Paul Gustafson
Texas A&M University - Math 482
Instructor: Dr. David Larson

# Cantor Sets in $\mathbb{R}$

Recall that a *perfect* set is a set for which every point is a limit point. A set $S$ is called *totally disconnected* if for every $x, y \in S$, there exist disjoint open sets $U, V \subset S$ such that $x \in U$, $y \in V$, and $U \cup V = S$.

**Definition 1.** *A* Cantor set *is a non-empty, totally disconnected, perfect, compact metric space.*

**Example 1.** *Let $C_0 := [0,1]$, $C_1 := [0,1/3] \cup [2/3,1]$, and $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. Similarly, for $i > 2$, let $C_i$ be the closed set given by removing the open middle third of each interval of $C_{i-1}$. The* ternary Cantor set

$$\Delta := \bigcap_{i=0}^{\infty} C_i$$

*is a Cantor set.*

*Proof.* Since $0 \in C_i$ for all $i$, $\Delta$ is non-empty. Since each interval in $C_i$ is of length $3^{-i}$, $\Delta$ is totally disconnected. It is closed and bounded, so compact by the Heine-Borel theorem.

To see that $\Delta$ is perfect, first note that the endpoints of any interval in any $C_i$ remain endpoints of intervals in $C_{i+1}$, and $C_{i+1} \subset C_i$. Hence, every point that is an endpoint of an interval in some $C_i$ is in $\Delta$. Now, fix $x \in \Delta$. Given $\epsilon > 0$, there exists a $C_i$ whose intervals are of length less than $\epsilon$. Hence, both endpoints of the interval in $C_i$ containing $x$ are within $\epsilon$ of $x$, and are members of $\Delta$. Thus, $x$ is a limit point, so $\Delta$ is perfect. $\square$

**Theorem 1.** *Let $K$ be a Cantor set. If $A \subset K$ is nonempty and clopen, then $A$ is Cantor.*

*Proof.* $A$ is compact since it is closed in $K$, and totally disconnected since it is open. To see that $A$ is perfect, let $x \in A$. Since $K$ is perfect, there exists a sequence $(x_n) \subset K$ such that $x_n \to x$. Since $A$ is open, all but a finite number of $x_n$ lie in A. $\square$

**Theorem 2.** *If $A \subset \mathbb{R}$ is a Cantor set, then there is a order-preserving homeomorphism $f : A \to \{0,1\}^{\mathbb{N}}$, where $\{0,1\}^{\mathbb{N}}$ is ordered lexicographically.*

*Proof.* Step 1. Let $a := \inf(A)$, and $d := \sup(A) - a = \operatorname{diam}(A)$. Since $A$ is totally disconnected, there exists $c \in [a + \frac{d}{4}, a + \frac{3d}{4}] \setminus A$. Then $M_0 := (-\infty, c) \cap A$ and $M_1 := (c, \infty) \cap A$ are clopen relative to $A$, hence Cantor sets by Theorem 1. Moreover, $\operatorname{diam}(M_i) \leq \frac{3}{4}\operatorname{diam}(A)$ for $i = 0, 1$.

Step 2. For $n > 1$, apply Step 1 to $M_t$ for each $t \in \{0,1\}^{n-1}$ to get Cantor sets $M_{t,0}, M_{t,1} \subset M_t$ with $M_{t,0} < M_{t,1}$ and $\mathrm{diam}(M_{t,i}) \leq \frac{3}{4}\mathrm{diam}(M_t)$ for $i = 0, 1$. By recursion on $n$, for all $r, s \in \{0,1\}^n$ we have $\mathrm{diam}(M_s) \leq \left(\frac{3}{4}\right)^n \mathrm{diam}(A)$, and if $r < s$ in the lexicographical ordering then $M_r < M_s$. Moreover, for any fixed $n$, $A = \bigcup_{s\in\{0,1\}^n} M_s$.

Step 3. Fix $x \in A$. The construction in Step 2 generates a descending sequence of sets $(M_{t_n})_{t_n\in\{0,1\}^n}$, each containing $x$. Since for all $n$ we have $t_{n+1} = t_n, i$ for some $i \in \{0,1\}$, this sequence of sets determines a unique element $f(x) \in \{0,1\}^{\mathbb{N}}$ such that, for any $n$, the first $n$ entries of $f(x)$ are $t_n$. To see that $f$ is bijective, note that if $t \in \{0,1\}^{\mathbb{N}}$ and $t_n = (t(1), t(2), ...t(n))$, then $f^{-1}(t) = \bigcup_{n=1}^{\infty} M_{t_n}$ contains exactly one point, since $M_{t_n}$ is a descending chain of compact sets.

To see that $f$ is continuous, note that $d(x,y)$ is small, so there exists $M_s$ containing $x$ and $y$ where $n := \mathrm{length}(s)$ is large, so $d(f(x), f(y)) \leq 2^{-n}$ is small. Since $A$ is compact, the continuity of $f$ implies $f^{-1}$ is also continuous.

To see that $f$ is order-preserving, if $x < y$ there exists $n$ so large that $x \in M_s, y \in M_t$ for $s, t$ of length $n$ with $s \neq t$. By Step 2, this implies $s < t$. Hence, $f(x) < f(t)$. □

**Theorem 3.** *If $S \subset R$ is a Cantor set, there exists a nondecreasing, onto, continuous function $g : S \to [0,1]$.*

*Proof.* Let $h : \{0,1\}^{\mathbb{N}} \to [0,1]$ be defined by $h(x) = \sum_{i=0}^{\infty} x(i)2^{-i}$. Defining $f$ as in Theorem 2, let $g = h \circ f$. It suffices to show that $h$ is nondecreasing, onto, and continuous.

Let $x, y \in \{0,1\}^{\mathbb{N}}$. Then $|h(x)-h(y)| = |\sum_{i=0}^{\infty}(x(i)-y(i))2^{-i}| \leq \sum_{i=0}^{\infty} |x(i)-y(i)|2^{-i} = d(x,y)$, so $h$ is continuous. If $x < y$, then there exists a minimal $n$ such that $x(n) \neq y(n)$. By the definition of lexicographical ordering, $x(n) = 0$ and $y(n) = 1$. Thus, $h(y) - h(x) = \sum_{i=n}^{\infty}(y(i) - x(i))2^{-i} = 2^{-n} + \sum_{i=n+1}^{\infty}(y(i)-x(i))2^{-i} \geq 2^{-n} + \sum_{i=n+1}^{\infty}(-1)2^{-i} = 0$. Hence, $h$ is nondecreasing. To see that $h$ is onto, let $E_n := \{x \in \{0,1\}^{\mathbb{N}} : x(i) = 0 \text{ for all } i > n\}$. Then each $h(E_n)$ is a $2^{-n+1}$-net for $[0,1]$, so the image of $h$ is dense in $[0,1]$. Since $S$ is compact, $h(S)$ is compact, so $h$ is onto. □

**Lemma 1.** *If $f : [a,b] \to [0,1]$ is nondecreasing and onto, then $f$ is continuous.*

*Proof.* Let $c \in (a,b]$. Since $f$ is nondecreasing, $\sup_{x<c} f(x) \leq f(c) = \inf_{x\geq c} f(x)$. Hence, since $f$ is onto, $\sup_{x<c} f(x) = f(c)$. To see that $f(c-) = f(c)$, set $\epsilon > 0$. By the definition of supremum, there exists $a < c$ such $f(c) - f(a) < \epsilon$. Then if $a < x < c$, since $f$ is nondecreasing, $f(c) - f(x) < \epsilon$. Hence, $f(c-) = f(c)$. The proof for right continuity is analogous. □

**Lemma 2.** *Every compact metric space $K$ can be written as $K = A \cup B$, where $A$ is perfect (hence compact), $B$ is countable, and $A \cap B = \emptyset$.*

*Proof.* Let $U$ be a countable base for $K$. Let $V := \{S \in U : S \text{ is countable}\}$, and $W := U \setminus V$. Then $B := \bigcup_{S\in V} S$ is countable and open. Let $A := K \setminus B$. Then A is closed, hence compact.

I claim that $W$ is a base for the topology of $A$ relative to $K$. Suppose $C \subset A$ is open in $A$, and $x \in C$. Then $C \cup B$ is open in $K$, so there exists $S \in U$ with $x \in S \subset (C \cup B)$. Since $x \notin B$, $S$ cannot be countable, so $S \in W$. Hence, $W$ is a base for $A$.

In particular, since every element of $W$ is uncountable, $A$ can have no isolated points. Hence, $A$ is perfect. ∎

**Definition 2.** *Given an nondecreasing function $\alpha : \mathbb{R} \to \mathbb{R}$, the* $\alpha$-exterior measure *of a set $E \subset \mathbb{R}$ is defined to be*

$$m_\alpha^*(E) := \inf\{\sum_{i=1}^{\infty} \alpha(b_i) - \alpha(a_i) : E \subset \bigcup_{i=1}^{\infty} (a_i, b_i)\}$$

**Theorem 4.** *If $E \subset R$ is closed and $m_\alpha^*(E) = 0$ for all nondecreasing, continous $\alpha : \mathbb{R} \to \mathbb{R}$, then $E$ is countable.*

*Proof.* Suppose $E$ were uncountable. If $E$ contains a nontrivial interval, then let $\alpha$ be the identity. Since $E$ contains an interval, it contains a compact set of the form $[a, b]$ for $a < b$. Hence, any cover of $E$ by open intervals must contain a finite subcover of $[a, b]$. The sum of the lengths of intervals in this subcover must be at least $b - a$, so $m_\alpha^*(E) \geq b - a > 0$, a contradiction.

Suppose $E$ does not contain any nontrivial intervals. Note that $E \cap [n, n+1]$ must be uncountable for some $n$, so WLOG, $E$ is compact. Then, by Lemma 2, $E = A \cup B$ where $A$ is a Cantor set and $B$ is countable. Since $A \subset E$, $m_\alpha^*(A) \leq m_\alpha^*(E)$, so it suffices to show that $m_\alpha^*(A) > 0$.

Let $f : A \to [0, 1]$ be the increasing, onto, continuous function defined in Theorem 3. Define

$$\alpha(x) = \begin{cases} 0 & : x \leq \inf(A) \\ \sup\{f(y) : y \in A \cap (-\infty, x)\} & : x > \inf(A) \end{cases}$$

Since $A$ is closed and $f$ is onto $[0, 1]$, $\alpha$ is onto $[0, 1]$. Also, $\alpha$ is clearly nondecreasing. Since $\alpha$ is constant outside $(\inf(A), \sup(A))$, Lemma 1 implies $\alpha$ is continuous.

Let $U$ be a cover of $A$ by open intervals. Since $A$ is compact, there exists a finite subcover $F \subset U$. Denote the elements of $F$ by $((a_i, b_i))_{i=1}^n$, sorted so that $a_i \leq a_{i+1}$ for all $i < n$. If $b_{i+1} < b_i$ for some $i < n$, then $(a_{i+1}, b_{i+1}) \subset (a_i, b_i)$. Since $F$ is finite, we can recursively throw out all such redundant sets. This procedure only reduces the sum of interval lengths of $F$, so we may assume $b_i \leq b_{i+1}$ for all $i < n$. For $i < n$, if $b_i \geq a_{i+1}$, then $\alpha(b_i) - \alpha(a_{i+1}) \geq 0$ since $\alpha$ is nondecreasing. On the other hand, if $b_i < a_{i+1}$, then $\alpha(b_i) - \alpha(a_{i+1}) = 0$ since $A \cap [b_i, a_{i+1}] = \emptyset$.

Thus, $\sum_{i=1}^n \alpha(b_i) - \alpha(a_i) \geq \alpha(b_n) - \alpha(a_1) = 1$. Hence, $m_\alpha^*(A) \geq 1$. ∎