

# FINITENESS FOR MAPPING CLASS GROUP REPRESENTATIONS FROM TWISTED DIJKGRAAF-WITTEN THEORY

PAUL GUSTAFSON

ABSTRACT. Any twisted Dijkgraaf-Witten representation of a mapping class group of an orientable, compact surface with boundary has finite image.

## 1. INTRODUCTION

Given a spherical fusion category  $\mathcal{A}$  over a field  $k$  and an oriented compact surface  $M$ , possibly with boundary, the Turaev-Viro-Barrett-Westbury (TVBW) construction gives a projective representation of the mapping class group  $\mathrm{MCG}(M)$  [?, ?]. A natural question is to determine the image of these representations. In particular, when does such a representation have finite image?

It is conjectured that these representations have finite image if and only if  $\mathcal{A}$  is weakly integral. This conjecture is a modification of the Property F conjecture [?, ?], which states that braid group representations coming from a braided monoidal category  $\mathcal{C}$  should have finite image if and only if  $\mathcal{C}$  is weakly integral. Instead of only considering braid group representations, one can consider mapping class groups of arbitrary orientable surfaces. In this case, the input categories to construct the representations must be more specialized than just braided monoidal. One can either apply the Reshetikhin-Turaev construction to a modular tensor category, or apply the TVBW construction to a spherical fusion category. The former is more general than the latter since the Reshetikhin-Turaev construction for the Drinfeld center  $Z(\mathcal{A})$  of a spherical fusion category  $\mathcal{A}$  yields the same representation as the TVBW construction for  $\mathcal{A}$ . However, for the case considered in this paper, the simpler TVBW construction suffices.

In this paper, our input category is  $\mathcal{A} = \mathrm{Vec}_G^\omega$ , the spherical fusion category of  $G$ -graded vector spaces with associativity modified by a cocycle  $\omega \in Z^3(G, k^\times)$ . In this case, the TVBW construction corresponds to the twisted Dijkgraaf-Witten theory of [?]. The category  $\mathrm{Vec}_G^\omega$  is integral, so one expects its associated mapping class group representations to have finite image. The main contribution of this paper is to verify this for arbitrary  $G$  and  $\omega$ .

**Acknowledgments.** This paper would not have been written without the guidance of my advisor, Eric Rowell. I am also grateful to Zhengnan Wang and my father Robert Gustafson for their advice.

## 2. RELATED WORK

The closest related work is a result of Fjelstad and Fuchs [?] showing that, given a surface with at most one boundary component, the mapping class group representations corresponding to the untwisted (i.e.  $\omega = 1$ ) Dijkgraaf-Witten theory have finite image. Their paper uses an algebraic method of Lyubashenko [?] that gives a projective mapping class group representation to any factorizable ribbon Hopf algebra, in their case, the double  $D(G)$ . In our case, we instead consider the mapping class group action on a vector space of  $\mathrm{Vec}_G^\omega$ -colored embedded graphs defined by Kirillov [?], yielding a simpler, more geometric proof.

In [?], Bantay defined representations of mapping class groups on the Hilbert space of an orbifold model associated to  $D^\omega(G)$ . These representations appear to coincide with the twisted Dijkgraaf-Witten representations. However, the precise details of the connection are not clear to me.

More is known when we fix a particular surface  $M$ . In the case where  $M$  is a torus, Ng and Schauenburg showed that any Reshetikhin-Turaev representation of the mapping class group of the torus is finite [?]. In the case where  $M$  is an  $n$ -punctured disk, the mapping class group of  $M$  relative to the boundary of the disk is the braid group  $B_n$ . In this case, Etingof, Rowell, and Witherspoon proved that the representations associated to  $\mathrm{Mod}(D^\omega(G))$  are finite [?].

## 3. DEFINITIONS

Let  $G$  be a finite group, and let  $\omega \in Z^3(G, k^\times)$  be a 3-cocycle. Let  $V_g \in \mathrm{Obj}(\mathrm{Vec}_G^\omega)$  denote a choice of 1-dimensional vector space in degree  $g$ . We will abuse notation by referring to the object  $V_g$  by the group element  $g$ . For

the structural morphisms of the spherical fusion category  $\text{Vec}_G^\omega$ , we follow [?]. The associator  $\alpha_{g,h,k} : (g \otimes h) \otimes k \rightarrow g \otimes (h \otimes k)$  is defined to be

$$\alpha_{g,h,k} = \omega(g, h, k) \text{id}_{ghk}.$$

The evaluator  $\text{ev}_g : g^* \otimes g \rightarrow 1$  is

$$\text{ev}_g = \omega(g^{-1}, g, g^{-1}) \text{id}_1.$$

The coevaluator  $\text{coev}_g : 1 \rightarrow g \otimes g^*$  is

$$\text{coev}_g = \text{id}_1.$$

The pivotal structure  $j_g : g^{**} \rightarrow g$  is

$$j_g = \omega(g^{-1}, g, g^{-1}) \text{id}_g.$$

The following definitions and theorem are from Kirillov's paper [?]. For any spherical fusion category  $\mathcal{A}$  and surface  $M$ , he gives the following presentation of the Levin-Wen model as a vector space of colored graphs modulo local relations. He also proves that this space is canonically isomorphic to the TVBW vector space associated to  $M$ . It is straightforward to check that this isomorphism, which amounts to replacing a triangulation with its dual graph, commutes with the mapping class group action.

We will denote by  $\text{Irr}(\mathcal{A})$  the set of isomorphism classes of simple objects; abusing the language, we will frequently use the same letter  $i$  for denoting both a simple object and its isomorphism class. In those cases where it can lead to confusion, we will use notation  $X_i$  for a representative of isomorphism class  $i$ .

We define the functor  $\mathcal{A}^{\boxtimes n} \rightarrow \text{Vec}$  by

$$(1) \quad \langle V_1, \dots, V_n \rangle = \text{Hom}_{\mathcal{A}}(1, V_1 \otimes \dots \otimes V_n)$$

for any collection  $V_1, \dots, V_n$  of objects of  $\mathcal{A}$ . Note that pivotal structure gives functorial isomorphisms

$$(2) \quad z : \langle V_1, \dots, V_n \rangle \simeq \langle V_n, V_1, \dots, V_{n-1} \rangle$$

such that  $z^n = \text{id}$  (see [?]\*Section 5.3); thus, up to a canonical isomorphism, the space  $\langle V_1, \dots, V_n \rangle$  only depends on the cyclic order of  $V_1, \dots, V_n$ .

We have a natural composition map

$$(3) \quad \begin{aligned} &\langle V_1, \dots, V_n, X \rangle \otimes \langle X^*, W_1, \dots, W_m \rangle \rightarrow \langle V_1, \dots, V_n, W_1, \dots, W_m \rangle \\ &\varphi \otimes \psi \mapsto \varphi \circ_X \psi = \text{ev}_X \circ (\varphi \otimes \psi) \end{aligned}$$

where  $\text{ev}_X : X \otimes X^* \rightarrow 1$  is the evaluation morphism.

**3.1. Colored graphs.** We will consider finite graphs embedded in an oriented surface  $\Sigma$  (which may have boundary); for such a graph  $\Gamma$ , let  $E(\Gamma)$  be the set of edges. Note that edges are not oriented. Let  $E^{or}$  be the set of oriented edges, i.e. pairs  $\mathbf{e} = (e, \text{orientation of } e)$ ; for such an oriented edge  $\mathbf{e}$ , we denote by  $\bar{\mathbf{e}}$  the edge with opposite orientation.

If  $\Sigma$  has a boundary, the graph is allowed to have uncolored one-valent vertices on  $\partial\Sigma$  but no other common points with  $\partial\Sigma$ ; all other vertices will be called interior. We will call the edges of  $\Gamma$  terminating at these one-valent vertices *legs*.

**Definition 3.1.** Let  $\Sigma$  be an oriented surface (possibly with boundary) and  $\Gamma \subset \Sigma$  — an embedded graph as defined above. A coloring of  $\Gamma$  is the following data:

- Choice of an object  $V(\mathbf{e}) \in \text{Obj } \mathcal{A}$  for every oriented edge  $\mathbf{e} \in E^{or}(\Gamma)$  so that  $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$ .
- Choice of a vector  $\varphi(v) \in \langle V(\mathbf{e}_1), \dots, V(\mathbf{e}_n) \rangle$  (see (1)) for every interior vertex  $v$ , where  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are edges incident to  $v$ , taken in counterclockwise order and with outward orientation (see Figure ??).

An isomorphism  $f$  of two coloring  $\{V(\mathbf{e}), \varphi(v)\}$ ,  $\{V'(\mathbf{e}), \varphi'(v)\}$  is a collection of isomorphisms  $f_{\mathbf{e}} : V(\mathbf{e}) \simeq V'(\mathbf{e})$  which agree with isomorphisms  $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$  and which identify  $\varphi', \varphi$ :  $\varphi'(v) = f \circ \varphi(v)$ .

We will denote the set of all colored graphs on a surface  $\Sigma$  by  $\text{Graph}(\Sigma)$ .

Note that if  $\Sigma$  has a boundary, then every colored graph  $\Gamma$  defines a collection of points  $B = \{b_1, \dots, b_n\} \subset \partial\Sigma$  (the endpoints of the legs of  $\Gamma$ ) and a collection of objects  $V_b \in \text{Obj } \mathcal{A}$  for every  $b \in B$ : the colors of the legs of  $\Gamma$  taken with outgoing orientation. We will denote the pair  $(B, \{V_b\})$  by  $\mathbf{V} = \Gamma \cap \partial\Sigma$  and call it *boundary value*. We will denote

$$\text{Graph}(\Sigma, \mathbf{V}) = \text{set of all colored graphs in } \Sigma \text{ with boundary value } \mathbf{V}.$$

We can also consider formal linear combinations of colored graphs. Namely, for fixed boundary value  $\mathbf{V}$  as above, we will denote

$$(4) \quad \text{VGraph}(\Sigma, \mathbf{V}) = \{\text{formal linear combinations of graphs } \Gamma \in \text{Graph}(\Sigma, \mathbf{V})\}$$

In particular, if  $\partial\Sigma = \emptyset$ , then the only possible boundary condition is trivial ( $B = \emptyset$ ); in this case, we will just write  $\text{VGraph}(\Sigma)$ .

The following theorem is a variation of result of Reshitikhin and Turaev.

**Theorem 3.2.** *There is a unique way to assign to every colored planar graph  $\Gamma$  in a disk  $D \subset \mathbb{R}^2$  a vector*

$$(5) \quad \langle \Gamma \rangle_D \in \text{Hom}(1, V(\mathbf{e}_1) \otimes \cdots \otimes V(\mathbf{e}_n))$$

where  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are the edges of  $\Gamma$  meeting the boundary of  $D$  (legs), taken in counterclockwise order and with outgoing orientation, so that the following conditions are satisfied:

- (1)  $\langle \Gamma \rangle$  only depends on the isotopy class of  $\Gamma$ .
- (2) If  $\Gamma$  is a single vertex colored by  $\phi \in \text{Hom}(1, V(\mathbf{e}_1) \otimes \cdots \otimes V(\mathbf{e}_n))$ , then  $\langle \Gamma \rangle = \phi$ .
- (3) Local relations shown in Figure 1 hold.

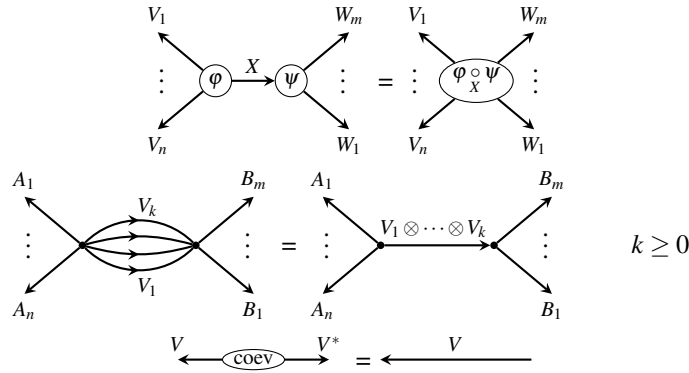


FIGURE 1. Local relations for colored graphs. Here  $\phi \circ_X \psi$  denotes the morphism  $1 \xrightarrow{\phi \otimes \psi} V_1 \otimes \cdots \otimes V_n \otimes X \otimes X^* \otimes W_1 \otimes \cdots \otimes W_m \xrightarrow{ev_X^*} V_1 \otimes \cdots \otimes V_n \otimes W_1 \otimes \cdots \otimes W_m$ .

Local relations should be understood as follows: for any pair  $\Gamma, \Gamma'$  of colored graphs which are identical outside a subdisk  $D' \subset D$ , and in this disk are homeomorphic to the graphs shown in Figure 1, we must have  $\langle \Gamma \rangle = \langle \Gamma' \rangle$ .

Moreover, so defined  $\langle \Gamma \rangle$  satisfies the following properties:

- (1)  $\langle \Gamma \rangle$  is linear in color of each vertex  $v$  (for fixed colors of edges and other vertices).
- (2)  $\langle \Gamma \rangle$  is additive in colors of edges as shown in Figure 2.

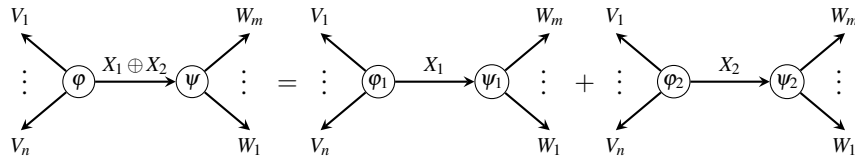


FIGURE 2. Linearity of  $\langle \Gamma \rangle$ . Here  $\phi_1, \phi_2$  are compositions of  $\phi$  with projector  $X_1 \oplus X_2 \rightarrow X_1$  (respectively,  $X_1 \oplus X_2 \rightarrow X_2$ ), and similarly for  $\psi_1, \psi_2$ .

- (3) If  $\Gamma, \Gamma'$  are two isomorphic colorings of the same graph, then  $\langle \Gamma \rangle = \langle \Gamma' \rangle$ .
- (4) Composition property: if  $D' \subset D$  is a subdisk such that  $\partial D'$  does not contain vertices of  $\Gamma$  and meets edges of  $\Gamma$  transversally, then  $\langle \Gamma \rangle_D$  will not change if we replace subgraph  $\Gamma \cap D'$  by a single vertex colored by  $\langle \Gamma \cap D' \rangle_{D'}$ .

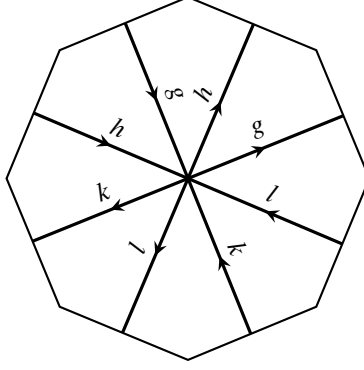


FIGURE 3. Element of the spanning set  $S$  for a genus 2 surface

The vector  $\langle \Gamma \rangle$  is called the evaluation of  $\Gamma$ .

To define local relations between graphs, Kirillov defines the space of null graphs as follows. Let  $\Gamma = c_1\Gamma_1 + \dots + c_n\Gamma_n$  be a formal linear combination of colored graphs in  $M$ . If there exists an embedded disk  $D \subset M$  such that

- (1)  $\Gamma$  is transversal to  $\partial D$  (i.e., no vertices of  $\Gamma_i$  are on the boundary of  $D$  and edges of each  $\Gamma_i$  meet  $\partial D$  transversally),
- (2) all  $\Gamma_i$  coincide outside of  $D$ ,
- (3) and  $\langle \Gamma \rangle_D = \sum c_i \langle \Gamma_i \cap D \rangle_D = 0$ ;

then  $\Gamma$  is called a null graph.

**Definition 3.3.** The representation space  $H(\Sigma, \mathbf{V})$  associated to a closed, oriented surface  $M$  by a spherical fusion category  $\mathcal{A}$  is the quotient space

$$H(\Sigma, \mathbf{V}) = \text{VGraph}(\Sigma, \mathbf{V}) / N(\Sigma, \mathbf{V})$$

where  $N(\Sigma, \mathbf{V})$  is the subspace spanned by null graphs (for all possible embedded disks  $D \subset \Sigma$ ).

#### 4. NO BOUNDARY CASE

**Theorem 4.1.** The image of any twisted Dijkgraaf-Witten representation of a mapping class group of an orientable, closed surface  $M$  is finite.

*Proof.* Let  $\Gamma$  be a  $\text{Vec}_G^{\omega}$ -colored graph embedding, and let  $g \geq 1$  be the genus of  $M$  (if  $g = 0$ , the mapping class group is trivial). Thinking of  $M$  as a quotient of its fundamental  $4g$ -gon, by isotopy we may assume that the vertices of  $\Gamma$  lie in the interior of the polygon, none of the edges of  $\Gamma$  intersect the corners of the polygon, and that the edges of  $\Gamma$  only meet the sides of the polygon transversally. Evaluating on the interior of the polygon shows that  $\Gamma$  is equivalent to a graph with a single vertex whose edges are simple closed curves, each of which intersect the boundary of the polygon precisely once. By using the local relations, we can replace all the edges intersecting a side with a single edge labeled by the tensor product of their labels. If there are no edges intersecting a side, we can insert a single edge labeled by the group identity into  $\Gamma$  that intersects only that side. Thus,  $\Gamma$  is equivalent to a colored graph with one vertex  $v$  and edges  $e_1, \dots, e_{2g}$  corresponding to the standard generators of  $\pi_1(M, v)$  as shown in Figure 3.

By Theorem 3.2 and the definition of the quotient map identifying the sides of the fundamental polygon, the vertex  $v$  is colored by an element  $\phi(v) \in \text{Hom}(1, \bigotimes_{i=1}^g V(e_{2i-1}) \otimes V(e_{2i}) \otimes V(e_{2i-1})^* \otimes V(e_{2i})^*)$ , where  $V(e_i) \in \text{Obj}(\text{Vec}_G^{\omega})$  is the coloring of the edge  $e_i$ . Since string-net evaluation is additive in the direct sum and linear in the vertex color, it follows that the representation space  $H$  is spanned by the set of such colored graphs  $\Gamma$  such that  $V(e_i) \in \text{Irr}(\text{Vec}_G^{\omega})$ .

such that each

$$S := \{\Gamma : \phi(v) = \text{id}_1\},$$

where  $\text{Irr}(\text{Vec}_G^{\omega})$  is the a complete set of representatives for the isomorphism classes of simple objects of  $\text{Vec}_G^{\omega}$ , which correspond to the elements of  $G$ .

The mapping class group of  $M$  is generated by the Lickorish generating set consisting of Dehn twists around  $3g - 1$  simple closed curves. These can be divided into two types of twists: the ones around a single hole (the blue and green curves in Figure 4), and the ones connecting two holes (the red curves).

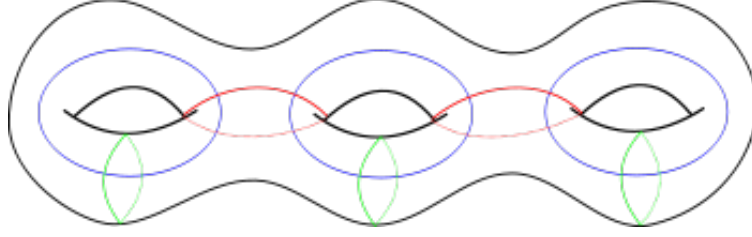


FIGURE 4. Lickorish generating set (Image source: [https://en.wikipedia.org/wiki/Dehn\\_twist](https://en.wikipedia.org/wiki/Dehn_twist))

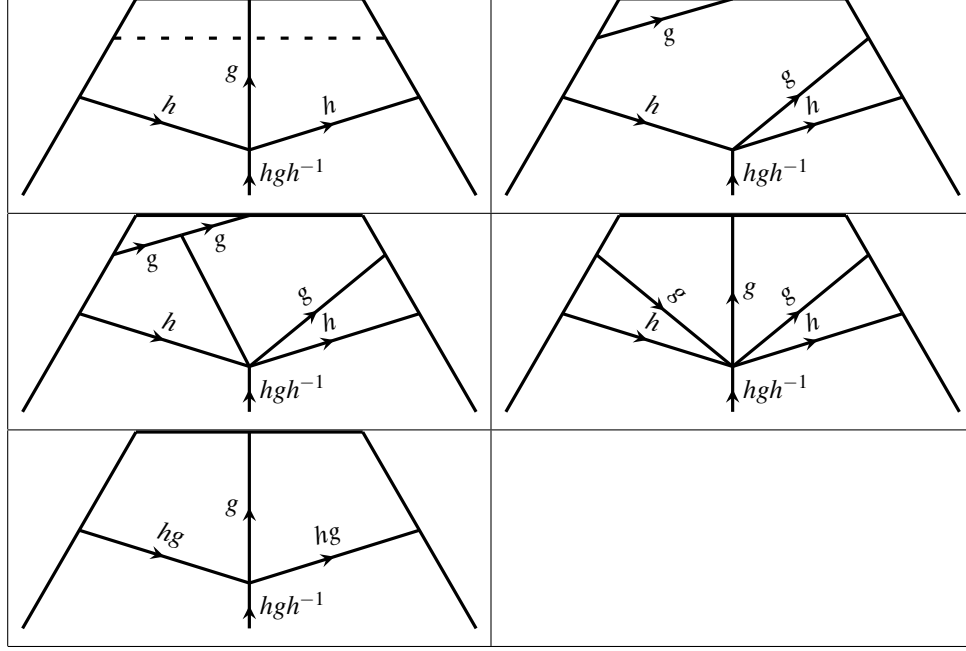


TABLE 1. First type of Dehn twist. Unlabeled interior edges are colored by the group identity element.

1 Using the local moves as in Tables 1 and 2, one sees that the result of each of these Dehn twists lies in  $\text{Im}(\omega)S$   
2 since the structural morphisms of  $\text{Vec}_G^\omega$  have coefficients in  $\text{Im}(\omega)$ . It is a basic result in group cohomology that, by  
3 replacing  $\omega$  with a cohomologous cocycle if necessary, the image of  $\omega$  lies in  $\mu_{|G|}$ , the set of  $|G|$ -th roots of unity.  
4 Since cohomologous cocycles give rise to equivalent spherical categories  $\text{Vec}_G^\omega$ , this replacement does not incur any  
5 loss in generality.

6 Thus, the image of any such mapping class group representation is a quotient of the group of permutations of the  
7 finite set  $\mu_{|G|}S$ , hence finite.  $\square$

8 **Remark 4.2.** When  $\omega = 1$  and  $M$  is closed, this representation is a permutation representation.

9 *Proof.* This fact follows from Theorem 2.6 in [?] under the plausible assumption that their representation coincides  
10 with ours. However, we can also see this directly by first noting that  $G$  acts on  $S$  by simultaneously conjugation of  
11 all edge labels by a single element  $g \in G$ . If  $s \in S$  and  $g \in G$ , then we can retrieve  $s$  from  $gs$  by separating two  
12 oppositely oriented,  $g$ -labeled edges from each edge in the embedded graph  $gs$ . This results in a loop labeled by  $g$ ,  
13 whose evaluation is 1. Thus  $gs$  is equivalent to  $s$ . Moreover, the cardinality  $|S/G| = |\text{Hom}(\pi_1(M), G)|/|G|$  is equal  
14 to the dimension of the untwisted Dijkgraaf-Witten representation space  $H$  [?]. Hence,  $S/G$  is a basis for  $H$ . The  
15 mapping class group action on  $S$  commutes with the  $G$ -action, so the mapping class group permutes  $S/G$ , i.e.  $H$  is a  
16 permutation representation.  $\square$

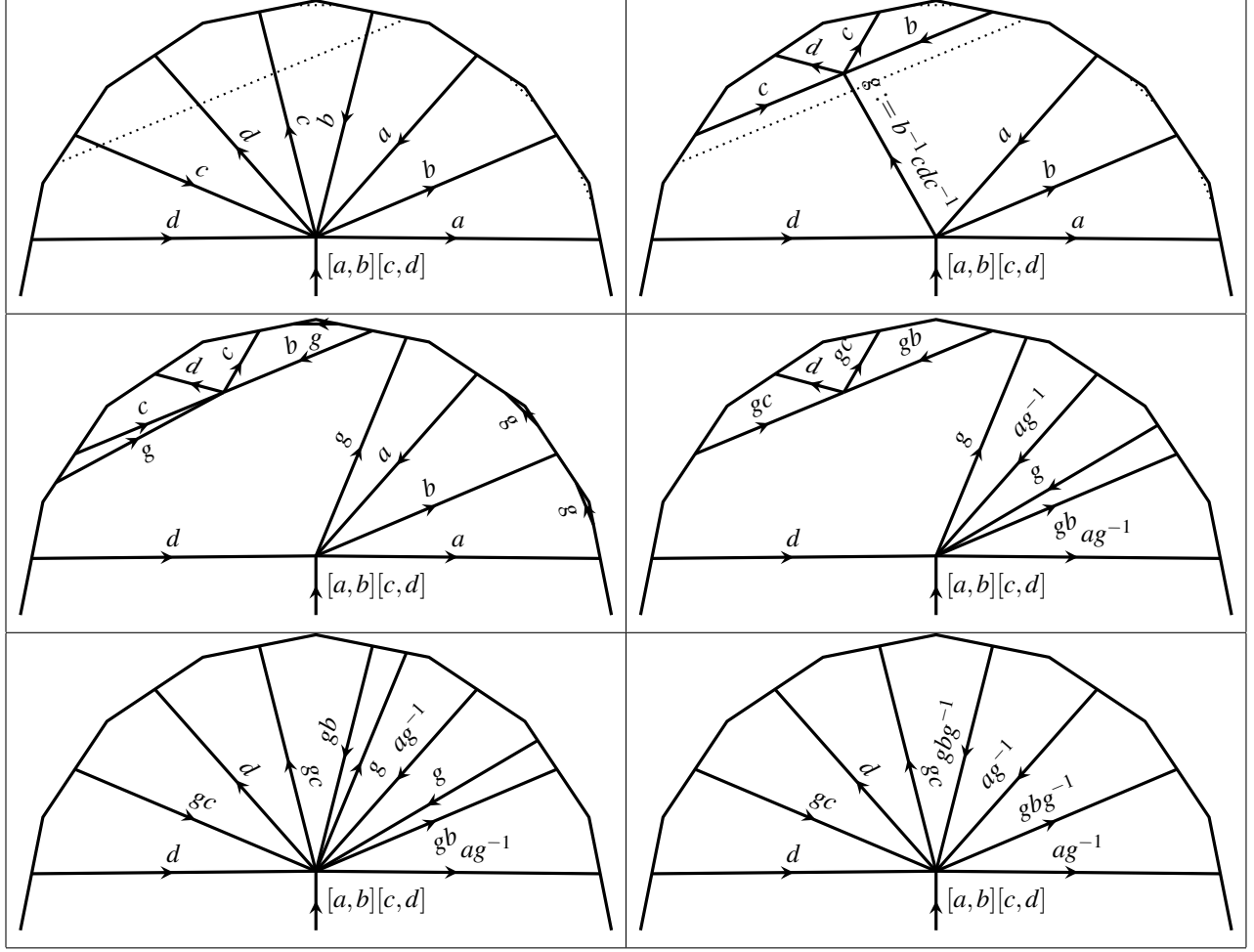


TABLE 2. Second type of Dehn twist.

## 5. BOUNDARY CASE

When  $M$  has boundary, we define the mapping class group  $\text{MCG}(M)$  to be the group of isotopy classes of homeomorphisms fixing the boundary of  $M$  setwise. There are mapping class group representations for every labelling of the boundary components  $\pi_0(\partial M) \rightarrow \text{Obj}(\mathcal{Z}(\text{Vec}_\omega^G))$ . Given such a labelling, the representation space is spanned by colored graphs for which the coloring of a leg ending in a boundary component coincides with the image of the forgetful functor  $F : \mathcal{Z}(\text{Vec}_\omega^G) \rightarrow \text{Vec}_\omega^G$  on the labelling of that boundary component. The same local relations are valid in this representation space [?].

By a similar argument as in the proof of the Theorem 4.1, any such representation space has a spanning set  $S$  of colored graphs with a single vertex, loops for each of the usual generators of the fundamental group of  $M$ , and a leg from the vertex to each of the boundary components. Each of the edges is labeled by a simple object, i.e. an element of the group  $G$ .

Let  $N$  denote the closed surface obtained by filling in all the boundary components of  $M$  with disks. The mapping class group  $\text{MCG}(M)$  is generated by the same Dehn twists as  $\text{MCG}(N)$ , as well as braids interchanging boundary components and mapping classes corresponding to dragging a boundary component along a representative of a standard generator of  $\pi_1(N)$  [?]. It is straightforward to check that applying any of these generators of  $\text{MCG}(M)$  to a colored graph in  $S$  yields an element in  $\text{Im}(\omega)S$  (see Tables 3 and 4). Since the braid group is also generated by such braids, we have the following theorem.

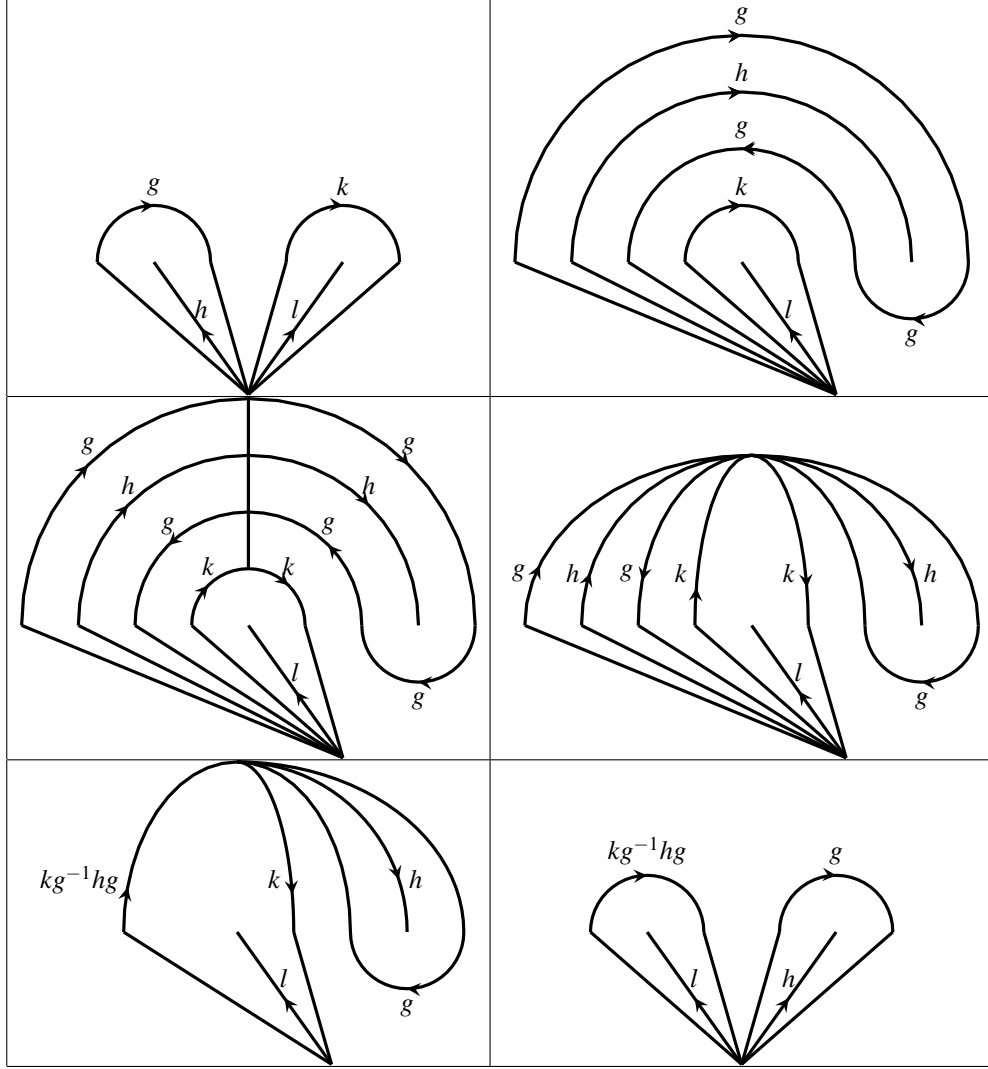


TABLE 3. A braid generator. Unlabeled interior edges are colored by the group identity element.

**Theorem 5.1.** *The image of any twisted Dijkgraaf-Witten representation of a mapping class group of an orientable, compact surface with boundary is finite. In particular, any such braid group representation is also finite.*

## 6. FURTHER DIRECTIONS

We have proved that every twisted Dijkgraaf-Witten representation of a mapping class group of a compact, orientable surface has finite image. This is a generalization of the results of [?] and [?], as well as another step towards the (modified) Property F conjecture. A potential next step would be to consider more complicated spherical categories than  $\text{Vec}_\omega^G$ . One candidate is the class of Tambara-Yamagami categories [?]. The main additional complication here is the appearance of multifusion channels, i.e. the tensor product of two simple objects can be a direct sum of multiple simple objects.

*E-mail address:* pgustafs@math.tamu.edu

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX U.S.A.

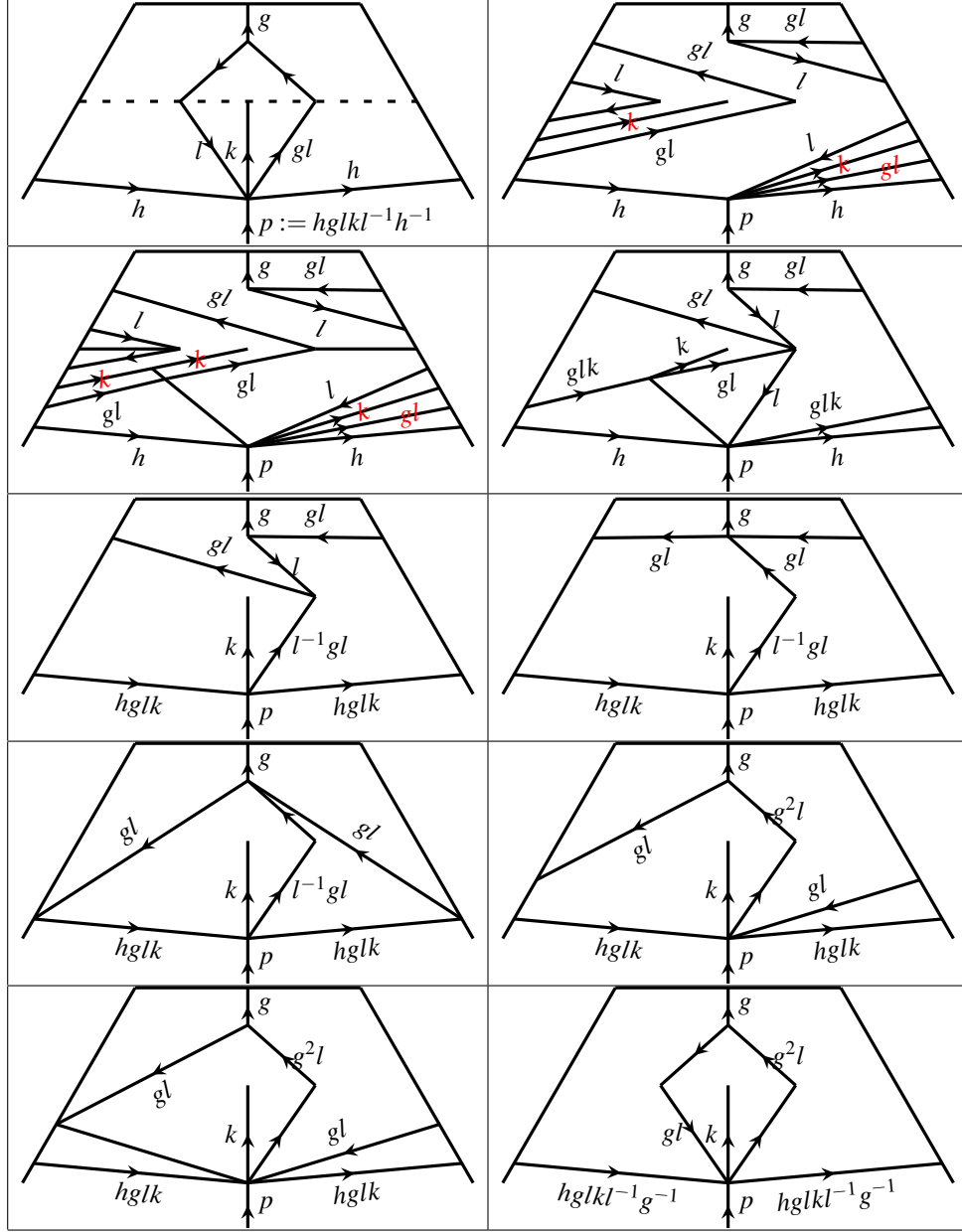


TABLE 4. Pulling a boundary component along a generator for the fundamental group of the corresponding closed surface. Unlabeled interior edges are colored by the group identity element.