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Math 643 - Algebraic Topology I

HW 2

1.6 Contractible sets and hence convex sets are connected.

Proof. Suppose not. Then there exists a contractible space X and a disconnection $X = U \cup V$. Let x be a point for which $1_X \simeq c_x$. WLOG $x \in U$. Let $F : X \times I \to X$ be a homotopy from 1_x to c_x.

Pick $y \in V$. Let $f(t) = F(y, t)$. Then by considering $f(I) \cap U$ and $f(I) \cap V$, we see that $f(I)$ is disconnected. Hence I is disconnected, a contradiction. □

1.8

(i) Give an example of a continuous image of a contractible space that is not contractible.

(ii) Show that a retract of a contractible space is contractible.

Proof. For (i), the circle is the image of the line under the winding map.

For (ii), let X be contractible and $r : X \to Y$ be a retraction. Pick any $y \in Y$. Since X is contractible, there exists $x \in X$ and a homotopy $F : 1_X \simeq c_x$.

Define $G : Y \times I \to Y$ by $G = r \circ F$. Then $G(y, 0) = r(F(y, 0)) = r(y) = y$, and $G(y, 1) = r(F(y, 1)) = r(x)$. Hence G is a homotopy from Y to $1_{r(x)}$. □

1.9 If $f : X \to Y$ is nullhomotopic and if $g : Y \to Z$ is continuous, then $g \circ f$ is null-homotopic.

Proof. Since f is nullhomotopic there exists a homotopy $F : f \simeq c_x$ for some constant map $c_x : X \to Y$. Define $G : X \times I \to Z$ by $G = g \circ F$. Then $G(x, 0) = g(F(x, 0)) = g(f(x))$ and $G(x, 1) = g(F(x, 1)) = g(c_x)$ is a constant map. Thus $g \circ f$ is nullhomotopic. □

1.10 Let $f : X \to Y$ be an identification, and let $g : Y \to Z$ be a continuous surjection. Then g is an identification iff gf is an identification.

Proof. Suppose g is an identification. Then gf is a continuous surjection. Suppose $U \subset Z$ with $(gf)^{-1}(U) = f^{-1}g^{-1}(U)$ open. Since f, g are identifications, we have $g^{-1}(U)$ is open and then U is open. Hence gf is an identification.

For the converse, suppose gf is an identification. Suppose $U \subset Z$ with $g^{-1}(U)$ open. Then $(gf)^{-1}(U) = f^{-1}g^{-1}(U)$ is open since f is continuous. Hence U is open since gf is an identifiction. Since g is already a continuous surjection, g is an identification. □

1.11 Let X and Y be spaces with equivalence relations $\sim$ and $\Box$, respectively, and let $f : X \to Y$ be a continuous map preserving the relations (if $x \sim x'$, then $f(x) \Box f(x')$). Prove that the induced map $\overline{f} : X/\sim \to Y/\Box$ is continuous; moreover, if f is an identification, then so is $\overline{f}$.

Proof. Let ψ and π be the identification maps for $\sim$ and $\square$, respectively. Suppose $U \subset Y/\square$ is open. Then $(\overline{f})^{-1}(U) = \psi f^{-1}\pi^{-1}(U)$ is open since ψ, π are identifications and f is continuous. Hence $\overline{f}$ is continuous.

If f is an identification, then $\overline{f}$ is surjection since f is a surjection. Suppose $(\overline{f})^{-1}(U) = \psi f^{-1}\pi^{-1}(U)$ is open. Then since ψ, f, π are identifications, we have that U is open. Hence, $\overline{f}$ is an identification. $\square$