Paul Gustafson
Texas A&M University - Math 666
Instructor: Igor Zelenko

## Bonus exercises

**1** Let $\lambda : [0,T] \to [0,1]$ be measurable. Let

$$A_n = \bigcup_{k=0}^{n-1} [kT/n, kT/n + \int_{kT/n}^{(k+1)T/n} \lambda(t)\, dt).$$

Show that for any $\phi \in L_1[0,T]$,

$$\int_0^t \chi_{A_n}(\tau)\phi(\tau)\, d\tau \to \int_0^t \lambda(\tau)\phi(\tau)\, d\tau,$$

uniformly on $[0,T]$.

*Proof.* We first consider the case $\phi = \chi_{(a,b)}$. Pick nonnegative integers $k_a, k_b \leq n$ such that $|k_aT/n - a| < T/n$ and $|b - k_bT/n| < T/n$. Then

$$\begin{aligned}
\left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))\phi(\tau)\, d\tau \right| &= \left| \int_a^b (\lambda(\tau) - \chi_{A_n}(\tau))\, d\tau \right| \\
&= \left| \left( \int_a^{k_aT/n} + \int_{k_bT/n}^b + \int_{k_aT/n}^{k_bT/n} \right) (\lambda(\tau) - \chi_{A_n}(\tau))\, d\tau \right| \\
&\leq |k_aT/n - a|(2) + |b - k_bT/n|(2) + 0 \\
&\leq 4T/n,
\end{aligned}$$

which goes to 0 uniformly in $t$.

By linearity, we get the same result for step functions.

Let $\epsilon > 0$ and $\phi \in L_1[0,T]$ be arbitrary. We can pick a step function $h$ such that $\|\phi - h\|_{L_1[0,T]} < \epsilon/(2T)$. Then

$$\begin{aligned}
\left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))\phi(\tau)\, d\tau \right| &\leq \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))h(\tau)\, d\tau \right| + \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))(h - \phi)(\tau)\, d\tau \right| \\
&\leq \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))h(\tau)\, d\tau \right| + 2t\epsilon/(2T) \\
&\leq \left| \int_0^t (\lambda(\tau) - \chi_{A_n}(\tau))h(\tau)\, d\tau \right| + \epsilon. \\
&\leq 2\epsilon,
\end{aligned}$$

uniformly in $t$ for $n$ sufficiently large by the step function case. $\square$

**2** Let $V$ be a finite dimensional complex vector space and $T : V \to V$ be a linear transformation. Let $p(x)$ denote the characteristic polynomial of $T$, and $m(x)$ denote the minimal polynomial of $T$. Find a necessary and sufficient condition on the Jordan Normal Form of $T$ for $p(x) = m(x)$.

*Proof.* I claim that $p = m$ if and only if the geometric multiplicity of each eigenvalue of $T$ is 1. This means that each Jordan block has a distinct eigenvalue.

Let $(\lambda_j)$ be an enumeration of the eigenvalues of $T$ without multiplicity. Let $s_j$ denote the size of the largest Jordan block corresponding to each eigenvalue $\lambda_j$. Then $(T - \lambda_j I)^{s_j}$ kills the generalized eigenspace $V_j$ for $\lambda_j$. Thus $m(x)$ divides $\prod_j (x - \lambda_j)^{s_j}$. Moreover, each $(x - \lambda_j)^{s_j}$ generates the $T$-annihilator for the basis vector acted on by the last column of the largest Jordan block for $\lambda_j$. Thus, each $(x - \lambda_j)^{s_j}$ divides $m(x)$. Hence $m(x) = \prod_j (x - \lambda_j)^{s_j}$. It follows that $p = m$ iff each $\lambda_j$ corresponds to exactly one Jordan block. $\square$