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Math 644

## Final

**1** Show that $H_c^{n+1}(X \times \mathbb{R}; G) \cong H_c^n(X; G)$ for all $n$.

*Proof.* To compute $\operatorname{colim} H^{n+1}(X \times \mathbb{R}, X \times \mathbb{R} - K'; G)$, it suffices to let $K'$ range over sets of the form $K' = K \times I$ for a compact $K \subset X$ and $I$ a compact interval. WLOG $I = [0, 1]$. Applying the relative Meyer-Vietoris sequence in the same way as in the proof of the suspension isomorphism gives the desired result. $\square$

**2** Show that for any connected oriented closed manifold $M$ of dimension $n$ there is a map $f : M \to S^n$ having degree 1.

*Proof.* Let $B \subset M$ be an open neighborhood homeomorphic to a ball in $\mathbb{R}^n$. Let $f$ be the quotient map $f : M \to M/(M - B) \simeq S^n$. Since $M$ is orientable, we have $H^n(M) \simeq H^n(M, M - B)$ in the long exact sequence of a pair. Applying $f$ to both sides, the naturality of the long exact sequence gives a commutative diagram

$$\begin{tikzcd}
H_n(M) \arrow[r] \arrow[d, "f_*"] & H^n(M, M - B) \arrow[d] \\
H_n(M/(M - B)) \arrow[r] & H_n(M/(M - B), (M - B)/(M - B))
\end{tikzcd}$$

The rightmost arrow is an isomorphism since $(M, M - U)$ is a good pair, and the bottom arrow is an obvious isomorphism. Thus, $f_*$ is an isomorphism, and in particular a degree $\pm 1$ map. If the degree is $-1$, compose $f$ with a reflection of $S^n$ through an equator to get a degree 1 map. $\square$

**3** Let $f : M \to N$ be a map between closed connected oriented manifolds of same dimension $n$.

a. Suppose there is a ball $B \subset N$ such that $f^{-1}(B)$ is the disjoint union of balls $B_i$ each mapped homeomorphically by $f$ onto $B$. Show the degree of $f$ is $\sum_i \epsilon_i$ where $\epsilon_i$ is 1 or $-1$ according to whether $f : B_i \to B$ preserves or reverses local orientations induced from given fundamental classes $[M]$ and $[N]$.

   *Proof.* Using the relative Mayer-Vietoris sequence, we have

   $$0 \to H_n(M, M - B_i - B_j) \to H_n(M, M - B_i) \oplus H_n(M, (M - B_i) \to H_n(M, (M - B_i) \cup (M - B_j))$$

   . For $i \neq j$, the last term is 0, so the middle terms are naturally isomorphic. Continuing in this way, we get $H_n(M | \bigcup_i B_i) \cong \bigoplus_i H_n(M | B_i)$. By the naturality of the long exact sequence, we have $f_* : H_n(M) \cong H_n(M | \bigcup_i B_i) \to$

$H_n(N|B) \cong H_n(N)$ is given by $f_* = \sum_i f_i$, where $f_i : H_n(M|B_i) \to H_n(N|B)$ are the maps induced by $f$. Since $f_{|B_i}$ is a homeomorphism onto $B$ for each $i$, each $f_i$ is an isomorphism and the sign of $f_i$ is determined by whether it reverses local orientations. □

b. Show that if $f$ is a $p$-sheeted covering projection then $\deg(f) = \pm p$.

*Proof.* It suffices to show that the set on which $f$ preserves local orientations is open. Let $x \in M$ such that $f$ preserves local orientations at $x$. Pick an evenly covered neighborhood $U$ of $f(x)$. Then $f$ also preserves local orientations on the component of $f^{-1}(U)$ containing $x$. □

c. If $M_g$ denotes the closed orientable surface of genus $g$, show that if $g \geq h$ there exists a map $f : M_g \to M_h$ of degree 1.

*Proof.* Think of $M_g$ as the connected sum of $M_h$ and $M_{g-h}$. Let $f$ be a map sending the $M_{g-h}$ part to a sphere and leaving the $M_h$ part alone. Applying part (a) to any untouched neighborhood in $M_h$ tells us that the degree of $f$ is 1. □

**4** If $g \geq 1$ show that for each nonzero $\alpha \in H^1(M_g;\mathbb{Z})$ there exists $\beta \in H^1(M_g;\mathbb{Z})$ with $\alpha \smile \beta \neq 0$. Use this fact to show that $M_g$ is not homotopy equivalent to a wedge $X \vee Y$ of CW-complexes with non trivial reduced homology.

*Proof.* We proved on a homework that $H^1(M_g) = \mathbb{Z}^{2g}$. Hatcher's Prop. 3.38 implies that the the cup product pairing $H^1(M_g) \times H^1(M_g) \to \mathbb{Z}$ given by evaluation of the cup product at the fundamental class is nonsingular. In particular, the first part of this problem holds.

For the second part, suppose that $M_g = X \vee Y$. Then $\mathbb{Z} = H_2(M_g) = H_2(X) \oplus H_2(Y)$. Hence WLOG $H_2(X) = \mathbb{Z}$ and $H_2(Y) = 0$. Let $\alpha \in H^1(Y)$. Pick $\beta \in H^1(M_g)$ such that $\alpha \smile \beta \neq 0$. Let $\phi \in C_2(X)$ be any chain representative of a generator of $H_2(X) = H_2(M_g)$. Since $\alpha \in H^1(Y)$, we have $(\alpha \smile \beta)(\phi) = 0$. Since this holds for a representative of a generator of $H_2(M_g)$, we have $\alpha \smile \beta = 0$, a contradiction. □

**5** Let $f : M \to N$ be a map between closed connected oriented manifolds of dimensions $m$ and $n$, respectively, and let $R$ be a commutative ring with 1.

a. Explain how this map makes $H^*(M;R)$ into an algebra over $H^*(N;R)$.

*Proof.* The map $H^*(M;R) \times H^*(N;R) \to H^*(M;R)$ defined by $(\alpha,\beta) \mapsto \alpha \smile f^*(\beta)$ gives the right action. Linearity is obvious, and associativity follows from the fact that $f^*(\beta \smile \gamma) = f^*(\alpha) \smile f^*(\gamma)$. □

b. Explain how to use Poincare duality to define group homomorphisms $f_! : H^i(M;R) \to H^{i+n-m}(N;R)$.

*Proof.* Poincare duality gives an isomorphisms $\phi : H^i(M; R) \simeq H_{m-i}(M; R)$, and $\psi : H_{m-i}(N; R) \simeq H^{i+n-m}(N; R)$. Let $f_! = \psi f_* \phi$. $\square$

c. Show that these maps assemble to give a homomorphism $f_! : H^*(M; R) \to H^*(N; R)$ of right $H^*(N; R)$-modules.

*Proof.* The assembled map is clearly a morphism of abelian groups. If $\alpha \in H^i(M; R)$ and $\beta \in H^j(N; R)$, we have $f_!(\alpha\beta) = f_!(\alpha \smile f^*(\beta)) = \psi f_* \phi(\alpha \smile f^*(\beta)) = f_!(\alpha) \smile f^*(\beta) = f_!(\alpha)\beta$, where we used the fact that $\psi, \phi$ commute with the cup product by the definition of the Poincare dual map as the cup product with the fundamental class. $\square$

**6** Let $p : E \to X$ be a vector bundle of rank $n$ over a paracompact space $X$.

(1) Show that if $E$ has $k$ sections $s_1, \ldots, s_k$ that are linearly independent at each $x \in X$ that it has a trivial subbundle of rank $k$.

*Proof.* Let $h : X \times \mathbb{R}^k \to E$ be defined by $h(x, t_1, \ldots, t_n) = \sum_i t_i s_i(x)$. Then $h$ is continuous (since it is continuous on each local trivialization), and a linear injection on each fiber since the $s_i$ are independent. Thus, by a lemma shown in class, $h$ is a subbundle map. $\square$

(2) Assume that $E$ has $k$ sections $s_1, \ldots, s_k$ such that at each $x$ the elements $s_1(x), \ldots, s_k(x)$ generate the fiber $E_x$ as a vector space. Show that $E$ is a quotient of a trivial bundle of rank $k$.

*Proof.* Define $h : X \times \mathbb{R}^k \to E$ by $h(x, t_1, \ldots, t_n) = \sum_i t_i s_i(x)$. This map is continuous and preserves fibres, hence a bundle map. It is also a linear surjection on fibers, so $E$ is a quotient of the trivial bundle of rank $k$. $\square$

(3) Under the assumptions of the previous question, let $x \in X$ and define $p_x : \mathbb{C}^k \to E_x$ as the surjective linear map sending $(\lambda_1, \ldots, \lambda_k)$ to $\sum_j \lambda_j s_j(x)$. Show that the map $\phi : X \to Gr_{k-n}(\mathbb{C}^k)$ sending $x \in X$ to $\ker p_x$ is a continuous map.

*Proof.* By restricting to a local trivialization, WLOG $E$ is a trivial bundle. Then $\ker p_x$ is the orthogonal complement of $\operatorname{span}\{s_1(x), \ldots, s_k(x)\}$. Since the $s_j$ are continuous and taking orthogonal complements of subspaces is a continous map from $Gr_{n-k}$ to $Gr_{k-n}$, the map $\phi$ is continuous. $\square$

(4) Let $q : \mathbb{Q}^n \to Gr_{k-n}(\mathbb{C}^k)$ denote the quotient bundle $\varepsilon^k / E_{k-n}$, where $E_{k-n}$ is the tautological bundle. Show that if $\phi$ is defined in the previous section then $\phi * Q_n \cong E$.

*Proof.* To show that $E$ is this pullback, we need to find a map $f : E \to Q^n$ mapping the fiber of each $x \in X$ to the fiber of $\phi(x)$ isomorphically. Define $f_x$ by $f_x(\sum_i \lambda_i s_i(x)) = \pi(\lambda_1, \ldots, \lambda_k)$, where $\pi$ is the orthogonal projection onto the orthogonal complement of $\ker p_x$ in $\mathbb{C}^k$. By the rank-nullity theorem, the projection $\pi$ is an isomorphism. $\square$

**7** Let $X$ be a finite CW-complex with only even-dimensional cells.

a. Show that $K(X)$ is a free abelian group on the set of cells of $X$ and that $K(SX) = \mathbb{Z}$. Explain why $K^{-1}(X) = 0$.

*Proof.* To see that $K(X)$ is free abelian on the cells of $X$, use induction on the number of cells. WLOG $X$ is connected. The base case, a point, is trivial. For the inductive step, assume that $X$ is constructed by attaching a $k$-cell to a subcomplex $A$, for some $k$. This gives a short exact sequence $\widetilde{K}^*(X/A) \to \widetilde{K}^*(X) \to \widetilde{K}^*(A)$. Since $X/A = S^k$, we have $\widetilde{K}(X/A) = \mathbb{Z}$. By induction $\widetilde{K}(A)$ is free, hence projective, so the s.e.s splits. Hence $\widetilde{K}(X)$ is free on the positive dimensional cells, so $K(X)$ is free on all the cells.

Since all the cells are even dimensional, the first term of the s.e.s. $K^1(X/A) \to K^1(X) \to K^1(A)$ is always 0. Thus, by induction on the number of cells, $K^1(X) = 0$. Thus $K(SX) = \mathbb{Z} \oplus \widetilde{K}(SX) = \mathbb{Z} \oplus K^1(X) = \mathbb{Z}$ and $K^{-1}(X) = K^1(X) = 0$. $\square$

b. Compute $K^*(\mathbb{CP}^n)$ and express as a module of $K^*(\mathrm{pt})$.

*Proof.* The CW-complex structure of $\mathbb{CP}^n$ has cell in each even dimension up to $2n$. So $K^0(\mathbb{CP}^n) = \mathbb{Z}^{n+1}$ and $K^1(\mathbb{CP}^n) = 0$. Since $K^0(\mathrm{pt}) = \mathbb{Z}$ and $K^1(\mathrm{pt}) = 0$, the action of $K^*(\mathrm{pt})$ is given by $1 \in K^0(\mathrm{pt}) = \mathbb{Z}$ maps to the $\mathrm{id} : K^*(\mathbb{CP}^n) \to K^*(\mathbb{CP}^n)$. $\square$