $c = 3$. Thus, the previous product is q^2 or q^{-2}, depending on $i(x_1)$. Since the choice of $i|_b$ is independent for each block b, we have

$$\begin{aligned}
\langle \xi_\pi, \xi_\sigma \rangle &= \beta^{-n} \prod_{b \in \pi \vee \sigma} (q^2 + q^{-2}) \\
&= \beta^{-n} \prod_{b \in \pi \vee \sigma} \beta \\
&= \beta^{|\pi \vee \sigma| - n}
\end{aligned}$$

Thus, the Gram matrix for τ_n wrt to $(D_\pi)_\pi$ is same as the Gram matrix for the vectors $(\xi_\pi)_\pi$. Thus, τ_n is positive semidefinite. □

Exercise 10 of Speicher Let $p, q \in B(H)$ be orthogonal projections on a separable complex Hilbert space H.

(a) Show that

$$\text{s-lim}_{n\to\infty} (pqp)^n = p \wedge q.$$

Proof. Since pqp is self-adjoint, the unital commutative C^*-algebra $C^*(1, pqp)$ is isometrically $*$-isomorphic to $C(\text{Spec}(pqp))$ via a map ϕ with $\phi(pqp) = \text{id}_{\text{Spec}(pqp)}$. It is easy to check that pqp is a contractive positive operator. Hence, $\text{Spec}(pqp) \subset [0, 1]$. Thus, $\phi((pqp)^n) = \text{id}^n_{\text{Spec}(pqp)} \to \chi_{\{1\}}$ strongly. Thus, $(pqp)^n$ converges strongly to some projection e.

We have $pe = \lim_n p(pqp)^n = e$, so $p \leq e$. We also have, for all $\xi \in H$,

$$\begin{aligned}
eqe\xi &= \lim_n (pqp)^n q \lim_m (pqp)^m \xi \\
&= \lim_n \lim_m (pqp)^n q (pqp)^m \xi \\
&= \lim_n \lim_m (pqp)^{n+m+1} \xi \\
&= \lim_n e\xi \\
&= e\xi.
\end{aligned}$$

Thus, for all $\xi \in H$,

$$\begin{aligned}\langle (e-q)\xi, \xi\rangle &= \langle (e^2 - eqe)\xi, \xi\rangle \\ &= \langle (1-q)e\xi, e\xi\rangle \\ &\geq 0,\end{aligned}$$

so $q \leq e$. Thus, $p \wedge q \leq e$

On the other hand, $(p \wedge q)e\xi = \lim_n (p \wedge q)(pqp)^n \xi = \lim_n (p \wedge q)\xi = p \wedge q\xi$. Thus, $e \leq p \wedge q$. Thus, $e = p \wedge q$. □

(b) Deduce that $\text{s-lim}_{n\to\infty}(pq)^n = p \wedge q$.

Proof. We have $(pq)^n \xi = (pqp)^{n-1} q\xi \to (p \wedge q)q\xi = p \wedge q\xi$ for all $\xi \in H$. □

(c) Discuss the statements (a) and (b) in the case $H = \mathbb{C}^3$ for the projections $p = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $q = u \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} u^*$, where $u = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$ for some $0 \leq \theta \leq \pi$.

Answer:

The matrix u is a rotation by θ about the y axis. The matrix q is a projection onto the space spanned by the y axis and rotation of the x-axis by θ around the y-axis. The matrix p projects back onto the x and y axis. Each time you do this the x-coordinate shrinks by a value of $\cos^2(\theta)$, but the y coordinate remains the same.

More explicitly, we have $(pq)^n = \begin{pmatrix} \cos(\theta)^{2n} & 0 & \cos(\theta)^{2n-1}\sin(\theta) \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $(pqp)^n = \begin{pmatrix} \cos(\theta)^{2n} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The limit in both cases is the projection onto the y-axis.

Exercise 11 of Speicher Let $(S_n)_{n=0}^{\infty}$ be the sequence of Chebyshev polynomials of the second kind, which are recursively defined by $S_0(x) = 1$, $S_1(x) = x$ and

$$xS_n(x) = S_{n+1}(x) + S_{n-1}(x) \qquad \text{for all } n \geq 1.$$

Prove the following statements.

(a) For all $n \geq 0$ and all $0 < \theta < \pi$, it holds true that

$$S_n(2\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

Proof. The base cases $n = 0, 1$ are easy to check. The inductive step reduces to checking the identity

$$\sin((n+2)\theta) = 2\cos(\theta)\sin((n+1)\theta) - \sin(n\theta).$$

Letting $q = e^{i\theta}$, this reduces to checking

$$q^{n+2} - q^{-(n+2)} = (q + q^{-1})(q^{n+1} - q^{-(n+1)}) - (q^n - q^{-n}).$$

□

(b) We have for all $n, m \geq 0$ that

$$\int_{-2}^{2} S_n(x)S_m(x)\frac{1}{2\pi}\sqrt{4-x^2}\,dx = \delta_{n,m}.$$

Proof. We have

$$\begin{aligned}
&\int_{-2}^{2} S_n(x)S_m(x)\frac{1}{2\pi}\sqrt{4-x^2}\,dx \\
&= \frac{2}{\pi}\int_{0}^{\pi} S_n(2\cos(\theta))S_m(2\cos(\theta))\sin^2(\theta)\,d\theta \\
&= \frac{2}{\pi}\int_{0}^{\pi} \sin((n+1)\theta)\sin((m+1)\theta)\,d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin((n+1)\theta)\sin((m+1)\theta)\,d\theta \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} (e^{i(n+1)\theta} - e^{-i(n+1)\theta})(e^{i(m+1)\theta} - e^{-i(m+1)\theta})\,d\theta, \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} e^{i(m-n)\theta} + e^{i(n-m)\theta}\,d\theta, \\
&= \delta_{nm},
\end{aligned}$$

using the fact that $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\,d\theta = \delta_{0k}$ for all integers k. □

(c) For all $x \in [-2, 2]$ and all $z \in \mathbb{C}$ with $|z| < 1$, we have

$$\frac{1}{1 - xz + z^2} = \sum_{n=0}^{\infty} S_n(x)z^n$$

Proof. The function $f(z) = 1 - xz + z^2$ has roots at $\frac{x\pm\sqrt{x^2-4}}{2}$. For $x \in [-2, 2]$, we have $\left|\frac{x\pm\sqrt{x^2-4}}{2}\right|^2 = \frac{1}{4}\left(x^2 + (4 - x^2)\right) = 1$. Thus, the power series for f centered at $z = 0$ has radius of convergence 1.

Letting $x = 2\cos(\theta)$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(x)z^n &= \sum_{n=0}^{\infty} \frac{\sin((n+1)\theta)}{\sin\theta} z^n \\
&= \frac{1}{2i\sin(\theta)} \sum_{n=0}^{\infty} (e^{in\theta} - e^{-in\theta})z^n \\
&= \frac{1}{2i\sin(\theta)} \sum_{n=0}^{\infty} (e^{i\theta}z)^n - (e^{-i\theta}z)^n \\
&= \frac{1}{2i\sin(\theta)} \left(\frac{1}{1-(e^{i\theta}z)} - \frac{1}{1-(e^{-i\theta}z)} \right) \\
&= \frac{1}{2i\sin(\theta)} \left(\frac{2i\sin(\theta)}{1-2\cos(\theta)z+z^2} \right) \\
&= \frac{1}{1-xz+z^2}.
\end{aligned}$$

□

(d) For $x, y \in [-2, 2]$ and all $n \geq 0$, we have

$$\frac{S_n(x) - S_n(y)}{x-y} = \sum_{k=1}^{n} S_{k-1}(x)S_{n-k}(y).$$

Proof. We have

$$\begin{aligned}
\sum_{n=0}^{\infty} S_n(x) - S_n(y)x - yz^n &= \frac{1}{x-y} \left(\frac{1}{1-xz+z^2} - \frac{1}{1-yz+z^2} \right) \\
&= \frac{1}{x-y} \left(\frac{(x-y)z}{(1-xz+z^2)(1-yz+z^2)} \right) \\
&= z \left(\sum_{n=0}^{\infty} S_n(x)z^n \right) \left(\sum_{n=0}^{\infty} S_n(y)z^n \right) \\
&= z \sum_{n=0}^{\infty} \sum_{k=0}^{n} S_k(x)S_{n-k}(y)z^n \\
&= \sum_{n=0}^{\infty} \sum_{k=1}^{n+1} S_{k-1}(x)S_{n-k+1}(y)z^{n+1} \\
&= \sum_{n=0}^{\infty} \sum_{k=1}^{n} S_{k-1}(x)S_{n-k}(y)z^n
\end{aligned}$$

□

Exercise 12 of Speicher

(a) Given f.d. von Neumann algebras $N \subset M \subset P$, show that

$$\Lambda_N^P = \Lambda_N^M \Lambda_M^P.$$

Proof. Let $(p_i), (q_j), (r_k)$ be the minimal central projections of N, M, and P respectively. For each i, let e_i be a minimal projection in $p_i N$. We have $e_i = \sum_j q_j e_i = \sum_{j,l} f_{ijl}$ for some minimal projections $f_{ij}^{(l)} \in q_j M$. Then, for any fixed i and j, we have $\sum_l 1 = \mathrm{tr}(\sum_l f_{ij}^{(l)}) = \mathrm{tr}(q_j e_i) = (\Lambda_N^M)_{ij}$. Thus,

$$\begin{aligned}
(\Lambda_N^P)_{ik} &= \mathrm{tr}(r_k e_i) \\
&= \sum_{j,l} \mathrm{tr}(r_k f_{ij}^{(l)}) \\
&= \sum_{j,l} (\Lambda_M^P)_{jk} \\
&= \sum_j (\Lambda_N^M)_{ij} (\Lambda_M^P)_{jk}
\end{aligned}$$

□

(b) Let s, t be trace vectors for f.d. von Neumann algebras N and M, respectively. Show that $\tau_M|_N = \tau_N$ if and only if $\Lambda_N^M t = s$.

Proof. We have $\tau_M|_N = \tau_N$ iff $\tau_M|_N(e) = \tau_N(e)$ for every minimal projection $e \in P(N)$. By definition, $\tau_N(e) = s_i$, where i is the index of the factor containing e. On the other hand, if (p_j) are minimal central projections for M,

$$\begin{aligned}
\tau_M(e) &= \sum_j \tau_M(p_j e) \\
&= \sum_j t_j \, \mathrm{tr}(p_j e) \\
&= \sum_j \Lambda_{ij} t_j
\end{aligned}$$

Thus, $\tau_N(e) = \tau_M|_N(e)$ iff $s_i = \sum_j \Lambda_{ij} t_j$. Thus, $\tau_N = \tau_M|_N$ iff $s = \Lambda t$.

□