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HW 8

1 Let $f : [0,1] \to \mathbb{R}$ be integrable (with respect to Lebesgue measure) and nonnegative. Define

$$G_- = \{(x,y) : 0 \le x \le 1, 0 \le y \le f(x)\}.$$

Show that G_- is measurable in $\mathbb{R} \times \mathbb{R}$ and that

$$m(G_-) = \int_0^1 f(x)\,dx.$$

Proof. Case f is simple. In standard form $f = \sum_{i=1}^n a_i \chi_{A_i}$. Hence $G_- = \bigcup_{i=1}^n A_i \times [0, a_i)$ is measurable. Since the $A_i \times [0, a_i)$ are disjoint we have $m(G_i) = \sum_i a_i m(A_i) = \int f\,dx$.

General case. There exists a sequence of simple functions $\phi_n \uparrow f$. Let $H_n = \{(x,y) : 0 \le x \le 1, 0 \le y \le \phi_n(x)\}$. Then by part (a), each H_n is measurable and $m(H_n) = \int \phi_n\,dx$. Hence $G_- = \bigcup_n H_n$ is measurable, and $m(G_-) = \lim_{n\to\infty} m(H_n) = \lim_{n\to\infty} \int \phi_n\,dx = \int f\,dx$, where the last equality follows from the MCT. □

2 Let f be Lebesgue integrable on $(0,1)$. For $0 < x < 1$ define

$$g(x) = \int_x^1 t^{-1} f(t)\,dt.$$

Prove that g is Lebesgue integrable on $(0,1)$ and that

$$\int_0^1 g(x)\,dx = \int_0^1 f(x)\,dx.$$

[Hint: first prove the case where $f \ge 0$.]

Proof. By the linearity of the integral on L_1 and splitting f into positive and negative parts, WLOG $f \ge 0$. Note that the function $(t,x) \mapsto \chi_{[x,1]} t^{-1} f(t)$ is nonnegative and measurable. Hence by the Tonelli theorem we have $g \in \mathcal{L}^+$ and $\int_0^1 g(x)dx = \int_0^1 \int_x^1 t^{-1} f(t)dtdx = \int_0^1 \int_0^t t^{-1} f(t)dxdt = \int_0^1 f(t)dt$. Hence g is integrable. □

3 Let $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$. Let μ be the Lebesgue measure on $\mathcal{M}$ and ν be the counting measure on $\mathcal{N}$. Show that for $D = \{(x,x) : x \in [0,1]\}$

a) $D \in \mathcal{M} \otimes \mathcal{N}$.

b) The numbers

$$\mu \otimes \nu(D), \int \int \chi_D \,d\mu d\nu, \text{ and } \int \int \chi_D \,d\nu d\mu$$

are all unequal.

c) Show that there is more than one measure π on $\mathbb{R}^2$ for which

$$\pi(A \times B) = \mu(A)\nu(B), \text{ whenever } A, B \in \mathcal{B}_{[0,1]}.$$

Proof. For (a), let $D_n = \bigcup_{k=0}^{2^n-1}[k2^{-n}, (k+1)2^{-n}] \times [k2^{-n}, (k+1)2^{-n}]$. Then $(D_n) \subset \mathcal{M} \otimes \mathcal{N}$ is decreasing, and $D \subset D_n$ for all n. Moreover $d(D, D_n^c) \to 0$, so $\bigcap_n D = D$. Hence $D \in \mathcal{M} \otimes \mathcal{N}$.

For (b), let (E_n) be a countable cover of D with each $E_n = M_n \times N_n$ for some nonempty $M_n \in \mathcal{M}$ and $N_n \in \mathcal{N}$. I claim that $\sum_n \mu \otimes \nu(E_n) = \infty$.

Suppose not. Let $F = \{n \in \mathcal{N} : \mu(M_n) > 0\}$. If $\bigcup_{n \in F} N_n$ is uncountable, then $\sum_{n \in \mathbb{N}} \mu \otimes \nu(E_n) \leq \sum_{n \in F} \mu \otimes \nu(E_n) \leq \sum_{n \in F} \mu(M_n)\nu(N_n) = \infty$, a contradiction. Hence $\bigcup_{n \in F} N_n$ is countable, so $\mu\left(\bigcup_{n \in F} N_n\right) = 0$.

If $x \in [0,1]$ then either $(x,x) \in \bigcup_{n \in F} E_n$, which implies that $x \in \bigcup_{n \in F} N_n$, or $(x,x) \in \bigcup_{n \in F^c} E_n$, which implies that $x \in \bigcup_{n \in F^c} M_n$. Thus $[0,1]$ is covered by $(N_n)_{n \in F} \cup (M_n)_{n \in F^c}$, all of which have μ-measure 0, a contradiction.

Therefore, $\sum_n \mu \otimes \nu(E_n) = \infty$ for all countable covers (E_n) of D where each $E_n \in \mathcal{M} \times \mathcal{N}$. Thus, $\mu \otimes \nu(D) = \infty$.

On the other hand, $\int \int \chi_D \, d\mu d\nu = \int 0 \, d\nu = 0$, and $\int \int \chi_D \, d\nu d\mu = \int 1, d\mu = 1$.

For (c), let $\pi_1 = \mu_1 \otimes \nu_1 : \mathcal{B}_{\mathbb{R}^2} \to [0, \infty]$ where μ_1 is the Lebesgue measure on $\mathbb{R}$ and ν_1 is the counting measure on $\mathbb{R}$. Let $\pi_2 = \mu_2 \otimes \nu_2 : \mathcal{B}_{\mathbb{R}^2} \to [0, \infty]$, where $\mu_2(M) = \mu(M \cap [0,1])$ and $\nu_2(N) = \nu(N \cap [0,1])$. To see that μ_2 is countably additive, if $(E_n) \subset \mathcal{B}_{\mathbb{R}}$ are pairwise disjoint we have $\mu_2(\bigcup_n E_n) = \mu(\bigcup_n E \cap E_n) = \sum_n \mu_2(E_n)$. The same argument applies to ν_2, so μ_2 and ν_2 are both measures. It is clear that $\pi_1(A \times B) = \mu(A)\nu(B) = \pi_2(A \times B)$ for $A, B \in \mathcal{B}_{[0,1]}$, but $\pi_1([2,3] \times \{0\}) = 1 \neq 0 = \pi_1([2,4] \times \{0\})$, for example. $\square$

4 Find a measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ measurable so that

a) $\int_{\mathbb{R}^2} |f(x,y)| \, dxdy = \infty$

b) $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \, dxdy$, and $\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \, dydx$ both exist but are unequal.

Proof. Let

$$a_{ij} = \begin{cases} 1 & j = i+1 \\ -1 & j = i-1 \\ 0 & \text{else} \end{cases}$$

Let $f(x) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} \chi_{[i,i+1) \times [j,j+1)}$. Then

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \, dxdy = \int_{\mathbb{R}} \begin{cases} 1 & 0 \leq y < 1 \\ 0 & \text{else} \end{cases} dy = 1,$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y) \, dydx = \int_{\mathbb{R}} \begin{cases} -1 & 0 \leq x < 1 \\ 0 & \text{else} \end{cases} dx = -1.$$

$\square$

5 Problem 49/Page 69. Prove Theorem 2.39 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas.

a. If $E \in \mathcal{M} \times \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.

b. If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x \, d\nu = \int f^y \, d\mu = 0$ for a.e. x and y. (Here the completeness of μ and ν is needed.)

Proof. For (a), suppose not. WLOG $\nu(E_x) > 0$ for $x \in F$ where $\mu(F) > 0$. Fix any $x \in F$. Since $E \in \mathcal{M} \times \mathcal{N}$, we have $E = E_x \times N$ for some $N \in \mathcal{N}$ with $F \subset N$. Thus $\mu \times \nu(E) \geq \mu(E_x)\nu(F) > 0$.

For (b), let $B = \{(x, y) : f(x, y) \neq 0\}$. It suffices to show that $\nu(B_x) = 0$ a.e. for a.e. x. We have $\mu \otimes \nu(B) = 0$.

To prove Theorem 2.39, use Proposition 2.12 to get an $\mathcal{M} \otimes \mathcal{N}$-measurable function g such that $f = g$ λ-a.e. Then apply (b) to $f - g$. $\square$

6 If $f \in L_1(\mathbb{R}^2)$ or $f \geq 0$ and mble and $c \in \mathbb{R} \setminus \{0\}$, then

$$\int f(cx, cy) dx dy = c^{-2} \int f(x, y) dx dy.$$

$$\int f(x + cy, cy) dx dy = \int f(x, y) dx dy.$$

(since this is part of the proof Theorem 2.44, you should not use that result, but you can use, without proof the formula of the area of a parallelogram).

Proof. If $f \in L_1(\mathbb{R}^2)$, both identities are linear in f, so by splitting f into positive and negative parts, WLOG $f \geq 0$. Thus, it suffices to prove the case $f \geq 0$ and measurable.

Moreover, if $\phi \geq 0$ is a simple measurable function then $\phi(x, y) \leq f(x, y)$ implies that the simple function $\phi(cx, cy) \leq f(cx, cy)$. Conversely, $\phi(x, y) \leq f(cx, cy)$ implies that $\phi(c^{-1}x, c^{-1}y) \leq f(x, y)$. A similar argument works for the identity involving $f(x + cy, cy)$. Hence WLOG f is a nonnegative simple function. By linearity again, WLOG $f = \chi_E$ for a measurable set $E \subset \mathbb{R}^2$.

Note that if (cR_n) is a collection of parallelograms containing E, then (R_n) are parallelograms containing E, and vice versa. A similar statement holds for the second identity. Ssince $m(E)$ is determined by the measures of its countable covers by rectangles hence by parallelograms, WLOG E is a parallelogram. But both identities are obvious for parallelograms (just calculate the determinant of each transformation). $\square$

7 Prove that for any $f \in L_1(\mathbb{R}^d)$ and any $\epsilon > 0$ there is a simple function

$$\phi = \sum_{j=1}^{n} \alpha_j \chi_{R_j},$$

where the R_j's are products of intervals, and $\|\phi - f\|_1 \leq \epsilon$.

Proof. Since there exist simple functions $0 \leq |\phi_n| \leq |f|$ with $\phi_n \to f$, by the DCT WLOG f is simple. Then if $f = \sum_i a_i \chi_{A_i}$ in standard form, it suffices to approximate each A_i by finite disjoint union of products of intervals.

Let A be a measurable set of finite measure in $\mathbb{R}^d$. By the outer regularity of Lebesgue measure, WLOG A is open. Let $E_n = \{x \in A : B_{1/n}(x) \in A\}$. Then since A is open, $A = \bigcup_{n=1}^{\infty} E_n$. Since (E_n) is increasing, we have $m(A) = \lim_{n\to\infty} m(E_n)$.

Let $\epsilon > 0$. Pick E_n such that $m(A \backslash E_n) < \epsilon$. Let $\mathcal{Q}$ be the collection of all R^d-cubes with half-open sides of length $\frac{1}{2\sqrt{3}n}$ and vertices at $\frac{1}{2\sqrt{3}n}\mathbb{Z}$-lattice points. Then $\mathcal{Q}$ is a pairwise disjoint covering of R^d. Let $U = \bigcup\{Q \in \mathbb{Q} : Q \cap E_n \neq \emptyset\}$. Then $E_n \subset U \subset A$, where the latter inclusion follows from the fact that the diameter of each cube is $\frac{1}{2n} < \frac{1}{n} \leq d(E_n, A^c)$.

Since U has finite measure, U is a finite disjoint union of products of intervals, and $m(A\Delta U) = m(A \setminus U) < \epsilon$. $\square$