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Math 643 - Algebraic Topology I

HW 6

4.9

(i) Using the explicit formula for β_{n+1}, show that

$$\partial_{n+1}\beta_{n+1} = (\lambda_1^\Delta{}_\# - \lambda_0^\Delta{}_\# - P_{n-1}^\Delta \partial_n)(\delta)$$

for $n = 0$ and $n = 1$.

Proof. For $n = 0$, we have

$$\begin{aligned}\partial_1\beta_1 &= \partial([a_0, b_0]) \\ &= [b_0] - [a_0] \\ &= (\lambda_1^\Delta{}_\# - \lambda_0^\Delta{}_\# - P_{-1}^\Delta \partial_1)(\delta)\end{aligned}$$

For $n = 1$, we have

$$\begin{aligned}\partial_2\beta_2 &= \partial([a_0, b_0, b_1] - [a_0, a_1, b_1]) \\ &= [b_0, b_1] - [a_0, b_1] + [a_0, b_0] - [a_1, b_1] + [a_0, b_1] - [a_0, a_1] \\ &= [b_0, b_1] - [a_0, a_1] + [a_0, b_0] - [a_1, b_1] \\ &= [b_0, b_1] - [a_0, a_1] - (\partial_2\delta \times 1)_\#\beta_1 \\ &= (\lambda_1^\Delta{}_\# - \lambda_0^\Delta{}_\# - P_0^\Delta \partial_2)(\delta)\end{aligned}$$

□

(ii) Give an explicit formula for $P_1^X(\sigma)$, where $\sigma : \Delta^1 \to X$ is a 1-simplex.

Proof. We have

$$P_1^X(\sigma) = (\sigma \times 1)_\#(\beta_2) = (\sigma \times 1)_\#([a_0, b_0, b_1] - [a_0, a_1, b_1])$$

□

4.10 Prove that P_n is natural.

Proof. The diagram holds when f is a simplex. Extend by linearity. □

4.11 If X is a deformation retract of Y, then $H_n(X) \simeq H_n(Y)$ for all $n \geq 0$. In fact, if $i : X \to Y$ is the inclusion, then $H_n(i)$ is an isomorphism.

Proof. Since X is a deformation retract of Y, $i \simeq 1_Y$. Hence $H_n(i) = H_n(1_Y) = 1$ is an isomorphism. In particular $H_n(X) \simeq H_n(Y)$ for all $n \geq 0$. □

4.13 Prove that the Hurewicz map ϕ is natural.

Proof. We have

$$\begin{aligned}\phi h_*[f] &= \phi[hf]\\ &= \operatorname{cls}(hf\eta)\\ &= h_* \operatorname{cls}(f\eta)\\ &= h_*\phi([f])\end{aligned}$$

□

4.16 If $f : S^1 \to S^1$ is continuous, define the degree of f to be m if the induced map $f_* : H_1(S^1) \to H_1(S^1)$ is multiplication by m. Show that this definition of degree coincides with the degree of a pointed map $(S^1, 1) \to (S^1, 1)$ defined in terms of $\pi_1(S_1, 1)$.

Proof. Let $f : (S^1, 1) \to (S^1, 1)$ be a pointed map with degree m with respect to the new definition. Let $i : (S^1, 1) \to (S^1, 1)$ be the identity map.

By Theorem 4.29, the Hurewicz map is an isomorphism $\phi : \pi_1(S^1, 1) \simeq H_1(S^1)$. Hence $\pi_1(f) = \phi^{-1}H_1(f) = \phi^{-1}H_1(f \circ i) = \phi^{-1}(mH_1(i)) = (\pi_1(i))^m$. Thus, the degree of f is m with respect to the old definition also. □