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HW 3

1 Let H be a Hilbert space and $x_n, x \in H$ such that $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$. Show that $x_n \xrightarrow{\|\cdot\|} x$.

Proof. We have $\|x_n - x\|^2 = \|x_n\|^2 - \langle x_n, x\rangle - \langle x, x_n\rangle + \|x\|^2 \to 2\|x\|^2 - 2\langle x, x\rangle = 0$. □

2 Let X be a vector space equipped with an inner product and (e_n) be an orthonormal sequence in X. If $x, y \in X$ show that $\sum_{k=1}^{\infty} |\langle x, e_k\rangle\langle y, e_k\rangle| \leq \|x\|\|y\|$.

Proof. Since the inner product on X is continuous, the completion of X is a Hilbert space extending the inner product on X. Hence WLOG X is Hilbert. We have

$$\begin{aligned}
\sum_{k=1}^{\infty} |\langle x, e_k\rangle\langle y, e_k\rangle| &= \lim_{N\to\infty} \left\langle \sum_{k=1}^{N} \epsilon_k \langle x, e_k\rangle e_k, \sum_{k=1}^{N} \langle y, e_k\rangle e_k \right\rangle \\
&\leq \lim_N \left\| \sum_{k=1}^{N} \epsilon_k \langle x, e_k\rangle e_k \right\| \left\| \sum_{k=1}^{N} \langle y, e_k\rangle e_k \right\| \\
&= \lim_N \|P_N x\| \|P_N y\| \\
&\leq \|x\|\|y\|
\end{aligned}$$

where $\epsilon_k = \pm 1$ for all k, and P_N is the projection onto $\operatorname{span}\{e_1, \ldots, e_N\}$. □

3 Let (e_n) be the usual basis of ℓ_2. Consider the set

$$A := \{e_m + m e_n : 1 \leq m < n\}.$$

Show that $0 \in \overline{A}^w$, but there is no sequence $a_k \in A$ such that $a_k \xrightarrow{w} 0$.

Proof. To show that $0 \in \overline{A}^w$, it suffices to show that $f^{-1}((-\delta, \delta))$ intersects A for every $f \in \ell_2^*$ and $\delta > 0$. By the Riesz Representation theorem, $f(\cdot) = \langle x, \cdot\rangle$ for some $x \in \ell_2$. We have $x = \sum_n x_n e_n$ for some scalars x_n. Thus, we need to find $m < n$ such that $|f(e_m + m e_n)| = |x_m + m x_n| < \delta$. This is easy since $x_k \to 0$ as $k \to \infty$. Simply pick m such that $|x_m| < \delta/2$, then pick $n > m$ such that $|x_n| < \delta/(2m)$.

For the other part of the problem, suppose there is a sequence $a_k \in A$ with $a_k \xrightarrow{w} 0$. We can write $a_k = e_{m_k} + m_k e_{n_k}$ for some $m_k < n_k$. If (m_k) is bounded, then by passing subsequence WLOG (m_k) is constant with $m_k = m$. Then $\langle a_k, e_m\rangle = 1$ for all k, a contradiction. Similarly, (n_k) cannot be bounded.

Hence we may assume (m_k) and (n_k) are unbounded. By passing to a subsequence WLOG $|m_k| \geq k$ and $n_{k+1} > n_k$ for all k. Then $\sum_k (1/k) e_{n_k} \in \ell_2$, and $|\langle a_k, \sum_k (1/k) e_{n_k} \rangle| = |m_k/k| \geq 1$ for all k, a contradiction. $\square$

4 Let H be a Hilbert space and $(x_n) \subset H$ such that $x_n \xrightarrow{w} 0$. Show that there exists a subsequence (x_{k_n}) such that

$$\left\| \frac{x_{k_1} + \ldots + x_{k_n}}{n} \right\| \to 0.$$

Proof. Let $k_1 = 1$. Given $k_1, \ldots k_{n-1}$, pick $k_n > k_{n-1}$ such that $|\langle x_{k_1} + \ldots + x_{k_{n-1}}, x_{k_n} \rangle| < 1$. Then

$$\begin{aligned} \|x_{k_1} + \ldots + x_{k_n}\|^2 &\leq 2 + \|x_{k_1} + \ldots + x_{k_{n-1}}\|^2 + \|x_{k_n}\|^2 \\ &\leq 4 + \|x_{k_1} + \ldots + x_{k_{n-2}}\|^2 + \|x_{k_{n-1}}\| + \|x_{k_n}\|^2 \\ &\ldots \\ &\leq 2n + \|x_{k_1}\|^2 + \ldots + \|x_{k_n}\|^2 \end{aligned}$$

Thus, it suffices to show that $(\|x_n\|)$ is bounded. Since $x_n \xrightarrow{w} 0$, we have $\sup_n |\langle x_n, y \rangle| < \infty$ for all $y \in H$. Thus by the uniform boundedness principle, $\sup_n \|\langle x_n, \cdot \rangle\|_{H^*} = \sup_n \|x_n\| < \infty$. $\square$

5 Let H be a Hilbert space and (x_n) be an orthogonal sequence in H. Show that $\sum_n x_n$ converges iff $\sum_n \|x_n\|^2$ converges.

Proof. For any $0 \leq M \leq N$ we have $\| \sum_{n=M}^N x_n \|^2 = \sum_{n=M}^N \|x_n\|^2$. Thus the partial sums of $\sum_n x_n$ are Cauchy iff the partial sums of $\sum_n \|x_n\|^2$ are Cauchy. $\square$

6 Let X be a vector space equipped with an inner product and $x_1, \ldots, x_n \in X$. Show that

$$\frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \| \sum_{i=1}^n \epsilon_i x_i \|^2 = \sum_{i=1}^n \|x_i\|^2.$$

Proof. We have

$$
\begin{aligned}
\frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \| \sum_{i=1}^{n} \epsilon_i x_i \|^2 &= \frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \left\langle \sum_{i=1}^{n} \epsilon_i x_i, \sum_{j=1}^{n} \epsilon_j x_j \right\rangle \\
&= \frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \sum_{i,j} \epsilon_i \epsilon_j \langle x_i, x_j \rangle \\
&= \frac{1}{2^n} \sum_{\epsilon_i = \pm 1} \sum_{i \neq j} \epsilon_i \epsilon_j \langle x_i, x_j \rangle + \sum_i \|x_i\|^2 \\
&= \sum_i \|x_i\|^2 + \frac{1}{2^n} \sum_{i \neq j} \sum_{\epsilon_i = \pm 1} \epsilon_i \epsilon_j \langle x_i, x_j \rangle \\
&= \sum_i \|x_i\|^2 + \frac{1}{2^n} \sum_{i \neq j} (1 + 1 - 1 - 1)(2^{n-2}) \langle x_i, x_j \rangle \\
&= \sum_i \|x_i\|^2
\end{aligned}
$$

$\square$