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## HW 3, due Thurs., February 14

**17.7** If $f : D \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, show that $g \circ f$ is measurable.

*Proof.* Let $U \subset \mathbb{R}$ be open. Then $g^{-1}(U)$ is open, so $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is measurable. □

**11** Let $G \subset [0, 1]$ be open, containing the interval rationals, and having $m(G) < 1/2$. Prove that $f = \chi_G$ is Borel measurable but cannot be equal a.e. to a Riemann integrable function.

*Proof.* $f$ is obviously Borel measurable. For the other conclusion, suppose $g = f$ on $[0, 1] \setminus N$ where $m(N) = 0$. Then $g = 1$ on $H := G \setminus N$, and $g = 0$ on $C := ([0, 1] \setminus G) \setminus N$. Note that $m(C) = 1 - m(G) > 1/2$. Hence, it suffices to show that $H$ is dense, for then every point of $C$ will be a point of discontinuity of $g$.

To see that $H$ is dense, let $U \subset [0, 1]$ be open. Since $G$ is open and dense, there exists an interval $I \subset (G \cap U)$. Since $m(N) = 0$, $N$ cannot contain $I$. Hence, $I \cap H = I \cap (G \setminus N) \neq \emptyset$. Thus, $H$ intersects $U$. □

**12** If $f : [a, b] \to \mathbb{R}$ is Lipschitz with constant $K$, and if $E \subset [a, b]$, show that $m^*(f(E)) \leq Km^*(E)$. In particular, $f$ maps null sets to null sets.

*Proof.* Consider the case when $E$ is an interval. Then, by the Lipschitz condition, $m^*(f(E)) \leq \sup_{x,y \in E} |f(x) - f(y)| \leq \sup_{x,y \in E} K|x - y| = Km(E)$.

For the general case, let $\epsilon > 0$, and let $(I_n)$ be a cover of $E$ by open intervals such that $\sum_n m(I_n) \leq m^*(E) + \epsilon$. Then $(f(I_n))$ covers $f(E)$, so by the special case above, $m^*(f(E)) \leq \sum_n m^*(f(I_n)) \leq \sum_n Km^*(I_n) \leq Km^*(E) + K\epsilon$. Letting $\epsilon \to 0$ gives the desired inequality. □

**17** If $f, g : \mathbb{R} \to \mathbb{R}$ are Borel measurable, show that $f \circ g$ is Borel measurable. If $f$ is Borel measurable and $g$ is Lebesgue measurable, show that $f \circ g$ is Borel measurable. If $f$ is Borel measurable and $g$ is Lebesgue measurable, show that $f \circ g$ is Lebesgue measurable.

*Proof.* Let $O$ denote the collection of all open sets in $\mathbb{R}$. By a theorem in class, for any function $g$, we have $\sigma(g^{-1}(O)) = g^{-1}(\sigma(O))$. Hence, if $U \subset \mathbb{R}$ is open and $f$ is Borel measurable, we have $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U)) \in g^{-1}(\sigma(O)) = \sigma(g^{-1}(O))$. □

**18(e)** Show that there is a Lebesgue measurable function $F$ and a continuous function $G$ such that $F \circ G$ is not Lebesgue measurable.

*Proof.* Let $0 < \alpha < 1$ and $\Delta_\alpha$ be the corresponding Cantor-like set. Since $m(\Delta_\alpha) > 0$, it contains an unmeasurable set $E$. Let $G : \Delta_\alpha \to \Delta$ be a homeomorphism (We proved $\Delta$ homeomorphic to $\{0,1\}^{\mathbb{N}}$ in 446; the same proof goes through for $\Delta_\alpha$). Let $F = \chi_{G(E)}$. Note that $m^*(G(E)) \leq m^*(\Delta) = 0$, so $G(E)$ is measurable. Hence, $F$ is measurable. However, $(F \circ G)^{-1}((1/2, \infty)) = G^{-1}(F^{-1}((1/2, \infty)) = G^{-1}(G(E)) = E$ is unmeasurable. □

**37** Give an example showing that a. u. convergence does not imply uniform convergence a.e.

*Proof.* Let $f_n : [0,1] \to [0,1]$ be defined by $f_n(x) = x^n$. On $[0,c]$, $f_n \leq c^n$ converges uniformly. Since $c$ can be picked arbitrarily close to 1, $f$ converges a.u.

On the other hand, suppose $f_n$ converged uniformly on $[0,1] \backslash N$ with $m(N) = 0$. Since $N$ cannot contain an interval, there exists a sequence $(x_n) \subset [0,1] \setminus N$ with $x_n \to 1$. By the uniform convergence, pick $N$ such that $\forall n \geq N$, $\sup_{[0,1)} |f_n| < 1/2$. Pick $M$ such that $x_M > (1/2)^{1/N}$. Then $f_N(x_M) > 1/2$, a contradiction. □

**40** If $f$ is Lebesgue measurable, prove that there is a Borel measurable function $g$ such that $f = g$ except, possibly, on a Borel set of measure zero. [Hint: Every null set is contained in a Borel set of measure zero.]

*Proof.* To justify the hint, if $N$ is a null set, there exist open sets $U_n \supset N$ with $m(U_n) = 1/n$. The required Borel set is $\bigcap_n U_n$.

Since $f$ is measurable, $f^+$ and $f^-$ are measurable. If we find Borel functions $g^+$, $g^-$ such that $f^+ = g^+$ except on a Borel set of measure zero and the same for $g^-$, then $f = f^+ - f^- = g^+ - g^-$ except on the union of Borel null sets, which is also a Borel null set. Moreover, $g^+ - g^-$ is Borel measurable since $g^+$ and $g^-$ are Borel (the proof that the Lebesgue functions form a vector space works verbatim for Borel functions). Thus, by breaking up $f$ into positive and negative parts, we may assume that $f > 0$.

Then there exist simple functions $f_n \to f$ with $0 \leq f_n \leq f_{n+1}$. Write $f_n = \sum_{i=0}^m a_i \chi_{A_i}$ in standard form. For each $i$, we have $A_i = B_i \cup N_{n,i}$ where $B_i$ is Borel, and $N_i$ is a null set. Let $\widetilde{N} = \bigcup_{n,i} N_{n,i}$. By the hint, there exists a Borel null set $N \supset \widetilde{N}$. Define $g_n := \sum_{i=0}^m a_i \chi_{A_i \setminus N}$. Since $A_i \setminus N = B_i \cup N_{n,i} \setminus N = B_i \setminus N$, each $g_n$ is Borel.

To see that $g := \lim_{n\to\infty} g_n$ is Borel, first note that $g^n$ is increasing on $N^c$ since $g_n = f_n$ on $N^c$. Moreover, $g^n = 0$ on $N$ for all $n$. Hence, $g^n$ are increasing on $\mathbb{R}$. Thus, for all $a \in \mathbb{R}$, we have $g^{-1}((a,\infty)) = \bigcup_n g_n^{-1}((a,\infty))$ is a Borel set, so $g$ is a Borel function.

Moreover, $g_n = f_n$ on $N^c$, so $g = f$ on $N^c$.

□

**41** Let $E \subset \mathbb{R}$ be closed, and let $f : E \to \mathbb{R}$ be continuous. Prove that $f$ extends to a continuous function $g : \mathbb{R} \to \mathbb{R}$ with $\sup_{\mathbb{R}} |g| \leq \sup_E |f|$.

*Proof.* $E^c$ is an open set, hence the union of disjoint open intervals $(I_n)$. If $E$ is empty, set $g := 0$. Otherwise, $(I_n)$ contains at most two semi-infinite intervals. Set the constant value of $g$ on any such interval to the value of $f$ at the finite endpoint. For any bounded interval $(a, b)$ of $(I_n)$, set $g$ to be the linear function interpolating the points $(a, f(a))$ and $(b, f(b))$.

By construction, $\sup_{\mathbb{R}} |g| \leq \sup_E |f|$, and $g$ is continuous on $E_c$. It is given that $g$ is continuous on the interior of $E$. The remaining case is to check that $g$ is continuous on the boundary of $E$.

Let $x \in E$ be a boundary point of $E$. I will check right continuity; the left continuity case is analogous.

If $x$ is an isolated point of $E \cap [x, \infty)$, then pick an open interval $U$ containing $x$ such that $U \cap E \cap [x, \infty) = x$. By construction, $g$ is linear hence continuous on $U \cap [x, \infty)$.

If $x$ is not an isolated point of $E \cap [x, \infty)$, let $\epsilon > 0$. By the continuity of $f$, pick $\delta > 0$ such that $\operatorname{osc}(f, (B_\delta(x) \cap E \cap [x, \infty)) < \epsilon$. Since $x$ is a limit point of $E \cap [x, \infty)$, there exists $y \in B_\delta(x) \cap E \cap (x, \infty)$. Note that $\operatorname{osc}(g, [x, y]) \leq \operatorname{osc}(f, [x, y]) \leq \epsilon$. Hence, $g$ is right continuous at $x$. □

**43** Let $f : [a, b] \to [-\infty, \infty]$ be a measurable and finite a.e., and let $\epsilon > 0$. Show that there is a polynomial $p$ such that $m(\{|f - p| \geq \epsilon\}) < \epsilon$.

*Proof.* By Theorem 17.20, there exists a continuous function $g : [a, b] \to \mathbb{R}$ such that $m(\{|f - g| \geq \epsilon/2\}) < \epsilon/2$. By the Weierstrauss approximation theorem, there exists $p$ such that $|g - p| < \epsilon/2$. Then, since $|f - p| \leq |f - g| + |g - p| = |f - g| + \epsilon/2$, we have

$m(\{|f - p| \geq \epsilon\}) \leq m(\{|f - g| + \epsilon/2 \geq \epsilon\}) \leq m(\{|f - g| \geq \epsilon/2\}) < \epsilon/2.$ □

**44** Let $f : [a, b] \to [-\infty, \infty]$ be a measurable and finite a.e. Prove that there is a sequence of polynomials $(g_n)$ on $[a, b]$ such that $g_n \to f$ a.e. on $[a, b]$. [Hint: For each $n$ choose $g_n$ so that the $E_n = \{|f - g_n| \geq 2^{-n}\}$ has $m(E_n) < 2^{-n}$. Now argue that $g_n \to f$ off the set $E = \limsup_{n\to\infty} E_n$.]

*Proof.* By (43), we can define $g_n$, $E_n$, and $E$ as in the hint. Since $\sum_n m(E_n) < \infty$, Corollary 16.24 implies $m(E) = 0$. Suppose $x \notin E$. Then there exists $N$ such that $x \notin \bigcup_{n=N}^{\infty} E_n$. Thus, for $n \geq N$, we have $|f(x) - g_n(x)| < 2^{-n}$. Hence, $g_n(x) \to f(x)$. □

**45** Let $f : [a, b] \to [-\infty, \infty]$ be a measurable and finite a.e., and let $\epsilon > 0$. Show that there is a continuous function $g$ on $[a, b]$ with $m\{f \neq g\} < \epsilon$. [Hint: Combine Exercises 41 and 44 and Egorov's theorem to find $(g_n)$ and a closed set $F$ with $m([a, b] \setminus F) < \epsilon$ and $g_n \to f$ uniformly on $F$. Now argue that $f|_F$ extends to a continuous function $g$.]

*Proof.* □

**46** (Luzin's Theorem) Show that $f : \mathbb{R} \to \mathbb{R}$ is measurable if and only if, for each $\epsilon > 0$, there is a measurable set $E$ with $m(E) < \epsilon$ such that the restriction of $f$ to $\mathbb{R} \setminus E$ is continuous.

*Proof.* Break $\mathbb{R}$ into countably many disjoint compact intervals $(I_n)_{n=1}^{\infty}$ such that $m((\bigcup_n I_n)^c) < \epsilon/2$. Apply (45) to $f$ on each $I_n$ to find continuous $g_n : I_n \to R$ and measurable $F_n \subset I_n$ with $m(I_n \setminus F_n) < \epsilon 2^{-n-1}$ such that $g_n = f$ on $F_n$. Let $E = \mathbb{R} \setminus \bigcup_n F_n$. Then $m(E) \leq m((\bigcup_n I_n)^c) + m(\bigcup_n I_n \setminus F_n) \leq \epsilon/2 + \epsilon/2$.

Note that since $F_n$ are disjoint, and $f = g_n$ on $F_n$ is continuous, we have $f$ is continuous on $\mathbb{R} \setminus E$. □

**48** Show that there is a measurable set $K \subset [0,1]$ such that $\chi_K$ is everywhere discontinuous in $[0,1] \setminus N$ for any null set $N$.

*Proof.* Let $A_n$ be a countable base for $[0,1]$ of open intervals. For each $A_n$, let $\Delta_n \subset A_n$ be a nowhere-dense, positive measure set (pick a generalized Cantor set out of $A_n$). Let $K := \bigcup_n \Delta_n$.

Let $N$ be a null set. To see that $K \setminus N$ is dense in $[0,1]$, let $U \subset \mathbb{R}$ be open and nonempty. Then for some $n$, $\Delta_n \subset A_n \subset U$. Since $\Delta_n$ has positive measure, $\Delta_n \setminus N$ is nonempty. Thus, $K \setminus N$ is dense in $[0,1]$, so $K$ is dense in $[0,1] \setminus N$.

Since $\Delta_n$ is nowhere dense, $K$ is first category. Thus, by Corollary 9.12, $[0,1] \setminus K$ is dense in $[0,1]$, so $[0,1] \setminus K$ is dense in $[0,1] \setminus N$. Thus, since $K \setminus N$ and $[0,1] \setminus (K \setminus N)$ are both dense in $[0,1] \setminus N$, we have that $\chi_K$ is everywhere discontinuous on $[0,1] \setminus N$. □