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Math 643 - Algebraic Topology I

HW 3

1.14 Prove that $X \mapsto CX$ defines a functor **Top** $\to$ **Top**. (Hint: use exercise 1.11)

Proof. If $f : X \to Y$ is continuous, then $f \times 1 : X \times I \to Y \times I$ is continuous. Since $f \times 1$ maps $(x, 1)$ to $(f(x), 1)$ for all x, it preserves the relation $\sim$ from the definition of the cone over a space. Hence, by Exercise 1.11, we get a map $\overline{f} : CX \to CY$.

To see that the association $f \mapsto \overline{f}$ is functorial, we need to check that it preserves identities and composition. It is clear from the definition that $\overline{1}_x = 1_{CX}$. For composition, for $t \neq 1$, we have $\overline{fg}(x,t) = ((fg)(x), t) = (\overline{f} \circ \overline{g})(x,t)$. For the other case, we have $\overline{fg}(x, 1) = (*, 1) = (\overline{f} \circ \overline{g})(*, 1)$ where $(*, 1)$ denotes the vertex of the cone. $\square$

1.19

(i) A space X is path connected iff every two constant maps $X \to X$ are homotopic.

(ii) If X is contractible and Y is path connected, then any two continuous maps $X \to Y$ are homotopic (and each is nullhomotopic).

Proof. For (i), first suppose X is path connected. Let $c_x, c_y : X \to X$ be the constant maps at x and y. Since X is path connected, there exists a path $p : I \to X$ from x to y. The map $H : X \times I \to X$ by $H(x,t) = p(t)$ is a homotopy from c_x to $c(y)$.

Conversely, every two constant maps $X \to X$ are homotopic. Let $x, y \in X$ and $H : c_x \simeq c_y$. Define $p : I \to X$ by $p(t) = H(x_0, t)$ for any fixed $x_0 \in X$. Then p is a path from x to y.

For (ii), let $f, g : X \to Y$. Since X is contractible there exists $x_0 \in X$ with a homotopy $H : 1_X \simeq c_{x_0}$. Let p be a path from $f(x_0)$ to $g(x_0)$. Let $G : X \times I \to Y$ be defined by the concatenation $(f \circ H) * p(t) * (g \circ H^{-1})$. Then $G : f \simeq g$. $\square$

1.23

(i) The $\sin(1/x)$ space X has exactly two path components: the vertical line A and the graph G.

(ii) Show that the graph G is not closed. Conclude that, in contrast to components (which are always closed), path components may not be closed.

(iii) Show that the natural map $\nu : X \to X/A$ is not an open map. (Hint: Let U be the open disk with center $(0/\frac{1}{2})$ and radius $\frac{1}{4}$; show that $\nu(X \cap U)$ is not open in $X/A (\approx [0, \frac{1}{2\pi}])$.)

Proof. (i) It is clear that A and G are both path connected. Hence it suffices to show that there is no path in X from $(0,1)$ to $(1/\pi, 0)$. Suppose such a path p exists. Then $\lim_{t\to 0} p(t) = (0,1)$. Hence, there exists t_0 such that the $y(p(t)) > 1/2$ for all $t < t_0$, where $y(\cdot)$ denotes the projection on the y-coordinate. Pick n so large that $(2\pi n)^{-1} < t_0$. Then since $\sin(1/x) < 0$ for $(2\pi n + 2\pi)^{-1} < x < (2\pi n + \pi)^{-1}$, the graph of p cannot meet this strip. Thus, the graph of p is disconnected, a contradiction.

(ii) The sequence $(1/(n\pi), 0)$ lies in G, but its limit is the origin.

(iii) Following the hint, let U be the open disk with center $(0, \frac{1}{2})$ and radius $\frac{1}{4}$. I claim that $\nu(X \cap U)$ is not open in X/A. If it were, then $\nu^{-1}\nu(X \cap U) = A \cup (X \cap U)$ is open in X. This is not true since any neighborhood of, say, $(0, -\frac{1}{2})$ must intersect G below the x-axis.

□

1.32 Assume that X, Y, and Z are spaces with $X \subset Y$. If X is a retract, then every continuous map $f : X \to Z$ can be extended to a continuous map $\tilde{f} : Y \to Z$, namely, $\tilde{f} = fr$, where $r : Y \to X$ is a retraction. Prove that if X is a retract of Y and if f_0 and f_1 are homotopic continuous maps $X \to Z$, then $\tilde{f}_0 \simeq \tilde{f}_1$.

Proof. Let $F : f_0 \simeq f_1$. Let $G : Y \times I \to Z$ be defined by $G(y,t) = F(r(y), t)$. Then $G(y, 0) = F(r(y), 0) = f_0 r(y) = \tilde{f}_0(y)$, and similarly for $G(y, 1)$. Hence G is the desired homotopy. □

1.34

(i) Define $i : X \to M_f$ by $i(x) = [x, 0]$ and $j : Y \to M_f$ by $j(y) = [y]$. Show that i and j are homeomorphisms to subspaces of M_f.

Proof. It is easy to see that i, j are continuous. The map i is injective since the relation $\sim$ does not affect its image. The map j is also injective since the fibers of f are disjoint (hence only one $y \in Y$ is in each equivalence class of $\sim$). Thus it suffices to show that i, j are open maps.

Let $U \subset X$ be open. Then $i(U) = \pi_X^{-1}(U) \cap i(X)$ is open, where $\pi_X : X \times I \to X$ is the canonical projection. Hence i is an open map.

Let $U \subset Y$ be open. Let $\nu : (X \times I) \sqcup Y \to M_f$ be the quotient map, and $i_1 : X \times I \to (X \times I) \sqcup Y$ and $i_2 : Y \to (X \times I) \sqcup Y$ be the canonical injections. Then $\nu^{-1} j(U) = i_1(f^{-1}(U) \times \{1\}) \cup i_2(U) = (i_1(f^{-1}(U)) \cap j(Y)) \cup i_2(U)$ is open, hence $j(U)$ is open. □

(ii) Define $r : M_f \to Y$ by $r[x,t] = f(x)$ for all $(x,t) \in X \times I$ and $r[y] = y$. Prove that r is a retraction $rj = 1_Y$.

Proof. If $y \in Y$ and $[x, 1] \sim y$, then $r[x, 1] = f(x) = y = r[y]$. Hence r is well-defined.

It is also clear that $rj = 1_Y$. To see that r is continuous, note that $r = \nu \circ ((f\pi_X) \sqcup id_Y)$, where ν is the quotient map defined above and $\pi_X : X \times I \to X$ is the canonical projection. $\square$

(iii) Prove that Y is a deformation retract of M_f. (Hint: Define $F : M_f \times I \to M_f$ by

$$\begin{aligned} F([x,t],s) &= [x,(1-s)t+s] \text{ if } x \in X, t \in I; \\ F([y],s) &= [y] \text{ if } y \in Y, s \in I \end{aligned}$$

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Proof. Let F be defined as in the hint. To see that F is a well-defined continous map, let $G : ((X \times I) \sqcup Y) \times I \to ((X \times I) \sqcup Y) \times I$ be defined by $G(x, t, s) = (x, (1 - s)t + s)$ and $G(y, s) = y$ for $(x, t) \in X \times I$ and $y \in Y$. Then G is continuous, and $G(x, 1, s) = (x, 1)$ and $G(y, s) = y$ for all $x \in X$ and $y \in Y$. Thus, G respects the identification map defining M_f. Hence F, the induced map, is a continuous map.

Moreover, $F([x, t], 0) = [x, t]$ and $F([y], 0) = [y]$, so $F(\cdot, 0) = 1_{M_f}$. Lastly, $F([x, t], 1) = [x, 1] \in Y$ and $F([y], 1) = [y]$. Thus, Y is a deformation retract of M_f. $\square$

(iv) Show that every continuous map $f : X \to Y$ is homotopic to $r \circ i$, where i is an injection and r is a homotopy equivalence.

Proof. Let i be the injection i from part (i). Let r be defined as in (ii). The proof of (iii) shows that $j \circ r \simeq 1_{M_f}$, so r is a homotopy equivalence. $\square$

2.8 Let $A \subset \mathbb{R}^n$ be an affine set and let $T : A \to \mathbb{R}^k$ be an affine map. If $X \subset A$ is affine (or convex), then $T(X) \subset \mathbb{R}^k$ is affine (or convex). In particular, if a, b are distinct points in A and if l is the line segment with endpoints a, b, then $T(l)$ is the line segment with points $T(a), T(b)$ if $T(a) \neq T(b)$, and $T(l)$ collapses to the point $T(a)$ if $T(a) = T(b)$.

Proof. To see that $T(X)$ is affine (convex), let $T(a), T(b) \in T(X)$. Then for any $\alpha + \beta = 1$ (with α, β nonnegative in the convex case), we have $\alpha T(a) + \beta T(b) = T(\alpha a + \beta b) \in T(X)$ since $\alpha a + \beta b \in X$ since X is affine (convex). $\square$