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HW 5

1 Let $(X,\mathcal{M})$ be a measurable space and $f_n : X \to \mathbb{R}$ be measurable for $n \in \mathbb{N}$. Show that

a) $\liminf_{n\to\infty} f_n$ is measurable.

b) $\{x \in X : \lim_{n\to\infty} f_n \text{ exists }\} \in \mathcal{M}$

Proof. For (a), I first show that if $g_n : X \to \mathbb{R}$ are measurable for $n \in \mathbb{N}$, then $g := \sup_n g_n$ is measurable. We have

$$\begin{aligned} x \in g^{-1}((a,\infty)) &\iff g(x) > a \\ &\iff \exists n \quad g_n(x) > a \\ &\iff \exists n \quad x \in g_n^{-1}((a,\infty)) \\ &\iff x \in \bigcup_n g_n^{-1}((a,\infty)), \end{aligned}$$

where the second equivalence follows from the fact that if $g_n(x) \le a$ for all n, then $\sup_n g_n(x) \le a$. Thus $g^{-1}((a,\infty)) = \bigcup_n g_n^{-1}((a,\infty))$, so g is measurable. Moreover note that, under the same conditions, $\inf_n g_n = -\sup_n -g_n$, so $\inf_n g_n$ is measurable also.

Hence $\inf_{k\ge n} f_k$ is measurable for each k, so $\liminf_{n\to\infty} f_n = \sup_n \inf_{k\ge n} f_k$ is measurable.

For (b), first note that the same argument applies to $\limsup_{n\to\infty} f_n$. Thus $h := \liminf_{n\to\infty} f_n - \limsup_{n\to\infty} f_n$ is measurable. Thus $h^{-1}(0) \in \mathcal{M}$, and this is precisely the set of points where the limit exists. □

2 Let $(X,\mathcal{M})$ and $(Y,\mathcal{N})$ be measure spaces. Assume that μ is a measure on $(X,\mathcal{M})$ and that $\phi : X \to Y$ is $(\mathcal{M},\mathcal{N})$ measurable. Then

$$\mu_\phi : \mathcal{N} \to [0,\infty], \quad A \mapsto \mu(\phi^{-1}(A))$$

is a measure on $(Y,\mathcal{N})$. It is called the image of μ under ϕ.

Proof. We need to show that $\mu_\phi(\emptyset) = 0$ and that μ_ϕ is countably additive. For the former, $\mu_\phi(\emptyset) = \mu(\phi^{-1}(\emptyset)) = \mu(\emptyset) = 0$. For the latter, suppose (A_n) is a countable collection of pairwise disjoint sets. Then $\mu(\phi^{-1}(\bigcup_n A_n)) = \mu(\bigcup_n \phi^{-1}(A_n)) = \sum_n \mu(\phi^{-1}(A_n)) = \sum_n \mu_\phi(A_n)$, where the second equality follows from the fact that $\phi^{-1}(A_j) \cap \phi^{-1}(A_k) = \phi^{-1}(A_j \cap A_k) = \emptyset$ for all $j \ne k$. □

3 Let $E \in \mathcal{L}$ with $m(E) > 0$. Then the set $E - E$ contains an open interval centered at 0.

Proof. By the inner regularity of m, there exists a compact $K \subset E$ with $m(K) > 0$. If $K - K$ contains an open interval, so does $E - E$. Hence WLOG E is compact.

Suppose that $E - E$ does not contain an open interval centered at 0. Then there exists a sequence $x_n \to 0$ such that $x_n \notin E - E$ for all n. Hence $E + x_n$ is disjoint from E for all n.

By the outer regularity of m, there exists open $U \supset E$ with $m(U) < 2\,m(E)$.

Since E is compact and U^c is closed, I claim that $d(E, U^c) := \inf\{|x - y| : x \in E, y \in U^c\} > 0$. Suppose $d(E, U^c) = 0$. Then there exists a sequence $(e_n) \subset E$ with $d(e_n, U) \to 0$. Since E is compact, by passing to a subsequence we may assume $e_n \to e$ for some $e \in E$. But then $d(e, U) = 0$, so e is a limit point of U. Hence $e \in U$ since U is closed. This contradicts the disjointness of E and U^c. Hence $d(E, U^c) > 0$.

Thus there exists x_n such that $E + x_n \subset U$. But then $m(U) \geq m(E \cup (E + x_n)) = m(E) + m(E + x_n) = 2\,m(E) > m(U)$, a contradiction. □

4 Let $(X, \mathcal{M}, \mu)$ be a measure space. We call $A \in \mathcal{M}$ an atom if $\mu(A) > 0$ and if $A = A_1 \cup A_2$ for $A_1, A_2 \in \mathcal{M}$ disjoint implies that $\mu(A_1) = 0$ or $\mu(A_2) = 0$.

Assume now that $(X, \mathcal{M}, \mu)$ is an atom-free measure space with $\mu(X) = 1$. Then there is for any $0 \leq r \leq 1$ an $A \in \mathcal{M}$ with $\mu(A) = r$. Hint: first show that there is a measurable set whose measure is between 1/3 and 1/2. Secondly show that there is a disjoint sequence of measurable sets (B_n) with $\mu(B_n) = 2^{-n}$. Write r as $r = \sum_{n=1}^{\infty} r_n 2^{-n}$ with $r_n \in \{0, 1\}$. Therefore $\mu(\bigcup_{r_n=1} B_n = r$.

Proof. Let $Y \subset X$ with $m(Y) > 0$. Since X is atom-free, there must exist disjoint sets Y_0 and Y_1 with $Y = Y_0 \cup Y_1$ and $\mu(Y_1) \geq \mu(Y_0) > 0$. Of all such Y_0 and Y_1, pick a pair such that $\mu(Y_1) - \mu(Y_0)$ is minimized.

I claim that $\mu(Y_1) = \mu(Y_0) = m(Y)/2$. Suppose not. By the atom-free assumption, there exists a nonempty collection $\mathcal{U} := \{U \subset Y_1 : \mu(U) > 0\}$.

I claim that $B := \inf_{U \in \mathcal{U}} \mu(U) = 0$. Suppose not. Pick $U \in \mathcal{U}$ with $m(U) < 2B$. Then $U = V \cup W$ for disjoint V, W with $\mu(V) \geq \mu(W) > 0$. But then $\mu(W) \leq \mu(U)/2 < B$, a contradiction. Hence, $B = 0$.

Thus there exists $U \in \mathcal{U}$ with $\mu(U) < (\mu(Y_1) - \mu(Y_0))/2$. Then $Y_1 \setminus U, Y_0 \cup U$ is a partition of Y. Moreover $m(Y_1 \setminus U) - m(Y_0 \cup U) = \mu(Y_1) - \mu(Y_0) - 2\mu(U) > 0$, so $m(Y_1 \setminus U) \geq m(Y_0 \cup U)$ and $m(Y_1 \setminus U) - m(Y_0 \cup U) < \mu(Y_1) - \mu(Y_2)$. This contradicts the minimality of $\mu(Y_1) - \mu(Y_0)$.

In summary, we have proved that if $Y \subset X$ with $m(Y) > 0$ there exist disjoint sets Y_0 and Y_1 with $Y = Y_0 \cup Y_1$ and $\mu(Y_0) = \mu(Y_1) = m(Y)/2$.

Applying this lemma to X, we get $X = X_1 \cup B_1$ for disjoint X_1 and B_1 with $\mu(X_1) = \mu(B_1) = 1/2$. Apply the lemma to X_1, to get $X_1 = X_2 \cup B_2$ for disjoint X_2 and B_2 with $\mu(X_2) = \mu(B_2) = 1/2$. Continuing in this way, we get a sequence (B_n) of pairwise disjoint sets such that $m(B_n) = 2^{-n}$ for each n. Following the hint, write r as $r = \sum_{n=1}^{\infty} r_n 2^{-n}$ with $r_n \in \{0, 1\}$. Then we have $\mu(\bigcup_{r_n=1} B_n) = r$.

□

5 Show that there is a measurable set $A \subset [0,1]$ such that $0 < m(A \cap I) < m(I)$ for all nondegenerate intervals I.

Proof. Let (U_n) be a countable base for the topology of $[0,1]$ consisting of bounded open intervals. For example, take all balls centered at the rationals of rational radius. Let $A_n \subset U_n$ be a fat Cantor set of measure $m(A_n)/2$. Each A_n is closed and nowhere dense, so by the Baire Category Theorem $A := \bigcup_n A_n$ is nowhere dense.

Let I be a nondegenerate interval. Since (U_n) is a base, there exists $A_n \subset U_n \subset I$. Hence $m(A \cap I) > m(A_n) > 0$. On the other hand, since A is nowhere dense, $(\overline{A})^c$ is open and dense. Hence $I \cap (\overline{A})^c$ is a nonempty open set, so contains some nondegenerate open interval J. Thus $J \subset I \cap A^c$, so $m(A \cap I) = m(I) - m(A^c \cap I) \leq m(I) - m(J) < m(I)$. $\square$