Paul Gustafson
Texas A&M University - Math 447
Instructor: Dr. Johnson

HW 4, due 2/21

18.3 Prove that $\int_1^\infty (1/x)\,dx = \infty$.

Proof. Let $f(x) = (\chi_{(1,\infty)}(x))(1/x)$. Define $\phi_m = \sum_{n=1}^m \frac{1}{n+1}\chi_{(n,n+1)}$. Then for all $m \geq 1$, we have $\phi_m \leq f$. Hence, by the monotonicity of the integral, $\int f\,dm \geq \int \phi_m\,dm = \sum_{n=1}^m \frac{1}{n+1} \to \infty$ as $m \to \infty$. □

4 Find (f_n) nonnegative measurable functions that converge uniformly to 0, but $\lim_{n\to\infty} \int f_n = 1$.

Proof. Let $f_n = (1/n)\chi_{(0,n)}$. □

6 Suppose (f_n) nonnegative, measurable decrease pointwise to f, and that $\int f_k < \infty$ for some k. Prove that $\int f = \lim_{n\to\infty} f_n$. Also, give an example showing that the condition $\int f_k < \infty$ is necessary.

Proof. For the counterexample, let $f_n = \chi_{(n,\infty)}$ for $n \geq 1$.

For the other part of the problem, for all $n \geq k$, let $g_n = f_k - f_n$. Since (f_n) is nonnegative and decreasing, $(g_n)_{n\geq k}$ is increasing and nonnegative. Since $g_n \leq f_k$, we have $\int g_n < \infty$ for all $n \geq k$. Hence, using the linearity of the integral on integrable functions and the MCT,

$$\begin{aligned}
\int f\,dm &= -\int (f - f_k)\,dm + \int f_k\,dm \\
&= -\int \lim_{n\to\infty} f_k - f_n\,dm + \int f_k\,dm \\
&= -\lim_{n\to\infty} \left(\int f_k - f_n\,dm \right) + \int f_k\,dm \\
&= -\lim_{n\to\infty} \int -f_n\,dm \\
&= \lim_{n\to\infty} \int f_n\,dm
\end{aligned}$$

□

7 Let $\mu : \mathcal{A} \to [0,\infty]$ be a nonnegative, finitely additive, set function defined on a σ-algebra $\mathcal{A}$. Prove that:

1. $\mu(E) \leq \mu(F)$ whenever $E, F \in \mathcal{A}$ satisfy $E \subset F$.
2. if $\mu(\emptyset) \neq 0$, then $\mu(E) = \infty$ for all $E \in \mathcal{A}$.

Proof. For (1), we have $\mu(F) = \mu(E) + \mu(E \setminus F) \geq \mu(E)$. For (2), if $\mu(\emptyset) \neq 0$, we have $\mu(E) = \mu(E \cup \bigcup_{i=1}^n \emptyset) = \mu(E) + n\mu(\emptyset) \to \infty$ as $n \to \infty$. □

8 Define μ and $\mathcal{A}$ as in (7). Prove that TFAE:

1. $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n)$ for every pairwise disjoint $(E_n) \subset \mathcal{A}$.
2. $\mu(\bigcup_{n=1}^{\infty} E_n) = \lim_{n\to\infty} \mu(E_n)$ for every increasing $(E_n) \subset \mathcal{A}$.

Proof. To prove (2) implies (1), let $F_k = \bigcup_{n=1}^{k} E_n$. Then (F_k) is an increasing sequences of sets in $\mathcal{A}$, so, by (2), $\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcup_{k=1}^{\infty} F_k) = \lim_{k\to\infty} \mu(F_k) = \lim_{k\to\infty} \sum_{n=1}^{k} \mu(E_n) = \sum_{n=1}^{\infty} \mu(E_n)$.

For (1) implies (2), let (F_n) be the disjointification of (E_n). That is, $F_n := E_n \backslash (\bigcup_{k<n} E_k)$, so for all N, we have $\bigcup_{n=1}^{N} F_n = \bigcup_{n=1}^{N} E_n$. Then, applying (1) to F_n, we have $\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcup_{n=1}^{\infty} F_n) = \sum_{n=1}^{\infty} \mu(F_n) = \lim_{N\to\infty} \sum_{n=1}^{N} \mu(F_n) = \lim_{N\to\infty} \mu(E_N)$. $\square$

15 Let f be nonnegative and measurable. Prove that $\int f < \infty$ if and only if $\sum_{i=-\infty}^{\infty} 2^k m\{f > 2^k\} < \infty$.

Proof. Suppose $\int f < \infty$. Then $\sum_{i=-N}^{N} 2^k m\{f > 2^k\} = \int \sum_{i=-N}^{N} 2^k \chi_{\{f>2^k\}} \leq \int f$. Letting $N \to \infty$, we see that $\sum_{i=-\infty}^{\infty} 2^k m\{f > 2^k\} \leq \int f < \infty$.

Conversely, suppose $\sum_{i=-\infty}^{\infty} 2^k m\{f > 2^k\} < \infty$. Let $\phi \leq f$ be an integrable simple function with standard representation $\phi(x) = \sum_{i=0}^{n} a_i \chi_{A_i}$. $\square$

16 Let $f \geq 0$ be integrable. Given $\epsilon > 0$, show that there is a measurable set E with $m(E) < \infty$ such that $\int_E f > \int f - \epsilon$. Moreover, show that E can be chosen so that f is bounded on E.

Proof. Pick an integrable, nonnegative, simple function $\phi \leq f$ such that $\int f - \int \phi \leq \epsilon/2$. Write ϕ in standard form as $\phi = \sum_{i=0}^{n} a_i \chi_{A_i}$ where $a_0 = 0$. Note that since ϕ is integrable, we have $m(A_0^c) = \sum_{i=1}^{n} m(A_i) \leq (\min_{i\geq 1} a_i)^{-1} \sum_{i=1}^{n} a_i m(A_i) = (\min_{i\geq 1} a_i)^{-1} \int \phi \leq \infty$. Hence, we have $\int_{A_0^c} f \geq \int_{A_0^c} \phi = \int_{\mathbb{R}} \phi \geq \int f - \epsilon/2$.

To get the bounded part, let $E_k := \{f > k\} \cap A_0^c$. Since f is integrable, it is finite a.e., so $m(\bigcap_k E_k) = 0$. Hence, since $m(E_1) \leq m(A_0^c) < \infty$, we have $lim_{k\to\infty} E_k = 0$. Pick K such that $m(E_k) < \epsilon/2$. Then, if $E = A_0^c$ $\square$

17 If f is nonnegative and integrable, prove that the function $F(x) = \int_{-\infty}^{x} f$ is continuous. In fact, even more is true: Given $\epsilon > 0$, show that there is a $\delta > 0$ such that $\int_E f < \epsilon$ whenever $m(E) < \delta$. [Hint: This is easy when f is bounded; see (16)]

Proof. $\square$

14 Define $f : [0, 1] \to [0, \infty)$ by $f(x) = 0$ if x is rational and $f(x) = 2^n$ if x is irrational with exactly $n = 0, 1, 2, \ldots$ leading zeros in its decimal expansion. Show that f is measurable, and find $\int_0^1 f$.

Proof. $\square$

J18.1 Suppose that f is a nonnegative integrable function and A is a measurable set. Define F on $\mathbb{R}$ by $F(t) = m_f(A+t)$. Show that F is a continuous function. Recall that $m_f(E) := \int \chi_E f \, dm$. (Hint: First treat the case where A is a bounded interval.)

Proof. $\square$