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1 $\mathbb{F}_2$ and the Fano plane

1.1 Introduction

The purpose of this paper is to answer Exercise 2.5 (p. 96) of Greenberg [1]:

> Let $\mathbb{F}_2$ be the field of two elements $\{0, 1\}$, whose multiplication and addition have the usual tables except that $1 + 1 = 0$. Show that $\mathbb{F}_2^2$ is isomorphic to the smallest affine plane. Show that $P^2(\mathbb{F}_2)$ is isomorphic to the Fano plane.

We will need a few preliminary definitions from Greenberg.

Definition 1. *An* ***incidence geometry*** $(\mathcal{P}, \mathcal{L}, \mathcal{I})$ *consists of a set of points* $\mathcal{P}$*, a set of lines* $\mathcal{L}$*, and an incidence relation* $\mathcal{I} \subset \mathcal{P} \times \mathcal{L}$ *such that:*

1. *Every pair of distinct points is incident to a unique line.*
2. *Every line is incident to at least two distinct points.*
3. *There exist three distinct noncollinear points.*

Definition 2. *Two lines are* ***parallel*** *if there is no point incident to both lines.*

Definition 3. *A* ***projective plane*** *is an incidence geometry in which:*

1. *No two lines are parallel.*
2. *Every line is incident to at least three distinct points.*

Definition 4. *An* ***affine plane*** *is an incidence geometry in which, for every line* l *and point* P *not incident to* l*, there exists a unique line* m *incident to* P *and parallel to* l*.*

1.2 The affine plane $\mathbb{F}_2^2$

As in $\mathbb{R}^2$, the points in $\mathbb{F}_2^2$ are simply the elements of the vector space $\mathbb{F}_2^2$, i.e. ordered pairs of elements of $\mathbb{F}_2$.

Also analogous to $\mathbb{R}^2$, the lines in $\mathbb{F}_2^2$ are cosets of 1-dimensional subspaces of $\mathbb{F}_2^2$. That is, every line in $\mathbb{F}_2^2$ can be written as $V + h$ for some 1-dimensional subspace $V \subset \mathbb{F}_2^2$ and $h \in \mathbb{F}_2^2$.

Incidence in $\mathbb{F}_2^2$ corresponds to inclusion. For example, the point $(1, 1) \in \mathbb{F}_2^2$ is incident to the line $\{(1, 0)t + (0, 1) : t \in \mathbb{F}_2\}$, since $(1, 1) = (1, 0)(1) + (0, 1)$.

As Greenberg notes, the smallest affine plane, call it $\mathcal{A}$, consists of a set of four points $\{A, B, C, D\}$ and a set of four lines $\{\{A, B\}, \{B, C\}, \{C, D\}, \{D, A\}\}$, where incidence corresponds to inclusion. For example, the point B is incident to the line $\{A, B\}$.

To see that $\mathcal{A}$ and $\mathbb{F}_2^2$ are isomorphic, first note that each 1-dimensional subspace over $\mathbb{F}_2$ has exactly 2 elements, so each line in $\mathbb{F}_2^2$ has 2 elements. Conversely, given two elements $a, b \in \mathbb{F}_2^2$, the line $L((b-a)t, a)$ passes through a and b. Thus, the lines in $\mathbb{F}_2^2$ are precisely the two-element subsets of $\mathbb{F}_2^2$.

Therefore, an arbitrary bijection f from the points of $\mathbb{F}_2^2$ to the points of $\mathcal{A}$ induces a bijection of lines (two-element subsets), and since inclusion is preserved under f, incidence is also preserved.

1.3 $P^2(\mathbb{F}_2)$ as the Fano plane

For an arbitrary field K, the points of the projective space $P^2(K)$ are the 1-dimensional subspaces of K^3. The lines are the 2-dimensional subspaces of K^3. Incidence corresponds to containment.

Projective points in $P^2(K)$ are usually denoted $(a{:}\, b{:}\, c)$ for some generator $(a, b, c) \in K^3 \setminus \{0\}$. Then $(a{:}\, b{:}\, c) = (d{:}\, e{:}\, f)$ iff (a, b, c) is a nonzero multiple of (d, e, f).

For example, the projective line $\{x + y + z = 0 : (x{:}\, y{:}\, z) \in P^2(\mathbb{F}_2)\}$ is incident to the point $(1{:}\, 0{:}\, 1) \in P^2(\mathbb{F}_2)$ since $1 + 0 + 1 = 0$.

Recall that each 1-dimensional subspace of an $\mathbb{F}_2$-vector space has only one non-zero element. Hence, a strange thing occurs in $P^2(\mathbb{F}_2)$: there is a correspondence between each point in $P^2(\mathbb{F}_2)$ and its unique nonzero element in $\mathbb{F}_2^3$. Since each non-zero element in $\mathbb{F}_2^3$ generates a 1-dimensional subspace of F_2^3, i.e. a projective point, this correspondence defines a bijection from $P^2(\mathbb{F}_2)$ to $\mathbb{F}_2^3 \setminus \{0\}$. Hence, there are $2^3 - 1 = 7$ points in $¶^2(\mathbb{F}_2)$.

Since every 2-dimensional subspace of $\mathbb{F}_2^3$ contains 0, a 1-dimensional subspace $V \subset \mathbb{F}_2^3$ lies within a 2-dimensional subspace $W \subset \mathbb{F}_2^3$ iff the unique nonzero element in V lies within W.

Note that each line in $P^2(\mathbb{F}_2)$ corresponds to a set of 3 $\mathbb{F}_2^3$-elements since it is a 2-dimensional $\mathbb{F}_2$-vector space. Thus, each line in $P^2(\mathbb{F}_2)$ is incident to precisely 3 projective points.

Similarly, the non-zero elements of $\mathbb{F}_2^3$ correspond to the pr

References

[1] Marvin J Greenberg. *Euclidean and non-Euclidean geometries: Development and history.* WH Freeman, 2007.

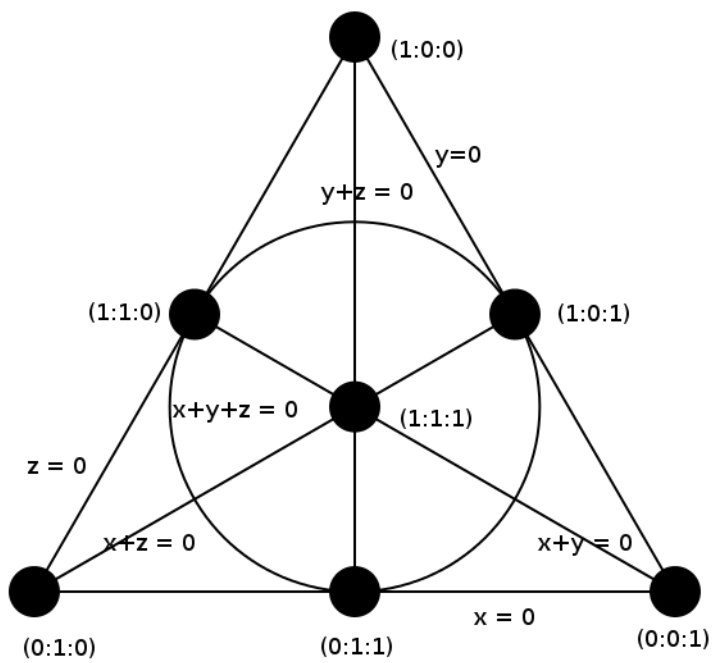

Figure 1: An isomorphism between $P^2(\mathbb{F}_2)$ and the Fano plane