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HW 1

1 Write the control system on $M = \mathbb{R}^2 \times \mathbb{T}^3$ corresponding to the car with two off-hook trailers system.

Proof. Let $n_i = (\cos\theta_i, \sin\theta_i)$ and $n_i' = (-\sin\theta_i, \cos\theta_i)$ for $0 \le i \le 2$. Then $n_i \cdot n_j = \cos(\theta_i - \theta_j) = n_i' \cdot n_j'$ and $n_i \cdot n_j' = \sin(\theta_i - \theta_j)$.

Let v_2 denote the velocity of the car, and v_i denote the velocity of the $(n-i)$-th trailer. Let $v_{1.5}$ denote the velocity of the first hook, and $v_{0.5}$ denote the velocity of the second hook. Let $\omega_i = \frac{\partial\theta_i}{\partial t}$.

In the case of linear motion of the car, we have $v_2 = vn_2$ and $\omega_2 = 0$. Hence,

$$v_{1.5} = vn_2$$

$$\begin{aligned} v_1 &= (v_{1.5} \cdot n_1)n_1 \\ &= (vn_2 \cdot n_1)n_1 \\ &= v\cos(\theta_2 - \theta_1)n_1 \end{aligned}$$

$$\begin{aligned} \omega_1 &= v_{1.5} \cdot n_1' \\ &= vn_2 \cdot n_1' \\ &= v\sin(\theta_2 - \theta_1) \end{aligned}$$

$$\begin{aligned} v_{0.5} &= v_1 - \omega_1 n_1' \\ &= v\cos(\theta_2 - \theta_1)n_1 - v\sin(\theta_2 - \theta_1)n_1' \end{aligned}$$

$$\begin{aligned} \omega_0 &= v_{0.5} \cdot n_0' \\ &= v\cos(\theta_2 - \theta_1)n_1 \cdot n_0' - v\sin(\theta_2 - \theta_1)n_1' \cdot n_0' \\ &= v\cos(\theta_2 - \theta_1)\sin(\theta_1 - \theta_0) - v\sin(\theta_2 - \theta_1)\cos(\theta_1 - \theta_0) \\ &= v\sin((\theta_1 - \theta_0) - (\theta_2 - \theta_1)) \\ &= v\sin(2\theta_1 - \theta_0 - \theta_2). \end{aligned}$$

For the case of the car turning, we have $v_2 = 0$ and $\omega_2 = \omega$. Hence,

$$v_{1.5} = -\omega n_2'$$

$$\begin{aligned} v_1 &= (v_{1.5} \cdot n_1)n_1 \\ &= (-\omega n_2' \cdot n_1)n_1 \\ &= \omega \sin(\theta_2 - \theta_1)n_1 \end{aligned}$$

$$\begin{aligned} \omega_1 &= v_{1.5} \cdot n_1' \\ &= -\omega n_2' \cdot n_1' \\ &= -\omega \cos(\theta_2 - \theta_1) \end{aligned}$$

$$\begin{aligned} v_{0.5} &= v_1 - \omega_1 n_1' \\ &= \omega \sin(\theta_2 - \theta_1)n_1 + \omega \cos(\theta_2 - \theta_1)n_1' \end{aligned}$$

$$\begin{aligned} \omega_0 &= v_{0.5} \cdot n_0' \\ &= \omega \sin(\theta_2 - \theta_1)n_1 \cdot n_0' + \omega \cos(\theta_2 - \theta_1)n_1' \cdot n_0' \\ &= \omega \sin(\theta_2 - \theta_1)\sin(\theta_1 - \theta_0) + \omega \cos(\theta_2 - \theta_1)\cos(\theta_1 - \theta_0) \\ &= \omega \cos(2\theta_1 - \theta_0 - \theta_2) \end{aligned}$$

Hence the control system for M is given by the family of vector fields $\mathcal{F} = \{\pm X_1, \pm X_2\}$, where

$$X_1 = \cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + A\frac{\partial}{\partial \theta_0}$$

with $A = \sin(2\theta_1 - \theta_0 - \theta_2)$, and

$$X_2 = \frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + B\frac{\partial}{\partial \theta_0}$$

with $B = \cos(2\theta_1 - \theta_0 - \theta_2)$. $\square$

2 Find all points $q \in M$ such that $\mathcal{F}$ is bracket-generating. At these points, calculate the degree of nonholonomy of $\mathcal{F}$.

Proof. We have

$$
\begin{aligned}
[X_1, X_2] = \Bigg[&\cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + A\frac{\partial}{\partial \theta_0}, \\
&\frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + B\frac{\partial}{\partial \theta_0}\Bigg] \\
&= \sin(\theta_2 - \theta_1)\left(-\sin(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial B}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + A\frac{\partial B}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
&\quad - \left(-\sin(\theta_2)\frac{\partial}{\partial x} + \cos(\theta_2)\frac{\partial}{\partial y} + \cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_2}\frac{\partial}{\partial \theta_0}\right) \\
&\quad + \cos(\theta_2 - \theta_1)\left(-\cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) - B\frac{\partial A}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} \\
&\quad + \left(-\sin^2(\theta_2 - \theta_1) - \cos(\theta_2 - \theta_1) - \cos^2(\theta_2 - \theta_1)\right)\frac{\partial}{\partial \theta_1} \\
&\quad + \left(\sin(\theta_2 - \theta_1)\frac{\partial B}{\partial \theta_1} + A\frac{\partial B}{\partial \theta_0} - \frac{\partial A}{\partial \theta_2} + \cos(\theta_2 - \theta_1)\frac{\partial A}{\partial \theta_1} - B\frac{\partial A}{\partial \theta_0}\right)\frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + (-1 - \cos(\theta_2 - \theta_1))\frac{\partial}{\partial \theta_1} \\
&\quad + \left(\sin(\theta_2 - \theta_1)(-2A) + A^2 + B + \cos(\theta_2 - \theta_1)(2B) - B(-B)\right)\frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + (-1 - \cos(\theta_2 - \theta_1))\frac{\partial}{\partial \theta_1} \\
&\quad + \left(2\cos((\theta_2 - \theta_1) + (2\theta_1 - \theta_0 - \theta_2)) + B + 1\right)\frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + C\frac{\partial}{\partial \theta_1} + D\frac{\partial}{\partial \theta_0},
\end{aligned}
$$

where $C = -1 - \cos(\theta_2 - \theta_1)$ and $D = 2\cos(\theta_1 - \theta_0) + \cos(2\theta_1 - \theta_0 - \theta_2) + 1$.

Hence,

$$
\begin{aligned}
\frac{\partial D}{\partial \theta_2} &= \sin(2\theta_1 - \theta_0 - \theta_2) \\
\frac{\partial D}{\partial \theta_1} &= -2\sin(\theta_1 - \theta_0) - 2\sin(2\theta_1 - \theta_0 - \theta_2) \\
\frac{\partial D}{\partial \theta_0} &= 2\sin(\theta_1 - \theta_0) + \sin(2\theta_1 - \theta_0 - \theta_2)
\end{aligned}
$$

Then

$$
\begin{aligned}
[X_1,[X_1,X_2]] = \Bigg[& \cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \sin(\theta_2-\theta_1)\frac{\partial}{\partial \theta_1} + A\frac{\partial}{\partial \theta_0}, \\
& \sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + C\frac{\partial}{\partial \theta_1} + D\frac{\partial}{\partial \theta_0}\Bigg] \\
= {} & \sin(\theta_2-\theta_1)\left(\frac{\partial C}{\partial \theta_1}\frac{\partial}{\partial \theta_1} + \frac{\partial D}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + A\frac{\partial D}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
& - C\left(-\cos(\theta_2-\theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + D\frac{\partial A}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
= {} & \left(\sin(\theta_2-\theta_1)\frac{\partial C}{\partial \theta_1} + C\cos(\theta_2-\theta_1)\right)\frac{\partial}{\partial \theta_1} \\
& \left(\sin(\theta_2-\theta_1)\frac{\partial D}{\partial \theta_1} + A\frac{\partial D}{\partial \theta_0} - C\frac{\partial A}{\partial \theta_1} + D\frac{\partial A}{\partial \theta_0}\right)\frac{\partial}{\partial \theta_0} \\
= {} & \left(\sin(\theta_2-\theta_1)\frac{\partial C}{\partial \theta_1} + C\cos(\theta_2-\theta_1)\right)\frac{\partial}{\partial \theta_1} \\
& \left(\sin(\theta_2-\theta_1)\frac{\partial D}{\partial \theta_1} + A\frac{\partial D}{\partial \theta_0} - C(2B) + D(-B)\right)\frac{\partial}{\partial \theta_0}
\end{aligned}
$$

and

$$
\begin{aligned}
[X_2,[X_1,X_2]] = \Bigg[&\frac{\partial}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial}{\partial\theta_1} + B\frac{\partial}{\partial\theta_0},\\
&\sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + C\frac{\partial}{\partial\theta_1} + D\frac{\partial}{\partial\theta_0}\Bigg]\\
={}& \cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \frac{\partial C}{\partial\theta_2}\frac{\partial}{\partial\theta_1} + \frac{\partial D}{\partial\theta_2}\frac{\partial}{\partial\theta_0}\\
&- \cos(\theta_2-\theta_1)\left(\frac{\partial C}{\partial\theta_1}\frac{\partial}{\partial\theta_1} + \frac{\partial D}{\partial\theta_1}\frac{\partial}{\partial\theta_0}\right) + B\frac{\partial D}{\partial\theta_0}\frac{\partial}{\partial\theta_0}\\
&- C\left(-\sin(\theta_2-\theta_1)\frac{\partial}{\partial\theta_1} + \frac{\partial B}{\partial\theta_1}\frac{\partial}{\partial\theta_0}\right) - D\frac{\partial B}{\partial\theta_0}\frac{\partial}{\partial\theta_0}\\
={}& \cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y}\\
&+ \left(\frac{\partial C}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial C}{\partial\theta_1} + C\sin(\theta_2-\theta_1)\right)\frac{\partial}{\partial\theta_1}\\
&+ \left(\frac{\partial D}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial D}{\partial\theta_1} + B\frac{\partial D}{\partial\theta_0} - C\frac{\partial B}{\partial\theta_1} - D\frac{\partial B}{\partial\theta_0}\right)\frac{\partial}{\partial\theta_0}\\
={}& \cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y}\\
&+ \left(\frac{\partial C}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial C}{\partial\theta_1} + C\sin(\theta_2-\theta_1)\right)\frac{\partial}{\partial\theta_1}\\
&+ \left(\frac{\partial D}{\partial\theta_2} - \cos(\theta_2-\theta_1)\frac{\partial D}{\partial\theta_1} + B\frac{\partial D}{\partial\theta_0} - C(-2A) - DA\right)\frac{\partial}{\partial\theta_0}.
\end{aligned}
$$

Letting T be the matrix with rows $X_1, X_2, [X_1,X_2], [X_1,[X_1,X_2]], [X_2,[X_1,X_2]]$, using MATLAB we find that $\det(T) = \sin(\theta_2-\theta_1) - \sin(\theta_1-\theta_0) + \sin(\theta_2 - 2\theta_1 + \theta_0)$.

If $\det(T) \neq 0$, then $Lie_q^3 = T_qM$, and the degree of nonholonomy at q is 3.

On the other hand, if $\det(T) = 0$ then let $\alpha = \theta_2 - \theta_1$ and $\beta = \theta_1 - \theta_0$. Then we have $0 = \det(T) = \sin(\alpha) - \sin(\beta) + \sin(\alpha-\beta) = \sin(\alpha) - \sin(\beta) + \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) = \sin(\alpha)(1+\cos(\beta)) - \sin(\beta)(1+\cos(\alpha))$. If either $\sin(\alpha) = 0$ or $\sin(\beta) = 0$, then $(\alpha,\beta) \in \{(0,0)\} \cup (\{\pi\}\times S^1) \cup (S^1 \times \{\pi\})$.

Otherwise, we have $\frac{1+\cos(\beta)}{\sin(\beta)} = \frac{1+\cos(\alpha)}{\sin(\alpha)}$. Let $f : (0,2\pi) \to \mathbb{R}$ be defined by $f(\pi) = 0$ and $f(x) = \frac{1+\cos(x)}{\sin(x)}$ otherwise. Note that $f'(x) = -1 - \frac{(1+\cos(x))\cos(x)}{\sin^2(x)} = -1 - \frac{\cos(x)}{1-\cos(x)} < 0$ for all x. Hence f is monotone decreasing. Thus, $\alpha = \beta$.

Thus, the points q such that $Lie_q^3 \neq T_qM$ are those points such that $\alpha = \pi$, or $\beta = \pi$, or $\beta - \alpha = 0$. In the original variables, this means $\theta_2 - \theta_1 = \pi$. or $\theta_1 - \theta_0 = \pi$, or $2\theta_1 - \theta_0 - \theta_2 = 0$.

Suppose $q \in M$ such that $Lie_q^4 \neq T_qM$. Using MATLAB, I found that the matrix with rows $X_1, X_2, [X_1,X_2], [X_1,[X_1[X_1,X_2]], [X_2,[X_1,X_2]]$ has determinant $\sin(\alpha)+\sin(\beta)+\sin(\alpha+\beta)$, which must be 0 at q. Hence if $\alpha=\beta$, then $0=2\sin(\alpha)+\sin(2\alpha)=2\sin(\alpha)(1+\cos(\alpha))$. Hence $\alpha\in\{0,\pi\}$ if $\alpha=\beta$.

From MATLAB, we also have $\det(X_1,X_2,[X_1,X_2],[X_1[X_1,X_2]],[X_2,[X_2,[X_1,X_2]])=2\cos(\beta)+\cos(\alpha+\beta)+2\cos(\alpha)+\cos(\alpha-\beta)+2$. If this determinant is zero, we cannot have $\alpha=\beta=0$.

The only remaining case is either $\alpha=\pi$ or $\beta=\pi$. Each of these subspaces of M is invariant under the family of controls $\mathcal{F}$. To see why, first suppose $\alpha=\theta_2-\theta_1=\pi$. Then $X_1=\cos(\theta_2)\frac{\partial}{\partial x}+\sin(\theta_2)\frac{\partial}{\partial y}-\sin(\theta_1-\theta_0)\frac{\partial}{\partial\theta_0}$. Hence, the value of α at q is the same as the value at $e^{X_1t}(q)$ for any t. A similar argument holds for X_2, and for the subspace $\beta=\pi$ in place of $\alpha=\pi$. Thus each of the subspaces is invariant under $\mathcal{F}$, hence under any Lie bracket of $\mathcal{F}$. In particular, $\frac{\partial}{\partial\theta_1}\notin Lie_q^n$ for some q such that $\alpha(q)=\pi$ or $\beta(q)=\pi$ and any n, since $\frac{\partial}{\partial\theta_1}\alpha\neq 0$ and $\frac{\partial}{\partial\theta_1}\beta\neq 0$. Thus, $\mathcal{F}$ is not bracket-generating at q.

In summary, the only non-bracket-generating points are those with $\theta_2-\theta_1=\pi$ or $\theta_1-\theta_0=\pi$. Out of the remaining points of M, the points with $\theta_1-\theta_0=\theta_2-\theta_1\neq\pi$ have degree of nonholomony 4. Everything else has degree of nonholonomy 3.

□

3 Let $\widetilde{M}$ denote the set of bracket-generating points of $\mathcal{F}$. Prove that the system is controllable on $\widetilde{M}$.

Proof. By the Rachevskii-Chow theorem, it suffices to show that $\widetilde{M}$ is connected. Let $q^1,q^2\in\widetilde{M}$ with $q^1=(x^1,y^1,\theta_2^1,\theta_1^1,\theta_0^1)$ and $q^2=(x^2,y^2,\theta_2^2,\theta_1^2,\theta_0^2)$. Let $I=[0,1]$. Define $p_1:I\to\widetilde{M}$ by $p_1(t)=q^1+(x^2-x^1,y^2-y^1,0,0,0)t$. Define $p_2:I\to\widetilde{M}$ by $p_2(t)=p_1(1)+(0,0,\theta_1^2-\theta_1^1,\theta_1^2-\theta_1^1,\theta_1^2-\theta_1^1)t$.

Define $p_3:I\to\widetilde{M}$ by holding all coordinates but θ_2 constant and letting the path of θ_2 in S_1 be a path that starts at the $\theta_2(p_2(1))$ and ends at θ_2^2 and does not pass through $\pi+\theta_1^2$. To see that such a path exists, first note that since $p_2(1)\in\widetilde{M}$ with $\theta_1(p_2(1))=\theta_1^2$, we have $\theta_2(p_2(1))-\theta_1^2\neq\pi$. Similarly, by the definition of $\widetilde{M}$, we have $\theta_2^2-\theta_2^1\neq\pi$. Since $S^1\setminus\{\pi+\theta_1^2\}$ is path-connected, there exists such a path p_3.

The same argument works to get a path p_4 from $p_3(1)$ to q^2, holding everything constant except the θ_0 coordinate.

Hence the concatenation of p_1, p_2, p_3, and p_4 is a path in $\widetilde{M}$ from q^1 to q^2. Thus $\widetilde{M}$ is path-connected, hence connected. □