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HW 2

1 a Show that if α, β are positive with $\alpha + \beta = 1$ then for all $u, v \geq 0$ we have

$$u^\alpha v^\beta \leq \alpha u + \beta v.$$

Proof. If $u = v = 0$, then the inequality holds. Since the inequality is symmetric in u and v, we may assume $v \neq 0$. Hence we wish to show

$$(\frac{u}{v})^\alpha \leq \alpha(\frac{u}{v}) + \beta$$

. Letting $x = \frac{u}{v}$, this is equivalent to showing that $f(x) \geq 0$, where $f(x) = \alpha x - x^\alpha + \beta$ and $x \geq 0$. Since $\alpha > 0$, we have $f'(x) = \alpha - \alpha x^{\alpha-1} = \alpha(1 - x^{\alpha-1})$ whose only zero in $[0, \infty)$ is at $x = 1$. Moreover, since $\alpha < 1$, we have $f''(1) = \alpha(\alpha-1)x^{\alpha-2}|_{x=1} = \alpha(\alpha - 1) < 0$. Hence, the maximum value of f on $[0, \infty)$ occurs at $x = 1$. We have $f(1) = \alpha - 1 + \beta = 0$, so $f(x) \leq 0$ for $x \geq 0$. $\square$

1 b Let $x, y \in \mathbb{R}^n$, and let $p > 1$ and define q by $q^{-1} = 1p^1$. Prove Hölder's inequality,

$$|\sum_j x_j y_j| \|x\|_p \|y\|_q.$$

Hint: Using the inequality in part (a). first prove it for $\|x\|_p = \|y\|_q = 1$. Scale to get the final inequality.

Proof. Suppose $\|x\|_p = \|y\|_q = 1$. Then

$$\begin{aligned}
|\sum_j x_j y_j| &\leq \sum_j |x_j||y_j| \\
&= \sum_j (|x_j|^p)^{1/p}((|y_j|)^q)^{1/q} \\
&\leq \sum_j \frac{1}{p}|x_j|^p + \frac{1}{q}|y_j|^q \\
&\leq \frac{1}{p}(\sum_j |x_j|^p) + \frac{1}{q}(\sum_j |y_j|^q) \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

For the general case, note that if $x = 0$ or $y = 0$ then the inequality holds. Hence we may assume both are nonzero. Let $x' = \frac{x}{\|}x\|_p$ and $y' = \frac{y}{\|}y\|_p$. We can now apply the special case to x' and y' then clear denominators to get the general inequality. $\square$

1 c Suppose $\phi = (y_1, \ldots, y_n) \in l_p^*$. Hölder's inequality implies that $\|\phi\|_{l_p^*} \leq \|y\|_q$. Show that we actually have $\|\phi\|_{l_p^*} = \|y\|_q$.

Proof. If $\|y\|_q = 0$ then $\phi = 0$, and $\|\phi\|_{l_p^*} = 0 = \|y\|_q$. Hence, we may assume $\|y\|_q \neq 0$. Let $x_i = \frac{|y_i|}{y_i} \frac{|y_i|^{q/p}}{\|y\|_q^{q/p}}$ for $1 \leq i \leq n$. Then $\|x\|_p = \sum_i \frac{|y_i|^q}{\|y\|_q^q} = 1$.

Then $\phi(x) = \sum_i x_i y_i = \sum_i \frac{|y_i|^{q/p}}{\|y\|_q^{q/p}} |y_i| = \frac{1}{\|y\|_q^{q/p}} \sum_i |y_i|^{\frac{p+q}{p}} = \frac{1}{\|y\|_q^{q/p}} \sum |y_i|^q = \|y\|_q^{q-q/p} = \|y\|_q$.

□

1 d Let $x, y \in \mathbb{R}^n$, and let $p > 1$. Prove Minkowski's inequality,

$$\|x + y\|_p \leq \|x\|_p + \|y\|_p.$$

Use this to show that $\|x\|_p$ defines a norm on $\mathbb{R}^n$. Hint: you will need to use Hölder's inequality, along with a trick.

Proof. Acknowledgement: I looked in Carother's Real Analysis book for a hint on this problem. Setting $1/p + 1/q = 1$, we have

$$\begin{aligned}
\|x + y\|_p^p &= \sum_i |x_i + y_i||x_i + y_i|^{p-1} \\
&\leq \sum_i |x_i||x_i + y_i|^{p-1} + \sum_i |y_i||x_i + y_i|^{p-1} \\
&\leq (\|x\|_p + \|y\|_p) \left(\sum_i |x_i + y_i|^{q(p-1)} \right)^{1/q} \\
&= (\|x\|_p + \|y\|_p) \left(\sum_i |x_i + y_i|^p \right)^{1-1/p} \\
&= (\|x\|_p + \|y\|_p) \|x + y\|_p^{p-1}
\end{aligned}$$

If $\|x+y\|_p^{p-1} \neq 0$, we can divide by it to get desired inequality. If $\|x+y\|_p = 0$ then the inequality follows from the fact that $\|x\|_p + \|y\|_p$ must be nonnegative by definition.

To show that $\|\cdot\|_p$ is a norm, it remains to show that it is homogeneous and positive definite. To see that $\|\cdot\|_p$ is homogeneous, let $c \in \mathbb{R}$ and $v \in \mathbb{R}^n$, then $\|cv\|_p = (\sum_i |cv_i|^p)^{1/p} = (|c|^p \sum_i |v_i|^p)^{1/p} = |c|\|v\|_p$. It is obvious that $\|v\|_p \geq 0$. If $\|v\|_p = 0$, then each component of v must be zero or else $\sum_i |v_i|^p > 0$. Hence $v = 0$.

□

2 L_2 minimization. Find the straight line $y = a+bx$ that minimizes $\int_0^1 (e^x abx)^2 \, dx$.

Proof. By HW 1, Problem 4, we know that $a+bx$ minimizes $\|e^x - a - bx\|_2$ iff

$$\begin{pmatrix} \langle e^x, 1\rangle \\ \langle e^x, x\rangle \end{pmatrix} = \begin{pmatrix} \langle 1,1\rangle & \langle x,1\rangle \\ \langle 1,x\rangle & \langle x,x\rangle \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

The one slightly tricky integral is $\langle e^x, x\rangle = \int_0^1 xe^x\,dx = xe^x|_{x=0}^1 - \int_0^1 e^x\,dx = e-(e-1) = 1$.

$$\begin{pmatrix} e-1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0.8731 \\ 1.6903 \end{pmatrix}$$

□

3 L_1 minimization. Find the straight line $y = a+bx$ that minimizes $\int_0^1 |e^x abx|\,dx$, by following these steps.

a. Whatever the minimizer is, geometric considerations show that e^x and $a+bx$ will cross at two points, $0 < s < t < 1$. Find these two points by minimizing, over a, b, the area A between $f(x)$ and $a+bx$:

$$A = \int_0^1 |e^x abx|\,dx = \int_0^s (e^x abx)\,dx + \int_s^t (a+bxe^x)\,dx + \int_t^1 (e^x abx)\,dx.$$

b. Use the crossing conditions $a+bs = e^s$ and $a+bt = e^t$ to find a and b.

Proof. a. Let $g_1(a,b,s) = e^s - a - bs$, and $g_2(a,b,t) = e^t - a - bt$. The method of Lagrange multipliers gives us the following necessary condition for (a,b,s,t) to minimize A given the constraints $g_1 = g_2 = 0$:

$$0 = \left(\frac{\partial}{\partial a}, \frac{\partial}{\partial b}, \frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)(A - \lambda_1 g_1 - \lambda_2 g_2)$$

$$\begin{aligned} 0 = {} & \Big(\int_0^s (-1)\,dx + \int_s^t 1\,dx + \int_t^1 (-1)\,dx - \lambda_1 - \lambda_2, \\ & \int_0^s (-x)\,dx + \int_s^t x\,dx + \int_t^1 (-x)\,dx - \lambda_1 s - \lambda_2 t, \\ & 2(e^s - a - bs) + \lambda_1(e^s - b), -2(e^t - a - bt) + \lambda(e^t - b)\Big) \end{aligned}$$

$$\begin{aligned} 0 = {} & (-s + (t-s) + (t-1) - \lambda_1 - \lambda_2, \\ & (-1/2)s^2 + (1/2)(t^2 - s^2) + (-1/2)(1-t^2) - \lambda_1 s - \lambda_2 t, \\ & \lambda_1(e^s - b), \lambda_2(e^t - b)) \end{aligned}$$

$$\begin{aligned} 0 = {} & (2t - 2s - 1 - \lambda_1 - \lambda_2, t^2 - s^2 - 1/2 - \lambda_1 s - \lambda_2 t, \\ & \lambda_1(e^s - b), \lambda_2(e^t - b)) \end{aligned} \tag{1}$$

From the last two components, we get four cases.

Case $e^s = e^t = b$. Since b is the slope of the line between (s, e^s) and (t, e^t), we have $b = \frac{e^t - e^s}{t-s} = 0$ which cannot correspond to a minimum.

Case $e^s = b$ *and* $\lambda_2 = 0$. From the first component of 1, we have $\lambda_1 = 2t - 2s - 1$. Substituting into the second component of 1, $0 = t^2 - s^2 - 1/2 - \lambda_1 s = t^2 - s^2 - 1/2 - (2t - 2s - 1)s = (t-s)^2 - (1/2 - s)$. Since $t - s > 0$, we have $t = s + \sqrt{1/2 - s}$. Using the case assumption, we have $e^s = b = \frac{e^t - e^s}{t-s} = e^s \frac{e^{\sqrt{1/2-s}} - 1}{\sqrt{1/2-s}}$. Thus if $u = \sqrt{1/2 - s}$, then $u = e^u - 1$. The only solution to this equation is $u = 0$. To see this, note that $f(u) := e^u - u - 1$ has derivative $e^u - 1$, hence f has a unique global minimum at 0.

Hence $s = 1/2$, so $t = 1/2$. This cannot correspond to a minimum.

Case $\lambda_1 = 0$ *and* $e^t = b$. From the first component of 1, we have $\lambda_2 = 2t - 2s - 1$. Substituting into the second component of 1, $0 = t^2 - s^2 - 1/2 - \lambda_2 t = t^2 - s^2 - 1/2 - (2t - 2s - 1)t = -(t-s)^2 + t - 1/2$. Since $t - s > 0$, we have $s = t - \sqrt{t - 1/2}$. Using the case assumption, we have $e^t = b = \frac{e^t - e^s}{t-s} = e^t 1 - e^{-\sqrt{t-1/2}} \sqrt{t - 1/2}$. Thus if $u = -\sqrt{t - 1/2}$, then $-u = 1 - e^u$. As in the previous case, the only solution to this equation is $u = 0$.

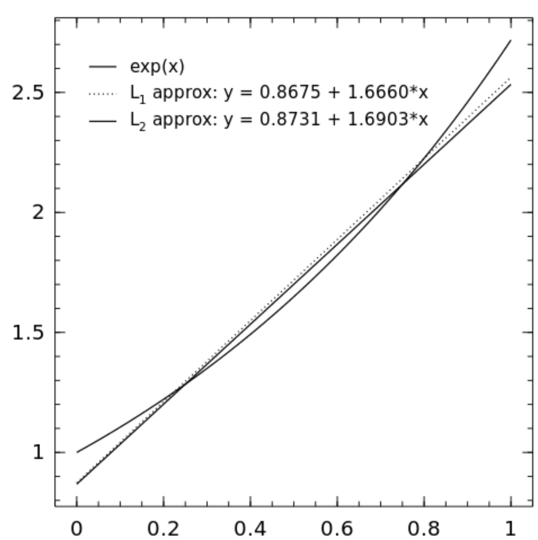
Hence $t = 1/2$, so $s = 1/2$. This cannot correspond to a minimum.

Case $\lambda_1 = \lambda_2 = 0$. We have $t = s + 1/2$, so $0 = (s + 1/2)^2 - s^2 - 1/2 = s - 1/4$. Hence $s = 1/4, t = 3/4$.

b. We have $a + b(1/4) = e^{(1/4)}$ and $a + b(3/4) = e^{(3/4)}$. Hence $a = 0.8675$ and $b = 1.6660$.

$\square$

3 Use your favorite software (mine is Matlab) and plot, on the same set of axes, e^x and the two minimization solutions found in the previous two problems.



4 Let V be a finite dimensional inner product space and let U be a subspace of V. Recall that the orthogonal complement of U is

$$U^{\perp} = \{v \in V | \langle v, u \rangle = 0 \text{ for all u } \in U\}.$$

Show that $V = U \oplus U^{\perp}$, where $\oplus$ symbolizes the direct sum of vector spaces. Also, show that $(U^{\perp})^{\perp} = U$.

Proof. By HW 1 (4)(b), the orthogonal projection $P : V \to U$ exists. Let $v \in V$. Then $v = Pv + (v - Pv)$. By HW 1 (3), $v - Pv \in U^{\perp}$. Hence, $V = U + U^{\perp}$. Moreover, if $w \in U \cap U^{\perp}$, then $\langle w, w \rangle = 0$ so $w = 0$. Thus, $v = U \oplus U^{\perp}$.

To see that $U \subset (U^{\perp})^{\perp}$, let $u \in U$. Then $\langle v, u \rangle = 0$ for all $v \in U^{\perp}$. Hence, $\langle u, v \rangle = 0$ for all $v \in U^{\perp}$. Thus, $u \in (U^{\perp})^{\perp}$.

Since $V = W \oplus W^{\perp}$ for any subspace W, we have $\dim(U) + \dim(U^{\perp}) = \dim(V) = \dim(U^{\perp}) + \dim((U^{\perp})^{\perp})$. Since $\dim(U^{\perp}) < \infty$, we have $\dim(U) = \dim((U^{\perp})^{\perp})$. Since $U \subset (U^{\perp})^{\perp}$ and they are finite dimensional, this implies that $U = (U^{\perp})^{\perp}$.

□