FINITENESS FOR MAPPING CLASS GROUP REPRESENTATIONS FROM TWISTED DIJKGRAAF-WITTEN THEORY

PAUL GUSTAFSON

ABSTRACT. Any twisted Dijkgraaf-Witten representation of a mapping class group of an orientable, compact surface with boundary has finite image.

1. INTRODUCTION

Given a spherical fusion category $\mathscr{A}$ over a field k and an oriented compact surface Σ, possibly with boundary, the Turaev-Viro-Barrett-Westbury (TVBW) construction gives a projective representation of the mapping class group $\mathrm{MCG}(\Sigma)$ [15, 3]. A natural question is to determine the image of these representations. In particular, when does such a representation have finite image?

It is conjectured that these representations have finite image if and only if $\mathscr{A}$ is weakly integral. This conjecture is a modification of the Property F conjecture [11, 7], which states that braid group representations coming from a braided monoidal category $\mathscr{C}$ should have finite image if and only if $\mathscr{C}$ is weakly integral. Instead of only considering braid group representations, one can consider mapping class groups of arbitrary orientable surfaces. In this case, the input categories to construct the representations must be more specialized than just braided monoidal. One can either apply the Reshitikhin-Turaev construction to a modular tensor category, or apply the TVBW construction to a spherical fusion category. The former is more general than the latter since the Reshitikhin-Turaev construction for the Drinfeld center $Z(\mathscr{A})$ of a spherical fusion category $\mathscr{A}$ yields the same representation as the TVBW construction for $\mathscr{A}$. However, for the case considered in this paper, the simpler TVBW construction suffices.

In this paper, our input category is $\mathscr{A} = \mathrm{Vec}_G^\omega$, the spherical fusion category of G-graded vector spaces with associativity modified by a cocycle $\omega \in Z^3(G, k^\times)$. In this case, the TVBW construction corresponds to the twisted Dijkgraaf-Witten theory of [5]. The category Vec_G^ω is integral, so the one expects its associated mapping class group representations to have finite image. The main contribution of this paper is to verify this for arbitrary G and ω.

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2. RELATED WORK

The closest related work is a result of Fjelstad and Fuchs [8] showing that, given a surface with at most one boundary component, the mapping class group representations corresponding to the untwisted (i.e. $\omega = 1$) Dijkgraaf-Witten theory have finite image. Their paper uses an algebraic method of Lyubashenko [10] that gives a projective mapping class group representation to any factorizable ribbon Hopf algebra, in their case, the double $D(G)$. In our case, we instead consider the mapping class group action on a vector space of Vec_G^ω-colored embedded graphs defined by Kirillov [9], yielding a simpler, more geometric proof.

In [2], Bantay defined representations of mapping class groups on the Hilbert space of an orbifold model associated to $D^\omega(G)$. These representations appear to coincide with the twisted Dijkgraaf-Witten representations. However, the precise details of the connection are not clear to me.

More is known when we fix a particular surface Σ. In the case where Σ is a torus, Ng and Schauenburg showed that any Reshitikhin-Turaev representation of the mapping class group of the torus is finite [12]. In the case where Σ is an n-punctured disk, the mapping class group of Σ relative to the boundary of the disk is the braid group B_n. In this case, Etingof, Rowell, and Witherspoon proved that the representations associated to $\mathrm{Mod}(D^\omega(G))$ are finite [7].

3. BACKGROUND

3.1. **The spherical fusion category Vec_G^ω.** The following defintions are well-known and can be found in, e.g., [6]. Let k be an algebraically closed field of characteristic 0, G a finite group, and $\omega \in Z^3(G, k^\times)$ a 3-cocycle. The spherical fusion category of G-graded k-vector spaces with associativity defined by ω is denoted Vec_G^ω. The objects of this category are vector spaces with a decomposition $V = \bigoplus_{g\in G} V_g$. Morphisms are linear maps preserving the grading. The tensor product is defined by

$$(V\otimes W)_g = \bigoplus_{x,y\in G, xy=g} V_x \otimes W_y.$$

For each $g\in G$, pick a 1-dimensional vector space $\delta_g \in \mathrm{Obj}(\mathrm{Vec}_G^\omega)$ concentrated in degree g. The set $\{\delta_g : g\in G\}$ is a complete set of pairwise non-isomorphic representatives for the isomorphism classes of simple objects of Vec_G^ω. We have $1\simeq \delta_0$ and $\delta_g^* \simeq \delta_{g^{-1}}$.

For the structural morphisms, we follow [13]. We will abuse notation by referring to an object δ_g by the group element g. The associator $\alpha_{g,h,k} : (g\otimes h)\otimes k \to g\otimes (h\otimes k)$ is defined by

$$\alpha_{g,h,k} = \omega(g,h,k)\,\mathrm{id}_{ghk}\,.$$

The evaluator $\mathrm{ev}_g : g^*\otimes g \to 1$ is

$$\mathrm{ev}_g = \omega(g^{-1},g,g^{-1})\,\mathrm{id}_1\,.$$

The coevaluator $\mathrm{coev}_g : 1\to g\otimes g^*$ is

$$\mathrm{coev}_g = \mathrm{id}_1\,.$$

The pivotal structure $j_g : g^{**}\to g$ is

$$j_g = \omega(g^{-1},g,g^{-1})\,\mathrm{id}_g\,.$$

3.2. **Colored graphs.** The following definitions and theorem are from Kirillov's paper [9], but recorded here for convenience. For any spherical fusion category $\mathscr{A}$ and surface Σ, he gives the following presentation of the Levin-Wen model as a vector space of colored graphs modulo local relations. He also proves that this space is canonically isomorphic to the TVBW vector space associated to Σ. It is straightforward to check that this isomorphism, which amounts to replacing a triangulation with its dual graph, commutes with the mapping class group action.

We use the convention that a tensor product of multiple objects with parentheses omitted correspond to the left-associative parenthesization.

We define the functor $\mathscr{A}^{\boxtimes n} \to \mathrm{Vec}$ by

$$\langle V_1,\dots,V_n\rangle = \mathrm{Hom}_{\mathscr{A}}(1, V_1\otimes\cdots\otimes V_n) \tag{1}$$

for any collection $V_1,\dots,V_n$ of objects of $\mathscr{A}$. Note that pivotal structure gives functorial isomorphisms

$$z\colon \langle V_1,\dots,V_n\rangle \simeq \langle V_n, V_1,\dots,V_{n-1}\rangle \tag{2}$$

where, up to associators and unitors,

$$z(\phi) = (j^{-1}_{**V}\otimes \mathrm{id}_{V_1\otimes\cdots\otimes V_{n-1}} \otimes \mathrm{ev}_{*V_n})\circ(\mathrm{id}_{**V_n}\otimes\phi\otimes\mathrm{id}_{*V_n})\circ \mathrm{coev}_{**V_n}$$

such that $z^n = \mathrm{id}$ (see [1, Section 5.3]); thus, up to a canonical isomorphism, the space $\langle V_1,\dots,V_n\rangle$ only depends on the cyclic order of $V_1,\dots,V_n$.

We have a natural composition map

$$\begin{aligned}&\langle V_1,\dots,V_n,X\rangle\otimes\langle {}^*X, W_1,\dots,W_m\rangle \to \langle V_1,\dots,V_n,W_1,\dots,W_m\rangle\\ &\varphi\otimes\psi \mapsto \varphi \underset{X}{\circ} \psi = \mathrm{ev}_{*X}\circ(\varphi\otimes\psi)\end{aligned} \tag{3}$$

where $\mathrm{ev}_{*X} : X\otimes {}^*X \to 1$ is the evaluation morphism.

We will consider finite graphs embedded in an oriented surface Σ (which may have boundary); for such a graph Γ, let $E(\Gamma)$ be the set of edges. Note that edges are not oriented. Let E^{or} be the set of oriented edges, i.e. pairs $\mathbf{e} = (e, \text{orientation of } e)$; for such an oriented edge $\mathbf{e}$, we denote by $\bar{\mathbf{e}}$ the edge with opposite orientation.

If Σ has a boundary, the graph is allowed to have uncolored one-valent vertices on $\partial\Sigma$ but no other common points with $\partial\Sigma$; all other vertices will be called interior. We will call the edges of Γ terminating at these one-valent vertices *legs*.

Definition 3.1. *Let Σ be an oriented surface (possibly with boundary) and $\Gamma\subset\Sigma$ — an embedded graph as defined above. A* coloring *of Γ is the following data:*

- *Choice of an object $V(\mathbf{e})\in \mathrm{Obj}\,\mathscr{A}$ for every oriented edge $\mathbf{e}\in E^{or}(\Gamma)$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.*
- *Choice of a vector $\varphi(v)\in\langle V(\mathbf{e}_1),\dots,V(\mathbf{e}_n)\rangle$ (see (1)) for every interior vertex v, where $\mathbf{e}_1,\dots,\mathbf{e}_n$ are edges incident to v, taken in counterclockwise order and with outward orientation.*

An isomorphism f *of two coloring* $\{V(\mathbf{e}), \varphi(v)\}$, $\{V'(\mathbf{e}), \varphi'(v)\}$ *is a collection of isomorphisms* $f_{\mathbf{e}}\colon V(\mathbf{e}) \simeq V'(\mathbf{e})$ *which agree with isomorphisms* $V(\overline{\mathbf{e}}) = V(\mathbf{e})^*$ *and which identify* φ', φ: $\varphi'(v) = f \circ \varphi(v)$.

We will denote the set of all colored graphs on a surface Σ *by* $\mathrm{Graph}(\Sigma)$.

Note that if Σ has a boundary, then every colored graph Γ defines a collection of points $B = \{b_1, \dots, b_n\} \subset \partial\Sigma$ (the endpoints of the legs of Γ) and a collection of objects $V_b \in \mathrm{Obj}\ \mathscr{A}$ for every $b \in B$: the colors of the legs of Γ taken with outgoing orientation. We will denote the pair $(B, \{V_b\})$ by $\mathbf{V} = \Gamma \cap \partial\Sigma$ and call it *boundary value*. We will denote

$$\mathrm{Graph}(\Sigma, \mathbf{V}) = \text{set of all colored graphs in } \Sigma \text{ with boundary value } \mathbf{V}.$$

We can also consider formal linear combinations of colored graphs. Namely, for fixed boundary value $\mathbf{V}$ as above, we will denote

$$\mathrm{VGraph}(\Sigma, \mathbf{V}) = \{\text{formal linear combinations of graphs } \Gamma \in \mathrm{Graph}(\Sigma, \mathbf{V})\} \tag{4}$$

In particular, if $\partial\Sigma = \varnothing$, then the only possible boundary condition is trivial ($B = \varnothing$); in this case, we wil just write $\mathrm{VGraph}(\Sigma)$.

The following theorem is a variation of result of Reshitikhin and Turaev.

Theorem 3.2. *There is a unique way to assign to every colored planar graph* Γ *in a disk* $D \subset \mathbb{R}^2$ *a vector*

$$\langle \Gamma \rangle_D \in \langle V(\mathbf{e}_1), \dots, V(\mathbf{e}_n) \rangle \tag{5}$$

where $\mathbf{e}_1, \dots, \mathbf{e}_n$ *are the edges of* Γ *meeting the boundary of* D *(legs), taken in counterclockwise order and with outgoing orientation, so that that following conditions are satisfied:*

(1) $\langle \Gamma \rangle$ *only depends on the isotopy class of* Γ.
(2) *If* Γ *is a single vertex colored by* $\varphi \in \langle V(\mathbf{e}_1), \dots, V(\mathbf{e}_n) \rangle$, *then* $\langle \Gamma \rangle = \varphi$.
(3) *Local relations shown in Figure* 1 *hold.*

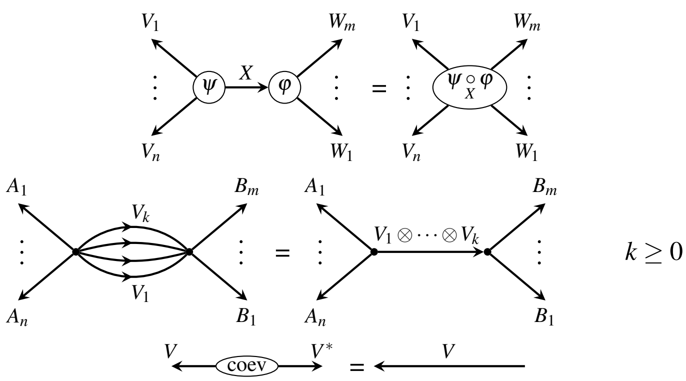

FIGURE 1. Local relations for colored graphs.

Local relations should be understood as follows: for any pair Γ, Γ' *of colored graphs which are identical outside a subdisk* $D' \subset D$, *and in this disk are homeomorphic to the graphs shown in Figure* 1, *we must have* $\langle \Gamma \rangle = \langle \Gamma' \rangle$.

Moreover, so defined $\langle \Gamma \rangle$ *satisfies the following properties:*

(1) $\langle \Gamma \rangle$ *is linear in color of each vertex* v (*for fixed colors of edges and other vertices*).
(2) $\langle \Gamma \rangle$ *is additive in colors of edges as shown in Figure* 2.
(3) *If* Γ, Γ' *are two isomorphic colorings of the same graph, then* $\langle \Gamma \rangle = \langle \Gamma' \rangle$.
(4) *Composition property: if* $D' \subset D$ *is a subdisk such that* $\partial D'$ *does not contain vertices of* Γ *and meets edges of* Γ *transversally, then* $\langle \Gamma \rangle_D$ *will not change if we replace subgraph* $\Gamma \cap D'$ *by a single vertex colored by* $\langle \Gamma \cap D' \rangle_{D'}$.

The vector $\langle \Gamma \rangle$ *is called the* evaluation *of* Γ.

To define local relations between graphs, Kirillov defines the space of null graphs as follows. Let $\Gamma = c_1\Gamma_1 + \cdots + c_n\Gamma_n$ be a formal linear combination of colored graphs in Σ. If there exists an embedded disk $D \subset M$ such that

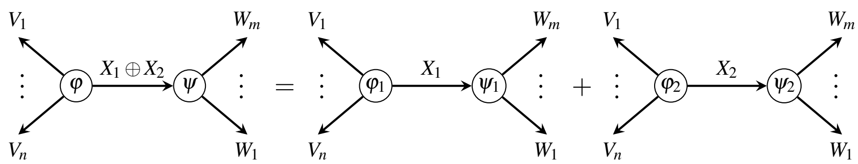

FIGURE 2. Linearity of $\langle\Gamma\rangle$. Here φ_1, φ_2 are compositions of φ with projector $X_1 \oplus X_2 \to X_1$ (respectively, $X_1 \oplus X_2 \to X_2$), and similarly for ψ_1, ψ_2.

(1) Γ is transversal to ∂D (i.e., no vertices of Γ_i are on the boundary of D and edges of each Γ_i meet ∂D transversally),
(2) all Γ_i coincide outside of D,
(3) and $\langle\Gamma\rangle_D = \sum c_i \langle\Gamma_i \cap D\rangle_D = 0$;

then Γ is called a null graph.

Definition 3.3. *The vector space $H := H(\Sigma, \mathbf{V})$ associated to a oriented surface Σ with boundary condition $\mathbf{V}$ by the spherical fusion category $\mathscr{A}$ is the quotient space*

$$H(\Sigma, \mathbf{V}) = \mathrm{VGraph}(\Sigma, \mathbf{V})/N(\Sigma, \mathbf{V})$$

where $N(\Sigma, \mathbf{V})$ is the subspace spanned by null graphs (for all possible embedded disks $D \subset \Sigma$).

4. RESULTS

Definition 4.1. *Let Γ be a graph embedded in a surface Σ. A Vec_G^ω coloring (V, ϕ) of Γ will be called* simple *if the following conditions both hold:*

(1) *For every oriented edge $\mathbf{e} \in E^{or}(\Gamma)$, there exists a group element $g(\mathbf{e}) \in G$ such that the coloring $V(e_i) = \delta_{g(\mathbf{e})}$.*
(2) *If v is an interior vertex of Γ and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are edges incident to v, taken in counterclockwise order and with outward orientation, then $\prod_{i=1}^n g(\mathbf{e}_i) = 1$ and $\phi(v) : \mathrm{Hom}(1, \bigotimes_{i=1}^n \delta_{g(\mathbf{e}_i)}$ is the canonical isomorphism*

Proposition 4.2. *Let Γ be a simple colored graph embedded in a surface Σ. Let Δ be the colored graph given by applying any local move in Figure 1 to Γ. Then*

(1) *each edge of Δ is labeled by δ_g for some $g \in G$, and*
(2) *there exists $\alpha \in \mathrm{Im}(\omega) \cup \{1\}$ such that*

$$\Delta - \alpha\Delta' \in N(\Sigma, \mathbf{V}),$$

where Δ' is the simple colored graph given by replacing each vertex label in Δ with the canonical isomorphism in condition (2) of Definition 4.1.

Proof. The proof of (1) follows from the definition of the local moves. For (2), we'll consider each local move separately. In each case, we need to show that Δ is equivalent to $\alpha\Delta'$ in H. This boils down to using the linearity of the evaluation map of Theorem 4.1 with respect to the vertex coloring.

For the first (edge contraction) local move in Figure 1, using the same notation as in the figure, we repeatedly apply associators and the cyclic z-morphism of Equation 2 so that the order of objects in the codomains of ϕ and ψ correspond to the orders on the left hand side of Equation 3 and that X and X^* are isolated (not contained in any parentheses). After applying the ev_X morphism, we reassociate until the new label $\varphi \underset{X}{\circ} \psi$ has the appropriate parenthesization. Since every edge is labeled by a simple object, each associator morphism is of the form $\alpha_{g,h,k}$ for some $g, h, k \in G$. For the same reason, every z-morphism is a composition of tensor products of morphism of the form id_g, ev_g, coev_g , $\alpha_{g,h,k}$, and j_g. By the definition of Vec_G^ω (in particular, the definitions of the structural morphisms on the simple objects), it follows that the result of the local moves is of the desired form.

For the second local move (tensoring parallel edges), there are two cases, $k = 0$ and $k > 0$. In the $k = 0$ case, we need to apply inverse unitors to introduce an edge labeled by the unit object, followed by some reassocation. In the $k > 0$ case, we need to reassociate to group togetherlabels of the parallel edges then reassociate at the end. For the same reasons as for the first move (every edge is labeled by a simple object), it follows that the result of this local moves is also of the desired form.

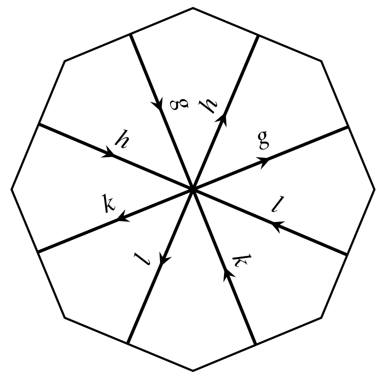

FIGURE 3. Element of the spanning set S for a genus 2 surface

For the third local move (adding a coev-labeled vertex), the colored graph given by direct application of the local move to a simple graph is already simple, so we can pick $\alpha = 1$. $\square$

4.1. **No Boundary Case.**

Theorem 4.3. *The image of any twisted Dijkgraaf-Witten representation of a mapping class group of an orientable, closed surface Σ is finite.*

Proof. Let Γ be a Vec_G^{ω}-colored graph embedding, and let $g \geq 1$ be the genus of Σ (if $g = 0$, the mapping class group is trivial). Thinking of Σ as a quotient of its fundamental $4g$-gon, by isotopy we may assume that the vertices of Γ lie in the interior of the polygon, none of the edges of Γ intersect the corners of the polygon, and that the edges of Γ only meet the sides of the polygon transversally. Evaluating on the interior of the polygon shows that Γ is equivalent to a graph with a single vertex whose edges are simple closed curves, each of which intersect the boundary of the polygon precisely once. By using the local relations, we can replace all the edges intersecting a side with a single edge labeled by the tensor product of their labels. If there are no edges intersecting a side, we can insert a single edge labeled by the group identity into Γ that intersects only that side. Thus, Γ is equivalent to a colored graph with one vertex v and edges $e_1, \ldots, e_{2g}$ corresponding to the standard generators of $\pi_1(M, v)$ as shown in Figure 3.

By Theorem 4.1 and the definition of the quotient map identifying the sides of the fundamental polygon, the vertex v is colored by an element $\phi(v) \in \mathrm{Hom}(1, \bigotimes_{i=1}^{g} V(e_{2i-1}) \otimes V(e_{2i}) \otimes V(e_{2i-1})^* \otimes V(e_{2i}))^*$, where $V(e_i) \in \mathrm{Obj}(\mathrm{Vec}_G^{\omega})$ is the coloring of the edge e_i.

Since the evaluation map of Theorem is additive in the direct sum, I claim that the representation space H is spanned by the set of such colored graphs Γ such that each $V(e_i)$ is simple. In fact, strictly speaking, we can only take advantage of the additivity on a disc, not on an edge e_i, which is a v-based loop. However, we can easily add a coev-labeled vertex to any edge e_i, apply the additivity on one of the two resulting edges (which lies in an embedded disc), and then contract on the other edge to get the decomposition we want.

Since isomorphic colorings give the same evaluation, we can play the same game to see that H is spanned by colored graphs Γ such that each $V(e_i) = \delta_{g_i}$ for some $g_i \in G$. For such Γ, the space of possible v-colors $\mathrm{Hom}(1, \bigotimes_{i=1}^{g} V(e_{2i-1}) \otimes V(e_{2i}) \otimes V(e_{2i-1})^* \otimes V(e_{2i}))^*$ is one-dimensional if $\prod_{i=1}^{g}[g_{2i-1}, g_{2i}] = 1$, and zero-dimensional otherwise.

By using the linearity in the vertex color, we can further restrict to colored graphs Γ such that $\varphi(v)$ is the canonical isomorphism $1 \to \bigotimes_{i=1}^{g} \delta_{g_{2i-1}} \otimes \delta_{g_{2i}} \otimes \delta_{g_{2i-1}}^* \otimes \delta_{g_{2i}}^*$.

Thus, the representation space H has a finite spanning set S consisting of all colored graphs Γ with one vertex v and edges $e_1, \ldots, e_{2g}$ corresponding to the standard generators of $\pi_1(M, v)$ such that each edge label $V(e_i) = \delta_{g_i}$ for some $g_i \in G$ such that $\prod_{i=1}^{g}[g_{2i-1}, g_{2i}] = 1$ and each vertex label is the canonical isomorphism $1 \to \bigotimes_{i=1}^{g} \delta_{g_{2i-1}} \otimes \delta_{g_{2i}} \otimes \delta_{g_{2i-1}}^* \otimes \delta_{g_{2i}}^*$.

The mapping class group of Σ is generated by the Lickorish generating set consisting of Dehn twists around $3g-1$ simple closed curves. These can be divided into two types of twists: the ones around a single hole (the blue and green curves in Figure 4), and the ones connecting two holes (the red curves). The action of a Dehn twist around a simple closed curve corresponds to cutting the manifold along the curve, holding one piece still and twisting the other piece by 2π radians in a clockwise direction, then gluing the two pieces back together.

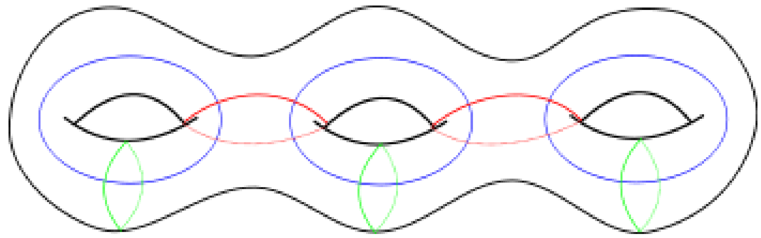

FIGURE 4. Lickorish generating set (Image source: `https://en.wikipedia.org/wiki/Dehn_twist`)

To understand the action of each type of Dehn twist on the representation space H, we will consider the action on the spanning set S. First, I claim that we can apply local moves to any element of S to get a colored graph of the form shown in the first figure of Table 1, where the unshown part of the fundamental polygon looks the same as in the definition of S. Indeed, to pass from an arbitrary element of S, to a colored graph of the form shown in the first figure of 1, we first add coevalution-labeled vertices to each edge intersecting the three shown sides of the fundamental polygon. Then connect the new vertices using 1-labeled edges (this corresponds to applying the second local move in Figure 1 with $k = 0$), contract the connections to get one new vertex, and tensor together the edges connecting the old vertex to the new vertex.

The action of the first type of Dehn twist on an arbitrary element of S is shown in the first two figures of Table 1. After applying the Dehn twist, we have the simple colored graph shown in the second figure of Table 1. We then apply local moves in the remaining figures.

By Proposition 4.2, it follows that the the result of each of this Dehn twist lies in $\mathrm{Im}(\omega)S$. It is a basic result in group cohomology that, by replacing ω with a cohomologous cocycle if necessary, the image of ω lies in $\mu_{|G|}$, the set of $|G|$-th roots of unity. Since cohomologous cocycles give rise to monoidally equivalent categories Vec_G^ω, this replacement does not incur any loss in generality.

An analagous proof works for the second type of Dehn twist shown in Table 2. Thus, the image of any such mapping class group representation is a quotient of the group of permutations of the finite set $\mu_{|G|}S$, hence finite. $\square$

Remark 4.4. *When $\omega = 1$ and Σ is closed, this representation is a permutation representation.*

Proof. Under the assumption that the representation in [8] coincides with ours, this fact follows from Theorem 2.6 in [8]. We can also see this directly by first noting that G acts on S by simultaneously conjugation of all edge labels by a single element $g \in G$. If $s \in S$ and $g \in G$, then we can retrieve s from gs by separating two oppositely oriented, g-labeled edges from each edge in the embedded graph gs. This results in a loop labeled by g, whose evaluation is 1. Thus gs is equivalent to s. Moreover, the cardinality $|S/G| = |\mathrm{Hom}(\pi_1(M), G)|/|G|$ is equal to the dimension of the untwisted Dijkgraaf-Witten representation space H [5]. Hence, S/G is a basis for H. The mapping class group action on S commutes with the G-action, so the mapping class group permutes S/G, i.e. H is a permutation representation. $\square$

4.2. **With Boundary Case.** When Σ has boundary, we denote by $\mathrm{MCG}(\Sigma)$ the group of isotopy classes of homeomorphisms fixing the boundary of Σ setwise. There are mapping class group representations for every labelling of the boundary $l : \pi_0(\partial M) \to \mathrm{Obj}(Z(\mathrm{Vec}_\omega^G))$. Given such a labelling, the representation space is $H(\Sigma, \mathbf{V})$ where the boundary condition $\mathbf{V} = F \circ l$, where F is the forgetful functor $F : Z(\mathrm{Vec}_\omega^G) \to \mathrm{Vec}_\omega^G$. The same local relations are valid in this representation space [9].

By a similar argument as in the proof of the Theorem 4.3, any such representation space has a spanning set S of colored graphs with a single vertex, loops for each of the usual generators of the fundamental group of Σ, and a leg from the vertex to each of the boundary components. Each of the edges is labeled by a simple object, i.e. an element of the group G.

Let N denote the closed surface obtained by filling in all the boundary components of Σ with disks. The mapping class group $\mathrm{MCG}(\Sigma)$ is generated by the same Dehn twists as $\mathrm{MCG}(N)$, as well as braids interchanging boundary components and mapping classes corresponding to dragging a boundary component along a representative of a standard generator of $\pi_1(N)$ [4]. As in the proof of Theorem 4.3, applying any of these generators of $\mathrm{MCG}(\Sigma)$ to a colored graph in S yields an element in $\mathrm{Im}(\omega)S$ (see Tables 3 and 4). Since the braid group is also generated by such braids, we have the following theorem.

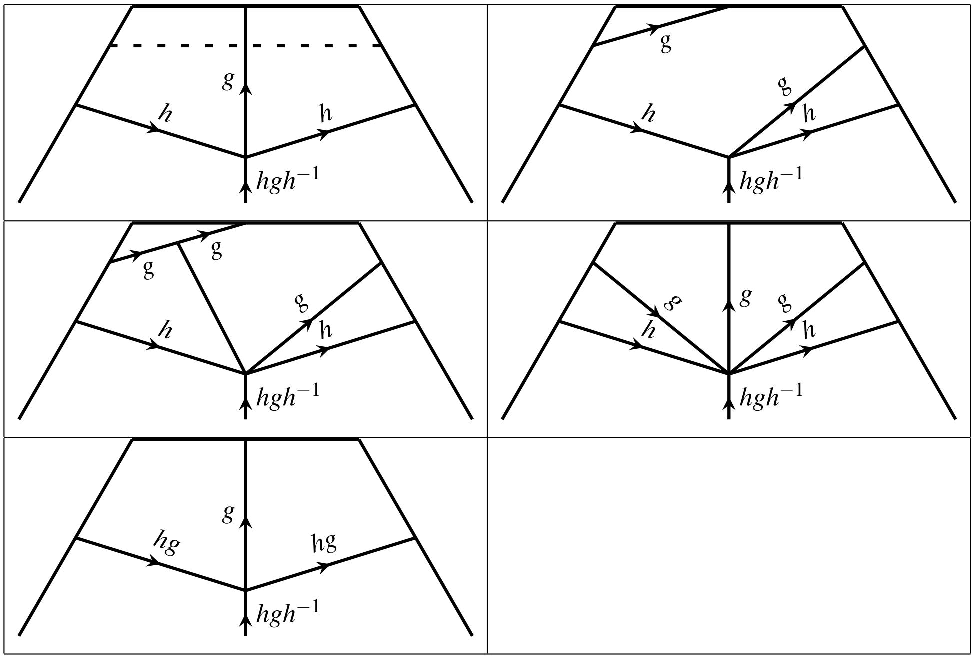

TABLE 1. First type of Dehn twist. Read from left to right, then top to bottom. The Dehn twist is performed along the dashed simple closed curve.

Theorem 4.5. *The image of any twisted Dijkgraaf-Witten representation of a mapping class group of an orientable, compact surface with boundary is finite. In particular, any such braid group representation is also finite.*

5. FURTHER DIRECTIONS

We have proved that every twisted Dijkgraaf-Witten representation of a mapping class group of a compact, orientable surface has finite image. This is a generalization of the results of [7] and [8], as well as another step towards the (modified) Property F conjecture. A potential next step would be to consider more complicated spherical categories than $\mathrm{Vec}_{\omega}^{G}$. One candidate is the class of Tambara-Yamagami categories [14]. The main additional complication here is the appearance of multifusion channels, i.e. the tensor product of two simple objects can be a direct sum of multiple simple objects.

REFERENCES

[1] B. Bakalov and A. Kirillov, Jr., *Lectures on Tensor Categories and Modular Functors*, University Lecture Series, vol. **21**, Amer. Math. Soc., 2001.
[2] P. Bantay, *Algebraic Aspects of Orbifold Models*, Int. J. Mod. Phys. **A9** (1994), 1443–1456.
[3] J. Barrett and B. Westbury. *Invariants of Piecewise-Linear 3-Manifolds*, Trans. Amer. Math. Soc. **348** (1996), 3997–4022.
[4] J. Birman. *Mapping class groups and their relationship to braid groups*, Comm. Pure Appl. Math. **22** (1969) 213–242.
[5] R. Dijkgraaf and E. Witten. *Topological gauge theories and group cohomology*, Comm. Math. Phys. **129** (1990), no. 2, 393–429.
[6] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Mathematical surveys and monographs **205** (2015).
[7] P. Etingof, E. C. Rowell, and S. Witherspoon, *Braid group representations from twisted quantum doubles of finite groups*, Pacific J. Math. **234** (2008), no. 1, 33–42.
[8] J. Fjelstad and J. Fuchs, *Mapping class group representations from Drinfeld doubles of finite groups*, Preprint (2015), arXiv:1506.03263.
[9] A. Kirillov, *String-net model of Turaev-Viro invariants*, Preprint (2011), arXiv:1106.6033.
[10] V. Lyubashenko, *Ribbon Abelian Categories as Modular Categories*, J. Knot Theory Ramifications **5** (1996), no. 3, 311–403.
[11] D. Naidu and E. C. Rowell. *A finiteness property for braided fusion categories*, Algebr. and Represent. Theor. **14** (2011), no. 5, 837–855.
[12] S.-H. Ng and P. Schauenburg, *Congruence Subgroups and Generalized Frobenius-Schur Indicators*, Comm. Math. Phys. **300** (2010), no. 1, 1–46.
[13] S.-H. Ng and P. Schauenburg, *Frobenius-Schur Indicators and Exponents of Spherical Categories*, Adv. Math. **211** (2007), no. 1, 34–71.
[14] D. Tambara and S. Yamagami, *Tensor Categories with Fusion Rules of Self-Duality for Finite Abelian Groups*, J. Algebra **209** (1998), 692–707.
[15] V. G. Turaev and O. Y. Viro. *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology **31** (1992), 865–902.

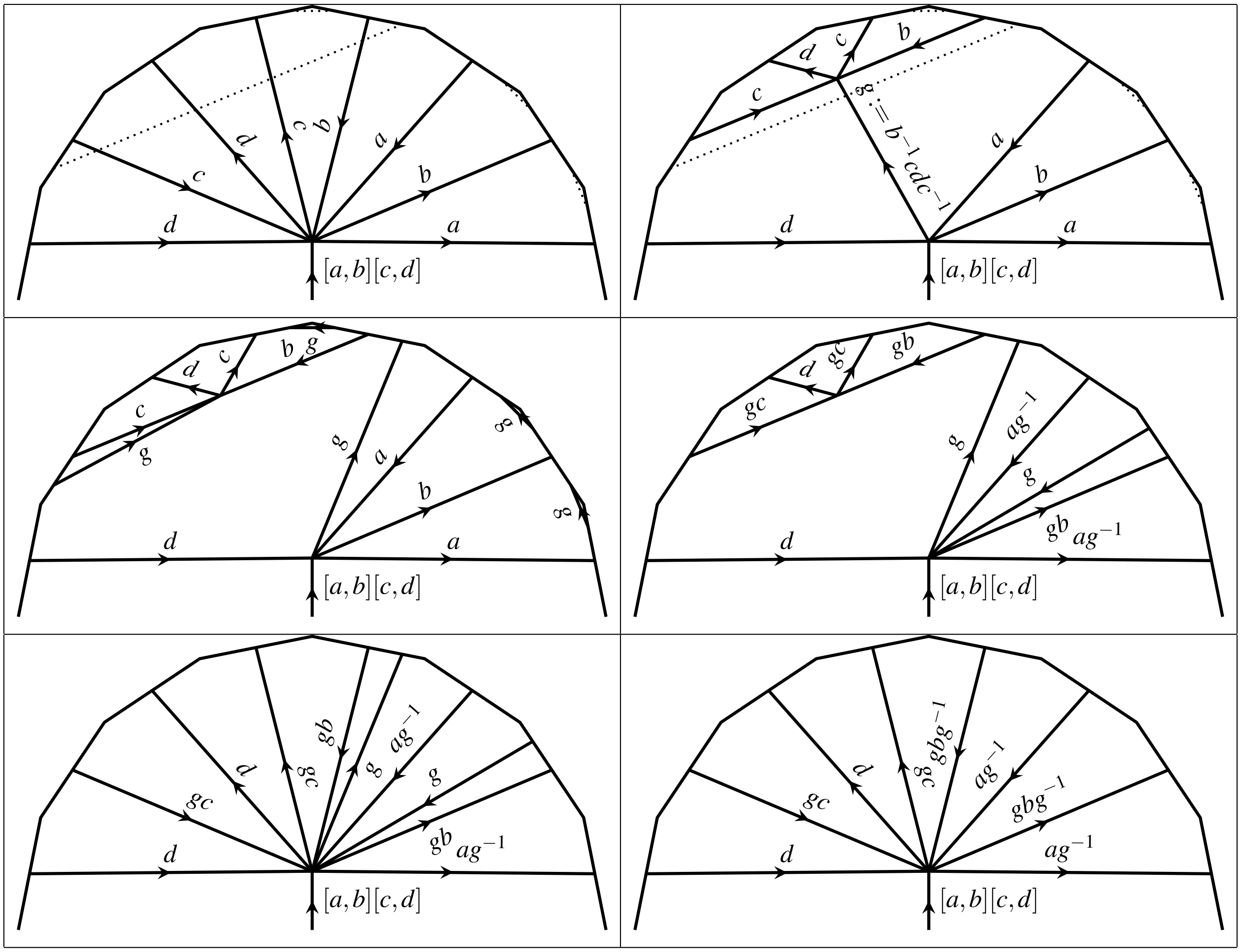

TABLE 2. Second type of Dehn twist. Read from left to right, then top to bottom. The Dehn twist is performed along the dashed simple closed curve.

E-mail address: pgustafs@math.tamu.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX U.S.A.

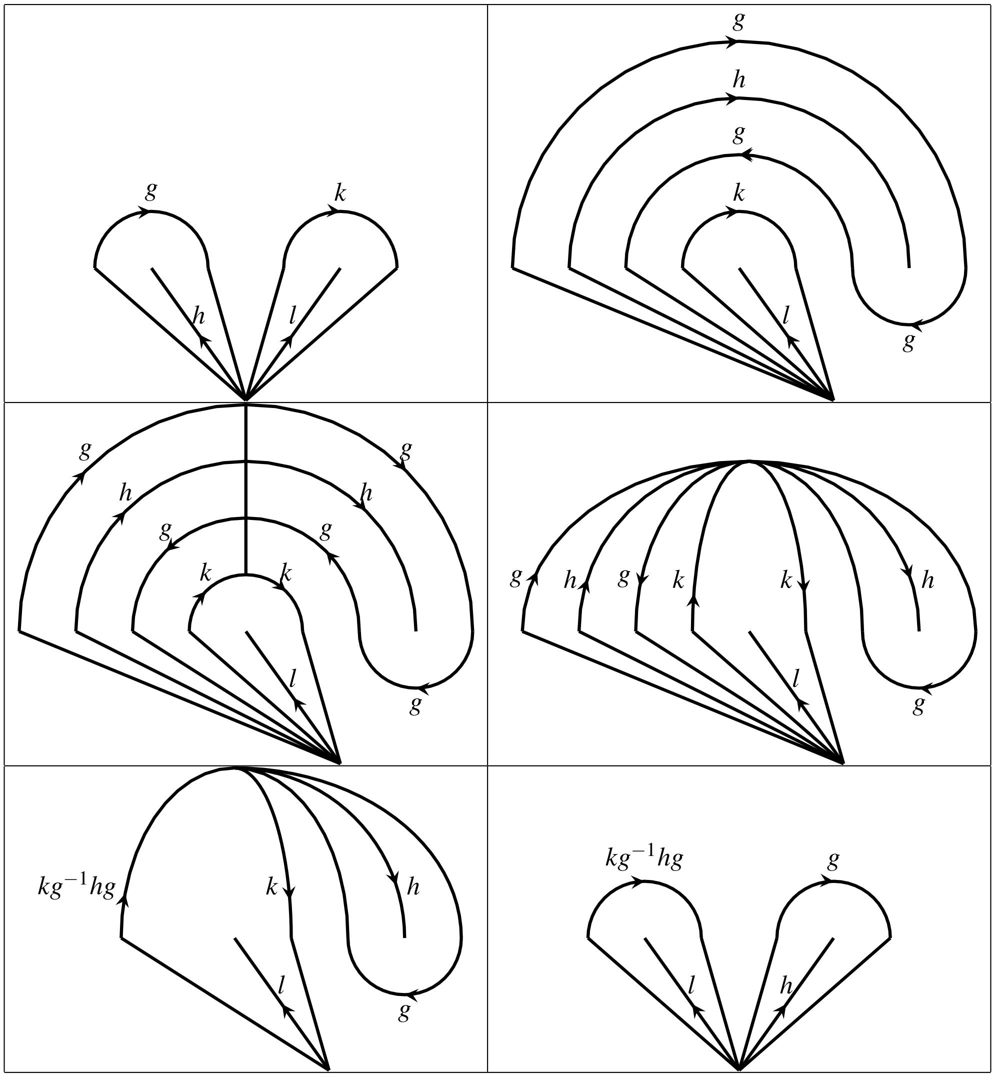

TABLE 3. A braid generator. Read from left to right, then top to bottom. Unlabeled interior edges are colored by the group identity element.

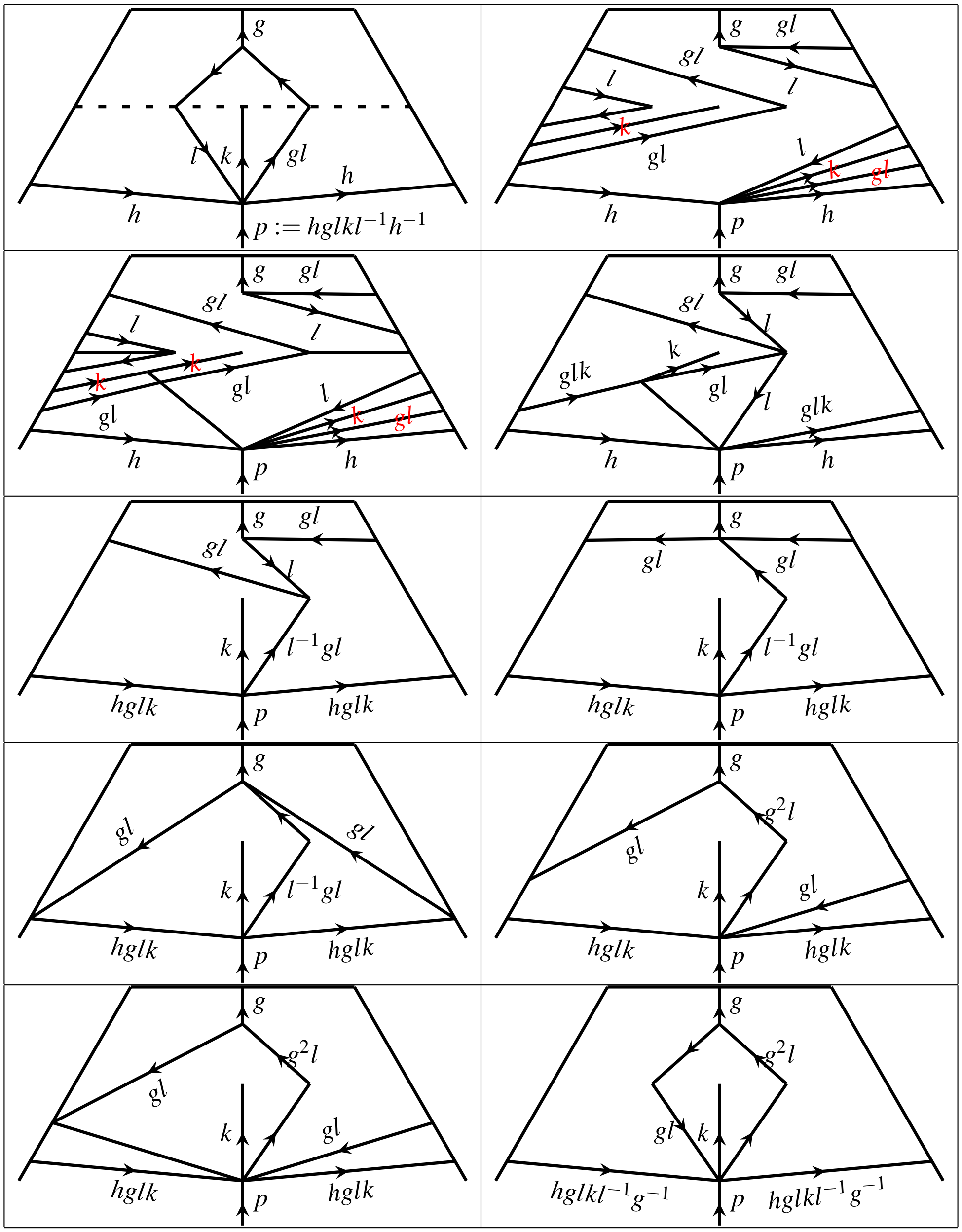

TABLE 4. Pulling a boundary component along a generator for the fundamental group of the corresponding closed surface. Read from left to right, then top to bottom. Unlabeled interior edges are colored by the group identity element.