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HW 9

12 16 It follows from the definition of absolute continuity that $AC[a,b] \subset C[a,b]$.

To check that $AC[a,b]$ is a subspace, let $\epsilon > 0$, $f, g \in AC[a,b]$. Pick $\delta > 0$ such that if $([a_i, b_i])$ are nonoverlapping with $\sum_i b_i - a_i < \delta$, then $\sum_i |f(b_i) - f(a_i)| < \epsilon/2$. Pick $\eta > 0$ similarly for g. If $([a_i, b_i])$ are nonoverlapping with $\sum_i [a_i, b_i] < \delta \wedge \eta$, then $\sum_i |(f+g)(b_i) - (f+g)(a_i)| \leq \sum_i |(f(b_i) - (f(a_i)| + |g(b_i) - g(a_i)| \leq \epsilon/2 + \epsilon/2$.

Let $\alpha \in \mathbb{R}$. If $\alpha = 0$, then αf is trivially absolutely continuous. If $|\alpha| > 0$, pick $\delta > 0$ such that if $([a_i, b_i])$ are nonoverlapping with $\sum_i [a_i, b_i] < \delta$, then $\sum_i |f(b_i) - f(a_i)| < \epsilon/|alpha|$. Hence if $([a_i, b_i])$ are nonoverlapping with $\sum_i b_i - a_i < \delta$, then $\sum_i |(\alpha f)(b_i) - (\alpha f)(a_i)| = |\alpha| \sum_i |f(b_i) - f(a_i)| < |\alpha|(\epsilon/|alpha|)$.

To see that $AC[a,b]$ is an algebra, it suffices to show that $f^2 \in AC[a,b]$ since $AC[a,b]$ is a vector space. But this follows from the estimate $\sum_i |f^2(b_i) - f^2(a_i)| = \sum_i |f(b_i) - f(a_i)||f(b_i) + f(a_i)| \leq 2||f||_\infty \sum_i |f(b_i) - f(a_i)|$.

To see that $AC[a,b]$ is closed in $C[a,b]$, suppose $f_n \to f$ with $(f_n) \subset AC[a,b]$. Let $\epsilon > 0$. Pick N such that $|f_N - f| < \frac{\epsilon}{2(b-a)}$. Pick δ such that if $((a_i, b_i))$ are disjoint with $\sum_i b_i - a_i < \delta$ then $\sum_i |f_N(b_i) - f_N(a_i)| < \epsilon/2$. Thus, if $((a_i, b_i))$ are disjoint with $\sum_i b_i - a_i < \delta$, then $\sum_i |f(b_i) - f(a_i)| \leq \sum_i |(f - f_N)(b_i) - (f - f_N)(a_i)| + \sum_i |f_N(b_i) - f_N(a_i)| < \sum_i \int_{a_i}$

17 By (16), if f can be written as the difference of two increasing, absolutely continuous functions, then f is absolutely continuous.

For the converse, by Prop 20.15, $f \in AC[a,b]$ implies $v(x) := V_a^x(f) \in AC[a,b]$. Define $p(x) = \frac{1}{2}(v(x) + f(x))$ and $n(x) = \frac{1}{2}(v(x) - f(x))$, so that $f = p - n$. By the same proof as that of Prop 13.11, p and n are increasing. By (16), $p, n \in AC[a,b]$.

18 Let $\epsilon > 0$. By the absolute continuity of f, pick δ so that if $((a_i, b_i))$ are disjoint with $\sum_i b_i - a_i < \delta$, then $\sum_i f(b_i) - f(a_i) = \sum_i |f(b_i) - f(a_i)| < \epsilon$. Since $m(E) = 0$, there exists a cover (a_i, b_i) of E by disjoint open intervals such that $\sum_i b_i - a_i < \delta$. By the IVT, for any $a < c < d < b$, $f([c,d]) \supset [f(c), f(d)]$. Thus, since f is monotonic, $f([c,d]) = [f(c), f(d)]$.

Thus, $f(E) \subset f(\bigcup_i [a_i, b_i]) = \bigcup_i f([a_i, b_i]) = \bigcup_i [f(a_i), f(b_i)]$. Hence, $m(f(E)) \leq \sum_i f(b_i) - f(a_i) \leq \epsilon$.