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Math 644

Final

1 Show that $H_c^{n+1}(X \times \mathbb{R}; G) \cong H_c^n(X; G)$ for all n.

Proof. To compute $\operatorname{colim}_{K \text{ compact}} H^{n+1}(X \times \mathbb{R}, X \times \mathbb{R} - K)$, it suffices to let K range over sets of the form $K' = K \times I$ for a compact $K \subset X$ and I a compact interval. Since the groups in the colimit do not vary with I, WLOG $I = [-2, 2]$.

To apply the relative Mayer-Vietoris sequence, let $A = X \times (-\infty, 1)$, $B = X \times (-1, \infty)$, $C = (X \times \mathbb{R} - K') \cap A$, and $D = (X \times \mathbb{R} - K') \cap B$. Then we have

$$\cdots \to H^n(X \times \mathbb{R}|K) \to H^n(A, C) \oplus H^n(B, D) \to H^n(A \cap B, C \cap D) \to \cdots$$

We also have that (A, C) strong deformation retracts onto $(X \times \{-3\}, X \times \{-3\})$, and a similar retraction holds for (B, D). Thus all the direct sum terms in the long exact sequence above are 0.

Moreover, $(A \cap B, C \cap D) = (X \times (-1, 1), (X - K') \times (-1, 1))$ strong deformation retracts to $(X, X - K')$. Thus, from the l.e.s., we get $H^n(X|K') \cong H^{n+1}(X \times \mathbb{R}|K')$. Passing to the colimit gives the desired isomorphism. $\square$

2 Show that for any connected oriented closed manifold M of dimension n there is a map $f : M \to S^n$ having degree 1.

Proof. Let $B \subset M$ be an open neighborhood homeomorphic to a ball in $\mathbb{R}^n$. Let f be the quotient map $f : M \to M/(M - B) \cong S^n$. Since M is orientable, we have $H^n(M) \cong H^n(M, M - B)$ in the long exact sequence of a pair. Applying f to both sides, the naturality of the long exact sequence gives a commutative diagram

$$\begin{tikzcd}
H_n(M) \arrow[r] \arrow[d, "f_*"] & H^n(M, M - B) \arrow[d] \\
H_n(M/(M - B)) \arrow[r] & H_n(M/(M - B), (M - B)/(M - B))
\end{tikzcd}$$

The rightmost arrow is an isomorphism since $(M, M - U)$ is a good pair, and the bottom arrow is an obvious isomorphism. Thus, f_* is an isomorphism, and in particular a degree ± 1 map. If the degree is -1, compose f with a reflection of S^n through an equator to get a degree 1 map. $\square$

3 Let $f : M \to N$ be a map between closed connected oriented manifolds of same dimension n.

a. Suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is the disjoint union of balls B_i each mapped homeomorphically by f onto B. Show the degree of f is $\sum_i \epsilon_i$ where ϵ_i is 1 or -1 according to whether $f : B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

Proof. Using the relative Mayer-Vietoris sequence, we have

$$0 \to H_n(M, M-B_i-B_j) \to H_n(M, M-B_i) \oplus H_n(M, (M-B_i) \to H_n(M, (M-B_i) \cup (M-B_j))$$

For $i \neq j$, the last term is 0, so the middle terms are naturally isomorphic. Continuing in this way, we get $H_n(M|\bigcup_i B_i) \cong \bigoplus_i H_n(M|B_i)$. By the naturality of the long exact sequence, we have $f_* : H_n(M) \cong H_n(M|\bigcup_i B_i) \to H_n(N|B) \cong H_n(N)$ is given by $f_* = \sum_i f_i$, where $f_i : H_n(M|B_i) \to H_n(N|B)$ are the maps induced by f. Since $f_{|B_i}$ is a homeomorphism onto B for each i, each f_i is an isomorphism and the sign of f_i is determined by whether it reverses local orientations. ☐

b. Show that if f is a p-sheeted covering projection then $\deg(f) = \pm p$.

Proof. It suffices to show that the set on which f preserves local orientations is open. Let $x \in M$ such that f preserves local orientations at x. Pick an evenly covered neighborhood U of $f(x)$. Then f also preserves local orientations on the component of $f^{-1}(U)$ containing x. ☐

c. If M_g denotes the closed orientable surface of genus g, show that if $g \geq h$ there exists a map $f : M_g \to M_h$ of degree 1.

Proof. Think of M_g as the connected sum of M_h and M_{g-h}. Let f be a map sending the M_{g-h} part to a sphere and leaving the M_h part alone. Applying part (a) to any untouched neighborhood in M_h tells us that the degree of f is 1. ☐

4 If $g \geq 1$ show that for each nonzero $\alpha \in H^1(M_g;\mathbb{Z})$ there exists $\beta \in H^1(M_g;\mathbb{Z})$ with $\alpha \smile \beta \neq 0$. Use this fact to show that M_g is not homotopy equivalent to a wedge $X \vee Y$ of CW-complexes with non trivial reduced homology.

Proof. We proved on a homework that $H^1(M_g) = \mathbb{Z}^{2g}$. Hatcher's Prop. 3.38 implies that the pairing $H^1(M_g) \times H^1(M_g) \to \mathbb{Z}$ given by evaluation of the cup product at the fundamental class is nonsingular. In particular, the first part of this problem holds.

For the second part, suppose that $M_g = X \vee Y$. Then $\mathbb{Z} = H_2(M_g) = H_2(X) \oplus H_2(Y)$. Hence WLOG $H_2(X) = \mathbb{Z}$ and $H_2(Y) = 0$. Let $\alpha \in H^1(Y)$. Pick $\beta \in H^1(M_g)$ such that $\alpha \smile \beta \neq 0$. Let $\phi \in C_2(X)$ be any chain representative of a generator of $H_2(X) = H_2(M_g)$. Since $\alpha \in H^1(Y)$, we have $(\alpha \smile \beta)(\phi) = 0$. Since this holds for a representative of a generator of $H_2(M_g)$, we have $\alpha \smile \beta = 0$, a contradiction. ☐

5 Let $f : M \to N$ be a map between closed connected oriented manifolds of dimensions m and n, respectively, and let R be a commutative ring with 1.

a. Explain how this map makes $H^*(M;R)$ into an algebra over $H^*(N;R)$.

Proof. The map $H^*(M;R) \times H^*(N;R) \to H^*(M;R)$ defined by $(\alpha, \beta) \mapsto \alpha \smile f^*(\beta)$ gives the right action. Linearity is obvious, and associativity follows from the fact that $f^*(\beta \smile \gamma) = f^*(\alpha) \smile f^*(\gamma)$. $\square$

b. Explain how to use Poincare duality to define group homomorphisms $f_! : H^i(M;R) \to H^{i+n-m}(N;R)$.

Proof. Poincare duality gives an isomorphisms $\phi : H^i(M;R) \cong H_{m-i}(M;R)$, and $\psi : H_{m-i}(N;R) \cong H^{i+n-m}(N;R)$. Let $f_! = \psi f_* \phi$. $\square$

c. Show that these maps assemble to give a homomorphism $f_! : H^*(M;R) \to H^*(N;R)$ of right $H^*(N;R)$-modules.

Proof. The assembled map is clearly a morphism of abelian groups. If $\alpha \in H^i(M;R)$ and $\beta \in H^j(N;R)$, we have $f_!(\alpha\beta) = f_!(\alpha \smile f^*(\beta)) = \psi f_* \phi(\alpha \smile f^*(\beta)) = f_!(\alpha) \smile f^*(\beta) = f_!(\alpha)\beta$, where we used the fact that ψ, ϕ commute with the cup product by the definition of the Poincare dual map as the cup product with the fundamental class. $\square$

6 Let $p : E \to X$ be a vector bundle of rank n over a paracompact space X.

(1) Show that if E has k sections $s_1, \ldots, s_k$ that are linearly independent at each $x \in X$ that it has a trivial subbundle of rank k.

Proof. Let $h : X \times \mathbb{R}^k \to E$ be defined by $h(x, t_1, \ldots, t_n) = \sum_i t_i s_i(x)$. Then h is continuous (since it is continuous on each local trivialization), and a linear injection on each fiber since the s_i are independent. Thus, by a lemma shown in class, h is a subbundle map. $\square$

(2) Assume that E has k sections $s_1, \ldots, s_k$ such that at each x the elements $s_1(x), \ldots, s_k(x)$ generate the fiber E_x as a vector space. Show that E is a quotient of a trivial bundle of rank k.

Proof. Define $h : X \times \mathbb{R}^k \to E$ by $h(x, t_1, \ldots, t_n) = \sum_i t_i s_i(x)$. This map is continuous and preserves fibres, hence a bundle map. It is also a linear surjection on fibers, so E is a quotient of the trivial bundle of rank k. $\square$

(3) Under the assumptions of the previous question, let $x \in X$ and define $p_x : \mathbb{C}^k \to E_x$ as the surjective linear map sending $(\lambda_1, \ldots, \lambda_k)$ to $\sum_j \lambda_j s_j(x)$. Show that the map $\phi : X \to Gr_{k-n}(\mathbb{C}^k)$ sending $x \in X$ to $\ker p_x$ is a continuous map.

Proof. By restricting to a local trivialization, WLOG E is a trivial bundle. Then $\ker p_x$ is the orthogonal complement of $\mathrm{span}\{s_1(x), \ldots, s_k(x)\}$. Since the s_j are continuous and taking orthogonal complements of subspaces is a continous map from Gr_{n-k} to Gr_{k-n}, the map ϕ is continuous. □

(4) Let $q : \mathbb{Q}^n \to Gr_{k-n}(\mathbb{C}^k)$ denote the quotient bundle ε^k / E_{k-n}, where E_{k-n} is the tautological bundle. Show that if ϕ is defined in the previous section then $\phi * Q_n \cong E$.

Proof. To show that E is this pullback, we need to find a map $f : E \to Q^n$ mapping the fiber of each $x \in X$ to the fiber of $\phi(x)$ isomorphically. Define f_x by $f_x(\sum_i \lambda_i s_i(x)) = \pi(\lambda_1, \ldots, \lambda_k)$, where π is the orthogonal projection onto the orthogonal complement of $\ker p_x$ in $\mathbb{C}^k$. By the rank-nullity theorem, the projection π is an isomorphism. □

7 Let X be a finite CW-complex with only even-dimensional cells.

a. Show that $K(X)$ is a free abelian group on the set of cells of X and that $K(SX) = \mathbb{Z}$. Explain why $K^{-1}(X) = 0$.

Proof. To see that $K(X)$ is free abelian on the cells of X, use induction on the number of cells. WLOG X is connected. The base case, a point, is trivial. For the inductive step, assume that X is constructed by attaching a k-cell to a subcomplex A, for some k. This gives a short exact sequence $\widetilde{K}^*(X/A) \to \widetilde{K}^*(X) \to \widetilde{K}^*(A)$. Since $X/A = S^k$, we have $\widetilde{K}(X/A) = \mathbb{Z}$. By induction $\widetilde{K}(A)$ is free, hence projective, so the s.e.s splits. Hence $\widetilde{K}(X)$ is free on the positive dimensional cells, so $K(X)$ is free on all the cells.

Since all the cells are even dimensional, the first term of the s.e.s. $K^1(X/A) \to K^1(X) \to K^1(A)$ is always 0. Thus, by induction on the number of cells, $K^1(X) = 0$. Thus $K(SX) = \mathbb{Z} \oplus \widetilde{K}(SX) = \mathbb{Z} \oplus K^1(X) = \mathbb{Z}$ and $K^{-1}(X) = K^1(X) = 0$. □

b. Compute $K^*(\mathbb{CP}^n)$ and express as a module of $K^*(\mathrm{pt})$.

Proof. The CW-complex structure of $\mathbb{CP}^n$ has cell in each even dimension up to $2n$. So $K^0(\mathbb{CP}^n) = \mathbb{Z}^{n+1}$ and $K^1(\mathbb{CP}^n) = 0$. Since $K^0(\mathrm{pt}) = \mathbb{Z}$ and $K^1(\mathrm{pt}) = 0$, the action of $K^*(\mathrm{pt})$ is given by $1 \in K^0(\mathrm{pt}) = \mathbb{Z}$ maps to the $\mathrm{id} : K^*(\mathbb{CP}^n) \to K^*(\mathbb{CP}^n)$. □