Paul Gustafson
Representations of Lie Algebras

HW 3

1 Prove that the roots systems for $\mathfrak{so}(5)$ and $\mathfrak{sp}(4)$ are isomorphic.

Proof. We already proved in class that the root system for $\mathfrak{so}(5)$ is isomorphic to

$$(\{(\pm1,0),(0,\pm1),(\pm1,\pm1)\},\mathbb{R}^2).$$

Using the representation of $\mathfrak{sp}(4)$ on pages 72-73 of Goodman and Wallach, the Lie algebra $\mathfrak{sp}(4)$ consists of all matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{11} \\ c_{11} & c_{12} & -a_{22} & -a_{12} \\ c_{21} & c_{11} & -a_{21} & -a_{11} \end{pmatrix}$$

with basis

$$\{e_{11}-e_{44}, e_{22}-e_{33}, e_{12}-e_{34}, e_{21}-e_{43}, e_{13}+e_{24}, e_{14}, e_{23}, e_{31}+e_{42}, e_{32}, e_{41}\}$$

The choice of Cartan subalgebra $\mathfrak{h}$ is the span of the first two basis elements, $x_1 := e_{11}-e_{44}$ and $x_2 := e_{22}-e_{33}$.

The adjoint action of the Cartan subalgebra $\mathfrak{h}$ on $\mathfrak{sp}(4)$ is

$$\begin{aligned} \mathrm{ad}(x_1) &= \mathrm{diag}(0,0,1,-1,1,2,0,-1,0,-2) \\ \mathrm{ad}(x_2) &= \mathrm{diag}(0,0,-1,1,1,0,2,-1,-2,0) \end{aligned}$$

Up to rearragement of basis vectors, this is the same as the adjoint action of the Cartan subalgebra of $\mathfrak{so}(5)$ on $\mathfrak{so}(5)$ that we calculated in class. Since the adjoint action of a choice of Cartan subalgebra on the full Lie algebra determines the root system, it follows that the corresponding root systems are isomorphic. ☐

2 Let (R,E) be a roots system, with Weyl group W. Show that W is a normal subgroup of the group of automorphisms of (R,E) (that is, the group of linear automorphisms of E, preserving R as a set, and preserving the Cartan integers).

Proof. This follows from the Lemma on page 43 of Humphreys. ☐

3 Fill in the details of the proofs of the results in 10.2 and 10.3 of Humphreys book.

Lemma 0.1. *If α is positive but not simple, then $\alpha-\beta$ is a root (necessarily positive) for some $\beta \in \Delta$*

Proof. If $(\alpha,\beta) \le 0$ for all $\beta \in \Delta$, suppose $0 = \sum_{\beta\in\Delta\cup\{\alpha\}} r_\beta \beta$. Let $\delta \in \mathfrak{C}(\Delta)$. Separating into sets for which $r_\beta \ge 0$ and $r_\beta \le 0$, we can rewrite this as $\sum s_\beta \beta = \sum t_\gamma \gamma$ where s_β and t_γ are nonnegative with disjoint support. Let ε denote the value of these two sums. Then

$$(\varepsilon,\varepsilon) = \sum_{\beta,\gamma\in\beta\in\Delta\cup\{\alpha\}} s_\beta t_\gamma (\beta,\gamma).$$

If $\beta \ne \gamma \in \Delta$, then $(\beta,\gamma) \le 0$ since Δ is a root base. By assumption, $(\alpha,\beta) \le 0$ for all $\beta \in \Delta$. Hence, $(\varepsilon,\varepsilon) \le 0$, so $\varepsilon = 0$ since the inner product is positive-definite. Since α is a positive root, $(\delta,\alpha) > 0$. Hence,

$$0 = (\delta,\varepsilon) = \sum_{\beta\in\Delta\cup\{\alpha\}} s_\beta(\delta,\beta) = \sum_{\gamma\in\Delta\cup\{\alpha\}} t_\gamma(\delta,\gamma).$$

Thus, $s_\beta = t_\gamma = 0$ for all $\beta, \gamma\Delta\cup\{\alpha\}$. Hence, $\Delta\cup\{\alpha\}$ is linearly independent, a contradiction.

Thus, there exists $\beta \in \Delta$ such that $(\alpha,\beta) > 0$. Hence, Lemma 9.4 implies that $\alpha-\beta$ is a root. ☐

Corollary 0.2. *Each $\beta \in \Phi^+$ can be written in the form $\alpha_1 + \ldots + \alpha_k$ ($\alpha_i \in \Delta$, not necessarily distinct) in such a way that each partial sum is a root.*

Proof. Use the lemma and induction on $\mathrm{ht}(\beta)$. Given $\beta \in \Phi^+$ not simple, write it as a sum of simple roots with $\mathrm{ht}(\beta)$ terms. Applying the lemma, we get another positive root with lower height. □

Lemma 0.3. *Let α be simple. Then σ_α permutes the positive roots other than α.*

Proof. The proof in the book shows that σ_α maps $\Phi^+ - \{\alpha\}$ to itself. Since σ_α is invertible, its restriction to $\Phi^+ - \{\alpha\}$ must be a permutation. □

Corollary 0.4. *Set $\delta = \frac{1}{2}\sum_{\beta \succ 0} \beta$. Then $\sigma_\alpha(\delta) = \delta - \alpha$ for all $\alpha \in \Delta$.*

Proof. The map σ_α permutes the roots other than α, but maps α to $-\alpha$. Thus $\sigma_\alpha(\delta) - \delta = \frac{1}{2}(-\alpha - \alpha) = -\alpha$. □

Lemma 0.5. *Let $\alpha_1, \dots, \alpha_t \in \Delta$ (not necessarily distinct. Write $\sigma_i = \sigma_{\alpha_i}$. If $\sigma_1 \dots \sigma_{t-1}(\alpha_t)$ is negative, then for some index $1 \le s < t$, $\sigma_1 \dots \sigma_t = \sigma_1 \dots \sigma_{s-1}\sigma_{s+1} \dots \sigma_{t-1}$.*

Proof. A slight clarification of the last sentence of the proof:

$$\begin{aligned}
\sigma_s &= \sigma_{\alpha_s} \\
&= \sigma_{\beta_s} \\
&= \sigma_{\sigma_{s+1} \dots \sigma_{t-1}(\alpha_t)} \\
&= \sigma_{s+1} \dots \sigma_{t-1} \sigma_{\alpha_t} (\sigma_{s+1} \dots \sigma_{t-1})^{-1} \\
&= \sigma_{s+1} \dots \sigma_{t-1} \sigma_t (\sigma_{s+1} \dots \sigma_{t-1})^{-1} \\
&= \sigma_{s+1} \dots \sigma_{t-1} (\sigma_{s+1} \dots \sigma_{t-1} \sigma_t)^{-1}
\end{aligned}$$

Substituting this value for σ_s into the product $\sigma_1 \dots \sigma_t$ gives the desired identity. □

Corollary 0.6. *If $\sigma = \sigma_1 \dots \sigma_2$ is an expression for $\sigma \in \mathscr{W}$ in terms of reflections corresponding to simple roots, with t as small as possible, then $\sigma(\alpha_t) \prec 0$.*

Proof. Suppose not. Then

$$\begin{aligned}
0 &\prec \sigma(\alpha_t) \\
&= \sigma_1 \dots \sigma_t(\alpha_t) \\
&= -\sigma_1 \dots \sigma_{t-1}(\alpha_t)
\end{aligned}$$

This implies that $\sigma_1 \dots \sigma_{t-1}(\alpha_t)$ is negative, which contradicts the lemma. □

Theorem 0.7. *Let Δ be a base of Φ.*

(a) If $\gamma \in E$, γ regular, there exists $\sigma \in \mathscr{W}$ such that $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \Delta$ (so $\mathscr{W}$ acts transitively on Weyl chambers).

(b) If Δ' is another base of Φ, then $\sigma(\Delta') = \Delta$ for some $\sigma \in \mathscr{W}$ (so $\mathscr{W}$ acts transitively on bases).

(c) If α is any root, there exists $\sigma \in \mathscr{W}$ such that $\sigma(\alpha) \in \Delta$.

(d) $\mathscr{W}$ is generated by the σ_α ($\alpha \in \Delta$).

(e) If $\sigma(\Delta = \Delta$, $\sigma \in \mathscr{W}$, then $\sigma = 1$ (so $\mathscr{W}$ acts simply transitively on bases.

Proof. (a) The only thing to add here is that the fact that σ_α is orthogonal follows from considering its action on $\mathbb{R}\alpha \oplus \alpha^\perp$.

(b) Weyl chambers are in 1-1 correspondence with bases. Since $\mathscr{W}'$ consists of orthogonal linear maps, positive roots for a Weyl chamber get mapped to positive roots for the image of that Weyl chamber, and simple roots get mapped to simple roots.

(c) Suppose α were decomposable with respect to the base $\Delta(\gamma')$. Then α is a $\mathbb{Z}_+$-linear combination of elements α of $\Delta(\gamma')$, each of which satisfies $(\alpha,\gamma') > \varepsilon$, a contradiction.

(d) The proof in Humphreys is clear.

(e) Suppose not. The we can write $\sigma = \sigma_1 \dots \sigma_t$, with α_t the simple root corresponding to σ_t, as in Lemma 10.2C and its corollary. Since $\sigma(\Delta) = \Delta$ and α_t is a positive root, $\sigma(\alpha_t)$ is a positive root by the orthogonality of σ. This contradicts the corrollary to Lemma 10.2C.

□

Lemma 0.8. *For all $\sigma \in \mathscr{W}$, $l(\sigma) = n(\sigma)$, where l is the length and $n(\sigma)$ is the number of positive roots for which $\sigma(\alpha) \prec 0$.*

Proof. To see that $n(\sigma\sigma_\alpha) = n(\sigma) - 1$, Lemma 10.2B says that σ_α permutes the positive roots other than α. Hence the only possible change between $n(\sigma)$ and $n(\sigma\sigma_\alpha)$ is due to α. As mentioned earlier in the proof, $\sigma(\alpha) \prec 0$. Thus $\sigma\sigma_\alpha(\alpha) = -\sigma(\alpha) \succ 0$. Hence $n(\sigma\sigma_\alpha) = n(\sigma) - 1$. □

Lemma 0.9. *Let $\lambda, \mu \in \overline{\mathfrak{C}(\Delta)}$. If $\sigma\lambda = \mu$ for some $\sigma \in \mathscr{W}$, then σ is a product of simple reflections which fix λ; in particular, $\lambda = \mu$.*

Proof. The proof in the book is clear. □