Quiz Part B - Question 4

Upload your typed/written solution to Quiz Part B - Question Four.

Question Four

Let $\mathbf{X} = (\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_n})$ be i.i.d. random variables, each with a density

$$f(x,\theta) = \begin{cases} \frac{1}{\sqrt{2\pi x\theta}} \exp\left(-\frac{1}{2}\left[\frac{\log(x)}{\theta}\right]^2\right), & x > 0 \\ 0 \quad \text{elsewhere} \end{cases}$$

where $\theta > 0$ is a parameter.

a) Prove that the family $L(\mathbf{X}, \theta)$ has a monotone likelihood ratio in $T = \sum_{i=1}^{n} (\log X_i)^2$.

b) Argue that there is a uniformly most powerful (UMP) $\alpha-$size test of the hypothesis

$$H_0 : \theta \leq \theta_0 \quad \text{against} \quad H_1 : \theta > \theta_0$$

and write down its structure.

c) Using the density transformation formula (or otherwise) show that

$$Y_i = \log X_i$$

has a $N(0, \theta^2)$ distribution.

Note: *Density transformation formula:* For $Y = W(X)$:

$$f_Y(y) = f_X(W^{-1}(y))|\frac{dW^{-1}(y)}{dy}| = f_X(x)|\frac{dx}{dy}|.$$

d) Using c) (or otherwise), find the threshold constant in the test and hence determine completely the uniformly most powerful $\alpha-$ size test φ^* of

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0.$$

e) Calculate the power function $E_\theta \varphi^*$ and sketch a graph of the power function as precisely as possible.