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HW 4

4 To Show: If $\angle BAC$ and $\angle B'A'C'$ are right angles and $AB \cong A'B'$ and $BC \cong B'C'$, then $\triangle ABC \cong \triangle A'B'C'$.

Proof. Following the hint, construct D on the ray opposite to $\overrightarrow{AC}$ such that $AD \cong A'C'$. Then by SAS, $\triangle DAB \cong \triangle C'A'B'$. Thus, $BD \cong BC$, so $\triangle DBC$ is isosceles with $\angle D \cong \angle C$. Hence, by SAA, $\triangle ABC \cong \triangle ABD \cong A'B'C'$. $\square$

30 To Show: If $\square ABCD$ is a convex quadrilateral and l is a line intersecting AB between A and B, then exactly one of the following holds:

1. There exists a point O such that $B * O * C$ and O is incident to l.
2. There exists a point O such that $C * O * D$ and O is incident to l.
3. There exists a point O such that $A * O * D$ and O is incident to l.
4. $O := C$ is incident to l.
5. $O := D$ is incident to l.

Proof. To see that at least one of these must hold, apply Pasch's theorem to $\triangle ABC$ to get that exactly one of (1),(4), and l intersects AC between A and C holds. In the last case, apply Pasch's theorem to $\triangle ACD$ to get that exactly one of (2), (3), and (5) holds.

From the preceding argument, it suffices to show that each of (2), (3), and (5) implies l intersects AC between A and C.

Let M denote the intersection of l and AB. Since $A * M * B$, M and B are on the same side of $\overleftrightarrow{AC}$. Note that by Exercise 28, B and D are on opposite sides of $\overleftrightarrow{AC}$. Thus, M and D are on opposite sides of $\overleftrightarrow{AC}$.

Note that in each of the cases (2), (3), and (5), O is on the same side of $\overleftrightarrow{AC}$ as D. Thus, M and O are on opposite sides of $\overleftrightarrow{AC}$.

Let I denote the point of intersection of MO and $\overleftrightarrow{AC}$. It suffices to show that I lies in the interior of $\square ABCD$, for then we have $A * I * C$.

To see that I lies in the interior, first note that $M * I * O$ implies that I and O lie on the same side of $\overleftrightarrow{AB}$. Since $\square ABCD$ is convex, O and D lie on the same side of $\overleftrightarrow{AB}$ in all cases. Thus, I, C, and D lie on the same side of $\overleftrightarrow{AB}$.

On the other hand, $M * I * O$ also implies M and I are on the same side of $\overleftrightarrow{CD}$. Thus, I,A, and B are on the same side of $\overleftrightarrow{CD}$. Since M,O,A, and D are on the same side of $\overleftrightarrow{BC}$, I is on the same side of $\overleftrightarrow{BC}$ as AD. Similarly, I is on the same side of $\overleftrightarrow{AD}$ as BC.

Thus, I lies in the interior of $\square ABCD$.

$\square$

32 Using Figure 4.33, note that $\angle A'B'B''$ is supplementary to $\angle A'B'B$. Moreover, $\angle ABB''$ is supplementary to $\angle B'BC$. Thus, since two angles are congruent iff their supplementary angles are congruent, $\angle A'B'B'' \cong \angle ABB''$ iff $\angle A'B'B \cong \angle B'BC$, one of the pairs of alternate interior angles. We get a similar equivalence between the other pair of corresponding angles and alternate interior angles.