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HW 10

1 Problem 11/page 92. Let μ be a positive measure on $(X, \mathcal{M})$. A collection of functions $(f_\alpha)_{\alpha \in A}$ is called *uniformly integrable* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\left|\int_E f_\alpha d\mu\right| < \epsilon$ for all $\alpha \in A$ whenever $\mu(E) < \delta$.

a) Any finite subset of L_1 is uniformly integrable.

b) A sequence (f_n) which is convergent in L_1 is uniformly integrable.

Proof. For (a), it suffices to show that a single function $f \in L_1$ is uniformly integrable. Pick an integrable simple function ϕ with $\|f - \phi\|_1 < \epsilon/2$. Pick $\delta > 0$ such that $\int_E |\phi| d\mu < \epsilon/2$ for all $\mu(E) < \delta$. Then $\left|\int_E f d\mu\right| \leq \int_E |f - \phi| d\mu + \int_E |\phi| d\mu < \epsilon$.

For (b), let $\epsilon > 0$ and let f be the L_1-limit of the sequence (f_n). Pick N such that $\|f_n - f\|_1 \leq \epsilon/2$ for all $n \geq N$. By part (a), pick $\delta > 0$ such that $\int_E |g| d\mu < \epsilon/2$ for all $\mu(E) < \delta$, $g \in \{f\} \cup \{f_n\}_{n<N}$. For $\mu(E) < \delta$ and $n \geq N$, we have $\left|\int_E f_n d\mu\right| \leq \int_E |f_n - f| d\mu + \int_E |f| d\mu \leq \epsilon$. □

2 Problem 13/page 92. Let $X = [0, 1], \mathcal{M} = \mathcal{B}_{\mathbb{R}}$ and μ be the counting measure $[0, 1]$.

a) $m << \mu$ but there is no $f \in L_0^+$ so that $dm = f d\mu$,

b) μ has no Lebesgue decomposition with respect to m.

Proof. For (a), the only null sets of μ are empty, so $m << \mu$. For the other part, suppose $dm = f d\mu$ for some $f \in L_0^+$. Then $0 = \int_{\{x\}} dm = \int_{\{x\}} f d\mu = f(x)$ for all $x \in [0, 1]$, a contradiction.

For (b), suppose $\mu = \lambda + \rho$ with $\lambda \perp m$ and $\rho << m$. Since $\lambda \perp m$, there exists a partition $E \cup F = [0, 1]$ with E being λ-null and F being m-null. Let $x \in [0, 1]$. Since $\rho << m$, $\rho(\{x\}) = 0$. Hence $1 = \mu(\{x\}) = \lambda(\{x\})$. Hence, $E = \emptyset$, since every nonempty subset of $[0, 1]$ has positive λ-measure. Thus $F = [0, 1]$, a contradiction. □

3 Assume that $(\Omega, \mathcal{M}, \mathbb{P})$ is a probability space and that $\tilde{\mathcal{M}} \subset \mathcal{M}$ is a sub-σ-algebra of $\mathcal{M}$. Let X be an integrable random variable. Then there exists a random variable $\tilde{X}$ so that:

a) $\tilde{X}$ is $\tilde{\mathcal{M}}$-measurable.

b) for all $A \in \tilde{\mathcal{M}}$,

$$\mathbb{E}_{\mathbb{P}}(\chi_A X) = \mathbb{E}_{\mathbb{P}}(\chi_A \tilde{X}).$$

Furthermore $\tilde{X}$ is unique, i.e. for every random variable Y which has properties (a) and (b) it follows that $Y = \tilde{X}$ almost surely. (Hint: consider the signed measure $d\nu = Xd\mathbb{P}$ and restrict that measure. Use the Radon Nikodym theorem.)

Proof. Define the signed measure ν on $\mathcal{M}$ by $d\nu = Xd\mathbb{P}$. Let $\tilde{\nu}$ be the restriction of ν to $\tilde{\mathcal{M}}$. Since X is integrable, ν is a finite signed measure. Moreover, $\nu << \tilde{\mathbb{P}}$, where $\tilde{\mathbb{P}}$ is the restriction of $\mathbb{P}$ to $\tilde{\mathcal{M}}$, since $\tilde{\mathbb{P}}(A) = 0 \implies \mathbb{P}(A) = 0 \implies \nu(A) = 0 \implies \tilde{\nu}(A) = 0$. Hence, by the Radon Nikodym theorem, there exists an $\tilde{\mathcal{M}}$-measureable random variable $\tilde{X}$ such that $d\tilde{\nu} = \tilde{X}d\tilde{\mathbb{P}}$. Moreover, if $A \in \tilde{\mathcal{M}}$, then

$$\mathbb{E}_{\mathbb{P}}(\chi_A X) = \nu(A) = \tilde{\nu}(A) = \mathbb{E}_{\mathbb{P}}(\chi_A \tilde{X}).$$

For the uniqueness, suppose $\tilde{Y}$ also satisfies (a) and (b). Suppose the uniqueness fails. WLOG we have that $\mathbb{P}(\{\tilde{X} - \tilde{Y} > 0\}) > 0$. Then there exists n such that $\mathbb{P}(\{\tilde{X} - \tilde{Y} > 1/n\}) > 0$. Let $A = \{\tilde{X} - \tilde{Y} > 1/n\}$. Then $\mathbb{E}_{\mathbb{P}}(\chi_A(\tilde{X} - \tilde{Y})) > 0$, so $\mathbb{E}_{\mathbb{P}}(\chi_A X) = \mathbb{E}_{\mathbb{P}}(\chi_A \tilde{X}) > \mathbb{E}_{\mathbb{P}}(\chi_A \tilde{Y}) = \mathbb{E}_{\mathbb{P}}(\chi_A X)$, a contradiction. ☐

4 Assume that $(\Omega, \mathcal{M}, \mathbb{P})$ is a probability space and that $\tilde{\mathcal{M}} \subset \mathcal{M}$ is a sub-σ-algbra of $\mathcal{M}$. Let X and Y be integrable random variables. Then

a) (Linearity)
$$\mathbb{E}(\alpha X + \beta Y|\tilde{\mathcal{M}}) = \alpha \mathbb{E}(X|\tilde{\mathcal{M}}) + \beta \mathbb{E}(Y|\tilde{\mathcal{M}}).$$

b) (Positivity)
$$X \leq Y \quad \mathbb{P}\text{-almost surely} \implies \mathbb{E}(X|\tilde{\mathcal{M}}) \leq \mathbb{E}(Y|\tilde{\mathcal{M}}) \quad \mathbb{P}\text{-a.s.}$$

c) (Tower-Property) Assume $\mathcal{N} \subset \tilde{\mathcal{M}}$ is a sub-σ-algebra of $\tilde{\mathcal{M}}$. Then
$$\mathbb{E}(\mathbb{E}(X|\tilde{\mathcal{M}})|\mathcal{N}) = \mathbb{E}(X|\mathcal{N}) \quad \mathbb{P}\text{-a.s.}$$

d) (Factorization) If Y is $\tilde{\mathcal{M}}$-measurable and XY is integrable, then
$$\mathbb{E}(Y|\tilde{\mathcal{M}}) = Y \text{ and } \mathbb{E}(YX|\tilde{\mathcal{M}}) = Y\mathbb{E}(X|\tilde{\mathcal{M}}) \quad \mathbb{P}\text{-a.s.}$$

e) (Absolute value)
$$|\mathbb{E}(X|\tilde{\mathcal{M}})| \leq \mathbb{E}(|X||\mathcal{M}) \quad \mathbb{P}\text{-a.s.}$$

Proof. For (a), if $A \in \tilde{\mathcal{M}}$ we have

$$\begin{aligned}
\mathbb{E}(\chi_A(\alpha\mathbb{E}(X|\tilde{\mathcal{M}}) + \beta\mathbb{E}(Y|\tilde{\mathcal{M}}))) &= \alpha\mathbb{E}(\chi_A\mathbb{E}(X|\tilde{\mathcal{M}})) + \beta\mathbb{E}(\chi_A\mathbb{E}(Y|\tilde{\mathcal{M}}))) \\
&= \alpha\mathbb{E}(\chi_A X) + \beta\mathbb{E}(\chi_A Y) \\
&= \mathbb{E}(\chi_A(\alpha X + \beta Y)). \\
&= \mathbb{E}(\chi_A\mathbb{E}(\alpha X + \beta Y|\tilde{\mathcal{M}})).
\end{aligned}$$

Hence, by the uniqueness part of exercise (3), we have the desired equality.

For (b), suppose not. Then there exists $n \in \mathbb{N}$ and $A \in \tilde{\mathcal{M}}$ with $m(A) > 0$ and $\mathbb{E}(X|\tilde{\mathcal{M}})(\omega) - \mathbb{E}(Y|\tilde{\mathcal{M}})(\omega) > 1/n$ for all $\omega \in A$. Thus, $0 < \mathbb{E}(\chi_A(\mathbb{E}(X|\tilde{\mathcal{M}}) - \mathbb{E}(Y|\tilde{\mathcal{M}})) = \mathbb{E}(\chi_A(X - Y)) \leq 0$, a contradiction.

For (c), let $A \in \mathcal{N}$. Then $\mathbb{E}(\chi_A \mathbb{E}(\mathbb{E}(X|\tilde{\mathcal{M}})|\mathcal{N})) = \mathbb{E}(\chi_A \mathbb{E}(X|\tilde{\mathcal{M}})) = \mathbb{E}(\chi_A X) = \mathbb{E}(\chi_A \mathbb{E}(X|\mathcal{N}))$.

For (d), let $A \in \tilde{\mathcal{M}}$. Then $\mathbb{E}(\chi_A \mathbb{E}(Y|\tilde{\mathcal{M}})) = \mathbb{E}(\chi_A Y)$. Thus, by the uniqueness in (3), we have $\mathbb{E}(Y|\tilde{\mathcal{M}}) = Y$.

For the second part of (d), first assume that Y is a characteristic function, then have $\mathbb{E}(\chi_A Y \mathbb{E}(X|\tilde{\mathcal{M}})) = \mathbb{E}(\chi_A Y X) = \mathbb{E}(\chi_A Y \mathbb{E}(X|\tilde{\mathcal{M}}))$ for all $A \in \tilde{\mathcal{M}}$. By part (a), we get the same equality for the case when Y is a simple function. Lastly, when Y is integrable, it is the limit of simple functions Y_n with $|Y_n| \leq |Y|$, so by the DCT for conditional expectations proved in (5), we have $\mathbb{E}(YX|\tilde{\mathcal{M}}) = \lim_n \mathbb{E}(Y_n X|\tilde{\mathcal{M}}) = \lim_n Y_n \mathbb{E}(X|\tilde{\mathcal{M}}) = Y\mathbb{E}(X|\tilde{\mathcal{M}})$.

For (e), we have $|\mathbb{E}(X|\tilde{\mathcal{M}})| = |\mathbb{E}(X^+ - X^-|\tilde{\mathcal{M}})| = \mathbb{E}(X^+|\tilde{\mathcal{M}}) + \mathbb{E}(X^-|\tilde{\mathcal{M}}) = \mathbb{E}(X^+ + X^-|\tilde{\mathcal{M}}) = \mathbb{E}(|X||\tilde{\mathcal{M}})$. □

5 State and prove the Monotone Convergence Theorem and the Dominated Convergence Theorem for conditional expectations.

MCT for conditional expectations: if $(\Omega, \mathcal{M}, \mathbb{P})$ is a probability space, $\tilde{\mathcal{M}}$ is a sub-σ-algebra of $\mathcal{M}$, and (X_n) is a sequence of positive random variables with $X_n \uparrow X$; then $\mathbb{E}(X_n|\tilde{\mathcal{M}}) \to \mathbb{E}(X|\tilde{\mathcal{M}})$.

DCT for conditional expectations: if $(\Omega, \mathcal{M}, \mathbb{P})$ is a probability space, $\tilde{\mathcal{M}}$ is a sub-σ-algebra of $\mathcal{M}$, Y is an integrable random varibale, and (X_n) is a sequence of random variables with $|X_n| \leq Y$ and $X_n \to X$; then $\mathbb{E}(X_n|\tilde{\mathcal{M}}) \to \mathbb{E}(X|\tilde{\mathcal{M}})$.

Proof. For the MCT, let $A \in \tilde{\mathcal{M}}$. Then $\mathbb{E}(\chi_A \mathbb{E}(X_n|\tilde{\mathcal{M}})) = \mathbb{E}(\chi_A X_n) \to \mathbb{E}(\chi_A X) = \mathbb{E}(\chi_A \mathbb{E}(X|\tilde{\mathcal{M}}))$ by the usual MCT. Hence, $\mathbb{E}(X_n|\tilde{\mathcal{M}}) \to \mathbb{E}(X|\tilde{\mathcal{M}})$.

For the DCT, let $A \in \tilde{\mathcal{M}}$. Then $\mathbb{E}(\chi_A \mathbb{E}(X_n|\tilde{\mathcal{M}})) = \mathbb{E}(\chi_A X_n) \to \mathbb{E}(\chi_A X) = \mathbb{E}(\chi_A \mathbb{E}(X|\tilde{\mathcal{M}}))$ by the usual DCT since $\chi_A |X_n| \leq |Y|$. Hence, $\mathbb{E}(X_n|\tilde{\mathcal{M}}) \to \mathbb{E}(X|\tilde{\mathcal{M}})$. □

6 Assume that $(\Omega, \mathcal{M}, \mathbb{P})$ is a probability space and that $\tilde{\mathcal{M}} \subset \mathcal{M}$ is a sub-σ-algebra of $\mathcal{M}$ generated by $A_1, A_2, \ldots, A_n \in \mathcal{M}$, a partition of Ω. Assume that X is an integrable random variable on $(\Omega, \mathcal{M}, \mathbb{P})$. Compute $\mathbb{E}(X|\tilde{\mathcal{M}})$.

Proof. Let $Y = \sum_{i=1}^n \frac{\mathbb{E}(\chi_{A_i} X)}{\mathbb{E}(\chi_{A_i})} \chi_{A_i}$. Then Y is clearly $\tilde{\mathcal{M}}$ measurable. Moreover, we have $\mathbb{E}(\chi_{A_i} Y) = \mathbb{E}(\frac{\mathbb{E}(\chi_{A_i} X)}{\mathbb{E}(\chi_{A_i})} \chi_{A_i}) = \mathbb{E}(\chi_{A_i} X)$.

Note that $\mathcal{M} = \{\bigcup_{j \in J} A_j : J \subset [n]\}$. If $A = \bigcup_{j \in J} A_j$, then $\mathbb{E}(\chi_A Y) = \sum_{j \in J} \mathbb{E}(\chi_{A_j} Y) = \sum_{j \in J} \mathbb{E}(\chi_{A_j} X) = \mathbb{E}(\chi_A X)$. Thus, by the uniqueness part of (3), we have $\mathbb{E}(X|\tilde{\mathcal{M}}) = Y$. □