Paul Gustafson
Texas A&M University - Math 607
Instructor: Thomas Schlumprecht

HW 7

1 Assume that $(f_n) \subset L_1(\mu)$ and $f_n \to f$ uniformly.
    a) If $\mu(X) < \infty$, then $f \in L_1$ and $\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu$.
    b) If $\mu(X) = \infty$, then the conclusion of (a) may fail.

Proof. For (a), we have

$$\left| \int f \, d\mu - \int f_n \, d\mu \right| \leq \int |f - f_n| \, d\mu$$
$$\leq \mu(X) \sup_{x \in X} |f(x) - f_n(x)|$$
$$\to 0.$$

    For (b), let $(f_n) = (1/n)\chi_{[0,n]}$. Then $f_n \to 0$ uniformly on $\mathbb{R}$, but $\int f_n \, dx = 1$ for all n. $\qquad\square$

2 Let $f_n, g_n, g \in L_1$, $n \in \mathbb{N}$, and assume that $f_n \to f$, f measurable, and $g_n \to g$ μ-a.e., and that $|f_n| \leq g_n$ and $\int g_n \, d\mu \to \int g \, d\mu$.
    Then $\int f_n \, d\mu \to \int f \, d\mu$.

Proof. Following the proof of the DCT, since $|f_n| \leq g_n$ we have $|f| \leq g$. Hence $f \in L_1$. We also have $g_n + f_n \geq 0$ a.e. and $g_n - f_n \geq 0$ a.e. Hence by Fatou's lemma and linearity of the integral on L_1,

$$\int g + \int f \leq \liminf \int (g_n + f_n) = \liminf \int g_n + \int f_n = \int g + \liminf \int f_n$$

The last equality follows from the fact that if $(a_n) \to a$ and $(b_n) \subset \mathbb{R}$, then $\liminf a_n + b_n = a + \liminf b_n$. To see this, pick $\epsilon > 0$ and N such that $|a - a_n| < \epsilon$ for all $n \geq N$. Hence $\liminf a_n + b_n = \liminf(a_n - a) + a + b_n \leq \liminf \epsilon + a + b_n = \epsilon + a + \liminf b_n$, and similarly $\liminf a_n + b_n \geq -\epsilon + a + \liminf b_n$. Hence $\liminf a_n + b_n = a + \liminf b_n$.
    Similarly,

$$\int g - \int f \leq \liminf \int (g_n - f_n) = \liminf \int g_n - \int f_n = \int g - \limsup \int f_n$$
.

    Hence $\limsup \int f_n \leq \int f \leq \liminf \int f_n$, so $\int f = \lim \int f_n$. $\qquad\square$

3 Suppose that for $n \in \mathbb{N}$, $f_n = \chi_{E_n}$ for some $E_n \subset \mathbb{R}$, and assume that $f(x) = \lim_{n\to\infty} f_n(x)$ exists a.e.
    a) Show that $f = \chi_E$ a.e. for some measurable set $E \subset \mathbb{R}$.

b) Show that for any $g \in L_1$:

$$\int_E g \, dx = \lim_{n \to \infty} \int_{E_n} g \, dx.$$

c) Establish a necessary and sufficient condition for $f_n \to f$ in L_1.

Proof. For (a), we have $\chi_{E_n} \to f$ on N^c for some null set N. Let $x \in N^c$. Since $(\chi_{E_n}(x))_n$ is a convergent discrete-valued sequence, it must be eventually constant. Thus, $f(x) \in \{0, 1\}$. Let $E = f^{-1}(1) \cap N^c$. Hence $f = \chi_E$ on N^c, so $f = \chi_E$ a.e. on $\mathbb{R}$. By a previous homework problem, f is measurable since it is the limit of measurable functions. Hence E is measurable.

For (b), we have $\chi_{E_n} g \to \chi_E g$ pointwise a.e. by part (a). Moreover, $\chi_{E_n} g \le |g| \in L_1$. Hence, by the DCT, we have the desired conclusion.

For (c), one such condition is that $m(E_n) \to m(E)$ with $m(E_n), m(E) < \infty$. Clearly, the latter condition is necessary for f_n, f to be in L_1. For the necessity of the former condition, suppose $f_n \to f$ in L_1. Then $|m(E_n) - m(E)| = |\int f_n - \int f| \le \int |f_n - f| \to 0$.

For sufficiency, suppose $m(E_n) \to m(E)$ with $m(E_n), m(E) < \infty$. Then $|f_n - f| \le |f_n| + |f| = \chi_{E_n} + \chi_E$ a.e. Moreover, $\chi_{E_n} + \chi_E \to 2\chi_E$ a.e., and $\int (\chi_{E_n} + \chi_E) = m(E_n) + m(E) \to 2m(E) = \int (2\chi_E)$. Hence, by the generalized DCT (Exercise 2), we have $\int |f_n - f| \to \int \lim_n |f_n - f| = 0$. $\qquad\square$

4 Let $L_0([0, 1])$ be the space of all measurable functions $f : [0, 1] \to \mathbb{R}$.
  a) for $f, g \in L_0([0, 1])$ put

$$d(f, g) = \int_0^1 \min\{1, |f - g|\} \, dx.$$

Show that $(L_0([0, 1]), d)$ is a metric space and that for $f, f_n \in L_0([0, 1])$:

$$f_n \to f \text{ in } (L_0([0, 1]), d) \iff f_n \to f \text{ in measure.}$$

b) Is there a metric d' on $L_0([0, 1])$ for which

$$f_n \to f \text{ in } (L_0([0, 1]), d') \iff f_n \to f \, a.e.$$

Proof. For (a), to see that d is a metric we need to show that d is positive definite, symmetric, and satisfies the triangle inequality. The function d is clearly nonnegative and $0 = d(f, g) = \int_0^1 \min\{1, |f - g|\} \, dx$ implies that $f = g$ a.e. The function d is obviously symmetric. For the triangle inequality, I first claim that for $x, y, z \in \mathbb{R}$ we have $\min\{1, |x - y|\} \le \min\{1, |x - z|\} + \min\{1, |y - z|\}$.

We have four cases from the RHS of the inequality.

Case $|x - z| \le 1$ and $|y - z| \le 1$. We have $\min\{1, |x - y|\} \le |x - y| \le |x - y| + |y - z| = \min\{1, |x - z|\} + \min\{1, |z - y|\}$.

Case $|x - z| \le 1$ and $|y - z| > 1$. We have $\min\{1, |x - y|\} \le \min\{1, |x - z| + |z - y|\} \le \min\{1, 1 + |z - y|\} = 1 + |z - y| = \min\{1, |x - z|\} + \min\{1, |z - y|\}$.

Case $|x - z| > 1$ and $|y - z| \leq 1$. Analogous to previous case.

Case $|x - z| > 1$ and $|y - z| > 1$. We have $\min\{1, |x - y|\} \leq \min\{1, |x - z| + |z - y|\} = 1 \leq \min\{1, |x - z|\} + \min\{1, |z - y|\}$.

Hence $\min\{1, |x - y|\} \leq \min\{1, |x - z|\} + \min\{1, |y - z|\}$ for all $x, y, z \in \mathbb{R}$. Thus, if $f, g, h \in L_0([0,1])$ then $d(f, g) = \int_0^1 \min\{1, |f - g|\}\, dx \leq \int_0^1 \min\{1, |f - h|\} + \min\{1, |h - g|\}\, dx = d(f, h) + d(g, h)$. Thus, d is a metric.

Suppose $f_n \to f$ in $(L_0([0,1]), d)$. Let $0 < \epsilon < 1$. Pick N such that $d(f, f_n) < \epsilon^2$ for all $n \geq N$. Then for all $n \geq N$, we have

$$
\begin{aligned}
m(\{|f - f_n| \geq \epsilon\}) &= \int_{\{\epsilon \leq |f - f_n| < 1\}} dx + \int_{\{|f - f_n| \geq 1\}} dx \\
&\leq \int_{\{\epsilon \leq |f - f_n| < 1\}} \frac{|f - f_n|}{\epsilon}\, dx + \int_{\{|f - f_n| \geq 1\}} \min\{1, |f - f_n|\}\, dx \\
&\leq \int_{\{\epsilon \leq |f - f_n| < 1\}} \epsilon^{-1} \min\{1, |f - f_n|\}\, dx + \int_{\{|f - f_n| \geq 1\}} \min\{1, |f - f_n|\}\, dx \\
&\leq \epsilon^{-1} \int \min\{1, |f - f_n|\}\, dx \\
&< \epsilon
\end{aligned}
$$

Conversely, suppose $f_n \to f$ in measure. Let $0 < \epsilon < 1$. Pick N such that $m(\{|f - f_n \geq \epsilon\}) \leq \epsilon$ for all $n \geq N$. Then for all $n \geq N$ we have

$$
\begin{aligned}
\int \min\{1, |f_n - f|\}\, dx &\leq \int_{\{0 \leq |f_n - f| < \epsilon\}} |f_n - f|\, dx + \int_{\{|f_n - f| \geq \epsilon\}} dx \\
&\leq \int_{\{0 \leq |f_n - f| < \epsilon\}} \epsilon\, dx + m(\{|f - f_n \geq \epsilon\}) \\
&\leq 2\epsilon.
\end{aligned}
$$

For (b), I use the following fact about convergence in metric spaces. Let (M, d) be a metric space, $(x_n)_{n \in \mathbb{N}} \subset M$, and $x \in M$. I claim that if every subsequence of (x_n) has a further subsequence converging to x, then $x_n \to x$.

Suppose $x_n \not\to x$. Then there exists $\epsilon > 0$ and a subsequence $(x_n)_{n \in N_1}$ such that $d(x, x_n) \geq \epsilon$ for all $n \in N_1$. This subsequence cannot have a further subsequence converging to x, contradicting the hypothesis.

Thus it suffices to find a sequence that does not converge pointwise a.e., but each subsequence has a subsequence that converges a.e. to 0. For $n \in \mathbb{N}$, write n as $n = 2^j + k$ for $j \geq 0$ and $0 \leq k < 2^j$. Let $E_n = [k2^{-j}, (k+1)2^{-j}]$ and $f_n = \chi_{E_n}$. Every element of $[0, 1]$ is contained in infinitely many E_n and infinitely many E_n^c, so (f_n) does not converge pointwise a.e.

On the other hand, suppose $(f_n)_{n \in N_1}$ is a subsequence of $(f_n)_{n \in \mathbb{N}}$. For each n, pick $x_n \in E_n$. Then by the sequential compactness of $[0, 1]$, there exists an infinite set $N_2 \subset N_1$ and $x_0 \in [0, 1]$ such that $x_n \to x_0$ as $n \to \infty, n \in N_2$. Since $\operatorname{diam}(E_n) \to 0$, we have $f_n(x) \to 0$ as $n \to \infty, n \in N_2$ for $x \neq x_0$. Hence, $f_n \to 0$ as $n \to \infty, n \in N_2$ pointwise a.e. $\qquad \square$

5

(a) $\displaystyle \lim_{n\to\infty} \int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx = 0$

(b) $\displaystyle \lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} = 0$

(c) $\displaystyle \lim_{n\to\infty} \int_0^\infty n\sin\left(\frac{x}{n}\right) [x(1+x^2)]^{-1} dx = \frac{\pi}{2}$

(d) $\displaystyle \lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2 x^2} dx = \begin{cases} 0 & \text{if } a > 0 \\ \pi/2 & \text{if } a = 0 \\ \pi & \text{if } a < 0 \end{cases}$

Proof. For (a), we have $\int_0^\infty \left(1+\frac{x}{n}\right)^{-n} \sin\left(\frac{x}{n}\right) dx = \int_0^\infty n^2 (1+nu)^{-n} \sin(nu)\, du$.

For $n \geq 2$, we have $\left| n^2(1+nu)^{-n} \sin(nu) \right| \leq n^2(1+nu)^{-2} \leq u^{-2}$, which is integrable on $(1,\infty)$. Hence, by the DCT,

$$\lim_{n\to\infty} \int_1^\infty n^2(1+nu)^{-n} \sin(nu)\, du = \int_1^\infty \lim_{n\to\infty} n^2(1+nu)^{-n} \sin(nu)\, du = 0$$

For the rest of the integral, we have

$$\left| \int_0^1 n^2(1+nu)^{-n} \sin(nu)\, du \right| \leq \int_0^1 n^2(1+nu)^{-n} nu\, du$$

$$= \left[\frac{n^2}{1-n} u(1+nu)^{1-n} \right]_{u=0}^1 - \int_0^1 \frac{n^2}{1-n}(1+nu)^{1-n}\, du$$

$$= \frac{n^2}{1-n}(1+n)^{1-n} - \left[\frac{n}{(1-n)(2-n)}(1+nu)^{2-n} \right]_{u=0}^1$$

$$\to 0$$

as $n \to \infty$. This proves (a).

For (b), for $x \geq 0$ we have $\frac{1+nx^2}{(1+x^2)^n} \leq 1$. Hence, by the DCT we have

$$\lim_{n\to\infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n}\, dx = \int_0^1 \lim_{n\to\infty} \frac{1+nx^2}{(1+x^2)^n}\, dx = 0.$$

For (c), for $x \in [0,\infty)$ we have $\left| n\sin\left(\frac{x}{n}\right)[x(1+x^2)]^{-1} \right| \leq n\left(\frac{x}{n}\right)[x(1+x^2)]^{-1} = (1+x^2)^{-1}$. We have $(1+x^2)^{-1} \in L_1([0,\infty))$ by applying the MCT to $\int_0^M (1+x^2)^{-1}\, dx$ as $M \to \infty$. Hence, by the DCT we have

$$\lim_{n\to\infty} \int_0^\infty n\sin\left(\frac{x}{n}\right) [x(1+x^2)]^{-1}\, dx = \int_0^\infty \lim_{n\to\infty} n\sin\left(\frac{x}{n}\right) [x(1+x^2)]^{-1}\, dx$$

$$= \int_0^\infty \lim_{m\to 0^+} \frac{\sin(m)}{m}(1+x^2)^{-1}\, dx$$

$$= \int_0^\infty (1+x^2)^{-1}\, dx$$

$$= \lim_{M\to\infty} \int_0^M (1+x^2)^{-1}\, dx$$

$$= \lim_{M\to\infty} \tan^{-1}(M)$$

$$= \pi/2,$$

where the fourth equality follows from the MCT.

For (d),

$$\lim_{n\to\infty} \int_a^\infty \frac{n}{1+n^2x^2}\, dx = \lim_{n\to\infty} \int_{na}^\infty \frac{du}{1+u^2}$$

$$= \lim_{n\to\infty} \pi/2 - \tan^{-1}(na)$$

$$= \begin{cases} 0 & \text{if } a > 0 \\ \pi/2 & \text{if } a = 0 \\ \pi & \text{if } a < 0 \end{cases},$$

where the second equality follows from the MCT applied to $\int_{na}^M \frac{du}{1+u^2}$ as $M\to\infty$.

$\square$