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HW 7

1 Let $f(x) = e^x$ for $-\pi < x < \pi$.

a. Find the complex form of the Fourier series for f.

b. Sketch three periods of the 2π-periodic function to which the series converges pointwise. (Hand-drawn is fine. No need to use a computer here.)

c. Find $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$.

d. Estimate the error $\|f - S_N\|_{L_2[-\pi,\pi]}$, where S_N is the partial sum of the Fourier series for f.

Proof. For (a), the Fourier series for f is $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ where

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx}\, dx \\ &= \frac{1}{2\pi(1-in)} \left[e^{(1-in)x} \right]_{x=-\pi}^{\pi} \\ &= \frac{(-1)^n (e^{\pi} - e^{-\pi})}{2\pi(1-in)} \end{aligned}$$

For (c), we have $|a_n|^2 = \frac{(e^{\pi}-e^{-\pi})^2}{(2\pi)^2(1+n^2)}$. Hence using Parseval's identity, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{1}{n^2+1} &= \frac{1}{2} + \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{n^2+1} \\ &= \frac{1}{2} + \frac{2\pi^2}{(e^{\pi} - e^{-\pi})^2} \sum_{n=-\infty}^{\infty} |a_n|^2 \\ &= \frac{1}{2} + \frac{\pi}{(e^{\pi} - e^{-\pi})^2} \int_{-\pi}^{\pi} e^{2x}\, dx \\ &= \frac{1}{2} + \frac{\pi}{(e^{\pi} - e^{-\pi})^2} \left(\frac{1}{2} \right) (e^{2\pi} - e^{-2\pi}) \\ &= \frac{1}{2} + \frac{\pi(e^{\pi} + e^{-\pi})}{2(e^{\pi} - e^{-\pi})} \\ &= \frac{1}{2} + \frac{\pi}{2} \coth(\pi) \end{aligned}$$

For (d), we have

$$\begin{aligned}
\|f - S_N\|^2_{L_2[-\pi,\pi]} &= \|\sum_{|n|>N} a_n e^{inx}\|^2_{L_2[-\pi,\pi]} \\
&= \sum_{|n|>N} |a_n|^2 \\
&= \frac{(e^\pi - e^{-\pi})^2}{(2\pi)^2} \sum_{|n|>N} (1+n^2)^{-1} \\
&= \frac{(e^\pi - e^{-\pi})^2}{(2\pi)^2} \sum_{|n|>N} \int_{n-1}^{n} (1+t^2)^{-1}\, dt \\
&= \frac{(e^\pi - e^{-\pi})^2}{2\pi^2} \int_N^\infty \frac{dt}{1+t^2} \\
&= \frac{(e^\pi - e^{-\pi})^2}{2\pi^2} \left(\frac{\pi}{2} - \tan^{-1}(N)\right)
\end{aligned}$$

□

2 Prove this: Let g be a 2π-periodic piecewise continuous function. Then, $\int_{-\pi+c}^{\pi+c} g(u)\, du$ is independent of c. (Remark: This holds for g integrable on each bounded interval of $\mathbb{R}$.)

Proof. Pick $k \in \mathbb{Z}$ such that $2\pi k \in [-\pi + c, \pi + c)$. We have

$$\begin{aligned}
\int_{-\pi+c}^{\pi+c} g(u)\, du &= \int_{-\pi+c}^{2\pi k} g(u)\, du + \int_{2\pi k}^{\pi+c} g(u)\, du \\
&= \int_{-2\pi(k-1)-\pi+c}^{2\pi} g(v + 2\pi(k-1))\, dv + \int_0^{\pi+c-2\pi k} g(v + 2\pi k)\, dv \\
&= \int_{-2\pi k+\pi+c}^{2\pi} g(v)\, dv + \int_0^{\pi+c-2\pi k} g(v)\, dv \\
&= \int_0^{2\pi} g(v)\, dv.
\end{aligned}$$

□

3 Use the previous result to show that if f is 2π-periodic and piecewise smooth, then it has the Fourier series $f(x) \sim a_0 + \sum_{n=1}^\infty a_n \cos(nx) + b_n \sin(nx)$ where

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\, dx \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\, dx.$$

Formulate a theorem on the pointwise convergence of the series.

Proof. The first part is just a rewriting of the definition of Fourier series. The theorem about pointwise convergence is that the series converges pointwise to f. This was proved in class by applying the Riemann-Lebesgue lemma to the convolution of f with the Dirichlet kernel. □

4 Find the Fourier series for $f(x) = x$, $0 < x < 2\pi$. Sketch three periods of the 2π-periodic function to which the series converges pointwise. (Hand-drawn is fine. No need to use a computer here.)

Proof. The Fourier series for f is $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ where $a_0 = \frac{1}{2\pi}\int_0^{2\pi} x\,dx = \frac{1}{2}$, and for $n \neq 0$

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_0^{2\pi} xe^{-inx}\,dx \\ &= \frac{1}{2\pi}\left[x\left(\frac{-1}{in}\right)e^{-inx}\right]_0^{2\pi} + \frac{1}{2\pi in}\int_0^{2\pi} e^{-inx}\,dx \\ &= \frac{1}{2\pi}\left(2\pi\left(\frac{-1}{in}\right)\right) + \frac{1}{2\pi in}\left[\frac{-1}{in}e^{-inx}\right]_0^{2\pi} \\ &= \frac{i}{n}. \end{aligned}$$

□

5 Find the Fourier series for $f(x) = \begin{cases} 1, & x \in [-\frac{\pi}{4}, \frac{\pi}{4}] \\ 0, & x \in (-\pi, -\frac{\pi}{4}) \cup (\frac{\pi}{4}, \pi) \end{cases}$.

Proof. The Fourier series is $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ where

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_{-\pi/4}^{\pi/4} e^{-inx}\,dx \\ &= \frac{1}{2\pi}\left[-\frac{e^{-inx}}{in}\right]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{\pi n}\sin(n\pi/4) \end{aligned}$$

□

6 Consider the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, where $\sum_{n=-\infty}^{\infty} |c_n| < \infty$. Show that the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges uniformly to a 2π-periodic continuous function $f(x)$ and the series is the Fourier series for f. Also, show that the series converges to f in $L_2[-\pi, \pi]$.

Proof. For $x \in \mathbb{R}$, we have $\sum_{n=-\infty}^{\infty} |c_n e^{inx}| = \sum_n |c_n| < \infty$. Hence $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges absolutely for all x and uniformly in x. Since f is the uniform limit of continuous functions, it is continuous. It is obvious that f is 2π-periodic and that the series is the Fourier series for f.

Let $f_N = \sum_{n=-N}^{N} c_n e^{inx}$. Then $\int_{-\pi}^{\pi} |f_N - f|^2\,dx \leq 2\pi\|f_N - f\|_\infty^2 \to 0$. Hence $f_N \to f$ in $L_2[-\pi, \pi]$. □