# FINITENESS FOR MAPPING CLASS GROUP REPRESENTATIONS FROM TWISTED DIJKGRAAF-WITTEN THEORY

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ABSTRACT. Any twisted Dijkgraaf-Witten representation of a mapping class group of a closed surface has finite image.

## 1. INTRODUCTION

Given a spherical category $\mathscr{A}$ over a field $k$ and an oriented compact surface $M$, possibly with boundary, the Turaev-Viro construction gives a projective representation of the mapping class group $\mathrm{MCG}(M)$ [1, 2]. A natural question is to determine the image of these representations. In particular, when does such a representation have finite image?

It is conjectured that these representations have finite image if and only if $\mathscr{A}$ is weakly integral. This conjecture is a mild modification of the Property F conjecture [4, 3], which states that braid group representations coming from a braided monoidal category $\mathscr{C}$ should have finite image if and only if $\mathscr{C}$ is weakly integral.

This paper considers the case when $\mathscr{A} = \mathrm{Vec}_G^{\omega}$, the monoidal category of $G$-graded vector spaces with associativity modified by a cocycle $\omega \in Z^3(G, k^{\times})$. In this case, the Turaev-Viro construction corresponds to the twisted Dijkgraaf-Witten theory [5]. The category $\mathrm{Vec}_G^{\omega}$ is integral, so the one expects that its associated mapping class group representations have finite image. Our main contribution is to verify this in the case when the surface $M$ is closed and ($G$, $\omega$) are arbitrary.

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## 2. RELATED WORK

This result is an extension of the result of Fjelstad and Fuchs [6] showing that, given a surface with at most one boundary component, the mapping class group representations corresponding to the untwisted (i.e. $\omega = 1$) case have finite image. Their paper uses an algebraic method of Lyubashenko [7] that gives a projective mapping class group representation to any factorizable ribbon Hopf algebra, in their case, the double $D(G)$. In our case, we will take the simpler, geometric approach of considering the mapping class group action on a vector space of $\mathrm{Vec}_G^{\omega}$-colored embedded graphs defined by Kirillov [8].

In [9], Bantay defined representations of mapping class groups on the Hilbert space of an orbifold model associated to $D^{\omega}(G)$. These representations appear to coincide with the twisted Dijkgraaf-Witten representations. However, the precise details of the connection are not clear to me.

More is known when we fix a particular surface $M$. In the case where $M$ is a torus, it turns out that any Reshitikhin-Turaev representation of the mapping class group of the torus is always finite [10]. In particular, this implies that the Turaev-Viro representations are all finite. In the case where $M$ is an $n$-punctured disk, the mapping class group of $M$ relative to the boundary of the disk is the braid group $B_n$. In this case, it has been proved that the Turaev-Viro representations coming from are finite [3].

## 3. DEFINITIONS

Let $M$ be a closed surface of genus $g$. Let $G$ be a finite group, and let $\mathrm{Vec}_G^{\omega}$ denote the category of $G$-graded vector spaces with associativity defined by the 3-cocycle $\omega \in Z^3(G, k^{\times})$. More explicitly, we will follow [11] in the choice of structural morphisms. The associator $\alpha_{g,h,k} : (g \otimes h) \otimes k \to g \otimes (h \otimes k)$ is defined to be

$$\alpha_{g,h,k} = \omega(g,h,k)\,\mathrm{id}_{ghk}\,.$$

The evaluator $\mathrm{ev}_g : g^* \otimes g \to 1$ is

$$\mathrm{ev}_g = \omega(g^{-1}, g, g^{-1})\,\mathrm{id}_1\,.$$

The coevaluator $\mathrm{coev}_g : g \otimes g^* \to 1$ is

$$\mathrm{coev}_g = \mathrm{id}_1 .$$

The pivotal structure $j_g : g^{**} \to g$ is

$$j_g = \omega(g^{-1}, g, g^{-1})\, \mathrm{id}_g .$$

The following definitions are due to Kirillov [8]. We will consider finite graphs embedded in the surface $M$; for such a graph $\Gamma$, let $E(\Gamma)$ be the set of edges. Note that edges are not oriented. Let $E^{or}$ be the set of oriented edges, i.e. pairs $\mathbf{e} = (e, \text{orientation of } e)$; for such an oriented edge $\mathbf{e}$, we denote by $\bar{\mathbf{e}}$ the edge with opposite orientation.

**Definition 3.1.** *Let $\Gamma \subset M$ be an embedded graph as defined above. A* coloring *of $\Gamma$ is the following data:*

- *Choice of an object $V(\mathbf{e}) \in \mathrm{Obj}\,\mathscr{A}$ for every oriented edge $\mathbf{e} \in E^{or}(\Gamma)$ so that $V(\bar{\mathbf{e}}) = V(\mathbf{e})^*$.*
- *Choice of a vector $\varphi(v) \in \mathrm{Hom}(1, V(\mathbf{e}_1) \otimes \cdots \otimes V(\mathbf{e}_n))$ for every interior vertex $v$, where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are edges incident to $v$, taken in counterclockwise order and with outward orientation.*

The following theorem is a variation of result of Reshetikhin and Turaev.

**Theorem 3.2.** *There is a unique way to assign to every colored planar graph $\Gamma$ in a disk $D \subset \mathbb{R}^2$ a vector*

$$\langle \Gamma \rangle_D \in \mathrm{Hom}(1, V(\mathbf{e}_1) \otimes \cdots \otimes V(\mathbf{e}_n)) \tag{1}$$

*where $\mathbf{e}_1, \dots, \mathbf{e}_n$ are the edges of $\Gamma$ meeting the boundary of $D$ (legs), taken in counterclockwise order and with outgoing orientation, so that that following conditions are satisfied:*

(1) *$\langle \Gamma \rangle$ only depends on the isotopy class of $\Gamma$.*
(2) *If $\Gamma$ is a single vertex colored by $\varphi \in \mathrm{Hom}(1, V(\mathbf{e}_1) \otimes \cdots \otimes V(\mathbf{e}_n))$, then $\langle \Gamma \rangle = \varphi$.*
(3) *Local relations shown in Figure 1 hold.*

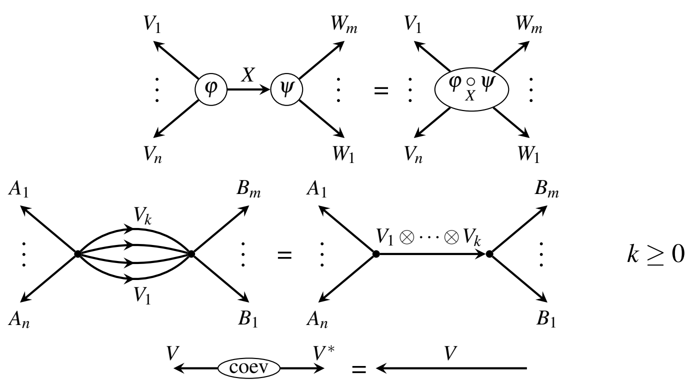


FIGURE 1. Local relations for colored graphs. Here $\varphi \underset{X}{\circ} \psi = (\varphi \otimes \psi) \circ \mathrm{ev}_X$.

*Local relations should be understood as follows: for any pair $\Gamma, \Gamma'$ of colored graphs which are identical outside a subdisk $D' \subset D$, and in this disk are homeomorphic to the graphs shown in Figure 1, we must have $\langle \Gamma \rangle = \langle \Gamma' \rangle$.*

*Moreover, so defined $\langle \Gamma \rangle$ satisfies the following properties:*

(1) *$\langle \Gamma \rangle$ is linear in color of each vertex $v$ (for fixed colors of edges and other vertices).*
(2) *$\langle \Gamma \rangle$ is additive in colors of edges as shown in Figure 2.*
(3) *Composition property: if $D' \subset D$ is a subdisk such that $\partial D'$ does not contain vertices of $\Gamma$ and meets edges of $\Gamma$ transversally, then $\langle \Gamma \rangle_D$ will not change if we replace subgraph $\Gamma \cap D'$ by a single vertex colored by $\langle \Gamma \cap D' \rangle_{D'}$.*

*The vector $\langle \Gamma \rangle$ is called the* evaluation *of $\Gamma$.*

To define local relations between graphs, Kirillov defines the space of null graphs as follows. Let $D \subset M$ be an embedded disk, and let $\Gamma = c_1\Gamma_1 + \cdots + c_n\Gamma_n$ be a linear combination of colored graphs in $M$ such that

(1) $\Gamma$ is transversal to $\partial D$ (i.e., no vertices of $\Gamma_i$ are on the boundary of $D$ and edges of each $\Gamma_i$ meet $\partial D$ transversally).

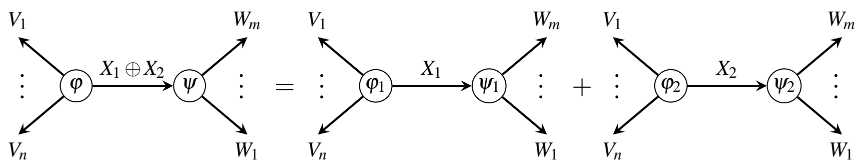

FIGURE 2. Linearity of $\langle\Gamma\rangle$. Here $\varphi_1, \varphi_2$ are compositions of $\varphi$ with projector $X_1 \oplus X_2 \to X_1$ (respectively, $X_1 \oplus X_2 \to X_2$), and similarly for $\psi_1, \psi_2$.

(2) All $\Gamma_i$ coincide outside of $D$.
(3) $\langle\Gamma\rangle_D = \sum c_i \langle \Gamma_i \cap D\rangle_D = 0$.

In this case $\Gamma$ is called a null graph.

**Definition 3.3.** *The representation space $H$ is the vector space of formal linear combinations of* $\mathrm{Vec}_G^{\omega}$*-colored graph embeddings modulo the subspace spanned by the null graphs.*

## 4. RESULT

**Theorem 4.1.** *The image of the twisted Dijkgraaf-Witten representation of a mapping class group of any closed surface $M$ is finite.*

*Proof.* Let $\Gamma$ be a $\mathrm{Vec}_G^{\omega}$-colored graph embedding . Thinking of $X$ as a quotient of its fundamental polygon, by isotopy we may assume vertices of $\Gamma$ lie in the interior of the polygon and that all the edges of $\Gamma$ do not intersect corners and meet the sides transversally. Evaluating on the interior of the polygon shows that $\Gamma$ is equivalent to a graph with a single vertex whose edges are simple closed curves, each of which intersect the boundary of the polygon precisely once. By using the local relations, we can replace all the edges intersecting a side with a single edge labeled by the tensor product of their labels. If there are no edges intersecting a side, we can insert a single edge labeled by the group identity into $\Gamma$ that intersects only that side. Thus, $\Gamma$ is equivalent to a colored graph with one vertex $v$ and $2n$ outgoing edges $e_1, \ldots e_{2n}$, each of which intersects precisely one side of the fundamental polygon.

By the definition of the evaluation of a string-net and the definition of the quotient map identifying the sides of the fundamental polygon, the vertex $v$ is colored by an element $\phi(v) \in \mathrm{Hom}(1, \bigotimes_{i=1}^{n} V(e_{2i-1}) \otimes V(e_{2i}) \otimes V(e_{2i-1}^*) \otimes V(e_{2i}^*))$, where $V(e_i) \in \mathrm{Obj}(\mathrm{Vec}_G^{\omega})$ is the coloring of the edge $e_i$. Since string-net evaluation is additive in the direct sum and linear in the vertex color, it follows that $H$ is spanned by the set of colored graphs

$$S := \{\Gamma \in H : V(e_i) \in \mathrm{Irr}(\mathrm{Vec}_G^{\omega}), \phi(v) = 1\},$$

where $\mathrm{Irr}(\mathrm{Vec}_G^{\omega})$ is the set of simple objects of $\mathrm{Vec}_G^{\omega}$, which correspond to the elements of $G$. See Figure 3 for a depiction.

The mapping class group of $M$ is generated by the Lickorish generating set consisting of Dehn twists around $3g-1$ simple closed curves (see Figure 4). These can be divided into two types of twists: the ones around a single hole (the blue and green curves in Figure 4), and the ones connecting two holes (the red curves in Figure 4).

Using the local moves as in Figures 5 to 15, one sees that the result of each of these Dehn twists lies in $\mathrm{Im}(\omega)S$. It is a basic result in group cohomology that, by replacing $\omega$ with a cohomologous cocycle if necessary, the image of $\omega$ lies in $\mu_{|G|}$, the set of $|G|$-th roots of unity. Since cohomologous cocycles give rise to monoidally equivalent categories $Vect_G^{\omega}$, this replacement does not incur any loss in generality.

Thus, the image of any such mapping class group representation is a quotient of the group of permutations of the finite set $\mu_{|G|}S$, hence finite. □

## 5. EXAMPLE CALCULATION

This section contains a calculation of the matrix coefficient for the first Dehn twist (shown in Figure 5). Assume that the main vertex is initially labeled with an element of $\mathrm{Hom}(1, h \otimes g \otimes 1 \otimes h^{-1} \otimes hg^{-1}h^{-1})$. In the following, we will abbreviate by saying that the vertex is "in state $h \otimes g \otimes 1 \otimes h^{-1} \otimes hg^{-1}h^{-1}$."

In Figure 7, we add the upper left vertex, which is labeled by $\mathrm{coev}_g$. We then connect the vertices with an unlabelled edge, which is shorthand for labelling by the object 1 . At this point, the vertices are in states $h \otimes g \otimes 1 \otimes h^{-1} \otimes hg^{-1}h^{-1}$

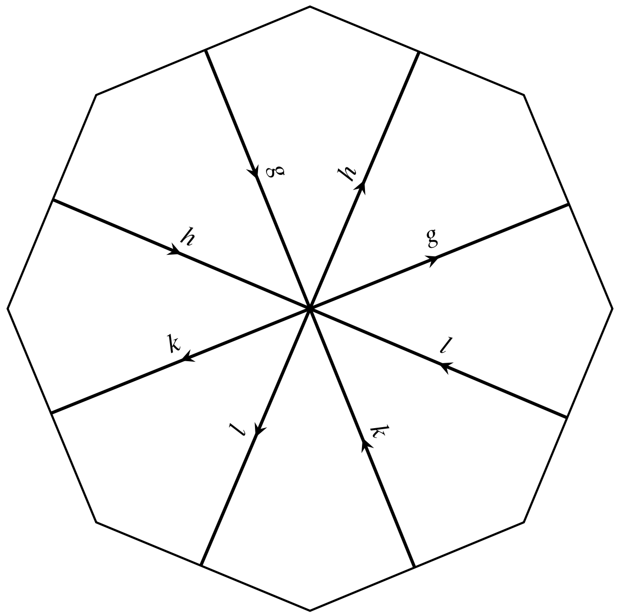


FIGURE 3. Element of the spanning set $S$ for a genus 2 surface

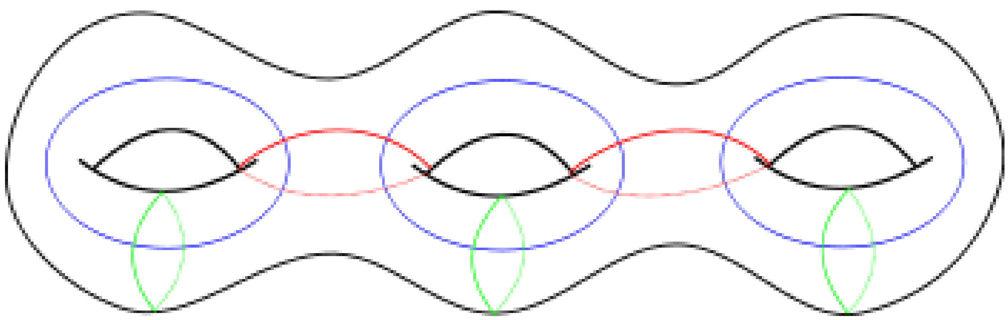

FIGURE 4. Lickorish generating set (Image source: `https://en.wikipedia.org/wiki/Dehn_twist`)

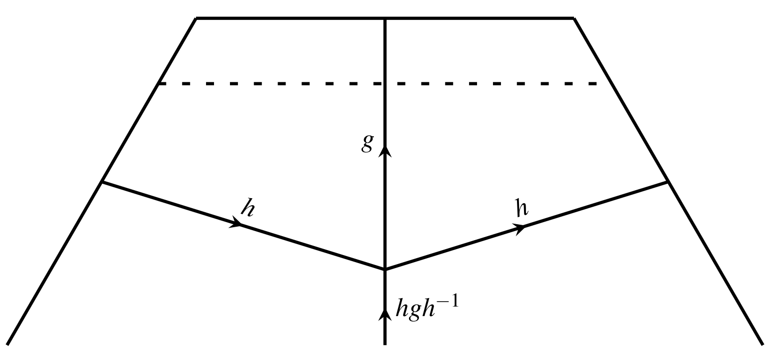


FIGURE 5. First type of Dehn twist

and $g \otimes g^{-1} \otimes 1$. To compose the two vertices, we use the spherical structure on the former vertex and reassociate until it is in the state $h^{-1} \otimes hg^{-1}h^{-1} \otimes h \otimes g \otimes 1$. In doing so, we pick up a factor of

$$\omega(h,g,h^{-1})\omega(h,gh^{-1},hg^{-1}h^{-1})\omega(g,h^{-1},hg^{-1}h^{-1})\omega(g,g^{-1}h^{-1},h)$$

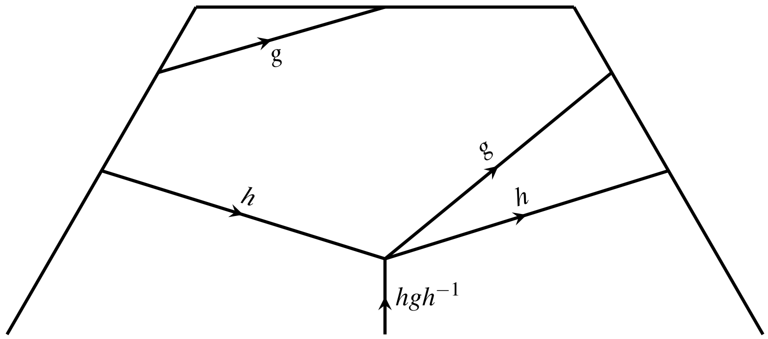


FIGURE 6. First type of Dehn twist

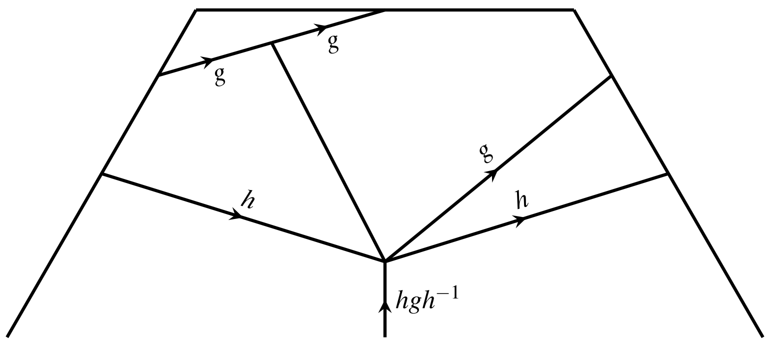


FIGURE 7. First type of Dehn twist

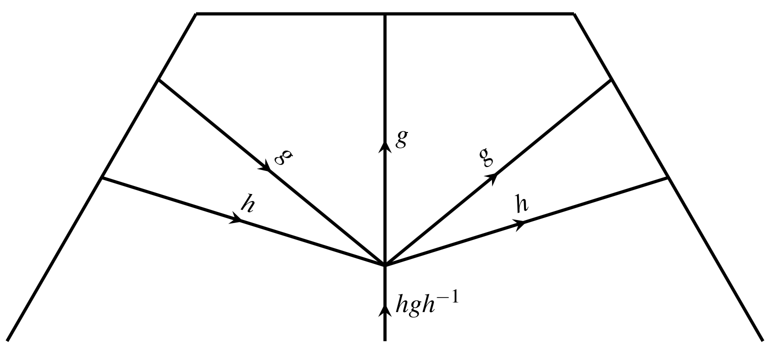


FIGURE 8. First type of Dehn twist

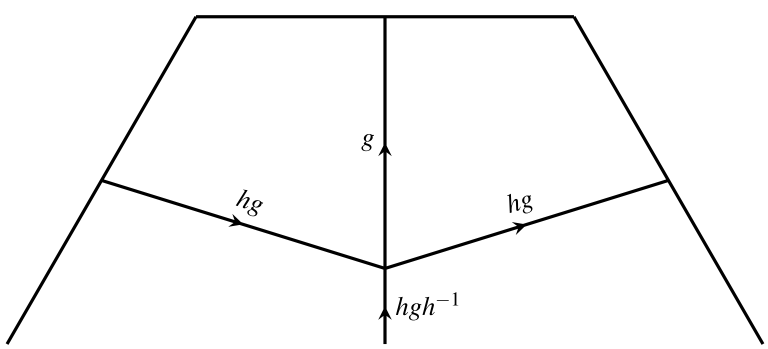


FIGURE 9. First type of Dehn twist

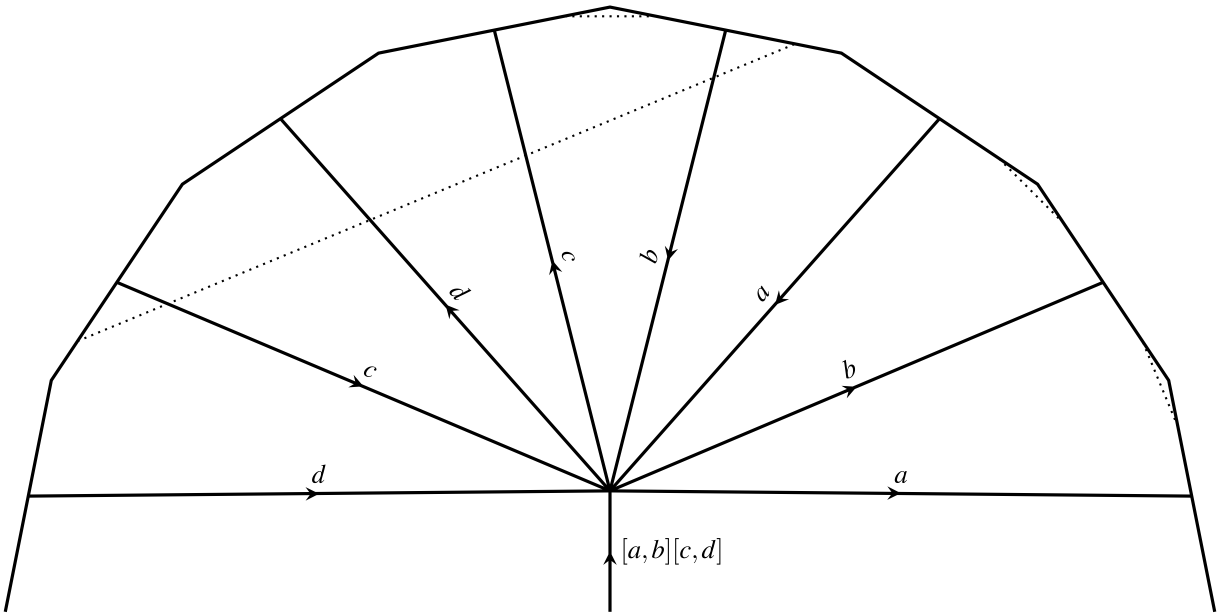


FIGURE 10. Second type of Dehn twist

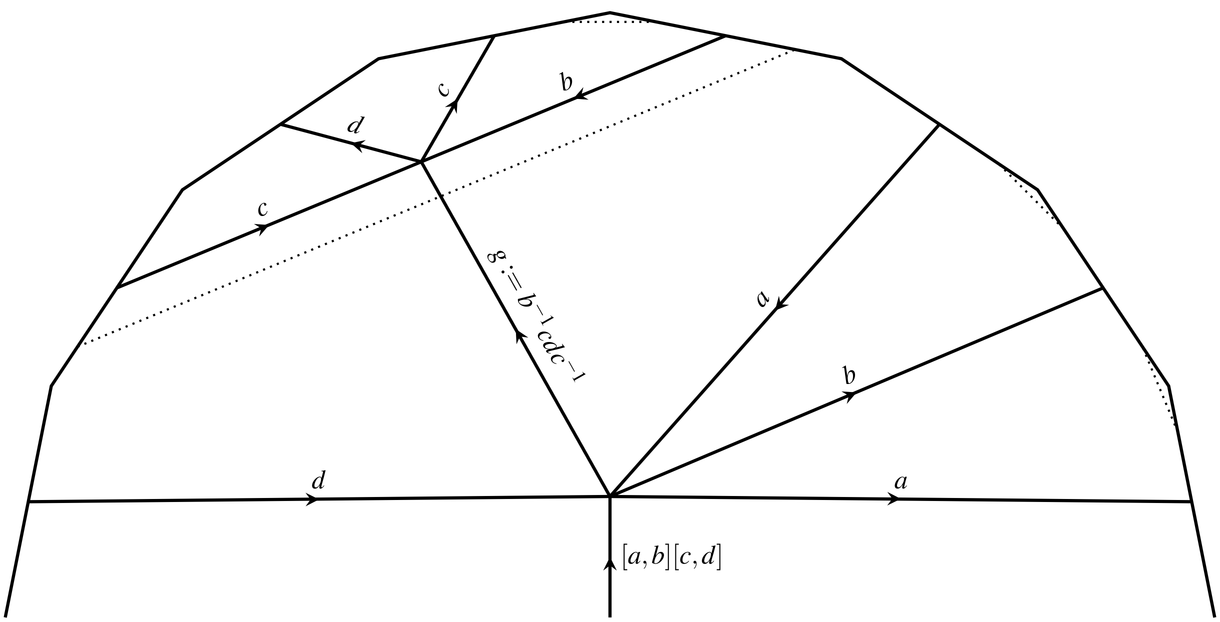


FIGURE 11. Second type of Dehn twist

After performing the composition, we are in the situation of Figure 8 with state $h^{-1} \otimes hg^{-1}h^{-1} \otimes h \otimes g \otimes (g \otimes g^{-1})$. To get rid of the last pair of parentheses, we get a factor of $\omega^{-1}(h^{-1}hg^{-1}h^{-1}hg, g, g^{-1}) = \omega^{-1}(1, g, g^{-1}) = 1$.

To tensor the parallel $g$ and $h$ edges together, we add $\mathrm{coev}_g$ and $\mathrm{coev}_h$ vertices in the middle of those edges and connect them with a 1. Composing along the 1, we get a vertex in state $g \otimes g^{-1} \otimes h^{-1} \otimes h$. To put this vertex in state $g^{-1}h^{-1} \otimes hg$ we pick up a factor of $\omega(g, g^{-1}, h^{-1})\omega(g, g^{-1}h^{-1}, h)\omega(g^{-1}h^{-1}, h, g)$. We also put the original vertex in the state $g^{-1}h^{-1} \otimes hg^{-1}h^{-1} \otimes hg \otimes g$ with a factor of

$$\omega^{-1}(g^{-1}, g^{-1}, g)\omega^{-1}(g^{-1}, g^{-1}h^{-1}, h)\omega^{-1}(g^{-1}, h^{-1}, hg^{-1}h^{-1})\omega(g^{-2}h^{-1}, h, g).$$

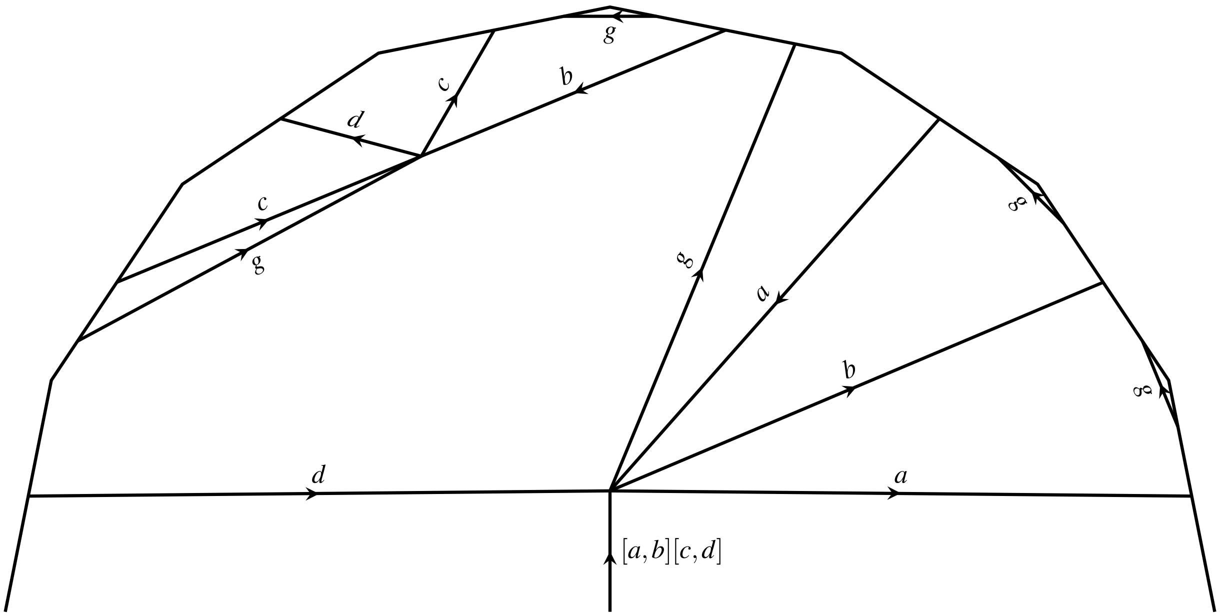


FIGURE 12. Second type of Dehn twist

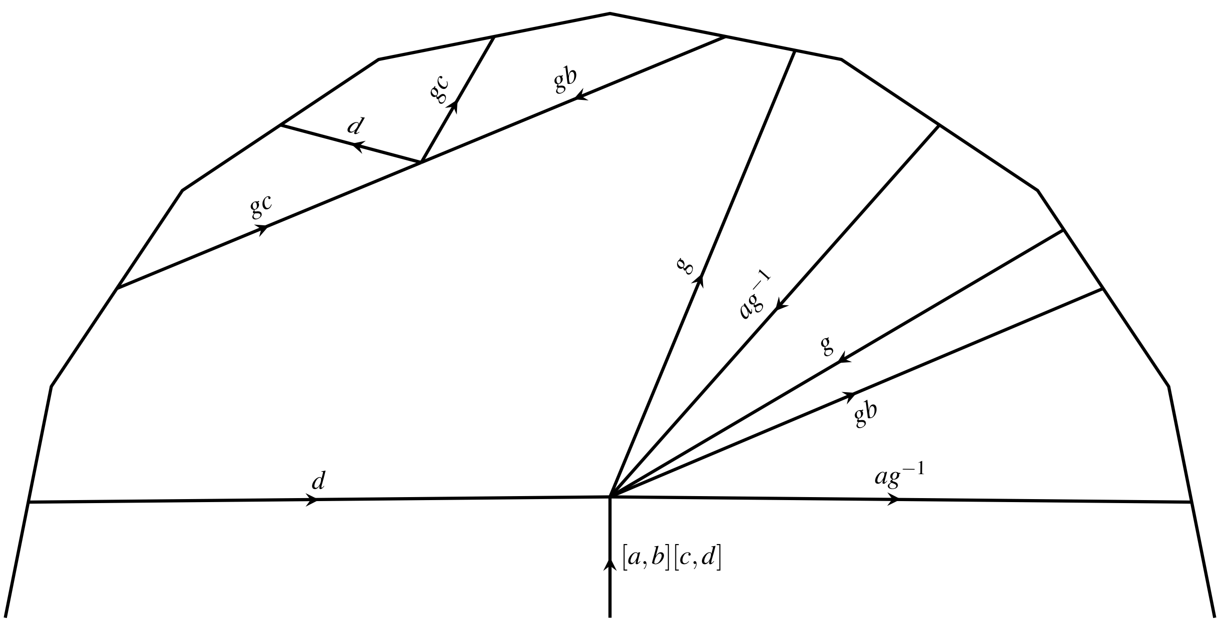


FIGURE 13. Second type of Dehn twist

To compose the two vertices, we rotate the original vertex to the state $g \otimes g^{-1}h^{-1} \otimes hg^{-1}h^{-1} \otimes hg$ which yields a factor of

$$\omega^{-1}(g, h^{-1}, hg)\omega^{-1}(g, g^{-1}h^{-1}, hg^{-1}h^{-1})\omega^{-1}(g, g^{-1}, h^{-1}).$$

We are then in a position to compose the two vertices, giving a factor $\omega(g^{-1}h^{-1}, hg, g^{-1}h^{-1})\,\mathrm{ev}_{g^{-1}h^{-1}} = 1$ and a vertex in state $g \otimes g^{-1}h^{-1} \otimes hg^{-1}h^{-1} \otimes hg$. Rotating the vertex into its initial configuration $hg \otimes g \otimes g^{-1}h^{-1} \otimes hg^{-1}h^{-1}$ gives a factor of

$$\omega^{-1}(hg, h^{-1}, hg^{-1}h^{-1})\omega^{-1}(hg, g, g^{-1}h^{-1}).$$

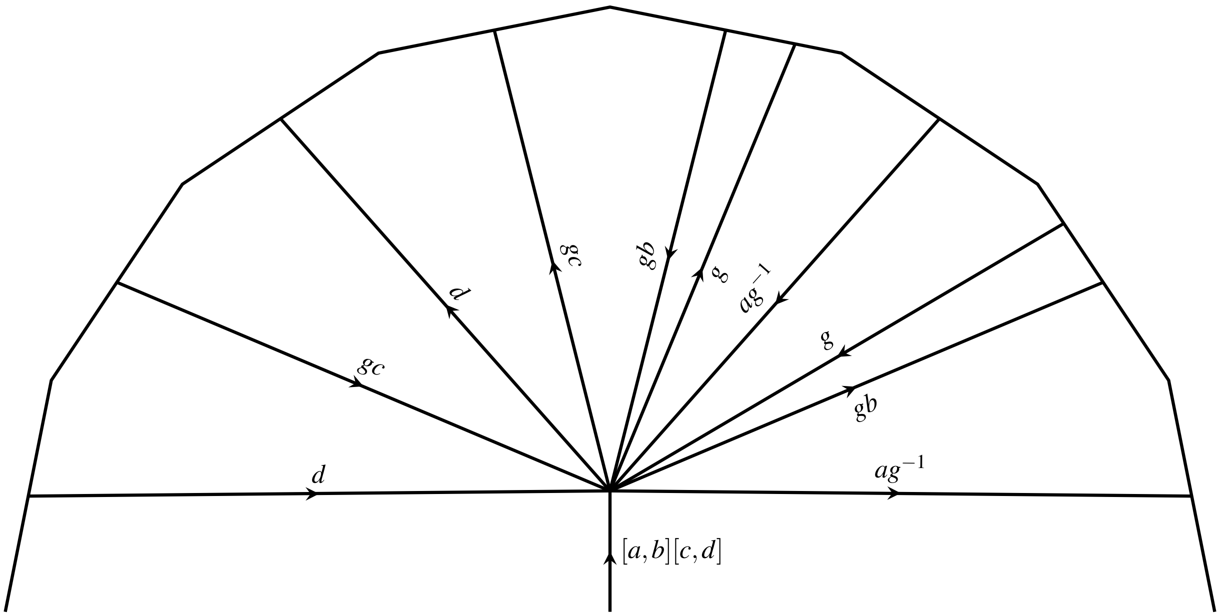


FIGURE 14. Second type of Dehn twist

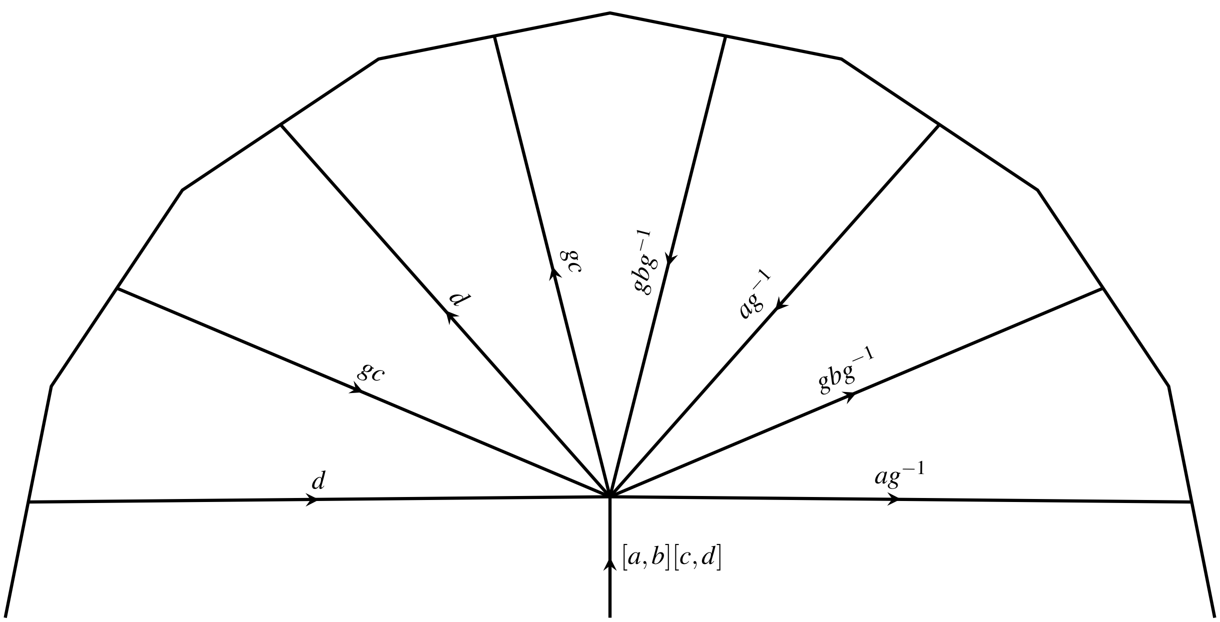


FIGURE 15. Second type of Dehn twist

Thus, we have an overall factor of

$$\frac{\omega(h,g,h^{-1})\omega(h,gh^{-1},hg^{-1}h^{-1})\omega(g,h^{-1},hg^{-1}h^{-1})\omega(g,g^{-1}h^{-1},h)}{\omega(g^{-1},g^{-1},g)\omega(g^{-1},g^{-1}h^{-1},h)\omega(g^{-1},h^{-1},hg^{-1}h^{-1})\omega(g,g^{-2}h^{-1},hg)}\cdot$$
$$\frac{\omega(g,g^{-1},h^{-1})\omega(g,g^{-1}h^{-1},h)\omega(g^{-1}h^{-1},h,g)\omega(g^{-2}h^{-1},h,g)}{\omega(g,g^{-1}h^{-1},hg^{-1}h^{-1})\omega(hg,h^{-1},hg^{-1}h^{-1})\omega(hg,g,g^{-1}h^{-1})}$$
$$=\frac{\omega(h,g,g^{-1}h^{-1})\omega(g,g^{-1}h^{-1},h)}{\omega(g^{-1},g^{-1},g)\omega(g^{-1},g^{-1}h^{-1},h)\omega(g^{-1},h^{-1},hg^{-1}h^{-1})\omega(g,g^{-2}h^{-1},hg)}.$$

$$
\begin{aligned}
&\frac{\omega(g,g^{-1},h^{-1})\omega(g,g^{-1}h^{-1},h)\omega(g^{-1}h^{-1},h,g)\omega(g^{-2}h^{-1},h,g)}{\omega(g,g^{-1}h^{-1},hg^{-1}h^{-1})\omega(hg,g,g^{-1}h^{-1})} \\
=&\frac{\omega(h,g,g^{-1}h^{-1})\omega(g,g^{-1}h^{-1},h)}{\omega(g^{-1},g^{-1},g)\omega(g^{-1},g^{-1}h^{-1},h)\omega(g,g^{-2}h^{-1},hg)}\cdot \\
&\frac{\omega^2(g,g^{-1},h^{-1})\omega(g,g^{-1}h^{-1},h)\omega(g^{-1}h^{-1},h,g)\omega(g^{-2}h^{-1},h,g)}{\omega(g,g^{-1},g^{-1}h^{-1})\omega(hg,g,g^{-1}h^{-1})} \\
=&\frac{\omega(h,g,g^{-1}h^{-1})\omega^2(g,g^{-1}h^{-1},h)\omega^2(g,g^{-1},h^{-1})\omega^2(g^{-1}h^{-1},h,g)}{\omega(g,g^{-2}h^{-1},hg)\omega(g,g^{-1},g^{-1}h^{-1})\omega(hg,g,g^{-1}h^{-1})\omega(g^{-1},g^{-1}h^{-1},hg)} \\
=&\frac{\omega(h,g,g^{-1}h^{-1})\omega^2(g,g^{-1}h^{-1},h)\omega^2(g,g^{-1},h^{-1})\omega^2(g^{-1}h^{-1},h,g)}{\omega(hg,g,g^{-1}h^{-1})} \\
=&\frac{\omega(h,g,g^{-1}h^{-1})\omega^2(g^{-1}h^{-1},h,g)}{\omega^2(g^{-1},h^{-1},h)\omega(hg,g,g^{-1}h^{-1})}
\end{aligned}
$$

## 6. Further Directions

Given the fact that the braid group representations associated to $D^\omega(G)$ are also finite [3], it should not be too much of a stretch to prove that mapping class group representations associated to $\mathrm{Vec}^\omega(G)$ for any surface, including boundary, are finite. The basic idea is that one labels the boundary components with elements of the double category, in this case $\mathrm{Mod}(D^\omega(G))$, and the $\mathrm{Vec}_G^\omega$-colored embedded graphs must be compatible with the boundary labels, as detailed in [8].

One could also consider more complicated spherical categories than $\mathrm{Vec}^\omega(G)$, such as the Tambara-Yamagami categories. The main additional complication here is the appearance of multiplicity, i.e. the tensor product of two simple objects can be a direct sum of multiple simple objects.

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