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MATH 663 - Subfactors, Knots, and Planar Algebras (Fall 2017)

HW 1

1 Recall the construction of the hyperfinite II_1-factor $R = \pi_\infty(A_\infty)''$, where $A_\infty = \bigcup_n A_n$, $A_n = \bigotimes_n M_2(\mathbb{C})$, and π_∞ is the GNS representation associated to the tracial state

$$\tau_\infty = \bigotimes_n \mathrm{tr}_{M_2(\mathbb{C})}(x) \qquad (x \in A_n \subset A_\infty)$$

(a) Let $p \in P(R) \setminus \{0, 1\}$. Explain why $pRp \cong R$.

Proof. If $A \subset R$ is a finite-dimensional $*$-subalgebra, then pAp is a finite-dimensional $*$-subalgebra. Moreover, I claim that $pRp = (\bigcup_n pA_np)''$. The $\supset$ inclusion is clear. For the other, suppose $x \in R$. Then there exists a net $(x_i) \subset A_\infty$ with $x_i \to x$ in the WOT. Since $\langle px_ip\xi, \eta\rangle = \langle x_ip\xi, p\eta\rangle \to \langle xp\xi, p\eta\rangle = \langle pxp\xi, \eta\rangle$, we have $px_ip \to pxp$ in the WOT. This implies that $pRp = (\bigcup_n pA_np)''$. Thus, pRp is AFD.

In exercise (3) of this homework we will show that a compression of a II_1 factor is still a II_1 factor. Thus, pRp is a hyperfinite II_1 factor, so by the uniqueness property, $pRp \cong R$. □

(b) Fix $\lambda \in (0, 1)$ and replace the canonical trace state $\mathrm{tr}_{M_2(\mathbb{C})}$ with the state

$$\phi_\lambda((x_{ij})) = \frac{\lambda x_{11} + x_{22}}{1 + \lambda}$$

Repeat the above GNS constrution for A_∞ with τ_∞ replaced by the state

$$\phi_{\lambda,\infty} : A_\infty \to \mathbb{C} \quad \phi_{\lambda,\infty}(x) = \bigotimes_n \phi_\lambda(x) \quad (x \in A_n \subset A_\infty).$$

Let $\pi_{\lambda,\infty}$ denote the corresponding GNS representation and let $R_\lambda := \pi_{\lambda,\infty}(A_\infty)''$. Show that R_λ is AFD and does not admit any faithful normal tracial state (hence R_λ is a type III AFD von Neumann algebra).

Proof. By construction, R_λ is AFD. Pick $2 < \alpha < \frac{\lambda+1}{\lambda}$. Let $(x_n) \subset A_\infty$ be the sequence defined by $x_n = \bigotimes_{i=1}^n \alpha e_{11}$, where $e_{11} \in M_2$ is the matrix unit. Then for $y = \bigotimes_i y^{(i)} \in A_N \subset L^2(A_\infty, \phi_{\lambda,\infty})$, we have $\|\pi_{\lambda,\infty}(x_n)y\|^2 = \phi(y^* x_n x_n y) = \left(\frac{\alpha\lambda}{1+\lambda}\right)^n C_y$ for some constant C_y for all $n \geq N$. Thus $\|\pi_{\lambda,\infty}(x_n)y\| \to 0$ for all $y \in A_\infty \subset L^2(A_\infty, \phi_{\lambda,\infty})$. This implies that $\pi_{\lambda,\infty}(x_n) \to 0$ in the SOT. Thus, if $\tau : R_\lambda \to \mathbb{C}$ is a faithful normal tracial state, then $\tau(x_n) \to 0$. On the other hand, the restriction of τ to each A_N must be the usual trace by the uniqueness of the trace on type I factors. Thus, $\tau(x_n) = \left(\frac{\alpha}{2}\right)^n \to \infty$ since $\alpha > 2$, a contradiction. □

2 Let M be a II_1-factor and let $(H_i)_{i\in\mathbb{N}}$ be M-modules. Prove that

$$\dim_M\left(\bigoplus_{i\in I} H_i\right) = \sum_i \dim_M(H_i)$$

Proof. For each i, let $v_i : H_i \to L^2(M)\otimes \ell^2(\mathbb{N})$ be an isometry such that $v_i x = (x\otimes 1)v_i$ for all $x \in M$. Then $v := \bigoplus_i v_i : \bigoplus_i H_i \to \bigoplus_i L^2(M)\otimes\ell^2(\mathbb{N}) \cong L^2(M)\otimes \ell^2(\mathbb{N})$ is an isometry such that $vx = (x\otimes 1)v$ for all $x \in M$. Thus $\dim_M\left(\bigoplus_{i\in I} H_i\right) = \mathrm{tr}(vv*) = \sum_i \mathrm{tr}(v_i v_i^*) = \sum_i \dim_M(H_i)$. $\square$

3 Let $M \subset B(\mathcal{H})$ be a von Neumann algebra on some Hilbert space $\mathcal{H}$ and let $p \in M$ be a non-zero projection. Prove the following statements:

(a) We have $pMp = (M'p)'$ and $(pMp)' = M'p$ as algebras of operators on the Hilbert space $p\mathcal{H} = \mathrm{ran}(p)$. Thus pMp and $M'p$ are both von Neumann algebras on $p\mathcal{H}$

Proof. To show that $(pM')' = pMp$, first we show that $pMp \subset (pM')'$. Suppose $x \in M$ and $y \in M'$. Then we have $pxp(py) = ppxpy = pypxp$. Thus $pMp \subset (pM')'$. For the other inclusion, suppose that $x \in (pM')'$. Then, for all $y \in M'$, we have $xpy = ypx$. Setting $y = 1$, we have $xp = px$. Substituting into the previous equation, we have $xpy = yxp$. Since $y \in M'$ was arbitrary, this implies that $xp \in M'' = M$. Thus $x = xp = p(xp)p \in pMp$ as operators on pH.

To show that $(pMp)' = M'p$, first we show that $(pMp)' \subset pM'$. Suppose $u \in (pMp)'$ is unitary. Define $\widetilde{u} : MpH \to MpH$ by $\widetilde{u} : \sum_{i=1}^n x_i\xi_i = \sum_{i=1}^n x_i u\xi_i$ for $x_i \in M$ and $\xi_i \in pH$. To see that $\widetilde{u}$ is well-defined, we have

$$\begin{aligned}
\|\widetilde{u}\sum_{i=1}^n x_i\xi_i\|^2 &= \sum_{i,j}\langle x_i u\xi_i, x_j u\xi_j\rangle \\
&= \sum_{i,j}\langle px_j^* x_i pu\xi_i, u\xi_j\rangle \\
&= \sum_{i,j}\langle upx_j^* x_i p\xi_i, u\xi_j\rangle \\
&= \sum_{i,j}\langle px_j^* x_i p\xi_i, \xi_j\rangle \\
&= \sum_{i,j}\langle x_i\xi_i, x_i\xi_j\rangle \\
&= \|x_i\xi_i\|^2.
\end{aligned}$$

Thus, if $\sum_i x_i\xi_i =: \xi = \eta := \sum_j y_j\eta_j$, then $u(\xi-\eta) = 0$. Thus, $\widetilde{u}$ is well-defined. Moreover, it can be extended an isometry on $K = \overline{MpH}$.

Let $q : H \to K$ be the orthogonal projection. It is clear that K is invariant under M and M'. Furthermore, we have if $\xi \in K^\perp$ and $x \in M\cup M'$, we have

$\langle x\xi, \eta\rangle = \langle \xi, x^*\eta\rangle = 0$ forall $\eta \in K$. Thus, $x\xi \in K^\perp$. Thus, both K and $K^\perp$ are invariant under M and M'. Thus, $q \in Z(M) = M \cap M'$. Thus, we have, for $\xi \in pH$, $\widetilde{u}q\xi = qu\xi = u\xi$. Thus, $u = \widetilde{u}q$ on pH. Moreover, if $x \in M$ and $\xi \in pH$, we have $\widetilde{u}qx\xi = qxu\xi = xqu\xi = x(\widetilde{u}q)\xi$, thus $u = \widetilde{u}q \in M'$.

The last inclusion to prove is that $pM' \subset (pMp)'$. But we already know that $pM' \subset (pM')'' = (pMp)'$ from the first part of the problem. $\square$

(b) If M is a factor, then pMp and pM' are both factors on $p\mathcal{H}$. Moreover, the map

$$\Phi : M' \to M'p, \quad x \mapsto xp$$

is a weakly continuous $*$-algebra isomorphism.

Proof. To see that $M'p$ is a factor, suppose $x \in M'p \cap (M'p)'$. Then we can write x as $x = yp$ for some $y \in M'$. Moreover, for all $z \in M'$, we have $yzp = ypzp = zpyp = zyp$. Thus, $yz = zy$ on pH. Since z was arbitrary, we have $y \in M' \cap M'' = M' \cap M = Z(M)$. Thus, $M'p$ is a factor. Since pMp is the commutant of $M'p$, this implies that pMp is also a factor.

To see that Φ is injective, suppose $xp = 0$ for some $x \in M'$. Then $xyp\xi = yxp\xi = 0$ for all $y \in M, \xi \in H$. Thus $xMpH = 0$. Using the same notation from part (a), the projection q onto $K = \overline{MpH}$ is in $Z(M)$ since M is a factor. Since $p \neq 0$, this implies that $q = 1$. Thus MpH is dense in H. Thus, $x = 0$.

The map Φ is linear, and $\Phi(xy) = xyp = xpyp = \Phi(x)\Phi(y)$. Similarly, easy to check the rest. $\square$

(c) If M is a factor and if $x \in M$ and $y \in M'$ are given, then $xy = 0$ implies that $x = 0$ or $y = 0$.

Proof. WLOG $x \neq 0$. Let p be the projection onto the closure of the range of x. We have $p \in M$ by the polar decomposition. Moreover, for $\xi \in H$ we have $yp = 0$ since y is zero on the range of x. Part (b) implies that $y = 0$. $\square$

(d) If M is a factor, then $M \cup M'$ generates $B(H)$ as a von Neumann algebra.

Proof. We have $\mathbb{C}1 \subset (M \cup M')' \subset M' \cap M = \mathbb{C}1$. Thus, $(M \cup M')' = \mathbb{C}1$. Thus, $M \cup M' = (M \cup M')'' = \mathcal{B}(H)$. $\square$

(e) If M is a type II_1 factor, then $pMp \subset B(p\mathcal{H})$ is also a type II_1 factor.

Proof. Let $\tau_{pMp} = \frac{1}{\tau_M(p)}\tau_m$ be the trace for pMp on pH. This is clearly unital normal tracial state. Faithfulness follows from the fact that $\tau_{pMp}((pxp)^*pxp) = 0$ is equivalent to $\tau((pxp)^*(pxp)) = 0$, which is equivalent to $pxp = 0$, for all $x \in M$. Thus, pMp is a finite factor.

Thus, it suffices to show that pMp has no minimal projections. Suppose that $\widetilde{e} \in pMp \subset B(pH)$ is a minimal projection. Let $e = \widetilde{e}p \in M \subset B(H)$. I claim that e is minimal. Suppose that $f \in P(M)$ with $f \leq e$. Then $\mathrm{ran}(f) \subset \mathrm{ran}(e) \subset pH$, so $f = fp \leq \widetilde{e}$ on pH. Since $\widetilde{e}$ is minimal, we have $fp = \widetilde{e} = ep$ or $f = 0$. Thus, $f = e$ or $f = 0$, so e is a minimal projection for the II_1 factor M, a contradiction. □

4 Let H and G be discrete i.c.c. groups, such that H is a subgroup of G. We denote by $[G : H]$ the group theoretic index of H in G, i.e. the number of (left or right) cosets of H in G. Recall that left and right cosets of H in G are of the form $gH = \{gh | h \in H\}$ and $Hg = \{hg | h \in H\}$ for $g \in G$, respectively, and that their number is always the same.

(a) Justify that $\ell^2(G)$ provides an $L(H)$-module and prove that its $L(H)$-dimension is given by

$$\dim_{L(H)}(\ell^2(G)) = [G : H]$$

Proof. Define $\pi : H \to B(\ell^2(G))$ to be the restriction of the left regular representation of $L(G)$ to $L(H)$. This is still a unital normal $*$-homomorphism, so $\ell^2(G)$ is an $L(H)$-module. We have

$$\ell^2(G) \cong \sum_{Hg \in H \backslash G} \ell^2(Hg) \cong \sum_{Hg \in H \backslash G} \ell^2(H),$$

as $L(H)$-modules. Thus, by exercise (2),

$$\dim_{L(H)} \ell^2(G) = [G : H] \dim_{L(H)} \ell^2(H) = [G : H]$$

□

(b) Consider the group factor $L(G)$ and denote by τ its cannonical trace. Show that

$$L^2(L(G), \tau) \text{ and } \ell^2(G)$$

are isomorphic as $L(G)$-modules.

Proof. The left regular representation defines an isometry $\lambda : \mathbb{C}G \to \lambda(\mathbb{C}G) \subset B(\ell^2(G))$. The set $\mathbb{C}G$ is dense in $\ell^2(G)$, and the set $\lambda(\mathbb{C}G)$ is dense in $L^2(L(G), \tau)$. Thus, it defines a unitary equivalence $\ell^2(G)$ to $L^2(L(G), \tau)$. Moreover, $\lambda(x\xi) = x\lambda(\xi)$ for all $x \in L(G)$ and $\xi \in \ell^2(G)$. □

(c) Show that $L(H)$ can be considered as a subfactor of $L(G)$ and deduce for the corresponding Jones index that

$$[L(G) : L(H)] = [G : H]$$

Proof. By part (a), we have $L(H) \subset L(G) \subset B(\ell^2(G))$, a unital inclusion. Thus, $L(H) \subset L(G)$ is a subfactor. Then, parts (a) and (b) together imply the conclusion. $\square$

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(a) Let M be a factor of type I_n. Prove that any subfactor N of M is of type I_m for some integer m dividing n. Moreover, show that all subfactors N of M of type I_m are uniquely determined, up to conjugation by unitaries in M, by the integer $k > 0$ such that pMp is a factor of type I_k for some minimal projection $p \in N$ and $mk = n$.

Proof. Suppose $N \subset M$ is a subfactor. Since N is finite dimensional, it must be of type I_m for some m. Pick a minimal projection $p \in P(N)$. As shown in class, pMp is a factor, obviously of type I_k for some k. Pick a minimal projection $q \in P(pMp)$. Then q is also minimal in M. Thus, $n = \frac{1}{\tau_M(q)} = \frac{1}{\tau_M(p)\tau_{pMp}(q)} = m \cdot k$, where we used the fact that $\tau_{pMp} = \frac{1}{\tau_M(p)}\tau_M$. Thus m divides k.

For the second part, suppose N and N' are of type I_m. We want to find a unitary U such that $N = UN'U^*$. Let (e_{ij}) be matrix units for N and (f_{ij}) be matrix units for N'. Pick a partial isometries u such that $uu^* = e_{11}$ and $u^*u = f_{11}$. Let $U = \sum_i e_{i1}uf_{1i}$. Then

$$\begin{aligned} U^*U &= \left(\sum_i f_{i1}u^*e_{1i}\right)\left(\sum_j e_{j1}uf_{1j}\right) \\ &= \sum_i f_{i1}u^*e_{11}uf_{1i} \\ &= \sum_i f_{i1}u^*uu^*uf_{1i} \\ &= \sum_i f_{i1}f_{11}f_{1i} \\ &= \sum_i f_{ii}, \end{aligned}$$

and similarly $U^*U = 1$.

Furthermore,

$$\begin{aligned} U^*e_{kl}U &= \left(\sum_i f_{i1}u^*e_{1i}\right)e_{kl}\left(\sum_j e_{j1}uf_{1j}\right) && (1) \\ &= f_{k1}u^*e_{11}uf_{1l} && (2) \\ &= f_{k1}u^*uu^*uf_{1l} && (3) \\ &= f_{k1}f_{11}f_{1l} && (4) \\ &= f_{kl}. && (5) \end{aligned}$$

Thus $N' = U^*NU$ for a unitary U. $\square$

(b) Let $N \subseteq M$ be finite dimensional von Neumann algebras. Let $p_1, \dots, p_m$ be the minimal central projections of M and $q_1, \dots, q_n$ those of N. For each $(i,j) \in \{1,\dots,n\} \times \{1,\dots,m\}$, $p_j q_i M q_i p_j$ yields a factor with subfactor $p_j q_i N$, to which we may associate an integer $k_{i,j}$ according to (a). We form the matrix

$$\Lambda = (k_{i,j})_{i=1,\dots,n.j=1,\dots,m}$$

Compute Λ for $M = M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$ and the subalgebra N of matrices of the form

$$\begin{pmatrix} X & 0 & 0 \\ 0 & X & 0 \\ 0 & 0 & z \end{pmatrix} \oplus \begin{pmatrix} X & 0 \\ 0 & z \end{pmatrix} \text{ with } z \in \mathbb{C} \text{ and } X \in M_2(\mathbb{C})$$

Proof. Let p_1 be the projection onto the M_5 component, p_2 the projection onto the M_3 component. Let q_1 be the projection onto the X component, and q_2 the projection onto the z component. Then, $p_1 q_1 M q_1 p_1 \cong M_4$ and $p_1 q_1 N \cong M_2$, so $k_{11} = 2$. Similarly, we get the rest of the entries of Λ:

$$\Lambda = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$\square$

(c) Show that $k_{i,j} = \mathrm{Tr}(p_j e_i)$ holds, if e_i is a minimal projection in the factor $q_i N$. Note that Tr denotes here the unnormalized trace on $p_j M p_j$, which is isomorphic to $M_{m_j}(\mathbb{C})$ for some $m_j \in (\mathbb{N})$

Proof. Since q_i is a minimal central projection of N, we have that $q_i N$ is a factor. Thus, exercise (3)(b) implies that, since $p_j \in (q_i N)'$, we have $q_i N \cong p_j q_i N$. Thus, $p_j e_i$ is a minimal projection of $p_j q_i N$. Thus, if $p_j q_i N$ is of type I_m and $p_j q_i M q_i p_j$ is of type I_n, we have

$$\begin{aligned} k_{ij} &= \frac{n}{m} \\ &= n \cdot \tau_{p_j q_i N}(p_j e_i) \\ &= \mathrm{tr}_{p_j q_i M q_i p_j}(p_j e_i) \\ &= \mathrm{tr}_{p_j M p_j}(p_j e_i) \end{aligned}$$

$\square$