Paul Gustafson
Representations of Lie Algebras

## HW 1

**1** Prove that over an algebraically closed field of characteristic not equal 2, if $B$ is a nondegenerate symmetric bilinear form over an f.d. vector space $V$, then $\mathfrak{so}(V, B) \simeq \mathfrak{so}(V)$.

*Proof.* I claim that there exists a vector $v \in V$ such that $B(v, v) \neq 0$. Suppose not. By nondegeneracy, there exist vectors $x, y$ with $B(x, y) \neq 0$. By assumption, $0 = B(x+y, x+y) = B(x, x) + 2B(x, y) + B(y, y) = 2B(x, y)$, a contradiction.

Thus, we may choose a vector $v$ such that $B(v, v) \neq 0$. By dividing if necessary, WLOG $B(v, v) = 1$. Let $v^\perp = \{w \in V : B(v, w) = 0\}$.

I claim that $v^\perp$ has codimension 1. Suppose $x \in V$. Let $p = B(x, v)v$. It suffices to show that $x - p \in v^\perp$. We have $B(x-p, v) = B(x, v) - B(p, v) = B(x, v) - B(x, v)B(v, v) = 0$. Thus $V = v^\perp + \lambda v$. Since $B(v, v) \neq 0$, $v^\perp \cap \lambda v = \{0\}$. Thus $V = v^\perp \oplus \lambda v$.

By induction, we get an orthonormal basis $v_1, \ldots, v_n$ for $V$ with respect to $B$. $\square$

**2** Show that $(\mathbb{R}^3, \times)$ is a real Lie algebra. Is this related to $\mathfrak{su}(2)$?

*Proof.* The cross-product is antisymmetric and bilinear by definition. To check that the Jacobi identity holds, it suffices to check it on basis vectors $i := e_1, j := e_2, k := e_3$. Moreover, the Jacobi identity is invariant under cyclic permutations of its arguments. Hence, it suffices to check the following cases, where $v, w$ are arbitrary:

$$i \times (j \times k) + k \times (i \times j) + j \times (k \times i) = i \times i + k \times k + j \times j = 0$$
$$j \times (i \times k) + k \times (j \times i) + i \times (k \times j) = -j \times j - k \times k - i \times i = 0$$
$$v \times (w \times w) + w \times (v \times w) + w \times (w \times v) = 0 + w \times (v \times w) - w \times (v \times w) = 0$$

For the comparison to $\mathfrak{su}(2)$, first we have the commutation relations

$$[i\sigma_x, i\sigma_y] = -2i\sigma_z$$
$$[i\sigma_y, i\sigma_z] = -2i\sigma_x$$
$$[i\sigma_z, i\sigma_x] = -2i\sigma_y$$

Thus, by swapping the cyclic orders, we get an isomorphism $(\mathbb{R}^3, \times) \simeq \mathfrak{su}(2)$. For example $i \mapsto \frac{i}{2}\sigma_x$, $j \mapsto \frac{i}{2}\sigma_z$, and $k \mapsto \frac{i}{2}\sigma_y$. $\square$

**3** Consider $\mathbb{H}$.

*Proof.* The quaternions are not a Lie algebra. For example, the Jacobi identity fails since $1ij + j1i + ij1 = k - k + k \neq 0$. However if we mod out the subspace spanned by 1, we have $\mathbb{H}/\mathbb{R} \simeq (\mathbb{R}^3, \times) \simeq \mathfrak{su}(2)$ as real Lie algebras. $\square$

**4** Check that the radical $\mathrm{rad}(K)$ of the Killing form of $\mathfrak{g}$ satisfies the conditions of Cartan's criterion for solvability.

*Proof.* Let $v \in \mathrm{rad}(K)$ and $w \in \mathrm{rad}(K)'$. By a lemma from class, the Killing form $K_{\mathrm{rad}(K)}$ of the Lie algebra $\mathrm{rad}(K)$ is the restriction of the Killing form $K$ of $\mathfrak{g}$ to $\mathrm{rad}(K) \times \mathrm{rad}(K)$. Thus $K_{\mathrm{rad}(K)}(v, w) = K(v, w) = 0$, by the definition of $\mathrm{rad}(K)$. $\square$