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1 $\mathbb{F}_2$ and the Fano plane

1.1 Introduction

The purpose of this paper is to answer Exercise 2.5 (p. 96) of Greenberg [1]:

> Let $\mathbb{F}_2$ be the field of two elements $\{0, 1\}$, whose multiplication and addition have the usual tables except that $1 + 1 = 0$. Show that $\mathbb{F}_2^2$ is isomorphic to the smallest affine plane. Show that $P^2(\mathbb{F}_2)$ is isomorphic to the Fano plane.

We will need a few preliminary definitions from Greenberg.

Definition 1. *An **incidence geometry** $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ consists of a set of points $\mathcal{P}$, a set of lines $\mathcal{L}$, and an incidence relation $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$ such that:*

1. *Every pair of distinct points is incident to a unique line.*
2. *Every line is incident to at least two distinct points.*
3. *There exist three distinct noncollinear points.*

Definition 2. *Two lines are **parallel** if there is no point incident to both lines.*

Definition 3. *A **projective plane** is an incidence geometry in which:*

1. *No two lines are parallel.*
2. *Every line is incident to at least three distinct points.*

Definition 4. *An **affine plane** is an incidence geometry in which, for every line l and point P not incident to l, there exists a unique line m incident to P and parallel to l.*

1.2 The affine plane $\mathbb{F}_2^2$

As in $\mathbb{R}^2$, the points in $\mathbb{F}_2^2$ are simply the elements of the vector space $\mathbb{F}_2^2$, i.e. ordered pairs of elements of $\mathbb{F}_2$.

Also analogously to $\mathbb{R}^2$, the lines in $\mathbb{F}_2^2$ are cosets of 1-dimensional subspaces of $\mathbb{F}_2^2$. That is, every line in $\mathbb{F}_2^2$ can be written as $V + h$ for some 1-dimensional subspace $V \subset \mathbb{F}_2^2$ and $h \in \mathbb{F}_2^2$.

Incidence in $\mathbb{F}_2^2$ corresponds to inclusion. For example, the point $(1, 1) \in \mathbb{F}_2^2$ is incident to the line $\{(1, 0)t + (0, 1) : t \in \mathbb{F}_2\}$, since $(1, 1) = (1, 0)(1) + (0, 1)$.

As Greenberg notes, the smallest affine plane, call it $\mathcal{A}$, consists of a set of four points $\{A, B, C, D\}$ and a set of four lines $\{\{A, B\}, \{B, C\}, \{C, D\}, \{D, A\}\}$, where incidence corresponds to inclusion. For example, the point B is incident to the line $\{A, B\}$.

To see that $\mathcal{A}$ and $\mathbb{F}_2^2$ are isomorphic, first note that each 1-dimensional subspace over $\mathbb{F}_2$ has exactly 2 elements, so each line in $\mathbb{F}_2^2$ has 2 elements. Conversely, given two elements $a, b \in \mathbb{F}_2^2$, the line $L((b-a)t, a)$ passes through a and b. Thus, the lines in $\mathbb{F}_2^2$ are precisely the two-element subsets of $\mathbb{F}_2^2$.

Therefore, an arbitrary bijection f from the points of $\mathbb{F}_2^2$ to the points of $\mathcal{A}$ induces a bijection of lines (two-element subsets), and since inclusion is preserved under bijections, incidence is also preserved.

1.3 $P^2(K)$

For an arbitrary field K, the points of the projective space $P^2(K)$ are the 1-dimensional subspaces of K^3. The lines are the 2-dimensional subspaces of K^3. Incidence corresponds to containment.

Projective points in $P^2(K)$ are usually denoted $(a{:}\,b{:}\,c)$ for some generator $(a, b, c) \in K^3 \setminus \{0\}$. Then $(a{:}\,b{:}\,c) = (d{:}\,e{:}\,f)$ iff (a, b, c) is a nonzero multiple of (d, e, f).

To label the projective lines, given an element $a \in K^3 \setminus \{0\}$, consider the rank-1 linear transformation $T(a) : K^3 \to K$ defined by $(T(a))(x) := \sum_{i=1}^{3} a_i x_i$. By the rank-nullity theorem, the nullity of $T(a)$ is 2.

Moreover, if $V \subset K^3$ is a 2-dimensional subspace, then I claim $V = \ker(T(a))$ for some $a \in K^3 \setminus \{0\}$. To see this, pick a basis $\{v, w\}$ for V. Since

$$\begin{aligned}
3 &\geq \dim(\ker(T(v)) + \ker(T(w))) \\
&= \dim(\ker(T(v)) + \dim(\ker(T(w)) - \dim(\ker(T(v)) \cap \ker(T(w))) \\
&= 4 - \dim(\ker(T(v)) \cap \ker(T(w))),
\end{aligned}$$

we have $\dim(\ker(T(v)) \cap \ker(T(w))) \geq 1$.

Hence, we may pick $a \in \ker(T(v)) \cap \ker(T(w)) \setminus \{0\}$. Thus, if $x \in V$, then $x = \alpha v + \beta w$ for some $\alpha, \beta \in K$, so

$$\begin{aligned}
(T(a))(x) &= \sum_{i=1}^{3} a_i(\alpha v_i + \beta w_i) \\
&= \alpha \sum_{i=1}^{3} v_i a_i + \beta \sum_{i=1}^{3} w_i a_i \\
&= \alpha(T(v))(a) + \beta(T(w))(a) \\
&= 0.
\end{aligned}$$

Therefore, $V \subset \ker(T(a))$, so by dimension counting, $V = \ker(T(a))$. Hence, the lines in $P^2(K)$, as 2-dimensional subspaces of K^3, are precisely the elements of the set $\{\ker(T(a)) : a \in K^3 \setminus \{0\}\}$.

Example 1. *The projective line* $\{x + y + z = 0 : (x{:}\,y{:}\,z) \in P^2(\mathbb{R})\}$ *is incident to the point* $(1{:}\,0{:}\,{-1}) \in P^2(\mathbb{R})$ *since* $t + 0 - t = 0$ *for all* $t \in \mathbb{R}$.

1.4 $P^2(\mathbb{F}_2)$ as the Fano plane

A simplification occurs in $P^2(\mathbb{F}_2)$: there is a correspondence between each point in $P^2(\mathbb{F}_2)$, as a 1-dimensional subspace of $\mathbb{F}_2^3$, and its unique nonzero element in $\mathbb{F}_2^3$. Since, on the other hand, each non-zero element in $\mathbb{F}_2^3$ generates a 1-dimensional subspace of F_2^3, this correspondence defines a bijection from $P^2(\mathbb{F}_2)$ to $\mathbb{F}_2^3 \setminus \{0\}$. Hence, there are $2^3 - 1 = 7$ points in $P^2(\mathbb{F}_2)$, and there is only one $(x\colon y\colon z)$-representation for each point.

Since every subspace of $\mathbb{F}_2^3$ contains 0, a 1-dimensional subspace $V \subset \mathbb{F}_2^3$ lies within a 2-dimensional subspace $W \subset \mathbb{F}_2^3$ iff the unique nonzero element in V lies within W. Hence, since each 2-dimensional subspace of $\mathbb{F}_2^3$ contains exactly $2^2 - 1 = 3$ nonzero $\mathbb{F}_2^3$-elements, each line in $P^2(\mathbb{F}_2)$ is incident to precisely 3 projective points.

Since we have seen that no two nonzero elements of an $\mathbb{F}_2$-vector space are linearly dependent, each pair of distinct nonzero elements in $\mathbb{F}_2^3$ determines a 2-dimensional subspace. Hence, since each 2-dimensional subspace of $\mathbb{F}_2^3$ contains exactly $\binom{3}{2} = 3$ distinct pairs of nonzero points, there are $(1/3)\binom{7}{2} = 7$ dimension-2 subspaces of $\mathbb{F}_2^3$, i.e. projective lines in $P^2(\mathbb{F}_2)$.

From the previous section, each projective line can be written as $\ker(T(a))$ for some $a \in \mathbb{F}_2^3 \setminus \{0\}$. Since there are 7 lines, the elements of $(\ker(T(a))_{a\in\mathbb{F}_2^3\setminus\{0\}}$ must be distinct. Thus, Figure 1 defines an explicit isomorphism between $P^2(\mathbb{F}_2)$ and the Fano plane.

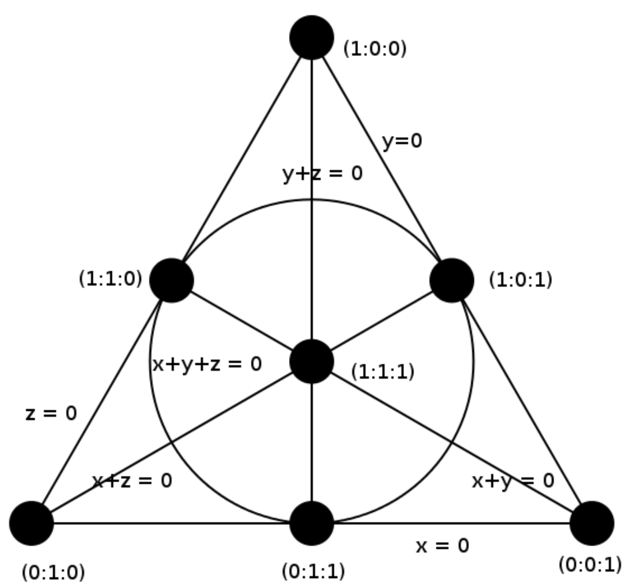

Figure 1: An isomorphism between $P^2(\mathbb{F}_2)$ and the Fano plane

However, this isomorphism is far from unique: the group of automorphisms of the Fano plane has order 168. Indeed, to see that that the order of this group is at most 168, first pick any 3 noncollinear points. Under any automorphism, these 3 points must map to 3 noncollinear points, so there are $(7)(6)(4) = 168$ choices for the images of these three points. However, each pair of these points determines a distinct line, and the sole other point on that line must remain collinear with the pair. Hence, since there are 3 such pairs, the images of 3 more points are fixed. But there are only 7 points, so the last point's image is also determined.

Conversely, consider the action of $GL(3, \mathbb{F}_2)$, the group of nonsingular linear transformations of $\mathbb{F}_2^3$, on $P^2(\mathbb{F}_2)$. This group action is well-defined since the action of $GL(3, \mathbb{F}_2)$ on $\mathbb{F}_2^3$ preserves subspaces and subspace dimension. Moreover, if $g \in GL(3, \mathbb{F}_2)$ fixes $P^2(\mathbb{F}_2)$, then, since every 1-dimensional subspace of $\mathbb{F}_3$ has only one nonzero point, g must fix $\mathbb{F}_2^3$. Thus, $GL(3, \mathbb{F}_2)$ acts faithfully on $P^2(\mathbb{F}_2)$, so is isomorphic to a subgroup of the automorphism group of the Fano plane. Finally, by counting column choices, $|GL(3, \mathbb{F}_2)| = (7)(6)(3) = 168$, so $|GL(3, \mathbb{F}_2)|$ must be the whole automorphism group.

References

[1] Marvin J Greenberg. *Euclidean and non-Euclidean geometries: Development and history.* WH Freeman, 2007.