Paul Gustafson
Texas A&M University - Math 608
Instructor: Grigoris Paouris

HW 4

6.3 If $1 \leq p < r \leq \infty$, $L^p \cap L^r$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$, and if $p < q < r$, the inclusion map $L^p \cap L^r \to L^q$ is continuous.

Proof. The restrictions of $\|\cdot\|_p$ and $\|\cdot\|_r$ to $L^p \cap L^r$ are norms, so their sum is a norm.

To see that $L^p \cap L^r$ is complete, suppose a sequence (f_n) is Cauchy in $L^p \cap L^r$. Then (f_n) is Cauchy in L^p, and hence converges to some $f \in L^p$. Moreover f is the pointwise a.e. limit of f_n, for otherwise there exists $\epsilon > 0$ and a set E of positive measure where $\limsup_n \chi_E |f - f_n| \geq \epsilon$. This implies $\limsup_n \|f - f_n\|_p^p \geq \limsup_n \int_E |f - f_n|^p \geq \mu(E)\epsilon^p$, a contradiction. Similarly, f is the limit of (f_n) in L^q. Thus, $f \in L^p \cap L^q$, and $\|f_n - f\| \leq \|f_n - f\|_p + \|f_n - f\|_q \to 0$.

To see that the inclusion map $L^p \cap L^r \to L^q$ is continuous, let $f \in L^p \cap L^r$ and pick λ as in Prop. 6.10. Then $\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda} \leq \|f\|^\lambda \|f\|^{1-\lambda} = \|f\|$. $\square$

4 If $1 \leq p < r \leq \infty$, $L^p + L^r$ is a Banach space with norm $\|f\| = \inf\{\|g\|_p + \|h\|_r : f = g + h\}$, and if $p < q < r$, the inclusion map $L^q \to L^p + L^r$ is continuous.

Proof. To see that $\|\cdot\|$ is positive definite, we must show that $\|f\| = 0$ implies $f = 0$ a.e. Suppose that $\mu(\{f > 0\}) > 0$. Then there exist a measurable set E and $\delta > 0$ such that $0 < \mu(E) < \infty$ and $f_{|E} \geq \delta$. Suppose $f = g + h$ for $g \in L^p$ and $h \in L^q$. Then

$$\begin{aligned}
\|g\|_p + \|h\|_r &\geq \|g_{|E}\|_p + \|h_{|E}\|_q \\
&\geq \|g_{|E}\|_p + \mu(E)^{1/p-1/q}\|h_{|E}\|_p \\
&\geq \min(\mu(E)^{1/p-1/q}, 1)(\|g_{|E}\|_p + \|h_{|E}\|_p) \\
&\geq \min(\mu(E)^{1/p-1/q}, 1)\|f_{|E}\|_p \\
&\geq \min(\mu(E)^{1/p-1/q}, 1)\delta^{1/p}
\end{aligned}$$

This implies that $\|f\| \geq \min(\mu(E)^{1/p-1/q}, 1)\delta^{1/p} > 0$.

The function $\|\cdot\|$ satisfies the homogeneity condition of a norm. For the triangle inequality, suppose $f_1, f_2 \in L^p + L^r$. Suppose $f_1 = g_1 + h_1$ and $f_2 = g_1 + h_2$ for some $g_1, g_2 \in L^p$ and $h_1, h_2 \in L^r$. Then $\|g_1\|_p + \|h_1\|_q + \|g_2\|_p + \|h_2\|_q \geq \|g_1 + g_2\|_p + \|h_1 + h_2\|_q \geq \|f_1 + f_2\|$. Thus, $\|f_1\| + \|f_2\| \geq \|f_1 + f_2\|$.

To see that $L^p + L^r$ is complete, suppose $f_n \in L^p + L^r$ and $\sum_n f_n$ converges absolutely. Pick $g_n \in L^p$ and $h_n \in L^r$ such that $\|g_n\|_p + \|h_n\|_r \leq \|f\| + 2^{-n}$. Then $\sum_n g_n$ and $\sum_n h_n$ converge absolutely in L^p and L^r respectively. Let

$g = \sum_n g_n$ and $h = \sum_n h_n$. Then $\sum_n f_n = g + h$ pointwise a.e. Moreover,

$$\begin{aligned} \| \sum_{n \geq N} f_n \| &\leq \sum_{n \geq N} \|f_n\| \\ &\leq \sum_{n \geq N} \|g_n\|_p + \|h_n\|_r \\ &\underset{N \to \infty}{\to} 0, \end{aligned}$$

so $\sum_n f_n = g + h$ in $L^p + L^r$. Hence $L^p + L^r$ is complete.

To see that the inclusion $L^q \to L^p + L^r$ is continuous, let $f \in L^q$ with $\|f\|_q = 1$.

Case f is simple. Let $E = \{x : f(x) \leq 1\}$. Then

$$\begin{aligned} \|f\| &\leq \|f\chi_{E^c}\|_p + \|f\chi_E\|_r \\ &= \left(\int_{E^c} |f|^p d\mu\right)^{1/p} + \left(\int_E |f|^r d\mu\right)^{1/r} \\ &\leq \left(\int_{E^c} |f|^q d\mu\right)^{1/p} + \left(\int_E |f|^q d\mu\right)^{1/r} \\ &\leq \left(\int |f|^q d\mu\right)^{1/p} + \left(\int |f|^q d\mu\right)^{1/r} \\ &= 2 \end{aligned}$$

General case. Let $E = \{x : f(x) \leq 1\}$, $g = f\chi_{E^c} \in L^p$, and $h = f\chi_E \in L^r$. Pick simple functions ϕ_n, ψ_n with $\phi_n \to g$ and $\psi_n \to h$ pointwise a.e., $|\phi_n| \leq |g|$, and $|\psi_n| \leq |h|$. Let $\theta_n = \phi_n + \psi_n$. We have

$$\begin{aligned} \|f - \theta_n\| &= \|g + h - \phi_n - \psi_n\| \\ &\leq \|g - \phi_n\|_p + \|h - \psi_n\|_r \\ &\to 0 \end{aligned}$$

Hence $\|f\| \leq \liminf_n \|\theta_n\| + \|\theta_n - f\| \leq 2$. □

5 Suppose $0 < p < q < \infty$. Then $L^p \not\subset L^q$ iff X contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ iff X contains sets of arbitrarily large finite measure. What about the case $q = \infty$? (Hint in book).

Proof. Fix $r \in \mathbb{R}$ with $p < r < q$.

Suppose X contains sets of arbitrarily small positive measure. Pick F_n with $0 < m(F_n) \leq 2^{-n}$. Let $E_n = F_n \setminus \bigcup_{j>n} F_j$. Then $0 < \mu(E_n) \leq 2^{-n}$ and the (E_n) are disjoint. Let $f = \sum_n (\mu(E_n))^{-1/r}\chi_{E_n}$. Then $\|f\|_p^p = \sum_n \mu(E_n)^{-p/r}\mu(E_n) \leq \sum_n \mu(E_n)^{1-p/r} \leq \sum_n 2^{-n(1-p/r)} < \infty$, but $\|f\|_q^q = \sum_n \mu(E_n)^{1-q/r} \geq \sum_n 2^{n(q/r-1)} = \infty$.

Conversely, suppose that there exists $\delta > 0$ such that if $\mu(E) > 0$ then $\mu(E) \geq \delta$. Suppose $f \in L^p$. Since $\int_{f \leq 1} |f|^q \leq \int_{f \leq 1} |f|^p$, WLOG $f \geq 1$. Then the support of f has finite measure.

I claim that there are only finitely many integers $n \geq 1$ such that $E_n := \{x : n \leq |f|^p < n+1\}$ has nonzero measure. Suppose not. Then $\int |f|^p = \sum_n \int_{E_n} |f|^p \geq \sum_{\mu(E_n)\neq 0} \delta n = \infty$, a contradiction.

Thus $\int |f|^q = \sum_n \int_{E_n} |f|^q$ is a finite sum. By definition $\mu(E_n)$ must be finite for all n, and $|f|$ is bounded on each E_n. Hence $f \in L^q$.

Now suppose X contains sets of arbitrarily large finite measure. Then, by disjointifying, it must contain a sequence of disjoint subsets $E_n \in \mathcal{M}$ with $1 \leq \mu(E_n) < \infty$. Let $f = \sum_n (n\mu(E_n))^{-1/r}\chi_{E_n}$. Then $\|f\|_q^q = \sum_n (n\mu(E_n))^{-q/r}\mu(E_n) \leq \sum_n n^{-q/r} < \infty$, but $\|f\|_p^p = \sum_n (n\mu(E_n))^{-p/r}\mu(E_n) \geq \sum_n n^{-p/r} = \infty$. Hence $L^q \not\subset L^p$.

Conversely, suppose there exists $K \geq 0$ such that every set $E \in \mathcal{M}$ of finite measure has $\mu(E) \leq K$. Let $f \in L^q$. Let $E = \{f \leq 1\}$. We have $\|f\|_p^p = \int_E |f|^p d\mu + \int_{E^c} |f|^p d\mu \leq \int_E |f|^p d\mu + \int_{E^c} |f|^q d\mu$.

Thus it suffices to show $\int_E |f|^p d\mu < \infty$. Let $\phi = \sum_{i=1}^n a_i \chi_{E_i}$ be a simple function with $0 \leq \phi \leq \chi_E |f|^p$. Then $\int \phi d\mu \leq \sum_i \mu(E_i) = \mu(\bigcup_i E_i) \leq K$. Hence $\int_E |f|^p d\mu \leq K$.

We now consider the case $q = \infty$. I claim $L_\infty \subset L_p$ iff $\mu(X) < \infty$. Suppose $\mu(X) < \infty$ and $f \in L_\infty$. Then $\|f\|_p^p = \int |f|^p \leq \mu(X)\||f|^p\|_\infty = \mu(X)\|f\|_\infty^p$. Conversely, suppose $\mu(X) = \infty$. Then $\chi_X \in L_\infty \setminus L_p$. $\square$

10 Suppose $1 \leq p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$. (Use Exercise 20 in 2.3)

Proof. Since $|\|f_n\|_p - \|f\|_p| \leq \|f_n - f\|_p$, one implication is clear.

For the converse, suppose $\|f_n\|_p \to \|f\|_p$. We have $|f_n - f|^p \leq (|f_n| + |f|)^p \leq 2^p(|f_n|^p + |f|^p)$. Letting $g_n = 2^p(|f_n|^p + |f|^p)$ and $g = 2^{p+1}|f|^p$, we have $g_n \to g$ a.e. and $\int g_n \to \int g$ since $\|f_n\|_p^p \to \|f\|_p^p$. Thus by the Generalized DCT, we have $\int |f_n - f|^p \to 0$. Hence $\|f_n - f\|_p \to 0$. $\square$

12 If $p \neq 2$, the L^p norm does not arise from an inner product on $L^p(X, \mathcal{M}, \mu)$, except in trivial cases when $\dim(L^p) \leq 1$.

Proof. Since $\dim(L^p) > 1$, there exist $E, F \in \mathcal{M}$ with $0 < \mu(E) < \infty$, $0 < \mu(F) < \infty$, and $E \cap F = \emptyset$. Then $\|\chi_E + \chi_F\|^2 + \|\chi_E - \chi_F\|^2 = 2(\mu(E) + \mu(F))^{2/p}$, whereas $2\|\chi_E\|^2 + 2\|\chi_F\|^2 = 2\mu(E)^{2/p} + 2\mu(F)^{2/p}$.

Let $s = 2/p$ and $\alpha = \mu(E)/(\mu(E) + \mu(F))$. Then we have $\frac{2\|\chi_E\|^2 + 2\|\chi_F\|^2}{\|\chi_E + \chi_F\|^2 + \|\chi_E - \chi_F\|^2} = \alpha^s + (1-\alpha)^s =: f(\alpha)$. We have $f''(\alpha) = s(s-1)(\alpha^s + (1-\alpha)^s)$, which has no roots in $[0,1]$ since $s \neq 0, 1$.

Suppose $f(\alpha_0) = 1$ for some $\alpha_0 \in (0,1)$. Since $f(0) = 1 = f(1)$, the mean value theorem implies there exist $\beta_1 \in (0, \alpha_0)$ and $\beta_2 \in (\alpha_0, 1)$ with $f'(\beta_1) = 0 = f'(\beta_2)$. Applying the MVT again implies that f'' has a root, a contradiction.

Thus $f(\alpha) \neq 1$ for $\alpha \in (0,1)$. Thus, $2\|\chi_E\|^2 + 2\|\chi_F\|^2 \neq \|\chi_E + \chi_F\|^2 + \|\chi_E - \chi_F\|^2$, so the parallelogram law fails. $\square$

13 $L^p(\mathbb{R}^n, m)$ is separable for $1 \leq p < \infty$. However, $L^\infty(\mathbb{R}^n, m)$ is not separable.

Proof. To see that $L^p(\mathbb{R}^n, m)$ is separable for $1 \leq p < \infty$, first recall that the simple functions are dense in L^p. Let $\phi = \sum_{i=1}^{j} a_i \chi_{E_i} \in L^p$ be simple. By the DCT and picking rational sequences converging to each a_i, WLOG $a_i \in \mathbb{Q}$ for all i (the set of such functions ϕ remains dense in L^p).

Since ϕ is bounded and supported on a set of finite measure, it suffices to approximate ϕ in measure. We have

$$m(E_i) = \inf\{\sum_{k=1}^{\infty} m(R_k) : l \in \mathbb{N}, R_k \text{ a rectangle with rational vertices}, E_i \subset \bigcup_k R_k\}.$$

Hence, WLOG $E_i = \bigcup_{k=1}^{l} R_k$ for a rational rectangle. The set of such functions ϕ is countable.

To see that $L^\infty(R^n, m)$ is not separable, for $i = \prod_k i_k \in \{0,1\}^{\mathbb{N}}$ let $E_i = \bigcup_{i_k=1} [i_k, i_k+1] \times \prod_{i=2}^{n} [0,1]$. Then if $i \neq j$, we have $\|\chi_{E_i} - \chi_{E_j}\|_\infty = 1$. Since $|\{0,1\}^{\mathbb{N}}| > \mathbb{N}$ it follows that $L^\infty(\mathbb{R}^n, m)$ is not separable. $\square$