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HW 1

1.6 Assume that k is infinite. Determine the function rings A_i ($i = 1, 2, 3$) of the plane curves whose equations are $F_1 = Y - X^2$, $F_2 = XY - 1$, $F_3 = X^2 + Y^2 - 1$. Show that A_1 is isomorphic to $k[T]$, and that A_2 is isomorphic to $k[T, T^{-1}]$. Show that A_1 and A_2 are not isomorphic. What can we say about A_3 relative to the other two rings?

Proof. To see that $A_1 \simeq k[T]$, I will first show that $I(F_1) = \langle Y - X^2 \rangle$. Clearly, $Y - X^2 \in I(F_1)$. For the reverse inclusion, let $f \in I(F_1)$. By dividing with respect to Y, we have $f = a(X, Y)(Y - X^2) + b(X)$. Thus $0 = f(t, t^2) = b(t)$ for any $t \in k$. Hence, since k is infinite, $b(X) \equiv 0$. Thus $I(F_1) = \langle Y - X^2 \rangle$, so $A_1 \simeq k[X, Y]/(Y - X^2)$.

Let $\phi : k[X, Y] \to k[T]$ be the k-algebra homomorphism defined by sending $X \mapsto T$ and $Y \mapsto T^2$. Clearly, $Y - X^2$ is in the kernel of ϕ. Thus, ϕ induces a map $\phi^* : A_1 = k[X, Y]/(Y - X^2) \to k[T]$. Note that the map $\beta : k[T] \to k[X, Y]/(Y - X^2)$ sending T to X is a left and right inverse of ϕ^*. Therefore, ϕ^* is an isomorphism. Hence $A_1 \simeq k[T]$.

To see that $A_2 = k[X, Y]/(XY - 1)$, suppose $f \in I(XY - 1)$. Then $f = a(X, Y)(XY - 1) + b(X) + c(Y)$. Evaluating at (t, t^{-1}) for $t \in k^\times$, we have $b(t) + c(t^{-1}) = 0$. Clearing denominators and recalling that k is infinite shows that $b(X) \equiv 0 \equiv c(Y)$. Thus $I(F_2) = \langle XY - 1 \rangle$, so $A_2 = k[X, Y]/(XY - 1) = k[T, T^{-1}]$.

For the last part, first suppose $\mathrm{char}(k) = 2$. Then $X^2 + Y^2 - 1 = (X + Y + 1)^2$. Thus $F_3 = X + Y + 1$. Let $f \in I(F_3)$. Then $f = a(X, Y)(X + Y + 1) + b(X)$. The same argument as before shows $b(X) \equiv 0$. Thus, $A_3 = k[X, Y]/(X + Y + 1) \simeq k[T]$, where the last isomorphism is shown in the same way as in the first part.

Now suppose $\mathrm{char}(k) \neq 2$. I claim that $I(F_3) = \langle X^2 + Y^2 - 1 \rangle$. Suppose $f \in I(F_3)$. Then $f = a(X, Y)(X^2 + Y^2 + 1) + b(X)Y + c(X)$. Then we have $0 = f(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}) = b(\frac{t^2-1}{t^2+1})\frac{2t}{t^2+1} + c(\frac{t^2-1}{t^2+1})$ for all $t \in k$. Clearing denominators, the right hand side must be identically 0 since k is infinite. Moreover, since the first term has only coefficients of odd degree and the second only has coefficients of even degree and $2 \neq 0$, we have $b(X) = c(X) \equiv 0$. Thus, $I(F_3) = \langle X^2 + Y^2 - 1 \rangle$, so $A_3 = k[X, Y]/(X^2 + Y^2 + 1)$.

Further suppose there exists $i \in k$ with $i^2 = -1$. Define a k-algebra map $\phi : k[X, Y] \to k[T, T^{-1}]$ by $(X, Y) \mapsto ((T + T^{-1})/2, (T - T^{-1})/(2i))$. Since $X^2 + Y^2 - 1 \in \ker(\phi)$, this induces a map $\phi^* : k[X, Y]/(X^2 + Y^2 - 1) \to k[T, T^{-1}]$. To construct an inverse, define $\psi : k[T, U] \to k[X, Y]/(X^2 + Y^2 - 1)$ by $(T, U) \mapsto (X + iY, X - iY)$. The kernel of ψ contains $TU - 1$, so we get a map $\psi^* : k[T, T^{-1}] \to k[X, Y]$. It is easy to see that ψ^* is a left and right inverse of ϕ^*, so $A_3 \simeq k[T, T^{-1}]$.

Now suppose that k does not contain a square root of -1. I claim that A_3 is not isomorphic to A_1 or A_2. Suppose $\phi : A_3 \to k[T]$ is an isomorphism. Then $\phi(X)^2 + \phi(Y)^2 = 1$. Since ϕ fixes k, $\deg(\phi(X)) > 0$ and $\deg(\phi(Y)) > 0$. Moreover, in order for the nonconstant terms of $\phi(X)^2 + \phi(Y)^2$ to disappear, at the very least $\deg(\phi(X)) = \deg(\phi(Y))$. Then the highest degree cofficients of X and Y (call them a and b) must satisfy $a^2 + b^2 = 0$. This equivalent to $\left(\frac{a}{b}\right)^2 = -1$, a contradiction.

The proof that A_3 is not isomorphic to A_2 is similar, except one also does the same for the term of lowest degree as well as the terms of highest degree. □

1.7 Let $f : k \to k^3$ be the map $t \mapsto (t, t^2, t^3)$ and let C be the image of F. Show that C is an affine algebraic set and calculate $I(C)$. Show that $\Gamma(C)$ is isomorphic to the ring of polynomials $k[T]$.

Proof. To see that C is an affine algebraic set, note that $C = V(X^3 - Z, X^2 - Y)$. Suppose $f \in I(C)$. By dividing by Y and Z, we have $f = a(X,Y,Z)(X^3 - Z) + b(X,Y,Z)(X^2 - Y) + c(X)$.

If k is infinite, $c(X) \equiv 0$ by the same argument as in the preceding problem. Thus $I(C) = \langle X^3 - Z, X^2 - Y\rangle$. Moreover, it is easy to check that the map $k[T] \to k[X,Y,Z]/\langle X^3 - Z, X^2 - Y\rangle = \Gamma(C)$ defined by $T \mapsto X$ is an isomorphism (by constructing the inverse).

If $k = \mathbb{F}_q$, then $c(X)$ is a multiple of $X^q - X$. Hence, $I(C) = \langle X^3 - Z, X^2 - Y, X^q - X\rangle$. Suppose $\phi : \Gamma(C) \to k[T]$ where a k-algebra isomorphishm. Then $\phi(X)^q - \phi(X) = 0$. Since ϕ fixes k, the degree of $\phi(X)$ must be greater than 0. This leads to a contradiction since the highest term of $\phi(X)^q - \phi(X)$ is simply the q-th power of the highest term of $\phi(X)$. □