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HW 2

1 a Show that if α, β are positive with $\alpha + \beta = 1$ then for all $u, v \geq 0$ we have

$$u^\alpha v^\beta \leq \alpha u + \beta v.$$

Proof. If $u = v = 0$, then the inequality holds. Since the inequality is symmetric in u and v, we may assume $v \neq 0$. Hence we wish to show

$$(\frac{u}{v})^\alpha \leq \alpha(\frac{u}{v}) + \beta$$

. Letting $x = \frac{u}{v}$, this is equivalent to showing that $f(x) \geq 0$, where $f(x) = \alpha x - x^\alpha + \beta$ and $x \geq 0$. Since $\alpha > 0$, we have $f'(x) = \alpha - \alpha x^{\alpha-1} = \alpha(1 - x^{\alpha-1})$ whose only zero in $[0, \infty)$ is at $x = 1$. Moreover, since $\alpha < 1$, we have $f''(1) = \alpha(\alpha-1)x^{\alpha-2}|_{x=1} = \alpha(\alpha-1) < 0$. Hence, the maximum value of f on $[0, \infty)$ occurs at $x = 1$. We have $f(1) = \alpha - 1 + \beta = 0$, so $f(x) \leq 0$ for $x \geq 0$. $\square$

1 b Let $x, y \in \mathbb{R}^n$, and let $p > 1$ and define q by $q^{-1} = 1 - p^{-1}$. Prove Hölder's inequality,

$$|\sum_j x_j y_j||\|x\|_p \|y\|_q.$$

Hint: Using the inequality in part (a). first prove it for $\|x\|_p = \|y\|_q = 1$. Scale to get the final inequality.

Proof. Suppose $\|x\|_p = \|y\|_q = 1$. Then

$$\begin{aligned}
|\sum_j x_j y_j| \leq & \sum_j |x_j||y_j| \\
= & \sum_j (|x_j|^p)^{1/p}((|y_j|)^q)^{1/q} \\
\leq & \sum_j \frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q \\
\leq & \frac{1}{p}(\sum_j |x_j|^p) + \frac{1}{q}(\sum_j |y_j|^q) \\
= & \frac{1}{p} + \frac{1}{q} \\
= & 1.
\end{aligned}$$

For the general case, note that if $x = 0$ or $y = 0$ then the inequality holds. Hence we may assume both are nonzero. Let $x' = \frac{x}{\|} x\|_p$ and $y' = \frac{y}{\|} y\|_p$. We can now apply the special case to x' and y' then clear denominators to get the general inequality. $\square$

1 c Suppose $\phi = (y_1, \ldots, y_n) \in l_p^*$. Hölder's inequality implies that $\|\phi\|_{l_p^*} \leq \|y\|_q$. Show that we actually have $\|\phi\|_{l_p^*} = \|y\|_q$.

Proof. If $\|y\|_q = 0$ then $\phi = 0$, and $\|\phi\|_{l_p^*} = 0 = \|y\|_q$. Hence, we may assume $\|y\|_q \neq 0$. Let $x_i = \mathrm{sign}(y_i)\frac{|y_i|^{q/p}}{\|y\|_q^{q/p}}$ for $1 \leq i \leq n$. Then $\|x\|_p = \sum_i \frac{|y_i|^q}{\|y\|_q^q} = 1$.

Then $\phi(x) = \sum_i x_i y_i = \sum_i \frac{|y_i|^{q/p}}{\|y\|_q^{q/p}}|y_i| = \frac{1}{\|y\|_q^{q/p}} \sum_i |y_i|^{\frac{p+q}{p}} = \frac{1}{\|y\|_q^{q/p}} \sum |y_i|^q = \|y\|_q^{q-q/p} = \|y\|_q$.

□

1 d Let $x, y \in \mathbb{R}^n$, and let $p > 1$. Prove Minkowski's inequality,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Use this to show that $\|x\|_p$ defines a norm on $\mathbb{R}^n$. Hint: you will need to use Hölder's inequality, along with a trick.

Proof. Acknowledgement: I looked in Carother's Real Analysis book for a hint on this problem. Setting $1/p + 1/q = 1$, we have

$$\begin{aligned}
\|x + y\|_p^p &= \sum_i |x_i + y_i||x_i + y_i|^{p-1} \\
&\leq \sum_i |x_i||x_i + y_i|^{p-1} + \sum_i |y_i||x_i + y_i|^{p-1} \\
&\leq (\|x\|_p + \|y\|_p)\left(\sum_i |x_i + y_i|^{q(p-1)}\right)^{1/q} \\
&= (\|x\|_p + \|y\|_p)\left(\sum_i |x_i + y_i|^{p}\right)^{1-1/p} \\
&= (\|x\|_p + \|y\|_p)\|x + y\|_p^{p-1}
\end{aligned}$$

If $\|x+y\|_p^{p-1} \neq 0$, we can divide by it to get desired inequality. If $\|x+y\|_p = 0$ then the inequality follows from the fact that $\|x\|_p + \|y\|_p$ must be nonnegative by definition.

To show that $\|\cdot\|_p$ is a norm, it remains to show that it is homogeneous and positive definite. To see that $\|\cdot\|_p$ is homogeneous, let $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then $\|cv\|_p = (\sum_i |cv_i|^p)^{1/p} = (|c|^p \sum_i |v_i|^p)^{1/p} = |c|\|v\|_p$. It is obvious that $\|v\|_p \geq 0$. If $\|v\|_p = 0$, then each component of v must be zero or else $\sum_i |v_i|^p > 0$. Hence $v = 0$.

□

2 L_2 minimization. Find the straight line $y = a + bx$ that minimizes $\int_0^1 (e^{-x} - a - bx)^2\, dx$.

Proof. By HW 1, Problem 4, we know that $a+bx$ minimizes $\|e^{-x}-a-bx\|_2$ iff

$$\begin{pmatrix} \langle e^{-x},1\rangle \\ \langle e^{-x},x\rangle \end{pmatrix} = \begin{pmatrix} \langle 1,1\rangle & \langle x,1\rangle \\ \langle 1,x\rangle & \langle x,x\rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The one slightly tricky integral is $\langle e^{-x},x\rangle = \int_0^1 xe^{-x}\,dx = x(-e^{-x})|_{x=0}^1 + \int_0^1 e^{-x}\,dx = -e^{-1} - (e^{-x})|_{x=0}^1 = -e^{-1} - (e^{-1}-1) = 1-2e^{-1}$.

$$\begin{pmatrix} 1-e^{-1} \\ 1-2e^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0.943036 \\ -0.62183 \end{pmatrix}$$

□

3 L_1 minimization. Find the straight line $y=a+bx$ that minimizes $\int_0^1 |e^{-x} - a - bx|\,dx$, by following these steps.

a. Whatever the minimizer is, geometric considerations show that e^{-x} and $a+bx$ will cross at two points, $0<s<t<1$. Find these two points by minimizing, over a,b, the area A between $f(x)$ and $a+bx$:

$$A = \int_0^1 |e^{-x}-a-bx|\,dx = \int_0^s (e^{-x}-a-bx)\,dx + \int_s^t (a+bx-e^{-x})\,dx + \int_t^1 (e^{-x}-a-bx)\,dx.$$

b. Use the crossing conditions $a+bs=e^{-s}$ and $a+bt=e^{-t}$ to find a and b.

Proof. a. Let $g_1(a,b,s)=e^{-s}-a-bs$, and $g_2(a,b,t)=e^{-t}-a-bt$. Geometric considerations imply that the global minimum of A with the constraint $g_1 = g_2 = 0$ must be a local minimum, not a boundary value. Hence, Lagrange multipliers gives us the following necessary condition for (a,b,s,t) to minimize

A given the constraints $g_1 = g_2 = 0$:

$$
\begin{aligned}
0 = \quad & (\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t})(A - \lambda_1 g_1 - \lambda_2 g_2) \\
0 = \quad & (\int_0^s (-1)\,dx + \int_s^t 1\,dx + \int_t^1 (-1)\,dx - \lambda_1 - \lambda_2, \\
& \int_0^s (-x)\,dx + \int_s^t x\,dx + \int_t^1 (-x)\,dx - \lambda_1 s - \lambda_2 t, \\
& 2(-e^{-s} - a - bs) + \lambda_1(-e^{-s} - b), -2(-e^{-t} - a - bt) + \lambda(-e^{-t} - b)) \\
0 = \quad & (-s + (t - s) + (t - 1) - \lambda_1 - \lambda_2, \\
& (-1/2)s^2 + (1/2)(t^2 - s^2) + (-1/2)(1 - t^2) - \lambda_1 s - \lambda_2 t, \\
& \lambda_1(-e^{-s} - b), \lambda_2(-e^{-t} - b)) \\
0 = \quad & (2t - 2s - 1 - \lambda_1 - \lambda_2, t^2 - s^2 - 1/2 - \lambda_1 s - \lambda_2 t, \\
& \lambda_1(-e^{-s} - b), \lambda_2(-e^{-t} - b))
\end{aligned} \tag{1}
$$

From the last two components, we get four cases.

Case $e^{-s} = e^{-t} = -b$. Since b is the slope of the line between (s, e^{-s}) and (t, e^{-t}), we have $b = \frac{e^{-t} - e^{-s}}{t - s} = 0$ which cannot correspond to a minimum.

Case $e^{-s} = -b$ *and* $\lambda_2 = 0$. From the first component of 1, we have $\lambda_1 = 2t - 2s - 1$. Substituting into the second component of 1, $0 = t^2 - s^2 - 1/2 - \lambda_1 s = t^2 - s^2 - 1/2 - (2t - 2s - 1)s = (t - s)^2 - (1/2 - s)$. Since $t - s > 0$, we have $t = s + \sqrt{1/2 - s}$. Using the case assumption, we have $e^{-s} = -b = -\frac{e^{-t} - e^{-s}}{t - s} = -e^{-s}\frac{e^{-\sqrt{1/2 - s}} - 1}{\sqrt{1/2 - s}}$. Thus if $u = -\sqrt{1/2 - s}$, then $u = e^u - 1$. The only solution to this equation is $u = 0$. To see this, note that $f(u) := e^u - u - 1$ has derivative $e^u - 1$, hence f has a unique global minimum at 0.

Hence $s = 1/2$, so $t = 1/2$. This cannot correspond to a minimum.

Case $\lambda_1 = 0$ *and* $e^{-t} = -b$. From the first component of 1, we have $\lambda_2 = 2t - 2s - 1$. Substituting into the second component of 1, $0 = t^2 - s^2 - 1/2 - \lambda_2 t = t^2 - s^2 - 1/2 - (2t - 2s - 1)t = -(t - s)^2 + t - 1/2$. Since $t - s > 0$, we have $s = t - \sqrt{t - 1/2}$. Using the case assumption, we have $e^{-t} = -b = -\frac{e^{-t} - e^{-s}}{t - s} = -e^{-t} 1 - e^{\sqrt{t - 1/2}} \sqrt{t - 1/2}$. Thus if $u = \sqrt{t - 1/2}$, then $u = e^u - 1$. As in the previous case, the only solution to this equation is $u = 0$.

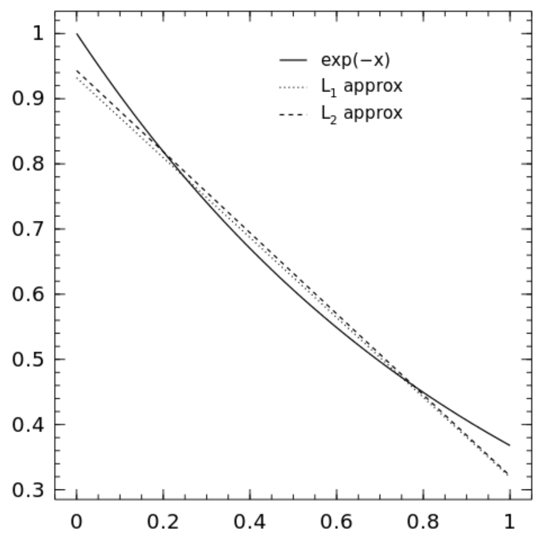
Hence $t = 1/2$, so $s = 1/2$. This cannot correspond to a minimum.

Case $\lambda_1 = \lambda_2 = 0$. We have $t = s + 1/2$, so $0 = (s + 1/2)^2 - s^2 - 1/2 = s - 1/4$. Hence $s = 1/4, t = 3/4$.

b. We have $a + b(1/4) = e^{-(1/4)}$ and $a + b(3/4) = e^{-(3/4)}$. Hence $a = 0.9320$ and $b = -0.6128$.

$\square$

3 Use your favorite software (mine is Matlab) and plot, on the same set of axes, e^{-x} and the two minimization solutions found in the previous two problems.



4 Let V be a finite dimensional inner product space and let U be a subspace of V. Recall that the orthogonal complement of U is

$$U^{\perp} = \{v \in V | \langle v, u \rangle = 0 \text{ for all u } \in U\}.$$

Show that $V = U \oplus U^{\perp}$, where $\oplus$ symbolizes the direct sum of vector spaces. Also, show that $(U^{\perp})^{\perp} = U$.

Proof. By HW 1 (4)(b), the orthogonal projection $P : V \to U$ exists. Let $v \in V$. Then $v = Pv + (v - Pv)$. By HW 1 (3), $v - Pv \in U^{\perp}$. Hence, $V = U + U^{\perp}$. Moreover, if $w \in U \cap U^{\perp}$, then $\langle w, w \rangle = 0$ so $w = 0$. Thus, $v = U \oplus U^{\perp}$.

To see that $U \subset (U^{\perp})^{\perp}$, let $u \in U$. Then $\langle v, u \rangle = 0$ for all $v \in U^{\perp}$. Hence, $\langle u, v \rangle = 0$ for all $v \in U^{\perp}$. Thus, $u \in (U^{\perp})^{\perp}$.

Since $V = W \oplus W^{\perp}$ for any subspace W, we have $\dim(U) + \dim(U^{\perp}) = \dim(V) = \dim(U^{\perp}) + \dim((U^{\perp})^{\perp})$. Since $\dim(U^{\perp}) < \infty$, we have $\dim(U) = \dim((U^{\perp})^{\perp})$. Since $U \subset (U^{\perp})^{\perp}$ and they are finite dimensional, this implies that $U = (U^{\perp})^{\perp}$.

$\square$