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HW 1

1 Show that for every symmetric convex body $K \subset \mathbb{R}^n$, one can define a norm $\|\cdot\|$ whose unit ball is K.

Proof. Define
$$\|x\| = \inf\{c : c > 0, x \in cK\}.$$

Since K contains some small ball B and $-B$, by convexity it must contain a small ball around 0. Thus, $0 \leq \|x\| < \infty$ for all x.

We have $\|0\| = 0$ since $0 \in cK$ for all c. If $x \neq 0$, we have $\|x\| \neq 0$ since K is bounded.

To see that the unit ball is K, let B denote the unit ball with respect to $\|\cdot\|$. Clearly $K \subset B$. Now suppose $x \in B \setminus K$. Note that $cK \subset K$ for $0 \leq c \leq 1$ by convexity since $0 \in K$. Hence, $\|x\| \geq 1$, so $\|x\| = 1$ since $x \in B$. By the definition of $\|\cdot\|$, there exists a sequence $(c_n) \to 1$ with $x \in c_n K$. Then $c_n^{-1} x \to x$ with $c_n^{-1} x \in K$. But K is closed, so $x \in K$, a contradiction.

To see that $\|\cdot\|$ is homogeneous, first note that if $\lambda = 0$, then $\|\lambda x\| = 0 = |\lambda| \|x\|$. If $\lambda \neq 0$, we have $\lambda x \in cK$ iff $|\lambda| x \in cK$ iff $x \in \frac{c}{|\lambda|} K$. Hence $\|\lambda x\| = \inf\{c : c > 0, x \in \frac{c}{|\lambda|} K\} = \inf\{|\lambda| c : c > 0, x \in cK\} = |\lambda| \|x\|$.

For the triangle inequality, it suffices to consider the case where $\|x\| + \|y\| = 1$ since the inequality is homogeneous. Then we have $x + y = \|x\|(x/\|x\|) + \|y\|(y/\|y\|) \in K$ since the RHS is a convex combination of elements of K. Thus $\|x + y\| \leq 1 = \|x\| + \|y\|$. □

2 Let X be a normed space and let $f : X \to \mathbb{R}$ be a nonzero linear functional. Show that the following are equivalent:

(i) f is not bounded

(ii) For every $x \in X$ and for every $r > 0$, $f(B(x, r)) = \mathbb{R}$.

(iii) $\ker(f)$ is a dense subspace of X.

Conclude the following: For every linear functional f either $\ker(f)$ is closed or $\ker(f)$ is dense.

Proof. (i) $\implies$ (ii): Suppose f is not bounded. Then there exists a sequence $(x_n) \subset B_X$ with $|f(x_n)| \to \infty$. Let $u \in \mathbb{R}$. Pick n such that $|u - f(x)|/|f(x_n)| < r$. Then $y := (\frac{u - f(x)}{f(x_n)} x_n + x) \in B(x, r)$, and $f(y) = u$.

(ii) $\implies$ (iii): 0 is in the preimage of every ball.

(iii) $\implies$ (i): Suppose f were bounded with $|f(x)| \leq M\|x\|$ for all $x \in X$. Pick $x_0 \in X$ such that $f(x) \neq 0$. By multiplying x_0 by a scalar, WLOG

$f(x_0) > M$. By (iii), pick $y \in B(x_0, 1)$ such that $f(y) = 0$. Then $f(x_0 - y) > M \geq M\|x_0 - y\|$, a contradiction.

The conclusion follows from the fact that if f is continuous then $f^{-1}(\{0\})$ is closed. □

3 Let X be a normed space. Show that the following are equivalent:

(i) Every linear functional f is bounded.

(ii) Every subspace of X is closed.

(iii) The unit ball of X, B_X is compact

(iv) X has finite dimension

Proof. (i) $\implies$ (ii): Suppose $Y \leq X$ is not closed. Pick $(x_n) \subset Y$ with $x_n \to x$ and $x \notin Y$. Let $V = \operatorname{span}(x)$. Then $X = V \oplus Z$ for some $Z \leq V$. Define a linear functional $\phi : X \to k$ by $\phi(\lambda x + z) = \lambda$ for $\lambda \in k$, $z \in Z$. Then for all n we have $\phi(x_n) = 0 \neq \phi(x)$, so ϕ is not continuous.

(ii) $\implies$ (i): Suppose there exists an unbounded linear functional $f : X \to k$. Then $\ker(f)$ is dense by Exercise 2. Since $\ker(f) \neq X$, it follows that $\ker(f)$ is not closed, a contradiction.

(i) $\implies$ (iii): Pick a basis $(e_i)_{i=1}^n$ for X. Define a norm $\|\cdot\|_2$ to be the l_2 norm with respect to this basis. Since X is finite-dimensional, $\|\cdot\|$ is equivalent to $\|\cdot\|_2$. In particular, $B_{X,\|\cdot\|}$ remains closed and bounded with respect to $\|\cdot\|_2$. Clearly $(X, \|\cdot\|_2)$ is isometrically isomorphic to k^n, so $B_{X,\|\cdot\|}$ is compact.

(iii) $\implies$ (iv): Suppose X is infinite dimensional. By repeatedly applying Riesz's lemma, we can pick a sequence $x_n \subset S_X$ with $\|x_n - x_m\| > 1/2$ for all $n \neq m$ (the space $\operatorname{span}(x_1, \ldots, x_k)$ is closed by $(iv) \implies (i) \implies (ii)$). Thus, B_X is not totally bounded, hence not compact.

(iv) $\implies$ (i): Let $\phi : X \to k$ be a linear functional, and $(e_i)_{i=1}^n$ be a basis for X. If $x = \sum_i a_i e_i$, then $\phi(x) \leq \max_i |\phi(e_i)| \sum_i |a_i| \leq C \max_i |\phi(e_i)| \|x\|$, where the last inequality is from the equivalence of the l_1 norm to $\|\cdot\|$ since X is finite dimensional. □

4 Let $(X, \|\cdot\|)$ be a normed space with $\dim(X) = \infty$. Show that

(i) There exists an unbounded injective linear operator from X onto X.

(ii) There exists a norm $\|\cdot\|_1$ in X such that $\|\cdot\|_1$ is not equivalent to $\|\cdot\|$ but the spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are isometric.

Proof. (i) Let $(x_n)_{n=1}^\infty$ be a Hamel basis for X. Let $T : X \to X$ be the linear transformation such that $x_n \mapsto nx_n$ for all n. The kernel of T is trivial, and T is onto. Since $\|Tx_n\| = n\|x_n\|$, T is unbounded.

(ii) Let (x_n) and T be defined as in (i). Let $y_n = Tx_n$. Let $\|\cdot\|_1$ be the max norm with respect to the basis (x_n). Let $\|\cdot\|_2$ be the max norm with respect to the basis (y_n). These norms are not equivalent because $\|y_n\|_1 = n = n\|y_n\|_2$ for all n. However, T is a linear isometry between $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$. □

5 Let X, Y be normed spaces and $T : X \to Y$ be a linear operator. Show that

(i) If for every sequence $(x_n) \subset X$ with $x_n \to 0$ the sequence $(Tx_n) \subset Y$ is bounded, then T is a bounded operator.

(ii) If for every absolutely convergent series $\sum_n x_n$ we have $\sum_n Tx_n$ converges, then T is bounded.

Proof. (i) Suppose T is not bounded. Then there exists a sequence $(x_n) \subset S_X$ with $\|Tx_n\| \to \infty$. Then $y_n := x_n \|Tx_n\|^{-1/2} \to 0$ but $\|Ty_n\| = \|Tx_n\|^{1/2} \to \infty$, a contradiction.

(ii) Suppose T is not bounded. There exists a sequence $(x_n) \subset S_X$ with $\|Tx_n\| \to \infty$. By passing to a subsequence, WLOG $Tx_n > n^2$ for all n. Let $y_n = x_n / \|Tx_n\|$. Then $\|y_n\| \leq 1/n^2$, so the series $\sum_n y_n$ converges absolutely. However, $\|Ty_n\| = 1$ for all n, so the series $\sum_n Ty_n$ does not converge, a contradiction. $\square$