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HW 2

1 a Show that if α, β are positive with $\alpha + \beta = 1$ then for all $u, v \geq 0$ we have

$$u^\alpha v^\beta \leq \alpha u + \beta v.$$

Proof. If $u = v = 0$, then the inequality holds. Since the inequality is symmetric in u and v, we may assume $v \neq 0$. Hence we wish to show

$$\left(\frac{u}{v}\right)^\alpha \leq \alpha\left(\frac{u}{v}\right) + \beta$$

. Letting $x = \frac{u}{v}$, this is equivalent to showing that $f(x) \geq 0$, where $f(x) = \alpha x - x^\alpha + \beta$ and $x \geq 0$. Since $\alpha > 0$, we have $f'(x) = \alpha - \alpha x^{\alpha-1} = \alpha(1 - x^{\alpha-1})$ whose only zero in $[0, \infty)$ is at $x = 1$. Moreover, since $\alpha < 1$, we have $f''(1) = \alpha(\alpha - 1) < 0$. Hence, the maximum value of f on $[0, \infty)$ occurs at $x = 1$. We have $f(1) = \alpha - 1 + \beta = 0$, so $f(x) \leq 0$ for $x \geq 0$. $\square$

1 b Let $x, y \in \mathbb{R}^n$, and let $p > 1$ and define q by $q^{-1} = 1p^1$. Prove Hlder's inequality,

$$|\sum_j x_j y_j| \|x\|_p \|y\|_q.$$

Hint: Using the inequality in part (a). first prove it for $\|x\|_p = \|y\|_q = 1$. Scale to get the final inequality.

Proof. Suppose $\|x\|_p = \|y\|_q = 1$. Then

$$
\begin{aligned}
|\sum_j x_j y_j| &\leq & \sum_j |x_j||y_j| \\
&= & \sum_j (|x_j|^p)^{1/p}((|y_j|)^q)^{1/q} \\
&\leq & \sum_j \frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q \\
&\leq & \frac{1}{p}(\sum_j |x_j|^p) + \frac{1}{q}(\sum_j |y_j|^q) \\
&= & \frac{1}{p} + \frac{1}{q} \\
&= & 1.
\end{aligned}
$$

For the general case, note that if $x = 0$ or $y = 0$ then the inequality holds. Hence we may assume both are nonzero. Let $x' = \frac{x}{\|}x\|_p$ and $y' = \frac{y}{\|}y\|_p$. We can now apply the special case to x' and y' then clear denominators to get the general inequality. $\square$

1 c Suppose $\phi = (y_1, \ldots, y_n) \in l_p^*$. Hlder's inequality implies that $\|\phi\|_{l_p^*} \leq \|y\|_q$. Show that we actually have $\|\phi\|_{l_p^*} = \|y\|_q$.

Proof. If $\|y\|_q = 0$ then $\phi = 0$, and $\|\phi\|_{l_p^*} = 0 = \|y\|_q$. Hence, we may assume $\|y\|_q \neq 0$. Let $x_i = \frac{|y_i|}{y_i} \frac{|y_i|^{q/p}}{\|y\|_q^{q/p}}$ for $1 \leq i \leq n$. Then $\|x\|_p = \sum_i \frac{|y_i|^q}{\|y\|_q^q} = 1$.

Then $\phi(x) = \sum_i x_i y_i = \sum_i \frac{|y_i|^{q/p}}{\|y\|_q^{q/p}} |y_i| = \frac{1}{\|y\|_q^{q/p}} \sum_i |y_i|^{\frac{p+q}{p}} = \frac{1}{\|y\|_q^{q/p}} \sum |y_i|^q = \|y\|_q^{q-q/p} = \|y\|_q$.

$\square$

1 d Let $x, y \in \mathbb{R}^n$, and let $p > 1$. Prove Minkowski's inequality,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Use this to show that $\|x\|_p$ defines a norm on $\mathbb{R}^n$. Hint: you will need to use Hlder's inequality, along with a trick.

2 L_2 minimization. Find the straight line $y = a + bx$ that minimizes $\int_0^1 (e^x a b x)^2 \, dx$.

Proof. By HW 1, Problem 4, we know that $a + bx$ minimizes $\|e^x - a - bx\|_2$ iff

$$\begin{pmatrix} \langle e^x, 1 \rangle \\ \langle e^x, x \rangle \end{pmatrix} = \begin{pmatrix} \langle 1, 1 \rangle & \langle x, 1 \rangle \\ \langle 1, x \rangle & \langle x, x \rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The one slightly tricky integral is $\langle e^x, x \rangle = \int_0^1 x e^x \, dx = x e^x |_{x=0}^1 - \int_0^1 e^x \, dx = e - (e - 1) = 1$.

$$\begin{pmatrix} e - 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0.8731 \\ 1.6903 \end{pmatrix}$$

$\square$

3 L_1 minimization. Find the straight line $y = a + bx$ that minimizes $\int_0^1 |e^x a b x| \, dx$, by following these steps.

a. Whatever the minimizer is, geometric considerations show that e^x and $a + bx$ will cross at two points, $0 < s < t < 1$. Find these two points by minimizing, over a, b, the area A between $f(x)$ and $a + bx$:

$$A = \int_0^1 |e^x a b x| \, dx = \int_0^s (e^x a b x) \, dx + \int_s^t (a + b x e^x) \, dx + \int_t^1 (e^x a b x) \, dx.$$

b. Use the crossing conditions $a + bs = e^s$ and $a + bt = e^t$ to find a and b.

Proof. a. Let $g_1(a, b, s) = e^s - a - bs$, and $g_2(a, b, t) = e^t - a - bt$. The method of Lagrange multipliers gives us the following necessary condition for (a, b, s, t) to minimize A given the constraints $g_1 = g_2 = 0$:

$$0 = (\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t})(A - \lambda_1 g_1 - \lambda_2 g_2)$$

$$0 = (\int_0^s (-1)\, dx + \int_s^t 1\, dx + \int_t^1 (-1)\, dx - \lambda_1 - \lambda_2,$$
$$\int_0^s (-x)\, dx + \int_s^t x\, dx + \int_t^1 (-x)\, dx - \lambda_1 s - \lambda_2 t,$$
$$2(e^s - a - bs) + \lambda_1(e^s - b), -2(e^t - a - bt) + \lambda(e^t - b))$$

$$0 = (-s + (t - s) + (t - 1) - \lambda_1 - \lambda_2,$$
$$(-1/2)s^2 + (1/2)(t^2 - s^2) + (-1/2)(1 - t^2) - \lambda_1 s - \lambda_2 t,$$
$$\lambda_1(e^s - b), \lambda_2(e^t - b))$$

$$0 = (2t - 2s - 1 - \lambda_1 - \lambda_2, t^2 - s^2 - 1/2 - \lambda_1 s - \lambda_2 t,$$
$$\lambda_1(e^s - b), \lambda_2(e^t - b)) \qquad (1)$$

From the last two components, we get four cases.

Case $e^s = e^t = b$. Since b is the slope of the line between (s, e^s) and (t, e^t), we have $b = \frac{e^t - e^s}{t - s} = 0$ which cannot correspond to a minimum.

Case $e^s = b$ and $\lambda_2 = 0$. From the first component of 1, we have $\lambda_1 = 2t - 2s - 1$. Substituting into the second component of 1, $0 = t^2 - s^2 - 1/2 - \lambda_1 s = t^2 - s^2 - 1/2 - (2t - 2s - 1)s = (t - s)^2 - (1/2 - s)$. Since $t - s > 0$, we have $t = s + \sqrt{1/2 - s}$. Using the case assumption, we have $e^s = b = \frac{e^t - e^s}{t - s} = e^s \frac{e^{\sqrt{1/2 - s}} - 1}{\sqrt{1/2 - s}}$. Thus if $u = \sqrt{1/2 - s}$, then $u = e^u - 1$. The only solution to this equation is $u = 0$. To see this, note that $f(u) := e^u - u - 1$ has derivative $e^u - 1$, hence f has a unique global minimum at 0.

Hence $s = 1/2$, so $t = 1/2$. This cannot correspond to a minimum.

Case $\lambda_1 = 0$ and $e^t = b$. From the first component of 1, we have $\lambda_2 = 2t - 2s - 1$. Substituting into the second component of 1, $0 = t^2 - s^2 - 1/2 - \lambda_2 t = t^2 - s^2 - 1/2 - (2t - 2s - 1)t = -(t - s)^2 + t - 1/2$. Since $t - s > 0$, we have $s = t - \sqrt{t - 1/2}$. Using the case assumption, we have $e^t = b = \frac{e^t - e^s}{t - s} = e^t \frac{1 - e^{-\sqrt{t - 1/2}}}{\sqrt{t - 1/2}}$. Thus if $u = -\sqrt{t - 1/2}$, then $-u = 1 - e^u$. As in the previous case, the only solution to this equation is $u = 0$.
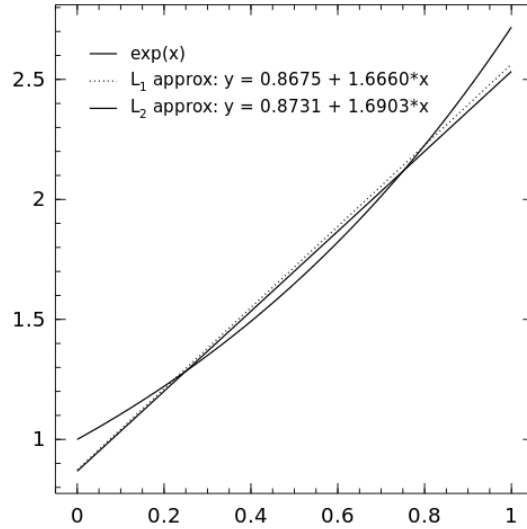
Hence $t = 1/2$, so $s = 1/2$. This cannot correspond to a minimum.

Case $\lambda_1 = \lambda_2 = 0$. We have $t = s + 1/2$, so $0 = (s + 1/2)^2 - s^2 - 1/2 = s - 1/4$. Hence $s = 1/4, t = 3/4$.

b. We have $a + b(1/4) = e^{(1/4)}$ and $a + b(3/4) = e^{(3/4)}$. Hence $a = 0.8675$ and $b = 1.6660$.

3

$\square$

3 Use your favorite software (mine is Matlab) and plot, on the same set of axes, e^x and the two minimization solutions found in the previous two problems.

4 Let V be a finite dimensional inner product space and let U be a subspace of V. Recall that the orthogonal complement of U is

$$U^\perp = \{v \in V | \langle v, u \rangle = 0 \ for \ all \ u \in U\}.$$

Show that $V = U \oplus U^\perp$, where $\oplus$ symbolizes the direct sum of vector spaces. Also, show that $(U^\perp)^\perp = U$.

Proof. By HW 1 (4)(b), the orthogonal projection $P : V \to U$ exists. Let $v \in V$. Then $v = Pv + (v - Pv)$. By HW 1 (3), $v - Pv \in U^\perp$. Hence, $V = U + U^\perp$. Moreover, if $w \in U \cap U^\perp$, then $\langle w, w \rangle = 0$ so $w = 0$. Thus, $v = U \oplus U^\perp$.

To see that $U \subset (U^\perp)^\perp$, let $u \in U$. Then $\langle v, u \rangle = 0$ for all $v \in U^\perp$. Hence, $\langle u, v \rangle = 0$ for all $v \in U^\perp$. Thus, $u \in (U^\perp)^\perp$.

Since $V = W \oplus W^\perp$ for any subspace W, we have $\dim(U) + \dim(U^\perp) = \dim(V) = \dim(U^\perp) + \dim((U^\perp)^\perp)$. Since $\dim(U^\perp) < \infty$, we have $\dim(U) = \dim((U^\perp)^\perp)$. Since $U \subset (U^\perp)^\perp$ and they are finite dimensional, this implies that $U = (U^\perp)^\perp$.

$\square$