

HW 7

5.3 Let

$$0 \rightarrow A \xrightarrow{i} B \xrightarrow{p} C \rightarrow 0$$

be a short exact sequence. Show that $iA \simeq A$ and $B/iA \simeq C$.

Proof. The kernel of i is trivial so the map is a bijection onto iA , hence an isomorphism $A \rightarrow iA$.

The kernel of $B \rightarrow C$ is iA and $B \rightarrow C$ is surjective, hence the induced map $B/iA \rightarrow C$ is an isomorphism. \square

5.5(ii) If $0 \rightarrow A_n \rightarrow A_{n-1} \rightarrow \dots \rightarrow A_1 \rightarrow A_0 \rightarrow 0$ is an exact sequence of f.g. abelian groups, then $\sum_{i=0}^n (-1)^i \text{span } A_i = 0$.

Proof. We use strong induction on n . Part (i) covers the cases $n \leq 2$. For $n > 3$, pick j to be greatest integer at most $n/2$. Then $0 \rightarrow A_n \rightarrow \dots \rightarrow A_j \rightarrow B \rightarrow 0$ and $0 \rightarrow B \rightarrow A_{j-1} \rightarrow \dots \rightarrow A_0 \rightarrow 0$ are exact sequences of shorter length than the original, where B is the image of the map $A_j \rightarrow A_{j-1}$. By the induction hypothesis,

$$0 = (-1)^j \text{span } B + \sum_{i=0}^{j-1} (-1)^i \text{span } A_i = \text{span } B + \sum_{i=j}^n (-1)^{i-j+1} \text{span } A_i.$$

By multiplying the last expression by $(-1)^{j-1}$ and adding it to the other, we get the desired equality. \square

5.11 If $U_* \subset T_* \subset S_*$ then

$$0 \rightarrow T_*/U_* \xrightarrow{i} S_*/U_* \xrightarrow{p} S_*/T_* \rightarrow 0,$$

where $i_n : t_n + U_n \mapsto t_n + U_n$ and $p_n(s_n + U_n) = s_n + T_n$.

Proof. By the third isomorphism theorem for groups, we get short exact sequences

$$T_n/U_n \xrightarrow{i_n} S_n/U_n \xrightarrow{p_n} S_n/T_n,$$

for every n .

Moreover $\bar{\delta}$ commutes with i_n and p_n , so we get the desired short exact sequence of complexes. \square