FINITENESS FOR MAPPING CLASS GROUP REPRESENTATIONS FROM TWISTED DIJKGRAAF-WITTEN THEORY

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ABSTRACT. Any twisted Dijkgraaf-Witten representation of a mapping class group of an orientable, compact surface with boundary has finite image.

1. INTRODUCTION

Given a spherical fusion category $\mathscr{A}$ over a field k and an oriented compact surface M, possibly with boundary, the Turaev-Viro-Barrett-Westbury (TVBW) construction gives a projective representation of the mapping class group $\mathrm{MCG}(M)$ [**?**, **?**]. A natural question is to determine the image of these representations. In particular, when does such a representation have finite image?

It is conjectured that these representations have finite image if and only if $\mathscr{A}$ is weakly integral. This conjecture is a modification of the Property F conjecture [**?**, **?**], which states that braid group representations coming from a braided monoidal category $\mathscr{C}$ should have finite image if and only if $\mathscr{C}$ is weakly integral. Instead of only considering braid group representations, one can consider mapping class groups of arbitrary orientable surfaces. In this case, the input categories to construct the representations must be more specialized than just braided monoidal. One can either apply the Reshitikhin-Turaev construction to a modular tensor category, or apply the TVBW construction to a spherical fusion category. The former is more general than the latter since the Reshitikhin-Turaev construction for the Drinfeld center $Z(\mathscr{A})$ of a spherical fusion category $\mathscr{A}$ yields the same representation as the TVBW construction for $\mathscr{A}$. However, for the case considered in this paper, the simpler TVBW construction suffices.

In this paper, our input category is $\mathscr{A} = \mathrm{Vec}_G^\omega$, the spherical fusion category of G-graded vector spaces with associativity modified by a cocycle $\omega \in Z^3(G, k^\times)$. In this case, the TVBW construction corresponds to the twisted Dijkgraaf-Witten theory of [**?**]. The category Vec_G^ω is integral, so the one expects its associated mapping class group representations to have finite image. The main contribution of this paper is to verify this for arbitrary G and ω.

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2. RELATED WORK

The closest related work is a result of Fjelstad and Fuchs [**?**] showing that, given a surface with at most one boundary component, the mapping class group representations corresponding to the untwisted (i.e. $\omega = 1$) Dijkgraaf-Witten theory have finite image. Their paper uses an algebraic method of Lyubashenko [**?**] that gives a projective mapping class group representation to any factorizable ribbon Hopf algebra, in their case, the double $D(G)$. In our case, we instead consider the mapping class group action on a vector space of Vec_G^ω-colored embedded graphs defined by Kirillov [**?**], yielding a simpler, more geometric proof.

In [**?**], Bantay defined representations of mapping class groups on the Hilbert space of an orbifold model associated to $D^\omega(G)$. These representations appear to coincide with the twisted Dijkgraaf-Witten representations. However, the precise details of the connection are not clear to me.

More is known when we fix a particular surface M. In the case where M is a torus, Ng and Schauenburg showed that any Reshitikhin-Turaev representation of the mapping class group of the torus is finite [**?**]. In the case where M is an n-punctured disk, the mapping class group of M relative to the boundary of the disk is the braid group B_n. In this case, Etingof, Rowell, and Witherspoon proved that the representations associated to $\mathrm{Mod}(D^\omega(G))$ are finite [**?**].

3. DEFINITIONS

Let G be a finite group, and a let $\omega \in Z^3(G, k^\times)$ be a 3-cocycle. Let $V_g \in \mathrm{Obj}(\mathrm{Vec}_G^\omega)$ denote a choice of 1-dimensional vector space in degree g. We will abuse notation by referring to the object V_g by the group element g. For