Paul Gustafson
Math 643 - Algebraic Topology I

HW 7

5.3 Let
$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$
be a short exact sequence. Show that $iA \simeq A$ and $B/iA \simeq C$.

Proof. The kernel of i is trivial so the map is a bijection onto iA, hence an isomorphism $A \to iA$.

The kernel of $B \to C$ is iA and $B \to C$ is surjective, hence the induced map $B/iA \to C$ is an isomorphism. $\square$

5.5(ii) If $0 \to A_n \to A_{n-1} \to \ldots \to A_1 \to A_0 \to 0$ is an exact sequence of f.g. abelian groups, then $\sum_{i=0}^{n}(-1)^i \operatorname{span} A_i = 0$.

Proof. We use strong induction on n. Part (i) covers the cases $n \leq 2$. For $n > 3$, pick j to be greatest integer at most $n/2$. Then $0 \to A_n \to \ldots toA_j \to B \to 0$ and $0 \to B \to A_{j-1} \to \ldots A_0 \to 0$ are exact sequences of shorter length than the original, where B is the image of the map $A_j \to A_{j-1}$. By the induction hypothesis,
$$0 = (-1)^j \operatorname{span} B + \sum_{i=0}^{j-1}(-1)^i \operatorname{span} A_i = \operatorname{span} B + \sum_{i=j}^{n}(-1)^{i-j+1} \operatorname{span} A_i.$$
By multiplying the last expression by $(-1)^{j-1}$ and adding it to the other, we get the desired equality. $\square$

5.11 If $U_* \subset T_* \subset S_*$ then
$$0 \to T_*/U_* \xrightarrow{i} S_*/U_* \xrightarrow{p} S_*/T_* \to 0,$$
where $i_n : t_n + U_n \mapsto t_n + U_n$ and $p_n(s_n + U_n) = s_n + T_n$.

Proof. By the third isomorphism theorem for groups, we get short exact sequences
$$T_n/U_n \xrightarrow{i_n} S_n/U_n \xrightarrow{p_n} S_n/T_n,$$
for every n.

Moreover $U_* \subset T_* \subset S_*$ implies that i and p are chain maps, so we get the desired short exact sequence of chain complexes. $\square$