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HW 2

1 Using the fact that $\mathcal{B}_{\mathbb{R}}$ is generated by the open intervals, show that:

$$\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\{[a,\infty) : a \text{ rational }\})$$

Proof. It suffices to show both that $\mathcal{B}_{\mathbb{R}}$ contains $[a,\infty)$ for each $a \in \mathbb{Q}$, and that every open interval (x,y) is in $\mathcal{M}(\{[a,\infty) : a \text{ rational }\})$. The former is obvious since $[a,\infty) = (infty, a)^c$ for each $a \in \mathbb{Q}$.

For the latter, suppose (x,y) is an arbitrary open interval. Pick $(x_n), (y_n) \subset \mathbb{Q}$ with $x_n \searrow x$ and $y_n \nearrow y$. Then $(x,y) = \bigcup_n (x_n, y_n)$. $\square$

2 Problem 1/Page 24. A *ring* is a nonempty family of sets closed under finite unions and differences. A ring that is closed under countable unions is called a *σ-ring.*

a. Rings (resp. σ-rings) are closed under finite (resp. countable) intersections.

b. If $\mathcal{R}$ is a ring (resp. σ-ring) , then R is an algebra (resp. σ-algebra) iff $X \in \mathcal{R}$.

c. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a σ-algebra.

d. If $\mathcal{R}$ is a σ-ring, then $\{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a σ-algebra.

Proof. For (a), let $\mathcal{R}$ be a ring, and $U, V \in \mathcal{R}$. Let $W = U \cup V$. Then $U \cap V = W \setminus ((W \setminus U) \cup (W \setminus V))$. This is just one of De Morgan's laws in the restricted universe W. A similar argument works for σ-rings with W the countable union of the sets involved.

For (b), let $\mathcal{R}$ be a ring (resp. σ-ring). Suppose $X \in \mathcal{A}$. Since (a) has been verified, we need only check that R contains complements. This is true since $E^c = X \setminus E$ for any set E. Conversely, suppose $\mathcal{R}$ is an algebra (resp. σ-algebra). Then $\mathcal{R}$ is nonempty, so there exists $E \in \mathcal{R}$. Thus, $X = E \cup E^c \in \mathcal{R}$.

For (c), let $\mathcal{M} = \{E \subset X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$. Since $\mathcal{R}$ is nonempty, so is $\mathcal{M}$. It is also clear that $\mathcal{M}$ is closed under complements. For closure under countable unions, let $(E_n) \subset \mathcal{M}$. Then $(E_n) = (A_n) \cup (B_n)$ for sequences $(A_n), (B_n)$ such that each $A_n \in \mathcal{R}$ and each $B_n^c \in \mathcal{R}$. Let $A = \bigcup A_n \in mathcalR$ and $B = \bigcap B_n^c \in \mathcal{R}$. Then $\bigcup_n E_n = \bigcup_n A_n \cup \bigcup B_n = A \cup B^c = (A^c \cap B)^c = (B \setminus A)^c \in \mathcal{M}$.

For (d), let $\mathcal{M} = \{E \subset X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$. Since $\mathcal{R}$ is nonempty, there exists $E \in \mathcal{R}$. Hence $\emptyset = E \setminus E \in \mathcal{R}$. Then it follows from the definition of $\mathcal{M}$ that $\emptyset \in \mathcal{M}$. In particular, $\mathcal{M}$ is nonempty. To see that $\mathcal{M}$ is closed under complements, suppose $E \in \mathcal{M}$. Let $F \in \mathcal{R}$. Then $E^c \cap F = F \setminus E \in \mathcal{R}$. Hence, $E^c \in \mathcal{M}$. For closure under countable unions, let $(E_n) \subset \mathcal{M}$. Let $F \in mathcalR$. Then $\bigcup_n (E_n) \cap F = \bigcup_n (E_n \cap F) \in \mathcal{R}$. Hence, $\bigcup_n E_n \in \mathcal{M}$. $\square$

3 Problem 5/Page 24. $\mathcal{M}(\mathcal{E})$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$.

Proof. Let

$$\mathcal{H} = \{\mathcal{M}(\mathcal{F}) : \mathcal{F} \subset \mathcal{E} \text{ and } \mathcal{F} \text{ is countable}\},$$

and $\mathcal{U} = \bigcup \mathcal{H}$.

Let $\mathcal{F} \subset \mathcal{E}$ be countable. Then $\mathcal{M}(\mathcal{F}) \subset \mathcal{M}(\mathcal{E}))$. Hence, $\mathcal{U} \subset \mathcal{M}(\mathcal{E}))$. For the reverse inclusion, it suffices to show that $\mathcal{U}$ is a σ-algebra, for then $\mathcal{U}$ is a σ-algebra containing (E), hence containing $\mathcal{M}(\mathcal{E})$.

To see that $\mathcal{U}$ is a σ-algebra, first note that $(\emptyset) \in \mathcal{H}$, so $\mathcal{U}$ is nonempty. To see that $\mathcal{U}$ is closed under taking complements, let $E \in \mathcal{U}$. Then $E \in \mathcal{M}(\mathcal{F})$ for some countable $\mathcal{F} \subset \mathcal{E}$, so $E^c \in \mathcal{M}(\mathcal{F}) \subset \mathcal{U}$.

For closure under countable union, let $(U_n) \subset \mathcal{U}$. Then each $U_n \in \mathcal{M}(\mathcal{F}_n)$ for some countable $\mathcal{F}_n \subset \mathcal{E}$. Let $(F_{nm})_m$ be an enumeration of $\mathcal{F}_n$, and Then $U_n =$

$\square$

4 Show that every σ-algebra has either finite or uncountable many elements.

Proof. Suppose that $\mathcal{M} \subset \mathcal{P}(X)$ is an infinite σ-algebra.

Case 1: suppose that every linearly inclusion-ordered subsets of $\mathcal{M}$ is finite. Let $L_1 \subset \mathcal{M}$ be a maximal chain. It is easy to see that $\operatorname{card} L_1 \geq 2$.

Inductively assume we are given disjoint finite chains $\mathcal{L}_1, \dots \mathcal{L}_n \subset \mathcal{M}$ with $\operatorname{card}(\mathcal{L}_i) \geq 2$ for all i. Let $\mathcal{N} = \mathcal{M} \setminus \bigcup_i \mathcal{L}_i$. Let $E \in \mathcal{M} \setminus$

Let $\mathcal{M}_1 = \mathcal{M} \setminus \{X\}$ and $E_1 = X$. Inductively, assume we have an infinite set $\mathcal{M}_n \subset \mathcal{M}$ and a sequence of disjoint sets $(E_k)_{k=1}^n \subset X$. Pick a chain $\mathcal{L}_n \subset \mathcal{M}_n$ that is maximal among chains in $\mathcal{M}_n$. Let L_n be the maximal element of $\mathcal{L}_n$. Since $\mathcal{M}_n$ is infinite, there exists $E_n \in \mathcal{M}_n \setminus (\mathcal{L}_n \cup \{\emptyset, X\})$.

Since L is maximal in $\mathcal{L}_n$, $E \cup L = X$. Hence, $E^c \subset L^c$. Hence, the set $\mathcal{M}_2 := \{M \in \mathcal{M} \setminus \{\emptyset\} : M \cap L^c = \emptyset\}$ is nonempty. Let $\mathcal{L}_2$ be maximal chain in $\mathcal{M}_2$. The cardinality of $\mathcal{L}_2$ must be finite, for otherwise L_2 could be extended to a maximal linearly ordered subset of $\mathcal{M}$, contradicting the construction of $\mathcal{L}$.

Let L_2 be the maximal element of $\mathcal{L}$. Since $\mathcal{M}$ is infinite, there exists $E_2 \in \mathcal{M} \setminus (\mathcal{L} \cup \mathcal{L}_2 \cup \{\emptyset, X\})$. Since L_2 is maximal in $\mathcal{L}_2$, $E_2 \cap$

For each n, let $F_n = E_n \setminus \left(\bigcup_{k=1}^{n-1} E_k\right)$.

$\square$

5 Let $(\Omega_j, \mathcal{M}_j)$ be measure spaces for $j \in [n]$. Show that

$$\mathcal{E} = \left\{ \prod_{j=1}^{n} E_j : E_j \in \mathcal{M}_j \forall j \right\}$$

is an elementary system.

Proof. Since $\emptyset \in \mathcal{M}_j$ for all j, we have $\emptyset = \prod_{j=1}^n \emptyset \in \mathcal{E}$. Now suppose $E, F \in \mathcal{E}$. Then $E = \prod_j E_j$ and $F = \prod_j F_j$ for $E_j, F_j \in \mathcal{M}_j$ for all j. Hence $E \cap F = \prod_j (E_j \cap F_j) \in \mathcal{E}$. Lastly, we need to check that E^c is the finite union of disjoint elements of $\mathcal{E}$. Let

$$\mathcal{U} = \{\prod U_j : U_j \in \{E_j, E_j^c\}\}$$

. Note that $\mathcal{U}$ is a partition of $\prod_j \Omega_j$, and $\mathcal{U} \subset \mathcal{E}$.. Hence $E^c = \bigcup(\mathcal{U} \setminus E)$ is a finite union of disjoint elements of $\mathcal{E}$. $\square$