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HW 2

3 Let a group G act on a topological space X by homeomorphisms. Show that the projection $\pi : X \to G\backslash X$ is an open map. As a corollary show that if X is locally compact so is $G\backslash X$.

Proof. To see that π is continuous, let $U \subset G\backslash X$ be open. Then $\pi^{-1}(U) = GU \subset X$ is open by the definition of the topology on $G\backslash X$.

To see that π is open, let $U \subset X$ be open. Then $\bigcup_{x\in U} Gx = \bigcup_{g\in G} gU$ is open since the RHS is a union of open sets. Thus $\pi(U)$ is open, so π is an open map.

Now suppose X is locally compact. Let $Gx \in G\backslash X$. Pick a compact neighborhood K_x of x. Then $\pi(K)$ is compact since it is the image of a compact set under a continuous map. Moreover, $\pi(\mathring{K})$ is a neighborhood of Gx within $\pi(K)$, so $\pi(K)$ is a compact neighborhood of Gx. Hence $G\backslash X$ is locally compact. $\square$

4 Let a group G act properly discontinuously on a locally compact, Hausdorff space X by homeomorphisms. Show that, for any two points $x, y \in X$, there exists an open set U_x containing x such that only finitely many elements of the orbit Gy are in U_x. As a corollary show that the orbit space $G\backslash X$ is Hausdorff.

Proof. Let K_x be a compact neighborhood of x. Since the action of G is properly discontinuous, $Gy \cap K_x$ is finite. Let U_x be the interior of K_x. Then $Gy \cap U_x$ is finite.

To show that $G\backslash X$ is Hausdorff, suppose $Gx \neq Gy$. From the above paragraph, pick a compact neighborhood K_x of x such that $Gy \cap K_x$ is finite. Since X is locally compact Hausdorff, we can intersect K_x with compact neighborhoods of x separating x from each point of $Gy \cap K_x$ to get a new compact neighborhood V_x of x such that $Gy \cap V_x = \emptyset$.

Let K_y be any compact neighborhood of y. Since G acts properly discontinuously, $F := \{g \in G : gV_x \cap K_y \neq \emptyset\}$ is finite. Hence $A := FV_x$ is closed. Since $y \notin GV_x$, the open set A^c is a neighborhood of y. Let $V_y = K_y \cap A^c$. Then $V_y \cap GV_x = K_y \cap A^c \cap GV_x \subset A^c \cap FV_x = \emptyset$, so GV_y and GV_x are disjoint neighborhoods of Gx and Gy in $G\backslash X$. $\square$