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HW 3

1 Let $\mathcal{F}$ be the constant presheaf associated to a field k on a topological space X. Find its sheafification.

Proof. For every open set $U \subset X$, let $\mathcal{G}(U) : U \to k$ denote the set of locally constant functions. I claim that $\mathcal{G}$ forms a sheaf, and that the morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ defined by the inclusions $\mathcal{F}(U) \to \mathcal{G}(U)$ is a sheafification.

Since the restriction of a locally constant function is locally constant, $\mathcal{G}$ forms a presheaf. For the gluing axiom, suppose $(U_i)_{i\in I}$ is an open cover of an open set $U \subset X$ and there exist functions $f_i \in U_i$ for all $i \in I$ such that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all $i, j \in I$. Then there exists a unique function $f : U \to k$ such that $f|_{U_i} = f_i$ for all i. To see that f is locally constant, let $x \in U$. Then $x \in U_i$ for some i. Since f_i is locally constant, there exists a neighborhood $x \in V \subset U_i$ with f_i constant on V. Since $f|_V = f_i|_V$, the function f is also constant on V. Thus f is locally constant, so $\mathcal{G}$ is a sheaf.

To check that ϕ is a sheafification, let $\mathcal{H}$ be a sheaf and let $\alpha : \mathcal{F} \to \mathcal{H}$ be a morphism of presheaves. Let $U \subset X$ be open with connected components $(U_i)_{i\in I}$. Let $\beta(U) : \mathcal{G}(U) \to \mathcal{H}(U)$ be defined as follows. Let $f \in \mathcal{G}(U)$. Then for every i, we have $f|_{U_i} \in \mathcal{F}(U_i)$ since U_i is connected. Hence $\alpha(U_i)(f|_{U_i})$ is well-defined. Let $\beta(U)(f)$ be the gluing of $(\alpha(U_i)(f|_{U_i}))_{i\in I}$.

If $f \in \mathcal{F}(U)$, then $(\beta \circ \phi)(U)(f)$ is the gluing of $(\alpha(U_i)(f|_{U_i}))_i = ((\alpha(U)(f))|_{U_i})_i$. Hence by the uniqueness of gluings $\alpha(U)(f) = (\beta \circ \phi)(U)(f)$. Hence $\beta \circ \phi = \alpha$.

For the uniqueness of β, suppose $\gamma \circ \phi = \alpha$ for some morphism of sheaves $\gamma : \mathcal{G} \to \mathcal{H}$. If $V \subset X$ is open and connected, then $\mathcal{F}(V) = \mathcal{G}(V)$ so $\phi(V)$ is an identity map, not just an inclusion. Hence $\gamma(V) = \gamma \circ \phi(V) = \alpha(V)$. If $U \subset X$ is open with connected components (U_i) and $f \in \mathcal{G}(U)$, we have $(\gamma(U)(f))|_{U_i} = \gamma(U_i)(f|_{U_i}) = \alpha(U_i)(f|_{U_i})$. Thus $\gamma(U)(f)$ is also the gluing of $(\alpha(U_i)(f))_{i\in I}$, so by the uniqueness of gluings $\gamma(U)(f) = \beta(U)(f)$. Thus $\gamma = \beta$, so ϕ is a sheafification. $\square$