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HW 5, due 3/21

16.3 Orbits: $\{1, 2, 3, 4\}, \{s_1, s_2, s_3, s_4\}, \{m_1, m_2\}, \{d_1, d_2\}, \{C\}, \{P_1, P_2, P_3, P_4\}$.
16.12 Let $g, h \in G_Y$. If $y \in Y$, then $ghy = gy = y$ and $g^{-1}y = g^{-1}(gy) = y$.
16.13 a. If $x \in \mathbb{R}^2$, then the action of 0 on x leaves x fixed by definition. For the associative property, write $x = (r, \theta)$ in polar form. Then $\alpha(\beta * x) = (r, \theta + \beta + \alpha) = (\alpha + \beta) * x$.

b. The orbit containing P is the circle centered at the origin passing through P.

c. $G_P = (2\pi\mathbb{Z}, +)$, since θ fixes P iff $\theta = 2\pi n$ for some $n \in \mathbb{Z}$.

17.2 G is the direct product of $\langle(1, 3)\rangle$ and $\langle(2, 4, 7)\rangle$ since these cycles are disjoint. Hence, $|G| = (2)(3) = 6$. By Corollary 17.2, the number of orbits is $(1/6)(2 * 3 + 3 * 2 + 3 * 6) = 5$. An easier way would be to just count the orbits directly.

17.4 As mentioned in the section, the group of rotations of the cube has 24 elements. Indeed, if we fix an orientation of the cube with a distinguished face and neighboring face, each rotation can be described by the 6 choices for the first face and then 4 choices for the neighboring face.

Hence, there are $\binom{8}{6}6!/24 = 840$ distinguishable colorings.

5 We can refine the hint by noting that there are 6 quarter-turns that leave opposite faces fixed and 3 half-turns. The 8 vertex rotations are third-turns, and the 6 edge rotations are half-turns. Thus, the Polya enumeration theorem implies that the number of distinguishable colorings is

$$\frac{1}{|G|}\sum_{g\in G} 8^{\lambda(g)} = (1/24)((8)^6 + 6(8)^3 + 3(8)^5 + 8(8)^2 + 6(8)^3) = 15296.$$

7 The group of symmetries is D_4 whose action on the edges is shown in Table 16.10. The cyclic group generated by each quarter turn acts transitively, and each reflection and half turn generates a group with two orbits. Thus, from Polya's enumeration theorem tells us that the number of distinguishable colorings by 6 colors with replacement (part b) is

$$\frac{1}{|G|}\sum_{g\in G} 6^{\lambda(g)} = \frac{1}{8}(6^4 + 2(6)^1 + 5(6)^2) = 186.$$

For part (a), since D_4 acts faithfully on the set of edges, we just get $\binom{6}{4}(4!/|D_8|) = (15)(24/8) = 45$ distinguishable colorings without replacement.

9 The group of rotations of this box is D_4, where each reflection in D_4 corresponds to a rotation around an axis through the 1x1 square cutting the box into two unit cubes. This action is clearly faithful, so for part (a) we get $6!/8 = 90$ distinguishable colorings without replacement. For part (b), we get

$(1/8)(6^6 + 4(6)^5 + 2(6)^3 + (6)^4) = 9936$ distinguishable colorings with replacement.