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HW 5, due 2/28

18.32 Let $(f_n), f$ be integrable. If $\int |f_n - f| \rightarrow 0$, show that $\int_E f_n \rightarrow \int_E f$ for all measurable sets E , and that $\int f_n^+ \rightarrow \int f^+$.

Proof. Note $\chi_E f_n \in L_1$ and $\chi_E f \in L_1$, so $|\int_E f - \int_E f_n| = |\int_E (f - f_n)| \leq \int_E |f - f_n| \leq \int |f - f_n| \rightarrow 0$.

Also, for any real function g , we have $g^+ = (|g| + g)/2$. Hence, $|\int (f_n^+ - f^+)| = (1/2) |\int (|f_n| - |f| + f_n - f)| \leq (1/2) \int (|f_n| - |f| + |f_n - f|) \leq \int |f_n - f| \rightarrow 0$. \square

40 Let $(f_n), (g_n)$, and g be integrable, and suppose that $f_n \rightarrow f$ a.e., $g_n \rightarrow g$ a.e., $|f_n| \leq g_n$ a.e., for all n , and that $\int g_n \rightarrow \int g$. Prove that $f \in L_1$ and that $\int f_n \rightarrow \int f$.

Proof. Just the proof of the DCT with the obvious substitutions. Since $|f_n| \leq g_n$, we have $|f| \leq g$, so $f \in L_1$. The only other interesting parts are the equality $\liminf_{n \rightarrow \infty} (\int g_n + \int f_n) = \int g + \liminf_{n \rightarrow \infty} \int f_n$ and the corresponding one for \limsup . This follows from the more general fact that if $a_n \rightarrow a$ and $(b_n) \subset \mathbb{R}$, then $\liminf_n (a_n + b_n) = a + \liminf_n b_n$.

Indeed, for $\epsilon > 0$, we have $|a_n - a| \leq \epsilon$ for all large n . Hence, $a - \epsilon + \liminf_n b_n \leq a + \liminf_n (a_n - a) + b_n \leq a + \epsilon + \liminf_n b_n$. Letting $\epsilon \rightarrow 0$, we have $\liminf_n (a_n + b_n) = a + \liminf_n (a_n - a) + b_n = a + \liminf_n b_n$. \square

43(a) Let f be measurable and finite a.e. on $[0, 1]$. If $\int_E f = 0$ for all measurable $E \subset [0, 1]$ with $m(E) = 1/2$, prove that $f = 0$ a.e. on $[0, 1]$.

Proof. Case $f \geq 0$. Suppose the conclusion fails. Then $m[f > 0] = m[\cup_n f \geq 1/n] > 0$, so $m[f \geq 1/n] > 0$ for some n . But then either $[0, 1/2] \cap [f \geq 1/n]$ or $[1/2, 1] \cap [f \geq 1/n]$ has positive measure. WLOG, suppose the former. Then $\int_0^{1/2} f \geq \int_0^{1/2} 1/n \chi_{[f \geq 1/n]} = 1/n m([0, 1/2] \cap [f \geq 1/n]) > 0$, a contradiction.

General case. Either $m[f \geq 0] \geq 1/2$ or $m[f \leq 0] \leq 1/2$. FIXME

Suppose not. Either $m[f > 0]$ or $m[f < 0]$ is positive; WLOG suppose the former is. Then $m[f > 0] = m[\cup_n f \geq 1/n] > 0$, so $m[f \geq 1/n] > 0$ for some n . If $m[f \geq 1/n] > 1/2$, then pick $E \subset [f \geq 1/n]$ with $m(E) = 1/2$, giving a contradiction. The existence of such an E follows the intermediate value theorem since $g(t) = m([f \geq 1/n] \cap [0, t])$ is continuous.

If $m[f \geq 1/n] < 1/2$, let $F \subset ([0, 1] \setminus [f \geq 1/n])$ with $m(F) = 1/2$. Let $G = F \cap [f \leq 0]$. \square

43(b) Let f be measurable and finite a.e. on $[0, 1]$. If $f > 0$ a.e., show that $\inf \{ \int_E f : m(E) \geq 1/2 \} > 0$.

Proof. \square

44(c) Show that $\lim_n \int_0^1 f_n = 0$ where $f_n(x) = \frac{nx \log x}{1+n^2x^2}$.

Proof. Note that $1 + n^2x^2 \geq 2nx$. Hence, $|f_n| \leq (1/2)|\log x|$. Note that $\int_0^1 |\log x| dx = \int_0^1 (-\log x) dx = [x - x \log x]_0^1 = 1 - \lim_{x \rightarrow 0} x \log x = 1 - \lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = 1$. Hence, by the DCT, $\lim_n \int_0^1 f_n = \int_0^1 \lim_{n \rightarrow \infty} f_n = 0$. \square

44(d) Show that $\lim_n \int_0^1 f_n = 0$ where $f_n(x) = \frac{n^{3/2}x}{1+n^2x^2}$.

Proof. Letting $u = 1+n^2x^2$, we have $\int_0^1 f_n = \int_1^{1+n^2} \frac{n^{-1/2}}{2u} du = (n^{-1/2}/2)[\log u]_1^{1+n^2} = (n^{-1/2}/2) \log(1+n^2)$.

Thus, $\lim_n \int_0^1 f_n = \lim_{n \rightarrow \infty} O(n^{-1/2} \log(1+n^2)) = \lim_{n \rightarrow \infty} O(\frac{2n/(1+n^2)}{n^{-1/2}}) = 0$. \square

47(b) Compute $\lim_{n \rightarrow \infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} dx$.

Proof. By the binomial theorem, $\frac{1+nx^2}{(1+x^2)^n} \leq 1$ for all n . Hence, by the DCT, $\lim_{n \rightarrow \infty} \int_0^1 \frac{1+nx^2}{(1+x^2)^n} dx = \int_0^1 \lim_{n \rightarrow \infty} \frac{1+nx^2}{(1+x^2)^n} dx = 0$. \square

47(d) Compute $\lim_{n \rightarrow \infty} \int_a^\infty \frac{n}{1+n^2x^2} dx$.

Proof. Let $u = nx$. Then $\int_a^\infty \frac{n}{1+n^2x^2} dx = \int_{na}^\infty \frac{1}{1+u^2} du = [\tan^{-1}(u)]_{na}^\infty = (\pi/2) - \tan^{-1}(na)$. As $n \rightarrow \infty$,

$$\int_a^\infty \frac{n}{1+n^2x^2} dx = (\pi/2) - \tan^{-1}(na) \rightarrow \begin{cases} 0, & a > 0 \\ \pi/2, & a = 0 \\ \pi, & a < 0 \end{cases}.$$

\square

49 For which $\alpha \in \mathbb{R}$ is $\sum_{n=1}^\infty xn^{-\alpha}e^{-nx}$ continuous on $[0, \infty)$? in $L_1[0, \infty)$?

Proof. \square

55 Prove that if f is integrable on \mathbb{R} , then $f(x) \cos(nx)$ is integrable and $\lim_{n \rightarrow \infty} \int_{-\infty}^\infty f(x) \cos(nx) dx = 0$. The same is true with \sin replacing \cos .

Proof. \square

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(a) Show that there is a sequence of polynomials (P_n) such that $P_n \rightarrow 0$ pointwise on $[0, 1]$, but with $\int_0^1 P_n(x) dx \rightarrow 3$.

(b) Find $\int_0^1 \sup_n |P_n(x)| dx$.

Proof. \square

51 Let (f_n) be a sequence of measurable functions with $|f_n| \leq g$ for all n , where $g \in L_1$. If $f_n \rightarrow f$ a.e., prove that $f_n \rightarrow f$ almost uniformly.

Proof.

□

56 Given $f \in L_1(\mathbb{R})$, define $g(x) = \int_{-\infty}^{\infty} f(t) \sin(xt) dt$ for $x \in \mathbb{R}$. Show that g is continuous on \mathbb{R} and that $g(x) \rightarrow 0$ as $x \rightarrow \pm\infty$; hence, g is uniformly continuous on \mathbb{R} .

Proof. Case $f = \chi_{(a,b)}$ for $a, b \in \mathbb{R}$. We have $g(x) = \int_a^b \sin(xt) dt = (1/x) \int_{xa}^{xb} \sin(t) dt = O(1/x)$ since $\int_r^{r+2n\pi} \sin(t) dt = 0$ for all $r \in \mathbb{R}$, $n \in \mathbb{Z}$. Hence, $g(x) \rightarrow 0$ as $|x| \rightarrow \infty$.

To see that g is continuous, fix x and suppose $x_n \rightarrow x$. Then $\sin(x_nt) \rightarrow \sin(xt)$ pointwise on (a, b) , hence in measure also. Let $\epsilon > 0$. Convergence in measure says that we can pick N such that, for all $n > N$, we have $m[|\sin(x_nt) - \sin(xt)| \geq \epsilon] < \epsilon$. Then for $n > N$, we have $|g(x) - g(x_n)| \leq \int_a^b |\sin(xt) - \sin(x_nt)| dt \leq 2m[|\sin(x_nt) - \sin(xt)| \geq \epsilon] + \epsilon m[|\sin(x_nt) - \sin(xt)| < \epsilon] \leq 2\epsilon + \epsilon(b-a)$. Hence, g is continuous.

Case f is a step function. Write f as $f = \sum_{i=1}^m a_i \chi_{A_i}$, where each A_i is an interval. Both conclusions follow from the linearity of the integral and the previous case.

General case. Let $\epsilon > 0$. Pick a step function h such that $\int |f - h| < \epsilon$. To see that $g(x) \rightarrow 0$ as $|x| \rightarrow \infty$, note that $|g(x)| = |\int h(t) \sin(xt) dt + \int (f(t) - h(t)) \sin(xt) dt| \leq |\int h(t) \sin(xt) dt| + \int |f - h| \rightarrow \epsilon$ as $|x| \rightarrow \infty$ by the step function case.

For the continuity, suppose $x_n \rightarrow x$. Then $|g(x_n) - g(x)| = |\int f(t)(\sin(x_nt) - \sin(xt)) dt| \leq |\int h(t)(\sin(x_nt) - \sin(xt)) dt| + 2 \int |f - h| \rightarrow 2\epsilon$ as $n \rightarrow \infty$. □