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Take Home Exam, Parts 2.1 and 2.2

TH3 Let $g(x) := x$. Suppose $f \in L_1(0,1)$ with $||f||_1 = 1$. Then $h(t) := \int_0^t |f(x)|\,dx$ is in $AC[0,1]$ with $h(0) = 0$ and $h(1) = 1$. By the IVT, there exists $t_0 \in (0,1)$ with $h(t_0) = 1/2$. Thus, $|\int fg| = |\int_0^1 xf\,dx| \leq \int_0^{t_0} |x||f|\,dx + \int_{t_0}^1 |x||f|\,dx \leq \int_0^{t_0} t_0|f|\,dx + \int_{t_0}^1 |f|\,dx = t_0(1/2) + (1/2) < 1 = ||g||_\infty$.

TH2.3 Let $N_{p,q} = \{x \in E = E_{p,q} : D_E(x) \neq 1\}$. By the Lebesgue Density Theorem, $m(N_{p,q}) = 0$. Let $N = \bigcup_{p,q \in \mathbb{Q}} N_{p,q}$. Then $m(N) = 0$.

Suppose $x \notin N$. Let $a < f(x) < b$. Pick $p, q \in \mathbb{Q}$ such that $a < p < f(x) < q < b$. Then $F := E_{p,q} \subset E_{a,b} =: E$. Hence, $D_E(x) \geq D_F(x) = 1$, so $D_E(x) = 1$.