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Representations of Lie Algebras

HW 1

1 Let $\mathfrak{g}$ be a real Lie algebra, and $\mathfrak{g}_{\mathbb{C}}$ its complexification: as a vector space it is $\mathfrak{g}$ tensored with $\mathbb{C}$ over $\mathbb{R}$. What is the bracket $[x, y]$ on $\mathfrak{g}_{\mathbb{C}}$?

Proof. The bracket on $\mathfrak{g}_{\mathbb{C}}$ is given by $[a_1 + ib_1, a_2 + ib_2] = [a_1, a_2] - [b_1, b_2] + i([b_1, a_2] + [a_1, b_2])$. This bracket is clearly $\mathbb{R}$-bilinear, antisymmetric, and restricts to the previous bracket of $\mathfrak{g}$. For $\mathbb{C}$-bilinearity, we have, for $x_1, x_2 \in \mathbb{R}$,

$$\begin{aligned}
[i(a_1 + ib_1), a_2 + ib_2] &= [-b_1 + ia_1, a_2 + ib_2] \\
&= -[b_1, a_2] - [a_1, b_2] + i([a_1, a_2] - [b_1, b_2]) \\
&= -[b_1, a_2] - [a_1, b_2] + i([a_1, a_2] - [b_1, b_2]) \\
&= -[b_1, a_2] - [a_1, b_2] + i([a_1, a_2] - [b_1, b_2]) \\
&= i([a_1, a_2] - [b_1, b_2] + i([b_1, a_2] + [a_1, b_2])) \\
&= [a_1 + ib_1, a_2 + ib_2]
\end{aligned}$$

The Jacobi identity follows from the real form. □

2 Determine the adjoint representation and Killing form for $\mathfrak{o}(S, V)$ with $V = \mathbb{C}^3$. Is the Killing form non-degenerate? Is the adjoint representation faithful?

Proof. The Lie algebra $\mathfrak{o}(3) \subset \mathfrak{gl}(3)$ consists of the endomorphisms E for which $S(Ev, w) + S(v, Ew) = 0$. Restricting to the standard basis, we get the equivalent condition $E_{ij} = -E_{ij}$. Hence, a basis for $\mathfrak{o}(3)$ is the set $f_{ij} := e_{ij} - e_{ji}$ for $i < j$. To calculate the adjoint representation, we have

$$\begin{aligned}
[f_{ij}, f_{kl}] &= [e_{ij} - e_{ji}, e_{kl} - e_{lk}] \\
&= [e_{ij}, e_{kl}] - [e_{ij}, e_{lk}] - [e_{ji}, e_{kl}] + [e_{ji}, e_{lk}] \\
&= (\delta_{jk} e_{il} - \delta_{li} e_{kj}) - (\delta_{jl} e_{ik} - \delta_{ki} e_{lj}) - (\delta_{ik} e_{jl} - \delta_{lj} e_{ki}) + (\delta_{il} e_{jk} - \delta_{kj} e_{li}) \\
&= \delta_{jk} f_{il} + \delta_{il} f_{jk} + \delta_{jl} f_{ki} + \delta_{ik} f_{lj}
\end{aligned}$$

Hence, in the basis (f_{12}, f_{13}, f_{23}), we have

$$\mathrm{ad}(f_{12}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \mathrm{ad}(f_{13}) = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \mathrm{ad}(f_{23}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For the Killing form, we have

$$K(f_{12}, f_{13}) = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$K(f_{12}, f_{23}) = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = 0$$

$$K(f_{13}, f_{23}) = \mathrm{Tr}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = 0,$$

and $K(f_{ij}, f_{ij}) = -2$. Thus, the Killing form is nondegenerate. Hence $\mathfrak{o}(S, V)$ is semisimple, so the adjoint representation is faithful. $\square$

3 Compute the Casimir for the 3-dimensional representation of o(S,V) with dim(V)=3 directly, by finding a basis and dual basis.

Proof. The set $(f_{ij})_{i<j}$ forms a basis, and the dual basis is $f_{ij}^* = -\frac{1}{2}f_{ij}$. Thus the Casimir element is

$$\begin{aligned} \Omega &= -\frac{1}{2}\sum_{i<j} f_{ij}^2 \\ &= -\frac{1}{2}\sum_{i<j} -e_{ii} - e_{jj} \\ &= \mathrm{Id} \end{aligned}$$

$\square$