Problem Set 6
CSCE 440/640

Due dates: Electronic submission of the pdf file of this homework is due on **11/2/2016 before 2:50pm** on ecampus.tamu.edu, a signed paper copy of the pdf file is due on **11/2/2016** at the beginning of class.

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Resources. I used Mathematica to calculate a couple singular value decompositions.

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

Signature: ____________________________

Problem 1. (20 points) Consider the mixed state

$$M = \left\{ \left(|0\rangle, \frac{1}{3} \right), \left(|1\rangle, \frac{2}{3} \right) \right\}.$$

(a) Determine the density matrix ρ of the mixed state M.
(b) Derive a different mixed state M' (which should not consist of computational basis states) that has the same density matrix ρ as M.

[This problem shows that density matrices are not in one-to-one correspondence with mixed states.]

Solution. (a)

$$\begin{aligned} \rho &= \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1| \\ &= \frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \end{aligned}$$

(b) First let's guess that there's a solution with real coefficients, so the derivation is slightly simpler. We're trying to find a mixed state

$$M' = \{(a'|0\rangle + b'|1\rangle, p), (c'|0\rangle + d'|1\rangle, q)\}$$

with density matrix ρ'. Let $a = \sqrt{p}a'$, $b = \sqrt{p}b'$, $c = \sqrt{q}c'$, and $d = \sqrt{q}d'$. Then the condition $\rho = \rho'$ becomes

$$\begin{aligned} a^2 + c^2 &= 1/3 \\ b^2 + d^2 &= 2/3 \\ ab + cd &= 0 \end{aligned}$$

The last equation implies that there exists a k such that $(a, c) = k(b, -d)$. Solving for k,

$$\begin{aligned} 1/3 &= a^2 + c^2 \\ &= k^2(b^2 + d^2) \\ &= k^2(2/3), \end{aligned}$$

so $k = \frac{1}{\sqrt{2}}$. A solution to the first system of equations is $a = c = \frac{1}{\sqrt{6}}$ and $b = -d = \frac{1}{\sqrt{3}}$. This corresponds to the mixed state

$$M' = \left\{ \left(\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle, \frac{1}{2} \right), \left(\frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle, \frac{1}{2} \right) \right\}.$$

Problem 2. (20 points)
(a) Do Exercise 3.5.1 (b) on page 55 of our textbook KLM.
(b) Do Exercise 3.5.1 (c) on page 55 of our textbook KLM.

Solution. (a)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(b)

$$\frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Problem 3. (20 points) Find the Schmidt decomposition of the states
(a) $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.
(b) $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$.
[Students of CSCE 440 only need to solve (a), and students of CSCE 640 should solve both (a) and (b).]

Solution. I used Mathematica to find the SVD of the corresponding matrices.
(a) $(\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle) \otimes (\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle)$
(b)

$$\begin{gathered} 0.934(0.851|0\rangle + 0.526|1\rangle) \otimes (0.526|0\rangle - 0.851|1\rangle) \\ +0.357(0.851|0\rangle + 0.526|1\rangle) \otimes (-0.526|0\rangle + 0.851|1\rangle) \end{gathered}$$

Problem 4. (20 points) Exercise 3.5.4 (a) on page 57 in our textbook KLM.

Solution. For any state of the form $|a\rangle \otimes |b\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$,

$$\begin{aligned} \operatorname{tr}_B((U \otimes I)|a\rangle\langle a| \otimes |b\rangle\langle b|(U^\dagger \otimes I)) &= \operatorname{tr}_B(U|a\rangle\langle a|U^\dagger \otimes |b\rangle\langle b|) \\ &= U|a\rangle\langle a|U^\dagger\langle b|b\rangle \\ &= U|a\rangle\langle a|\langle b|b\rangle U^\dagger \\ &= U \operatorname{tr}_B(|a\rangle\langle a| \otimes |b\rangle\langle b|)U^\dagger) \end{aligned}$$

Extending linearly, the same identity holds for any state.

Problem 5. (20 points) Choi has shown that for all matrices $V_j \in \mathbf{C}^{n\times m}$, the map $T\colon M_n(\mathbf{C}) \to M_m(\mathbf{C})$ given by

$$T(\rho) = \sum_{j=1}^{\ell} V_j^* \rho V_j$$

is completely positive. Show that if the matrices V_j satisfy the condition

$$\sum_{j=1}^{\ell} V_j V_j^* = I,$$

where I denotes the identity matrix, then T is trace preserving, so $\operatorname{tr} T(A) = \operatorname{tr} A$. [Hint: the matrix trace satisfies $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$.]

Solution.

$$\begin{aligned}
\operatorname{tr} T(A) &= \operatorname{tr}\left(\sum_{j=1}^{\ell} V_j^* A V_j\right) \\
&= \sum_{j=1}^{\ell} \operatorname{tr}(V_j^* A V_j) \\
&= \sum_{j=1}^{\ell} \operatorname{tr}(V_j V_j^* A) \\
&= \operatorname{tr}\left(\sum_{j=1}^{\ell} V_j V_j^* A\right) \\
&= \operatorname{tr} A
\end{aligned}$$

Checklist:

☐ Did you add your name?
☐ Did you disclose all resources that you have used?
(This includes all people, books, websites, etc. that you have consulted)
☐ Did you sign that you followed the Aggie honor code?
☐ Did you solve all problems?
☐ Did you submit the pdf file resulting from your latex source file on ecampus?
☐ Did you submit a hardcopy of the pdf file in class?