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Midterm

1 Use the Courant-Fischer mini-max theorem to show that $\lambda_2 < 0$ for the matrix

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

Proof. The characteristic polynomial for A is $f(x) := x^3 + 6 + 6 - 9x - 4x - x = x^3 - 14x + 12$. We have $\lim_{x \rightarrow -\infty} f(x) < 0$, $f(0) > 0$, $f(1) < 0$, and $\lim_{x \rightarrow \infty} f(x) > 0$. Thus $\lambda_2 < 0$. \square

2 Let A be an $n \times n$ complex matrix that satisfies $A^*A = AA^*$. Show that A is diagonalizable and that there is a unitary matrix U for which $U^*AU = \Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$.

Proof. Step 1: A and A^ are simultaneously diagonalizable.* Let $J \in M_n(\mathbb{C})$ be the Jordan Normal Form for A . I claim that J is diagonal. Suppose not. Then J contains an $m \times m$ Jordan block B for $1 < m \leq n$. If λ is the generalized eigenvalue corresponding for B , then we have $[B, B^*]_{11} = (BB^*)_{11} - (B^*B)_{11} = (|\lambda|^2 + 1) - |\lambda|^2 \neq 0$. Hence $[J, J^*] \neq 0$, so $[A, A^*] \neq 0$, a contradiction. Thus, J is diagonal. The matrix $J^* = \overline{J}^T$ is clearly diagonal also.

Step 2: A is unitarily diagonalizable. The proof is by induction on n . The base case is trivial. For the inductive step, recall that A must have an eigenvector. Let v be an normalized eigenvector of A . Let $w \in v^\perp$. Then $\langle v, Aw \rangle = \langle A^*v, w \rangle = 0$ since v is an eigenvector of both A and A^* by Step 1. Hence v^\perp is an invariant subspace of A , and we can apply the inductive hypothesis to $A|_{v^\perp}$. \square

3 Let f be continuous on $[0, 1]$, with $f(0) = f(1) = 0$ and let $s \in S^{1/n}(1, 0)$ be the linear spline interpolant to f , with knots at $x_j = \frac{j}{n}$.

(a) Let $\lambda \in \mathbb{R}$. Show that $\left| \int_0^1 s(x) e^{i\lambda x} dx \right| \leq \frac{2n^2}{\lambda^2} \omega(f, 1/n)$.

Proof. We have

$$\begin{aligned}
\left| \int_0^1 s(x) e^{i\lambda x} dx \right| &= \left| \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} s(x) e^{i\lambda x} dx \right| \\
&= \left| \sum_{k=0}^{n-1} \left[\frac{1}{i\lambda} s(x) e^{i\lambda x} \right]_{x=k/n}^{(k+1)/n} - \frac{1}{i\lambda} \int_{k/n}^{(k+1)/n} s'(x) e^{i\lambda x} dx \right| \\
&= \left| \sum_{k=0}^{n-1} \frac{1}{i\lambda} \int_{k/n}^{(k+1)/n} s'(x) e^{i\lambda x} dx \right| \\
&= \left| -\frac{1}{\lambda^2} \sum_{k=0}^{n-1} [s'(x) e^{i\lambda x}]_{x=k/n}^{(k+1)/n} \right| \\
&\leq \frac{1}{\lambda^2} \sum_{k=0}^{n-1} \left| s' \left(\frac{k+1}{n} \right) \right| + \left| s' \left(\frac{k}{n} \right) \right| \\
&\leq \frac{1}{\lambda^2} \sum_{k=0}^{n-1} 2n\omega(f, 1/n) \\
&= \frac{2n^2}{\lambda^2} \omega(f, 1/n).
\end{aligned}$$

□

(b) Use the previous part to show that $\left| \int_0^1 f(x) e^{i\lambda x} dx \right| \leq \omega(f, 1/n) + \frac{2n^2}{\lambda^2} \omega(f, 1/n)$.

Proof. We have

$$\begin{aligned}
\left| \int_0^1 f(x) e^{i\lambda x} dx \right| &\leq \left| \int_0^1 f(x) - s(x) e^{i\lambda x} dx \right| + \left| \int_0^1 s(x) e^{i\lambda x} dx \right| \\
&\leq \int_0^1 |f(x) - s(x)| dx + \frac{2n^2}{\lambda^2} \omega(f, 1/n) \\
&\leq \int_0^1 \omega(f, 1/n) dx + \frac{2n^2}{\lambda^2} \omega(f, 1/n) \\
&\leq \omega(f, 1/n) + \frac{2n^2}{\lambda^2} \omega(f, 1/n)
\end{aligned}$$

□

4 Let $\{\phi_n(x)\}_{n=0}^\infty$ be a set of polynomials orthogonal with respect to a weight function $w(x)$ on a domain $[a, b]$. Assume that the degree of ϕ_n is n , and that the coefficient of x^n in $\phi_n(x)$ is $k_n > 0$. In addition, suppose that the continuous functions are dense in $L_w^2[a, b] = \{f : [a, b] \rightarrow \mathbb{C} : \int_a^b |f(x)|^2 w(x) dx < \infty\}$.

(a) Show that ϕ_n is orthogonal to all polynomials of degree $n-1$ or less.

Proof. The set $\{\phi_k\}_{1 \leq k < n}$ spans the polynomials of degree less than $n - 1$. \square

- (b) Show that $\{\phi_n\}_{n=0}^\infty$ is complete in $L_w^2[a, b]$.

Proof. WLOG the ϕ_n are normalized. Let $g \in L_w^2[a, b]$ be continuous. Let $\epsilon > 0$. By the Weierstrauss Approximation theorem, pick a polynomial p such that $\|g - p\|_{C[a, b]} < \epsilon$. Then $\|g - p\|_{L_w^2[a, b]}^2 = \int_a^b |g - p|^2 w dx \leq \epsilon^2 \int_a^b w dx$. Since $1 = \phi_0 \in L_w^2[a, b]$, this last integral is finite. Hence, the polynomials are dense in $L_w^2[a, b]$.

Now suppose $\{\phi_n\}_{n=0}^\infty$ is not complete. By a previous homework problem, there exists a normalized function $f \in L_w^2[a, b]$ with $\langle f, \phi_n \rangle = 0$ for all n . Thus for any polynomial p , we have $\|f - p\|_{L_w^2[a, b]}^2 = \|f\|_{L_w^2[a, b]}^2 + \|p\|_{L_w^2[a, b]}^2 \geq 1$. This contradicts the fact that the polynomials are dense in $L_w^2[a, b]$. \square

- (c) Show that the polynomials satisfy the recurrence relation $\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) + C_n\phi_{n-1}(x)$. Find A_n in terms of the k_n 's.

Proof. \square

5 Suppose that $f(\theta)$ is a 2π -periodic function in $C^m(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and 2π -periodic. Here $m > 0$ is a fixed integer. Let c_k denote the k -th (complex) Fourier coefficient for f and let $c_k^{(j)}$ denote the k -th Fourier coefficient for $f^{(j)}$.

- (a) Show that $c_k^{(j)} = (ik)^j c_k$ for $1 \leq j \leq m + 1$.

Proof. Integrate by parts j times. \square

- (b) For $k \neq 0$, show that the Fourier coefficient c_k satisfies the bound

$$|c_k| \leq \frac{1}{2\pi|k|^{m+1}} \|f^{(m+1)}\|_{L_1[0, 2\pi]}$$

Proof. Integrate by parts k times. \square

- (c) Let $S_n(\theta) = \sum_{k=-n}^n c_k e^{ik\theta}$ be the n -th partial sum of the Fourier series for f , $n \geq 1$. Show that both of the following hold for f :

$$\|f - S_n\|_{L_2} \leq C \frac{\|f^{(m+1)}\|_{L_1}}{n^{m+\frac{1}{2}}} \text{ and } \|f - S_n\|_{C[0, 2\pi]} \leq C' \frac{\|f^{(m+1)}\|_{L_1}}{n^m}.$$

Proof. By Parseval's theorem, we have

$$\begin{aligned}
\|f - S_n\|_{L_2} &= \left(\sum_{k>n} |c_k|^2 \right)^{-1/2} \\
&\leq \left(\sum_{k>n} \frac{C}{|k|^{2m+2}} \|f^{(m+1)}\|_{L_1[0,2\pi]}^2 \right)^{-1/2} \\
&= \left(\sum_{k>n} \frac{C}{|k|^{2m+2}} \right)^{-1/2} \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&\leq \left(\int_{k>n} \frac{C_1}{|k|^{2m+2}} dk \right)^{-1/2} \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&= \left(\frac{C_2}{n^{2m+1}} \right)^{-1/2} \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&= \frac{C_3}{n^{m+1/2}} \|f^{(m+1)}\|_{L_1[0,2\pi]}
\end{aligned}$$

and

$$\begin{aligned}
\|f - S_n\|_{C[0,2\pi]} &= \sup_{x \in [0,2\pi]} \left| \sum_{k>n} c_k(x) e^{ikx} \right| \\
&\leq \sum_{k>n} \frac{1}{2\pi |k|^{m+1}} \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&\leq \int_{k>n} \frac{C'}{|k|^{m+1}} dk \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&= \frac{C'}{n^m} \|f^{(m+1)}\|_{L_1[0,2\pi]}.
\end{aligned}$$

□

- (d) Let $f(x)$ be the 2π -periodic function that equals $x^2(2\pi - x)^2$ when $x \in [0, 2\pi]$. Verify that f satisfies the conditions above with $m = 1$. With the help of (a), calculate the Fourier coefficients for f . (Hint: look at f'' .)

To see that f satisfies the conditions with $m = 1$, we need to check that $f'(0+) = f'(2\pi-)$ and f'' is piecewise continuous (f'' is 2π -periodic since f is). The former follows from the fact that f has double roots at 0 and 2π . The latter is obvious.

Proof. For $x \in [0, 2\pi]$, we have

$$\begin{aligned}
f(x) &= x^4 - 4\pi x^3 + 4\pi^2 x^2 \\
f'(x) &= 4x^3 - 12\pi x^2 + 8\pi^2 x \\
f''(x) &= 12x^2 - 24\pi x + 8\pi^2
\end{aligned}$$

From (a), the Fourier coefficient c_k for f is

$$\begin{aligned}
 c_k &= (ik)^{-2} c_k^{(j)} \\
 &= -\frac{1}{2\pi k^2} \int_0^{2\pi} f''(x) e^{-ikx} dx \\
 &= -\frac{1}{2\pi k^2} \int_0^{2\pi} (12x^2 - 24x) e^{-ikx} dx \\
 &= -\frac{1}{2\pi k^2} \left(\left[\frac{12x^2 - 24x}{-ik} e^{-ikx} \right]_0^{2\pi} + \frac{1}{ik} \int_0^{2\pi} (24x - 24) e^{-ikx} dx \right) \\
 &= -\frac{1}{2\pi k^2} \frac{2}{4} ik \int_0^{2\pi} x e^{-ikx} dx \\
 &= \frac{24i}{2\pi k^3} \int_0^{2\pi} x e^{-ikx} dx \\
 &= \frac{24i}{k^3} \left(\frac{i}{k} \right) \\
 &= -\frac{24}{k^4},
 \end{aligned}$$

where the penultimate equality uses the homework problem calculating the Fourier series of $g(x) = x, 0 \leq x < 2\pi$. \square