

**Problem Set 6**  
CSCE 440/640

**Due dates:** Electronic submission of the pdf file of this homework is due on **11/2/2016 before 2:50pm** on [ecampus.tamu.edu](http://ecampus.tamu.edu), a signed paper copy of the pdf file is due on **11/2/2016** at the beginning of class.

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**Resources.** I used Mathematica to calculate a couple singular value decompositions.

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

**Signature:** \_\_\_\_\_

**Problem 1.** (20 points) Consider the mixed state

$$M = \left\{ \left( |0\rangle, \frac{1}{3} \right), \left( |1\rangle, \frac{2}{3} \right) \right\}.$$

- (a) Determine the density matrix  $\rho$  of the mixed state  $M$ .
- (b) Derive a different mixed state  $M'$  (which should not consist of computational basis states) that has the same density matrix  $\rho$  as  $M$ .

[This problem shows that density matrices are not in one-to-one correspondence with mixed states.]

**Solution.** (a)

$$\begin{aligned} \rho &= \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1| \\ &= \frac{1}{3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \end{aligned}$$

- (b) First let's guess that there's a solution with real coefficients, so the derivation is slightly simpler. We're trying to find a mixed state

$$M' = \{(a'|0\rangle + b'|1\rangle, p), (c'|0\rangle + d'|1\rangle, q)\}$$

with density matrix  $\rho'$ . Let  $a = \sqrt{p}a'$ ,  $b = \sqrt{p}b'$ ,  $c = \sqrt{q}c'$ , and  $d = \sqrt{q}d'$ . Then the condition  $\rho = \rho'$  becomes

$$\begin{aligned} a^2 + c^2 &= 1/3 \\ b^2 + d^2 &= 2/3 \\ ab + cd &= 0 \end{aligned}$$

The last equation implies that there exists a  $k$  such that  $(a, c) = k(b, -d)$ . Solving for  $k$ ,

$$\begin{aligned} 1/3 &= a^2 + c^2 \\ &= k^2(b^2 + d^2) \\ &= k^2(2/3), \end{aligned}$$

so  $k = \frac{1}{\sqrt{2}}$ . A solution to the first system of equations is  $a = c = \frac{1}{\sqrt{6}}$  and  $b = -d = \frac{1}{\sqrt{3}}$ . This corresponds to the mixed state

$$M' = \left\{ \left( \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle, \frac{1}{2} \right), \left( \frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle, \frac{1}{2} \right) \right\}.$$

**Problem 2.** (20 points)

(a) Do Exercise 3.5.1 (b) on page 55 of our textbook KLM.

(b) Do Exercise 3.5.1 (c) on page 55 of our textbook KLM.

**Solution.** (a)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(b)

$$\frac{1}{2} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

**Problem 3.** (20 points) Find the Schmidt decomposition of the states

(a)  $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ .

(b)  $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$ .

[Students of CSCE 440 only need to solve (a), and students of CSCE 640 should solve both (a) and (b).]

**Solution.** I used Mathematica to find the SVD of the corresponding matrices.

(a)  $(\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle) \otimes (\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle)$

(b)

$$\begin{aligned} & \sqrt{\frac{1}{6}(\sqrt{5}+3)} \left( \frac{\sqrt{5}+3}{2\sqrt{2\sqrt{5}+5}}|0\rangle + \frac{3-\sqrt{5}}{2\sqrt{5}-2\sqrt{5}}|1\rangle \right) \otimes \left( \frac{\sqrt{5}+1}{\sqrt{2}(\sqrt{5}+5)}|0\rangle + \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}}|1\rangle \right) \\ & + \sqrt{\frac{1}{6}(\sqrt{5}-3)} \left( \frac{\sqrt{5}+1}{2\sqrt{2\sqrt{5}+5}}|0\rangle + \frac{1-\sqrt{5}}{2\sqrt{5}-2\sqrt{5}}|1\rangle \right) \otimes \left( \frac{2}{\sqrt{2}(\sqrt{5}+5)}|0\rangle + \sqrt{\frac{1}{10}(\sqrt{5}+5)}|1\rangle \right) \end{aligned}$$

which is approximately

$$\begin{aligned} & 0.934(0.851|0\rangle + 0.526|1\rangle) \otimes (0.526|0\rangle - 0.851|1\rangle) \\ & + 0.357(0.851|0\rangle + 0.526|1\rangle) \otimes (-0.526|0\rangle + 0.851|1\rangle) \end{aligned}$$

**Problem 4.** (20 points) Exercise 3.5.4 (a) on page 57 in our textbook KLM.

**Solution.** For any state of the form  $|a\rangle \otimes |b\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ ,

$$\begin{aligned} \text{tr}_B((U \otimes I)|a\rangle\langle a| \otimes |b\rangle\langle b|(U^\dagger \otimes I)) &= \text{tr}_B(U|a\rangle\langle a|U^\dagger \otimes |b\rangle\langle b|) \\ &= U|a\rangle\langle a|U^\dagger \langle b|b\rangle \\ &= U|a\rangle\langle a|\langle b|b\rangle U^\dagger \\ &= U \text{tr}_B(|a\rangle\langle a| \otimes |b\rangle\langle b|)U^\dagger \end{aligned}$$

Extending linearly, the same identity holds for any state.

**Problem 5.** (20 points) Choi has shown that for all matrices  $V_j \in \mathbf{C}^{n \times m}$ , the map  $T: M_n(\mathbf{C}) \rightarrow M_m(\mathbf{C})$  given by

$$T(\rho) = \sum_{j=1}^{\ell} V_j^* \rho V_j$$

is completely positive. Show that if the matrices  $V_j$  satisfy the condition

$$\sum_{j=1}^{\ell} V_j V_j^* = I,$$

where  $I$  denotes the identity matrix, then  $T$  is trace preserving, so  $\text{tr } T(A) = \text{tr } A$ . [Hint: the matrix trace satisfies  $\text{tr}(ABC) = \text{tr}(CAB)$ .]

**Solution.**

$$\begin{aligned} \text{tr } T(A) &= \text{tr} \left( \sum_{j=1}^{\ell} V_j^* A V_j \right) \\ &= \sum_{j=1}^{\ell} \text{tr}(V_j^* A V_j) \\ &= \sum_{j=1}^{\ell} \text{tr}(V_j V_j^* A) \\ &= \text{tr} \left( \sum_{j=1}^{\ell} V_j V_j^* A \right) \\ &= \text{tr } A \end{aligned}$$

**Checklist:**

- ☐ Did you add your name?
- ☐ Did you disclose all resources that you have used?  
(This includes all people, books, websites, etc. that you have consulted)
- ☐ Did you sign that you followed the Aggie honor code?
- ☐ Did you solve all problems?
- ☐ Did you submit the pdf file resulting from your latex source file on ecampus?
- ☐ Did you submit a hardcopy of the pdf file in class?