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HW 5

1 Let $g \in C^2[a,b]$, and $h = b - a$. Show that if $g(a) = g(b) = 0$, then

$$\|g\|_{C[a,b]} \leq (h^2/8)\|g''\|_{C[a,b]}.$$

Give an example showing that $1/8$ is the best possible constant.

Proof. Since $g(a) = g(b) = 0$, there exists a point $c \in (a,b)$ such that $|g(c)| = \|g\|$. Also $g'(c) = 0$. By the fundamental theorem of calculus, we have

$$\begin{aligned}
|g(c)| &= \left| g(a) + \int_a^c g'(x)\,dx \right| \\
&= \left| \int_a^c \left(g'(c) + \int_c^t g''(x)\,dt \right)\,dx \right| \\
&= \left| \int_a^c \int_t^c g''(x)\,dt\,dx \right| \\
&\leq \int_a^c \int_t^c |g''(t)|\,dt\,dx \\
&\leq \int_a^c \int_t^c \|g''\|\,dt\,dx \\
&= \int_a^c (c-t)\|g''\|\,dx \\
&= \left[ct - t^2/2\right]_{t=a}^c \|g''\| \\
&= (c^2 - c^2/2 - ac + a^2/2)\|g''\| \\
&= (1/2)(c-a)^2\|g''\|
\end{aligned}$$

Similarly, using b in place of a, we get $|g(c)| \leq (1/2)(b-c)^2\|g''\|$. Since $c \in (a,b)$, we have $\min(c-a, b-c) \leq (b-a)/2$. Hence $\|g\| = |g(c)| \leq (1/2)((b-a)/2)^2\|g''\| = (h^2/8)\|g''\|$.

An example showing that $1/8$ is the best possible constant is $g(x) = x^2 - 1$ on $[-1,1]$. To see this, note that $\|g\| = 1 = ((2)^2/8)(2) = (h^2/8)\|g''\|$. □

2 Use the previous problem to show that if $f \in C^2[0,1]$, then the equally spaced linear spline interpolant f_n satisfies

$$\|f - f_n\|_{C[a,b]} \leq (8n^2)^{-1}\|f''\|_{C[a,b]}.$$

Proof. For $1 \leq k \leq n-1$, we have $f - f_n \in C^2[k/n,(k+1)/n]$ with $f - f_n = 0$ at the endpoints. Hence $\|f - f_n\|_{C[k/n,(k+1)/n]} \leq (8n^2)^{-1}\|(f-f_n)''\|_{C[k/n,(k+1)/n]} = (8n^2)^{-1}\|f''\|_{C[k/n,(k+1)/n]} \leq (8n^2)^{-1}\|f''\|_{C[0,1]}$. Since $([k/n,(k+1)/n])_k$ covers the interval $[0,1]$, we have $\|f - f_n\|_{C[0,1]} \leq (8n^2)^{-1}\|f''\|_{C[0,1]}$. □

3 Let $0 < \alpha < 1$ be fixed. Define $f(x) = x^\alpha, x \in [0,1]$. Show that $\omega(f;\delta) \le C\delta^\alpha$ where C is independent of δ.

Proof. Let $\delta > 0$. Note that $f'' < 0$ so f is convex. Moreover f is increasing. Hence, if $s < t$ with $t - s \le \delta$, we have $|f(s) - f(t)| = f(t) - f(s) \le f(\delta + s) - f(s) \le \frac{\delta}{\delta+s} f(\delta) + \frac{s}{\delta+s} f(s) - f(s) \le f(\delta) + f(s) - f(s) = \delta^\alpha$. □

4 Let V be a Banach space. Suppose that there is an uncountable set of vectors U and $\epsilon_0 > 0$ such that for all $u, v \in U$ with $u \ne v$, $\|u - v\| \ge \epsilon_0$. Prove that V is not separable. Use this to show that $L^\infty[0,1]$ is not separable.

Proof. Suppose V is separable. Let D be a countable dense set. Then for every $u \in U$, we have $B_{\epsilon_0/2}(u) \cap D \ne \emptyset$. However, if $u, v \in U$ with $u \ne v$, by the triangle inequality $B_{\epsilon_0/2}(u) \cap B_{\epsilon_0/2}(v) = \emptyset$. Thus there is an injection from $\{B_{\epsilon_0/2}(u) : u \in U\}$ to D. This contradicts the countability of D.

To see that $L^\infty[0,1]$ is not separable, let (A_n) be a sequence of disjoint subsets of $[0,1]$ of positive measure. For example, take $A_n = (1/(n+1), 1/n)$. For each $N \subset \mathbb{N}$, let $U_N = \bigcup_{n\in N} A_n$ and $f_N = \chi_{U_N}$. If $N, M \subset \mathbb{N}$ with $N \ne M$, then WLOG there exists $n \in N \setminus M$. Then for all $x \in A_n$, we have $f_N(x) - f_M(x) = 1$. Thus $\|f_N - f_M\|_\infty = 1$. Hence $(f_N)_{N\in\mathcal{P}(\mathbb{N})}$ is an uncountable family of elements of $L^\infty[0,1]$ with $\|f_N - f_M\| = 1$ for $N \ne M$, so by the lemma $L^\infty[0,1]$ is not separable. □

5 Recall that the B-splines N_m satisfy the recurrence relation

$$N_m(x) = \frac{x}{m-1} N_{m-1}(x) + \frac{m-x}{m-1} N_{m-1}(x-1),\ m \ge 2.$$

Use this to show $N_3(x) = \frac{1}{2}\big((x)_+^2 - 3(x-1)_+^2 + 3(x-2)_+^2 - (x-3)_+^2\big)$. Hint: $(x-a)((x-a)_+)^k = ((x-a)_+)^{k+1}$ for $k \ge 1$.

Proof. To prove the hint, if $x - a \le 0$ then $(x-a)_+ = 0$, so $(x-a)((x-a)_+)^k = 0 = ((x-a)_+)^{k+1}$. If $x - a > 0$, then $(x-a)((x-a)_+)^k = (x-a)^{k+1} = ((x-a)_+)^{k+1}$.

We have $N_1 = \chi_{[0,1)}$. Hence,

$$\begin{aligned} N_2(x) &= \frac{x}{2-1}\chi_{[0,1)}(x) + \frac{2-x}{2-1}\chi_{[0,1)}(x-1) \\ &= x\chi_{[0,1)}(x) + (2-x)\chi_{[1,2)}(x) \\ &= x^+ - 2(x-1)_+ + (x-2)_+ \end{aligned}$$

Thus,

$$
\begin{aligned}
N_3(x) &= \frac{x}{2}(x_+ - 2(x-1)_+ + (x-2)_+) + \frac{3-x}{2}((x-1)_+ - 2(x-2)_+ + (x-3)_+) \\
&= \frac{1}{2}((x)x_+ - 2(x-1)(x-1)_+ - 2(x-1)_+ + (x-2)(x-2)_+ + 2(x-2)_+ \\
&\quad - (x-3)((x-1)_+ - 2(x-2)_+ + (x-3)_+)) \\
&= \frac{1}{2}((x_+)^2 - 2((x-1)_+)^2 - 2(x-1)_+ + ((x-2)_+)^2 + 2(x-2)_+ \\
&\quad - ((x-1)(x-1)_+ - 2(x-1)_+ - 2(x-2)(x-2)_+ + 2(x-2)_+ + (x+3)(x+3)_+)) \\
&= \frac{1}{2}((x_+)^2 - 2((x-1)_+)^2 + ((x-2)_+)^2 - (((x-1)_+)^2 - 2((x-2)_+)^2 + ((x+3)_+)^2)) \\
&= \frac{1}{2}((x_+)^2 - 3((x-1)_+)^2 + 3((x-2)_+)^2 - ((x+3)_+)^2)
\end{aligned}
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□