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HW 9

1 If $f \in L_1(0,\infty)$, define

$$g(s) = \int_0^\infty e^{-st} f(t)\,dt, \quad 0 < s < \infty.$$

Prove that $g(s)$ is differentiable on $(0,\infty)$ and that

$$g'(s) = -\int_0^\infty te^{-st} f(t)\,dx, \quad 0 < s < \infty.$$

Proof. Let $s \in (0,\infty)$ and $0 \le |h| \le s/2$. We have $|e^{-st}f(t)| \le |f(t)|$, so $e^{-st}f(t) \in L_1$. Hence

$$\frac{g(s+h)-g(s)}{h} = \int_0^\infty \frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\,dt.$$

By the Mean Value theorem, we have

$$\begin{aligned}\left|\frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\right| &\le \sup_{h \in (-s/2, s/2)} |-te^{-(s+h)t}|\,|f(t)| \\ &= te^{-(s/2)t}|f(t)| \\ &\le C_s|f(t)|\end{aligned}$$

Hence, by the DCT,

$$\lim_{h\to 0} \frac{g(s+h)-g(s)}{h} = \int_0^\infty \frac{d}{ds} e^{-st} f(t) = -\int_0^\infty te^{-st} f(t)\,dx.$$

□

2 Let (Ω, μ, Σ) be a finite measure space and (f_n) be a sequence of measurable functions on Ω. Suppose that for each $\omega \in \Omega$ there is an $M_\omega \in \mathbb{R}$ so that for all $k \in \mathbb{N}$, $|f_k(\omega)| \le M_\omega$. Let $\epsilon > 0$. Show that there is a measurable $A \subset \Omega$ and an $M \in \mathbb{R}$ so that $\mu(\Omega \setminus A) < \epsilon$ and $f_k(\omega) < M$ for all $k \in \mathbb{N}$ and all $\omega \in A$.

Proof. Let $\epsilon > 0$ and $E_j := \bigcap_n \{f_n < j\}$. Then (E_j) is increasing and $\bigcup_j E_j = \Omega$. Hence $\lim_j \mu(E_j) = \mu(\Omega)$. Since $\mu(Omega) < \infty$, we can pick M such that $\mu(\Omega \setminus E_M) = \mu(\Omega) - \mu(E_M) < \epsilon$. Moreover, if $\omega \in E_M$, then $f_k(\omega) < M$ for all k. □

3 57/page 77. Show that $\int_0^\infty e^{-sx}x^{-1}\sin x\,dx = \arctan(s^{-1})$ for $s > 0$ by integrating $e^{-sxy}\sin x$ with respect to x and y. (Hints: $\tan(\frac{\pi}{2} - \theta) = \cot\theta$ and Exercise 31d.)

Proof. We have

$$\int_0^\infty e^{-sx}x^{-1}\sin x\,dx = s\int_0^\infty\int_1^\infty e^{-sxy}\sin x\,dydx$$
$$= s\int_1^\infty\int_0^\infty e^{-sxy}\sin x\,dxdy$$

□

4 60/page 77. $\Gamma(x)\Gamma(y)/\Gamma(x+y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$. (Recall that Γ was defined in Section 2.3. Write $\Gamma(x)\Gamma(y)$ as a double integral and use the argument of the exponential as a new variable of integration.)
5 Given a bounded function $f : [a,b] \to \mathbb{R}$, define

$$H(x) = \lim_{\delta\to 0}\sup_{|x-y|\le\delta} f(y), \text{ and } h(x) = \lim_{\delta\to 0}\inf_{|x-y|\le\delta} f(y)$$

a) For $x \in [a,b]$, f continuous at $x \iff H(x) = h(x)$.

b) Assume now that (P_k) is an increasing sequence of partitions of $[a,b]$ for which the mesh converges to zero. Write $P_k = (t_0^{(k)}, t_1^{(k)}, \ldots, t_{n_k}^{(k)})$. Define for $x \in [a,b]$,

$$G(x) = \lim_{k\to\infty} G_{P_k}(x) \text{ and } g(x) = \lim_{k\to\infty} g_{P_k}(x),$$

where for a partition $P = (t_0, t_1, \ldots, t_n)$

$$G_P = \sum_{i=1}^n \chi_{(t_{i-1},t_i]}\sup_{t\in(t_{i-1},t_i]} f(t) \text{ and } g_P = \sum_{i=1}^n \chi_{(t_{i-1},t_i]}\inf_{t\in(t_{i-1},t_i]} f(t).$$

Prove that $H = G$ and $h = g$ m-a.e.

c) Show that f is Riemann integrable $\iff$ the set of discontinuities of f has Lebesgue measure zero.

6 Problem 30/page 60. Hint: AM-GM. Show that $\lim_{k\to\infty}\int_0^k x^n(1-k^{-1}x)^k\,dx = n!$.
7 Problem 1/88. Let ν be a signed measure on $(X,\mathcal{M})$. If (E_j) is an increasing sequence in $\mathcal{M}$, the $\nu(\bigcup_1^\infty E_j) = \lim_{j\to\infty}\nu(E_j)$. If (E_j) is a decreasing sequence in $\mathcal{M}$ and $\nu(E_1)$ is finite, then $\nu(\bigcap_1^\infty E_j) = \lim_{j\to\infty}\nu(E_j)$.
8 Problem 4/88. If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \ge \nu^+$ and $\mu \ge \nu^-$.
9 Problem 7/88. Suppose that ν is a signed measure on $(X,\mathcal{M})$ and $E \in \mathcal{M}$.

a. $\nu^+(E) = \sup\{\nu(F) : F \in \mathcal{M}, F \subset E\}$ and $\nu^-(E) = -\inf\{\nu(F) : F \in \mathcal{M}, F \subset E\}$.

b. $|\nu|(E) = \sup\{\sum_1^n |\nu(E_j)| : n \in \mathbb{N}, E_1, \ldots, E_n \text{ are disjoint, and } \bigcup_1^n E_j = E\}$.