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HW 8

1 Let $f : [0,1] \to \mathbb{R}$ be integrable (with respect to Lebesgue measure) and nonnegative. Define

$$G_- = \{(x,y) : 0 \le x \le 1, 0 \le y \le f(x)\}.$$

Show that G_- is measurable in $\mathbb{R} \times \mathbb{R}$ and that

$$m(G_-) = \int_0^1 f(x)\,dx.$$

Proof. Case f is simple. In standard form $f = \sum_{i=1}^n a_i \chi_{A_i}$. Hence $G_- = \bigcup_{i=1}^n A_i \times [0, a_i)$ is measurable. Since the $A_i \times [0, a_i)$ are disjoint we have $m(G_i) = \sum_i a_i m(A_i) = \int f\,dx$.

General case. There exists a sequence of simple functions $\phi_n \uparrow f$. Let $H_n = \{(x,y) : 0 \le x \le 1, 0 \le y \le \phi_n(x)\}$. Then by part (a), each H_n is measurable and $m(H_n) = \int \phi_n\,dx$. Hence $G_- = \bigcup_n H_n$ is measurable, and $m(G_-) = \lim_{n\to\infty} m(H_n) = \lim_{n\to\infty} \int \phi_n\,dx = \int f\,dx$, where the last equality follows from the MCT. □

2 Let f be Lebesgue integrable on $(0,1)$. For $0 < x < 1$ define

$$g(x) = \int_x^1 t^{-1} f(t)\,dt.$$

Prove that g is Lebesgue integrable on $(0,1)$ and that

$$\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx.$$

[Hint: first prove the case where $f \ge 0$.]

Proof. Case $f \ge 0$

□

3 Let $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$. Let μ be the Lebesgue measure on $\mathcal{M}$ and ν be the counting measure on $\mathcal{N}$. Show that for $D = \{(x,x) : x \in [0,1]\}$

a) $D \in \mathcal{M} \otimes \mathcal{N}$.

b) The numbers

$$\mu \otimes v(D), \int\int \chi_D\,d\mu d\nu, \text{ and } \int\int \chi_D\,d\nu d\mu$$

are all unequal.

c) Show that there is more than one measure π on $\mathbb{R}^2$ for which

$$\pi(A \times B) = \mu(A)\nu(B), \text{ whenever } A, B \in \mathcal{B}[0,1].$$

p4 Find a measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ measurable so that
a) $\int_{\mathbb{R}^2} |f(x,y)|\, dxdy = \infty$
b) $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, dxdy$, and $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, dydx$ both exist but are unequal.

Proof. Let

$$a_{ij} = \begin{cases} 1 & j = i+1 \\ -1 & j = i-1 \\ 0 & \text{else} \end{cases}$$

Let $f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \chi_{[i,i+1)\times[j,j+1)}$. Then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, dxdy = \int_{\mathbb{R}} \begin{cases} 1 & 0 \le y < 1 \\ 0 & \text{else} \end{cases} dy = 1,$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)\, dydx = \int_{\mathbb{R}} \begin{cases} -1 & 0 \le x < 1 \\ 0 & \text{else} \end{cases} dx = -1.$$

$\square$

5 Problem 49/Page 69. Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas.

a. If $E \in \mathcal{M} \times \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.

b. If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x\, d\nu = \int f^y\, d\mu = 0$ for a.e. x and y. (Here the completeness of μ and ν is needed.)

6 If $f \in L_1(\mathbb{R}^2)$ or $f \ge 0$ and mble and $c \in \mathbb{R} \setminus \{0\}$, then

$$\int f(cx, cy) dxdy = c^{-2} \int f(x,y) dxdy.$$

$$\int f(x + cy, cy) dxdy = \int f(x,y) dxdy.$$

7 Prove that for any $f \in L_1(\mathbb{R}^d)$ and any $\epsilon > 0$ there is a simple function

$$\phi = \sum_{j=1}^{n} \alpha_j \chi_{R_j},$$

where the R_j's are products of intervals, and $\|\phi - f\|_1 \le \epsilon$.

Proof. Since there exist simple functions $0 \le |\phi_n| \le |f|$ with $\phi_n \to f$, by the DCT WLOG f is simple. Then if $f = \sum_i a_i \chi_{A_i}$ in standard form, it suffices to approximate each A_i by finite disjoint union of products of intervals.

Let A be a measurable set of finite measure in $\mathbb{R}^d$. By the outer regularity of Lebesgue measure, WLOG A is open. Let $E_n = \{x \in A : B_{1/n}(x) \in A\}$. Then since A is open, $A = \bigcup_{n=1}^{\infty} E_n$. Since (E_n) is increasing, we have $m(A) = \lim_{n\to\infty} m(E_n)$.

Let $\epsilon > 0$. Pick E_n such that $m(A \backslash E_n) < \epsilon$. Let $\mathcal{Q}$ be the collection of all R^d-cubes with half-open sides of length $\frac{1}{2\sqrt{3}n}$ and vertices at $\frac{1}{2\sqrt{3}n}\mathbb{Z}$-lattice points. Then $\mathcal{Q}$ is a pairwise disjoint covering of R^d. Let $U = \bigcup\{Q \in \mathbb{Q} : Q \cap E_n \neq \emptyset\}$. Then $E_n \subset U \subset A$, where the latter inclusion follows from the fact that the diameter of each cube is $\frac{1}{2n} < 1/n \leq d(E_n, A^c)$.

Since U has finite measure, U is a finite disjoint union of products of intervals, and $m(A \Delta U) = m(A \setminus U) < \epsilon$. $\square$