The derivation of variational form for dynamic nonlinear elasticity is derived. The method described is detailed in chapters 28 and 29 of the FEniCS book.

The structure body in question is idealized to be described by Ω points in the body. The problem is considered in a Lagrangian description with reference positions $X \in \Omega$. We introduce φ, a continuously differentiable one to one map, to describe the displacement field

$$u(X,t) = \varphi(X,t) - X \tag{1}$$

where the displacement u for a given point in time is measured relative to the known reference position X. $\partial\Omega$ is the boundary of Ω.

The solution of our displacement field depends on three categories of boundary conditions:

- Body forces that act on the entire volume of the body: $B(X,t)$
- Traction forces, force per unit area on the Neumann boundary: $\partial\Omega_N$
- Displacement, prescribed displacement boundary conditions on the Dirichlet boundary: $\partial\Omega_D$

Figure 1 depicts a generic structure body and associated boundary conditions.

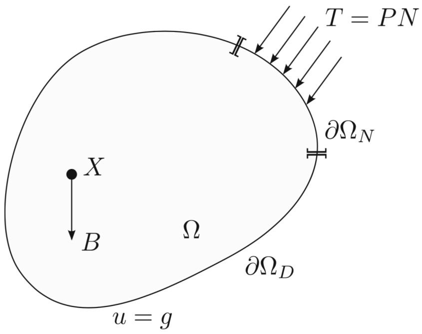

Figure 1: Structure body subject to body force, traction on the Neumann boundary and prescribed displacement on the Dirichlet boundary. [1]

Demonstrated in Figure 1, we assume that the Neumann and Dirichlet boundaries do not coincide.

Beginning with the Lagrangian form of the balance of linear momentum:

$$\rho \frac{\partial^2 u}{\partial t^2} = Div(P) + B \text{ in } \Omega \tag{2}$$

where ρ is the density of the structure, P the first Piola-Kirchoff stress tensor and B the body force per unit volume.

We define initial conditions $u(X,0) = u_0(X)$ and $\frac{\partial u}{\partial t}(X,0) = v_0(X)$ and boundary conditions $u(X,t) = g(X,t)$ on $\partial\Omega_D$ and $P(X,t)N(X) = T(X,t)$ on $\partial\Omega_N$. $N(X)$ is the outward normal to the boundary at point X.

We take the dot product of 2 with a test function $v \in \hat{V}$ and integrate over the reference domain and time to arrive at

$$\int_0^T \int_\Omega \rho \frac{\partial^2 u}{\partial t^2} \cdot v \; dx \; dt = \int_0^T \int_\Omega Div(P) \cdot v \; dx \; dt + \int_0^T \int_\Omega B \cdot v \; dx \; dt \tag{3}$$

Next, apply the divergence theorem to arrive at the weak form the balance of linear momentum. We introduce the traction vector $T = PN$ on the Neumann boundary $\partial\Omega_N$ and use $v = 0$ on $\partial\Omega_D$, by definition of the Dirichlet boundary. This yields a weak form. Find $u \in V$ such that $\forall \; v \in \hat{V}$:

$$\int_0^T \int_\Omega \rho \frac{\partial^2 u}{\partial t^2} \cdot v \; dx \; dt + \int_0^T \int_\Omega P : Grad(v) \; dx \; dt = \int_0^T \int_\Omega B \cdot v \; dx \; dt + \int_0^T \int_{\partial\Omega_N} T \cdot v \; ds \; dt \tag{4}$$

The dynamic approach taken here is CG_1 meaning the finite element space V is continuous and piecewise linear in time. We rewrite 4, which is 2nd order in time, as a system of first order equations with the variable $w = \frac{\partial u}{\partial t}$. This yields a new weak form. Find $(u, w) \in V$, such that $\forall \; (v, r) \in \hat{V}$:

$$\int_0^T \int_\Omega \rho \frac{\partial w}{\partial t} \cdot v \; dx \; dt + \int_0^T \int_\Omega P : Grad(v) \; dx \; dt = \int_0^T \int_\Omega B \cdot v \; dx \; dt + \int_0^T \int_{\partial\Omega_N} T \cdot v \; ds \; dt \quad \text{and} \tag{5}$$

$$\int_0^T \int_\Omega \frac{\partial u}{\partial t} \cdot r \; dx \; dt = \int_0^T \int_\Omega w \cdot r \; dx \; dt. \tag{6}$$

with the initial conditions established earlier as $u(X, 0) = u_0(X)$ and $w(X, 0)\frac{\partial u}{\partial t}(X, 0) = v_0(X)$ in Ω and boundary conditions $u(X, t) = g(X, t)$ on $\partial\Omega_D$. While the displacement finite element approximation space is CG_1, the velocity space $\hat{V}$ is DG_0, discontinuous and piecewise constant in time. This produces

$$\int_\Omega \rho \frac{(w_{n+1} - w_n)}{\Delta t} \cdot v \; dx + \int_\Omega P(u_{mid}) : Grad(v) \; dx = \int_\Omega B \cdot v \; dx + \int_{\partial\Omega_N} T \cdot v \; ds \quad \text{and} \tag{7}$$

$$\int_\Omega \frac{(u_{n+1} - u_n)}{\Delta t} \cdot r \; dx = \int_\Omega w_{mid} \cdot r \; dx \tag{8}$$

where $(\cdot)_n$ and $(\cdot)_{n+1}$ denote a quantity at the previous and current time step, and $(\cdot)_{mid} = \frac{((\cdot)_n + (\cdot)_{n+1})}{2}$. In FEniCS this equation is solved in a mixed function space with Netwon's method. Each term is multiplied by Δt and $Piola - Kirchoff$ stress is calculated from $(u)_{mid}$.

To arrive at this variational form the first Piola-Kirchhoff stress tensor must be found. The necessary kinematic terms to determine the first Piola-Kirchhoff stress are presented in Table 1

Table 1: Kinematic terms

Deformation Gradient	$F = I + Grad(u)$
Right Cauchy-Green tensor	$C = F^T F$
Green-Lagrange strain tensor	$E = \frac{1}{2}(C - I)$

For the benchmark FSI problem, the material model employed is St Venant Kirchhoff with a stored strain energy of

$$\Psi = \frac{\lambda}{2} tr(E^2) + \mu tr(E^2) \tag{9}$$

where μ and λ are the structure first and second lame constants respectively. Given Ψ is dependent on the Green-Lagrange tensor the second Piola-Kirchhoff stress can be calculated as.

$$S = \frac{\partial \Psi(E)}{\partial E} \tag{10}$$

From this the first Piola-Kirchoff stress is found as

$$P = FS \tag{11}$$

Excluding the traction, this implementation has been validated against a benchmark in FEniCS. I'm therefore confident of the derivation and implementation of the first Piola-Kirchohhoff stress in the dynamic code.

Addressing the traction.

The Cauchy stress tensor of the fluid problem is calculated as

$$\sigma = -pI + \mu_f(grad(u_f) + grad(u_f)^T) \tag{12}$$

where p is the fluid pressure, μ_f the dynamic viscosity and u_f the fluid velocity. This stress is projected onto a tensor function space in the structure domain.

The traction is then calculated.

The fluid traction is transferred to the structure via the Piola map

$$(j\sigma_F \cdot f^{-T}) \cdot n_f = -\sigma_s \cdot n_s$$

where $j = det(f)$ is the determinant of the Jacobi matrix $f = grad(\varphi)$ and n_f and n_s are the respective fluid and structure mesh facet normals. φ is defined earlier relating to the structure displacement as $u(X,t) = \varphi(X,t) - X$. My current FEniCS implementation reads $f = grad(\varphi) = grad(u) + I$. I'm not confident on the reasoning implicit here that $grad(X) = I$.

My understanding is that the implementation I have in FEniCS follows the book closely. At the moment when I run

References

[1] Anders Logg, Kent-Andre Mardal, and Garth Wells. *Automated solution of differential equations by the finite element method: The FEniCS book*, volume 84. Springer Science & Business Media, 2012.