Problem Set 6
CSCE 440/640

Due dates: Electronic submission of the pdf file of this homework is due on **11/2/2016 before 2:50pm** on ecampus.tamu.edu, a signed paper copy of the pdf file is due on **11/2/2016** at the beginning of class.

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Resources. I used Mathematica to calculate a couple singular value decompositions.

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

Signature: ___________________________

Problem 1. (20 points) Consider the mixed state

$$M = \left\{ \left(|0\rangle, \frac{1}{3} \right), \left(|1\rangle, \frac{2}{3} \right) \right\}.$$

(a) Determine the density matrix ρ of the mixed state M.

(b) Derive a different mixed state M' (which should not consist of computational basis states) that has the same density matrix ρ as M.

[This problem shows that density matrices are not in one-to-one correspondence with mixed states.]

Solution. (a)

$$\begin{aligned}
\rho &= \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1| \\
&= \frac{1}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{2}{3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}
\end{aligned}$$

(b) First let's guess that there's a solution with real coefficients, so the derivation is slightly simpler. We're trying to find a mixed state

$$M' = \{(a'|0\rangle + b'|1\rangle, p), (c'|0\rangle + d'|1\rangle, q)\}$$

with density matrix ρ'. Let $a = \sqrt{p}a'$, $b = \sqrt{p}b'$, $c = \sqrt{q}c'$, and $d = \sqrt{q}d'$. Then the condition $\rho = \rho'$ becomes

$$\begin{aligned}
a^2 + c^2 &= 1/3 \\
b^2 + d^2 &= 2/3 \\
ab + cd &= 0
\end{aligned}$$

The last equation implies that there exists a k such that $(a, c) = k(b, -d)$. Solving for k,

$$\begin{aligned}
1/3 &= a^2 + c^2 \\
&= k^2(b^2 + d^2) \\
&= k^2(2/3),
\end{aligned}$$

so $k = \frac{1}{\sqrt{2}}$. A solution to the first system of equations is $a = c = \frac{1}{\sqrt{6}}$ and $b = -d = \frac{1}{\sqrt{3}}$. This corresponds to the mixed state

$$M' = \left\{ \left(\frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle, \frac{1}{2} \right), \left(\frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle, \frac{1}{2} \right) \right\}.$$

Problem 2. (20 points)
(a) Do Exercise 3.5.1 (b) on page 55 of our textbook KLM.
(b) Do Exercise 3.5.1 (c) on page 55 of our textbook KLM.

Solution. (a)

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(b)

$$\frac{1}{2}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Problem 3. (20 points) Find the Schmidt decomposition of the states
(a) $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.
(b) $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$.
[Students of CSCE 440 only need to solve (a), and students of CSCE 640 should solve both (a) and (b).]

Solution. I used Mathematica to find the SVD of the corresponding matrices.
(a) $(\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle) \otimes (\frac{\sqrt{2}}{2}|0\rangle + \frac{\sqrt{2}}{2}|1\rangle)$
(b)

$$\sqrt{\frac{1}{6}\left(\sqrt{5}+3\right)}\left(\frac{\sqrt{5}+3}{2\sqrt{2\sqrt{5}+5}}|0\rangle + \frac{3-\sqrt{5}}{2\sqrt{5-2\sqrt{5}}}|1\rangle\right) \otimes \left(\frac{\sqrt{5}+1}{\sqrt{2\left(\sqrt{5}+5\right)}}|0\rangle + \frac{1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}}|1\rangle\right)$$
$$+\sqrt{\frac{1}{6}\left(\sqrt{5}-3\right)}\left(\frac{\sqrt{5}+1}{2\sqrt{2\sqrt{5}+5}}|0\rangle + \frac{1-\sqrt{5}}{2\sqrt{5-2\sqrt{5}}}|1\rangle\right) \otimes \left(\frac{2}{\sqrt{2\left(\sqrt{5}+5\right)}}|0\rangle + \sqrt{\frac{1}{10}\left(\sqrt{5}+5\right)}|1\rangle\right)$$

which is approximately

$$\begin{aligned} &0.934(0.851|0\rangle + 0.526|1\rangle) \otimes (0.526|0\rangle - 0.851|1\rangle) \\ &+0.357(0.851|0\rangle + 0.526|1\rangle) \otimes (-0.526|0\rangle + 0.851|1\rangle) \end{aligned}$$

Problem 4. (20 points) Exercise 3.5.4 (a) on page 57 in our textbook KLM.

Solution. For any pure state $|a\rangle \otimes |b\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$,

$$\begin{aligned} \mathrm{tr}_B((U \otimes I)|a\rangle\langle a| \otimes |b\rangle\langle b|(U^\dagger \otimes I)) &= \mathrm{tr}_B(U|a\rangle\langle a|U^\dagger \otimes |b\rangle\langle b|) \\ &= U|a\rangle\langle a|U^\dagger)\langle b|b\rangle \\ &= U|a\rangle\langle a|\langle b|b\rangle U^\dagger) \\ &= U\,\mathrm{tr}_B(|a\rangle\langle a| \otimes |b\rangle\langle b|)U^\dagger) \end{aligned}$$

Since the trace tr_B is linear, the same identity holds for any linear combination of density matrices of pure states. Hence, it holds for all states.

Problem 5. (20 points) Choi has shown that for all matrices $V_j \in \mathbf{C}^{n\times m}$, the map $T\colon M_n(\mathbf{C}) \to M_m(\mathbf{C})$ given by

$$T(\rho) = \sum_{j=1}^{\ell} V_j^* \rho V_j$$

is completely positive. Show that if the matrices V_j satisfy the condition

$$\sum_{j=1}^{\ell} V_j V_j^* = I,$$

where I denotes the identity matrix, then T is trace preserving, so $\operatorname{tr} T(A) = \operatorname{tr} A$. [Hint: the matrix trace satisfies $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$.]

Solution.

$$\begin{aligned}
\operatorname{tr} T(A) &= \operatorname{tr}\left(\sum_{j=1}^{\ell} V_j^* A V_j\right) \\
&= \sum_{j=1}^{\ell} \operatorname{tr}(V_j^* A V_j) \\
&= \sum_{j=1}^{\ell} \operatorname{tr}(V_j V_j^* A) \\
&= \operatorname{tr}\left(\sum_{j=1}^{\ell} V_j V_j^* A\right) \\
&= \operatorname{tr} A
\end{aligned}$$

Checklist:

☐ Did you add your name?
☐ Did you disclose all resources that you have used?
(This includes all people, books, websites, etc. that you have consulted)
☐ Did you sign that you followed the Aggie honor code?
☐ Did you solve all problems?
☐ Did you submit the pdf file resulting from your latex source file on ecampus?
☐ Did you submit a hardcopy of the pdf file in class?