Towards finiteness for mapping class group representations from group-theoretical categories

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Preliminary Exam
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Definition of the mapping class group

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- Examples
  - $\mathrm{MCG}(\Sigma_{0,1}^m) = B_m$
  - $\mathrm{MCG}(\Sigma_{1,0}^0) = SL(2, \mathbb{Z})$

Introduction to the problem

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- They also asked if, more generally, all mapping class group representations associated to $\mathrm{Mod}(D^\omega(G))$ have finite image.
- In this talk, I'll work through the genus 2 case.

Other Related Work

Theorem (Ng–Schauenberg [?Ng2010])

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Theorem (Fjelstad–Fuchs [?fjfu])

Every mapping class group representation of a closed surface with at most one marked point associated to $\mathrm{Mod}(D(G))$ *has finite image.*

- Fjelstad and Fuchs use Lyubashenko's method of constructing projective representations of mapping class groups from factorizable ribbon Hopf algebras (in this case $D(G)$).
- We will use a different construction due to Kirillov. In our case, this construction corresponds to the twisted Dijkgraaf-Witten theory.

Outline

1. Input data
2. Generators for the mapping class group
3. The representation space
4. A spanning set for the representation space
5. Action of the generators on the spanning set
6. Sketch of a proof of finiteness
7. Future Directions

Input data

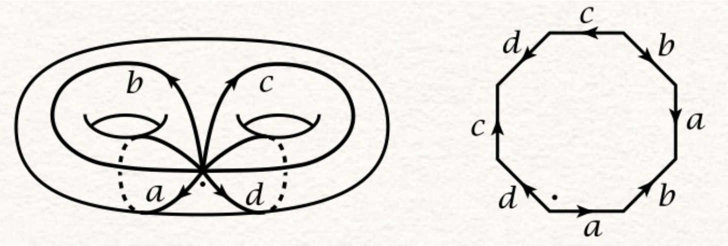

Figure: A genus 2 surface Σ as a quotient of its fundamental polygon. Image source: Hatcher's *Algebraic Topology*.

- Oriented closed surface Σ of genus 2

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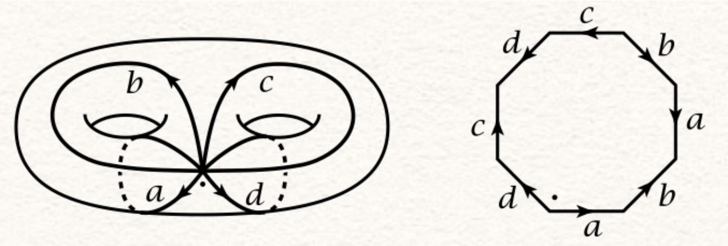

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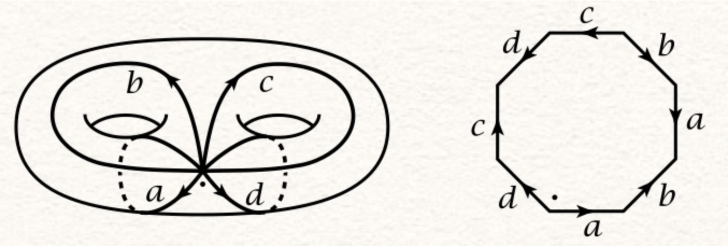

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- Oriented closed surface Σ of genus 2
- Finite group G
- Normalized 3-cocycle $\omega : G \times G \times G \to U(1)$.

- A theorem of Lickorish [?lickorish1964finite] implies that $MCG(\Sigma)$ is generated by the Dehn twists T_a, T_b, T_c, T_d, $T_{a^{-1}d}$.

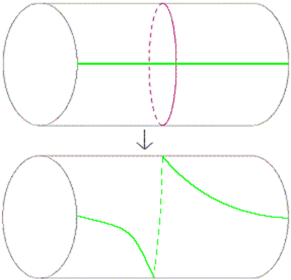

Figure: A Dehn twist with respect to the red curve. Image source: Wikipedia article on Dehn twists.

The representation space

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- The Drinfel'd center $\mathcal{Z}(\text{Vect}_G^\omega)$ is braided monoidally equivalent to $\text{Mod}(D^\omega(G))$ (well-known according to [?0704.0195]).
- Hence, the mapping class group representation on H should be equivalent to the mapping class group representation associated to $\text{Mod}(D^\omega(G))$ by the Reshitikhin-Turaev construction [?1012.0560, preprint].

The spherical category Vect_G^ω

- The spherical fusion category Vect_G^ω is the category of G-graded finite-dimensional vector spaces with the following modified structural morphisms from [?math/0601012], where V_g is the simple object:
  - The associator $a_{g,h,k} : (V_g \otimes V_h) \otimes V_k \to V_g \otimes (V_h \otimes V_k)$
$$a_{g,h,k} = \omega(g, h, k)$$
  - The evaluator $ev_g : V_g^* \otimes V_g \to 1$
$$ev_g = \omega(g^{-1}, g, g^{-1})$$
  - The pivotal structure $j_g : V_g^{**} \to V_g$
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- A *coloring* of Γ is the following data:
    - Choice of an object $V(\mathbf{e}) \in \operatorname{Obj} \mathcal{A}$ for every oriented edge $\mathbf{e} \in E^{or}$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.

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    - Choice of an object $V(\mathbf{e}) \in \operatorname{Obj} \mathcal{A}$ for every oriented edge $\mathbf{e} \in E^{or}$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.
    - Choice of a vector $\varphi(v) \in \mathrm{Hom}_{\mathcal{A}}(1, V_1 \otimes \cdots \otimes V_n)$ for every interior vertex v, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are edges incident to v, taken in counterclockwise order and with outward orientation.

Local relations

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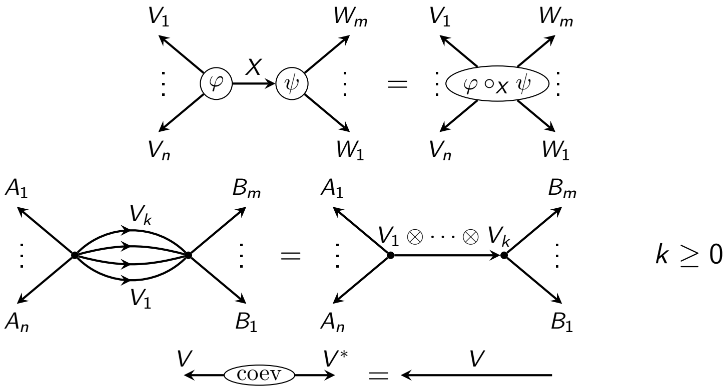

Figure: The remaining local relations. Image source: [?kirillovStringNets].

Consequences of the local relations

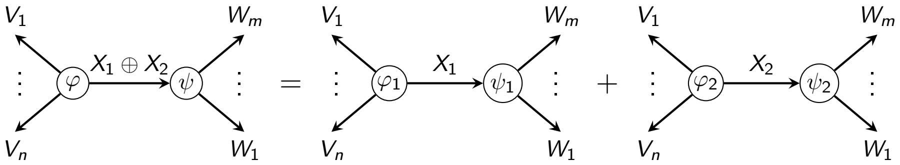

Figure: Additivity in edge colorings. Here φ_1, φ_2 are compositions of φ with projector $X_1 \oplus X_2 \to X_1$ (respectively, $X_1 \oplus X_2 \to X_2$), and similarly for ψ_1, ψ_2. Image source: [?kirillovStringNets].

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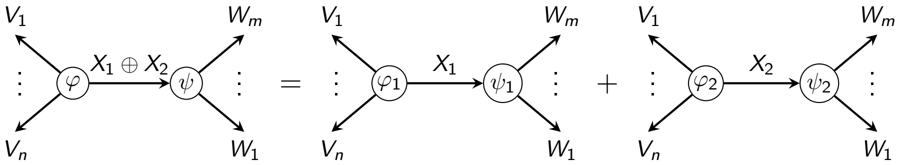

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- Additivity in edge colorings
- A colored graph may be evaluated on any disk $D \subset S$, giving an equivalent colored graph Γ' such that Γ' is identical to Γ outside of D, has the same colored edges crossing ∂D, and contains at most one colored vertex within D.

A spanning set for the representation space

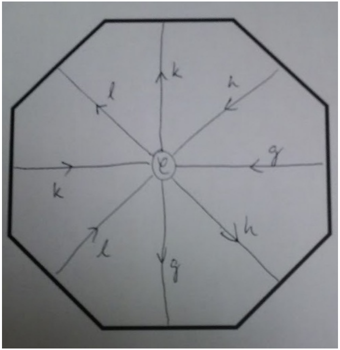


Figure: The spanning set S consists of all such colored graphs, where the edge labels vary over all 4-tuples $g, h, k, l \in G$ satisfying $[g, h][k, l] = 1$ and $\varphi := \varphi_{g,h,k,l}$ is the canonical basis element of the one-dimensional space $\mathrm{Hom}(1, ((\cdots((V_g \otimes V_h) \otimes V_g^{-1}) \otimes \cdots \otimes V_l^{-1})$.

Action of the Dehn twist T_a on the spanning set I

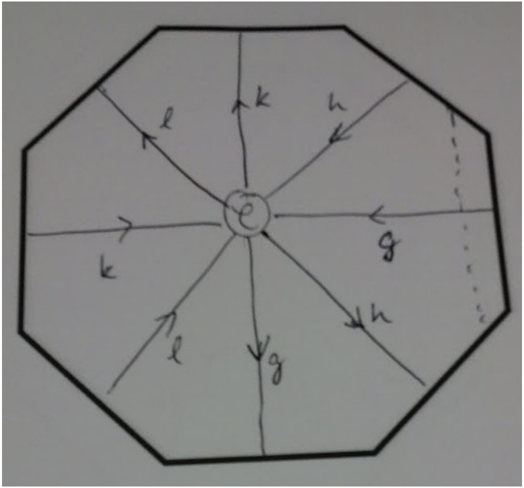


Figure: The dashed line is a simple closed curve isotopic to a.

Action of the Dehn twist T_a on the spanning set II

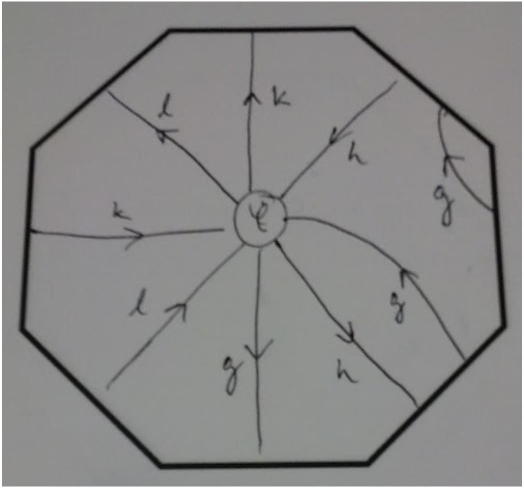


Figure: The result of the twist T_a.

Action of the Dehn twist T_a on the spanning set III

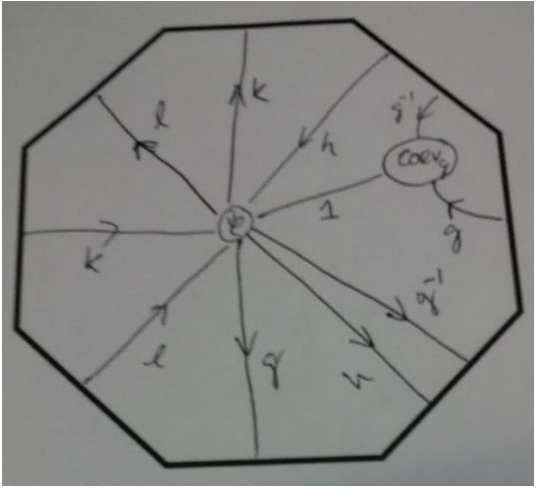


Figure: Using the local relations.

Action of the Dehn twist T_a on the spanning set IV

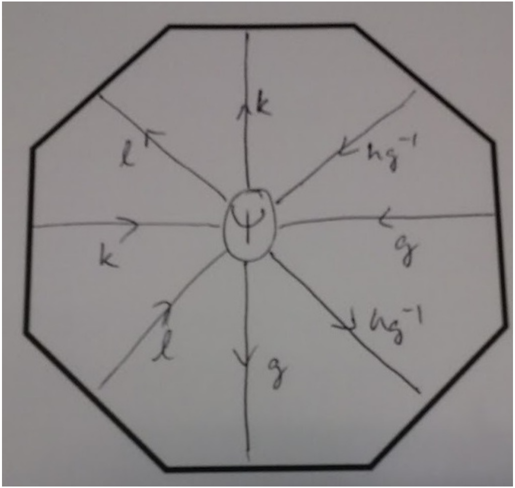


Figure: The result. The map ψ differs from $\phi_{g,hg^{-1},k,l}$ by a product of factors in $\mathrm{Im}(\omega)$.

Action of the Dehn twist $T_{a^{-1}d}$ on the spanning set I

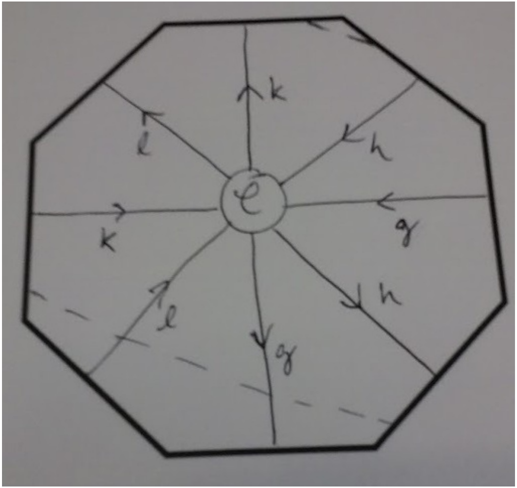


Figure: The dashed line is a simple closed curve isotopic to $a^{-1}d$.

Action of the Dehn twist $T_{a^{-1}d}$ on the spanning set II

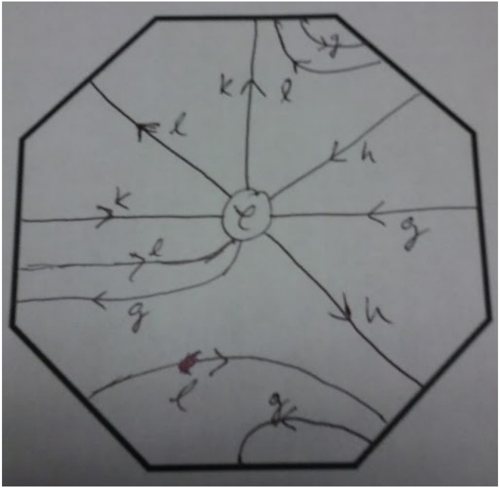
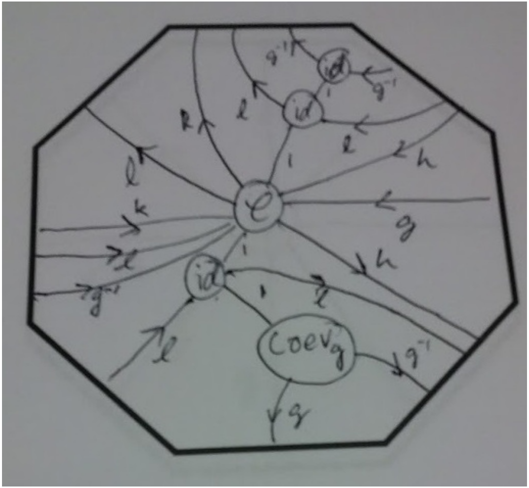


Figure: The result of the twist $T_{a^{-1}d}$.

Action of the Dehn twist $T_{a^{-1}d}$ on the spanning set III



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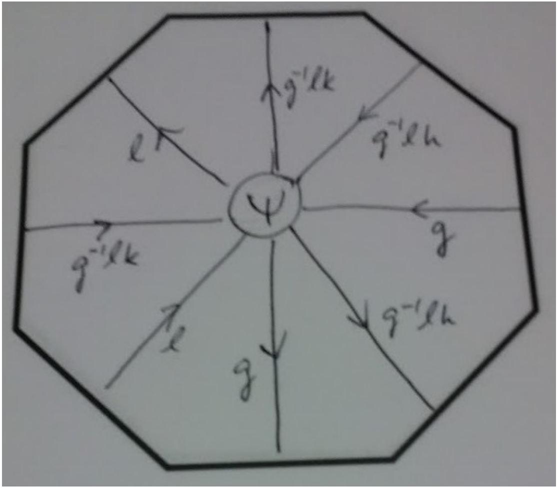

Figure: The result. Again, ψ differs from $\phi_{g,g^{-1}lh,g^{-1}lk,l}$ by a product of factors in $\mathrm{Im}(\omega)$.

Sketch of a proof of finiteness

Proposition

Let $\rho : \mathsf{MCG}(\Sigma) \to \mathsf{PGL}(H)$ *be the representation defined above. Then* $|\mathrm{Im}(\rho)| < \infty$.

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- For any k, let R denote the set of $|G|$-th roots of unity. Then ω is cohomologous to a cocycle taking values in R (follows from [?weibel1995introduction, Theorem 6.58]). Hence, WLOG ω takes values in R.

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Future Directions

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- Add marked points using the Birman exact sequence.
- Look at the simplest undetermined cases of weakly integral modular categories.

Acknowledgements

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- Thanks for listening!

References I