Finiteness of mapping class group representations from twisted Dijkgraaf-Witten theory

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Mapping class groups

- The mapping class group of a compact surface Σ, $\mathrm{MCG}(\Sigma)$, is the group of isotopy classes of orientation-preserving self-homeomorphisms of Σ
  - $\mathrm{MCG}(\mathbf{D}$ with n marked points$) = B_m$
  - $\mathrm{MCG}(\mathbf{T}^2) = SL(2, \mathbb{Z})$

Property F conjecture for mapping class groups (Rowell)

The Turaev-Viro-Barrett-Westbury (TVBW) mapping class group representation associated to a compact surface Σ and spherical fusion category $\mathcal{A}$ has finite image iff $\mathcal{A}$ is weakly integral.

The spherical fusion category Vect_G^ω

Vect_G^ω, the category G-graded vector spaces twisted by a 3-cocycle ω has the following structural morphisms:

- The associator $\alpha_{g,h,k} : (V_g \otimes V_h) \otimes V_k \to V_g \otimes (V_h \otimes V_k)$
$$\alpha_{g,h,k} = \omega(g, h, k)$$
- The evaluator $\mathrm{ev}_g : V_g^* \otimes V_g \to 1$
$$ev_g = \omega(g^{-1}, g, g^{-1})$$
- The coevaluator $\mathrm{coev}_g : V_g \otimes V_g^* \to 1$
$$\mathrm{coev}_g = 1$$
- The pivotal structure $j_g : V_g^{**} \to V_g$
$$j_g = \omega(g^{-1}, g, g^{-1})$$

Related Work

- All Vect_G^ω braid group representations have finite images (Etingof–Rowell–Witherspoon)
- If $\omega = 1$, every mapping class group representation of a closed surface with ≤ 1 marked point has finite image (Fjestad–Fuchs)
- Every $SL(2, \mathbf{Z})$ representation from any modular category has finite image (Ng–Schauenberg)

Main result

The image of any Vect_G^ω TVBW representation of a mapping class group of an orientable, compact surface with boundary is finite.

Proof outline:

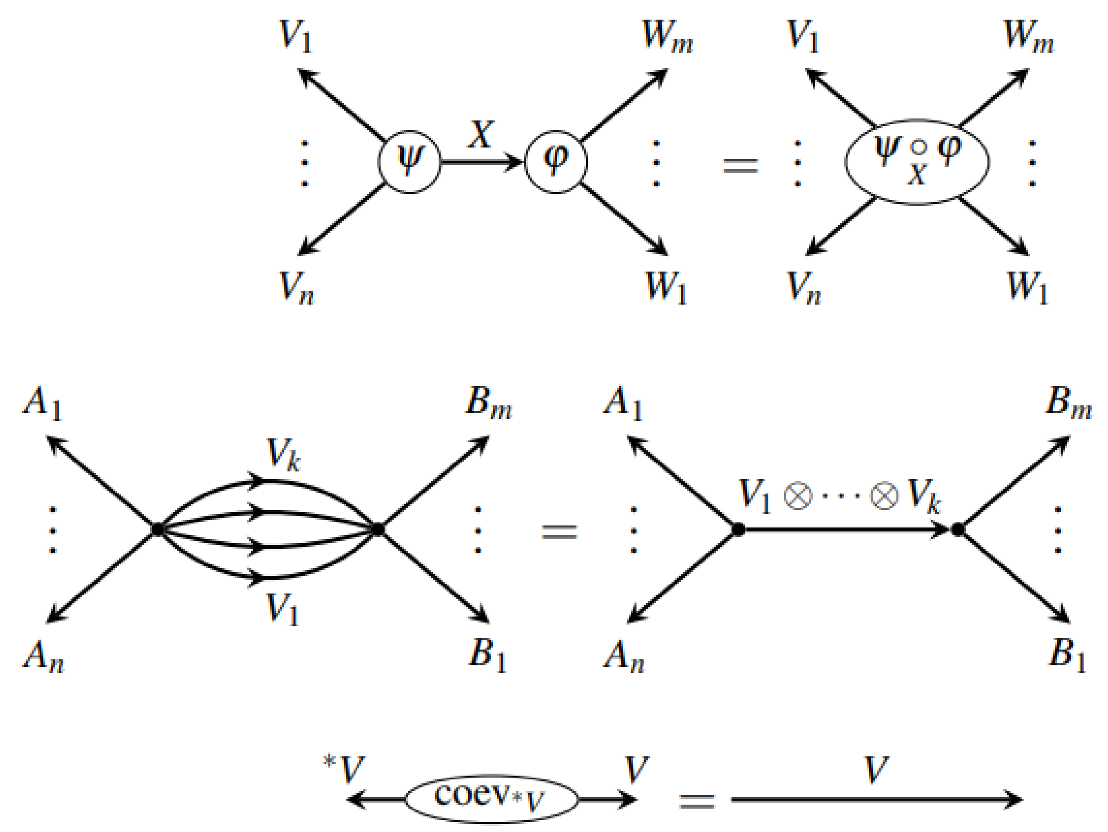
- Describe a tractable presentation of the representation space
- Find a good finite spanning set S for the representation space
- Calculate the action of each Birman generator on S
- Show that the representation of each Birman generator lies in a quotient of a finite group of monomial matrices.

The TVBW space associated to a surface

- The TVBW representation space is canonically isomorphic to a vector space of formal linear combinations of $\mathcal{A}$-colored graphs in Σ modulo certain local relations (Kirillov).

Local relations

- Isotopy of the graph embedding
- Linearity in the vertex colorings
- And the following:



$k \geq 0$

Spanning set for genus 2 closed surface

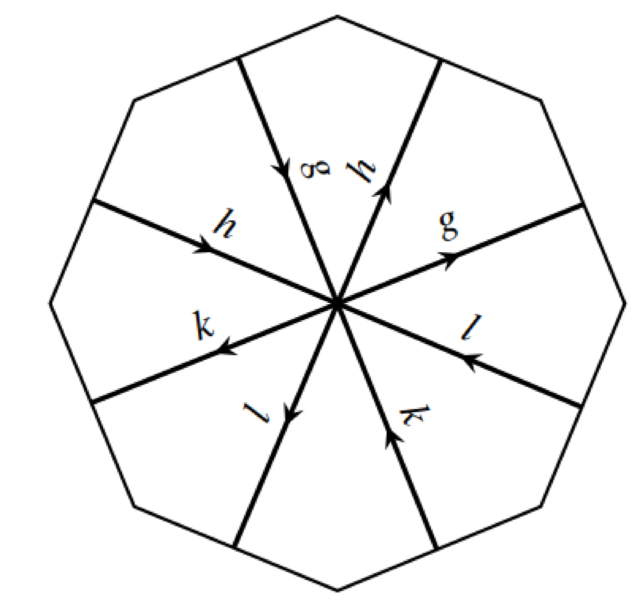


Figure: Element of the spanning set for a genus 2 surface. Here $[g, h][k, l] = 1$, and the vertex is labeled by a "simple" morphism (a $|G|$-th root of unity times a canonical morphism)

One of the generators: a Dehn twist

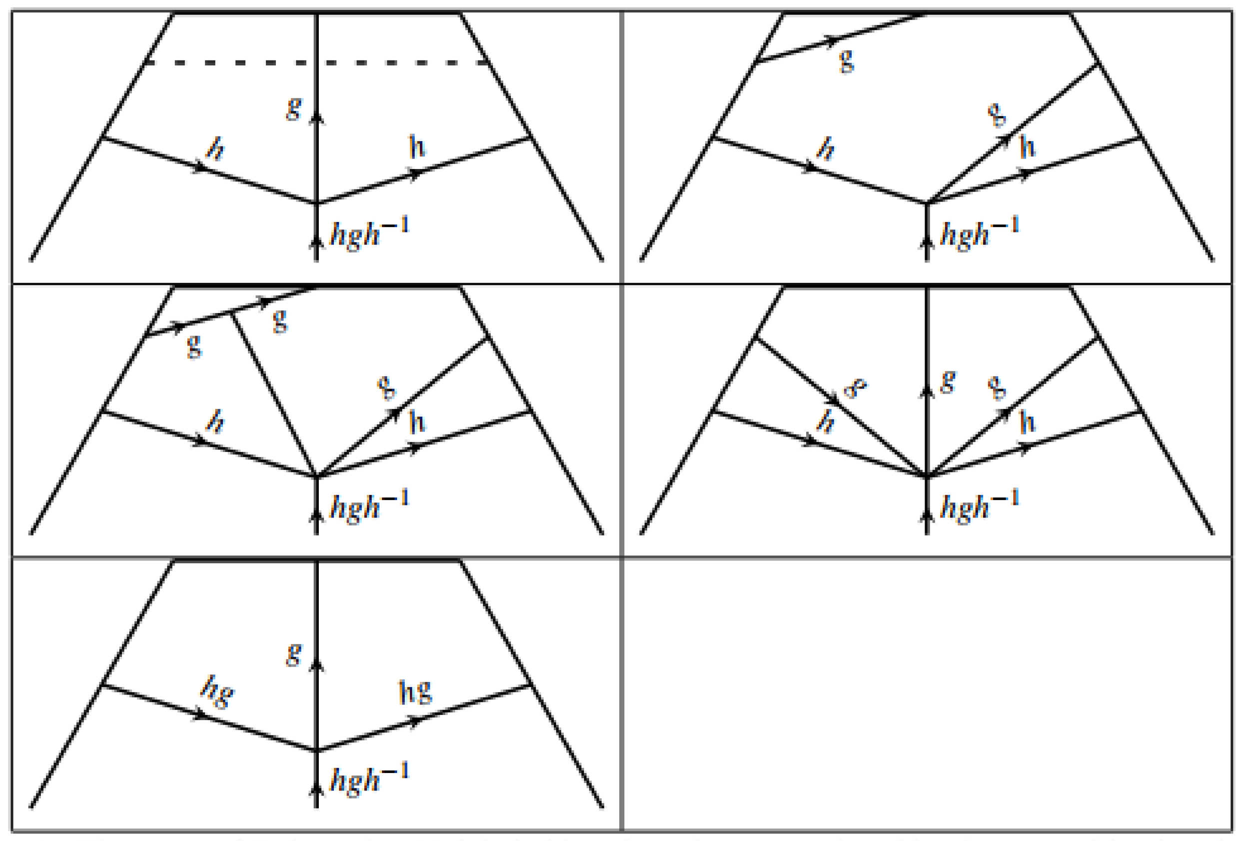


TABLE 1. First type of Dehn twist. Unlabeled interior edges are colored by the group identity element.

References

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