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Midterm

1 Use the Courant-Fischer mini-max theorem to show that $\lambda_2 < 0$ for the matrix

$$A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix}$$

Proof. The characteristic polynomial for A is $f(x) := x^3 + 6 + 6 - 9x - 4x - x = x^3 - 14x + 12$. We have $\lim_{x\to-\infty} f(x) < 0$, $f(0) > 0$, $f(1) < 0$, and $\lim_{x\to\infty} f(x) > 0$. Thus $\lambda_2 < 0$. $\square$

2 Let A be an $n \times n$ complex matrix that satisfies $A^*A = AA^*$. Show that A is diagonalizable and that there is a unitary matrix U for which $U^*AU = \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$.

Proof. Step 1: A and A^* are simultaneously diagonalizable. Let $J \in M_n(\mathbb{C})$ be the Jordan Normal Form for A. I claim that J is diagonal. Suppose not. Then J contains an $m \times m$ Jordan block B for $1 < m \leq n$. If λ is the generalized eigenvalue corresponding for B, then we have $[B, B^*]_{11} = (BB^*)_{11} - (B^*B)_{11} = (|\lambda|^2 + 1) - |\lambda|^2 \neq 0$. Hence $[J, J^*] \neq 0$, so $[A, A^*] \neq 0$, a contradiction. Thus, J is diagonal. The matrix $J^* = \overline{J}^T$ is clearly diagonal also.

Step 2: A is unitarily diagonalizable. The proof is by induction on n. The base case is trivial. For the inductive step, recall that A must have an eigenvector. Let v be an normalized eigenvector of A. Let $w \in v^\perp$. Then $\langle v, Aw \rangle = \langle A^*v, w \rangle = 0$ since v is an eigenvector of both A and A^* by Step 1. Hence $v^\perp$ is an invariant subspace of A, and we can apply the inductive hypothesis to $A|_{v^\perp}$. $\square$

3 Let f be continuous on $[0, 1]$, with $f(0) = f(1) = 0$ and let $s \in S^{1/n}(1, 0)$ be the linear spline interpolant to f, with knots at $x_j = \frac{j}{n}$.

(a) Let $\lambda \in \mathbb{R}$. Show that $\left| \int_0^1 s(x) e^{i\lambda x} dx \right| \leq \frac{2n^2}{\lambda^2} \omega(f, 1/n)$.

Proof. We have

$$\begin{aligned}
\left|\int_0^1 s(x)e^{i\lambda x}dx\right| &= \left|\sum_{k=0}^{n-1}\int_{k/n}^{(k+1)/n} s(x)e^{i\lambda x}dx\right| \\
&= \left|\sum_{k=0}^{n-1}\left[\frac{1}{i\lambda}s(x)e^{i\lambda x}\right]_{x=k/n}^{(k+1)/n} - \frac{1}{i\lambda}\int_{k/n}^{(k+1)/n} s'(x)e^{i\lambda x}dx\right| \\
&= \left|\sum_{k=0}^{n-1}\frac{1}{i\lambda}\int_{k/n}^{(k+1)/n} s'(x)e^{i\lambda x}dx\right| \\
&= \left|-\frac{1}{\lambda^2}\sum_{k=0}^{n-1}\left[s'(x)e^{i\lambda x}\right]_{x=k/n}^{(k+1)/n}\right| \\
&\leq \frac{1}{\lambda^2}\sum_{k=0}^{n-1}\left|s'\left(\frac{k+1}{n}-\right)\right| + \left|s'\left(\frac{k}{n}+\right)\right| \\
&\leq \frac{1}{\lambda^2}\sum_{k=0}^{n-1} 2n\omega(f,1/n) \\
&= \frac{2n^2}{\lambda^2}\omega(f,1/n).
\end{aligned}$$

□

(b) Use the previous part to show that $\left|\int_0^1 f(x)e^{i\lambda x}dx\right| \leq \omega(f,1/n)+\frac{2n^2}{\lambda^2}\omega(f,1/n)$.

Proof. We have

$$\begin{aligned}
\left|\int_0^1 f(x)e^{i\lambda x}dx\right| &\leq \left|\int_0^1 f(x)-s(x)e^{i\lambda x}dx\right| + \left|\int_0^1 s(x)e^{i\lambda x}dx\right| \\
&\leq \int_0^1 |f(x)-s(x)|dx + \frac{2n^2}{\lambda^2}\omega(f,1/n) \\
&\leq \int_0^1 \omega(f,1/n)dx + \frac{2n^2}{\lambda^2}\omega(f,1/n) \\
&\leq \omega(f,1/n) + \frac{2n^2}{\lambda^2}\omega(f,1/n)
\end{aligned}$$

□

4 Let $\{\phi_n(x)\}_{n=0}^\infty$ be a set of polynomials orthogonal with respect to a weight function $w(x)$ on a domain $[a,b]$. Assume that the degree of ϕ_n is n, and that the coeffiction of x^n in $\phi_n(x)$ is $k_n > 0$. In addition, suppose that the continuous functions are dense in $L^2_w[a,b] = \{f : [a,b] \to \mathbb{C} : \int_a^b |f(x)|^2 w(x)dx < \infty\}$.

(a) Show that ϕ_n is orthogonal to all polynomials of degree $n-1$ or less.

Proof. The set $\{\phi_k\}_{0\le k<n}$ spans the polynomials of degree less than $n-1$. $\square$

(b) Show that $\{\phi_n\}_{n=0}^{\infty}$ is complete in $L^2_w[a,b]$.

Proof. Let $g \in L^2_w[a,b]$ be continuous. Let $\epsilon > 0$. By the Weierstrauss Approximation Theorem, pick a polynomial p such that $\|g-p\|_{C[a,b]} < \epsilon$. Then $\|g-p\|_{L^2_w[a,b]} = \int_a^b |g-p|^2 w dx \le \epsilon^2 \int_a^b w dx$. Since $\phi_0 \in L^2_w[a,b]$, this last integral is finite. Hence, the polynomials are dense in $L^2_w[a,b]$.

Now suppose $\{\phi_n\}_{n=0}^{\infty}$ is not complete. By a previous homework problem, there exists a normalized function $f \in L^2_w[a,b]$ with $\langle f, \phi_n\rangle = 0$ for all n. Thus for any polynomial p, we have $\|f-p\|^2_{L^2_w[a,b]} = \|f\|^2_{L^2_w[a,b]} + \|p\|^2_{L^2_w[a,b]} \ge 1$. This contradicts the fact that the polynomials are dense in $L^2_w[a,b]$.

$\square$

(c) Show that the polynomials satisfy the recurrence relation $\phi_{n+1}(x) = (A_n x + B_n)\phi_n(x) + C_n\phi_{n-1}(x)$. Find A_n in terms of the k_n's.

Proof. We have $\phi_{n+1} = A_n x\phi_n + \sum_{j=0}^{n} a_j\phi_j$ for some unique A_n and $(a_j)_{j=1}^{n}$.

For $1 \le l \le n-2$, part (a) implies that

$$\begin{aligned}
0 &= \langle \phi_{n+1}, \phi_l\rangle \\
&= \left\langle A_n x\phi_n + \sum_{j=0}^{n} a_j\phi_j, \phi_l \right\rangle \\
&= \langle A_n x\phi_n, \phi_l\rangle + a_l\langle \phi_l, \phi_l\rangle \\
&= A_n\langle \phi_n, x\phi_l\rangle + a_l\langle \phi_l, \phi_l\rangle \\
&= a_l\langle \phi_l, \phi_l\rangle
\end{aligned}$$

Hence $a_l = 0$ for $1 \le l \le n-2$, so $\phi_{n+1} = A_n x\phi_n + B_n\phi_n + C_n\phi_{n-1}$.

By comparing leading coefficients, $A_n = \frac{k_{n+1}}{k_n}$. $\square$

5 Suppose that $f(\theta)$ is a 2π-periodic function in $C^m(\mathbb{R})$, and that $f^{(m+1)}$ is piecewise continuous and 2π-periodic. Here $m > 0$ is a fixed integer. Let c_k denote the k-th (complex) Fourier coefficient for f and let $c_k^{(j)}$ denote the k-th Fourier coefficient for $f^{(j)}$.

(a) Show that $c_k^{(j)} = (ik)^j c_k$ for $1 \le j \le m+1$.

Proof. Integrating by parts j times, we have

$$\begin{aligned}
c_k^{(j)} &= \frac{1}{2\pi}\int_0^{2\pi} f^{(j)}(x)e^{-ikx}dx \\
&= \frac{1}{2\pi}\left(\left[f^{(j-1)}(x)e^{2\pi kx}\right]_0^{2\pi} + (ik)\int_0^{2\pi} f(x)e^{2\pi kx}dx\right) \\
&= \frac{ik}{2\pi}\left(\int_0^{2\pi} f^{(j-1)}(x)e^{2\pi kx}dx\right) \\
&\vdots \\
&= \frac{(ik)^j}{2\pi}\left(\int_0^{2\pi} f(x)e^{2\pi kx}dx\right) \\
&= (ik)^j c_k
\end{aligned}$$

□

(b) For $k \neq 0$, show that the Fourier coefficient c_k satisifies the bound

$$|c_k| \leq \frac{1}{2\pi|k|^{m+1}}\|f^{(m+1)}\|_{L_1[0,2\pi]}$$

Proof. Integrating by parts $m+1$ times, we have

$$\begin{aligned}
|c_k| &= \left|\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}dx\right| \\
&= \left|\frac{1}{2\pi(ik)^{m+1}}\int_0^{2\pi} f^{(m+1)}(x)e^{-ikx}dx\right| \\
&\leq \frac{1}{2\pi|k|^{m+1}}\|f^{(m+1)}\|_{L_1[0,2\pi]}
\end{aligned}$$

□

(c) Let $S_n(\theta) = \sum_{k=-n}^{n} c_k e^{ik\theta}$ be the n-th partial sum of the Fourier series for f, $n \geq 1$. Show that both of the following hold for f:

$$\|f - S_n\|_{L_2} \leq C\frac{\|f^{(m+1)}\|_{L_1}}{n^{m+\frac{1}{2}}} \text{ and } \|f - S_n\|_{C[0,2\pi]} \leq C'\frac{\|f^{(m+1)}\|_{L_1}}{n^m}.$$

Proof. By Parseval's theorem, we have

$$\begin{aligned}
\|f - S_n\|_{L_2} &= \left(\sum_{k>n} |c_k|^2\right)^{-1/2} \\
&\le \left(\sum_{k>n} \frac{C}{|k|^{2m+2}} \|f^{(m+1)}\|^2_{L_1[0,2\pi]}\right)^{-1/2} \\
&= \left(\sum_{k>n} \frac{C}{|k|^{2m+2}}\right)^{-1/2} \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&\le \left(\int_{k>n} \frac{C_1}{|k|^{2m+2}} dk\right)^{-1/2} \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&= \left(\frac{C_2}{n^{2m+1}}\right)^{-1/2} \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&= \frac{C_3}{n^{m+1/2}} \|f^{(m+1)}\|_{L_1[0,2\pi]}
\end{aligned}$$

Using part (b), we have

$$\begin{aligned}
\|f - S_n\|_{C[0,2\pi]} &= \sup_{x\in[0,2\pi]} \left|\sum_{k>n} c_k(x) e^{ikx}\right| \\
&\le \sup_{x\in[0,2\pi]} \sum_{k>n} |c_k(x)| \\
&\le \sum_{k>n} \frac{1}{2\pi|k|^{m+1}} \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&\le \left(\int_{k>n} \frac{C'}{|k|^{m+1}} dk\right) \|f^{(m+1)}\|_{L_1[0,2\pi]} \\
&= \frac{C''}{n^m} \|f^{(m+1)}\|_{L_1[0,2\pi]}.
\end{aligned}$$

□

(d) Let $f(x)$ be the 2π-periodic function that equals $x^2(2\pi - x)^2$ when $x \in [0, 2\pi]$. Verify that f satisfies the conditions above with $m = 1$. With the help of (a), calculate the Fourier coefficients for f. (Hint: look at f''.)

Proof. To see that f satisfies the conditions with $m = 1$, we need to check that $f'(0+) = f'(2\pi-)$ and f'' is piecewise continuous (f'' is 2π-periodic since f is). The former follows from the fact that f has double roots at 0 and 2π. The latter is obvious.

For $x \in (0, 2\pi)$, we have

$$\begin{aligned} f(x) &= x^4 - 4\pi x^3 + 4\pi^2 x^2 \\ f'(x) &= 4x^3 - 12\pi x^2 + 8\pi^2 x \\ f''(x) &= 12x^2 - 24\pi x + 8\pi^2 \end{aligned}$$

From (a), the Fourier coefficient c_k for f is

$$\begin{aligned} c_k &= (ik)^{-2} c_k^{(j)} \\ &= -\frac{1}{2\pi k^2} \int_0^{2\pi} f''(x) e^{-ikx} dx \\ &= -\frac{1}{2\pi k^2} \int_0^{2\pi} (12x^2 - 24\pi x) e^{-ikx} dx \\ &= -\frac{1}{2\pi k^2} \left(\left[\frac{12x^2 - 24\pi x}{-ik} e^{-ikx} \right]_0^{2\pi} + \frac{1}{ik} \int_0^{2\pi} (24x - 24\pi) e^{-ikx} dx \right) \\ &= -\frac{1}{2\pi k^2} \left(\frac{24}{ik} \right) \int_0^{2\pi} x e^{-ikx} dx \\ &= \frac{24i}{2\pi k^3} \int_0^{2\pi} x e^{-ikx} dx \\ &= \frac{24i}{k^3} \left(\frac{i}{k} \right) \\ &= -\frac{24}{k^4}, \end{aligned}$$

where the penultimate equality uses the homework problem calculating the Fourier series of $g(x) = x, 0 \le x < 2\pi$. $\square$