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HW 4

6.3 If $1 \leq p < r \leq \infty$, $L^p \cap L^r$ is a Banach space with norm $\|f\| = \|f\|_p + \|f\|_r$, and if $p < q < r$, the inclusion map $L^p \cap L^r \to L^q$ is continuous.

Proof. The restrictions of $\|\cdot\|_p$ and $\|\cdot\|_r$ to $L^p \cap L^r$ are norms, so their sum is a norm. To see that $L^p \cap L^r$ is complete, suppose $\sum_n f_n$ converges absolutely with respect to $\|\cdot\|$ for $f_n \in L^p \cap L^r$. Then the same series converges absolutely in L^p and L^q. Thus the pointwise limit of the series exists a.e. and lies in $L^p \cap L^q$.

To see that the inclusion map $L^p \cap L^r \to L^q$ is continuous, let $f \in L^p \cap L^r$ and pick λ as in Prop. 6.10. Then $\|f\|_q \leq \|f\|_p^\lambda \|f\|_q^{1-\lambda} \leq \|f\|^\lambda \|f\|^{1-\lambda} = \|f\|$. □

4 If $1 \leq p < r \leq \infty$, $L^p + L^r$ is a Banach space with norm $\|f\| = \inf\{\|g\|_p + \|h\|_r : f = g + h\}$, and if $p < q < r$, the inclusion map $L^q \to L^p + L^r$ is continuous.

Proof. To see that $\|\cdot\|$ is positive definite, we must show that $\|f\| = 0$ implies $f = 0$ a.e. Suppose that $\mu(\{f > 0\}) > 0$. Then there exist a measurable set E and $\delta > 0$ such that $\mu(E) > 0$ and $f_{|E} \geq \delta$. Suppose $f = g + h$ for $g \in L^p$ and $h \in L^q$. Then

$$\begin{aligned}
\|g\|_p + \|h\|_r &\geq \|g_{|E}\|_p + \|h_{|E}\|_q \\
&\geq \|g_{|E}\|_p + \mu(E)^{1/p-1/q} \|h_{|E}\|_p \\
&\geq \min(\mu(E)^{1/p-1/q}, 1)(\|g_{|E}\|_p + \|h_{|E}\|_p) \\
&\geq \min(\mu(E)^{1/p-1/q}, 1)\|f_{|E}\|_p \\
&\geq \min(\mu(E)^{1/p-1/q}, 1)\delta^{1/p}
\end{aligned}$$

This implies that $\|f\| \geq \min(\mu(E)^{1/p-1/q}, 1)\delta^{1/p} > 0$.

The function $\|\cdot\|$ satisfies the homogeneity condition of a norm. For the triangle inequality, suppose $f_1, f_2 \in L^p + L^r$. Suppose $f_1 = g_1 + h_1$ and $f_2 = g_1 + h_2$ for some $g_1, g_2 \in L^p$ and $h_1, h_2 \in L^r$. Then $\|g_1\|_p + \|h_1\|_q + \|g_2\|_p + \|h_2\|_q \geq \|g_1 + g_2\|_p + \|h_1 + h_2\|_q \geq \|f_1 + f_2\|$. Thus, $\|f_1\| + \|f_2\| \geq \|f_1 + f_2\|$.

To see that $L^p + L^r$ is complete, suppose $f_n \in L^p + L^r$ and $\sum_n f_n$ converges absolutely. Pick $g_n \in L^p$ and $h_n \in L^r$ such that $\|g_n\|_p + \|h_n\|_r \leq \|f\| + 2^{-n}$. Then $\sum_n g_n$ and $\sum_n h_n$ converge absolutely in L^p and L^r respectively. Let $g = \sum_n g_n$ and $h = \sum_n h_n$. Then $\sum_n f_n = g + h$ pointwise a.e. Moreover,

$$\begin{aligned}
\|\sum_{n \geq N} f_n\| &\leq \sum_{n \geq N} \|f_n\| \\
&\leq \sum_{n \geq N} \|g_n\|_p + \|h_n\|_r \\
&\underset{N \to \infty}{\to} 0,
\end{aligned}$$

so $\sum_n f_n = g + h$ in $L^p + L^r$. Hence $L^p + L^r$ is complete.

Lemma: if $|f| \leq 1$ and $1 < q < r < \infty$, then $\|f\|_r \leq \|f\|_q$.

To see that the inclusion $L^q \to L^p + L^r$ is continuous, let $f \in L^q$. Let $E = \{x : f(x) > 1\}$. Then $\mu(E) < \infty$, $f\chi_E \in L^p$ and $f\chi_{E^c} \in L^r$. Hence

$$\|f\| \leq \|f\chi_E\|_p + \|f\chi_{E^c}\|_r \quad \leq \mu(E)^{1/p-1/q}\|f\chi_E\|_q + \left(\int_{E^c} |f|^r d\mu\right)^{1/r} \leq \mu(E)^{1/p-1/q}\|f\|_q +$$

$\square$

5 Suppose $0 < p < q < \infty$. Then $L^p \not\subset L^q$ iff X contains sets of arbitrarily small positive measure, and $L^q \not\subset L^p$ iff X contains sets of arbitrarily large finite measure. (Hint in book).

10 Suppose $1 \leq p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $\|f_n - f\|_p \to 0$ iff $\|f_n\|_p \to \|f\|_p$. (Use Exercise 20 in 2.3)

12 If $p \neq 2$, the L^p norm does not aries on L^p, except in trivial cases when $\dim(L^p) \leq 1$.

13 $L^p(\mathbb{R}^n, m)$ is separable for $1 \leq p < \infty$. However, $L^\infty(\mathbb{R}^n, m)$ is not separable.