Paul Gustafson
Texas A&M University - Math 641
Instructor - Fran Narcowich

HW 6

1 Let $(y_j)_{j=0}^n \subset \mathbb{R}$. Consider the cubic-Hermite spline $s(x) \in S^h(3,1)$, with $h = 1/n$, that satisfies $s(j/n) = y_j$ and minimizes $\int_0^1 (s'')^2\, dx$. Show that $s(x) \in S^h(3,2)$.

Proof. Let $V = \{f \in S^h(3,1) : f(0) = f(1) = 0 \text{ for all } j\}$. Define $\langle \cdot, \cdot \rangle_V : V \times V \to \mathbb{R}_{\geq 0}$ by $\langle f, g \rangle_V = \int_0^1 f''(x) g''(x)\, dx$. This function is bilinear, and $\langle f, f \rangle_V = 0$ implies $f'' = 0$ which implies that $f = 0$ since $f(0) = f(1) = 0$. Hence, $\langle \cdot, \cdot \rangle_V$ is an inner product.

Let $W = \{f \in S^h(3,1) : f(j/n) = 0 \text{ for all } j\}$. Pick any $l \in S^h(3,1)$ interpolating the points $((j/n, y_j))_j$, then $l + W$ is the space of all such interpolants.

We have $s = w + v$ for some $w \in W$ and $v \in W^\perp$. Hence $v = s - w \in l + W$, and $\|s\|_V^2 = \|w\|_V^2 + \|v\|_V^2$. Hence, $\|v\|_V \leq \|s\|_V$ with equality iff $w = 0$. Thus, since s minimizes $\|s\|_V$ over $l + W$, we must have $s = v$. Thus $s \in W^\perp$.

Now suppose that $s \notin S^h(3,2)$. Let j_0 be a knot for which s'' is discontinuous. I claim there exists $u \in W$ such that $u'(j_0) = 1$ and $u'(j) = 0$ for all $j \neq j_0$. To construct such a u, for $x \in [(j_0 - 1)/n, j_0/n]$ let $u(x) = \int_{(j_0-1)/n}^x A(t - (j_0 - 1)/n)^2 + B(t - (j_0 - 1)/n)\, dt$ where real numbers A, B are chosen such that $u'(j_0/n) = A(1/n)^2 + B(1/n) = 0$, and $u(j_0) = A/(3n^3) + B/(2n^2) = 0$. We can do similarly for $x \in [j_0/n, (j_0 + 1)/n]$. Lastly, set $u(x) = 0$ for all other x.

Then we have

$$
\begin{aligned}
\langle s, u \rangle &= \int_0^1 s'' u''\, dx \\
&= \sum_j \int_{j/n}^{(j+1)/n} s'' u''\, dx \\
&= \sum_j [s'' u']_{x=j/n}^{(j+1)/n} - \int_{j/n}^{(j+1)/n} s''' u'\, dx \\
&= \sum_j [s'' u']_{x=j/n}^{(j+1)/n} - s''' \int_{j/n}^{(j+1)/n} u'\, dx \\
&= \sum_j [s'' u']_{x=j/n}^{(j+1)/n} \\
&= \lim_{x \to j_0^+} s'' - \lim_{x \to j_0^-} s'' \\
&\neq 0,
\end{aligned}
$$

which contradicts the fact that $s \in W^\perp$. □

2 We want to solve the boundary value problem (BVP):

$$-u'' = f(x), u(0) = u(1) = 0, f \in C[0,1]$$

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a. Let H be the set of all continuous functions vanishing at $x = 0$ and $x = 1$, and having L^2 derivatives. Also let H have the inner product

$$\langle f, g \rangle_H = \int_0^1 f'(x) g'(x)\, dx.$$

Use integration by parts to convert the BVP into its "weak" form:

$$\langle u, v \rangle_H = \int_0^1 f(x) v(x)\, dx \text{ for all } v \in H.$$

Proof. We have $\langle u, v \rangle_H = \int_0^1 u'v'\, dx = -\int_0^1 u''v\, dx = \int_0^1 fv\, dx$ for all $v \in \mathcal{H}$. $\square$

b. Conversely, suppose that $u \in H$ is also in $C^2[0,1]$ and

$$\langle u, v \rangle_H = \int_0^1 f(x) v(x)\, dx \text{ for all } v \in H.$$

Show that u satisfies the BVP.

Proof. Suppose not. Then $-u''(x_0) \neq f(x_0)$ for some $x_0 \in (0,1)$. WLOG assume $u''(x_0) + f(x_0) > 0$. By continuity, this inequality holds on some interval $(x_0 - \delta, x_0 + \delta)$. Let $v \in H$ be a nonnegative with support in $(x_0 - \delta, x_0 + \delta)$ and $v(x_0) > 0$. Then $\int fv\, dx - \langle u, v \rangle_H = \int fv\, dx + \int u''v\, dx > 0$, a contradiction. $\square$

c. Let $V = S^h(1,0)$, with $h = 1/n$. Thus, V is spanned by $\phi_j(x) := N_2(nx - j + 1), 1 \leq j \leq n-1$ Show that the least squares approximation to u from V is $y = \sum_j \alpha_j \phi_j(x) \in V$, where $G\alpha = \beta$ for $\beta_j = \langle y, \phi_j \rangle_H$ and $G_{kj} = \langle \phi_j, \phi_k \rangle_H$.

Proof. This is just the definition of the normal equations, which was proved in HW 1. $\square$

d. Show that $G_{kj} = \langle \phi_j, \phi_k \rangle_H$ is given by $G_{j,j} = 2n$, $Gj, j-1 = -n$, and $G_{j,j+1} = n$ and $G_{j,k} = 0$ for all other k.

Proof. Recall that $N_2(x) = x_+ - 2(x-1)_+ + (x-2)_+$. Thus, for all j, we have $\phi_j' = n\chi_{[(j-1)h, jh]} - n\chi_{[jh,(j+1)h]}$. Hence,

$$\begin{aligned}
\langle \phi_j, \phi_k \rangle &= \int (n\chi_{[(j-1)h,jh]} - n\chi_{[jh,(j+1)h]})(n\chi_{[(k-1)h,kh]} - n\chi_{[kh,(k+1)h]})\, dx \\
&= \int (n^2\delta_{j,k} - n^2\delta_{j-1,k})\chi_{[(j-1)h,jh]} + (-n^2\delta j, k-1 + n^2\delta_{j,k})\chi_{[jh,(j+1)h]}\, dx \\
&= 2n\delta_{j,k} - n\delta_{j-1,k} - n\delta_{j,k-1}
\end{aligned}$$

$\square$

3 Let $S = \{s \in C^2(\mathbb{R}) : \forall j \in \mathbb{Z} \quad s \text{ is a cubic on } [j, j+1]\}$. Suppose that $s \in S$ has compact support in $[0, M]$. Determine the smallest value of M such that $s \not\equiv 0$.

Proof. Suppose $s \in S$ with nonnegative compact support. Since s''' is piecewise constant with knots at the integers, the support of s must be an interval of the form $[a, b]$ for $a, b \in \mathbb{N}$. Further suppose that $s \not\equiv 0$ and $M = b$ is minimized by s. Then $a = 0$ since otherwise $s(x + a) \in S$ with support $[0, M - a]$. Hence the support of s is precisely $[0, M]$ for some $M \in \mathbb{N}$.

For $j \in \mathbb{Z}$, let s_j be the cubic defined by s on $[j, j+1]$. Since $s \in C^2$, we have $s_j''(j) - s_{j-1}''(j) = s_j'(j) - s_{j-1}'(j) = s_j(j) - s_{j-1}(j) = 0$ for all j. Thus, by comparing Taylor expansions at $x = j$, we have $s_j = s_{j-1} + a_j(x-j)^3$ for some $a_j \in \mathbb{R}$. Since $s_{-1} = 0$, we have $s_j = \sum_{k=0}^{j} a_k(x-k)^3$ for $j \geq 0$.

Let P^3 be the vector space of polynomials of degree 3 or less. Let $T : P^3 \to P^3$ be defined by the matrix $T_{j,k} = (-1)^{k-j}\binom{k}{j}$ for $0 \leq j, k \leq 3$ with respect to the basis $(1, x, x^2, x^3)$. Then $Tx^k = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}x^j = (x-1)^k$ for all $0 \leq k \leq 3$. Thus $s_j = \sum_{k=0}^{j} a_k T^k x^3$.

Thus, M is equal to the degree of the minimal nonzero polynomial f such that $f(T^k)x^3 = 0$. Note that $(T - I)x^k = (x-1)^k - x^k$, so if $\deg(g) = k > 0$ then $\deg((T - I)g) = k - 1$. Thus, $((T - I)^n x^3)_{n=0}^{3}$ forms a basis for P^3. Hence $(T^n x^3)_{n=0}^{3}$ forms a basis for P^3. Hence $M = \deg(f) = 4$. □

4 Let $U := \{u_j\}_{j=1}^{\infty}$ be an orthonormal set in a Hilbert space $\mathcal{H}$. Show that the following are equivalent:

i. U is maximal in the sense that there is no non-zero vector in $\mathcal{H}$ that is orthogonal to U.

ii. Every vector in $\mathcal{H}$ may be uniquely represented as the series $f = \sum_{j=1}^{\infty}\langle f, u_j\rangle u_j$.

Proof. Suppose (i) holds but (ii) does not. Then there exists $f \in \mathcal{H}$ with $f \neq \sum_{j=1}^{\infty}\langle f, u_j\rangle u_j =: v$. Let $g = f - v$. Then if $u_j \in U$, then $\langle g, u_j\rangle = \langle f, u_j\rangle - \langle v, u_j\rangle = \langle f, u_j\rangle - \langle f, u_j\rangle = 0$. Thus g is a non-zero vector orthogonal to U, a contradiction.

For the converse, suppose (ii) holds but (i) does not. Let w be a nonzero vector orthogonal to U. By (ii), $w = \sum_{j=1}^{\infty}\langle w, u_j\rangle u_j = 0$, a contradiction. □

5 Show that every separable Hilbert space $\mathcal{H}$ has an o.n. basis.

Proof. Let $\mathcal{U}$ be the poset of all o.n. subsets of mH. If $\mathcal{L}$ is a chain in $\mathcal{U}$, then $\bigcup \mathcal{L}$ is orthonormal. Hence, by Zorn's Lemma, $\mathcal{U}$ contains a maximal element U.

I claim that U is countable. Suppose not. If $u, v \in U$ with $u \neq v$, then $\|u - v\|^2 = \|u\|^2 - \langle u, v\rangle - \langle v, u\rangle + \|v\|^2 = 2$. This implies that U is not separable, a contradiction.

Hence U satisfies condition (i) of Exercise 4, so U is a basis. □