

Paul Gustafson
Texas A&M University - Math 666
Instructor: Igor Zelenko

HW 1

1 Write the control system on $M = \mathbb{R}^2 \times \mathbb{T}^3$ corresponding to the car with two off-hook trailers system.

Proof. Let $n_i = (\cos \theta_i, \sin \theta_i)$ and $n'_i = (-\sin \theta_i, \cos \theta_i)$ for $0 \leq i \leq 2$. Then $n_i \cdot n_j = \cos(\theta_i - \theta_j) = n'_i \cdot n'_j$ and $n_i \cdot n'_j = \sin(\theta_i - \theta_j)$.

Let v_2 denote the velocity of the car, and v_i denote the velocity of the $(n-i)$ -th trailer. Let $v_{1.5}$ denote the velocity of the first hook, and $v_{0.5}$ denote the velocity of the second hook. Let $\omega_i = \frac{\partial \theta_i}{\partial t}$.

In the case of linear motion of the car, we have $v_2 = vn_2$ and $\omega_2 = 0$. Hence,

$$v_{1.5} = vn_2$$

$$\begin{aligned} v_1 &= (v_{1.5} \cdot n_1)n_1 \\ &= (vn_2 \cdot n_1)n_1 \\ &= v \cos(\theta_2 - \theta_1)n_1 \end{aligned}$$

$$\begin{aligned} \omega_1 &= v_{1.5} \cdot n'_1 \\ &= vn_2 \cdot n'_1 \\ &= v \sin(\theta_2 - \theta_1) \end{aligned}$$

$$\begin{aligned} v_{0.5} &= v_1 - \omega_1 n'_1 \\ &= v \cos(\theta_2 - \theta_1)n_1 - v \sin(\theta_2 - \theta_1)n'_1 \end{aligned}$$

$$\begin{aligned} \omega_0 &= v_{0.5} \cdot n'_0 \\ &= v \cos(\theta_2 - \theta_1)n_1 \cdot n'_0 - v \sin(\theta_2 - \theta_1)n'_1 \cdot n'_0 \\ &= v \cos(\theta_2 - \theta_1) \sin(\theta_1 - \theta_0) - v \sin(\theta_2 - \theta_1) \cos(\theta_1 - \theta_0) \\ &= v \sin((\theta_1 - \theta_0) - (\theta_2 - \theta_1)) \\ &= v \sin(2\theta_1 - \theta_0 - \theta_2). \end{aligned}$$

For the case of the car turning, we have $v_2 = 0$ and $\omega_2 = \omega$. Hence,

$$v_{1.5} = -\omega n_2'$$

$$\begin{aligned} v_1 &= (v_{1.5} \cdot n_1) n_1 \\ &= (-\omega n_2' \cdot n_1) n_1 \\ &= \omega \sin(\theta_2 - \theta_1) n_1 \end{aligned}$$

$$\begin{aligned} \omega_1 &= v_{1.5} \cdot n_1' \\ &= -\omega n_2' \cdot n_1' \\ &= -\omega \cos(\theta_2 - \theta_1) \end{aligned}$$

$$\begin{aligned} v_{0.5} &= v_1 - \omega_1 n_1' \\ &= \omega \sin(\theta_2 - \theta_1) n_1 + \omega \cos(\theta_2 - \theta_1) n_1' \end{aligned}$$

$$\begin{aligned} \omega_0 &= v_{0.5} \cdot n_0' \\ &= \omega \sin(\theta_2 - \theta_1) n_1 \cdot n_0' + \omega \cos(\theta_2 - \theta_1) n_1' \cdot n_0' \\ &= \omega \sin(\theta_2 - \theta_1) \sin(\theta_1 - \theta_0) + \omega \cos(\theta_2 - \theta_1) \cos(\theta_1 - \theta_0) \\ &= \omega \cos(2\theta_1 - \theta_0 - \theta_2) \end{aligned}$$

Hence the control system for M is given by the family of vector fields $\mathcal{F} = \{\pm X_1, \pm X_2\}$, where

$$X_1 = \cos(\theta_2) \frac{\partial}{\partial x} + \sin(\theta_2) \frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + A \frac{\partial}{\partial \theta_0}$$

with $A = \sin(2\theta_1 - \theta_0 - \theta_2)$, and

$$X_2 = \frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + B \frac{\partial}{\partial \theta_0}$$

with $B = \cos(2\theta_1 - \theta_0 - \theta_2)$. □

2 Find all points $q \in M$ such that \mathcal{F} is bracket-generating. At these points, calculate the degree of nonholonomy of \mathcal{F} .

Proof. Hence,

$$\begin{aligned}
[X_1, X_2] &= \left[\cos(\theta_2) \frac{\partial}{\partial x} + \sin(\theta_2) \frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + A \frac{\partial}{\partial \theta_0}, \right. \\
&\quad \left. \frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + B \frac{\partial}{\partial \theta_0} \right] \\
&= \sin(\theta_2 - \theta_1) \left(-\sin(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + \frac{\partial B}{\partial \theta_1} \frac{\partial}{\partial \theta_0} \right) + A \frac{\partial B}{\partial \theta_0} \frac{\partial}{\partial \theta_0} \\
&\quad - \left(-\sin(\theta_2) \frac{\partial}{\partial x} + \cos(\theta_2) \frac{\partial}{\partial y} + \cos(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_2} \frac{\partial}{\partial \theta_0} \right) \\
&\quad + \cos(\theta_2 - \theta_1) \left(-\cos(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_1} \frac{\partial}{\partial \theta_0} \right) - B \frac{\partial A}{\partial \theta_0} \frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2) \frac{\partial}{\partial x} - \cos(\theta_2) \frac{\partial}{\partial y} \\
&\quad + \left(-\sin^2(\theta_2 - \theta_1) - \cos(\theta_2 - \theta_1) - \cos^2(\theta_2 - \theta_1) \right) \frac{\partial}{\partial \theta_1} \\
&\quad + \left(\sin(\theta_2 - \theta_1) \frac{\partial B}{\partial \theta_1} + A \frac{\partial B}{\partial \theta_0} - \frac{\partial A}{\partial \theta_2} + \cos(\theta_2 - \theta_1) \frac{\partial A}{\partial \theta_1} - B \frac{\partial A}{\partial \theta_0} \right) \frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2) \frac{\partial}{\partial x} - \cos(\theta_2) \frac{\partial}{\partial y} + (-1 - \cos(\theta_2 - \theta_1)) \frac{\partial}{\partial \theta_1} \\
&\quad + \left(\sin(\theta_2 - \theta_1)(-2A) + A^2 + B + \cos(\theta_2 - \theta_1)(2B) - B(-B) \right) \frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2) \frac{\partial}{\partial x} - \cos(\theta_2) \frac{\partial}{\partial y} + (-1 - \cos(\theta_2 - \theta_1)) \frac{\partial}{\partial \theta_1} \\
&\quad + (2 \cos((\theta_2 - \theta_1) + (2\theta_1 - \theta_0 - \theta_2)) + B + 1) \frac{\partial}{\partial \theta_0} \\
&= \sin(\theta_2) \frac{\partial}{\partial x} - \cos(\theta_2) \frac{\partial}{\partial y} + C \frac{\partial}{\partial \theta_1} + D \frac{\partial}{\partial \theta_0},
\end{aligned}$$

where $C = -1 - \cos(\theta_2 - \theta_1)$ and $D = 2 \cos(\theta_1 - \theta_0) + \cos(2\theta_1 - \theta_0 - \theta_2) + 1$.

Hence

$$\begin{aligned}
\frac{\partial D}{\partial \theta_2} &= \sin(2\theta_1 - \theta_0 - \theta_2) \\
\frac{\partial D}{\partial \theta_1} &= -2 \sin(\theta_1 - \theta_0) - 2 \sin(2\theta_1 - \theta_0 - \theta_2) \\
\frac{\partial D}{\partial \theta_0} &= 2 \sin(\theta_1 - \theta_0) + \sin(2\theta_1 - \theta_0 - \theta_2)
\end{aligned}$$

Then

$$\begin{aligned}
[X_1, [X_1, X_2]] &= \left[\cos(\theta_2) \frac{\partial}{\partial x} + \sin(\theta_2) \frac{\partial}{\partial y} + \sin(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + A \frac{\partial}{\partial \theta_0}, \right. \\
&\quad \left. \sin(\theta_2) \frac{\partial}{\partial x} - \cos(\theta_2) \frac{\partial}{\partial y} + C \frac{\partial}{\partial \theta_1} + D \frac{\partial}{\partial \theta_0} \right] \\
&= \sin(\theta_2 - \theta_1) \left(\frac{\partial C}{\partial \theta_1} \frac{\partial}{\partial \theta_1} + \frac{\partial D}{\partial \theta_1} \frac{\partial}{\partial \theta_0} \right) + A \frac{\partial D}{\partial \theta_0} \frac{\partial}{\partial \theta_0} \\
&\quad - C \left(-\cos(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + \frac{\partial A}{\partial \theta_1} \frac{\partial}{\partial \theta_0} \right) + D \frac{\partial A}{\partial \theta_0} \frac{\partial}{\partial \theta_0} \\
&= \left(\sin(\theta_2 - \theta_1) \frac{\partial C}{\partial \theta_1} + C \cos(\theta_2 - \theta_1) \right) \frac{\partial}{\partial \theta_1} \\
&\quad \left(\sin(\theta_2 - \theta_1) \frac{\partial D}{\partial \theta_1} + A \frac{\partial D}{\partial \theta_0} - C \frac{\partial A}{\partial \theta_1} + D \frac{\partial A}{\partial \theta_0} \right) \frac{\partial}{\partial \theta_0} \\
&= \left(\sin(\theta_2 - \theta_1) \frac{\partial C}{\partial \theta_1} + C \cos(\theta_2 - \theta_1) \right) \frac{\partial}{\partial \theta_1} \\
&\quad \left(\sin(\theta_2 - \theta_1) \frac{\partial D}{\partial \theta_1} + A \frac{\partial D}{\partial \theta_0} - C(2B) + D(-B) \right) \frac{\partial}{\partial \theta_0}
\end{aligned}$$

and

$$\begin{aligned}
[X_2, [X_1, X_2]] &= \left[\frac{\partial}{\partial \theta_2} - \cos(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + B \frac{\partial}{\partial \theta_0}, \right. \\
&\quad \left. \sin(\theta_2) \frac{\partial}{\partial x} - \cos(\theta_2) \frac{\partial}{\partial y} + C \frac{\partial}{\partial \theta_1} + D \frac{\partial}{\partial \theta_0} \right] \\
&= \cos(\theta_2) \frac{\partial}{\partial x} + \sin(\theta_2) \frac{\partial}{\partial y} + \frac{\partial C}{\partial \theta_2} \frac{\partial}{\partial \theta_1} + \frac{\partial D}{\partial \theta_2} \frac{\partial}{\partial \theta_0} \\
&\quad - \cos(\theta_2 - \theta_1) \left(\frac{\partial C}{\partial \theta_1} \frac{\partial}{\partial \theta_1} + \frac{\partial D}{\partial \theta_1} \frac{\partial}{\partial \theta_0} \right) + B \frac{\partial D}{\partial \theta_0} \frac{\partial}{\partial \theta_0} \\
&\quad - C \left(-\sin(\theta_2 - \theta_1) \frac{\partial}{\partial \theta_1} + \frac{\partial B}{\partial \theta_1} \frac{\partial}{\partial \theta_0} \right) - D \frac{\partial B}{\partial \theta_0} \frac{\partial}{\partial \theta_0} \\
&= \cos(\theta_2) \frac{\partial}{\partial x} + \sin(\theta_2) \frac{\partial}{\partial y} \\
&\quad + \left(\frac{\partial C}{\partial \theta_2} - \cos(\theta_2 - \theta_1) \frac{\partial C}{\partial \theta_1} + C \sin(\theta_2 - \theta_1) \right) \frac{\partial}{\partial \theta_1} \\
&\quad + \left(\frac{\partial D}{\partial \theta_2} - \cos(\theta_2 - \theta_1) \frac{\partial D}{\partial \theta_1} + B \frac{\partial D}{\partial \theta_0} - C \frac{\partial B}{\partial \theta_1} - D \frac{\partial B}{\partial \theta_0} \right) \frac{\partial}{\partial \theta_0} \\
&= \cos(\theta_2) \frac{\partial}{\partial x} + \sin(\theta_2) \frac{\partial}{\partial y} \\
&\quad + \left(\frac{\partial C}{\partial \theta_2} - \cos(\theta_2 - \theta_1) \frac{\partial C}{\partial \theta_1} + C \sin(\theta_2 - \theta_1) \right) \frac{\partial}{\partial \theta_1} \\
&\quad + \left(\frac{\partial D}{\partial \theta_2} - \cos(\theta_2 - \theta_1) \frac{\partial D}{\partial \theta_1} + B \frac{\partial D}{\partial \theta_0} - C(-2A) - DA \right) \frac{\partial}{\partial \theta_0}.
\end{aligned}$$

Letting T be the matrix with rows $X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]], [X_2, [X_1, X_2]]$, using MATLAB we find that $\det(T) = \sin(\theta_2 - \theta_1) - \sin(\theta_1 - \theta_0) + \sin(\theta_2 - 2\theta_1 + \theta_0)$.

If $\det(T) \neq 0$, then $Lie_q^3 = T_q M$, and the degree of nonholonomy at q is 3.

On the other hand, if $\det(T) = 0$ then let $\alpha = \theta_2 - \theta_1$ and $\beta = \theta_1 - \theta_0$. Then we have $0 = \det(T) = \sin(\alpha) - \sin(\beta) + \sin(\alpha - \beta) = \sin(\alpha) - \sin(\beta) + \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha) = \sin(\alpha)(1 + \cos(\beta)) - \sin(\beta)(1 + \cos(\alpha))$. If either $\sin(\alpha) = 0$ or $\sin(\beta) = 0$, then $(\alpha, \beta) \in \{(0, 0)\} \cup (\{\pi\} \times S^1) \cup (S^1 \times \pi)$.

Otherwise, we have $\frac{1+\cos(\beta)}{\sin(\beta)} = \frac{1+\cos(\alpha)}{\sin(\alpha)}$. Let $f : (0, 2\pi) \rightarrow \mathbb{R}$ be defined by $f(\pi) = 0$ and $f(x) = \frac{1+\cos(x)}{\sin(x)}$ otherwise. Then $f(\alpha) = f(\beta)$ if we identify $[0, 2\pi)$ with S^1 . Note that f is continuous, and $f'(x) = -1 - \frac{(1+\cos(x))\cos(x)}{\sin^2(x)} = -1 - \frac{\cos(x)}{1-\cos(x)} < 0$ for all x . Hence f is monotone decreasing. Thus, $\alpha = \beta$.

Thus, the points q such that $Lie_q^3 \neq T_q M$ are those points such that $\alpha = \pi$, or $\beta = \pi$, or $\beta - \alpha = 0$. In the original variables, this means $\theta_2 - \theta_1 = \pi$, or $\theta_1 - \theta_0 = 0$, or $2\theta_1 - \theta_0 - \theta_2 = 0$.

□

3 Let \widetilde{M} denote the set of bracket-generating points of \mathcal{F} . Prove that the system is controllable on \widetilde{M} .