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HW 8

J20.1 Suppose f is Lipschitz with constant M. Suppose $h_k \to 0$ with $h_k \neq 0$ for all k, and $\frac{f(x+h_k)-f(x)}{h_k} \to \alpha$. Then since $|\frac{f(x+h_k)-f(x)}{h_k}| \leq M$ for all k, we have $|\alpha| \leq M$.

Conversely, suppose $Df(x) \subset [-M, M]$ for all x. Define g by $g(x) = f(x) + Mx$. Then $Dg(x) \subset [0, 2M]$ for all x, so by (10), g is increasing. Hence, if $x < y$, then $f(x) + Mx \leq f(y) + My$. This implies $-M(y-x) \leq (f(y) - f(x))$.

Let $h(x) := -f(x) + Mx$. Then for all x, $Dh(x) = -Df(x) + M \subset [0, 2M]$. Hence, by (10), h is increasing. Thus, for $x < y$, $-f(x) + Mx \leq -f(y) + My$, so $(f(y) - f(x)) \leq M(y-x)$.

Thus, for all $x < y$, $|f(y) - f(x)| \leq M|y-x|$. By swapping y with x, this also holds for $x > y$.

7 Suppose $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ is differentiable, hence continuous. Let $f_k(x) = (f(x + 1/k) - f(x))k$. Then each f_k is Borel measurable and $f_k \to f'$, so f is Borel measurable.

If f is differentiable a.e., then by the above case, it is Borel measurable on $\mathbb{R} \setminus N$ for some null set N. Thus, it is Lebesgue measurable on $\mathbb{R}$.

10 As noted in class, it suffices to prove this in the case that there exists $\epsilon_0 > 0$ such that, for all x, $Df(x) \subset [\epsilon_0, \infty]$. Indeed, in the general case, define $f_n(x) := f + (1/n)x$. Assuming the special case, each f_n is increasing, so f is increasing since $f_n \to f$.

To prove the special case, suppose f is not increasing. Then there exists $x < y_0$ such that $f(x) < f(y_0)$. Let $B = \{t : t > x \text{ and } f(t) < f(x)\}$ and $y = \inf B$.

Case $f(y) < f(x)$. There exists a sequence $y_k \to y$ from the left with $y_k > x$ for all k. Since $f(y_k) \geq f(x)$ for all k, $\frac{f(y)-f(y_k)}{y-y_k} \leq \frac{f(y)-f(x)}{y-y_k} \to -\infty$ as $k \to \infty$, a contradiction.

Case $f(y) \geq f(x)$. Note that in this case y must be a limit point of B. Pick a sequence $(y_k) \subset B$ with $y_k \to y$. Then $q_k := \frac{f(y_k)-f(y)}{y_k-y} \leq \frac{f(x)-f(x)}{y_k-y} = 0$. Pick any subsequence of q_k with a limit in the extended reals to get a contradiction.

11 Following the hint, let (U_n) be a decreasing sequence of open sets with each $U_n \supset E$ and $m(U_n) < 2^{-n}$. Let $f_n(x) = m((-\infty, x) \cap U_n)$, and $f = \sum_n f_n$.

Each f_n is Lipschitz of constant 1 (proved on the exam) and increasing. Moreover, $f_n \leq m(U_n) < 2^{-n}$, so $\sum_n f_n$ converges uniformly. Thus, f is continuous and increasing.

Note that if $x \in E$, then U_n contains a neighborhood of x. Hence, if $x_k \to x$, then for all large k, $\frac{f_n(x_k)-f_n(x)}{x_k-x} = \frac{x_k-x}{x_k-x} = 1$. Thus, each f_n is differentiable at x with derivative 1.

Hence, if $x \in E$ and $x_k \to x$, then for every $N > 0$, $\frac{f(x_k)-f(x)}{x_k-x} = \sum_n \frac{f_n(x_k)-f_n(x)}{x_k-x} \geq \sum_{n=1}^N \frac{f_n(x_k)-f_n(x)}{x_k-x} \to N$ as $k \to \infty$. Hence, $Df(x) = \{\infty\}$.