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## HW 8

**1** Let $f(x) = x^2$ for $-1 \le x \le 2$. Find two simple functions $s_1 \le f \le s_2$ and

$$\int_{-1}^{2} s_2(x)dx - \int_{-1}^{2} s_1(x)dx < 0.01.$$

How well do these integrals compare with $\int_{-1}^{2} f(x)dx$?

*Proof.* Let $t_j = -1 + 0.001j$ for $0 \le j \le 3000$. Let

$$s_1(x) = 4\chi_{\{2\}} + \sum_{j=0}^{2999} \left( \inf_{x \in [t_j, t_{j+1})} f(x) \right) \chi_{[t_j, t_{j+1})}$$

and

$$s_2(x) = 4\chi_{\{2\}} + \sum_{j=0}^{2999} \left( \sup_{x \in [t_j, t_{j+1})} f(x) \right) \chi_{[t_j, t_{j+1})}.$$

Then $s_1 \le f \le s_2$. We have

$$\begin{aligned}
\int_{-1}^{2} s_1 dx &= \sum_{j=0}^{999} (0.001) f(t_{j+1}) + \sum_{j=1000}^{2999} (0.001) f(t_j) \\
&= \sum_{j=1}^{1000} (0.001)(-1 + 0.001j)^2 + \sum_{j=1000}^{2999} (0.001)(-1 + 0.001j)^2 \\
&= 2.9975
\end{aligned}$$

and

$$\begin{aligned}
\int_{-1}^{2} s_2 dx &= \sum_{j=0}^{999} (0.001) f(t_j) + \sum_{j=1000}^{2999} (0.001) f(t_{j+1}) \\
&= \sum_{j=0}^{999} (0.001)(-1 + 0.001j)^2 + \sum_{j=1001}^{3000} (0.001)(-1 + 0.001j)^2 \\
&= 3.0025,
\end{aligned}$$

so $\int_{-1}^{2} s_2(x)dx - \int_{-1}^{2} s_1(x)dx = 0.005$.

We also have $\int_{-1}^{2} f(x)dx = \int_{-1}^{2} x^2 dx = \left[ \frac{x^3}{3} \right]_{-1}^{2} = \frac{8}{3} + \frac{1}{3} = 3.$ □

**2** Let $F(s) = \int_0^\infty e^{-st} f(t)dt$ be the Laplace transform of $f \in L^1([0, \infty))$. Use the DCT to show that $F$ is continuous from the right as $s = 0$.

*Proof.* Note that for $s, t > 0$ we have $|e^{-st} f(t)| \leq |f(t)|$. Hence, by the DCT, $\lim_{s\to 0^+} F(s) = \lim_{s\to 0^+} \int_0^\infty e^{-st} f(t)dt = \int_0^\infty \lim_{s\to 0^+} e^{-st} f(t)dt = \int_0^\infty f(t)dt = F(0)$. $\square$

**3** Let $f_n = n^{3/2} x e^{-nx}$, where $x \in [0,1]$ and $n = 1, 2, 3, \ldots$

a. Verify that the pointwise limit of $f_n$ is $f = 0$.

b. Show that $\|f_n\|_{C[0,1]} \to \infty$ as $n \to \infty$, so that $f_n$ does not converge uniformly to 0.

c. Find a constant $C$ such that for all $n$ and $x$ fixed $f_n(x) \leq Cx^{-1/2}, x \in (0, 1]$.

d. Use the DCT to show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0.$$

*Proof.* For (a), note that $f_n(0) = 0$ for all $n$. For fixed $x > 0$, we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} xe^{-nx+(3/2)\log n} = x \lim_{u\to -\infty} e^u = 0$.

For (b), we have $\sup_{x\in[0,1]} n^{3/2} xe^{-nx} = n^{1/2} \sup_{u\in[0,n]} ue^{-u} \leq n^{1/2} \sup_{u\in[0,\infty]} ue^{-u} \to \infty$ as $n \to \infty$.

For (c), for $x > 0$ we have $\frac{f_n(x)}{x^{-1/2}} = (nx)^{3/2} e^{-nx} \leq \sup_{u\in[0,\infty]} ue^{-u}$.

For (d), since $x^{-1/2} \in L_1(0, 1)$, part (c) and the DCT imply

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx = 0.$$

$\square$

**4** Let $L$ be a bounded linear operator on Hilbert space $\mathcal{H}$. Show that the two formulas for $\|L\|$ are equivalent:

i. $\|L\| = \sup\{\|Lu\| : u \in \mathcal{H}, \|u\| = 1\}$

ii. $\|L\| = \sup\{|\langle Lu, v\rangle| : u, v \in \mathcal{H}, \|u\| = \|v\| = 1\}$

*Proof.* Fix $u \in \mathcal{H}$ with $\|u\| = 1$. If $Lu = 0$, then $\|Lu\| = 0 = \langle Lu, v\rangle$ for all $v$. If $Lu \neq 0$, we have $\|Lu\| = \langle Lu, \frac{Lu}{\|Lu\|}\rangle$. Hence, in either case $\|Lu\| \leq |\langle Lu, v\rangle|$ for some $\|v\| = 1$. Hence, $\sup\{\|Lu\| : u \in \mathcal{H}, \|u\| = 1\} \leq \sup\{|\langle Lu, v\rangle| : u, v \in \mathcal{H}, \|u\| = \|v\| = 1\}$.

On the other hand, $|\langle Lu, v\rangle| \leq \|Lu\|$ for all $\|v\| = 1$ by Cauchy-Schwartz. Thus, $\sup\{|\langle Lu, v\rangle| : u, v \in \mathcal{H}, \|u\| = \|v\| = 1\} = \sup\{\|Lu\| : u \in \mathcal{H}, \|u\| = 1\}$. $\square$

**5** Let $V$ be a Banach space and let $L : V \to V$ be linear. Show that $L$ is bounded iff $L$ is continuous.

*Proof.* Suppose $L$ is bounded. Let $\epsilon > 0$. If $\|w - v\| < \epsilon/\|L\|$ then $\|Lw - Lv\| \leq \|L\|\|w - v\| < \epsilon$.

Suppose $L$ is continuous. Pick $\delta > 0$ such that $\|Lv\| < 1$ for all $\|v\| \leq \delta$. Then for all $\|w\| \leq 1$, we have $\|Lw\| = \delta^{-1}\|L(\delta w)\| < \delta^{-1}$. $\square$

**6** Consider the BVP $-u''(x) = f(x)$ for $0 \leq x \leq 1, f \in C[0,1], u(0) = 0$ and $u'(1) = 0$.

a. Verify that the solution is given by $u(x) = \int_0^1 k(x,y)f(y)dy$, where

$$k(x,y) = \begin{cases} y, & 0 \leq y \leq x \\ x, & x \leq y \leq 1 \end{cases}$$

b. Let $L$ be the integral operator $Lf = \int_0^1 k(x,y)f(y)dy$. Show that $L : C[0,1] \to C[0,1]$ is bounded and that the norm $\|L\|_{C[0,1]\to C[0,1]} \leq 1$. Try to show that $\|L\|_{C[0,1]\to C[0,1]} = 1/2$.

c. Show that $k(x,y)$ is a Hilbert-Schmidt kernel and that $\|L\|_{L^2\to L^2} \leq \sqrt{\frac{3}{20}}$.

*Proof.* To see that $u(x) = \int_0^1 k(x,y)f(y)dy$ is a solution of the BVP, first note that $\left|\frac{k(x+h,y)-k(x,y)}{h}f(y)\right| \leq \frac{k(x+h,y)-k(x,y)}{h}\|f\|_\infty \leq \|f\|_\infty$ for all $h$ for which the quotient is defined. The last inequality follows from case analysis on $k$ (one can consider all slopes of secant lines of $k(x,y)$ for any fixed $y$).

Hence, by the DCT we have $u' = \int_0^1 \frac{\partial k}{\partial x} f(y)dy = \int_x^1 f(y)dy$. Thus, by the Fundamental Theorem of Calculus, $u'' = -f(y)$.

For uniqueness, suppose $v$ satisfies the BVP. Then $(u-v)'' = 0$ and $(u-v)(0) = 0$ and $(u-v)'(1) = 0$. Thus, $u = v$.

For (b), for $x \in [0,1]$ we have $|Lf(x)| \leq \int_0^1 |k(x,y)||f(y)|dy \leq \int_0^1 k(x,y)\|f(y)\|_{C[0,1]}dy = \left(\frac{x^2}{2} + x(1-x)\right)\|f(y)\|_{C[0,1]} = (x - x^2/2)\|f(y)\|_{C[0,1]}$. Thus $L$ is bounded and of norm no greater than $\sup_{x\in[0,1]} x - x^2/2 = 1/2$. Moreover, this bound is attained if $f$ is a constant function. Hence $\|L\|_{C[0,1]\to C[0,1]} = 1/2$.

For (c), $k(x,y)$ is bounded, so it must have finite $L^2([0,1]^2)$-norm. Hence $k(x,y)$ is a Hilbert-Schmidt kernel.

Moreover,

$$
\begin{aligned}
\|Lu\|_2^2 &= \int_0^1 \left| \int_0^1 k(x,y)f(y)dy \right|^2 dx \\
&\leq \int_0^1 \left( \int_0^1 k(x,y)|f(y)|dy \right)^2 dx \\
&= \int_0^1 \left( \int_0^x y|f(y)|dy + \int_x^1 x|f(y)|dy \right)^2 dx \\
&\leq \int_0^1 \left( \|f\|_2 \int_0^x y^2 dy + \|f\|_2 \int_x^1 x^2 dy \right)^2 dx \\
&= \|f\|_2^2 \int_0^1 \left( \frac{1}{3}x^3 + x^2 - x^3 \right)^2 dx \\
&= \|f\|_2^2 \int_0^1 \left( -\frac{2}{3}x^3 + x^2 \right)^2 dx \\
&= \|f\|_2^2 \int_0^1 \frac{4}{9}x^6 - \frac{4}{3}x^5 + x^4 \, dx \\
&= \|f\|_2^2 \left( \frac{4}{63} - \frac{4}{18} + \frac{1}{5} \right) \\
&= \frac{13}{315} \|f\|_2^2
\end{aligned}
$$

Hence $\|L\|_{L^2 \to L^2} \leq \sqrt{\frac{13}{315}} \leq \sqrt{\frac{3}{20}}$. $\square$