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Math 643 - Algebraic Topology I

Problem Set 1

1.12 Let X, Z be compact Hausdorff spaces, and let $h : X \to Z$ be a continuous surjection. Prove that $\phi : X/\ker h \to Z$, defined by $[x] \to h(x)$, is a homeomorphism.

Proof. Since $X/\ker h$ is a continuous image of the compact set X, it is compact. Since Z is Hausdorff and ϕ is a continuous bijection, this implies ϕ is a homeomorphism (ϕ maps compact sets to compact sets, hence closed sets to closed sets). $\square$

1.13 For fixed t with $0 \leq t < 1$, prove that $f : x \mapsto [x, t]$ defines a homeomorphism from a space X to a subspace of CX.

Proof. This map f is continuous since the map $x \mapsto (x, t) \subset X \times I$ is continuous and respects the equivalence relation $\sim$. The map f is also injective since $\sim$ only identifies points of the form $[x, 1]$.

To see that f^{-1} is continuous, let $U \subset X$ be open. Then $U \times [0, (t+1)/2) \subset X \times I$ is open. Let $\pi : X \times I \to CX$ be the canonical quotient map. Then $f(U) = \pi(U \times [0, (t+1)/2)) \cap f(X)$ is open in $f(X)$. $\square$

2.9 If $\{p_0, p_1, \dots, p_m\}$ is affine independent with barycenter b, then $\{b, p_0, \cdots, \hat{p}_i, \dots, p_m\}$ is affine independent for each i.

Proof. Fix $0 \leq i \leq m$. Suppose $sb + \sum_{j \neq i} s_j p_j = 0$ with $s + \sum_{j \neq i} s_j = 0$ for some $s, s_j \in \mathbb{R}$. Then we have

$$\begin{aligned} 0 &= sb + \sum_{j \neq i} s_j p_j \\ &= \frac{s}{m+1} \sum_j p_j + \sum_{j \neq i} s_j p_j \\ &= \sum_j t_j p_j, \end{aligned}$$

where $t_j = \frac{s}{m+1} + s_j$ for $j \neq i$, and $t_i = \frac{s}{m+1}$. Hence $\sum_j t_j = s + \sum_{j \neq i} s_j = 0$. Since $\{p_i\}_i$ is affine independent, this implies $t_j = 0$ for all j. Thus $s = 0$ and $s_j = 0$ for all $j \neq i$. Thus $\{b, p_0, \cdots, \hat{p}_i, \dots, p_m\}$ is affine independent.

It is false that the diameter of a simplex with the barycenter replacing a vertex is always strictly smaller than the diameter of the original simplex. Take an equilateral triangle as an example. $\square$

2.10 Show that for $0 \leq i \leq m$, $[p_0, \dots, p_m]$ is homeomorphic to the cone $C[p_0, \dots, \hat{p}_i, \dots, p_m]$ with vertex p_i.

Proof. Since $C[p_0, \dots, \hat{p}_i, \dots, p_m]$ is the continuous image of the compact set $[p_0, \dots, \hat{p}_i, \dots, p_m]$, it is compact. Also $[p_0, \dots, p_m]$ is Hausdorff. Hence it suffices to find a continuous bijection from $C[p_0, \dots, \hat{p}_i, \dots, p_m]$ to $[p_0, \dots, p_m]$.

Define $\phi : [p_0, \dots, \hat{p}_i, \dots, p_m] \times I \to [p_0, \dots, p_m]$ by $\phi(\sum_{j \neq i} s_j p_j, t) = tp_i + \sum_{j \neq i} s_j(1-t)p_j$. Since $\phi(x, 1) = p_i$ for all x, ϕ induces a continuous map $\bar{\phi} : C[p_0, \dots, \hat{p}_i, \dots, p_m] \to [p_0, \dots, p_m]$. Since ϕ is surjective, so is $\bar{\phi}$.

To see that $\bar{\phi}$ is injective, first note that ϕ maps only the vertex to the point p_i. Next if $\bar{\phi}[\sum_{j \neq i} s_j p_j, t] = \bar{\phi}[\sum_{j \neq i} r_j p_j, u] \neq p_i$, then $(u-t)p_i + \sum_{j \neq i}(r_j - s_j)p_j$. Hence, $u = t$ and $r_j = s_j$ for all $j \neq i$ since the p_j are affine independent. Thus $\bar{\phi}$ is injective. $\square$

5 Show that if x deformation retracts to A in the weak sense, then the inclusion map $A \to X$ is a homotopy equivalence.

Proof. Let $i : A \to X$ be the inclusion. Let $f : X \to A$ be defined by $f(x) = F(x, 1)$, where F is the weak deformation retraction. Then $F : 1_X \simeq (i \circ f)$. Moreover $F_{|A \times I} : 1_A \simeq f \circ i$ makes sense because $F(A, I) \subset A$. $\square$