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MATH 663 - Subfactors, Knots, and Planar Algebras (Fall 2017)

HW 5

15 Let $n \in \mathbb{N}$ with $n \geq 2$ be fixed. Consider the symmetric matrix $\Lambda \in M_n(\mathbb{C})$ defined by

$$\Lambda_{ij} = \begin{cases} 1, & \text{if } |i-j| = 1 \\ 0, & \text{else} \end{cases}$$

(a) Prove that the eigenvalues of Λ are precisely the zeros of the n-th Chebyshev polynomial S_n of the second kind, i.e.

$$\left\{ 2\cos\left(\frac{k\pi}{n+1}\right) \mid k = 1, \ldots, n \right\},$$

where an eigenvector corresponding to the eigenvalue $\lambda_k := 2\cos\left(\frac{k\pi}{n+1}\right)$ is given by

$$t_k = \left(\sin\left(\frac{k\pi}{n+1}\right), \sin\left(\frac{2k\pi}{n+1}\right), \ldots, \sin\left(\frac{nk\pi}{n+1}\right) \right)^T$$

Proof. The double angle formula for $\sin(x)$ gives $(\Lambda t_k)_1 = 2\cos\left(\frac{k\pi}{n+1}\right)\sin(\frac{k\pi}{n+1}) = (\lambda_k t_k)_1$. Moreover, we have $(\Lambda t_k)_n = \sin(k\pi - \frac{2k\pi}{n+1}) = (-1)^{k+1} 2\cos\left(\frac{k\pi}{n+1}\right)\sin(\frac{k\pi}{n+1}) = (\lambda_k t_k)_n$.

Let $q = e^{\frac{k\pi i}{n+1}}$. For $1 < j < n$, we have

$$\begin{aligned}
(\Lambda t_k)_j &= \frac{1}{2i}(q^{j-1} - q^{1-j} + q^{j+1} - q^{-j-1}) \\
&= \frac{1}{2i}(q + q^{-1})(q^j - q^{-j}) \\
&= \lambda_k (t_k)_j
\end{aligned}$$

□

(b) Deduce that all values in

$$\left\{ 4\cos^2\left(\frac{\pi}{n+1}\right) \mid n \geq 2 \right\}$$

show up as the Jones index for some subfactor of the hyperfinite II_1 factor.

Proof. We showed how to do this in class (using a theorem of Jones about Markov traces + the basic construction). □

16 Let a real matrix $P \in M_n(\mathbb{R})$ be a real symmetric matrix with nonnegative entries. Suppose there exists a real eigenvector $y \in \mathbb{R}^n$ of P with positive entries and corresponding eigenvalue $\lambda \geq 0$.

(a) On the set

$$\Gamma_n := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1, \dots x_n > 0\}$$

consider the function

$$L : \Gamma_n \to [0, \infty), x \mapsto \max\{s \geq 0 \mid sx \leq Px\},$$

where $x \leq x'$ means that it holds entry-wise. Prove that

$$\sup_{x \in \Gamma_n} L(x) = \lambda = L(y).$$

Proof. Since $\lambda y = Py$, we have $\sup_{x \in \Gamma_n} L(x) \geq L(y) \geq \lambda$. To see that $\sup_x L(x) \leq \lambda$, let $x \in \Gamma_n$. Suppose $s \geq 0$ with $sx \leq Px$. Then we have

$$\begin{aligned} \langle sx, y \rangle &\leq \langle Px, y \rangle \\ &= \langle x, Py \rangle \\ &= \lambda \langle x, y \rangle \end{aligned}$$

Thus, $s \leq \lambda$, so $L(x) \leq \lambda$. Thus, $\sup_x L(x) \leq \lambda$. $\square$

(b) Deduce that $\|P\| = \lambda$.

Proof. Note that the same proof as in (a) works for

$$\Gamma_n' := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \setminus \{0\} \mid x_1, \dots x_n > 0\}.$$

One important thing to note is that $\langle x, y \rangle > 0$ for $x \in \Gamma'$ since $x \neq 0$ and y has positive entries. Let $L' : \Gamma_n' \to [0, \infty)$ denote the corresponding function.

Let β denote the eigenvalue of P such that $\|P\| = |\beta|$. Let x be an eigenvector for β. Define $\hat{x} := (|x_1|, |x_2|, \dots, |x_n|)$. Then for all i, we have $\|P\|\hat{x}_i = |\lambda x_i| = |(Px)_i| \leq (P\hat{x})_i$. Thus $\|P\|\hat{x} \leq \hat{x}$. Thus $\|P\| \leq L'(\hat{x}) \leq \lambda$. Thus $\|P\| = \lambda$. $\square$

17 Find braids whose closures are the given links, and their associated Jones polynomials.

Soln: A braid for the Hopf link is $b := \sigma^2$. The Jones polynomial is

$$\begin{aligned}
V_{\hat{b}}(t) &= (\sqrt{t} + \sqrt{t}^{-1})^{n-1}(\sqrt{t})^{\mathrm{wr}(b)}\tau(\pi_t(b)) \\
&= (\sqrt{t} + \sqrt{t}^{-1})^{2-1}(\sqrt{t})^{2}\tau((1-(1+t)e)^2) \\
&= (\sqrt{t} + \sqrt{t}^{-1})t\tau(1-2(1+t)e + (1+2t+t^2)e) \\
&= \sqrt{t}(t+1)\tau(1+(t^2-1)e) \\
&= \sqrt{t}(t+1)(1+(t^2-1)\frac{t}{(t+1)^2}) \\
&= \sqrt{t}(t+1+(t-1)t) \\
&= t^{5/2} - t^{1/2}
\end{aligned}$$

18 Let $\mathcal{H}$ be a separable complex Hilbert space and let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator. Prove that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n\xi = \pi(\xi)$$

holds for any $\xi \in \mathcal{H}$, where π denotes the orthogonal projection from $\mathcal{H}$ onto the closed subspace $\mathcal{H}^U$ of all U-invariant vectors in $\mathcal{H}$.

Proof. Let $\mathcal{W} := \{U\xi - \xi \mid \xi \in \mathcal{H}\}$. To see that $\mathcal{H}^U$ is orthogonal to $\mathcal{W}$, suppose $\eta, \xi \in \mathcal{H}$. Then we have $\langle \eta, U\xi - \xi\rangle = \langle U\eta, U\xi\rangle - \langle \eta, \xi\rangle = 0$.

The formula for the mean ergodic theorem obviously holds for $\xi \in \mathcal{H}^U$. Moreover, if $\xi \in \mathcal{H}$, we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n(U\xi - \xi) &= \lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^{n+1}\xi - U^n\xi \\
&= \lim_{N\to\infty} \frac{1}{N}(U^N\xi - \xi) \to 0.
\end{aligned}$$

Since $\mathcal{H}^U$ is orthogonal to $\mathcal{W}$, we have $\pi(U\xi - \xi) = 0$ also. Thus, the formula holds on $\mathcal{H}^U + \mathcal{W}$.

Now suppose the formula holds for some sequence $(\xi_i)_i \subset \mathcal{H}$ with $\xi_i \to \xi$ for some ξ. Then we have

$$\|\frac{1}{N}\sum_{n=0}^{N-1} U^n(\xi - \xi_i)\| \le \frac{1}{N}\sum_{n=0}^{N-1} \|U^n\|\|\xi - \xi_i\| \le \|\xi - \xi_i\|.$$

Hence,

$$\begin{aligned}
\|\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n\xi - \pi(\xi)\| &\le \|\xi - \xi_i\| + \|\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n\xi_i - \pi(\xi - \xi_i) - \pi(\xi_i)\| \\
&\le 2\|\xi - \xi_i\| + \|\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n\xi_i - \pi(\xi_i)\| \\
&\to 0.
\end{aligned}$$

Thus, the formula holds on $\overline{\mathcal{H}^U + \mathcal{W}}$.

To see that $\mathcal{H} = \overline{\mathcal{H}^U + \mathcal{W}}$, suppose not. Then there exists a nonzero vector $\xi \in (\mathcal{H}^U + \mathcal{W})^\perp$. Since ξ is orthogonal to $\mathcal{W}$, we have $\langle \xi, U\xi - \xi \rangle = 0$. Thus,

$$\begin{aligned}
\|U\xi - \xi\|^2 &= \langle U\xi, U\xi \rangle - \langle U\xi, \xi \rangle - \langle \xi, U\xi \rangle + \langle \xi, \xi \rangle \\
&= 2\langle \xi, \xi \rangle - \langle U\xi, \xi \rangle - \langle \xi, U\xi \rangle \\
&= \langle \xi - U\xi, \xi \rangle + \langle \xi, \xi - U\xi \rangle \\
&= 0.
\end{aligned}$$

Thus, $\xi \in \mathcal{H}^U$, a contradiction. $\square$