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HW 9

1 Finish the proof of the Projection Theorem: If for every $f \in \mathcal{H}$ there is a $p \in V$ such that $\|p - f\| = \min_{v \in V} \|v - f\|$ the V is closed.

Proof. Let $(f_n) \subset V$ with $f_n \to f$. Suppose $f \notin V$. Let p be the projection of f to V. Then $\|f_n - f\| \geq \|f - p\| > 0$ for all n. Letting $n \to \infty$, we get $0 \geq \|f - p\| > 0$, a contradiction. $\square$

2 If $L : \mathcal{H} \to \mathcal{H}$ is a bounded linear transformation, then $\overline{R(L)} = N(L^*)^\perp$.

Proof. Suppose $(Lv_n)_n \subset R(L)$ with $Lv_n \to w$. Then if $u \in N(L^*)$ we have $\langle u, Lv_n \rangle = \langle Lu, v_n \rangle = 0$. Letting $n \to \infty$, we have $\langle u, w \rangle = 0$. Hence, $\overline{R(L)} \subset N(L^*)^\perp$.

For the reverse inclusion, suppose $v \in N(L^*)^\perp$. Let P be the orthogonal projection operator mapping onto $\overline{R(L)}$. Let $u = v - Pv \in \overline{R(L)}^\perp$. Hence, $0 = \langle LL^*u, u \rangle = \|L^*u\|$, so $L^*u = 0$. Moreover $v, Pv \in N(L^*)^\perp$, so $u \in N(L^*)^\perp$. Thus $u = 0$, so $v \in \overline{R(L)}$. $\square$

3 Let $\mathcal{H}$ be a Hilbert space of functions that are defined on [0,1]. In addition, suppose that $\mathcal{H} \subset C[0,1]$, with $\|f\|_{C[0,1]} \leq K\|f\|_H$ for all $f \in \mathcal{H}$. (The Sobolev space H^1 has this property.)

a. Show that the point-evaluation functional $\phi_x(f) = f(x)$ is a bounded linear functional on $\mathcal{H}$.

b. Let x be fixed. Show that there is a kernel $k(x, y) \in \mathcal{H}$ such that

$$\phi_x(f) = f(x) = \langle f, k(x, \cdot) \rangle$$

(The kernel $k(x, y)$ is called a reproducing kernel and $\mathcal{H}$ is called a reproducing kernel Hilbert space.)

c. For x, z fixed, show that $k(z, x) = \langle k(z, \cdot), k(x, \cdot) \rangle$. In addition, let $(x_j)_{j=1}^n$ be any finite set of distinct points in $[0, 1]$. Show that the matrix $G_{jk} = k(x_k, x_j)$ is positive semidefinite; that is $\sum_{j,k} c_k \overline{c_j} k(x_k, x_j) \geq 0$.

d. Suppose the matrix G is positive definite and therefore invertible. Let $f \in \mathcal{H}$. Show that there are unique coefficients $(c_j)_{j=1}^n$ such that $s(x) = \sum_{j=1}^n k(x_j, x)c_j$ interpolates f at the x_j's.

Proof. For (a), we have $|\phi_x(f)| \leq \|f\|_{C[0,1]} \leq K\|f\|_{\mathcal{H}}$.

(b) follows from (a) and the Riesz Representation Theorem.

For (c), we have $k(z,x) = \phi_x(k(z,\cdot)) = \langle k(z,\cdot), k(x, cdot)\rangle$. For the other part, we have

$$\begin{aligned}
\sum_{j,k} c_k \overline{c_j} k(x_k, x_j) &= \sum_{j,k} c_k \overline{c_j} \langle k(x_k,\cdot), k(x_j, cdot)\rangle \\
&= \sum_{j,k} \langle c_k k(x_k,\cdot), c_j k(x_j, cdot)\rangle \\
&= \left\langle \sum_k c_k k(x_k,\cdot), \sum_j c_j k(x_j, cdot)\right\rangle \\
&\geq 0.
\end{aligned}$$

For (d), let $v = (f(x_1), f(x_2), \ldots, f(x_n))$. A coefficient vector c interpolates f at the x_j's iff it is the solution to the matrix equation $cG = v$. This has a unique solution since G is invertible. □

4 Consider the finite rank (degenerate) kernel $k(x,y) = \phi_1(x)\overline{\psi_1}(y) + \phi_2(x)\overline{\psi_2}(y)$, where $\phi_1 = 2x-1, \phi_2 = x^2, \psi_1 = 1, \psi_2 = 4x-3$. Let $Ku = \int_0^1 k(x,y)u(y)dy$. Assume that $L := I - \lambda K$ has closed range.

a. For what values of λ does the integral equation

$$u(x) - \lambda \int_0^1 k(x,y)u(y)dy = f(x)$$

have a solution for all $f \in L^2[0,1]$.

b. For these values, find the solution $u = (I - \lambda K)^{-1}F$ – i.e., find the resolvent.

c. For the values of λ for which the equation does not have a solution for all f, find a condition on f that guarantees a solution exists. Will the solution be unique?

Proof. For (a), we have $u - \lambda \sum_i \langle u, \psi_i\rangle \phi_i = f$. Hence $\langle u, \psi_j\rangle - \sum_i \langle u, \psi_i\rangle\langle \phi_i, \psi_j\rangle = \langle f, \psi_j\rangle$. Hence $\sum_i (\delta_{ij} - \lambda\langle \phi_i, \psi_j\rangle)\langle u, \psi_i\rangle = \langle f, \psi_j\rangle$. Using the alternative value for ϕ_2, the determinant of $\delta_{ij} - \lambda\langle \phi_j, \psi_j$ is 1. Thus, we can find solutions for all λ.

For (b), the solution is given by $\langle u, \psi_1\rangle = \langle f, \psi_1\rangle + \lambda/3\langle f, \psi_2\rangle$ and $\langle u, \psi_2\rangle = \langle f, \psi_2\rangle$. □

5 Let $S = \{(a_j) \in \ell^2 : \sum_j (1+j^2)|a_j|^2 \leq 1\}$. Show that S is a compact subset of ℓ^2.

Proof. It suffices to show that S is closed and totally bounded. To see that S is closed, suppose $(a^(n)) \subset S$ with $a^(n) \to a$ in ℓ^2. Suppose $a \notin S$. Then

$\sum_{j=1}^{N}(1+j^2)|a_j|^2 > 1+\epsilon$ for some N and some $\epsilon > 0$. But then we have

$$\begin{aligned}\epsilon &\le \sum_{j=1}^{N}(1+j^2)|a_j|^2 - \sum_{j=1}^{\infty}(1+j)^2|a_j^{(n)}|^2 \\ &\le \sum_{j=1}^{N}(1+j^2)(|a_j|^2 - |a_j^{(n)}|^2) \to 0\end{aligned}$$

as $n \to \infty$, a contradiction. Hence S is closed.

To see that S is totally bounded, let $\epsilon > 0$. We have $\sum_{j=N}^{\infty}|a_j|^2 \le N^{-2}\sum_{j=N}^{\infty}(1+j^2)|a_j|^2 \le N^{-}2$ for $(a_j) \in \ell^2$. Pick N such that $N^{-}2 < \epsilon$. Note that $\|a\| \le 1$ for all $a \in S$. Recall that $B_1(\ell_2^N)$ is totally bounded. The rest of S is within ϵ of $B_1(\ell_2^N)$, so S is totally bounded. □