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HW 1

1 Write the control system on $M = \mathbb{R}^2 \times \mathbb{T}^3$ corresponding to the car with two off-hook trailers system.

Proof. Let $n_i = (\cos\theta_i, \sin\theta_i)$ and $n_i' = (-\sin\theta_i, \cos\theta_i)$ for $0 \le i \le 2$. Then $n_i \cdot n_j = \cos(\theta_i - \theta_j) = n_i' \cdot n_j'$ and $n_i \cdot n_j' = \sin(\theta_i - \theta_j)$.

Let v_2 denote the velocity of the car, and v_i denote the velocity of the $(n-i)$-th trailer. Let $v_{1.5}$ denote the velocity of the first hook, and $v_{0.5}$ denote the velocity of the second hook. Let $\omega_i = \frac{\partial\theta_i}{\partial t}$.

In the case of linear motion of the car, we have $v_2 = vn_2$ and $\omega_2 = 0$. Hence,

$$v_{1.5} = vn_2$$

$$\begin{aligned} v_1 &= (v_{1.5} \cdot n_1)n_1 \\ &= (vn_2 \cdot n_1)n_1 \\ &= \cos(\theta_2 - \theta_1)n_1 \end{aligned}$$

$$\begin{aligned} \omega_1 &= v_{1.5} \cdot n_1' \\ &= vn_2 \cdot n_1' \\ &= v\sin(\theta_2 - \theta_1) \end{aligned}$$

$$\begin{aligned} v_{0.5} &= v_1 - \omega_1 n_1' \\ &= v\cos(\theta_2 - \theta_1)n_1 - v\sin(\theta_2 - \theta_1)n_1' \end{aligned}$$

$$\begin{aligned} \omega_0 &= v_{0.5} \cdot n_0' \\ &= v\cos(\theta_2 - \theta_1)n_1 \cdot n_0' - v\sin(\theta_2 - \theta_1)n_1' \cdot n_0' \\ &= v\cos(\theta_2 - \theta_1)\sin(\theta_1 - \theta_0) - v\sin(\theta_2 - \theta_1)\cos(\theta_1 - \theta_0) \\ &= v\sin((\theta_1 - \theta_0) - (\theta_2 - \theta_1)) \\ &= v\sin(2\theta_1 - \theta_0 - \theta_2). \end{aligned}$$

For the case of the car turning, we have $v_2 = 0$ and $\omega_2 = \omega$. Hence,

$$v_{1.5} = -\omega n_2'$$

$$\begin{aligned} v_1 &= (v_{1.5} \cdot n_1) n_1 \\ &= (-\omega n_2' \cdot n_1) n_1 \\ &= \omega \sin(\theta_2 - \theta_1) n_1 \end{aligned}$$

$$\begin{aligned} \omega_1 &= v_{1.5} \cdot n_1' \\ &= -\omega n_2' \cdot n_1' \\ &= -\omega \cos(\theta_2 - \theta_1) \end{aligned}$$

$$\begin{aligned} v_{0.5} &= v_1 - \omega_1 n_1' \\ &= \omega \sin(\theta_2 - \theta_1) n_1 + \omega \cos(\theta_2 - \theta_1) n_1' \end{aligned}$$

$$\begin{aligned} \omega_0 &= v_{0.5} \cdot n_0' \\ &= \omega \sin(\theta_2 - \theta_1) n_1 \cdot n_0' + \omega \cos(\theta_2 - \theta_1) n_1' \cdot n_0' \\ &= \omega \sin(\theta_2 - \theta_1) \sin(\theta_1 - \theta_0) + \omega \cos(\theta_2 - \theta_1) \cos(\theta_1 - \theta_0) \\ &= \omega \cos(2\theta_1 - \theta_0 - \theta_2) \end{aligned}$$

Hence the control system for M is given by the family of vector fields $\mathcal{F} = \{\pm X_1, \pm X_2\}$, where

$$X_1 = \cos(\theta_2) \frac{\partial}{\partial x} + \sin(\theta_0) \frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + A \frac{\partial}{\partial \theta_0}$$

with $A = \sin(2\theta_1 - \theta_0 - \theta_2)$, and

$$X_2 = \frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + B \frac{\partial}{\partial \theta_0}$$

with $B = \cos(2\theta_1 - \theta_0 - \theta_2)$. □

2 Find all points $q \in M$ such that $\mathcal{F}$ is bracket-generating. At these points, calculate the degree of nonholonomy of $\mathcal{F}$.

Proof. Hence,

$$
\begin{aligned}
[X_1, X_2] = &\left[\cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + A\frac{\partial}{\partial \theta_0},\right. \\
&\quad \left.\frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + B\frac{\partial}{\partial \theta_0}\right] \\
&= \sin(\theta_2 - \theta_1)\left(-\sin(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial B}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + A\frac{\partial B}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
&\quad - \left(-\sin(\theta_2)\frac{\partial}{\partial x} + \cos(\theta_2)\frac{\partial}{\partial y} + \cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) \\
&\quad + \cos(\theta_2 - \theta_1)\left(-\cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) - B\frac{\partial A}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + \\
&\quad \left(-\sin^2(\theta_2 - \theta_1) - \cos(\theta_2 - \theta_1) + \cos^2(\theta_2 - \theta_1)\right)\frac{\partial}{\partial \theta_1} + \\
&\quad + \left(\sin(\theta_2 - \theta_1)\frac{\partial B}{\partial \theta_1} + A\frac{\partial B}{\partial \theta_0} + \frac{\partial A}{\partial \theta_1} + \cos(\theta_2 - \theta_1)\frac{\partial A}{\partial \theta_1} - B\frac{\partial A}{\partial \theta_0}\right)\frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + (\cos(2\theta_2 - 2\theta_1) - \cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} \\
&\quad + \left(\sin(\theta_2 - \theta_1)(-2A) + A^2 + 2B + \cos(\theta_2 - \theta_1)(2B) - B(-B)\right)\frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + (\cos(2\theta_2 - 2\theta_1) - \cos(\theta_2 - \theta_1))\frac{\partial}{\partial \theta_1} \\
&\quad + (2\cos((\theta_2 - \theta_1) + (2\theta_1 - \theta_0 - \theta_2)) + 2B + 1)\frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + C\frac{\partial}{\partial \theta_1} + D\frac{\partial}{\partial \theta_0},
\end{aligned}
$$

where $C = \cos(2\theta_2 - 2\theta_1) - \cos(\theta_2 - \theta_1)$ and $D = 2\cos(\theta_1 - \theta_0) + 2\cos(2\theta_1 - \theta_0 - \theta_2) + 1$.

Then

$$
\begin{aligned}
[X_1, [X_1, X_2]] = \Bigg[&\cos(\theta_2)\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + A\frac{\partial}{\partial \theta_0}, \\
&\sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + C\frac{\partial}{\partial \theta_1} + D\frac{\partial}{\partial \theta_0}\Bigg] \\
= \sin(\theta_2 - \theta_1)&\left(\frac{\partial C}{\partial \theta_1}\frac{\partial}{\partial \theta_1} + \frac{\partial D}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + A\frac{\partial D}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
&- C\left(-\cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + D\frac{\partial A}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
= \Bigg(&\sin(\theta_2 - \theta_1)\frac{\partial C}{\partial \theta_1} + C\cos(\theta_2 - \theta_1)\Bigg)\frac{\partial}{\partial \theta_1} \\
&\left(\sin(\theta_2 - \theta_1)\frac{\partial D}{\partial \theta_1} + A\frac{\partial D}{\partial \theta_0} - C\frac{\partial A}{\partial \theta_1} + D\frac{\partial A}{\partial \theta_0}\right)\frac{\partial}{\partial \theta_0}
\end{aligned}
$$

and

$$
\begin{aligned}
[X_2, [X_1, X_2]] = \Bigg[&\frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + B\frac{\partial}{\partial \theta_0}, \\
&\sin(\theta_2)\frac{\partial}{\partial x} - \cos(\theta_2)\frac{\partial}{\partial y} + C\frac{\partial}{\partial \theta_1} + D\frac{\partial}{\partial \theta_0}\Bigg] \\
= \cos(\theta_2)&\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} + \frac{\partial C}{\partial \theta_2}\frac{\partial}{\partial \theta_1} + \frac{\partial D}{\partial \theta_2}\frac{\partial}{\partial \theta_0} \\
&- \cos(\theta_2 - \theta_1)\left(\frac{\partial C}{\partial \theta_1}\frac{\partial}{\partial \theta_1} + \frac{\partial D}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) + B\frac{\partial D}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
&- C\left(-\sin(\theta_2 - \theta_1)\frac{\partial}{\partial \theta_1} + \frac{\partial B}{\partial \theta_1}\frac{\partial}{\partial \theta_0}\right) - D\frac{\partial B}{\partial \theta_0}\frac{\partial}{\partial \theta_0} \\
= \cos(\theta_2)&\frac{\partial}{\partial x} + \sin(\theta_2)\frac{\partial}{\partial y} \\
&+ \left(\frac{\partial C}{\partial \theta_2} - \cos(\theta_2 - \theta_1)\frac{\partial C}{\partial \theta_1} + C\sin(\theta_2 - \theta_1)\right)\frac{\partial}{\partial \theta_1} \\
&+ \left(\frac{\partial D}{\partial \theta_2} - \cos(\theta_2 - \theta_1)\frac{\partial D}{\partial \theta_1} + B\frac{\partial D}{\partial \theta_0} - C\frac{\partial B}{\partial \theta_1} - D\frac{\partial B}{\partial \theta_0}\right)\frac{\partial}{\partial \theta_0}
\end{aligned}
$$

□

3 Let $\widetilde{M}$ denote the set of bracket-generating points of $\mathcal{F}$. Prove that the system is controllable on $\widetilde{M}$.