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HW 6

7.10 If μ is a Radon measure and $f \in L^1(\mu)$ is real-valued, for every $\epsilon > 0$ there exist and LSC function g and a USC function H such that $h \leq f \leq g$ and $\int (g - h)\, d\mu < \epsilon$.

Proof. By Proposition 7.14, we can pick LSC functions g_1 and h_2 with $g_1 \geq f^+$, $h_2 \geq f^-$, $\int (g_1 - f^+) < \epsilon$, and $\int (h_2 - f^-) < \epsilon$. Similarly, we can pick USC functions g_1, g_2 with $0 \leq h_1 \leq f^+$, $0 \leq g_2 \leq f^-$, $\int (f^+ - h_1) < \epsilon$, and $\int (f^- - g_2) < \epsilon$.

Let $g = g_1 - g_2$ and $h = h_1 - h_2$. Then g is LSC and h is USC. Moreover, we have $h = h_1 - h_2 \leq f^+ - f^- = f$ and $g = g_1 - g_2 \geq f^+ - f^- = f$. Lastly, $\int (g - h) = \int (g_1 - f^+ + h^2 - f^- + f^- - g_2 + f^+ - h_2) < 4\epsilon$. □

7.11 Suppose that μ is a Radon measure on X such that $\mu(\{x\}) = 0$ for all $x \in X$, and $A \in \mathcal{B}_X$ satisfies $0 < \mu(A) < \infty$. Then for any α such that $0 < \alpha < \mu(A)$ there is a Borel set $B \subset A$ such that $\mu(B) = \alpha$.

Proof. Let
$$\mathcal{E} := \{E \subset A : E \text{ is open in } A \text{ and } \mu(E) \leq \alpha\}.$$
To see that chains are bounded in $\mathcal{E}$, let (E_λ) be an increasing (with respect to inclusion) chain in $\mathcal{E}$. Let $E = \bigcup_\lambda E_\lambda$. Then E is open in A.

Moreover, $\mu(E) \leq \mu(A) < \infty$, so by the inner regularity of μ on σ-finite sets
$$\begin{aligned}\mu(E) &= \sup\{\mu(K) : K \subset E, K \text{ compact}\}\\ &= \sup\{\mu(K) : K \subset E_\lambda \text{ for some } \lambda, K \text{ compact}\}\\ &\leq \alpha\end{aligned}$$
, where the second step uses the compactness of E to pick a finite cover of K by the E_λ and then fixes λ to be the maximum index of this finite cover. Hence, $E \in \mathcal{E}$.

Thus every chain in $\mathcal{E}$ is bounded, so by Zorn's lemma $\mathcal{E}$ contains a maximal element B.

I claim that $\mu(B) = \alpha$. Suppose not. Then $\mu(B) < \alpha < \mu(A)$. Pick $x \in A \setminus B$. By outer regularity, there exists an open set U such that $x \in U$ and $\mu(U) < \alpha - \mu(B)$. Then $C := B \cup (U \cap A)$ is open in A, and $\mu(C) \leq \alpha$. Since $x \in C \setminus B$, this contradicts the maximality of B in $\mathcal{E}$. □

7.17 If μ is a positive Radon measure on X with $\mu(X) = \infty$, there exists $f \in C_0(X)$ such that $\int f\, d\mu = \infty$. Consequently, every positive linear functional on $C_0(X)$ is bounded.

Proof. By the inner regularity of μ at X, we can pick an ascending chain of compact sets K_n with $\mu(K_n) \geq n$ for all n. By picking a subsequence, WLOG $\mu(K_n) \geq \mu(K_{n-1}) + 1$. Let $E_n = K_n \setminus K_{n-1}$. Then for every n, the set E_n has compact closure, $1 \leq \mu(E_n) < \infty$, and all the E_n are disjoint. Let $f = \sum_n 1/n\chi_{E_n}$. □

7.22 A sequence (f_n) in $C_0(X)$ converges weakly to $f \in C_0(X)$ iff $\sup \|f_n\|_u < \infty$ and $f_n \to f$ pointwise.

Proof. If $M = \sup_n \|f_n\|_u < \infty$ and $f_n \to f$, then $\|f\|_u \leq M$. Hence, for every finite Radon measure μ, we have $\int f_n d\mu \to \int f d\mu$ by the DCT. Thus, f_n weakly converges to f in $C_0(X)$.

For the converse, first suppose $f_n \not\to f$ pointwise. Then there exists an $x \in X$ for which $f_n(x) \not\to f(x)$. Define a Radon measure μ by $\mu(E) = 1$ if $x \in E$ and $\mu(E) = 0$ otherwise. Then $\int f_n d\mu = f(x_0) \neq f(x) = \int f d\mu$, so f_n does not converge weakly to f.

Now suppose $\sup \|f_n\|_u = \infty$. For each n, pick x_n such that $f(x_n) = \|f_n\|_u$. If (x_n) has an accumulation point x, then $|f_n(x)| \to \infty$, a contradiction. □

7.25 Let μ be a Radon measure on X such that every nonempty open set has positive measure. For each $x \in X$ there is a net (f_α) in $L^1(\mu)$ that converges vaguely in $M(X)$ to the point mass at x. If X is first countable, the net can be taken to be a sequence. (Consider functions of the form $\mu(U)^{-1}\chi_U$.)