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HW 9

1 If $f \in L_1(0, \infty)$, define

$$g(s) = \int_0^\infty e^{-st} f(t)\, dt, \quad 0 < s < \infty.$$

Prove that $g(s)$ is differentiable on $(0, \infty)$ and that

$$g'(s) = -\int_0^\infty t e^{-st} f(t)\, dx, \quad 0 < s < \infty.$$

Proof. Let $s, t \in (0, \infty)$ and $0 < |h| \le s/2$. We have $|e^{-st} f(t)| \le |f(t)|$, so $e^{-st} f(t) \in L_1$. Hence

$$\frac{g(s+h) - g(s)}{h} = \int_0^\infty \frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\, dt.$$

By the Mean Value theorem, we have

$$\begin{aligned} \left| \frac{e^{-(s+h)t} - e^{-st}}{h} \right| &\le \sup_{h \in (-s/2, s/2)} |-te^{-(s+h)t}| \\ &= te^{-(s/2)t} \\ &\le C_s \end{aligned}$$

Hence, $\left| \frac{e^{-(s+h)t} - e^{-st}}{h} f(x) \right| \le C_s |f(x)| \in L_1(0, \infty)$. Thus, by the DCT

$$g'(s) = \int_0^\infty \frac{d}{ds} e^{-st} f(t) = -\int_0^\infty t e^{-st} f(t)\, dx.$$

□

2 Let (Ω, μ, Σ) be a finite measure space and (f_n) be a sequence of measurable functions on Ω. Suppose that for each $\omega \in \Omega$ there is an $M_\omega \in \mathbb{R}$ so that for all $k \in \mathbb{N}$, $|f_k(\omega)| \le M_\omega$. Let $\epsilon > 0$. Show that there is a measurable $A \subset \Omega$ and an $M \in \mathbb{R}$ so that $\mu(\Omega \setminus A) < \epsilon$ and $f_k(\omega) < M$ for all $k \in \mathbb{N}$ and all $\omega \in A$.

Proof. Let $\epsilon > 0$ and $E_j := \bigcap_n \{f_n < j\}$. Then (E_j) is increasing and $\bigcup_j E_j = \Omega$. Hence $\lim_j \mu(E_j) = \mu(\Omega)$. Since $\mu(\Omega) < \infty$, we can pick M such that $\mu(\Omega \setminus E_M) = \mu(\Omega) - \mu(E_M) < \epsilon$. Moreover, if $\omega \in E_M$, then $f_k(\omega) < M$ for all k. □

3 57/page 77. Show that $\int_0^\infty e^{-sx} x^{-1} \sin x\, dx = \arctan(s^{-1})$ for $s > 0$ by integrating $e^{-sxy} \sin x$ with respect to x and y. (Hints: $\tan(\frac{\pi}{2} - \theta) = \cot \theta$ and Exercise 31d.)

Proof. We have

$$\begin{aligned}
\int_0^\infty \int_1^\infty \left|e^{-sxy}\sin x\right|\, dydx &\le \int_0^{\pi/2}\int_1^\infty e^{-sxy}\sin x\, dydx + \int_{\pi/2}^\infty \int_1^\infty e^{-sxy}\, dydx \\
&= \int_0^{\pi/2} \frac{\sin x}{sx} e^{-sxy} dx + C_s \\
&\le \int_0^{\pi/2} \frac{1}{s} e^{-sxy} dx + C_s \\
&< \infty,
\end{aligned}$$

so $f(x,y) = e^{-sxy}\sin x$ is in $L_1((0,\infty)\times(1,\infty))$.

Thus, we have

$$\begin{aligned}
\int_0^\infty e^{-sx}x^{-1}\sin x\, dx &= s\int_0^\infty \int_1^\infty e^{-sxy}\sin x\, dydx \\
&= s\int_1^\infty \int_0^\infty e^{-sxy}\sin x\, dxdy \\
&= s\int_1^\infty \int_0^\infty e^{-sxy}\sin x\, dxdy \\
&= \frac{s}{2i}\int_1^\infty \int_0^\infty e^{(i-sy)x} - e^{(-i-sy)x}\, dxdy \\
&= \frac{s}{2i}\int_1^\infty \left[\frac{1}{i-sy}e^{(i-sy)x} + \frac{1}{i+sy}e^{(-i-sy)x}\right]_{x=0}^\infty dy \\
&= -\frac{s}{2i}\int_1^\infty \frac{1}{i-sy} + \frac{1}{i+sy}\, dy \\
&= \int_1^\infty \frac{s}{1+s^2y^2}\, dy \\
&= \int_s^\infty \frac{1}{1+u^2}\, du \\
&= \frac{\pi}{2} - \arctan(s) \\
&= \arctan\cot\arctan(s) \\
&= \arctan(s^{-1})
\end{aligned}$$

□

4 60/page 77. $\Gamma(x)\Gamma(y)/\Gamma(x+y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$ for $x, y > 0$. (Recall that Γ was defined in Section 2.3. Write $\Gamma(x)\Gamma(y)$ as a double integral and use the argument of the exponential as a new variable of integration.)

Proof. We have $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$ for $\Re z > 0$. Thus

$$\begin{aligned}
\Gamma(x)\Gamma(y) &= \left(\int_0^\infty s^{x-1}e^{-s}\,ds\right)\left(\int_0^\infty t^{y-1}e^{-t}\,dt\right)\\
&= \int_0^\infty \int_0^\infty s^{x-1}t^{y-1}e^{-s-t}\,dsdt\\
&= \int_0^\infty \int_s^\infty s^{x-1}(u-s)^{y-1}e^{-u}\,duds\\
&= \int_0^\infty \int_0^u s^{x-1}(u-s)^{y-1}e^{-u}\,dsdu\\
&= \int_0^\infty \int_0^1 (uv)^{x-1}(u-uv)^{y-1}e^{-u}u\,dvdu\\
&= \left(\int_0^\infty u^{x+y-1}e^{-u}\,du\right)\left(\int_0^1 v^{x-1}(1-v)^{y-1}\,dv\right)\\
&= \Gamma(x+y)\int_0^1 v^{x-1}(1-v)^{y-1}\,dv,
\end{aligned}$$

where the intechange of integration is justified by the fact that the integrand is positive and the double integral is finite. □

5 Given a bounded function $f : [a,b] \to \mathbb{R}$, define

$$H(x) = \lim_{\delta\to 0} \sup_{|x-y|\le\delta} f(y), \text{ and } h(x) = \lim_{\delta\to 0} \inf_{|x-y|\le\delta} f(y)$$

a) For $x \in [a,b]$, f continuous at $x \iff H(x) = h(x)$.

b) Assume now that (P_k) is an increasing sequence of partitions of $[a,b]$ for which the mesh converges to zero. Write $P_k = (t_0^{(k)}, t_1^{(k)}, \ldots, t_{n_k}^{(k)})$. Define for $x \in [a,b]$,

$$G(x) = \lim_{k\to\infty} G_{P_k}(x) \text{ and } g(x) = \lim_{k\to\infty} g_{P_k}(x),$$

where for a partition $P = (t_0, t_1, \ldots, t_n)$

$$G_P = \sum_{i=1}^n \chi_{(t_{i-1},t_i]} \sup_{t\in(t_{i-1},t_i]} f(t) \text{ and } g_P = \sum_{i=1}^n \chi_{(t_{i-1},t_i]} \inf_{t\in(t_{i-1},t_i]} f(t).$$

Prove that $H = G$ and $h = g$ m-a.e.

c) Show that f is Riemann integrable $\iff$ the set of discontinuities of f has Lebesgue measure zero.

Proof. Let $H_\delta(x) = \sup_{|x-y|\le\delta} f(y)$ and $h_\delta(x) = \inf_{|x-y|\le\delta} f(y)$. For fixed x, note that $H_\delta(x)$ is an increasing function of δ, and $h_\delta(x)$ is a decreasing function of δ.

a) Suppose f is continuous at x. Let $\epsilon > 0$. Pick $\gamma > 0$ such that $|f(x)-f(y)| < \epsilon$ for all y with $|x-y| \le \gamma$. Then $f(x) \le H_\gamma(x) \le f(x)+\epsilon$ and $f(x)-\epsilon \le h_\gamma(x) \le f(x)$. By the monotinicity of H_δ and h_δ in δ, it follows that for $0 < \delta \le \gamma$ we have $f(x) \le H_\delta(x) \le f(x)+\epsilon$ and $f(x)-\epsilon \le h_\delta(x) \le f(x)$. Thus $\lim_{\delta\to 0} H_\delta(x) = f(x) = \lim_{\delta \to 0} h_\delta(x)$.

For the converse, we assume that $H(x) = h(x)$. Suppose $f(x) \ne H(x)$. Note that $h(x) \le f(x) \le H(x)$. Hence $h(x) < H(x)$, a contradiction. Hence $f(x) = H(x) = h(x)$.

Let $\epsilon > 0$. Pick $\delta > 0$ such that $H_\delta(x) - f(x) < \epsilon$ and $f(x) - h_\delta(x) < \epsilon$. Then if $|x-y| < \delta$, we have $f(y) - f(x) \le H_\delta(x) - f(x) < \epsilon$ and $f(y) - f(x) \ge h_\delta(x) - f(x) > -\epsilon$.

b) For any P_k with mesh size less than δ, we have $G_{P_k} \le H_\delta$. Hence, $G \le H_\delta$ for every δ, so $G \le H$.

For the reverse inequality, let $B = \{(t_i^{(k)} : \forall k, i\}$ be the set of all mesh points of all partitions P_k.

Fix $k \in \mathbb{N}$. If $x \notin B$ then x is not one of the mesh points of P_k, so $H_\delta(x) \le G_{P_k}(x)$ for δ sufficiently small. Hence $H(x) \le G_{P_k}(x)$.

Hence if $x \notin B$, then $H(x) \le G_{P_k}(x)$ for all k. Hence $H(x) \le G(x)$, so $H(x) = G(x)$. Since B is countable, we have $H = G$ a.e. A similar argument (or taking negatives) gives $h = g$ a.e.

c) Recall that f is Riemann integrable if and only if for every partition (P_k) with mesh converging to zero we have $\int G_{P_k} - g_{P_k} \to 0$. Since $G_{P_k} - g_{P_k}$ is decreasing in k, by the DCT we have $\lim_k \int G_{P_k} - g_{P_k} = \int \lim_k G_{P_k} - g_{P_k} = \int H - h$. Thus f is Riemann integrable if and only if $H = h$ a.e. By part (a), $H = h$ a.e. if and only if the set of discontinuities of f has measure zero.

$\square$

6 Problem 30/page 60. Hint: AM-GM. Show that $\lim_{k\to\infty} \int_0^k x^n (1-k^{-1}x)^k\,dx = n!$.

Proof. Using Exercise (4), for $k \in \mathbb{N}$ we have

$$\begin{aligned}
\int_0^k x^n(1-k^{-1}x)^k\,dx &= k^{n+1}\int_0^1 u^n(1-u)^k\,du \\
&= k^{n+1}\frac{\Gamma(n+1)\Gamma(k+1)}{\Gamma(n+k+2)} \\
&= n!\left(\frac{k}{k+1}\right)\left(\frac{k}{k+2}\right)\cdots\left(\frac{k}{k+n+1}\right) \\
&\to n!
\end{aligned}$$

as $k \to \infty$. $\square$

7 Problem 1/88. Let ν be a signed measure on $(X, \mathcal{M})$. If (E_j) is an increasing sequence in $\mathcal{M}$, the $\nu(\bigcup_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$. If (E_j) is a decreasing sequence in $\mathcal{M}$ and $\nu(E_1)$ is finite, then $\nu(\bigcap_1^\infty E_j) = \lim_{j\to\infty} \nu(E_j)$.

Proof. For the first part, we have

$$\begin{aligned}
\nu(\bigcup_1^\infty E_j) &= \nu(E_1 \cup \bigcup_1^\infty E_{j+1} \setminus E_j) \\
&= \nu(E_1) + \sum_1^\infty \nu(E_{j+1} \setminus E_j) \\
&= \lim_{J\to\infty} \nu(E_1) + \sum_1^J \nu(E_{j+1} \setminus E_j) \\
&= \lim_{J\to\infty} \nu(E_1 + \bigcup_1^J \nu(E_{j+1} \setminus E_j) \\
&= \lim_{J\to\infty} \nu(E_{J+1}).
\end{aligned}$$

For the second part, first note that if $A \subset E_1$ with $A \in \mathcal{M}$ then $\nu(A) + \nu(E_1 \setminus A) = \nu(E_1)$. Since $\nu(E_1)$ is finite, $\nu(A)$ must be finite. Hence $\nu(E_1 \setminus A) = \nu(E_1) - \nu(A)$. Using this fact and the previous part, we have

$$\begin{aligned}
\nu(\bigcap_1^\infty E_j) &= \nu(E_1 \setminus \bigcup_1^\infty (E_1 \setminus E_j)) \\
&= \nu(E_1) - \lim_{j\to\infty} \nu(E_1 \setminus E_j) \\
&= \nu(E_1) - \lim_{j\to\infty} \nu(E_1) - \nu(E_j) \\
&= \lim_{j\to\infty} \nu(E_j)
\end{aligned}$$

□

8 Problem 4/88. If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \geq \nu^+$ and $\mu \geq \nu^-$.

Proof. Suppose not. WLOG there exists a measurable set A such that $\lambda(A) < \nu^+(A)$. From the Haar decomposition, there exists a partition $P \cup N = X$ where P contains the support of ν^+ and N contains the support of ν^-. Then $\lambda(A \cap P) \leq \lambda(A) < \nu^+(A) = \nu(A \cap P) = \lambda(A \cap P) - \mu(A \cap P) < \lambda(A \cap P)$, a contradiction. □

9 Problem 7/88. Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.

a. $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$.

b. $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n$ are disjoint, and $\bigcup_1^n E_j = E\}$.

Proof. For (a), we have the partition $X = P \cup N$ where P and N contain the support of ν^+ and ν^- respectively. Hence $\nu^+(E) = \nu(E \cap P) \le \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$. On the other hand, if $F \in \mathcal{M}, F \subset E$ then $\nu(F) = \nu^+(F) - \nu^-(F) \le \nu^+(F) \le \nu^+(E)$. The ν^- part follows from applying the first part to $-\nu$.

For (b), let $K = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n$ are disjoint, and $\bigcup_1^n E_j = E\}$. We have $|\nu|(E) = \nu^+(E) + \nu^-(E) = |\nu(E \cap P)| + |\nu(E \cap N)| \le K$. On the other hand, if $(E_j)_1^n$ is a partition of E then $\sum_1^n |\nu(E_j)| = \sum_1^n |\nu^+(E_j) - \nu^-(E_j)| \le \sum_1^n |\nu|(E_j) = |\nu|(E)$. $\square$