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HW 4

1 Define

$$\mathcal{A}^{(\mathbb{Q})} = \left\{ \bigcup_{i=1}^{n} [a_i, b_i) \cap \mathbb{Q} : \begin{array}{c} n \in \mathbb{N}, \{a_i, b_i : 1 \le i \le n\} \subset \mathbb{Q} \cup \{\pm\infty\}, \\ \text{and } a_1 < b_1 < a_2 < \ldots < b_n \end{array} \right\}.$$

For $A = \bigcup_{i=1}^{n} [a_i, b_i) \cap \mathbb{Q}$ with $-\infty \le a_1 < b_1 < a_2 < \ldots < b_n \le \infty$ put

$$\mu_0(A) = \sum_{i=1}^{n} b_i - a_i.$$

a) $\mathcal{A}^{(\mathbb{Q})}$ is an algebra on $\mathbb{Q}$ and μ_0 is a finitely additive measure on $\mathcal{A}^{(\mathbb{Q})}$.
b) Show that μ_0 is not a premeasure.

Proof. For (a), suppose $E, F \in \mathcal{A}^{(\mathbb{Q})}$ with $E = \bigcup_{i=1}^{n} [a_i, b_i) \subset \mathbb{Q}$ and $F = \bigcup_{i=1}^{m} [c_i, d_i) \subset \mathbb{Q}$ for $a_i, b_i, c_i, d_i \in \mathbb{R}$ for all i. We have $\emptyset \in \mathcal{A}^{(\mathbb{Q})}$, so to show that $\mathcal{A}^{(\mathbb{Q})}$ is an algebra, we only need to show that E^c and $E \cup F$ are in $\mathcal{A}^{(\mathbb{Q})}$. For the former, we have $E^c = [-\infty, a_1) \cup [b_n, \infty) \cup \bigcup_{i=1}^{n-1} [b_i, a_{i+1})$, so $E^c \in \mathcal{A}^{(\mathbb{Q})}$.

For the latter, we have $E \cup F = \bigcup_{i=1}^{n+m} [e_i, f_i)$ for $([e_i, f_i))_i$ a reordering of the concatenation of $([a_i, b_i))$ and $([c_i, d_i))$ such that $e_1 \le e_2 \le \ldots \le e_{n+m}$. Suppose $f_i > e_{i+1}$ for some i. Then $[e_i, f_i) \cup [e_{i+1}, f_{i+1}) = [e_i, f_{i+1})$. Hence, $E \cap F = \bigcup_{i=1}^{n+m-1} [e'_j, f'_j)$ where $[e'_j, f'_j) = ([e_j, f_j))$ for $j < i$, $[e'_i, f'_i) = [e_i, f_{i+1})$, and $[e'_j, f'_j) = [e_{j+1}, f_{j+1}$ for $j > i$. Then $e'_1 \le e'_2 \le \ldots \le e'_{n+m-1}$. We can continue this process until we get $E \cup F = \bigcup_{i=1}^{l} [g_i, h_i)$ for some l, with $g_i \le g_{i+1}$ for all i and $h_i \le g_{i+1}$ for all i. Note that $g_i \le h_i$ by construction. This implies that $E \cup F \in \mathcal{A}^{(\mathbb{Q})}$.

To see that μ_0 is finitely additive on $\mathcal{A}^{(\mathbb{Q})}$, we need to show that $\mu_0(E \cup F) = \mu_0(E) + \mu_0(F)$ if E, F are disjoint. Using the same notation as above, we have $E \cup F = \bigcup_{i=1}^{n+m} [e_i, f_i)$ for $([e_i, f_i))_i$ a reordering of the concatenation of $([a_i, b_i))$ and $([c_i, d_i))$ such that $e_1 \le e_2 \le \ldots \le e_{n+m}$. If $f_i > e_{i+1}$ for some i, then we contradict $b_j \le a_{j+1}$, $d_j \le c_{j+1}$, or the disjointness of E and F. Hence $e_i \le f_i$ and $f_i \le e_{i+1}$ for all i, so $\mu_0(E \cup F) = \sum_{i=1}^{n+m} f_i - e_i = \sum_{i=1}^{n} b_i - a_i + \sum_{i=1}^{m} d_i - c_i = \mu_0(E \cup F)$.

For (b), suppose μ_0 is a premeasure. It extends to a measure μ on $\mathcal{M}(\mathcal{A})$. Let $q \in \mathbb{Q}$. Pick any real-valued sequences $a_n \uparrow q$ and $b_n \downarrow q$. Then $q = \bigcap_n (a_n, b_n] \cap \mathbb{Q}$ and $\mu(b_1 - a_1) < \infty$, so $\mu(q) = \lim_{n \to \infty} b_n - a_n = 0$. Since every element in $\mathcal{A}$ is the union of its countably many rational elements, this implies that every element of $\mathcal{A}$ has measure 0, a contradiction. $\square$

2 Let $d \in \mathbb{N}$ and

$$\mathcal{E} = \left\{ \prod_{i=1}^{d} [a_i, b_i) : -\infty \leq a_i \leq b_i \leq \infty \text{ for } i = 1, 2, \ldots n \right\}.$$

(if $a_i = \infty$, replace $[a_i, b_i)$ with (a_i, b_i)). Let $\mathcal{A}$ be the algebra generated by $\mathcal{E}$.

a) Show that

$$\mathcal{A} = \left\{ \bigcup_{i=1}^{n} E_i : n \in \mathbb{N}, E_i \in \mathcal{E} \text{ are pairwise disjoint } \right\}.$$

b) Show that there is a measure μ on $\mathcal{M}(\mathcal{A})$ so that

$$\mu(\prod_{i=1}^{d} [a_i, b_i)) = \prod_{j=1}^{d} (b_i - a_i) \text{ whenever } -\infty \leq a_i \leq b_i \leq \infty \text{ for } i = 1, 2, \ldots, n.$$

Proof. For (a), let $\mathcal{B} = \{\bigcup_{i=1}^{n} E_i : n \in \mathbb{N}, E_i \in \mathcal{E} \text{ are pairwise disjoint }\}$. Clearly $\mathcal{B} \subset \mathcal{A}$, so it suffices to show that $\mathcal{B}$ is an algebra. Since $\mathcal{E}$ is nonempty, $\mathcal{B}$ must be nonempty.

To see that $\mathcal{B}$ is closed under taking finite intersections, let $B, C \in \mathcal{B}$. Then $B = \bigcup_{j=1}^{m} B_j$ for some $m \in \mathbb{N}$ and disjoint $(B_j) \subset \mathcal{E}$, and $C = \bigcup_{k=1}^{n} C_k$ for some $n \in \mathbb{N}$ and disjoint $(C_k) \subset \mathcal{E}$. Then $B \cap C = \bigcup_{j,k} B_j \cap C_k$. To see that the sets $(B_j \cap C_k)_{j,k}$ are disjoint, suppose $(j, k) \neq (j', k')$. WLOG $j \neq j'$. Then $(B_j \cap C_k) \cap (B_{j'} \cap C_{k'}) = (B_j \cap B_{j'}) \cap (C_k \cap C_{k'}) = \emptyset$ since the (B_j) are disjoint. Hence $(B_j \cap C_k)_{jk}$ are disjoint, so it suffices to break an arbitrary $B_j \cap C_k$ into disjoint elements of $\mathcal{E}$.

Write $B_j = \prod_{i=1}^{d} [a_i, b_i)$ and $C_k = \prod_{i=1}^{d} [c_i, d_i)$. Then $B_j \cap C_k = \prod_{i=1}^{d} [a_i, b_i) \cap [c_i, d_i)$. For each i, we have $[a_i, b_i) \cap [c_i, d_i) = [e_i, f_i)$ for some $-\infty \leq e_i \leq f_i \leq \infty$ by case analysis on the order of a_i, b_i, c_i, d_i. Hence, $B_j \cap C_k \in \mathcal{E}$.

To see that $\mathcal{B}$ is closed under taking complements, let $B \in \mathcal{B}$. Then $B = \bigcup_{i=1}^{n} E_i$ for $E_i \in \mathcal{E}$, and $B^c = \bigcap_i E_i^c$. Since we know that $\mathcal{B}$ is closed under finite intersections, it suffices to show that each $E_i^c \in \mathcal{B}$. Writing E_i as $E_i = \prod_{j=1}^{d} [a_j, b_j)$, let $\mathcal{U} = \{\prod_{j=1}^{d} U_j : \forall j \quad U_j \in \{(-\infty, a_j), [a_j, b_j), [b_j, \infty)\}\}$. Then $\mathcal{U} \subset \mathcal{E}$ is a finite partition of $\mathbb{R}^d$, and $E_i^c = \bigcup (\mathcal{U} \setminus E)$. Hence $E_i^c \in \mathcal{B}$.

For (b), we first define $V : \mathcal{E} \to [0, \infty]$ by $V(\prod_{i=1}^{d} [a_i, b_i)) = \prod_{i=1}^{d} b_i - a_i$. Define V in the same way for open and closed boxes. Let $\mu_0 : \mathcal{A} \to [0, \infty]$ be defined by $\mu_0(A) = \inf\{\sum_{j=1}^{\infty} V(E_j) : \bigcup_j E_j \supset A\}$. By the extension of premeasures theorem, it suffices to show that $\mu_0(E) = V(E)$ for $E \in \mathcal{E}$ and that μ_0 is a premeasure.

Step 1: If $E \in \mathcal{E}$ and $\mathcal{P} \subset \mathcal{E}$ is a finite partition of E, then $V(E) = \sum_{P \in \mathcal{P}} V(P)$. This will be proved by induction on the dimension d.

Step 2: If $E \in \mathcal{E}$, then $\mu_0(E) = V(E)$. Since $\{E\}$ is a cover of E, we have $\mu_0(E) \leq V(E)$.

Case 1: E is bounded. Let $\epsilon > 0$. Let (E_j) be a countable $\mathcal{E}$-cover of E. Enlarge each E_j slightly to get open boxes $\widetilde{E}_j$ with $V(\widetilde{E}_j) - V(E_j) < 2^{-j}\epsilon$. We can also pick a closed box $\widetilde{E} \subset E$ with $V(E) - V(\widetilde{E}) < \epsilon$. Since $\widetilde{E}$ is compact, there exists a finite set F such that $(\widetilde{E}_j)_{j \in F}$ covers $\widetilde{E}$. Let $(F_j)_{j\ inF} \subset \mathcal{E}$ be defined by letting each $F_j = \prod_i [a_{ji}, b_{ji}$ where $\prod_i (a_{ji}, b_{ji}) = \widetilde{E}_j$. Similarly, let $F \in \mathcal{E}$ be defined by $F = \prod_i [a_i, b_i)$ where $\prod_i [a_i, b_i] = \widetilde{E}$. Hence $(F_j)_{j \in F}$ is a finite $\mathcal{E}$-cover of F with $\sum_{j \in F} V(F_j) < \sum_{j \in F} V(E_j) + 2^{-j}\epsilon \leq \epsilon + \sum_{j=1}^{\infty} V(E_j)$ and $V(F) - V(E) < \epsilon$. Therefore, it suffices to show that $V(F) \leq \sum_{j \in F} V(F_j)$, for then $V(E) < \epsilon + V(F) \leq \epsilon + \sum_{j \in F} V(F_j) \leq 2\epsilon + \sum_{j=1}^{\infty} V(E_j)$.

Case 2: E is unbounded. Intersect E and $\mathcal{P}$ against the boxes $[-N, N]$ and use Case 1.

Step 3: μ_0 is a premeasure. By definition, $\mu_0(\emptyset) = 0$. I still need to show countable additivity.

□

3 Let μ be a finite measure on $\mathcal{B}_{\mathbb{R}}$. Show that for all $\epsilon > 0$ and all $A \in \mathcal{B}_{\mathbb{R}}$, there is an open set U and a closed set F so that $F \subset A \subset U$ and $\mu(U \setminus F) < \epsilon$. Prove this by showing that

$$\widetilde{\mathcal{M}} := \{A \in \mathcal{B}_{\mathbb{R}} : \forall \epsilon > 0 \exists U \text{ open } \exists C \text{ closed} \quad C \subset A \subset U \text{ and } \mu(U \setminus C) < \epsilon\}$$

is a σ-algebra.

Proof. I first show that $\widetilde{\mathcal{M}}$ contains the open sets in $\mathbb{R}$. Let $U \subset \mathbb{R}$ be open, and $\epsilon > 0$. We have $U = \bigcup_{n=1}^{\infty} I_n$ for disjoint open intervals I_n. Thus $\mu(U) = \sum_n \mu(I_n)$. Since $\mu(U) < \infty$, we can pick N such that $\mu(U) - \sum_{n=1}^{N} \mu(I_n) < \epsilon/2$. For each open interval I_n we can pick an ascending sequence (F_m) of closed intervals such that $\bigcup_m F_m = I_n$. Hence $\mu(I_n) = \lim_m \mu(F_m)$, so there exists $C_n \in (F_m)$ such that $\mu(I_n \setminus C_n) < \epsilon/(2N)$. Hence $C = \bigcup_{n=1}^{N} C_n$ is closed, and $\mu(U \setminus C) = \mu(U \setminus \bigcap_{n=1}^{N} I_n) + \mu(\bigcap_{n=1}^{N} I_n \setminus C_n) < \epsilon/2 + N(\epsilon/(2N)) = \epsilon$.

Thus $\widetilde{\mathcal{M}}$ contains all the open sets in $\mathbb{R}$, so it suffices to show that $\widetilde{\mathcal{M}}$ is a σ-algebra. Clearly $\widetilde{\mathcal{M}}$ is nonempty. Suppose $M \in \widetilde{\mathcal{M}}$. Let $\epsilon > 0$. There exist F closed and U open such that $F \subset M \subset U$ and $\mu(U \setminus F) < \epsilon$. We have U^c closed and F^c open with $U^c \subset M^c \subset F^c$ and $\mu(F^c \setminus U^c) = \mu(F^c \cap U) = \mu(U \setminus F) < \epsilon$, so $M^c \in \widetilde{\mathcal{M}}$.

For closure under countable unions, let $(M_n) \subset \widetilde{\mathcal{M}}$. Let $\epsilon > 0$, $M = \bigcup_{n=1}^{\infty} M_n$ and $S_N = \bigcup_{n=1}^{N} M_n$. Then $\mu(M) = \lim_N \mu(S_N)$, so we can pick N such that $\mu(\bigcup_n M_n) - \mu(S_N) < \epsilon$. For each $n \in \mathbb{N}$, pick closed F_n and open U_n such that $F_n \subset M_n \subset U_n$ and $\mu(U_n \setminus F_n) < 2^{-n}\epsilon$. Let $U = \bigcup_{n=1}^{\infty} U_n$ and $F = \bigcup_{n=1}^{N} F_n$. Then U is open and F is closed with $F \subset S_N \subset M \subset U$. Moreover, $\mu(U \setminus F) \leq \mu(\bigcup_{n=N+1}^{\infty} U_n) + \mu((\bigcup_{n=1}^{N} U_n) \setminus \bigcup_{n=1}^{N} F_n) < 2^{-N}\epsilon + \mu(\bigcup_{n=1}^{N} U_n \setminus F_n) < 2^{-N}\epsilon + \epsilon < 2\epsilon$. □

4 If $E \in \mathcal{L}$ (the Lebesgue sets) and $m(E) > 0$ then there is for any $\alpha < 1$ and open interval I such that $m(E \cap I) > \alpha m(I)$.

Proof. Let $\epsilon > 0$. Since m is outer regular, we can pick an open set $U \supset E$ with $m(U \setminus E) < \epsilon$. We can write $U = \bigcup_{n=1}^{\infty} I_n$ for disjoint open intervals

I_n. Suppose $m(E \cap I_n) \leq \alpha\, m(I_n)$ for all n. Then $m(E) = m(\bigcup_n E \cap I_n) = \sum_n E \cap I_n \leq \alpha \sum_n I_n = \alpha(U) = \alpha(m(E) + m(U \setminus E)) < \alpha(m(E) + \epsilon)$. Letting $\epsilon \to 0$, we have $m(E) \leq \alpha\, m(E)$, a contradiction. Hence, $m(E \cap I_n) > \alpha\, m(I_n)$ for some n. $\square$