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## HW 6

1 Assume  $(X, \mathcal{M}, \mu)$  is a complete measure space.

- a) If  $f : X \rightarrow \mathbb{R}$  is  $\mathcal{M}$ -measurable and  $f = g$   $\mu$ -a.e., then  $g$  is also measurable.
- b) If  $f_n : X \rightarrow \mathbb{R}$  is  $\mathcal{M}$ -measurable and  $\lim_{n \rightarrow \infty} f_n = f$   $\mu$ -a.e., then  $f$  is measurable.

*Proof.* For (a), let  $N = \{x : f(x) \neq g(x)\}$ . Since  $\mathcal{M}$  is complete,  $N \in \mathcal{M}$ . Let  $a \in \mathbb{R}$ . We need to show that  $A := g^{-1}((a, \infty))$  is in  $\mathcal{M}$ . To see this, note that  $A \cap N^c = f^{-1}((a, \infty)) \cap N^c$ , which is in  $\mathcal{M}$ . Hence  $A = (A \cap N) \cup (A \cap N^c)$  is in  $\mathcal{M}$  since  $A \cap N$  is a null set.

For (b), let  $N = \{x : f(x) \neq \lim_n f_n\} \in \mathcal{M}$ . Then  $f_n \chi_{N^c} \rightarrow f \chi_{N^c}$  pointwise. Hence  $f \chi_{N^c}$  is measurable by the first exercise of the last homework set. Thus, by (a),  $f$  is measurable.  $\square$

2 If  $f \in \mathcal{L}^+$  and  $\int f d\mu < \infty$ , then for any  $\epsilon > 0$ , there is an  $E \in \mathcal{M}$ , so that  $\mu(E) < \infty$  and  $\int_E f d\mu > \int f d\mu - \epsilon$ .

*Proof.* There exists a simple function  $0 \leq \phi \leq f$  with  $\int f - \int \phi \leq \epsilon$ . Write  $\phi$  in standard form as  $\phi = \sum_{n=1}^N a_n \chi_{A_n}$ , where  $a_n \neq 0$ . Since  $\phi$  is integrable,  $\mu(A_n) < \infty$  for all  $n$ . Thus if  $E := \bigcup_n A_n$ , then  $\mu(E) = \sum_n \mu(A_n) < \infty$ . Also, we have  $\int f - \int_E f \leq \int f - \int_E \phi = \int f - \int \phi \leq \epsilon$ .  $\square$

3 Let  $f \in \mathcal{L}^+(\mu)$ . If  $\int f d\mu < \infty$ , then  $\mu(\{x \in X : f(x) = \infty\}) = 0$ .

*Proof.* Let  $E := \{x \in X : f(x) = \infty\}$ . Suppose  $\mu(E) \neq 0$ . Let  $\phi_n = n \chi_E$ . Then  $\int f \geq \int \phi_n \rightarrow \infty$ , a contradiction.  $\square$

4 Prove Fatou's Lemma without using the Monotone Convergence Theorem, and deduce the MCT from Fatou's Lemma. Fatou's lemma states that if  $(f_n) \subset \mathcal{L}^+$ , then  $\int \liminf f_n \leq \liminf \int f_n$ .

*Proof. Step 1.* Suppose  $\rho$  is a measure on  $\mathcal{M}$ , and  $(E_n) \subset \mathcal{M}$ . Then  $\rho(\liminf E_n) = \rho(\bigcup_n \bigcap_{k \geq n} E_k) = \lim_n \rho(\bigcap_{k \geq n} E_k) \leq \lim_n \inf_{k \geq n} \rho(E_k) = \liminf_n \rho(E_n)$ .

*Step 2.* Let  $\phi$  be a simple function such that  $0 \leq \phi \leq \liminf f_n$ . Let  $E_n = \{x : \phi(x) \leq f_n(x)\}$ . Then

$$\begin{aligned} \liminf E_n &= \bigcup_n \bigcap_{k \geq n} E_k \\ &= \{x : \exists n \forall k \geq n \quad \phi(x) \leq f_k(x)\} \\ &\supset \{x : \exists n \quad \phi(x) \leq \inf_{k \geq n} f_k(x)\} \\ &\supset \{x : \phi(x) \leq \sup_n \inf_{k \geq n} f_k(x)\} \\ &= \mathbb{R}. \end{aligned}$$

By Step 1 applied to the measure  $A \mapsto \int_A \phi$ , we have  $\int \phi = \int_{\liminf E_n} \phi \leq \liminf \int_{E_n} \phi \leq \liminf \int_{E_n} f_n \leq \liminf \int f_n$ . Hence  $\int \liminf f_n \leq \liminf \int f_n$ .

For the proof of the MCT, let  $(f_n) \subset \mathcal{L}^+$  with  $f_n \uparrow f$ . First note that  $\lim \int f_n$  must exist since  $(\int f_n)$  is an increasing sequence. Hence  $\int f = \int \lim f_n \leq \liminf \int f_n = \lim \int f_n$ . For the reverse inequality, we have  $f_n \leq f$  so  $\int f_n \leq \int f$ . Thus,  $\lim \int f_n \leq \int f$ .  $\square$

**5** Let  $f : [0, 1] \rightarrow [0, 1]$  be the Cantor function, and  $C$  be the Cantor set. Define  $g(x) = f(x) + x$  for  $x \in [0, 1]$ .

a)  $g$  is a bijection from  $[0, 1]$  to  $[0, 2]$  and  $g^{-1}$  is continuous.

b)  $m(g(C)) = 1$ .

c) Using Exercise 29/Chapter 1 show that for some nonmeasurable  $A \subset g(C)$ ,  $B = g^{-1}(A)$  is Lebesgue measurable but not Borel measurable.

d) There is a Lebesgue measurable function on  $\mathcal{R}$  which is not Borel measurable.

*Proof.* To see that the Cantor function  $f$  is increasing, recall that if  $0 \leq a < b \leq 1$  with  $a = \sum_n a_n r^{-n}$  and  $b = \sum_n b_n r^{-n}$  base- $r$  expansions, then  $(a_n) < (b_n)$  in the lexicographical ordering. Conversely, if  $(a_n) < (b_n)$  in the lexicographical ordering, then  $a \leq b$ . Hence  $f$  is increasing on  $C$ .

To see that  $f$  is increasing on  $[0, 1]$ , let  $0 \leq x < y \leq 1$ . If  $x \in C^c$ , replace it with the lower endpoint  $x'$  of the interval in  $C^c$  containing  $x$ . We have  $f(x) = f(x')$  and  $x' < x < y$ . Hence WLOG  $x \in C$ . We can do similarly for  $y$  using an upper endpoint instead. Thus  $f(x) \leq f(y)$  since  $f$  is increasing on  $C$ . Thus  $f$  is increasing on  $[0, 1]$ .

To see that  $f$  is continuous, let  $a \in [0, 1]$ . If  $a \in C^c$ , then  $f$  is continuous at  $a$  since  $C^c$  is open and  $f$  is locally constant on  $C^c$ .

If  $a \in C$ , fix  $\epsilon > 0$ . There exists  $n$  such that  $2^{-n} < \epsilon$ . Suppose  $|a - b| < 3^{-n}$  for some  $b$  in  $[0, 1]$ . Then  $a_j = b_j$  for  $j \leq n$ . Hence,  $|f(a) - f(b)| \leq \sum_{j=n+1}^{\infty} 2^{-j} = 2^{-n} < \epsilon$ . Thus  $f$  is continuous at  $a$  for all  $a \in C$ . Hence  $f$  is continuous.

Since  $g$  is the sum of an increasing function and a strictly increasing function,  $g$  is strictly increasing. In particular,  $g$  is injective. Since  $f$  is continuous, so is  $g$ . Since  $g(0) = 0$  and  $g(1) = 2$ , the intermediate value theorem implies that  $g$  surjects onto  $[0, 2]$ .

To see that  $g^{-1}$  is continuous, suppose  $F \subset [0, 1]$  is closed. Then  $F$  is compact. Let  $(U_\alpha)_{\alpha \in A}$  be an open cover of  $g(F)$ . Then  $(g^{-1}(U_\alpha))$  is an open cover of  $F$ . Let  $(g^{-1}(U_\alpha))_{\alpha \in G}$  be a finite subcover of  $F$ . Then  $(U_\alpha)_{\alpha \in G}$  is a subcover of  $g(F)$ , since  $g$  is a bijection. Thus  $g(F)$  is compact, hence closed. Thus  $g^{-1}$  is continuous.

For (b), from the definition of the Cantor set we have  $C^c = \bigcup_n I_n$  for some countable collection of disjoint intervals  $I_n$  with  $\sum_n m(I_n) = 1$ . The construction of  $f$  implies that, for each  $n$ ,  $f(I_n) = \{x_n\}$  for some singleton  $\{x_n\}$ . Hence  $m(g(I_n)) = m(x_n + I_n) = m(I_n)$ . Since  $g$  is a bijection,  $(g(I_n))_n$  remain pairwise disjoint. Thus,  $m(g(C)) = 2 - m(g(C^c)) = 2 - \sum_n m(g(I_n)) = 2 - \sum_n m(I_n) = 1$ .

For (c), Exercise 29/Chapter 1 implies that there exists a nonmeasurable set  $A \subset g(C)$ . The proof of this exercise is simple (pick an interval  $[i, i + 1]$  for which  $E \cap [i, i + 1]$  has positive measure and then do the Vitali construction using  $E \cap N_r$  instead of  $N_r$ ). Since  $m(C) = 0$  and the Lebesgue measure is complete,  $B := g^{-1}(A)$  is Lebesgue measurable. However, if  $B$  were Borel measurable, then  $A = g(B)$  would be Borel since  $g^{-1}$  is continuous hence Borel measurable. Thus,  $B$  is not Borel.

For (d), such a function is  $\chi_B$ . □