Blog Post

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How I see things right now:

1. **Intro/Motivation (1 and 2 below)**
2. **How it will be done (3, 4)**
3. **Actually doing it using simple example (5, 6, 7)**
4. **Concluding on what it gives and what is next (8, 9)**

1 Revised Outline

1. Intro/Motivation
   (a) relationship between categories and TT. How does the paper fit in this context?
   (b) examples from linear type theories and *-autonomous categories, HTT and infinity 1 toposes, MLTT (Martin-Lof)
   (c) Which of those are practical?
   (d) What is this blog post about?
   (e) two requirements we focus on 1) Logically well-behaved (initiality theorem) and 2) leverage intuition (sets with elements)
   (f) illustrate this using dual pair example. (and introduce this example here)
2. Transition between intro and the rest of the blog (i.e. dual example): What maps to what between PTT and SMC.
3. Dual example
   (a) A lot goes here!
   (b) Free dual pairs, coequaliser
   (c) instances of generator rules
4. Translation of the type theory (ie how to use) with examples
5. Conclusion about getting more info/reading the paper/how this paper is structured: Diagram figure of how things are connected

2 (Old) Outline

1. A paragraph about what a good, i.e. practical type theory is, in particular, focus on having a term calculus, and typing and other judgements such that careful elementary reasoning is equivalent to categorical reasoning. Say something about this being the real magic of "synthetic mathematics"

   - Explain what the goal of this post is. i.e. The paper goes through a lot of examples about how to use the newly constructed PTT, we don't plan on regoing through all of that or on repeating the proofs provided in the paper. Rather we will approach the construction of the PTT from a different perspective, going through the basics and motivation and context, etc.
   - talk about: initiality, what it really means,
   - talk about: terms and types
   - leave out the stuff which compares his PTT's to linear logica and linear type theory, i.e. same tensor on both sides etc. anyone who knows/wants to know about that will be better served by reading the paper.

2. Explain that mike Shulman's type theory does exactly this for SMC's. (What is "this" here?)

3. Behold the coequalizer: recall that coequalizers in categorical universal algebra serve the same role as congruences in classical universal algebra, with examples of congruences being eq relations for sets, natural eq. relations on hom sets for small categories, etc.

4. Say we'll use this universal form to extract a type theory whose:

   (a) term judgemnts come from the second thing in the co-eq,

   (b) typing judgements come from the second this in the co-eq

   (c) equality judgemnts come from the first things in the co-eq

   - Cite relevant theorems from the paper
   - Also put in the "Grand Outline" figure and explain it.

5. Build example of free dual pair as a quotient of free props. Don't make a big deal about props, just say how they are smcs whose objects are the free the monoid generated by a signature (define those also).

6. Show the generator rule and identity rule give us the composites relevant to the triangle identities

   - Also add some details about how they "take care of everything that needed to be taken care of"

7. show how we get axioms for equality from the the first thing in the co-eq for free props.

- explain how we build a PTT for free props then "lift it" to obtain the desired PTT.

8. Compare the complexity of the proofs of cyclicity of trace.

    - Explain that actually, it is not necessarily that useful for this simple example since string diagrams are often best for structures that "don't change the topology" but for things like coalgebraic and Hopf-type structures it can be better to reason using type theories.

9. Conclude with some comments about bigger context, usefulness of this PTT, and possible future work.

3 Draft

3.1 Introducing the free dual pair example

To illustrate the ideas of this paper, we will look at how they take form in the free prop generated by dual pairs (example 7.1 of the paper).

First, let's introduce the notions of duality and free dual pair. A duality in a SMC $\mathbf{C}$ consists firstly of a dual pair, that is a pair of objects D and D^*. The object D can be thought of as "things of functions" and D^* as being "things of functionals". A duality also contains two maps, the evaluation and coevaluation, which are defined as follows.

$$\mathsf{coev} : \mathbf{1} \longrightarrow D \otimes D^\star \qquad\qquad \mathsf{ev} : D^\star \otimes D \longrightarrow \mathbf{1}$$

and such that the following diagrams commute:

$$\begin{tikzcd}
\mathbf{1} \otimes D \arrow[r, "\mathsf{coev} \otimes D"] \arrow[d, "\mathsf{id}"] & (D \otimes D^\star) \otimes D \arrow[d, "\alpha"] \\
D \otimes \mathbf{1} & D \otimes (D^\star \otimes D) \arrow[l, "D \otimes \mathsf{ev}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
D^\star \otimes \mathbf{1} \arrow[r, "D^\star \otimes \mathsf{coev}"] \arrow[d, "\mathsf{id}"] & D^\star \otimes (D \otimes D^\star) \arrow[d, "\alpha^{-1}"] \\
\mathbf{1} \otimes D^\star & (D^\star \otimes D) \otimes A^\star \arrow[l, "\mathsf{ev} \otimes D^\star"']
\end{tikzcd}$$

For any duality $(D, D^*, \mathsf{ev}, \mathsf{coev})$, we consider the minimal SMC that contains the duality, which we will call the free dual pair $\mathcal{D}$ associated with that duality.

3.2 Idea of the content of the paper

What is the general idea for obtaining this practical type theory?

1. *Start with some input data (which we will call signature)*
   - Our input data for constructing type theories will consist of a set of objects together with a set of arrows whose domain and codomain consist of finite lists of objets.
2. *We build a type theory for the free prop generated by a signature*
   - This is done by defining rules for terms and rules for typing judgements.
   - Much care is taken while defining these rules to ensure, among other things, that the composition and the exchange rules are admissible, therefore ensuring that any judgement has a unique derivation. This is a key requirement to prove the initiality theorem that we discussed earlier.
3. *The term model of this type theory can be proven to be the prop freely generated by the input signature. Hence we now have the initiality theorem*

- Taking the contexts of this type theory as objects and the derivable term judgement (modulo an equality rule) as morphisms forms a strict symmetric monoidal category, which we denote $\mathfrak{F}\mathcal{G}$. It is also easy to show that $\mathfrak{F}\mathcal{G}$ is in fact also a prop.
- We can then show that $\mathfrak{F}\mathcal{G}$ is actually the free prop generate by $\mathcal{G}$.

Great. Now we have a type theory for props freely generated by a signature. But, what about all the other props? Or in other words, how do we deal with SMC that have additional equality relations? Well, since we know that the category of prop is monadic over the category of signatures, we have that every prop $\mathcal{P}$ admit a presentation in terms of signatures (i.e. a coequaliser diagram $\mathfrak{F}\mathcal{R} \rightrightarrows \mathfrak{F}\mathcal{G} \to \mathcal{P}$), where $\mathcal{R}$ is a signature that provides equality axioms.

4. *We then essentially redo the two previous steps, but this time we also quotient by these additional equality axioms provided by the signature $\mathcal{R}$ in the presentation of the prop*

That's it! Following these steps gives us a type theory for the prop presented by $(\mathcal{G}, \mathcal{R})$, which, as proven in the paper, allows us to reason about structures in any props, and hence also in any symmetric monoidal category.

3.3 Illustrating the content of the paper using the free dual pair example

Start with the duality $(D, D^*, \mathsf{ev}, \mathsf{coev})$ (and its associated free dual pair $\mathcal{D}$) given above.

1. *Start with some input data*

   - Starting with the duality $(D, D^*, \mathsf{ev}, \mathsf{coev})$, we define some input data (signature) $\mathcal{G}$ as follows:
     - Objects: $\{D, D^*\}$
     - Arrows: $\{\mathsf{ev} : (D^*, D) \to (), \quad \mathsf{coev} : () \to (D, D^*)\}$

2. *Build a type theory from the input data*

   - The arrows in the input data become the rules in the type theory $\mathbf{T}_{\mathcal{G}}$:

$$\begin{array}{ccc} \mathsf{ev} : (D^\star, D) \longrightarrow () & \qquad & \mathsf{coev} : () \longrightarrow (D^\star, D) \\ \downarrow & & \downarrow \\ \dfrac{\Gamma \vdash x : (D^\star, D)}{\Gamma \vdash \mathsf{ev}(x) : ()} & & \dfrac{\Gamma \vdash x : ()}{\Gamma \vdash \mathsf{coev}(x) : (D^\star, D)} \end{array}$$

   ***HOW CAN WE EXPRESS THE EQUALITY RULE IN THIS EXAMPLE?

3. *Prove initiality theorem*

   - By the theorems in the paper the context and derivable typing judgements in the type theory $\mathbf{T}_{\mathcal{G}}$ form a prop which is in fact the free prop generated by the input data $\mathcal{G}$.

4. *Account for equality relations by using a presentation as a coequaliser*

- The free dual pair $\mathcal{D}$ admits a presentation as the following colimit:

$$\varinjlim \{\mathfrak{F}\mathcal{R} \rightrightarrows \mathfrak{F}\mathcal{G}\} \xrightarrow{\sim} \mathcal{D}$$

where R is the signature of relations that imposes the following two axioms (these correspond to the commutative diagrams introduced above):

$$(\mathsf{ev} \otimes \mathsf{id}_D) \circ (\mathsf{id}_D \otimes \mathsf{coev}) = \mathsf{id}_D \qquad (\mathsf{ev} \otimes \mathsf{id}_D) \circ (\mathsf{id}_D \otimes \mathsf{coev}) = \mathsf{id}_D$$

First translation:

$$x : D \vdash (\eta_{(1)} \mid \varepsilon(\eta_{(2)}, x)) = x : D \qquad y : D^* \vdash (\eta_{(2)} \mid \varepsilon(y, \eta_{(1)})) = y : D^*$$

Second translation:

$$x : D \vdash (u \mid \lambda^D u \triangleleft x) = x : D \qquad y : D^* \vdash (\lambda^D u \mid y \triangleleft x) = D^*$$

***NEEDS FIXING!!!