Supplementary Materials: Input uncertainty propagation through trained neural networks

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1. Introduction

This document is the supplementary materials file of the main paper "Input uncertainty propagation through trained neural networks". It contains the derivation proofs for analytical formula of first two moments of a multivariate Gaussian distribution through ReLU, Leaky Relu and ELU activation functions, provided in A.

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A. Moment Propagation

A.1. ReLU activation function

Proposition A.1. *Let X be a random variable with $X \sim \mathcal{N}_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{\mu} \in \mathbb{R}^d$, $\boldsymbol{\Sigma} \in \mathbb{S}_d^+(\mathbb{R})$ with rank$(\boldsymbol{\Sigma}) = r$, $r \in \mathbb{N}^*$ and f be the rectified linear function, then for the random variable $\boldsymbol{Y} = (Y_1, ..., Y_d) = f(\boldsymbol{X}) = (f(X_1), ..., f(X_d))$, $\forall k, k' \in \{1...d\}$:*

$$\begin{aligned}
\mathbb{E}[Y_k] &= c_k\phi_k + \mu_k\Phi_k && (1)\\
\mathbb{E}[Y_k^2] &= (\mu_k^2 + c_k^2)\Phi_k + \mu_k c_k \phi_k && (2)\\
\mathbb{E}[Y_k Y_{k'}] &= c_k\alpha\phi_{k'}\phi_{NS} + (\mu_k\mu_{k'} + c_k\beta)\left(\Phi_{k'} - \Phi_{kk'}\right) + \mu_k c_{k'}\phi_{k'}\Phi_S + \mu_{k'}c_k\phi_k\Phi_{NS} && (3)
\end{aligned}$$

where: $\mu_k = \mathbb{E}[X_k]$, the mean of the k^{th} component of the random variable $\boldsymbol{X}$, $Q_r = (q_k)_{k\in\{1...d\}} \in \mathcal{M}_{r,d}(\mathbb{R})$ the r first line of Q such that $\Sigma = Q^T\Lambda Q = Q_r^T\Lambda_r Q_r$, and $\Lambda_r \in \mathcal{D}_r(\mathbb{R})$. Then, we have $c_k = \left\|\sqrt{\Lambda_r} q_k\right\|$, $c_{k-k'} = \left\|\sqrt{\Lambda_r}(q_k - q_{k'})\right\|$, $\beta = \frac{c_k^2 + c_j^2 - c_{k-k'}^2}{2c_k}$, $\alpha = \sqrt{c_j^2 - \beta^2}$. Finally, we note $\phi_k = \phi_{0,1}\left(\frac{\mu_k}{c_k}\right)$, $\Phi_{k'} = \Phi_{0,1}\left(\frac{\mu_{k'}}{c_{k'}}\right)$, $\Phi_{NS} = \Phi_{0,1}\left(\frac{\mu_{k'}}{\alpha} - \frac{\beta\mu_k}{\alpha c_k}\right)$, $\Phi_S = \Phi_{0,1}\left(\frac{c_{k'}\mu_k}{c_k\alpha} - \frac{\beta\mu_{k'}}{\alpha c_{k'}}\right)$, $\phi_{NS} = \phi_{0,1}\left(\frac{\beta\mu_{k'}}{\alpha c_{k'}} - \frac{\mu_k c_{k'}}{c_k\alpha}\right)$ and $\Phi_{kk'} = \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_{k'}}\\ -\frac{\beta}{c_{k'}} & 1\end{bmatrix}}\left(\left[\frac{\mu_{k'}}{c_{k'}}, -\frac{\mu_k}{c_k}\right]^T\right)$ and ϕ and Φ are respectively the PDF and CDF of the standard normal distribution. Proof can be found in appendix B.

A.2. Leaky ReLU activation function

Proposition A.2. *Let X be a random variable with $X \sim \mathcal{N}_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{\mu} \in \mathbb{R}^d$, $\boldsymbol{\Sigma} \in \mathbb{S}_d^+(\mathbb{R})$ with rank$(\boldsymbol{\Sigma}) = r$, $r \in \mathbb{N}^*$ and f be the leaky rectified linear function of parameters $\lambda \in [0, 1[$, then for the random variable $\boldsymbol{Y} = (Y_1, ..., Y_d) = f(\boldsymbol{X}) = (f(X_1), ..., f(X_d))$, $\forall k, k' \in \{1...d\}$:*

$$\begin{aligned}
\mathbb{E}[Y_k] &= (1-\lambda)c_k\phi_k + (1-\lambda)\mu_k\Phi_k + \lambda\mu_k && (4)\\
\mathbb{E}[Y_k^2] &= (1-\lambda^2)(\mu_k^2 + c_k^2)\Phi_k + (1-\lambda^2)\mu_k c_k\phi_k + \lambda^2(c_k^2 + \mu_k^2) && (5)\\
\mathbb{E}[Y_k Y_{k'}] &= (1-\lambda)^2 c_k\alpha\phi_{k'}\phi_{NS} \\
&+ (\mu_k\mu_{k'} + c_k\beta)\left((1-\lambda)\Phi_{k'} - (1-\lambda)^2\Phi_{kk'} + \lambda(1-\lambda)\Phi_k + \lambda^2\right) \\
&+ (1-\lambda)^2\mu_k c_{k'}\phi_{k'}\Phi_S \\
&+ (1-\lambda)^2\mu_{k'}c_k\phi_k\Phi_{NS} \\
&+ \lambda(1-\lambda)(\mu_{k'}c_k\phi_k + \mu_k c_{k'}\phi_{k'}) && (6)
\end{aligned}$$

where: $\mu_k = \mathbb{E}[X_k]$, the mean of the k^{th} component of the random variable $\boldsymbol{X}$, $Q_r = (q_k)_{k\in\{1...d\}} \in \mathcal{M}_{r,d}(\mathbb{R})$ the r first line of Q such that $\Sigma = Q^T\Lambda Q = Q_r^T\Lambda_r Q_r$, and $\Lambda_r \in \mathcal{D}_r(\mathbb{R})$. Then, we have $c_k = \left\|\sqrt{\Lambda_r} q_k\right\|$, $c_{k-k'} = \left\|\sqrt{\Lambda_r}(q_k - q_{k'})\right\|$, $\beta = \frac{c_k^2 + c_j^2 - c_{k-k'}^2}{2c_k}$, $\alpha = \sqrt{c_j^2 - \beta^2}$. Finally, we note $\phi_k = \phi_{0,1}\left(\frac{\mu_k}{c_k}\right)$, $\Phi_{k'} = \Phi_{0,1}\left(\frac{\mu_{k'}}{c_{k'}}\right)$, $\Phi_{NS} = \Phi_{0,1}\left(\frac{\mu_{k'}}{\alpha} - \frac{\beta\mu_k}{\alpha c_k}\right)$, $\Phi_S = \Phi_{0,1}\left(\frac{c_{k'}\mu_k}{c_k\alpha} - \frac{\beta\mu_{k'}}{\alpha c_{k'}}\right)$, $\phi_{NS} = \phi_{0,1}\left(\frac{\beta\mu_{k'}}{\alpha c_{k'}} - \frac{\mu_k c_{k'}}{c_k\alpha}\right)$ and $\Phi_{kk'} = \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_{k'}}\\ -\frac{\beta}{c_{k'}} & 1\end{bmatrix}}\left(\left[\frac{\mu_{k'}}{c_{k'}}, -\frac{\mu_k}{c_k}\right]^T\right)$ and ϕ and Φ are respectively the PDF and CDF of the standard normal distribution.

Proof can be found in appendix C.

A.3. ELU activation function

Proposition A.3. *Let X be a random variable with $X \sim \mathcal{N}_d(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{\mu} \in \mathbb{R}^d$, $\boldsymbol{\Sigma} \in \mathbb{S}_d^+(\mathbb{R})$ with rank$(\boldsymbol{\Sigma}) = r$, $r \in \mathbb{N}^*$ and f be the exponential linear function, then for the random variable $\boldsymbol{Y} = (Y_1, ..., Y_d) = f(\boldsymbol{X}) = (f(X_1), ..., f(X_d))$, $\forall k, k' \in \{1...d\}$:*

$$
\begin{aligned}
\mathbb{E}[Y_k] &= c_k\phi_k + (\mu_k + \lambda)\Phi_k + \lambda e^{\mu_k + \frac{c_k^2}{2}}(1 - \Phi_{kc_k}) - \lambda && (7)\\
\mathbb{E}[Y_k^2] &= (\mu_i^2 + c_i^2 - \lambda^2)\Phi_i + \mu_i c_i \phi_i + \lambda^2 e^{2(c_i^2 + \mu_i)}(1 - \Phi_{2c_i}) - 2\lambda^2 e^{\mu_i + \frac{c_i^2}{2}}(1 - \Phi_{c_i})\\
&+ \lambda^2 && (8)\\
\mathbb{E}[Y_k Y_{k'}] &= c_i \alpha \phi_j \phi_{NS}\\
&+ (\mu_i\mu_j + c_i\beta + \lambda\mu_i)\,\Phi_j\\
&- \left(\mu_i\mu_j + c_i\beta + \lambda(\mu_i + \mu_j) + \lambda^2\right)\Phi_{ij}\\
&+ (\mu_i c_j \phi_j + \lambda c_j \phi_j + \frac{\lambda\beta c_i}{c_j}((\phi_j - e^{\mu_j + \frac{c_j^2}{2}}\phi_{c_j}))\Phi_S\\
&+ (\mu_j c_i \phi_i + \lambda c_i \phi_i + \lambda\beta(\phi_i - e^{\mu_i + \frac{c_i^2}{2}}\phi_{c_i}))\Phi_{NS}\\
&+ \lambda c_i e^{\mu_j + \frac{c_j^2}{2}}\phi_{\beta_i}\Phi_\alpha\\
&+ \lambda c_j e^{\mu_i + \frac{c_i^2}{2}}\phi_a\Phi_e\\
&+ \lambda^2 e^{\mu_i + \frac{c_i^2}{2}}\Phi_{c_i}\\
&- \lambda e^{\mu_j + \frac{c_j^2}{2}}(\mu_i + \beta c_i)\Phi_{c_j}\\
&+ \lambda e^{\mu_i + \frac{c_i^2}{2}}(\lambda + \beta c_i + \mu_j)\Phi_A\\
&+ \lambda e^{\mu_j + \frac{c_j^2}{2}}(\lambda + \beta c_i + \mu_i)(\Phi_{\beta_i} + \Phi_B)\\
&+ \lambda^2 e^{\mu_i + \mu_j + \frac{c_i^2 + c_j^2}{2} + c_i\beta}(1 - \Phi_{\beta + c_i} - \Phi_C)\\
&- \lambda(\lambda e^{\mu_i + \frac{c_i^2}{2}} + \lambda e^{\mu_j + \frac{c_j^2}{2}} + c_i\phi_i + c_j\phi_j - \lambda) && (9)
\end{aligned}
$$

where: $\mu_k = \mathbb{E}[X_k]$, the mean of the k^{th} component of the random variable $\boldsymbol{X}$, $Q_r = (q_k)_{k \in \{1...d\}} \in \mathcal{M}_{r,d}(\mathbb{R})$ the r first line of Q such that $\Sigma = Q^T \Lambda Q = Q_r^T \Lambda_r Q_r$, and $\Lambda_r \in \mathcal{D}_r(\mathbb{R})$. Then, we have $c_k = \left\|\sqrt{\Lambda_r} q_k\right\|$, $c_{k-k'} = \left\|\sqrt{\Lambda_r}(q_k - q_{k'})\right\|$, $\beta = \frac{c_k^2 + c_j^2 - c_{k-k'}^2}{2c_k}$, $\alpha = \sqrt{c_j^2 - \beta^2}$. Finally, we note $\phi_k = \phi_{0,1}\left(\frac{\mu_k}{c_k}\right)$, $\Phi_{k'} = \Phi_{0,1}\left(\frac{\mu_{k'}}{c_{k'}}\right)$, $\Phi_{NS} = \Phi_{0,1}\left(\frac{\mu_{k'}}{\alpha} - \frac{\beta\mu_k}{\alpha c_k}\right)$, $\Phi_S = \Phi_{0,1}\left(\frac{c_{k'}\mu_k}{c_k\alpha} - \frac{\beta\mu_{k'}}{\alpha c_{k'}}\right)$, $\phi_{NS} = \phi_{0,1}\left(\frac{\beta\mu_{k'}}{\alpha c_{k'}} - \frac{\mu_k c_{k'}}{c_k\alpha}\right)$, $\Phi_{kk'} = \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_{k'}}\\ -\frac{\beta}{c_{k'}} & 1\end{bmatrix}}\left(\left[\frac{\mu_{k'}}{c_{k'}}, -\frac{\mu_k}{c_k}\right]^T\right)$, $\phi_a = \phi_{0,1}\left(\frac{\mu_j + \beta c_i}{c_j}\right)$, $\Phi_{\beta_i} = \Phi_{0,1}\left(\frac{\mu_i}{c_i} + \beta\right)$, $\Phi_e = \Phi_{0,1}\left(\frac{\beta}{\alpha}\frac{\mu_j + \beta c_i}{c_j} - \frac{c_j\mu_i}{\alpha c_i} - \frac{c_j c_i}{\alpha}\right)$, $\Phi_\alpha = \Phi_{0,1}\left(\frac{\beta\mu_i}{\alpha c_i} - \frac{\mu_j}{\alpha} - \alpha\right)$, $\Phi_{\beta + c_i} = \Phi_{0,1}(\frac{\mu_i}{c_i} + c_i + \beta)$, $\Phi_{c_i} = \Phi_{0,1}\left(\frac{\mu_i}{c_i} + c_i\right)$, $\Phi_{2c_i} = \Phi_{0,1}\left(\frac{\mu_i}{c_i} + 2c_i\right)$, $\Phi_A = \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j + \beta c_i}{c_j}, -\frac{\mu_i}{c_i} - c_i\right]^T\right)$,

$\Phi_B = \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j}{c_j} + c_j, -\frac{\mu_i}{c_i} - \beta\right]^T\right)$,

and $\Phi_C = \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j + c_j^2 + \beta c_i}{c_j}, -(\frac{\mu_i}{c_i} + c_i + \beta)\right]^T\right))$,

Finally, ϕ and Φ are respectively the PDF and CDF of the standard normal distribution.

Proof can be found in appendix D.

B. Derivation for Relu

B.1. Expected value estimation

We wish to estimate the moments of the distribution when passed through an elementary block composed of a linear operation and a ReLU function. Let's note $W \in \mathbb{R}^{n\times d}$ and $B \in \mathbb{R}^d$ the parameters of the linear operation such that $X_l = WX + B \in \mathbb{R}^n$ and the ReLU function such that: $Y = \mathrm{ReLU}(X_l) \in \mathbb{R}^n$

The linear operation transforms a multivariate Gaussian to an other multivariate Gaussian distribution, so:

$$X_l \sim \mathcal{N}(\mu_l, \Sigma_l),$$

$$\begin{cases} \mu_l = W\mu + B, \\ \Sigma_l = W\Sigma W^T, \end{cases}$$

with $\mathrm{rank}(\Sigma_l) \leq \min(d, \mathrm{rank}(W))$. $\Sigma_l \in \mathcal{S}_n^+(\mathbb{R})$, so by diagonalisation we have

$$\Sigma_l = Q\Lambda Q^T,$$

where $Q \in \mathbb{O}_n(\mathbb{R})$. Let's note r the rank of Σ_l, then, there exists a matrix Q_r such that:

$$\Sigma_l = Q_r\Lambda_r Q_r^T,$$

where $Q_r \in \mathbb{R}^{n\times r}$ is the matrix composed of the r first columns of Q and $\Lambda \in \mathbb{R}^{r\times r}$ the diagonal matrix containing the r strictly positive eigenvalues of Σ_l.
We note the random variable $R = Q_r Z + \mu_l$ where $Z \sim \mathcal{N}(0, \Lambda_r)$ ie $R \sim \mathcal{N}(\mu_l, \Sigma_l)$. X_l and R are equal in term of distribution.
Let's note f_i such that:

$$\begin{aligned} f_i \colon \mathbb{R}^n &\to \mathbb{R}_+ \\ x &\mapsto x_i \mathbb{1}_{x_i \geq 0} \end{aligned}$$

Then:

$$\begin{aligned} \mathbb{E}[Y_i] &= \mathbb{E}_{z\sim p(Z)}\left[f_i(Q_r z + \mu)\right] \\ &= \mathbb{E}_{z\sim p(Z)}\left[(z^T q_i + \mu_i)\mathbb{1}_{(z^T q_i + \mu_i \geq 0)}\right] \\ &= \int_{-\infty}^{+\infty} (z^T q_i + \mu_i)\mathbb{1}_{(z^T q_i + \mu_i \geq 0)}\phi_{0,\Lambda_r}(z)dz \end{aligned} \tag{10}$$

where ϕ_{0,Λ_r} is the probability density function of the multivariate Gaussian distribution with zero mean and covariance matrix Λ_r.

Let's note

$$Z_i^+ = \left\{z \in \mathbb{R}^r : z^T q_i + \mu_i \geq 0\right\},$$

then

$$\mathbb{E}[Y_i] = \int_{Z_i^+} z^T q_i \phi_{0,\Lambda_r}(z)dz + \mu_i \int_{Z_i^+} \phi_{0,\Lambda_r}(z)dz \tag{11}$$

With:

$$\int_{Z_i^+} z^T q_i \phi_{0,\Lambda_r}(z)dz = (2\pi)^{-\frac{r}{2}} |\Lambda_r|^{-\frac{1}{2}} \int_{Z_i^+} z^T q_i \exp\left(-\frac{1}{2} z^T \Lambda_r z\right) dz \tag{12}$$

We note $a = \sqrt{\Lambda_r^{-1}} z$, ie $z = \sqrt{\Lambda_r} a$; $|J| = |\Lambda_r|^{\frac{1}{2}}$ and

$$\bar{Z}_i^+ = \left\{ a \in \mathbb{R}^r : a^T \sqrt{\Lambda_r} q_i + \mu_i \geq 0 \right\}$$

We need to perform an integration over an half space of $\mathbb{R}^r$. Then:

$$\int_{Z_i^+} z^T q_i \phi_{0,\Lambda_r}(z)dz = (2\pi)^{-\frac{r}{2}} \int_{\bar{Z}_i^+} a^T \sqrt{\Lambda_r} q_i \exp\left(-\frac{1}{2} a^T a\right) da \tag{13}$$

By performing a rotation with respect to the hyperplan equation $a^T q_i = 0$, we find an unitary matrix B such that $B\sqrt{\Lambda_r} qi = \left\| \sqrt{\Lambda_r} q_i \right\| e_r$ where $e_r = [0, 0, ..., 1]^T \in \mathbb{R}^r$, then

$$\begin{aligned} \tilde{Z}_i^+ &= \left\{ b \in \mathbb{R}^r : b \left\| \sqrt{\Lambda_r} q_i \right\| e_r + \mu_i \geq 0 \right\} \\ &= \left\{ b \in \mathbb{R}^r : b_r \geq -\frac{\mu_i}{\left\| \sqrt{\Lambda_r} q_i \right\|} \right\} \end{aligned} \tag{14}$$

$$\begin{aligned} \int_{Z_i^+} z^T q_i \phi_{0,\Lambda_r}(z)dz &= (2\pi)^{-\frac{r}{2}} \int_{\tilde{Z}_i^+} b^T \left\| \sqrt{\Lambda_r} q_i \right\| e_r \exp\left(-\frac{1}{2} b^T b\right) db \\ &= (2\pi)^{-\frac{r}{2}} \int_{-\infty}^{+\infty} ... \int_{-\infty}^{+\infty} \int_{-\frac{\mu_i}{\left\| \sqrt{\Lambda_r} q_i \right\|}}^{+\infty} \left\| \sqrt{\Lambda_r} q_i \right\| b_r \exp\left(-\frac{1}{2} b^T b\right) db_r \\ &= \left\| \sqrt{\Lambda_r} q_i \right\| (2\pi)^{-\frac{1}{2}} \int_{-\frac{\mu_i}{\left\| \sqrt{\Lambda_r} q_i \right\|}}^{+\infty} b_r \exp\left(-\frac{1}{2} b_r^2\right) db_r \\ &= \left\| \sqrt{\Lambda_r} q_i \right\| \phi_{0,1}\left(\frac{\mu_i}{\left\| \sqrt{\Lambda_r} q_i \right\|}\right) \end{aligned} \tag{15}$$

Similarly:

$$\begin{aligned} \int_{Z_i^+} \phi_{0,\Lambda_r}(z)dz &= \int_{-\frac{\mu_i}{\left\| \sqrt{\Lambda_r} q_i \right\|}}^{+\infty} \phi_{0,1}(b_r) db_r \\ &= \Phi_{0,1}\left(\frac{\mu_i}{\left\| \sqrt{\Lambda_r} q_i \right\|}\right) \end{aligned} \tag{16}$$

Then, by noting $c_i = \left\| \sqrt{\Lambda_r} q_i \right\|$, we have:

$$\boxed{\mathbb{E}[Y_i] = c_i \phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \mu_i \Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)} \tag{17}$$

B.2. Covariance value estimation

Let's note f_{ij} such that:

$$
\begin{aligned}
f_{ij} \colon & \mathbb{R}^n \to \mathbb{R}_+ \\
& x \mapsto x_i \mathbb{1}_{x_i \geq 0} x_j \mathbb{1}_{x_j \geq 0}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y_i Y_j] &= \mathbb{E}_{z \sim p(Z)} \left[f_{ij}(Q_r z + \mu) \right] \\
&= \mathbb{E}_{z \sim p(Z)} \left[(z^T q_i + \mu_i) \mathbb{1}_{(z^T q_i + \mu_i \geq 0)} (z^T q_j + \mu_j) \mathbb{1}_{(z^T q_j + \mu_j \geq 0)} \right] \\
&= \int_{-\infty}^{+\infty} (z^T q_i + \mu_i) \mathbb{1}_{(z^T q_i + \mu_i \geq 0)} (z^T q_j + \mu_j) \mathbb{1}_{(z^T q_j + \mu_j \geq 0)} \phi_{0,\Lambda_r}(z) dz
\end{aligned} \tag{18}
$$

Let's note

$$
Z_{ij}^+ = \left\{ z \in \mathbb{R}^r : z^T q_i + \mu_i \geq 0,\ z^T q_j + \mu_j \geq 0 \right\}
$$

$$
\begin{aligned}
\mathbb{E}[Y_i Y_j] &= \int_{Z_{ij}^+} z^T q_i z^T q_j \phi_{0,\Lambda_r}(z) dz \\
&+ \mu_i \int_{Z_{ij}^+} z^T q_j \phi_{0,\Lambda_r}(z) dz \\
&+ \mu_j \int_{Z_{ij}^+} z^T q_i \phi_{0,\Lambda_r}(z) dz \\
&+ \mu_i \mu_j \int_{Z_{ij}^+} \phi_{0,\Lambda_r}(z) dz
\end{aligned} \tag{19}
$$

$$
\mathbb{E}[Y_i Y_j] = (*) + \mu_i (**)_j + \mu_j (**)_i + \mu_i \mu_j (***) \tag{20}
$$

B.2.1. Special case where i=j: (Variance estimation)

$$
\mathbb{E}[Y_i^2] = \int_{Z_i^+} (z^T q_i)^2 \phi_{0,\Lambda_r}(z) dz + 2\mu_i \int_{Z_i^+} z^T q_i \phi_{0,\Lambda_r}(z) dz + \mu_i^2 P(Z \in Z_i^+) \tag{21}
$$

with

$$
\int_{Z_i^+} (z^T q_i)^2 \phi_{0,\Lambda_r}(z) dz = c_i^2 (2\pi)^{-\frac{1}{2}} \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r^2 \exp\left(-\frac{1}{2} b_r^2 \right) db_r \tag{22}
$$

Then by integration by parts we have:

$$
\boxed{\mathbb{E}[Y_i^2] = (\mu_i^2 + c_i^2) \Phi_{0,1}\left(\frac{\mu_i}{c_i} \right) + \mu_i c_i \phi_{0,1}\left(\frac{\mu_i}{c_i} \right)} \tag{23}
$$

B.2.2. General case where $i \neq$ j: (Covariance estimation)

We need to perform an integration over a quarter space of $\mathbb{R}^r$. By noting as previously $a = \sqrt{\Lambda_r^{-1}}z$, we have

$$Z_{ij}^{+} = \left\{a \in \mathbb{R}^r : a^T\sqrt{\Lambda_r}q_i + \mu_i \geq 0,\ a^T\sqrt{\Lambda_r}q_j + \mu_j \geq 0\right\}$$

It is useful to note that $Z_{ij}^{+} = Z_i^{+} \cap Z_j^{+}$

Once again, by rotation we can find a unitary matrix B such that:

$$\begin{cases} B\sqrt{\Lambda_r}q_i = c_i e_r \\ B\sqrt{\Lambda_r}q_j = \alpha e_{r-1} + \beta e_r \end{cases} \tag{24}$$

From now on, we suppose that $\alpha \neq 0$. Otherwise, if $\alpha = 0$, then $\beta = c_j$ and

$$\tilde{Z}_{ij}^{+} = \left\{b \in \mathbb{R}^r : b_r \geq m = \max\left(-\frac{\mu_i}{c_i}, -\frac{\mu_j}{c_j}\right)\right\},$$

which corresponds to the previous case of integration over an half space of R^r:

$$\boxed{\mathbb{E}[Y_iY_j] = (\mu_i\mu_j + c_ic_j)\Phi_{0,1}\left(-m\right) + (\mu_ic_j + \mu_jc_i + mc_ic_j)\phi_{0,1}\left(m\right)} \tag{25}$$

We can find the matrix B by choosing r-2 orthonormal vectors in $Z_i^{+} \cap Z_j^{+}$, one vector in Z_i^{+} and one orthogonal vector to complete the base (by Gram Schmidt for example).

We know that unitary matrix does preserve the L_2 norm and thus:

$$\begin{cases} \alpha^2 + \beta^2 = c_j^2 \\ \left\|B(\sqrt{\Lambda_r}q_i - \sqrt{\Lambda_r}q_j)\right\| = \left\|\sqrt{\Lambda_r}q_i - \sqrt{\Lambda_r}q_j\right\| \end{cases}$$

We obtain:

$$\begin{cases} \beta = \frac{c_i^2 + c_j^2 - c_{i-j}^2}{2c_i} \\ \alpha = \pm\sqrt{c_j^2 - \beta^2} \end{cases}$$

where

$$\begin{cases} c_i = \left\|\sqrt{\Lambda_r}q_i\right\| \\ c_j = \left\|\sqrt{\Lambda_r}q_j\right\| \\ c_{i-j} = \left\|\sqrt{\Lambda_r}q_i - \sqrt{\Lambda_r}q_j\right\| \end{cases}$$

WLOG, we can choose $\alpha = \sqrt{c_j^2 - \beta^2}$. We have then:

$$\begin{aligned} \tilde{Z}_{ij}^{+} &= \left\{b \in \mathbb{R}^r : b_r \geq -\frac{\mu_i}{ci}\ ,\ \alpha b_{r-1} + \beta b_r \geq -\mu_j\right\} \\ &= \left\{b \in \mathbb{R}^r : b_r \geq -\frac{\mu_i}{ci}\ ,\ b_{r-1} \geq -\frac{(\mu_j + \beta b_r)}{\alpha}\right\} \end{aligned} \tag{26}$$

$$
\begin{aligned}
(***) &= \int_{Z_{ij}^+} \phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\frac{(\mu_j+\beta b_r)}{\alpha}}^{+\infty} \phi_{0,1}\left(b_r\right)\phi_{0,1}\left(b_{r-1}\right)db_{r-1}db_r \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}\left(b_r\right)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= \int_{-\infty}^{+\infty} \phi_{0,1}\left(b_r\right)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r - \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}\left(b_r\right)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= \Phi_{0,1}\left(\frac{\mu_j}{c_j}\right) - \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j}{c_j}, -\frac{\mu_i}{c_i}\right]^T\right)
\end{aligned}
\tag{27}
$$

The last equation is obtained by Owen (1980) (equations 10,010.1 and 10,010.8).

$$
\begin{aligned}
(**)_i &= \int_{Z_{ij}^+} z^T q_i \phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\frac{(\mu_j+\beta b_r)}{\alpha}}^{+\infty} c_i b_r \phi_{0,1}\left(b_r\right)\phi_{0,1}\left(b_{r-1}\right)db_{r-1}db_r \\
&= c_i \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}\left(b_r\right)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r
\end{aligned}
\tag{28}
$$

By integration by parts:

$$
(**)_i = c_i\left[\left[-\phi_{0,1}\left(b_r\right)\Phi_{0,1}\left(\tfrac{\mu_j+\beta b_r}{\alpha}\right)\right]_{-\frac{\mu_i}{c_i}}^{+\infty} + \frac{\beta}{\alpha}\int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}(b_r)\phi_{0,1}(\tfrac{\mu_j+\beta b_r}{\alpha})db_r\right]
\tag{29}
$$

with

$$
\phi_{0,1}\left(b_r\right)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) = \phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\phi_{0,1}\left(\frac{c_j}{\alpha}b_r + \frac{\beta\mu_j}{c_j\alpha})\right)
$$

and

$$
\begin{aligned}
\int_{-A}^{+\infty} \phi_{0,1}\left(ax+b\right)dx &= \frac{1}{a}\int_{-aA+b}^{\mathrm{sgn}(a)\infty} \phi_{0,1}\left(X\right)dX \\
&= \frac{\mathrm{sgn}(a)}{a}\Phi_{0,1}(\mathrm{sgn}(a)(aA-b))
\end{aligned}
\tag{30}
$$

Then

$$
\begin{aligned}
(**)_i &= c_i \left[\phi_{0,1}\left(\frac{\mu_i}{c_i}\right) \Phi_{0,1}\left(\frac{\mu_j - \beta\frac{\mu_i}{c_i}}{\alpha}\right) + \frac{\beta}{\alpha}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \frac{c_j}{\alpha} \Phi_{0,1}\left(\left(\frac{c_j\mu_i}{c_i\alpha} - \frac{\beta\mu_j}{\alpha c_j}\right)\right)\right] && (31)\\
&= c_i \phi_{0,1}\left(\frac{\mu_i}{c_i}\right) \Phi_{0,1}\left(\frac{\mu_j - \beta\frac{\mu_i}{c_i}}{\alpha}\right) + \frac{\beta c_i}{c_j}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \Phi_{0,1}\left(\left(\frac{c_j\mu_i}{c_i\alpha} - \frac{\beta\mu_j}{\alpha c_j}\right)\right) && (32)
\end{aligned}
$$

$$
\begin{aligned}
(**)_j &= \int_{Z_{ij}^+} z^T q_j \phi_{0,\Lambda_r}(z) dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\frac{(\mu_j+\beta b_r)}{\alpha}}^{+\infty} (\alpha b_{r-1} + \beta b_r)\phi_{0,1}(b_r)\,\phi_{0,1}(b_{r-1})\, db_{r-1} db_r \\
&= \alpha \int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}(b_r)\,\phi_{0,1}\left(-\frac{\mu_j + \beta b_r}{\alpha}\right) db_r + \beta \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}(b_r)\,\Phi_{0,1}\left(\frac{\mu_j + \beta b_r}{\alpha}\right) db_r \\
&= \alpha\phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}\left(\frac{c_j}{\alpha}\left(br + \frac{\beta\mu_j}{c_j^2}\right)\right) db_r + \frac{\beta}{c_i}(**)_i \\
&= c_j\phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \Phi_{0,1}\left(\left(\frac{c_j\mu_i}{c_i\alpha} - \frac{\beta\mu_j}{\alpha c_j}\right)\right) + \beta\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi_{0,1}\left(\frac{\mu_j}{\alpha} - \frac{\mu_i\beta}{c_i\alpha}\right) && (33)
\end{aligned}
$$

$$
\begin{aligned}
(*) &= \int_{Z_{ij}^+} z^T q_i z^T q_j \phi_{0,\Lambda_r}(z) dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\frac{(\mu_j+\beta b_r)}{\alpha}}^{+\infty} (c_i b_r)(\alpha b_{r-1} + \beta b_r)\phi_{0,1}(b_r)\,\phi_{0,1}(b_{r-1})\, db_{r-1} db_r \\
&= \beta c_i \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r^2 \phi_{0,1}(b_r) \int_{-\frac{(\mu_j+\beta b_r)}{\alpha}}^{+\infty} \phi_{0,1}(b_{r-1})\, db_{r-1} db_r \\
&+ \alpha c_i \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}(b_r) \int_{-\frac{(\mu_j+\beta b_r)}{\alpha}}^{+\infty} b_{r-1}\phi_{0,1}(b_{r-1})\, db_{r-1} db_r \\
&= \beta c_i \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r^2 \phi_{0,1}(b_r)\,\Phi_{0,1}\left(\frac{1}{\alpha}(\mu_j + \beta b_r)\right) db_r \\
&+ \alpha c_i \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}(b_r)\,\phi_{0,1}\left(\frac{\mu_j + \beta b_r}{\alpha}\right) db_r && (34)
\end{aligned}
$$

the second term of the sum in equation (34) can be rewritten as:

$$
\begin{aligned}
&\int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}(b_r)\,\phi_{0,1}\left(\frac{\mu_j + \beta b_r}{\alpha}\right) db_r \\
&= \phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}\left(\frac{c_j}{\alpha} b_r + \frac{\beta\mu_j}{c_j\alpha}\right) db_r \\
&= \phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \left[\frac{\alpha^2}{c_i^2}\phi_{0,1}\left(\frac{\beta\mu_j}{\alpha c_j} - \frac{\mu_i c_j}{c_i\alpha}\right) - \frac{\alpha\beta\mu_j}{c_j^3}\Phi_{0,1}\left(\left(\frac{c_j\mu_i}{c_i\alpha} - \frac{\beta\mu_j}{\alpha c_j}\right)\right)\right] && (35)
\end{aligned}
$$

and the first term as:

$$
\begin{aligned}
& \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r^2 \phi_{0,1}(b_r)\, \Phi_{0,1}\left(\frac{1}{\alpha}(\mu_j + \beta b_r)\right) db_r = \\
= \quad & \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}(b_r)\, b_r \Phi_{0,1}\left(\frac{1}{\alpha}(\mu_j + \beta b_r)\right) db_r \\
= \quad & \left[-b_r \phi_{0,1}(b_r)\, \Phi_{0,1}\left(\frac{\mu_j + \beta b_r}{\alpha}\right)\right]_{-\frac{\mu_i}{c_i}}^{+\infty} \\
+ \quad & \int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}(b_r)\, \Phi_{0,1}\left(\frac{1}{\alpha}(\mu_j + \beta b_r)\right) db_r \\
+ \quad & \frac{\beta}{\alpha} \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}(b_r)\, \phi_{0,1}\left(\frac{1}{\alpha}(\mu_j + \beta b_r)\right) db_r \\
= \quad & -\frac{\mu_i}{c_i} \phi_{0,1}\left(\frac{\mu_i}{c_i}\right) \Phi\left(\frac{1}{\alpha}\left(\mu_j - \beta\frac{\mu_i}{c_i}\right)\right) + \Phi_{0,1}\left(\frac{\mu_j}{c_j}\right) - \Phi_{ij} \\
+ \quad & \frac{\beta}{\alpha} \phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}\left(\frac{c_j}{\alpha}\left(b_r + \frac{\beta\mu_j}{c_j^2}\right)\right) db_r
\end{aligned}
\tag{36}
$$

Recombining the two terms of equations (35)- (36) yields to:

$$
\begin{aligned}
(*) \quad = \quad & c_i \alpha \phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \phi_{0,1}\left(\frac{\beta\mu_j}{\alpha c_j} - \frac{\mu_i c_j}{c_i \alpha}\right) \\
- \quad & \phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \frac{c_i \beta \mu_j}{c_j} \Phi_{0,1}\left(\left(\frac{c_j \mu_i}{c_i \alpha} - \frac{\beta\mu_j}{\alpha c_j}\right)\right) \\
+ \quad & c_i \beta (***) \\
- \quad & \beta \mu_i \phi_{0,1}\left(\frac{\mu_i}{c_i}\right) \Phi_{0,1}\left(\frac{\mu_j}{\alpha} - \frac{\beta\mu_i}{\alpha c_i}\right)
\end{aligned}
\tag{37}
$$

Using the following notations:

$$\phi_j = \phi_{0,1}\left(\frac{\mu_j}{c_j}\right), \tag{38}$$

$$\phi_i = \phi_{0,1}\left(\frac{\mu_i}{c_i}\right), \tag{39}$$

$$\Phi_j = \Phi_{0,1}\left(\frac{\mu_j}{c_j}\right), \tag{40}$$

$$\Phi_{ij} = \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}} \left(\left[\frac{\mu_j}{c_j}, -\frac{\mu_i}{c_i}\right]^T\right), \tag{41}$$

$$\Phi_{NS} = \Phi_{0,1}\left(\frac{\mu_j}{\alpha} - \frac{\beta\mu_i}{\alpha c_i}\right), \tag{42}$$

$$\Phi_S = \Phi_{0,1}\left(\left(\frac{c_j \mu_i}{c_i \alpha} - \frac{\beta\mu_j}{\alpha c_j}\right)\right), \tag{43}$$

$$\phi_{NS} = \phi_{0,1}\left(\frac{\beta\mu_j}{\alpha c_j} - \frac{\mu_i c_j}{c_i \alpha}\right), \tag{44}$$

we obtain:

$$
\begin{aligned}
\mathbb{E}[Y_iY_j] &= c_i\alpha\phi_j\phi_{NS} \\
&- \frac{c_i\beta\mu_j}{c_j}\phi_j\Phi_S \\
&+ (\mu_i\mu_j + c_i\beta)\left(\Phi_j - \Phi_{ij}\right) \\
&- \beta\mu_i\phi_i\Phi_{NS} \\
&+ \mu_i c_j\phi_j\Phi_S \\
&+ \beta\mu_i\phi_i\Phi_{NS} \\
&+ \mu_j c_i\phi_i\Phi_{NS} \\
&+ \frac{c_i\beta\mu_j}{c_j}\phi_j\Phi_S
\end{aligned} \tag{45}
$$

$$
\boxed{\mathbb{E}[Y_iY_j] = c_i\alpha\phi_j\phi_{NS} + (\mu_i\mu_j + c_i\beta)\left(\Phi_j - \Phi_{ij}\right) + \mu_i c_j\phi_j\Phi_S + \mu_j c_i\phi_i\Phi_{NS}} \tag{46}
$$

C. Derivation for Leaky Relu

C.1. Expected value estimation

Let's note f_i such that:

$$
\begin{aligned}
f_i \colon \mathbb{R}^n &\to \mathbb{R}_+ \\
x &\mapsto \begin{cases} x_i \text{ if } x_i \geq 0 \\ \lambda x_i \text{ otherwise, } \lambda \in [0,1] \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y_i] &= \mathbb{E}_{z\sim p(Z)}\left[f_i(Q_r z + \mu)\right] \\
&= \int_{-\infty}^{+\infty} f_i(Q_r z + \mu)\phi_{0,\Lambda_r}(z)dz
\end{aligned} \tag{47}
$$

Let's note

$$
Z_i^+ = \left\{z \in \mathbb{R}^r : z^T q_i + \mu_i \geq 0\right\},
$$

and:

$$
Z_i^- = \left\{z \in \mathbb{R}^r : z^T q_i + \mu_i < 0\right\},
$$

then

$$
\begin{aligned}
\mathbb{E}[Y_i] &= \int_{Z_i^+} f_i(Q_r z + \mu)\phi_{0,\Lambda_r}(z)dz + \int_{Z_i^-} f_i(Q_r z + \mu)\phi_{0,\Lambda_r}(z)dz \\
&= \int_{Z_i^+} (z^T q_i + \mu_i)\phi_{0,\Lambda_r}(z)dz + \lambda\int_{Z_i^-} (z^T q_i + \mu_i)\phi_{0,\Lambda_r}(z)dz
\end{aligned} \tag{48}
$$

Yet, from the equation (17), we have:

$$\int_{Z_i^+} (z^T q_i + \mu_i)\phi_{0,\Lambda_r}(z)dz = c_i\phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \mu_i\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)$$

and:

$$\begin{aligned}
\int_{Z_i^-} (z^T q_i + \mu_i)\phi_{0,\Lambda_r}(z)dz &= \int_{Z_i^-} (z^T q_i)\phi_{0,\Lambda_r}(z)dz + \mu_i \int_{Z_i^-} \phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} c_i b_r \phi_{0,1}(b_r)db_r + \mu_i \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)db_r \\
&= c_i\left[[-\phi_{0,1}(x)]_{-\infty}^{-\frac{\mu_i}{c_i}}\right] + \mu_i\Phi_{0,1}\left(-\frac{\mu_i}{c_i}\right) \\
&= -c_i\phi_{0,1}(\frac{\mu_i}{c_i}) + \mu_i\left(1 - \Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\right) \\
&= -\left(c_i\phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \mu_i\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\right) + \mu_i
\end{aligned} \tag{49}$$

Thus, we obtain:

$$\boxed{\mathbb{E}[Y_i] = (1-\lambda)c_i\phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + (1-\lambda)\mu_i\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \lambda\mu_i} \tag{50}$$

C.2. Variance value estimation

$$\begin{aligned}
\mathbb{E}[Y_i^2] &= \mathbb{E}_{z\sim p(Z)}\left[f_i(Q_r z + \mu)^2\right] \\
&= \int_{-\infty}^{+\infty} f_i(Q_r z + \mu)^2 \phi_{0,\Lambda_r}(z)dz \\
&= \int_{Z_i^+} (z^T q_i + \mu_i)^2 \phi_{0,\Lambda_r}(z)dz + \lambda^2 \int_{Z_i^-} (z^T q_i + \mu_i)^2 \phi_{0,\Lambda_r}(z)dz
\end{aligned}$$

Yet, by the equation (23):

$$\int_{Z_i^+} (z^T q_i + \mu_i)^2 \phi_{0,\Lambda_r}(z)dz = (\mu_i^2 + c_i^2)\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \mu_i c_i \phi_{0,1}\left(\frac{\mu_i}{c_i}\right)$$

Now:

$$\begin{aligned}
\int_{Z_i^-} (z^T q_i + \mu_i)^2 \phi_{0,\Lambda_r}(z)dz = \int_{Z_i^-} (z^T q_i)^2 \phi_{0,\Lambda_r}(z)dz &+ 2\mu_i \int_{Z_i^-} z^T q_i \phi_{0,\Lambda_r}(z)dz \\
&+ \mu_i^2 \int_{Z_i^-} \phi_{0,\Lambda_r}(z)dz
\end{aligned}$$

with:

$$\int_{Z_i^-} \phi_{0,\Lambda_r}(z)dz = 1 - \Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) \tag{51}$$

$$\int_{Z_i^-} z^T q_i \phi_{0,\Lambda_r}(z)dz = -c_i\phi_{0,1}\left(\frac{\mu_i}{c_i}\right) \tag{52}$$

$$\begin{aligned}\int_{Z_i^-} (z^T q_i)^2 + &= c_i^2 \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r^2 \phi_{0,1}(b_r)db_r \\ &= c_i^2 \left([-x\phi_{0,1}(x)]_{-\infty}^{-\frac{\mu_i}{c_i}} + \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)db_r \right) \\ &= \mu_i c_i \phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + c_i^2 \left(1 - \Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\right)\end{aligned} \tag{53}$$

Then:

$$\int_{Z_i^-} (z^T q_i + \mu_i)^2 \phi_{0,\Lambda_r}(z)dz = -\left((\mu_i^2 + c_i^2)\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \mu_i c_i \phi_{0,1}\left(\frac{\mu_i}{c_i}\right) \right) + c_i^2 + \mu_i^2 \tag{54}$$

We finally obtain:

$$\boxed{\mathbb{E}[Y_i^2] = (1-\lambda^2)(\mu_i^2 + c_i^2)\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + (1-\lambda^2)\mu_i c_i \phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \lambda^2(c_i^2 + \mu_i^2)} \tag{55}$$

C.3. Covariance value estimation

Let's note f_{ij} such that:

$$\begin{aligned} f_{ij} \colon \mathbb{R}^n &\to \mathbb{R}_+ \\ x &\mapsto f_i(x) f_j(x) \end{aligned}$$

with f_i defined as in (47).

$$\begin{aligned}\mathbb{E}[Y_i Y_j] &= \mathbb{E}_{z\sim p(Z)}\left[f_{ij}(Q_r z + \mu)\right] \\ &= \mathbb{E}_{z\sim p(Z)}\left[f_i(Q_r z + \mu) f_j(Q_r z + \mu)\right]\end{aligned} \tag{56}$$

By noting:

$$\begin{aligned} Z_{ij}^{+} &= \left\{ z \in \mathbb{R}^r : z^T q_i + \mu_i \geq 0,\ z^T q_j + \mu_j \geq 0 \right\} \\ Z_{ij}^{-} &= \left\{ z \in \mathbb{R}^r : z^T q_i + \mu_i < 0,\ z^T q_j + \mu_j < 0 \right\} \\ Z_{ij}^{-+} &= \left\{ z \in \mathbb{R}^r : z^T q_i + \mu_i \geq 0,\ z^T q_j + \mu_j < 0 \right\} \end{aligned}$$

$$Z_{ij}^{+-} = \left\{ z \in \mathbb{R}^r : z^T q_i + \mu_i < 0,\ z^T q_j + \mu_j \geq 0 \right\}$$

We have:

$$\begin{aligned}
\mathbb{E}[Y_i Y_j] &= \int_{z_{ij}^+} f_{ij}(Q_r z + \mu)\phi_{0,\Lambda_r}(z)dz \\
&+ \int_{z_{ij}^-} f_{ij}(Q_r z + \mu)\phi_{0,\Lambda_r}(z)dz \\
&+ \int_{z_{ij}^{+-}} f_{ij}(Q_r z + \mu)\phi_{0,\Lambda_r}(z)dz \\
&+ \int_{z_{ij}^{-+}} f_{ij}(Q_r z + \mu)\phi_{0,\Lambda_r}(z)dz \\
&= (+) + (-) + (+-) + (-+)
\end{aligned} \tag{57}$$

C.3.1. Derivation over Z_{ij}^-:

$$\begin{aligned}
(-) &= \int_{z_{ij}^-} f_{ij}(Q_r z + \mu)\phi_{0,\Lambda_r}(z)dz \\
&= \lambda^2 \int_{z_{ij}^-} (z^T q_i + \mu_i)(z^T q_j + \mu_j)\phi_{0,\Lambda_r}(z)dz \\
&= \lambda^2 [\int_{z_{ij}^-} z^T q_i z^T q_j \phi_{0,\Lambda_r}(z)dz + \mu_j \int_{z_{ij}^-} z^T q_i \phi_{0,\Lambda_r}(z)dz + \mu_i \int_{z_{ij}^-} z^T q_j \phi_{0,\Lambda_r}(z)dz \\
&+ \mu_i\mu_j \int_{z_{ij}^-} \phi_{0,\Lambda_r}(z)dz] \\
&= \lambda^2[(-*) + \mu_j(-**)_i + \mu_i(-**)_j + \mu_i\mu_j(-***)]
\end{aligned} \tag{58}$$

With:

$$\begin{aligned}
(-***) &= \int_{z_{ij}^-} \phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_r db_{r-1} \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)\Phi_{0,1}(-\frac{\mu_j + \beta b_r}{\alpha})db_r \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{01}(b_r)db_r - \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)\Phi_{0,1}(\frac{\mu_j + \beta b_r}{\alpha})db_r \\
&= \Phi_{0,1}\left(-\frac{\mu_i}{c_i}\right) - \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}} \left(\left[\frac{\mu_j}{c_j}, -\frac{\mu_i}{c_i}\right]^T \right) \\
&= 1 - \Phi_i - \Phi_{ij}
\end{aligned} \tag{59}$$

and:

$$
\begin{aligned}
(-**)_i &= \int_{z_{ij}^-} z^T q_i \phi_{0,\Lambda_r}(z) dz \\
&= c_i \int_{-\infty}^{-\frac{\mu_i}{c_i}} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} b_r \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_r db_{r-1} \\
&= c_i \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&= c_i \Big[\left[-\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)\right]_{-\infty}^{-\frac{\mu_i}{c_i}} \\
&- \frac{\beta}{\alpha}\int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \Big] \\
&= c_i \Big[-\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi_{0,1}\left(-\frac{\mu_j}{\alpha}+\frac{\beta\mu_i}{\alpha c_i}\right) \\
&- \frac{\beta}{\alpha}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \Big] \\
&= -c_i\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi_{0,1}\left(-\frac{\mu_j}{\alpha}+\frac{\beta\mu_i}{\alpha c_i}\right) - \frac{c_i\beta}{c_j}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\Phi_{0,1}\left(\frac{\beta\mu_j}{\alpha c_j}-\frac{c_j\mu_i}{\alpha c_i}\right) \\
&= -c_i\phi_i(1-\Phi_{NS}) - \frac{c_i\beta}{c_j}\phi_j(1-\Phi_S)
\end{aligned} \tag{60}
$$

and:

$$
\begin{aligned}
(-**)_j &= \int_{z_{ij}^-} z^T q_j \phi_{0,\Lambda_r}(z) dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} (\alpha b_{r-1}+\beta b_r)\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_r db_{r-1} \\
&= \beta \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&- \alpha \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r\phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&= \beta \left[-\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)\right]_{-\infty}^{-\frac{\mu_i}{c_i}} \\
&- \left(\frac{\beta^2}{\alpha}+\alpha\right)\int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r\phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&= -\beta\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi_{0,1}\left(-\frac{\mu_j}{\alpha}+\frac{\beta\mu_i}{\alpha c_i}\right) - \frac{c_j^2}{\alpha}\int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&= -\beta\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi_{0,1}\left(\frac{\beta\mu_i}{\alpha c_i}-\frac{\mu_j}{\alpha}\right) - c_j\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\Phi_{0,1}\left(\frac{\beta\mu_j}{\alpha c_j}-\frac{c_j\mu_i}{\alpha c_i}\right) \\
&= -\beta\phi_i(1-\Phi_{NS}) - c_j\phi_j(1-\Phi_S)
\end{aligned} \tag{61}
$$

and:

$$
\begin{aligned}
(-*) &= \int_{z_{ij}^{-}} z^T q_i z^T q_j \phi_{0,\Lambda_r}(z) dz \\
&= c_i\beta \int_{-\infty}^{-\frac{\mu_i}{c_i}} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} b_r^2 \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_r db_{r-1} \\
&+ \alpha c_i \int_{-\infty}^{-\frac{\mu_i}{c_i}} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} b_{r-1} \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_r db_{r-1} \\
&= c_i\beta \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r^2 \phi_{0,1}(b_r) \Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&- \alpha c_i \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_{r-1} \phi_{0,1}(b_r) \phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r
\end{aligned} \tag{62}
$$

With:

$$
\begin{aligned}
&\int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r \phi_{0,1}(b_r) \phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r = \phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r \phi_{0,1}\left(\frac{c_j}{\alpha} b_r + \frac{\beta\mu_j}{\alpha c_j}\right) db_r \\
=\; & \phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \left[\frac{\alpha^2}{c_j^2} \int_{-\infty}^{-\frac{c_j\mu_i}{\alpha c_i}+\frac{\beta\mu_j}{\alpha c_j}} x\phi_{0,1}(x) dx - \frac{\alpha\beta\mu_j}{c_j^3} \int_{-\infty}^{-\frac{c_j\mu_i}{\alpha c_i}+\frac{\beta\mu_j}{\alpha c_j}} \phi_{0,1}(x) dx \right] \\
=\; & -\frac{\alpha^2}{c_j^2} \phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \phi_{0,1}\left(\frac{\beta\mu_j}{\alpha c_j} - \frac{c_j\mu_i}{\alpha c_i}\right) - \frac{\alpha\beta\mu_j}{c_j^3} \phi_{0,1}\left(\frac{\mu_j}{c_j}\right) \Phi_{0,1}\left(\frac{\beta\mu_j}{\alpha c_j} - \frac{c_j\mu_i}{\alpha c_i}\right) \\
=\; & -\frac{\alpha^2}{c_j^2} \phi_j \phi_{NS} - \frac{\alpha\beta\mu_j}{c_j^3} \phi_j (1-\Phi_S)
\end{aligned} \tag{63}
$$

and:

$$
\begin{aligned}
& \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r^2 \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right) db_r = \left[-b_r\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)\right]_{-\infty}^{-\frac{\mu_i}{c_i}} \\
+ \quad & \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right) db_r - \frac{\beta}{\alpha}\int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r\phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
= \quad & \frac{\mu_i}{c_i}\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi_{0,1}\left(-\frac{\mu_j}{\alpha}+\frac{\beta\mu_i}{\alpha c_i}\right) + (-***) + \frac{\beta\alpha}{c_j^2}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\phi_{0,1}\left(-\frac{c_j\mu_i}{\alpha c_i}+\frac{\beta\mu_j}{\alpha c_j}\right) \\
+ \quad & \frac{\beta^2\mu_j}{c_j^3}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\Phi_{0,1}\left(-\left(\frac{c_j\mu_i}{\alpha c_i}-\frac{\beta\mu_j}{\alpha c_j}\right)\right) \\
= \quad & \frac{\mu_i}{c_i}\phi_i(1-\Phi_{NS}) + (-***) + \frac{\beta\alpha}{c_j^2}\phi_j\phi_{NS} + \frac{\beta^2\mu_j}{c_j^3}\phi_j(1-\Phi_S)
\end{aligned} \tag{64}
$$

Finally:

$$
(-*) = \beta\mu_i\phi_i(1-\Phi_{NS}) + \beta c_i(-***) + \alpha c_i\phi_j\phi_{NS} + \frac{\beta\mu_j c_i}{c_j}\phi_j(1-\Phi_S) \tag{65}
$$

Then:

$$
\begin{aligned}
(-) \quad = \quad & \lambda^2[\beta\mu_i\phi_i(1-\Phi_{NS}) + (\beta c_i+\mu_i\mu_j)(1-\Phi_i-\Phi_{ij}) + \alpha c_i\phi_j\phi_{NS} \\
+ \quad & \frac{\beta\mu_j c_i}{c_j}\phi_j(1-\Phi_S) - \mu_i\beta\phi_i(1-\Phi_{NS}) - \mu_i c_j\phi_j(1-\Phi_S) - \mu_j c_i\phi_i(1-\Phi_{NS}) \\
- \quad & \frac{c_i\beta\mu_j}{c_j}\phi_j(1-\Phi_S)] \\
= \quad & \lambda^2[(\beta c_i+\mu_i\mu_j)(1-\Phi_i-\Phi_{ij}) + \alpha c_i\phi_j\phi_{NS} - \mu_j c_i\phi_i(1-\Phi_{NS}) \\
- \quad & \mu_i c_j\phi_j(1-\Phi_S)]
\end{aligned} \tag{66}
$$

C.3.2. Derivation over Z_{ij}^{+-}:

$$
\begin{aligned}
(+-) \quad = \quad & \int_{z_{ij}^{+-}} f_{ij}(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz \\
= \quad & \lambda\int_{z_{ij}^{+-}} (z^T q_i+\mu_i)(z^T q_j+\mu_j)\phi_{0,\Lambda_r}(z)dz \\
= \quad & \lambda[\int_{z_{ij}^{+-}} z^T q_i z^T q_j\phi_{0,\Lambda_r}(z)dz + \mu_j\int_{z_{ij}^{+-}} z^T q_i\phi_{0,\Lambda_r}(z)dz \\
+ \quad & \mu_i\int_{z_{ij}^{+-}} z^T q_j\phi_{0,\Lambda_r}(z)dz + \mu_i\mu_j\int_{z_{ij}^{+-}} \phi_{0,\Lambda_r}(z)dz] \\
= \quad & \lambda[(+-*) + \mu_j(+-**)_i + \mu_i(+-**)_j + \mu_i\mu_j(+-***)]
\end{aligned} \tag{67}
$$

With:

$$
\begin{aligned}
(+-*) &= \int_{z_{ij}^{+-}} z^T q_i z^T q_j \phi_{0,\Lambda_r}(z) dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} (c_i b_r)(\alpha b_{r-1} + \beta b_r)\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_r db_{r-1} \\
&= \beta c_i \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r^2 \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&- \alpha c_i \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r
\end{aligned} \tag{68}
$$

with:

$$
\begin{aligned}
\int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r^2 \quad & \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right) db_r = \left[-b_r\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)\right]_{-\frac{\mu_i}{c_i}}^{+\infty} \\
+ \quad & \int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right) db_r - \frac{\beta}{\alpha}\int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r\phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
= \quad & -\frac{\mu_i}{c_i}\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi_{0,1}\left(-\frac{\mu_j}{\alpha}+\frac{\beta\mu_i}{\alpha c_i}\right) + (+-***) \\
- \quad & \frac{\beta}{\alpha}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r\phi_{0,1}\left(\frac{c_j}{\alpha}b_r + \frac{\beta\mu_j}{\alpha c_j}\right) db_r \\
= \quad & -\frac{\mu_i}{c_i}\phi_i(1-\Phi_{NS}) + (+-***) - \frac{\beta\alpha}{c_j^2}\phi_j\phi_{NS} + \frac{\beta^2\mu_j}{c_j^3}\phi_j\Phi_S
\end{aligned} \tag{69}
$$

and:

$$
\begin{aligned}
\int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r\phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r &= \phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r\phi_{0,1}\left(\frac{c_j}{\alpha}b_r + \frac{\beta\mu_j}{\alpha c_j}\right) db_r \\
&= \frac{\alpha^2}{c_j^2}\phi_j\phi_{NS} - \frac{\alpha\beta\mu_j}{c_j^3}\phi_j\Phi_S
\end{aligned} \tag{70}
$$

$$ \tag{71} $$

Then:

$$
\begin{aligned}
(+-*) &= -\beta\mu_i\phi_i(1-\Phi_{NS}) + \beta c_i(+-***) - \frac{\beta^2\alpha c_i}{c_j^2}\phi_j\phi_{NS} \\
&+ \frac{\beta^3 c_i\mu_j}{c_j^3}\phi_j\Phi_S - \frac{\alpha^3 c_i}{c_j^2}\phi_j\phi_{NS} + \frac{\beta\alpha^2 c_i\mu_j}{c_j^3}\phi_j\Phi_S \\
&= -\beta\mu_i\phi_i(1-\Phi_{NS}) + \beta c_i(+-***) - \alpha c_i\phi_j\phi_{NS} + \frac{\beta c_i\mu_j}{c_j}\phi_j\Phi_S
\end{aligned} \tag{72}
$$

$$ \tag{73} $$

and:

$$
\begin{aligned}
(+-**)_i &= \int_{z_{ij}^{+-}} z^T q_i \phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} c_i b_r \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_r db_{r-1} \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} c_i b_r \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= c_i\left[\left[-\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)\right]_{-\frac{\mu_i}{c_i}}^{+\infty}\right] - \frac{\beta}{\alpha}\int_{-\frac{\mu_i}{c_i}}^{+\infty}\phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= c_i\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi\left(-\frac{\mu_j}{\alpha}+\frac{\beta\mu_j}{\alpha c_i}\right) - \frac{\beta c_i}{c_j}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\Phi_{0,1}\left(\left(\frac{c_j\mu_i}{c_i\alpha}-\frac{\beta\mu_j}{\alpha c_j}\right)\right) \\
&= c_i\phi_i(1-\Phi_{NS}) - \frac{\beta c_i}{c_j}\phi_j\Phi_S
\end{aligned}
\tag{74}
$$

and:

$$
\begin{aligned}
(+-**)_j &= \int_{z_{ij}^{+-}} z^T q_j \phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} (\alpha b_{r-1}+\beta b_r)\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_r db_{r-1} \\
&= -\alpha\int_{-\frac{\mu_i}{c_i}}^{+\infty}\phi_{0,1}(b_r)\phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&+ \beta\int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= -\alpha\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\int_{-\frac{\mu_i}{c_i}}^{+\infty}\phi_{0,1}\left(\frac{c_j}{\alpha}b_r+\frac{\beta\mu_j}{\alpha c_j}\right)db_r \\
&+ \beta\Big[\left[-\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)\right]_{-\frac{\mu_i}{c_i}}^{+\infty} \\
&- \frac{\beta}{\alpha}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\int_{-\frac{\mu_i}{c_i}}^{+\infty}\phi_{0,1}\left(\frac{c_j}{\alpha}b_r+\frac{\beta\mu_j}{\alpha c_j}\right)db_r\Big] \\
&= \beta\phi_i(1-\Phi_{NS}) - c_j\phi_j\Phi_S
\end{aligned}
\tag{75}
$$

and:

$$
\begin{aligned}
(+-***) &= \int_{z_{ij}^{+-}} \phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_r db_{r-1} \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}(b_r)db_r - \int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= \Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) - \int_{-\infty}^{+\infty} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&+ \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= \Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) - \Phi_{0,1}\left(\frac{\mu_j}{c_j}\right) + \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j}{c_j}, -\frac{\mu_i}{c_i}\right]^T\right) \\
&= \Phi_i - \Phi_j + \Phi_{ij}
\end{aligned} \tag{76}
$$

Then:

$$
\begin{aligned}
(+-) &= \lambda[(+-*) + \mu_j(+-**)_i + \mu_i(+-**)_j + \mu_i\mu_j(+-***)] \\
&= \lambda[-\beta\mu_i\phi_i(1-\Phi_{NS}) + \beta c_i(\Phi_i - \Phi_j + \Phi_{ij}) - \alpha c_i\phi_j\phi_{NS} + \frac{\beta c_i\mu_j}{c_j}\phi_j\Phi_S \\
&+ c_i\mu_j\phi_i(1-\Phi_{NS}) - \frac{\beta c_i\mu_j}{c_j}\phi_j\Phi_S \\
&+ \beta\mu_i\phi_i(1-\Phi_{NS}) - c_j\mu_i\phi_j\Phi_S \\
&+ \mu_i\mu_j(\Phi_i - \Phi_j + \Phi_{ij})] \\
&= \lambda[(\beta c_i + \mu_i\mu_j)(\Phi_i - \Phi_j + \Phi_{ij}) - \alpha c_i\phi_j\phi_{NS} - c_j\mu_i\phi_j\Phi_S \\
&+ c_i\mu_j\phi_i(1-\Phi_{NS})]
\end{aligned} \tag{77}
$$

C.3.3. Derivation over Z_{ij}^{-+}:

$$
\begin{aligned}
(-+) &= \int_{z_{ij}^{-+}} f_{ij}(Q_r z + \mu)\phi_{0,\Lambda_r}(z)dz \\
&= \lambda\int_{z_{ij}^{-+}} (z^T q_i + \mu_i)(z^T q_j + \mu_j)\phi_{0,\Lambda_r}(z)dz \\
&= \lambda[\int_{z_{ij}^{-+}} z^T q_i z^T q_j \phi_{0,\Lambda_r}(z)dz + \mu_j\int_{z_{ij}^{-+}} z^T q_i\phi_{0,\Lambda_r}(z)dz \\
&+ \mu_i\int_{z_{ij}^{-+}} z^T q_j\phi_{0,\Lambda_r}(z)dz + \mu_i\mu_j\int_{z_{ij}^{-+}} \phi_{0,\Lambda_r}(z)dz] \\
&= \lambda[(-+*) + \mu_j(-+**)_i + \mu_i(-+**)_j + \mu_i\mu_j(-+***)]
\end{aligned} \tag{78}
$$

With:

$$
\begin{aligned}
(-+*) &= \int_{z_{ij}^{-+}} z^T q_i z^T q_j \phi_{0,\Lambda_r}(z) dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} \int_{-\frac{\mu_j+\beta b_r}{\alpha}}^{+\infty} (c_i b_r)(\alpha b_{r-1} + \beta b_r)\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_r db_{r-1} \\
&= \beta c_i \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r^2 \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&+ \alpha c_i \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r \phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r
\end{aligned} \tag{79}
$$

with:

$$
\begin{aligned}
\int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r^2 \quad & \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r = \left[-b_r\phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)\right]_{-\infty}^{-\frac{\mu_i}{c_i}} \\
&+ \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r + \frac{\beta}{\alpha}\int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r\phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&= \frac{\mu_i}{c_i}\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi_{0,1}\left(\frac{\mu_j}{\alpha} - \frac{\beta\mu_i}{\alpha c_i}\right) + (-+***) \\
&+ \frac{\beta}{\alpha}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r\phi_{0,1}\left(\frac{c_j}{\alpha}b_r + \frac{\beta\mu_j}{\alpha c_j}\right) db_r \\
&= \frac{\mu_i}{c_i}\phi_i\Phi_{NS} + (-+***) - \frac{\beta\alpha}{c_j^2}\phi_j\phi_{NS} - \frac{\beta^2\mu_j}{c_j^3}\phi_j(1-\Phi_S)
\end{aligned} \tag{80}
$$

and:

$$
\begin{aligned}
\int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r\phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r &= \phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r\phi_{0,1}\left(\frac{c_j}{\alpha}b_r + \frac{\beta\mu_j}{\alpha c_j}\right) db_r \\
&= -\frac{\alpha^2}{c_j^2}\phi_j\phi_{NS} - \frac{\alpha\beta\mu_j}{c_j^3}\phi_j(1-\Phi_S)
\end{aligned} \tag{81}
$$

Then:

$$
\begin{aligned}
(-+*) &= \beta\mu_i\phi_i\Phi_{NS} + \beta c_i(-+***) - \frac{\beta^2\alpha c_i}{c_j^2}\phi_j\phi_{NS} \\
&- \frac{\beta^3 c_i\mu_j}{c_j^3}\phi_j(1-\Phi_S) - \frac{\alpha^3 c_i}{c_j^2}\phi_j\phi_{NS} - \frac{\beta\alpha^2 c_i\mu_j}{c_j^3}\phi_j(1-\Phi_S) \\
&= \beta\mu_i\phi_i\Phi_{NS} + \beta c_i(-+***) - \alpha c_i\phi_j\phi_{NS} - \frac{\beta c_i\mu_j}{c_j}\phi_j(1-\Phi_S)
\end{aligned} \tag{82}
$$

and:

$$
\begin{aligned}
(-+**)_i &= \int_{z_{ij}^{-+}} z^T q_i \phi_{0,\Lambda_r}(z) dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} \int_{-\frac{\mu_j+\beta b_r}{\alpha}}^{+\infty} c_i b_r \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_r db_{r-1} \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} c_i b_r \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&= c_i \left[-\phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)\right]_{+\infty}^{-\frac{\mu_i}{c_i}} \\
&+ c_i \frac{\beta}{\alpha} \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&= -c_i \phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\Phi\left(\frac{\mu_j}{\alpha} - \frac{\beta\mu_j}{\alpha c_i}\right) + \frac{\beta c_i}{c_j}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\Phi_{0,1}\left(\frac{\beta\mu_j}{\alpha c_j} - \frac{c_j\mu_i}{c_i\alpha}\right) \\
&= -c_i\phi_i\Phi_{NS} + \frac{\beta c_i}{c_j}\phi_j(1-\Phi_S)
\end{aligned} \tag{83}
$$

and:

$$
\begin{aligned}
(-+**)_j &= \int_{z_{ij}^{-+}} z^T q_j \phi_{0,\Lambda_r}(z) dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} \int_{-\frac{\mu_j+\beta b_r}{\alpha}}^{+\infty} (\alpha b_{r-1} + \beta b_r)\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_r db_{r-1} \\
&= \alpha \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&+ \beta \int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right) db_r \\
&= \alpha\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}\left(\frac{c_j}{\alpha}b_r + \frac{\beta\mu_j}{\alpha c_j}\right) db_r \\
&+ \beta[\left[-\phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)\right]_{-\infty}^{-\frac{\mu_i}{c_i}} \\
&+ \frac{\beta}{\alpha}\phi_{0,1}\left(\frac{\mu_j}{c_j}\right)\int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}\left(\frac{c_j}{\alpha}b_r + \frac{\beta\mu_j}{\alpha c_j}\right) db_r] \\
&= -\beta\phi_i\Phi_{NS} + c_j\phi_j(1-\Phi_S)
\end{aligned} \tag{84}
$$

and:

$$
\begin{aligned}
(-+***) &= \int_{z_{ij}^{-+}} \phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} \int_{-\frac{\mu_j+\beta b_r}{\alpha}}^{+\infty} \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_r db_{r-1} \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= \Phi_{ij}
\end{aligned} \tag{85}
$$

Then:

$$
\begin{aligned}
(-+) &= \lambda[(-+*) + \mu_j(-+**)_i + \mu_i(-+**)_j + \mu_i\mu_j(-+***)] \\
&= \lambda[\beta\mu_i\phi_i\Phi_{NS} + \beta c_i\Phi_{ij} - \alpha c_i\phi_j\phi_{NS} - \frac{\beta c_i\mu_j}{c_j}\phi_j(1-\Phi_S) \\
&- \quad c_i\mu_j\phi_i\Phi_{NS} + \frac{\beta c_i\mu_j}{c_j}\phi_j(1-\Phi_S) \\
&- \quad \beta\mu_i\phi_i\Phi_{NS} + c_j\mu_i\phi_j(1-\Phi_S) \\
&+ \quad \mu_i\mu_j\Phi_{ij}] \\
&= \lambda[(\beta c_i + \mu_i\mu_j)\Phi_{ij} - \alpha c_i\phi_j\phi_{NS} + c_j\mu_i\phi_j(1-\Phi_S) - c_i\mu_j\phi_i\Phi_{NS}]
\end{aligned} \tag{86}
$$

C.3.4. Final derivation:

Finally, by the equation (46), we have:

$$
(+) = \mathbb{E}[Y_iY_j] = c_i\alpha\phi_j\phi_{NS} + (\mu_i\mu_j + c_i\beta)(\Phi_j - \Phi_{ij}) + \mu_i c_j\phi_j\Phi_S + \mu_j c_i\phi_i\Phi_{NS}
$$

Then:

$$
\begin{aligned}
\mathbb{E}[Y_iY_j] &= (+) + (-) + (+-) + (-+) \\
&= (\beta c_i + \mu_i\mu_j)(\Phi_j - \Phi_{ij}) + c_i\alpha\phi_j\phi_{NS} + \mu_i c_j\phi_j\Phi_S + \mu_j c_i\phi_i\Phi_{NS} \\
&+ \quad \lambda^2(\beta c_i + \mu_i\mu_j)(1-\Phi_i-\Phi_{ij}) + \lambda^2\alpha c_i\phi_j\phi_{NS} - \lambda^2\mu_j c_i\phi_i(1-\Phi_{NS}) \\
&- \quad \lambda^2\mu_i c_j\phi_j(1-\Phi_S) \\
&+ \quad \lambda(\beta c_i + \mu_i\mu_j)(\Phi_i - \Phi_j + \Phi_{ij}) - \lambda\alpha c_i\phi_j\phi_{NS} - \lambda c_j\mu_i\phi_j\Phi_S + \lambda c_i\mu_j\phi_i(1-\Phi_{NS}) \\
&+ \quad \lambda(\beta c_i + \mu_i\mu_j)\Phi_{ij} - \lambda\alpha c_i\phi_j\phi_{NS} + \lambda c_j\mu_i\phi_j(1-\Phi_S) - \lambda c_i\mu_j\phi_i\Phi_{NS}
\end{aligned} \tag{87}
$$

Finally, we obtain:

$$
\begin{aligned}
\mathbb{E}[Y_iY_j] &= (1-\lambda)^2 c_i\alpha\phi_j\phi_{NS} \\
&+ \quad (\mu_i\mu_j + c_i\beta)\left((1-\lambda)\Phi_j - (1-\lambda)^2\Phi_{ij} + \lambda(1-\lambda)\Phi_i + \lambda^2\right) \\
&+ \quad (1-\lambda)^2\mu_i c_j\phi_j\Phi_S \\
&+ \quad (1-\lambda)^2\mu_j c_i\phi_i\Phi_{NS} \\
&+ \quad \lambda(1-\lambda)(\mu_j c_i\phi_i + \mu_i c_j\phi_j)
\end{aligned} \tag{88}
$$

D. Derivation for Elu

D.1. Expected value estimation

Let's note f_i such that:

$$
\begin{aligned}
f_i : \mathbb{R}^n &\to \mathbb{R}_+ \\
x &\mapsto \begin{cases} x_i \text{ if } x_i \geq 0 \\ \lambda\left(e^{x_i}-1\right) \text{ otherwise}, \lambda \in [0,1] \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[Y_i] &= \mathbb{E}_{z\sim p(Z)}\left[f_i(Q_r z+\mu)\right] \\
&= \int_{-\infty}^{+\infty} f_i(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz
\end{aligned} \tag{89}
$$

Let's note

$$
Z_i^+ = \left\{z\in\mathbb{R}^r : z^T q_i + \mu_i \geq 0\right\},
$$

and:

$$
Z_i^- = \left\{z\in\mathbb{R}^r : z^T q_i + \mu_i < 0\right\},
$$

then

$$
\begin{aligned}
\mathbb{E}[Y_i] &= \int_{Z_i^+} f_i(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz + \int_{Z_i^-} f_i(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz \\
&= \int_{Z_i^+} (z^T q_i+\mu_i)\phi_{0,\Lambda_r}(z)dz + \lambda\int_{Z_i^-} (e^{z^T q_i+\mu_i}-1)\phi_{0,\Lambda_r}(z)dz
\end{aligned} \tag{90}
$$

Yet, from the equation (17), we have:

$$
\int_{Z_i^+} (z^T q_i+\mu_i)\phi_{0,\Lambda_r}(z)dz = c_i\phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \mu_i\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)
$$

and:

$$
\begin{aligned}
\int_{Z_i^-} (e^{z^T q_i+\mu_i}-1)\phi_{0,\Lambda_r}(z)dz &= e^{\mu_i}\int_{Z_i^-} e^{z^T q_i}\phi_{0,\Lambda_r}(z)dz - \int_{Z_i^-}\phi_{0,\Lambda_r}(z)dz \\
&= e^{\mu_i}\int_{-\infty}^{-\frac{\mu_i}{c_i}} e^{c_i b_r}\phi_{0,1}(b_r)db_r - \int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r)db_r \\
&= e^{\mu_i+\frac{c_i^2}{2}}\int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r-c_i)db_r - (1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)) \\
&= e^{\mu_i+\frac{c_i^2}{2}}(1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}+c_i\right)) - 1 + \Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)
\end{aligned} \tag{91}
$$

Thus, we obtain:

$$\begin{aligned}\mathbb{E}[Y_i] &= c_i\phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \mu_i\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \lambda e^{\mu_i+\frac{c_i^2}{2}}(1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}+c_i\right)) \\ &- \lambda(1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)) \end{aligned} \tag{92}$$

By noting: $\phi_i = \phi_{0,1}\left(\frac{\mu_i}{c_i}\right)$, $\Phi_i = \Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)$ and $\Phi_{c_i} = \Phi_{0,1}\left(\frac{\mu_i}{c_i}+c_i\right)$ we have:

$$\boxed{\mathbb{E}[Y_i] = c_i\phi_i + (\mu_i+\lambda)\Phi_i + \lambda e^{\mu_i+\frac{c_i^2}{2}}(1-\Phi_{c_i}) - \lambda} \tag{93}$$

D.2. Variance value estimation

$$\begin{aligned}\mathbb{E}[Y_i^2] &= \mathbb{E}_{z\sim p(Z)}\left[f_i(Q_r z+\mu)^2\right] \\ &= \int_{-\infty}^{+\infty} f_i(Q_r z+\mu)^2\phi_{0,\Lambda_r}(z)dz \\ &= \int_{Z_i^+}(z^T q_i+\mu_i)^2\phi_{0,\Lambda_r}(z)dz + \lambda^2\int_{Z_i^-}(e^{z^T q_i+\mu_i}-1)^2\phi_{0,\Lambda_r}(z)dz\end{aligned}$$

Yet, by the equation (23):

$$\int_{Z_i^+}(z^T q_i+\mu_i)^2\phi_{0,\Lambda_r}(z)dz = (\mu_i^2+c_i^2)\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) + \mu_i c_i\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)$$

Now:

$$\begin{aligned}\int_{Z_i^-}(e^{z^T q_i+\mu_i}-1)^2\phi_{0,\Lambda_r}(z)dz &= e^{2\mu_i}\int_{Z_i^-} e^{2z^T q_i}\phi_{0,\Lambda_r}(z)dz \\ &- 2e^{\mu_i}\int_{Z_i^-} e^{z^T q_i}\phi_{0,\Lambda_r}(z)dz + \int_{Z_i^-}\phi_{0,\Lambda_r}(z)dz\end{aligned} \tag{94}$$

with:

$$\int_{Z_i^-}\phi_{0,\Lambda_r}(z)dz = 1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right) \tag{95}$$

$$\begin{aligned}\int_{Z_i^-} e^{z^T q_i}\phi_{0,\Lambda_r}(z)dz &= \int_{-\infty}^{-\frac{\mu_i}{c_i}} e^{c_i b_r}\phi_{0,1}(b_r)db_r \\ &= e^{\frac{c_i^2}{2}}\int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r-c_i)db_r \\ &= e^{\frac{c_i^2}{2}}(1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}+c_i\right))\end{aligned} \tag{96}$$

$$\begin{aligned}\int_{Z_i^-} e^{2z^T q_i}\phi_{0,\Lambda_r}(z)dz &= e^{2c_i^2}\int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r-2c_i)db_r \\ &= e^{2c_i^2}(1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}+2c_i\right))\end{aligned} \tag{97}$$

Then:

$$\begin{aligned}\int_{Z_i^-}(e^{z^T q_i+\mu_i}-1)^2\phi_{0,\Lambda_r}(z)dz &= e^{2(c_i^2+\mu_i)}(1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}+2c_i\right)) \\ &- 2e^{\mu_i+\frac{c_i^2}{2}}(1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}+c_i\right))+1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)\end{aligned}$$

By noting: $\phi_i=\phi_{0,1}\left(\frac{\mu_i}{c_i}\right)$, $\Phi_i=\Phi_{0,1}\left(\frac{\mu_i}{c_i}\right)$, $\Phi_{c_i}=\Phi_{0,1}\left(\frac{\mu_i}{c_i}+c_i\right)$ and $\Phi_{2c_i}=\Phi_{0,1}\left(\frac{\mu_i}{c_i}+2c_i\right)$ we have:

$$\begin{aligned}\mathbb{E}[Y_i^2] &= (\mu_i^2+c_i^2)\Phi_i+\mu_i c_i\phi_i+\lambda^2 e^{2(c_i^2+\mu_i)}(1-\Phi_{2c_i})-2\lambda^2 e^{\mu_i+\frac{c_i^2}{2}}(1-\Phi_{c_i}) \\ &+ \lambda^2(1-\Phi_i)\end{aligned} \tag{98}$$

We finally obtain:

$$\boxed{\mathbb{E}[Y_i^2]=(\mu_i^2+c_i^2-\lambda^2)\Phi_i+\mu_i c_i\phi_i+\lambda^2 e^{2(c_i^2+\mu_i)}(1-\Phi_{2c_i})-2\lambda^2 e^{\mu_i+\frac{c_i^2}{2}}(1-\Phi_{c_i})+\lambda^2} \tag{99}$$

D.3. Covariance value estimation

Let's note f_{ij} such that:

$$\begin{aligned}f_{ij}\colon \mathbb{R}^n &\to \mathbb{R}_+ \\ x &\mapsto f_i(x)f_j(x)\end{aligned}$$

with f_i defined as in (89).

$$\begin{aligned}\mathbb{E}[Y_iY_j] &= \mathbb{E}_{z\sim p(Z)}\left[f_{ij}(Q_r z+\mu)\right] \\ &= \mathbb{E}_{z\sim p(Z)}\left[f_i(Q_r z+\mu)f_j(Q_r z+\mu)\right]\end{aligned} \tag{100}$$

By noting:

$$\begin{aligned}Z_{ij}^+ &= \left\{z\in\mathbb{R}^r : z^T q_i+\mu_i\geq 0,\ z^T q_j+\mu_j\geq 0\right\} \\ Z_{ij}^- &= \left\{z\in\mathbb{R}^r : z^T q_i+\mu_i< 0,\ z^T q_j+\mu_j< 0\right\} \\ Z_{ij}^{-+} &= \left\{z\in\mathbb{R}^r : z^T q_i+\mu_i\geq 0,\ z^T q_j+\mu_j< 0\right\} \\ Z_{ij}^{+-} &= \left\{z\in\mathbb{R}^r : z^T q_i+\mu_i< 0,\ z^T q_j+\mu_j\geq 0\right\}\end{aligned}$$

We have:

$$
\begin{aligned}
\mathbb{E}[Y_iY_j] &= \int_{z_{ij}^{+}} f_{ij}(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz \\
&+ \int_{z_{ij}^{-}} f_{ij}(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz \\
&+ \int_{z_{ij}^{+-}} f_{ij}(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz \\
&+ \int_{z_{ij}^{-+}} f_{ij}(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz \qquad (101)\\
&= (+)+(-)+(+-)+(-+) \qquad (102)
\end{aligned}
$$

D.3.1. DERIVATION OVER Z_{ij}^{-}:

$$
\begin{aligned}
(-) &= \int_{z_{ij}^{-}} f_{ij}(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz \\
&= \lambda^2\int_{z_{ij}^{-}} (e^{z^Tq_i+\mu_i}-1)(e^{z^Tq_j+\mu_j}-1)\phi_{0,\Lambda_r}(z)dz \\
&= \lambda^2[e^{\mu_i+\mu_j}\int_{z_{ij}^{-}} e^{z^Tq_i+z^Tq_j}\phi_{0,\Lambda_r}(z)dz - e^{\mu_i}\int_{z_{ij}^{-}} e^{z^Tq_i}\phi_{0,\Lambda_r}(z)dz \\
&- e^{\mu_j}\int_{z_{ij}^{-}} e^{z^Tq_j}\phi_{0,\Lambda_r}(z)dz + \int_{z_{ij}^{-}}\phi_{0,\Lambda_r}(z)dz] \\
&= \lambda^2[e^{\mu_i+\mu_j}(-*) - e^{\mu_i}(-**)_i - e^{\mu_j}(-**)_j + (-***)] \qquad (103)
\end{aligned}
$$

By equation (59), we have:

$$
(-***) = 1-\Phi_i-\Phi_{ij} \qquad (104)
$$

$$
\begin{aligned}
(-**)_i &= \int_{z_{ij}^{-}} e^{z^Tq_i}\phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}}\int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} e^{c_ib_r}\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_rdb_{r-1} \\
&= e^{\frac{c_i^2}{2}}\int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r-c_i)\Phi_{0,1}\left(-\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\
&= e^{\frac{c_i^2}{2}}\left[\int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i}\phi_{0,1}(b_r)db_r - \int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i}\phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta c_i+\beta b_r}{\alpha}\right)db_r\right] \\
&= e^{\frac{c_i^2}{2}}\left(1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}+c_i\right)-\Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j+\beta c_i}{c_j},-\frac{\mu_i}{c_i}-c_i\right]^T\right)\right) \\
&= e^{\frac{c_i^2}{2}}\left(1-\Phi_{c_i}-\Phi_A\right) \qquad (105)
\end{aligned}
$$

where $\Phi_A = \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j+\beta c_i}{c_j},-\frac{\mu_i}{c_i}-c_i\right]^T\right)$

$$\begin{aligned}
(-**)_j &= \int_{z_{ij}^-} e^{z^T q_j}\phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}}\int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} e^{\alpha b_{r-1}+\beta b_r}\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_{r-1}db_r \\
&= e^{\frac{\alpha^2+\beta^2}{2}}\int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r-\beta)\int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}}\phi_{0,1}(b_{r-1}-\alpha)db_{r-1}db_r \\
&= e^{\frac{c_j^2}{2}}\int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r-\beta)\Phi_{0,1}\left(-\frac{\mu_j+\alpha^2+\beta b_r}{\alpha}\right)db_r \\
&= e^{\frac{c_j^2}{2}}\int_{-\infty}^{-\frac{\mu_i}{c_i}-\beta}\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+c_j^2+\beta b_r}{\alpha}\right)db_r \\
&= e^{\frac{c_j^2}{2}}\left(1-\Phi_{0,1}\left(\frac{\mu_i}{c_i}+\beta\right)-\Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\-\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j}{c_j}+c_j,-\frac{\mu_i}{c_i}-\beta\right]^T\right)\right) \\
&= e^{\frac{c_j^2}{2}}\left(1-\Phi_{\beta_i}-\Phi_B\right)
\end{aligned} \tag{106}$$

where $\Phi_{\beta_i}=\Phi_{0,1}\left(\frac{\mu_i}{c_i}+\beta\right)$, $\Phi_B=\Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\-\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j}{c_j}+c_j,-\frac{\mu_i}{c_i}-\beta\right]^T\right)$ and:

$$\begin{aligned}
(-*) &= \int_{z_{ij}^-} e^{z^T q_i+z^T q_j}\phi_{0,\Lambda_r}(z)dz \\
&= \int_{-\infty}^{-\frac{\mu_i}{c_i}}\int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} e^{c_i b_r+\alpha b_{r-1}+\beta b_r}\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_{r-1}db_r \\
&= e^{\frac{\alpha^2+(c_i+\beta)^2}{2}}\int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r-(c_i+\beta))\int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}}\phi_{0,1}(b_{r-1}-\alpha)db_{r-1}db_r \\
&= e^{\frac{c_j^2+c_i^2+2c_i\beta}{2}}\int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r-(c_i+\beta))\Phi_{0,1}\left(-\frac{\mu_j+\alpha^2+\beta b_r}{\alpha}\right)db_r \\
&= e^{\frac{c_j^2+c_i^2}{2}+c_i\beta}\int_{-\infty}^{-\frac{\mu_i}{c_i}-(c_i+\beta)}\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+c_j^2+\beta c_i+\beta b_r}{\alpha}\right)db_r \\
&= e^{\frac{c_j^2+c_i^2}{2}+c_i\beta}(1-\Phi_{0,1}(\frac{\mu_i}{c_i}+c_i+\beta) \\
&- \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\-\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j+c_j^2+\beta c_i}{c_j},-(\frac{\mu_i}{c_i}+c_i+\beta)\right]^T\right))
\end{aligned} \tag{107}$$

$$= e^{\frac{c_j^2+c_i^2}{2}+c_i\beta}(1-\Phi_{\beta+c_i}-\Phi_C) \tag{108}$$

where: $\Phi_{\beta+c_i}=\Phi_{0,1}(\frac{\mu_i}{c_i}+c_i+\beta)$ and

$\Phi_C = \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j+c_j^2+\beta c_i}{c_j}, -(\frac{\mu_i}{c_i}+c_i+\beta)\right]^T\right)$

Finally:

$$\begin{aligned}
(-) &= \lambda^2[e^{\mu_i+\mu_j}(-*) - e^{\mu_i}(-**)_i - e^{\mu_j}(-**)_j + (-***)] \\
&= \lambda^2[e^{\mu_i+\mu_j+\frac{c_j^2+c_i^2}{2}+c_i\beta}(1-\Phi_{\beta+c_i}-\Phi_C) \\
&- e^{\mu_i+\frac{c_i^2}{2}}(1-\Phi_{c_i}-\Phi_A) - e^{\mu_j+\frac{c_j^2}{2}}(1-\Phi_{\beta_i}-\Phi_B) + 1 - \Phi_i - \Phi_{ij}]
\end{aligned} \tag{109}$$

where $\Phi_A = \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j+\beta c_i}{c_j}, -\frac{\mu_i}{c_i}-c_i\right]^T\right)$,
$\Phi_B = \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j}{c_j}+c_j, -\frac{\mu_i}{c_i}-\beta\right]^T\right)$,
$\Phi_C = \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j+c_j^2+\beta c_i}{c_j}, -(\frac{\mu_i}{c_i}+c_i+\beta)\right]^T\right))$,
and $\Phi_{\beta+c_i} = \Phi_{0,1}(\frac{\mu_i}{c_i}+c_i+\beta)$, $\Phi_{\beta_i} = \Phi_{0,1}\left(\frac{\mu_i}{c_i}+\beta\right)$ and $\Phi_{c_i} = \Phi_{0,1}\left(\frac{\mu_i}{c_i}+c_i\right)$

D.3.2. DERIVATION OVER Z_{ij}^{+-}:

$$\begin{aligned}
(+-) &= \int_{z_{ij}^{+-}} f_{ij}(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz \\
&= \lambda\int_{z_{ij}^{+-}} (z^Tq_i+\mu_i)(e^{z^Tq_j+\mu_j}-1)\phi_{0,\Lambda_r}(z)dz \\
&= \lambda[e^{\mu_j}\int_{z_{ij}^{+-}} z^Tq_ie^{z^Tq_j}\phi_{0,\Lambda_r}(z)dz - \int_{z_{ij}^{+-}} z^Tq_i\phi_{0,\Lambda_r}(z)dz \\
&+ \mu_ie^{\mu_j}\int_{z_{ij}^{+-}} e^{z^Tq_j}\phi_{0,\Lambda_r}(z)dz - \mu_i\int_{z_{ij}^{+-}}\phi_{0,\Lambda_r}(z)dz] \\
&= \lambda[e^{\mu_j}(+-*) - (+-**)_i + \mu_ie^{\mu_j}(+-**)_j - \mu_i(+-***)]
\end{aligned} \tag{110}$$

With (76), we have:

$$(+-***) = \Phi_i - \Phi_j + \Phi_{ij} \tag{111}$$

By the equation (74), we have:

$$\begin{aligned}
(+-**)_i &= \int_{z_{ij}^{+-}} z^Tq_i\phi_{0,\Lambda_r}(z)dz \\
&= c_i\phi_i(1-\Phi_{NS}) - \frac{\beta c_i}{c_j}\phi_j\Phi_S
\end{aligned} \tag{112}$$

and:

$$
\begin{aligned}
(+-**)_j &= \int_{z_{ij}^{+-}} e^{z^T q_j} \phi_{0,\Lambda_r}(z) dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} e^{\alpha b_{r-1}+\beta b_r} \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_{r-1} db_r \\
&= e^{\frac{c_j^2}{2}} \int_{-\frac{\mu_i}{c_i}}^{+\infty} \phi_{0,1}(b_r-\beta) \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} \phi_{0,1}(b_{r-1}-\alpha) db_{r-1} db_r \\
&= e^{\frac{c_j^2}{2}} \int_{-\frac{\mu_i}{c_i}-\beta}^{+\infty} \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+c_j^2+\beta b_r}{\alpha}\right) db_r \\
&= e^{\frac{c_j^2}{2}} (\Phi_{0,1}\left(\frac{\mu_i}{c_i}+\beta\right) - \Phi_{0,1}\left(\frac{\mu_j}{c_j}+c_j\right) \\
&+ \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j}{c_j}+c_j, -(\frac{\mu_i}{c_i}+\beta)\right]^T\right)) \qquad (113) \\
&= e^{\frac{c_j^2}{2}} (\Phi_{\beta_i} - \Phi_{c_j} + \Phi_B) \qquad (114)
\end{aligned}
$$

and:

$$
\begin{aligned}
(+-*) &= \int_{z_{ij}^{+-}} z^T q_i e^{z^T q_j} \phi_{0,\Lambda_r}(z) dz \\
&= \int_{-\frac{\mu_i}{c_i}}^{+\infty} \int_{-\infty}^{-\frac{\mu_j+\beta b_r}{\alpha}} (c_i b_r) e^{\alpha b_{r-1}+\beta b_r} \phi_{0,1}(b_r)\phi_{0,1}(b_{r-1}) db_r db_{r-1} \\
&= c_i e^{\frac{c_j^2}{2}} \int_{-\frac{\mu_i}{c_i}}^{+\infty} b_r \phi_{0,1}(b_r-\beta)\Phi_{0,1}\left(-\frac{\mu_j+\alpha^2+\beta b_r}{\alpha}\right) db_r \\
&= c_i e^{\frac{c_j^2}{2}} [\int_{-\frac{\mu_i}{c_i}-\beta}^{+\infty} b_r \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+c_j^2+\beta b_r}{\alpha}\right) db_r \\
&+ \beta \int_{-\frac{\mu_i}{c_i}-\beta}^{+\infty} \phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j+c_j^2+\beta b_r}{\alpha}\right) db_r] \\
&= c_i e^{\frac{c_j^2}{2}} [(+-*)_1 + \beta(+-*)_2] \qquad (115)
\end{aligned}
$$

With:

$$\begin{aligned}
(+-*)_2 &= \int_{-\frac{\mu_i}{c_i}-\beta}^{+\infty} \phi_{0,1}(b_r)db_r - \int_{-\frac{\mu_i}{c_i}-\beta}^{+\infty} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j + c_j^2 + \beta b_r}{\alpha}\right) db_r \\
&= \int_{-\frac{\mu_i}{c_i}-\beta}^{+\infty} \phi_{0,1}(b_r)db_r - \int_{-\infty}^{+\infty} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j + c_j^2 + \beta b_r}{\alpha}\right) db_r \\
&+ \int_{-\infty}^{-\frac{\mu_i}{c_i}-\beta} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j + c_j^2 + \beta b_r}{\alpha}\right) db_r \\
&= \Phi_{0,1}\left(\frac{\mu_i}{c_i} + \beta\right) - \Phi_{0,1}\left(\frac{\mu_j}{c_j} + c_j\right) \\
&+ \Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j}{c_j} + c_j, -(\frac{\mu_i}{c_i} + \beta)\right]^T\right) \\
&= \Phi_{\beta_i} - \Phi_{c_j} + \Phi_B
\end{aligned} \tag{116}$$

and:

$$\begin{aligned}
(+-*)_1 &= \left[-\phi_{0,1}(b_r)\Phi_{0,1}\left(-\frac{\mu_j + c_j^2 + \beta b_r}{\alpha}\right)\right]_{-\frac{\mu_i}{c_i}-\beta}^{+\infty} \\
&- \frac{\beta}{\alpha}\int_{-\frac{\mu_i}{c_i}-\beta}^{+\infty} \phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j + c_j^2 + \beta b_r}{\alpha}\right) db_r \\
&= \phi_{0,1}\left(\frac{\mu_i}{c_i} + \beta\right)\Phi_{0,1}\left(-\frac{\mu_j + c_j^2 - \beta^2}{\alpha} + \frac{\beta\mu_i}{\alpha c_i}\right) \\
&- \frac{\beta}{c_j}\phi_{0,1}\left(\frac{\mu_j}{c_j} + c_j\right)\Phi_{0,1}\left(\frac{c_j\mu_i}{\alpha c_i} + \frac{\beta c_j}{\alpha} - \frac{\beta}{\alpha}(\frac{\mu_j}{c_j} + c_j)\right) \\
&= \phi_{\beta_i}\Phi_{0,1}\left(\frac{\beta\mu_i}{\alpha c_i} - \frac{\mu_j}{\alpha} - \alpha\right) - \frac{\beta}{c_j}\Phi_{0,1}\left(\frac{c_j\mu_i}{\alpha c_i} - \frac{\beta\mu_j}{\alpha c_j}\right) \\
&= \phi_{\beta_i}\Phi_\alpha - \frac{\beta}{c_j}\phi_{c_j}\Phi_S
\end{aligned} \tag{117}$$

With $\Phi_\alpha = \Phi_{0,1}\left(\frac{\beta\mu_i}{\alpha c_i} - \frac{\mu_j}{\alpha} - \alpha\right)$.

Then:

$$(+-*) = c_i e^{\frac{c_j^2}{2}}[\phi_{\beta_i}\Phi_\alpha - \frac{\beta}{c_j}\phi_{c_j}\Phi_S + \beta(\Phi_{\beta_i} - \Phi_{c_j} + \Phi_B)] \tag{118}$$

Thus, we have:

$$\begin{aligned}
(+-) &= \lambda[e^{\mu_j}(+-*) - (+-**)_i + \mu_i e^{\mu_j}(+-**)_j - \mu_i(+-***)] \\
&= \lambda[c_i e^{\mu_j + \frac{c_j^2}{2}}(\phi_{\beta_i}\Phi_\alpha - \frac{\beta}{c_j}\phi_{c_j}\Phi_S + \beta(\Phi_{\beta_i} - \Phi_{c_j} + \Phi_B)) - c_i\phi_i(1 - \Phi_{NS}) \\
&+ \frac{\beta c_i}{c_j}\phi_j\Phi_S + \mu_i e^{\mu_j + \frac{c_j^2}{2}}(\Phi_{\beta_i} - \Phi_{c_j} + \Phi_B) - \mu_i(\Phi_i - \Phi_j + \Phi_{ij})] \\
&= \lambda[e^{\mu_j + \frac{c_j^2}{2}}(\mu_i + \beta c_i)(\Phi_{\beta_i} - \Phi_{c_j} + \Phi_B) + \frac{\beta c_i}{c_j}(\phi_j - e^{\mu_j + \frac{c_j^2}{2}}\phi_{c_j})\Phi_S \\
&+ c_i e^{\mu_j + \frac{c_j^2}{2}}\phi_{\beta_i}\Phi_\alpha - c_i\phi_i(1 - \Phi_{NS}) - \mu_i(\Phi_i - \Phi_j + \Phi_{ij})]
\end{aligned} \tag{119}$$

D.3.3. DERIVATION OVER Z_{ij}^{-+}:

$$\begin{aligned}(-+) &= \int_{z_{ij}^{-+}} f_{ij}(Q_r z+\mu)\phi_{0,\Lambda_r}(z)dz \\ &= \lambda\int_{z_{ij}^{-+}} (e^{z^Tq_i+\mu_i}-1)(z^Tq_j+\mu_j)\phi_{0,\Lambda_r}(z)dz \\ &= \lambda[e^{\mu_i}\int_{z_{ij}^{-+}} e^{z^Tq_i}z^Tq_j\phi_{0,\Lambda_r}(z)dz - \int_{z_{ij}^{-+}} z^Tq_j\phi_{0,\Lambda_r}(z)dz \\ &+ \mu_j e^{\mu_i}\int_{z_{ij}^{-+}} e^{z^Tq_i}\phi_{0,\Lambda_r}(z)dz - \mu_j\int_{z_{ij}^{-+}}\phi_{0,\Lambda_r}(z)dz]\end{aligned} \tag{120}$$

With (85), we have:

$$(-+***) = \Phi_{ij} \tag{121}$$

By the equation (84), we have:

$$\begin{aligned}(-+**)_j &= \int_{z_{ij}^{-+}} z^Tq_j\phi_{0,\Lambda_r}(z)dz \\ &= -\beta\phi_i\Phi_{NS}+c_j\phi_j(1-\Phi_S)\end{aligned} \tag{122}$$

and:

$$\begin{aligned}(-+**)_i &= \int_{z_{ij}^{-+}} e^{z^Tq_i}\phi_{0,\Lambda_r}(z)dz \\ &= \int_{-\infty}^{-\frac{\mu_i}{c_i}}\int_{-\frac{\mu_j+\beta b_r}{\alpha}}^{+\infty} e^{c_ib_r}\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_{r-1}db_r \\ &= e^{\frac{c_i^2}{2}}\int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i}\phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j+\beta c_i+\beta b_r}{\alpha}\right)db_r \\ &= e^{\frac{c_i^2}{2}}\Phi_{\begin{bmatrix}0\\0\end{bmatrix},\begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j+\beta c_i}{c_j}, -(\frac{\mu_i}{c_i}+c_i)\right]^T\right) \\ &= e^{\frac{c_i^2}{2}}\Phi_A\end{aligned} \tag{123}$$

Finally,

$$\begin{aligned}(-+*) &= \int_{z_{ij}^{-+}} z^Tq_je^{z^Tq_i}\phi_{0,\Lambda_r}(z)dz \\ &= \int_{-\infty}^{-\frac{\mu_i}{c_i}}\int_{-\frac{\mu_j+\beta b_r}{\alpha}}^{+\infty}(\alpha b_{r-1}+\beta b_r)e^{c_ib_r}\phi_{0,1}(b_r)\phi_{0,1}(b_{r-1})db_{r-1}db_r \\ &= e^{\frac{c_i^2}{2}}[\alpha\int_{-\infty}^{-\frac{\mu_i}{c_i}}\phi_{0,1}(b_r-c_i)\phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r \\ &+ \beta\int_{-\infty}^{-\frac{\mu_i}{c_i}} b_r\phi_{0,1}(b_r-c_i)\Phi_{0,1}\left(\frac{\mu_j+\beta b_r}{\alpha}\right)db_r]\end{aligned} \tag{124}$$

With:

$$\begin{aligned}
\int_{-\infty}^{-\frac{\mu_i}{c_i}} & \phi_{0,1}(b_r - c_i)\phi_{0,1}\left(\frac{\mu_j + \beta b_r}{\alpha}\right) db_r = \int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i} \phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j + \beta c_i + \beta b_r}{\alpha}\right) db_r \\
= & \quad \phi_{0,1}\left(\frac{\mu_j + \beta c_i}{c_j}\right) \int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i} \phi_{0,1}\left(\frac{c_j}{\alpha} b_r + \frac{\beta}{\alpha}\frac{\mu_j + \beta c_i}{c_j}\right) db_r \\
= & \quad \frac{\alpha}{c_j}\phi_{0,1}\left(\frac{\mu_j + \beta c_i}{c_j}\right) \Phi_{0,1}\left(\frac{\beta}{\alpha}\frac{\mu_j + \beta c_i}{c_j} - \frac{c_j \mu_i}{\alpha c_i} - \frac{c_j c_i}{\alpha}\right)
\end{aligned} \tag{125}$$

and:

$$\begin{aligned}
\int_{-\infty}^{-\frac{\mu_i}{c_i}} & b_r \phi_{0,1}(b_r - c_i)\Phi_{0,1}\left(\frac{\mu_j + \beta b_r}{\alpha}\right) db_r = \int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i} b_r \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j + \beta c_i + \beta b_r}{\alpha}\right) db_r \\
+ & \quad c_i \int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j + \beta c_i + \beta b_r}{\alpha}\right) db_r
\end{aligned}$$

With:

$$\int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i} \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j + \beta c_i + \beta b_r}{\alpha}\right) db_r \quad = \quad \Phi_A \tag{126}$$

and:

$$\begin{aligned}
\int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i} & b_r \phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j + \beta c_i + \beta b_r}{\alpha}\right) db_r = \left[-\phi_{0,1}(b_r)\Phi_{0,1}\left(\frac{\mu_j + \beta c_i + \beta b_r}{\alpha}\right)\right]_{-\infty}^{-\frac{\mu_i}{c_i}-c_i} \\
+ & \quad \frac{\beta}{\alpha}\int_{-\infty}^{-\frac{\mu_i}{c_i}-c_i} \phi_{0,1}(b_r)\phi_{0,1}\left(\frac{\mu_j + \beta c_i + \beta b_r}{\alpha}\right) db_r \\
= & \quad -\phi_{0,1}\left(\frac{\mu_i}{c_i} + c_i\right)\Phi_{0,1}\left(\frac{\mu_j}{\alpha} - \frac{\beta\mu_i}{\alpha c_i}\right) \\
+ & \quad \frac{\beta}{c_j}\phi_{0,1}\left(\frac{\mu_j + \beta c_i}{c_j}\right)\Phi_{0,1}\left(\frac{\beta}{\alpha}\frac{\mu_j + \beta c_i}{c_j} - \frac{c_j \mu_i}{\alpha c_i} - \frac{c_j c_i}{\alpha}\right)
\end{aligned}$$

Then:

$$\begin{aligned}
(-+*) \quad = & \quad e^{\frac{c_i^2}{2}}[\frac{\alpha^2 + \beta^2}{c_j}\phi_{0,1}\left(\frac{\mu_j + \beta c_i}{c_j}\right)\Phi_{0,1}\left(\frac{\beta}{\alpha}\frac{\mu_j + \beta c_i}{c_j} - \frac{c_j \mu_i}{\alpha c_i} - \frac{c_j c_i}{\alpha}\right) \\
- & \quad \beta\phi_{0,1}\left(\frac{\mu_i}{c_i} + c_i\right)\Phi_{0,1}\left(\frac{\mu_j}{\alpha} - \frac{\beta\mu_i}{\alpha c_i}\right) \\
+ & \quad \beta c_i \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j + \beta c_i}{c_j}, -(\frac{\mu_i}{c_i} + c_i)\right]^T\right)] \\
= & \quad e^{\frac{c_i^2}{2}}[c_j\phi_{0,1}\left(\frac{\mu_j + \beta c_i}{c_j}\right)\Phi_{0,1}\left(\frac{\beta}{\alpha}\frac{\mu_j + \beta c_i}{c_j} - \frac{c_j \mu_i}{\alpha c_i} - \frac{c_j c_i}{\alpha}\right) \\
- & \quad \beta\phi_{0,1}\left(\frac{\mu_i}{c_i} + c_i\right)\Phi_{0,1}\left(\frac{\mu_j}{\alpha} - \frac{\beta\mu_i}{\alpha c_i}\right) \\
+ & \quad \beta c_i \Phi_{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1 & -\frac{\beta}{c_j}\\ -\frac{\beta}{c_j} & 1\end{bmatrix}}\left(\left[\frac{\mu_j + \beta c_i}{c_j}, -(\frac{\mu_i}{c_i} + c_i)\right]^T\right)] \\
= & \quad e^{\frac{c_i^2}{2}}[c_j\phi_a\Phi_e - \beta\phi_{c_i}\Phi_{NS} + \beta c_i \Phi_A]
\end{aligned} \tag{127}$$

where $\phi_a = \phi_{0,1}\left(\frac{\mu_j+\beta c_i}{c_j}\right)$ and $\Phi_e = \Phi_{0,1}\left(\frac{\beta}{\alpha}\frac{\mu_j+\beta c_i}{c_j} - \frac{c_j\mu_i}{\alpha c_i} - \frac{c_j c_i}{\alpha}\right)$.

Then:

$$\begin{aligned}
(-+) &= \lambda[e^{\mu_i}(-+*) - (-+**)_j + \mu_j e^{\mu_i}(-+**)_i - \mu_j(-+***)] \\
&= \lambda[e^{\mu_i+\frac{c_i^2}{2}}(c_j\phi_a\Phi_e - \beta\phi_{c_i}\Phi_{NS} + \beta c_i\Phi_A) + \beta\phi_i\Phi_{NS} - c_j\phi_j(1-\Phi_S) \\
&+ \mu_j e^{\mu_i+\frac{c_i^2}{2}}\Phi_A - \mu_j\Phi_{ij}] \\
&= \lambda[e^{\mu_i+\frac{c_i^2}{2}}(\beta c_i+\mu_j)\Phi_A + \beta(\phi_i - e^{\mu_i+\frac{c_i^2}{2}}\phi_{c_i})\Phi_{NS} + c_j e^{\mu_i+\frac{c_i^2}{2}}\phi_a\Phi_e \\
&- c_j\phi_j(1-\Phi_S) - \mu_j\Phi_{ij}]
\end{aligned} \tag{128}$$

D.3.4. Final derivation:

Finally, by the equation (46), we have:

$$(+) = \mathbb{E}[Y_iY_j] = c_i\alpha\phi_j\phi_{NS} + (\mu_i\mu_j + c_i\beta)(\Phi_j - \Phi_{ij}) + \mu_i c_j\phi_j\Phi_S + \mu_j c_i\phi_i\Phi_{NS}$$

Then:

$$\mathbb{E}[Y_iY_j] = (+) + (-) + (+-) + (-+) \tag{129}$$

Finally, we obtain:

$$\begin{aligned}
\mathbb{E}[Y_iY_j] &= c_i\alpha\phi_j\phi_{NS} + (\mu_i\mu_j + c_i\beta)(\Phi_j - \Phi_{ij}) + \mu_i c_j\phi_j\Phi_S + \mu_j c_i\phi_i\Phi_{NS} \\
&+ \lambda^2[e^{\mu_i+\mu_j+\frac{c_j^2+c_i^2}{2}+c_i\beta}(1-\Phi_{\beta+c_i} - \Phi_C) \\
&- e^{\mu_i+\frac{c_i^2}{2}}(1-\Phi_{c_i} - \Phi_A) - e^{\mu_j+\frac{c_j^2}{2}}(1-\Phi_{\beta_i} - \Phi_B) + 1 - \Phi_i - \Phi_{ij}] \\
&+ \lambda[e^{\mu_j+\frac{c_j^2}{2}}(\mu_i+\beta c_i)(\Phi_{\beta_i} - \Phi_{c_j} + \Phi_B) + \frac{\beta c_i}{c_j}(\phi_j - e^{\mu_j+\frac{c_j^2}{2}}\phi_{c_j})\Phi_S \\
&+ c_i e^{\mu_j+\frac{c_j^2}{2}}\phi_{\beta_i}\Phi_\alpha - c_i\phi_i(1-\Phi_{NS}) - \mu_i(\Phi_i - \Phi_j + \Phi_{ij})] \\
&+ \lambda[e^{\mu_i+\frac{c_i^2}{2}}(\beta c_i+\mu_j)\Phi_A + \beta(\phi_i - e^{\mu_i+\frac{c_i^2}{2}}\phi_{c_i})\Phi_{NS} + c_j e^{\mu_i+\frac{c_i^2}{2}}\phi_a\Phi_e \\
&- c_j\phi_j(1-\Phi_S) - \mu_j\Phi_{ij}]
\end{aligned} \tag{130}$$

Thus:

$$
\begin{aligned}
\mathbb{E}[Y_iY_j] \;=\;& c_i\alpha\phi_j\phi_{NS} \\
+\;& (\mu_i\mu_j + c_i\beta + \lambda\mu_i)\,\Phi_j \\
-\;& \left(\mu_i\mu_j + c_i\beta + \lambda(\mu_i+\mu_j) + \lambda^2\right)\Phi_{ij} \\
+\;& (\mu_i c_j\phi_j + \lambda c_j\phi_j + \frac{\lambda\beta c_i}{c_j}((\phi_j - e^{\mu_j+\frac{c_j^2}{2}}\phi_{c_j}))\Phi_S \\
+\;& (\mu_j c_i\phi_i + \lambda c_i\phi_i + \lambda\beta(\phi_i - e^{\mu_i+\frac{c_i^2}{2}}\phi_{c_i}))\Phi_{NS} \\
+\;& \lambda c_i e^{\mu_j+\frac{c_j^2}{2}}\phi_{\beta_i}\Phi_\alpha \\
+\;& \lambda c_j e^{\mu_i+\frac{c_i^2}{2}}\phi_a\Phi_e \\
+\;& \lambda^2 e^{\mu_i+\frac{c_i^2}{2}}\Phi_{c_i} \\
-\;& \lambda e^{\mu_j+\frac{c_j^2}{2}}(\mu_i+\beta c_i)\Phi_{c_j} \\
+\;& \lambda e^{\mu_i+\frac{c_i^2}{2}}(\lambda+\beta c_i+\mu_j)\Phi_A \\
+\;& \lambda e^{\mu_j+\frac{c_j^2}{2}}(\lambda+\beta c_i+\mu_i)(\Phi_{\beta_i}+\Phi_B) \\
+\;& \lambda^2 e^{\mu_i+\mu_j+\frac{c_i^2+c_j^2}{2}+c_i\beta}(1-\Phi_{\beta+c_i}-\Phi_C) \\
-\;& \lambda(\lambda e^{\mu_i+\frac{c_i^2}{2}} + \lambda e^{\mu_j+\frac{c_j^2}{2}} + c_i\phi_i + c_j\phi_j - \lambda)
\end{aligned}
\tag{131}
$$

References

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