Paul Gustafson
Texas A&M University - Math 641
Instructor - Fran Narcowich

HW 4

1 Let A and B be self-adjoint matrices, which may be real or complex. We say that $A \le B$ if and only if $\langle A\mathbf{x}, \mathbf{x}\rangle \le \langle B\mathbf{x}, \mathbf{x}\rangle$ for all $\mathbf{x}$.

a. If $\lambda_1 \ge \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A and $\tilde{\lambda}_1 \ge \tilde{\lambda}_2, \ldots, \tilde{\lambda}_n$ are the eigenvalues of B, then show that $\lambda_k \le \tilde{\lambda}_k$.

b. Show that $\mathrm{Trace}(A) \le \mathrm{Trace}(B)$ if $A \le B$.

c. Show that if we increase a diagonal entry of A, then the resulting matrix B satisfies $A \le B$.

d. (Keener, problem 1.3(b)). Use the previous part to estimate the lowest eigenvalue of the matrix below. Keener gets $-\frac{1}{3}$. Using matlab you get less than about -2. Can you beat $-\frac{1}{3}$?

$$A = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & -4 \\ 4 & -4 & 3 \end{pmatrix}$$

Proof. For (a), $\square$

2 Let A be a self-adjoint matrix with eigenvalues $\lambda_1 \ge \lambda_2, \ldots, \ge \lambda_n$. Show that for $2 \le k < n$ we have

$$\max_U \sum_{j=1}^{k} \langle A u_j, u_j\rangle = \sum_{j=1}^{k} \lambda_j,$$

where $U = \{u_1, \ldots, u_k\}$ is any o.n. set. (Hint: Put A in diagonal form and use a judicious choice of B.)

3 Show that ℓ^∞ is a Banach space under the norm $\|\{x_j\}\| = \sup_j |x_j|$

4 Show that ℓ^2 is a Hilbert space under the inner product

$$\langle \{x_j\}, \{y_j\}\rangle := \sum_{j=1}^{\infty} \bar{y}_j x_j.$$

5 Let $0 \le \delta \le 1$. We define the modulus of continuity for $f \in C[0,1]$ by

$$\omega(f;\delta) := \sup_{|s-t| \le \delta} |f(s) - f(t)|, \text{ where } s, t \in [0,1].$$

a. Explain why $\omega(f;\delta)$ exists for every $f \in C[0,1]$.

b. Fix δ. Let $S_\delta = \{\epsilon > 0 : |f(t) - f(s)| < \epsilon \text{ for all } |s-t| \le \delta\}$. Show that $\omega(f;\delta) = \inf S_\delta$.

c. Show that $\omega(f;\delta)$ is nondecreasing as a function of δ.

d. Show that $\lim_{\delta \downarrow 0} \omega(f;\delta) = 0$.