Paul Gustafson
Math 643 - Algebraic Topology I

Problem Set 2

3.20 Let X be a space with basepoint x_0, and let $\{U_j : j \in J\}$ be an open cover of X by path connected subspaces such that:

(i) $x_0 \in U_j$ for all j;

(ii) $U_j \cap U_k$ is path connected for all j, k.

Prove that $\pi_1(X, x_0)$ is generated by the subgroups $\operatorname{im} i_{j*}$ where $i_j : (U_j, x_0) \to (X, x_0)$ is the inclusion.

Proof. The set $\{f^{-1}(U_j)\}_j$ is an open cover of I. Let δ be its Lebesgue number. Pick N such that $1/N < \delta$.

For each $1 \leq i \leq N$, there exists U_{j_i} such that $f([(i-1)/N), i/N]) \subset U_{j_i}$ by the definition of Lebesgue number. For $1 \leq i < N$, the point $f(iN)$ lies in $U_i \cap U_{i+1}$, a path connected space. Let p_i be a path from x_0 to $f(iN)$ in $U_i \cap U_{i+1}$. If $i = 0$ or $i = N$, let p_i be the constant path at x_0. Then for $1 \leq i \leq N$ Then $g_i := p_{i-1} * f_{|[(i-1)/N),i/N]} * p_i^{-1}$ defines a path in U_i with endpoints at x_0. The fact that $[f] = [g_1] * [g_2] * \ldots * [g_N]$ proves the desired result. □

3.21 If $n \geq 2$, prove that S^n is simply connected.

Proof. Let $U_1, U_2 \subset S^n$ be the complements of the north and south poles, respectively. By Exercise 3.20, it suffices to show that both U_1 and U_2 are simply connected. However, the stereographic projection from the north pole defines a homeomorphism from U_1 to $\mathbb{R}^n$ which is simply connected. Thus U_1 is simply connected. A similar argument works for U_2. □

3.24 Let G be a simply connnected topological group and let H be a discrete closed normal subgroup. Prove that $\pi_1(G/H, 1) \simeq H$.

Proof. Let $\nu : G \to G/H$ be the canonical quotient map. Let f and (X, x_0) be defined as in Lemma 3.14 with codomain $(G/H, 1)$ instead of $(S^1, 1)$. I will prove the analog of Lemma 3.14 with ν replacing exp and G replacing $\mathbb{R}$.

Since H is discrete, there exists an open $U \subset G$ with $U \cap H = 1$.

I claim that there exists $V \subset U$, a symmetric open neighborhood of 1 with $V * V \subset U$. Indeed, there exists open neighborhoods A, B of 1 with $A * B \subset U$ by the continuity of group multiplication. Therefore $W := A \cap B$ is an open neighborhood of 1 with $W * W \subset U$. Moreover W^{-1} is an open neighborhood of 1. Hence $V := W \cap W^{-1}$ works.

Note that Vg is the inverse image of V under multiplication by g^{-1} on the right, so is open for all $g \in G$. Therefore $\{f^{-1}(Vg)\}_g$ is an open cover of X. Let ϵ be the Lebesgue number of this cover. It follows that if $\|x - x'\| < \epsilon$ then $f(x)f(x')^{-1} \in Vgg^{-1}V = V$.

The rest of the proof of Lemma 3.14 follows with V replacing $(-\frac{1}{2}, \frac{1}{2})$ and $\nu(V)$ replacing $S^1 - \{-1\}$. The only sticky part is showing that $\nu_{|V}$ is a homeomorphism onto its image.

First $\nu_{|V}$ is injective since $H \cap V = \{1\}$. Thus $\nu_{|V}$ is a continuous bijection onto its image, so it suffices to show that ν is an open map. Suppose $U \subset G$ is open. Then $\nu^{-1}(\nu(U)) = \bigcup_{h \in H} Vh$ is the union of open sets, hence open. Hence $\nu(U)$ is open since ν is a quotient map. Thus $\nu_{|V}$ is a homeomorphism onto its image.

The rest of the proof follows with the obvious modifications (replacing $+$ with the group multiplication, and exp with ν). □

4

Proof. The new naturality condition is

$$(\sigma \times 1)_\# P_{n-1}^{\Delta^n} = P_{n-1}^X \sigma_\#.$$

This still holds for $n = 0$ since $P_{-1}^X = 0$ by definition for all X. The key formula on p. 77 remains unchanged. To verify that the new naturality condition holds for $P_n^{\Delta^{n+1}}$, the same argument as in the text works, except we choose $\tau : \Delta^n \to \Delta^{n+1}$ and $\sigma : \Delta^{n+1} \to X$. □

5.14

(i) *Proof.* Since i_{n-1} is injective, we have $0 = \ker A_{n-1} = \operatorname{im}(C_n \to A_{n-1})$ by the exactness of the long sequence at A_{n-1}. Hence, by the exactness at C_n in the long sequence, we have $\operatorname{im} p_n = C_n$.

The exactness at B_n in the long sequence implies exactness at B_n in the short sequence. Since i_n is injective, we get exactness at A_n in the short sequence. □

(ii) If A is a retract of X, prove that for all $n \geq 0$,

$$H_n(X) \simeq H_n(A) \oplus H_n(X, A).$$

Proof. Since A is a retract of X, the s.e.s $0 \to A \to X \to X/Ato0$ splits.

Therefore $0 \to S_*(A) \xrightarrow{i} S_*(X) \xrightarrow{p} S_*(X/A)to0$ also splits.

Applying Theorem 5.6, we get an exact sequence

$$\ldots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{p_*} H_n(X, A) \to H_{n+1}(A) \to \ldots.$$

Since i has a left inverse, so does i_* for all n since H_n is a functor. In particular, i_* is invertible for all n.

Thus by (i), we have the split exact sequence

$$0 \to H_n(A) \to H_n(X) \to H_n(X, A) \to 0.$$

□

(iii) if A is a deformation retract of X, then $H_n(X, A) = 0$ for all $n \geq 0$.

Proof. Since A is a deformation retract of X, the inclusion $i : A \to X$ is a homotopy equivalence. Since H_n factors through **hTop**, the map $i_* := H_n(i)$ is an isomorphism, and in particular surjective. The map p_* in the s.e.s.

$$0 \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{p_*} H_n(X, A) \to 0,$$

is must be the 0-map and also be surjective. Hence $H_n(X, A) = 0$. $\square$