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Inequalities for Overlapping Circles

The purpose of this paper is to answer Major Exercise 3.3 (p. 155) of Greenberg [1], paraphrased as follows:

> In a Hilbert plane, let γ be a circle with center O and radius r. Let γ' be another circle with center $O' \neq O$ and radius r', and let $d = |OO'|$. Suppose there exists a point M of γ' inside γ and another point N of γ' outside γ. Show that the following three inequalities hold: $r + r' > d$, $r + d > r'$, and $r' + d > r$. You may assume the triangle inequality in a Hilbert plane: if A, B, C are not collinear, then $|AB| + |BC| > |AC|$.

I will use the following alternative to Greenberg's Triangle Inequality.

Theorem 1. (Weak Triangle Inequality) *Given points A, B, C in a Hilbert plane, then $|AB| + |BC| \geq |AC|$.*

Proof. If A, B, C are noncollinear, then the conclusion follows from Greenberg's Triangle Inequality.

If $A = B$, then $|AB| + |BC| = |BC| = |AC|$. If $B = C$, then $|AB| + |BC| = |AB| = |AC|$. If $A = C$, then $|AB| + |BC| \geq |AC|$ trivially.

The last case is when A, B, C are distinct and collinear. Then one of the three points must lie between the other two. If $A * C * B$, then $AB > AC$, so $AB + BC > AC$. Similarly, if $B * A * C$, then $BC > AC$, so $AB + BC > AC$. Lastly, if $A * B * C$, then $AB + BC \cong AC$ by the definition of addition for line segments. $\square$

To prove the original problem, first note that by the definition of "inside γ", we have $|OM| < r$. Hence, $r + r' > |OM| + r' = |OM| + |MO'| \geq |OO'| = d$ by the weak triangle inequality. We also have $r + d > |MO| + d = |MO| + |OO'| \geq |MO'| = r'$.

By the definition of "outside γ", we have $|NO| > r$. Hence, $r' + d = |NO'| + |O'O| \geq |NO| > r$.

References

[1] Marvin J Greenberg. *Euclidean and non-Euclidean geometries: Development and history.* WH Freeman, 2007.