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HW 2, due February 7

16.58 Suppose that $m^*(E) < \infty$. Prove that E is measurable if and only if, for every $\epsilon > 0$, there is a finite union of bounded intervals A such that $m^*(E \triangle A) < \epsilon$ (where $E \triangle A$ is the symmetric difference of E and A).

Proof.

Lemma 1. *If S, T, U are sets, then $S \triangle U \subset (S \triangle T) \cup (T \triangle U)$.*

Proof.

$$\begin{aligned} S \triangle U &= (S \setminus U) \cup (U \setminus S) \\ &\subset (((S \setminus T) \cup T) \setminus U) \cup (((U \setminus T) \cup T) \setminus S) \\ &\subset (S \setminus T) \cup (T \setminus U) \cup (U \setminus T) \cup (T \setminus S) \\ &= (S \triangle T) \cup (T \triangle U) \end{aligned}$$

□

Suppose E is measurable. Let $\epsilon > 0$. Pick an open set $U \supset E$ with $m(U \setminus E) < \epsilon/2$. Since $m(U) < \infty$, $U = \bigcup_{n=1}^{\infty} I_n$ where the I_n are disjoint bounded open intervals. Pick N such that $\sum_{n=N+1}^{\infty} m(I_n) < \epsilon/2$. Let $A := \bigcup_{n=1}^{N} I_n$. Then, by the lemma, $m(A \triangle E) \leq m(A \triangle U) + m(U \triangle E) = m(U \setminus A) + m(U \setminus E) < \epsilon$.

Conversely, let $\epsilon > 0$ and suppose such an A exists. Let $U \supset E \triangle A$ be an open set such that $m(U) < \epsilon$. Since A is measurable, there exists an open set $J \supset A$ such that $m(J \setminus A) < \epsilon$. Then $G := U \cup J$ is open, and $G \supset (E \triangle A) \cup A \supset E$. Moreover,

$$\begin{aligned} m^*(G \setminus E) &\leq m(U) + m^*(J \setminus E) \\ &\leq \epsilon + m^*(J \triangle E) \\ &\leq \epsilon + m(J \triangle A) + m^*(A \triangle E) \\ &= \epsilon + m(J \setminus A) + m^*(A \triangle E) \\ &< 3\epsilon \end{aligned}$$

□

16.60 If E is a measurable set, show that $E + x$ and rE are measurable for any $x, r \in \mathbb{R}$. [Hint: Use Theorem 16.21].

Proof. If $r = 0$, $rE = \{0\}$ is measurable, so we may assume $r \neq 0$. Let $\epsilon > 0$, and let $U \supset E$ be open with $m(U \setminus E) < \epsilon$.

I claim that I is in open interval iff $rI + x$ is an open interval. Let $f(y) := ry + x$. Since $r \neq 0$, f is a homeomorphism. It also preserves betweenness since $a < b < c$ implies $f(a) < f(b) < f(c)$ if $r > 0$, and $f(a) > f(b) > f(c)$ if $r > 0$. If $f(I)$ is an interval, let $a < b < c$ with a, c in I. Then $f(b)$ is between $f(a)$ and $f(c)$, so $f(b) \in f(I)$ which implies $b \in I$. The converse follows from the fact that f^{-1} has the same form as f.

Hence, since U is a countable union of open intervals, $rU + x$ is open. Moreover, (I_n) is a cover of $U \setminus E$ by open intervals iff $(rI_n + x)$ covers $(rU + x) \setminus (rE + x)$. Therefore, $m^*((rU + x) \setminus (rE + x)) = rm(U \setminus E)$. □

16.70 Prove that an arbitrary union of positive-length intervals is measurable. [Hint: Let $\mathcal{C}$ be the collection of all closed intervals J such that $J \subset I_\alpha$ for some α.]

Proof. Let (I_α) be an arbitrary collection of positive length intervals, and $U = \bigcup_\alpha I_\alpha$. Let $\mathcal{C}$ be the collection of all closed intervals J such that $J \subset I_\alpha$ for some α.

Let (q_n) be a countable dense-in-$\mathbb{R}$ set. Let $E_n = \bigcup\{J \in \mathcal{C} : q_n \in J\}$. I claim E_n is an interval. Suppose $a < b < c$ with $a, c \in E_n$. Then there exists $J_a \in \mathcal{C}$ with $q_n, a \in J_a$ and J_c with $q_n, c \in J_c$. Hence, if $q_n \leq b$, then $b \in J_c$; and if $q_n > b$, then $b \in J_a$. Hence, E_n is an interval.

To see that $E := \bigcup_n E_n \supset U$, suppose $x \in U$. Then $x \in I_\alpha$ for some α. There exists a closed interval I such that $x \in I \subset I_\alpha$. There exists some q_n such that $q_n \in I$. Hence, $x \in I \subset E_n$.

The reverse inclusion, $E \subset U$, follows from the fact that each $E_n \subset U$. Hence, $U = E$ is the countable union of intervals.

□

16.78 If E is a measurable subset of A, show that $m^*(A) = m(E) + m^*(A \setminus E)$. Thus $m^*(A \setminus E) = m^*(A) - m(E)$ provided that $m(E) < \infty$.

Proof. Let $\epsilon > 0$. There exists an open set $U \supset A$ such that $m(U) \leq m^*(A) + \epsilon$. Thus, $m^*(A) \geq m(U) + \epsilon = m(E) + m(U \setminus E) + \epsilon \geq m(E) + m^*(A \setminus E) + \epsilon$. □

J16.4 Prove that every set of positive outer measure contains a nonmeasurable subset.

Proof. Following Carothers' hint, let $A \subset \mathbb{R}$ with $m^*(A) > 0$. Then since $m^*(A) \geq \sum_n A \cap [n, n+1)$, some $A \cap [n, n+1)$ has positive outer measure. Let $N_r \subset [0, 1)$ be defined as in Carothers' construction of an unmeasurable set. Since $\bigcup_r N_r + n = [n, n+1)$, some $A \cap (N_s + n) =: E$ must have positive outer measure.

Suppose E were measurable. Then $E - n \subset N_s$ has positive measure. Let $F_r := E - n + r (\mathrm{mod}\ 1)$. Then $m(F_r) = m(E)$, and $F_r \subset N_r$. Hence, the F_r are disjoint, so $m([0, 1)) \geq m(\bigcup_r F_r) = \sum_r m(F_r) = \sum_r m(E) = \infty$, a contradiction. □

J16.5 Prove that m is Lipschitz with constant 1 on $(\mathcal{M}_1, d)$, where $\mathcal{M}_1$ denotes the measurable subsets of $[0,1]$, and $d(E,F) = m(E \triangle F)$. Prove that $(\mathcal{M}_1, d)$ is complete. [Hint: If (E_n) is d-Cauchy, then, by passing to a subsequence, you may assume that $d(E_n, E_{n+1}) < 2^{-n}$. Now argue that (E_n) converges to, say, $\limsup_{n\to\infty} E_n$.]

Proof. Let $E, F \in \mathcal{M}_1$. WLOG, $m(E) \le m(F)$. Then $|m(F) - m(E)| = m(F) - m(E) \le m(F \setminus E) \le m(E \triangle F) = d(E,F)$.

For the completeness, suppose (E_n) is d-Cauchy. By passing to a subsequence, assume $d(E_n, E_{n+1}) < 2^{-n}$. Let $E = \limsup_{n\to\infty} E_n$. For any k, we have

$$
\begin{aligned}
d(E_k, E) &= m(E_k \triangle \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} E_n) \\
&= m((\bigcup_{m=1}^{\infty} \bigcap_{n\ge m} E_n \setminus E_k) \cup ((\bigcup_{m=1}^{\infty} \bigcap_{n\ge m} E_n)^c \setminus E_k^c)) \\
&\le m(\bigcup_{m=1}^{\infty} \bigcap_{n\ge m} E_n \setminus E_k) + m(\bigcap_{m=1}^{\infty} \bigcup_{n\ge m} E_n^c \setminus E_k^c) \\
&\le m(\bigcup_{n>k} E_n \setminus E_k) + m((\bigcup_{n>k} E_n^c) \setminus E_k^c) \\
&\le m(\bigcup_{n>k} (E_n \setminus E_{n-1}) \setminus E_k) + m((\bigcup_{n>k} E_n^c \setminus E_{n-1}^c) \setminus E_k^c) \\
&\le m(\bigcup_{n>k} E_n \setminus E_{n-1}) + m(\bigcup_{n>k} E_n^c \setminus E_{n-1}^c) \\
&\le m(\bigcup_{n>k} E_n \triangle E_{n-1}) + m(\bigcup_{n>k} E_n \triangle E_{n-1}) \\
&\le 2^{-k} + 2^{-k}
\end{aligned}
$$

so $E_k \to E$.

□

J16.6 Let X be a metric space, $E \subset X$, and $\mathcal{B} := \{B_{r(x)}(x) : x \in S\}$ be a cover of E such that $\sup_{x\in S} r(x) < \infty$. Prove that there is a (finite or infinite) sequence $\{B_{r(x_i)}(x_i)\}_{i=1}^N$ of disjoint balls in $\mathcal{B}$ so that either

1. $N = \infty$ and $\inf_i r(x_i) > 0$, or
2. $E \subset \bigcup_{n=1}^N B_{5r(x_i)}(x_i)$. ($N$ can be either finite or infinite in this case.)

Hint: Greed is good.

Proof. Let $S_1 := S$. Recursively, for $i = 1, 2, \ldots$,

1. Choose $x_i \in S_i$ such that $r(x_i) > 1/2 \sup_{x\in S_i} r(x)$.

2. Let $S_{i+1} = \{x \in S : B_{r(x)}(x) \cap \bigcup_{j=1}^{i} B_{r(x_j)}(x_j) = \emptyset\}$.

3. If $S_{i+1} = \emptyset$, stop.

By (2), $\{B_{5r(x_i)}(x_i)\}_{n=1}^{N}$ are disjoint.

Let $y \in E$. There exists $x \in S$ such that $y \in B_{r(x)}(x)$.

Case $N < \infty$: Since $S_{N+1} = \emptyset$, we have $B_{r(x)}(x) \cap \bigcup_{j=1}^{N} B_{r(x_j)} \neq \emptyset$. Thus, for some minimal j, $B_{r(x_j)}(x_J) \cap B_{r(x)}(x) \neq \emptyset$. By construction step (1), since j is minimal, $r(x) < 2r(x_j)$. Thus, by the triangle inequality, $B_{r(x)}(x) \subset B_{5r(x_j)}(x_j)$.

Case $\inf_i r(x_i) = 0$: Suppose $B_{r(x)}(x) \cap \bigcup_{j=1}^{N} B_{r(x_j)} = \emptyset$. Since $\inf_i r(x_i) = 0$, there exists j such that $r(x_j) < 1/2r(x_i)$. But this contradicts the choice of x_j in construction step (1).

Hence, as in the previous case, $B_{r(x)}(x) \subset B_{5r(x_j)}(x_j)$ for some j.

$\square$