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HW 5

1 Let $g \in C^2[a,b]$, and $h = b - a$. Show that if $g(a) = g(b) = 0$, then

$$\|g\|_{C[a,b]} \leq (h^2/8)\|g''\|_{C[a,b]}.$$

Give an example showing that $1/8$ is the best possible constant.

Proof. We have

$$\begin{aligned}
g(x) &= \int_a^x g'(t)\,dt \\
&= \int_a^x g'(a) + \int_a^t g''(s)\,ds\,dt \\
&\leq (x-a)g'(a) + \int_a^x (t-a)\|g''\|\,dt \\
&= (x-a)g'(a) + \left[(t^2/2 - at)\|g''\|\right]_{t=a}^x \\
&= (x-a)g'(a) + (x^2/2 - ax - a^2/2 + a^2)\|g''\| \\
&= (x-a)g'(a) + \frac{1}{2}(x-a)^2\|g''\|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
g(x) &= \int_b^x g'(t)\,dt \\
&= \int_b^x g'(b) + \int_b^t g''(s)\,ds\,dt \\
&\leq (x-b)g'(b) + \int_b^x (t-b)\|g''\|\,dt \\
&= (x-b)g'(b) + \left[(t^2/2 - bt)\|g''\|\right]_{t=b}^x \\
&= (x-b)g'(b) + (x^2/2 - bx - b^2/2 + b^2)\|g''\| \\
&= (x-b)g'(b) + \frac{1}{2}(x-b)^2\|g''\|.
\end{aligned}$$

Adding the inequalities and dividing by 2, we have

□