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HW 1

16.10 Prove that $m^*(\bigcup_{n=1}^\infty U_n) = \sum_{n=1}^\infty m^*(U_n)$ for any sequence (U_n) of pairwise disjoint open sets.

Proof. Let $U := \bigcup_{n=1}^\infty U_n$. Note that since each U_n is the disjoint union of countably many open intervals, the general case reduces to the case where each U_n is an open interval. If $\ell(U_n) = \infty$ for some n, then $m(U) = \infty$ by monotonicity. Hence, we may also assume $\ell(U_n) < \infty$ for all n.

By the definition of exterior measure, $m^*(U) \leq \sum_{n=1}^\infty \ell(U_n)$. For the opposite inequality, let $\epsilon > 0$. There exist disjoint compact intervals $K_n \subset U_n$ such that $\ell(U_n) - \ell(K_n) < \epsilon 2^{-n}$. Let $S_i := \bigcup_{n=1}^i K_n$. Since each S_i is compact, each cover of S_i by open intervals admits a finite subcover. Since the K_n are disjoint compact intervals, we can do the usual monkeying to show that $m(S_i) = \sum_{n=1}^i \ell(K_n)$. Thus, $m(U) \geq \lim_{i\to\infty} S_i = \sum_{n=1}^\infty \ell(K_n) = (\sum_{n=1}^\infty \ell(U_n)) - \epsilon$. □

16.27 For each n, let G_n be an open subset of $[0,1]$ containing the rationals in $[0,1]$ with $m^*(G_n) < 1/n$, and let $H = \bigcap_{n=1}^\infty G_n$. Prove that $m^*(H) = 0$ and that $[0,1] \setminus H$ is a first category set in $[0,1]$. Thus, $[0,1]$ is the disjoint union of two "small" sets.

Proof. For any n, $m^*(H) < m^*(G_n) = 1/n$ since $H \subset G_n$. Thus, $m^*(H) = 0$.

Note that $[0,1] \setminus H = \bigcup_{n=0}^\infty [0,1] \setminus G_n$. For each n, $[0,1] \setminus G_n$ is closed, and cannot have interior since it does not intersect the rationals. Thus, $[0,1] \setminus H$ is first category. □

16.28 Fix α with $0 < \alpha < 1$ and repeat our "middle thirds" construction for the Cantor set except that now, at the nth stage, each of the 2^{n-1} open intervals we discard from $[0,1]$ is to have length $(1-\alpha)3^{-n}$. The limit, Δ_α, of this process is called a generalized Cantor set. Check that $m^*(\Delta_\alpha) = \alpha$.

Proof. Let $C_0 = [0,1]$, and C_n denote the $n$the stage of the construction. Then $\Delta_\alpha = \bigcap_n C_n$. Note that each stage removes intervals of total length $2(1-\alpha)(\frac{2}{3})^n$. Thus, since C_n is the disjoint union of compact intervals, $m^*(C_n) = 1 - \sum_{i=1}^n \frac{1-\alpha}{2}(\frac{2}{3})^n = 1 - (\frac{1-\alpha}{3})\frac{1-(\frac{2}{3})^{n+1}}{1/3} = \alpha + (1-\alpha)(\frac{2}{3})^{n+1}$. Thus, since $\Delta_\alpha \subset C_n$ for all n, we have $m^*(\Delta_\alpha) \leq \lim_{n\to\infty} m^*(C_n) = \alpha$.

Let U be a cover of Δ_α by open intervals. Since Δ_α is compact, there exists a finite subcover $F \subset U$ of nonempty intervals. By replacing overlapping intervals with their union and adding the in-between point to every pair of abutting intervals, WLOG each pair of intervals in F is disjoint and nonabutting. Then $[0,1] \setminus F$ is a disjoint finite collection of closed intervals of positive length. Let $B_n := C_n \setminus C_{n-1}$. By construction, each B_n is the union of disjoint, open intervals of length $(1-\alpha)3^{-n}$, and the B_n are themselves disjoint. Thus, since

$[0,1] \setminus \Delta_\alpha = \bigcup_{n=1}^\infty B_n$ and the length of the intervals of $B_n \to 0$ as $n \to \infty$, there exists N such that $[0,1] \setminus F \subset \bigcup_{n=1}^N B_n$. Hence, $C_N \subset F$. This implies $\alpha = \lim_{n\to\infty} C_n \le \Delta_\alpha$, so $m^*(\Delta_\alpha) = \alpha$. □

16.29 Check that $\bigcup_{n=1}^\infty \Delta_{1-1/n}$ has outer measure 1. Use this to give another proof that $[0,1]$ can be written as the disjoint union of a set of first category and a set of zero measure.

Proof. Let $S := \bigcup_{n=1}^\infty \Delta_{1-1/n}$. Then since for all n, $\Delta_{1-1/n} \subset S \subset [0,1]$, we have $m^*(S) = 1$. Since $m^*(S) + m^*([0,1] \setminus S) \le 1$, we have $m^*([0,1] \setminus S) = 0$. To see that S is first category, note that, for any $0 < \alpha < 1$, Δ_α contains no intervals. Hence, it has empty interior. Thus, since each Δ_α is closed, S is the countable union of nowhere dense sets.

□

16.42 Suppose that E is measurable with $m(E) = 1$. Show that:

1. There is a measurable set $F \subset E$ such that $m(F) = 1/2$. (Hint: Consider the function $f(x) = m(E \cap (-\infty, x])$.)
2. There is a closed set F, consisting entirely of irrationals, such that $F \subset E$ and $m(F) = 1/2$.
3. There is a compact set F with empty interior such that $F \subset E$ and $m(F) = 1/2$.

Proof. By the inner regularity of m, there exists compact $K \subset E$ with $m(K) = 0.99$. Let (q_n) be an enumeration of the rationals. Note that $G := K \setminus \bigcup_{i=1}^\infty B_{0.01/2^{-n}}(q_n)$ is compact, and $m(G) \ge 0.98$.

Let $f(x) = m(G \cap (-\infty, x])$. To see that f is continuous, note that, if $xleqy$, $f(y) - f(x) = m(G \cap (-\infty, y]) - m(G \cap (-\infty, x]) = m(G \cap (x, y]) \le y - x$. Since G is bounded, $f(x) = 0$ for all large negative x, and $f(x) = m(G) \ge 0.98$ for all large positive x. Thus, by the intermediate value theorem, there exists x such that $f(x) = 1/2$. Hence, $F := G \cap (-\infty, x]$ satisfies all three requirements.

□

16.48 Let $\mathcal{E}$ be any collection of subsets of $\mathbb{R}$. Show that there is always a smallest σ-algebra $\mathcal{A}$ containing $\mathcal{E}$.

Proof. Let $\{\mathcal{B}_\alpha\}$ be the collection of all σ-algebras containing $\mathcal{E}$, and $\mathcal{A} = \bigcap_\alpha \mathcal{B}_\alpha$. To see that $\mathcal{A}$ is a σ-algebra, let $(S_n) \subset \mathcal{A}$. Since $S_1 \in \mathcal{A}$, it is in every $\mathcal{B}_\alpha$, so $S_1^c \in \bigcap_\alpha \mathcal{B}_\alpha$. Thus, A is closed under complements. Similarly, $\bigcup_{n=1}^\infty S_n \in \bigcap_\alpha \mathcal{B}_\alpha = A$, and $\bigcap_{n=1}^\infty S_n \in \bigcap_\alpha \mathcal{B}_\alpha = A$. □

16.49 The smallest σ-algebra containing $\mathcal{E}$ is called the σ-algebra generated by $\mathcal{E}$ and is denoted $\sigma(\mathcal{E})$. If $\mathcal{E} \subset \mathcal{F}$, prove that $\sigma(\mathcal{E}) \subset \sigma\mathcal{F}$.

Proof. Every σ-algebra containing $\mathcal{F}$ also contains $\mathcal{E}$. Hence, if $\{\mathcal{B}_\alpha\}$ is the collection of all σ-algebras containing $\mathcal{E}$, and $\{\mathcal{C}_\alpha\}$ is the same for $\mathcal{F}$, then $\{\mathcal{C}_\alpha\} \subset \{\mathcal{B}_\alpha\}$. Hence, $\sigma(\mathcal{E}) = \bigcap_\alpha \mathcal{C}_\alpha \subset \bigcap_\alpha \mathcal{B}_\alpha = \sigma(\mathcal{F})$. □

16.53 Show that the Borel σ-algebra $\mathcal{B}$ is generated by each of the following:

1. The open intervals $\mathcal{E}_1 := \{(a,b) : a < b\}$
2. The closed intervals $\mathcal{E}_1 := \{[a,b] : a < b\}$
3. The half-open intervals $\mathcal{E}_1 := \{(a,b], [a,b) : a < b\}$
4. The open rays $\mathcal{E}_1 := \{(a,\infty), (-\infty,a) : a \in \mathbb{R}\}$
5. The closed rays $\mathcal{E}_1 := \{[a,\infty), (-\infty,a] : a \in \mathbb{R}\}$

Proof. Since each of these collections is a subset of the Borel sets, we only need to show that each collection generates the Borel sets. For 1, note that $(a,\infty) = \bigcap_{b=a+1}^{\infty}(a,b)$ and similarly for $(-\infty,a)$. Hence, (1) generates all the open intervals, so all the open sets since every open set is the countable union of open intervals.

For 2, note that $(a,b) = \bigcup_n [a + \frac{b-a}{n+5}, b - \frac{b-a}{n+5}]$. Hence, (2) generates (1). The rest are similar. $\square$

16.25 Suppose that $m^*(E) > 0$. Given $0 < \alpha < 1$, show that there exists an open interval I such that $m^*(E \cap I) > \alpha m^*(I)$. (Hint: It is enough to consider the case that $m^*(E) < \infty$. Now suppose the conclusion fails.)

Proof. If $m^*(E) = \infty$, then $m(E) \leq \sum_{i=0}^{\infty} E \cap (-i,i)$ implies that, for some $i > 0$, $m^*(E \cap (-i,i)) > 0$. Then if we have the finite case proved below, apply it to $E \cap (-i,i)$ to get an interval such that $m^*(E \cap (-i,i) \cap I) > \alpha m^*(I)$. This implies $m^*(E \cap I) > \alpha m^*(I)$.

In the case that $m^*(E) < \infty$, suppose the conclusion fails. That is, for every open interval I, $m^*(E \cap I) \leq \alpha m^*(I)$. Let (I_n) be a cover of E by open intervals such that $\sum_n \ell(I_n) < \alpha^{-1} m^*(E)$. Then $m^*(E) \leq m^*(\bigcup_{n=1}^{\infty} E \cap I_n) \leq \alpha \sum_{n=1}^{\infty} m^*(I_n) < m^*(E)$, a contradiction.

$\square$

16.44 Let E be a measurable set with $m(E) > 0$. Prove that $E - E$ contains an interval centered at 0. (Hint: Take I as in Exercise 25 for $\alpha = 3/4$. If $|x| < m(I)/2$, note that $I \cup (I + x)$ has measure at most $3m(I)/2$. Thus, $E \cap I$ and $(E \cap I) + x$ cannot be disjoint. Finally, $(E + x) \cap E \neq \emptyset$ means that $x \in E - E$; that is, $E - E \subset (-m(I)/2, m(I)/2)$.)

Proof. The hint is the proof. One elaboration: to see that $E \cap I$ and $(E \cap I) + x$ cannot be disjoint, suppose for the sake of contradiction they were disjoint. Then $3/4m(I) + 3/4m(I) < m(E \cap I) + m(E \cap I) < m(E \cap I) + m((E \cap I) + x) = m((E \cap I) \cup ((E \cap I) + x)) = m(E \cap (I \cup (I + x))) \leq m(I \cup I + x) \leq 3/2m(I)$, a contradiction. $\square$

J16.1.2 Suppose f_n and f are Riemann integrable on $[a,b]$ and $f_n \to f$ pointwise on $[a,b]$. Prove that $\int_a^b f_n(t)\,dt \to \int_a^b f(t)\,dt$

Proof. By subtracting f from f_n, we may assume $f_n \to 0$. Since f_n are Riemann integrable, $f_n^+ := f_n \vee 0$ and $f_n^- := -(f_n \wedge 0)$ are also Riemann integrable, and go to 0 pointwise. Since $f_n = f_n^+ - f_n^-$, we only need to prove the conclusion for nonnegative functions. Hence, we may also assume $f \geq 0$.

Assume $\int_a^b f_n(t) \not\to 0$. By passing to a subsequence, we have, for some fixed $\epsilon > 0$, $\int_a^b f_n(t) > \epsilon$ for all n. In particular, for every n there exists a finite partition P_n of $[a, b]$ such that $L(f, P) > \epsilon/2$. Let $Q_m := \bigcup_{n=1}^m P_n$. Then for every n, $\liminf_{m\to\infty} L(f_n, Q_m) > \epsilon/2$.

□

J16.2 Construct ϕ_n in $C[0, 1]$ s.t. $0 \leq \phi_n \leq 1$, $\phi_1 \geq \phi_2 \geq \ldots$, $\phi_n \to \phi$ pointwise on $[0, 1]$, but ϕ is not Riemann integrable on $[0, 1]$. (Hint: The function ϕ can be the characteristic function of a "fat Cantor set" that you construct in 16.28. Why is it not Riemann integrable?)

Proof. Let $\phi = \Delta_\alpha$ for some $0 < \alpha < 1$. To see that ϕ is not Riemann integrable, note that for any partition P, we have $U(\phi, P) - L(\phi, P) = \sum_{I\in P} \omega(f, I)\ell(I) \geq \sum_{I\in P, I\cap\Delta_\alpha\neq\emptyset} \omega(f, I)\ell(I) = \sum_{I\in P, I\cap\Delta_\alpha\neq\emptyset} \ell(I) \geq m * (\Delta_\alpha) = \alpha$.

Let C_n denote the set at the nth stage of the construction of Δ_α. Let ϕ_n be the piecewise linear function defined to be 1 on C_n, 0 on the middle $\frac{n+3}{n+5}$th of each interval of C_n^c, and the line segment connecting the two on each $\frac{1}{n+5}$th end of each such interval. It is easy to check that (ϕ_n) satisfies all the requirements. □

J16.3 If $f \in \mathcal{R}[a, b]$ and $\int_a^b |f| = 0$, then $f = 0$ a.e.

Proof. Suppose $S := \{x : f(x) \neq 0\}$ has $m^*(S) > 0$. Then if $D(f)$ denotes the set of discontinuities of f, we have $m^*(D(f)) = 0$ since f is Riemann integrable. Hence, $m^*(S \setminus D(f)) \geq m^*(S) - m^*(D(f)) > 0$.

In particular, there exists $x_0 \in S \setminus D(f)$. By the continuity of f at x_0, there exists $c > 0$ and $\delta > 0$ such that $|f| > c$ in $B_\delta(x_0)$. This contradicts the assumption that $\int_a^b |f| = 0$. □