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HW 3

5 Let $f : X \to Y$ be a map of topological spaces. Show that f is a homotopy equivalence iff there exists maps $g, h : Y \to X$ such that $gf \simeq 1_X$ and $fh \simeq 1_Y$.

Proof. We have $fg \simeq fgfh \simeq fh \simeq 1$. $\square$

6 (a) Prove the Borsuk-Ulam Theorem in dimension 1, i.e., prove that for every map $f : S^1 \to \mathbb{R}$ there exists a pair of antipodal points x and $-x$ in S^1 such that $f(x) = f(-x)$.

(b) Is the following version of the Borsuk-Ulam Theorem for the torus correct? For every map $f : S^1 \times S^1 \to R^2$ there exists a pair of antipodal points (x, y) and $(-x, -y)$ in $T^2 = S^1 \times S^1$ such that $f(x, y) = f(-x, -y)$.

Proof. (a) Let $g : [0, 1] \to S^1$ be defined by $g(x) = e^{i\pi x}$. Let $h(x) = f(x) - f(-x)$. Let $\phi = h \circ g$. Then $\phi(0) = f(1) - f(-1)$, and $\phi(1) = f(-1) - f(1)$. One of these values must be nonnegative and the other nonpositive. It follows from the intermediate value theorem that ϕ has a root. Hence h has a root, so there exists $x \in S^1$ with $f(x) = f(-x)$.

(b) No, let $\pi : S^1 \times S^1$ be the projection on the first coordinate, and $\phi : S^1 \to R^2$ be the usual inclusion. Let $f = \phi \circ \pi$. For any $(x, y) \in S^1 \times S^1$, we have $\pi(x, y) = x \neq -x = \pi(-x, -y)$. Hence $f(x, y) \neq f(-x, -y)$ since ϕ is an injection. $\square$