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Math 644

HW 1

1 To see that (L, ϕ^*) is compatible with the directed system, suppose $i \prec j$. Let $x \in C_i$. We have $\phi^j \circ \phi^i_j(x) = q(\phi^i_j(x)) = q(x) = \phi^i(x)$.

To see that L is the colimit, suppose (D, f^*) is compatible with the directed system. Let $\widetilde{f} : \oplus_{i \in I} C_i \to D$ be the unique map such that $f^i = \widetilde{f} \circ \iota_i$ for all i, where $\iota_i : C_i \to \oplus_{i \in I} C_i$ is the inclusion. To see that $\widetilde{f}$ descends to L, let $x \in C_i$ for some i, and suppose $i \prec j$. Then $\widetilde{f}(\phi^i_j(x) - x) = f_j \circ \phi^i_j(x) - f^i(x) = 0$. Hence $\widetilde{f}$ is 0 on H_ϕ, so $\widetilde{f}$ descends to a function $f : L \to D$. The uniqueness of f follows from the uniqueness of $\widetilde{f}$ (if an f' replaced f, then $f' \circ q$ would coincide with $\widetilde{f}$).

For the second part of the problem, pick any $\widetilde{l} \in q^{-1}(l)$. Pick $j \in I$ dominating the finite support of $\widetilde{l}$. Let $x = \sum_{i \in I} \phi^i_j(\widetilde{l})$ where ϕ^i_j is extended to be 0 outside of C_i. Then $x - \widetilde{l} \in H_\phi$, so $\phi^j(x) = l$.

2 Let (L, ψ^*) be the colimit as defined in Problem 1. Define $\alpha : \oplus_i C_i \to \mathbb{Q}$ by $\alpha \circ \iota_i(a) = a/i$. This map is a surjective homomorphism, so it suffices to show that its kernel is H_ϕ. To see that $H_\phi \subset \ker \alpha$, suppose $x \in C_i$ and $i \prec j$. Then $\alpha(\iota_i x - \iota_j \phi^i_j(x)) = x/i - (j/i)(x/j) = 0$. For the reverse inclusion, suppose $\alpha(l) = 0$. Let k dominate the the support of l (i.e. k is a multiple of all indices of nonzero coordinates of l). We have $0 = \alpha(l) = \sum_{i \in I} l_i/i = \sum_{i \in I} (k/i) l_i/k$. Hence $0 = \sum_{i \in I} (k/i) l_i = \sum_{i \in I} \phi^i_k(l_i)$. Hence $l \in H_\phi$.

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i. In the inclusion order, property (b) implies that the partial order is directed. In the reverse inclusion order, property (a) implies that the empty set is in Φ, which dominates every other set.

ii. A closed subset of a compact Hausdorff set is compact. A finite union of compact sets is compact.

iii. If $F \subset G$, we have a map of pairs $(X, X - G) \to (X, X - F)$ given by id_X. Let ϕ^F_G be the image of this map under the contravariant functor H^n.

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i. Let $\phi^i : C_i \to \operatorname{colim}_{i \in I} C_i$ and $\psi^k : C_k \to \operatorname{colim}_{k \in K} C_k$ be the compatible families of morphisms satisfying the respective universal properties. Let $g : \operatorname{colim}_{k \in K} C_k \to \operatorname{colim}_{i \in I} C_i$ be the unique map such that $\phi^k = g\psi^k$ for all $k \in K$.

   For each $i \in I$, pick any $k \in K$ with $i \prec k$, and define $f^i = \psi^k \phi^i_k$. To see that the particular choice of k does not matter, let $k' \in K$ with $i \prec k'$. Pick any $j \in K$ with $k, k' \prec j$. We have $\psi^k \phi^i_k = \psi^j \phi^k_j \phi^i_k = \psi^j \phi^i_j$, and the same for k' in place of k. Hence f^i is independent of the choice of k.

I claim $(f^i)_{i\in I}$ is compatible with the directed system $(C_i)_{i\in I}$. Indeed if $i \prec j \in I$, pick $k \in K$ such that $i, j \prec k$. Then $f^j \phi^i_j = \psi^k \phi^j_k \phi^i_j = \psi^k \phi^i_k = f^i$.

Thus there exists a unique map $f : \operatorname{colim}_{i\in I} C_i \to \operatorname{colim}_{k\in K} C_k$ such that $f^i = f\phi^i$ for all $i \in I$.

I claim that f is a left and right inverse for g. For all $k \in K$, we have $fg\psi^k = f\phi^k = f^k = \psi^k$. Hence $fg = \mathrm{id}$, by the uniqueness of the map in the colimit universal property.

Similarly, for all $i \in I$, we have $gf\phi^i = gf^i = g\psi^k\phi^i_k = \phi^k\phi^i_k = \phi^i$, where $k \in K$ is any index such that $i \prec k$. This implies $gf = \mathrm{id}$.

Hence f and g are isomorphisms.

ii. Any compact subset of $\mathbb{R}^n$ is bounded.

iii. From (ii), we have $H^r_c(\mathbb{R}^n; R) = \operatorname{colim}_{i\in\mathbb{N}} H^r(\mathbb{R}^n, \mathbb{R}^n - B_i(0); R)$, where the ϕ^i_j are induced by inclusions. From the long exact sequence of a pair, we get $\widetilde{H}^{r-1}(\mathbb{R}^n - B_i(0); R) \cong \widetilde{H}^r(\mathbb{R}^n, \mathbb{R}^n - B_i(0); R)$ for all r. Thus $H^r(\mathbb{R}^n, \mathbb{R}^n - B_i(0); R) = \widetilde{H}^r(\mathbb{R}^n, \mathbb{R}^n - B_i(0); R) = R$ if $r = n$, and 0 otherwise.

Since the inclusion $\mathbb{R}^n - B_j \to \mathbb{R}^n - B_i$ is a homotopy equivalence for any $i < j$, the maps ϕ^i_j are isomorphisms. It follows that $H^r_c(\mathbb{R}^n; R) = \operatorname{colim}_{i\in\mathbb{N}} H^r(\mathbb{R}^n, \mathbb{R}^n - B_i(0); R) = H^r(\mathbb{R}^n, \mathbb{R}^n - B_1(0); R) = R$ if $r = n$, and 0 otherwise.

5 Let Φ be the cofinal system of $\mathcal{K}_U$ defined by taking the complement of each neighborhood in the cofinal system mentioned in the problem.

We have $H^r_c(U; R) = \operatorname{colim}_{F\in\Phi} H^r(U, U - F; R) = \operatorname{colim}_{F\in\Phi} H^r(X, X - F; R)$, where the last equality is by excision since $Y = \overline{Y} \subset (X - F)^\circ = X - F \subset X$. Since $X - F$ strong deformation retracts to Y for all F, any inclusion $X - F \to X - F'$ for $F, F' \in \Phi$ maps to id_Y under the homotopy equivalences induced by the deformation retractions. Thus, $H^r_c(U; R) = \operatorname{colim}_{F\in\Phi} H^r(X, Y; R)$, where the morphisms of the directed system are all isomorphisms. Thus, $H^r_c(U; R) = \operatorname{colim}_{F\in\Phi} H^r(X, Y; R)$.

6 3.2.2. Since A and B are contractible, we have an isomorphism $f^* : \widetilde{H}^k(X; R) \to H^k(X, A; R)$ induced from a map $f : (X, a) \to (X, A)$ for some $a \in A$. Similarly, we have an isomorphism $g^* : \widetilde{H}^l(X; R) \to H^l(X, A; R)$. Also, the cup product $H^k(X, A; R) \times H^l(X, B; R) \to H^{k+l}(X, A \cup B; R) = H^{k+l}(X, X; R) = 0$ must be 0. Hence, by relative cohomology version of Proposition 3.10, if $\alpha \in H^k(X; R)$ and $\beta \in H^l(X; R)$ for $k, l > 0$, we have $\alpha \cup \beta = 0$. The same proof works for the general case.

7 3.2.3.

(a) Suppose such a map $f : \mathbb{R}P^n \to \mathbb{R}P^m$ exists. We have $H^*(\mathbb{R}P^n; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/\alpha^{n+1}$ with $|\alpha| = 1$. Hence $f^* : \mathbb{Z}_2[\alpha]/\alpha^{m+1} \to \mathbb{Z}_2[\beta]/\beta^{n+1}$ is a ring homomorphism mapping α to β. This implies $\beta^{m+1} = 0$, a contradiction.

In the complex case, $|\alpha| = 2$, so the corresponding result would replace H^1 with H^2.

(b) To see that g is nonzero, use the same proof as the proof of Prop. 2B.6 on p. 175 with g replacing f, using cohomology instead of homology.

8 3.2.7. The $\mathbb{Z}_2$-cohomology ring of $\mathbb{R}P^3$ is $Z_2[\alpha]/\alpha^4$ where $|\alpha| = 1$. The reduced $\mathbb{Z}_2$-cohomology ring of $\mathbb{R}P^2 \vee S^3$ is $\widetilde{H}^*(\mathbb{R}P^2) \oplus \widetilde{H}^*(S^3)$. In particular, the cube of any element of $H^1(\mathbb{R}P^2 \vee S^3)$ is 0, which does not hold for $H^1(\mathbb{R}P^3)$.
9 3.2.8. We have $H^*(\mathbb{C}P^2; \mathbb{Z}) = \mathbb{Z}[\alpha]/\alpha^3$ where $|\alpha| = 2$. The attachment of the 3-cell removes the H^2 cohomology and makes $H^3(X) = \mathbb{Z}^p$.

On the other hand, $\widetilde{H}^*(M(\mathbb{Z}_p, 2) \vee S^4; \mathbb{Z}) = \widetilde{H}^*(M(\mathbb{Z}_p, 2)) \oplus \widetilde{H}^*(S^4)$. By the universal coefficient theorem, $\widetilde{H}^r(M(\mathbb{Z}_p, 2)) = \mathbb{Z}_p$ if $r = 3$ and 0 otherwise.

Both reduced cohomology rings only have cohomology in dimensions 3 and 4, so the cup product is zero. Hence, mapping 1 to 1 for H^0 and a generator to a generator for H^3 and H^4 defines an isomorphism of the two unreduced cohomology rings.

Using the universal coefficient theorem in the $\mathbb{Z}_p$-coefficients case, we get $H^r(X) = \mathbb{Z}_p$ if $r = 0, 2, 4$, and the same for $H^r((M(\mathbb{Z}_p, 2) \vee S^4)$. In the latter case, multiplying any two elements of H^2 gives 0. In fact, the square of any element of the reduced cohomology ring is 0.

On the other hand, the square of a generator of $H^2(X; Z_p)$ is a non-zero element of $H^4(X; Z_p)$, so the rings cannot be isomorphic.
10 3.2.9 Let I be a basis for $H_n(X; \mathbb{Z})$. Using the universal coefficient theorem for the first and last equalities, we have

$$\begin{aligned} H^n(X; \mathbb{Z}_p) &= \mathrm{Hom}(H_n(X; \mathbb{Z}), \mathbb{Z}_p) \\ &= \mathrm{Hom}(\bigoplus_{i \in I} \mathbb{Z}, \mathbb{Z}_p) \\ &= \bigoplus_{i \in I} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_p) \\ &= \bigoplus_{i \in I} \mathbb{Z}_p \\ &= \bigoplus_{i \in I} \mathbb{Z} \otimes \mathbb{Z}_p \\ &= \mathrm{Hom}(H_n(X; \mathbb{Z}), \mathbb{Z}) \otimes \mathbb{Z}_p \\ &= H^n(X; \mathbb{Z}) \otimes \mathbb{Z}_p. \end{aligned}$$