Paul Gustafson
Math 644

HW 1

1 Given a (left) R-module show:

i. The covariant functor $\mathrm{Hom}_R(M,-)$ is a left-exact functor.

Proof. Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be a short exact sequence. Application of the functor gives a complex

$$0 \to \mathrm{Hom}_R(M,A) \xrightarrow{f_*} \mathrm{Hom}_R(M,B) \xrightarrow{g_*} \mathrm{Hom}_R(M,C) \to 0.$$

For exactness at $\mathrm{Hom}_R(M,A)$, suppose $f_*(\alpha) = 0$ for some $\alpha : M \to A$. Then $f(\alpha(m)) = 0$ for all $m \in M$. Thus, $\alpha(m) = 0$ for all $m \in M$ since f is injective. Thus f_* is injective.

For the exactness at $\mathrm{Hom}_R(M,B)$, suppose $g_*(\beta) = 0$ for some $\beta : M \to B$. Then $\mathrm{im}(\beta) \subset \ker(g)$. Since f is an isomorphism from A to $\mathrm{im}(A)$, the map $f^{-1}\beta : M \to A$ is well-defined. Thus, $\beta = f_*(f^{-1}\beta)$ is in the image of f_*. Thus $\ker(g_*) = \mathrm{im}(f_*)$. □

ii. This functor is right-exact iff M is a projective R-module.

Proof. In view of part (i), for the functor to be right-exact is the same as saying that g_* surjects onto $\mathrm{Hom}_R(M,C)$ for every surjection $g : B \to C$. This is the same as saying that every map $M \to C$ lifts through every surjection $B \to C$, i.e. M satisfies the definition of projective R-module. □

2 Given an R-module M and a short exact sequence of R-modules

$$0 \to A \to B \to C \to 0,$$

use the previous problem to show that the sequence induces a long exact sequence:

$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to Ext^1_R(M,A) \to \cdots$$

Proof. Let P_* be a projective resolution of M. Since the P_i are projective we get a s.e.s. of chain complexes $0 \to \mathrm{Hom}_R(P_*,A) \to \mathrm{Hom}_R(P_*,B) \to \mathrm{Hom}_R(P_*,C) \to 0$. Applying the cohomology functor and the snake lemma gives the desired long exact sequence. □

3 Regarding $\mathbb{Z}_2$ as a module over the ring $\mathbb{Z}_4$, construct a resolution of $\mathbb{Z}_2$ by free modules over $\mathbb{Z}_4$ and use this to show that $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2,\mathbb{Z}_2)$ is nonzero for all n.

Proof. A free resolution is the following:

$$\cdots \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\bmod 2} \mathbb{Z}_2 \to 0.$$

Applying the $\mathrm{Hom}_{\mathbb{Z}_4}(-, \mathbb{Z}_2)$ functor, we get

$$\cdots \xleftarrow{0} \mathbb{Z}_2 \xleftarrow{0} \mathbb{Z}_2 \xleftarrow{0} \mathbb{Z}_2 \xleftarrow{\mathrm{id}} \mathbb{Z}_2 \leftarrow 0.$$

Thus $\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) = \mathbb{Z}_2$ for all $n \geq 1$. $\square$