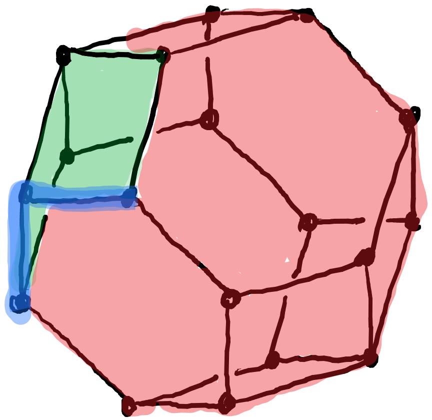
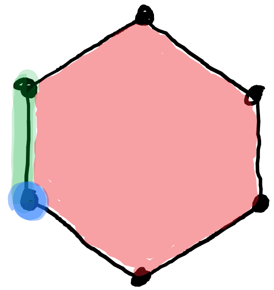
Topology of supersolvable oriented matroids



$$\mathcal{L}^\vee \xrightarrow{\rho_X} \mathcal{L}_X^\vee$$



$$\begin{tikzcd}
 & (\tilde{\rho}_X \downarrow (0,T)) & \\
(\tilde{\rho}_X \downarrow a) \arrow[ur, hook, "\simeq"] \arrow[rr, hook, "\simeq"] & & (\tilde{\rho}_X \downarrow b) \arrow[ul, hook', "\simeq"']
\end{tikzcd}$$

1. Motivation & Overview
2. Main Result & Consequences
3. About the proof

1 Motivation & Overview

- $\mathcal{A}$ Arrangement in $V \cong \mathbb{C}^\ell$, $M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H$ the complement
- $L(\mathcal{A}) := \{ \bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A} \}$ intersection lattice (order : $X \leq Y \Leftrightarrow X \supseteq Y$)
- $X \in L(\mathcal{A})$ modular $:\Leftrightarrow \forall Y \in L(\mathcal{A}) : X + Y \in L(\mathcal{A})$

  [$\overset{\text{comb.}}{\Leftrightarrow}$ $\forall\ Y, Z \in L(\mathcal{A})$ with $Y \leq Z$: $Z \vee (X \wedge Y) = (Z \vee X) \wedge Y$

  where • $X_1 \vee X_2 := X_1 \cap X_2$

  • $X_1 \wedge X_2 := \sup \{ Z \in L(\mathcal{A}) \mid X_1 \supseteq Z$ and $X_2 \supseteq Z \}$]

- $L(\mathcal{A})$ resp. $\mathcal{A}$ is supersolvable

  $:\Leftrightarrow \exists \quad X_0 < X_1 < \ldots < X_r$ max chain in $L(\mathcal{A})$

  s.t. X_i modular $\forall\ i = 0, \ldots, r$.

__Thm__ [Falk-Randell '85, Terao '86]

If $X \in L(\mathcal{A})$ is modular with $\operatorname{codim}(X) = \operatorname{rk}(\mathcal{A}) - 1$ then the map $V \xrightarrow{p} V/X$ restricted to $M(\mathcal{A})$:

[$\mathcal{A}_X := \{ H \in \mathcal{A} \mid X \subseteq H \}$ localization]

$$p|_{M(\mathcal{A})} : M(\mathcal{A}) \longrightarrow M(\mathcal{A}_X / X)$$

is a fibre bundle map with fibre $\cong \mathbb{C} \setminus \{ z_1, \ldots, z_r \}$ $(\cong M(d\mathcal{B}), \mathcal{B}$ rk $2)$

__Corollary__: $\mathcal{A}$ supersolvable $\Rightarrow$ $M(\mathcal{A})$ is a $K(\pi,1)$-space [$:\Leftrightarrow \widetilde{M(\mathcal{A})}$ contractible]

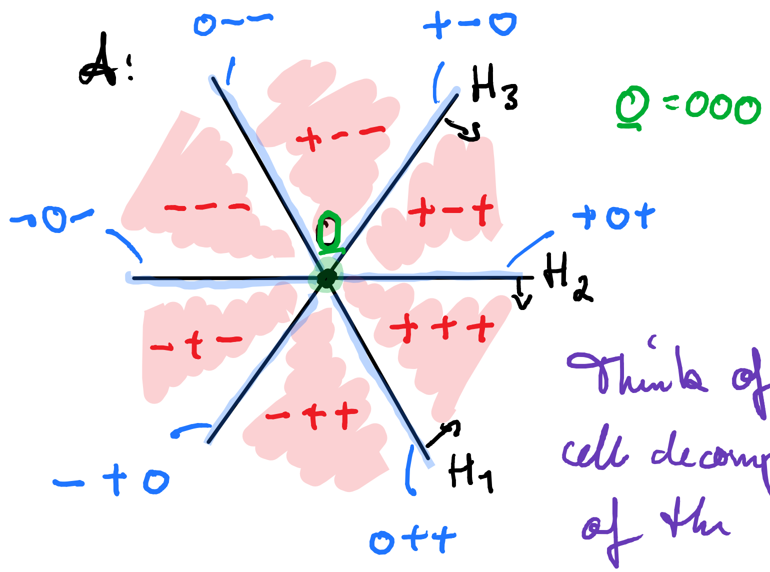
1.2 Oriented matroids

- $\mathcal{A}$ is an arrangement in $V \cong \mathbb{R}^\ell$, fix $\alpha_H \in V^*$ for each $H \in \mathcal{A}$
- Def. $\mathcal{L}(\mathcal{A}) := \{ (\mathrm{sgn}(\alpha_H(v)) \mid H \in \mathcal{A}) \mid v \in V \} \subseteq \{+,-,0\}^{\mathcal{A}}$

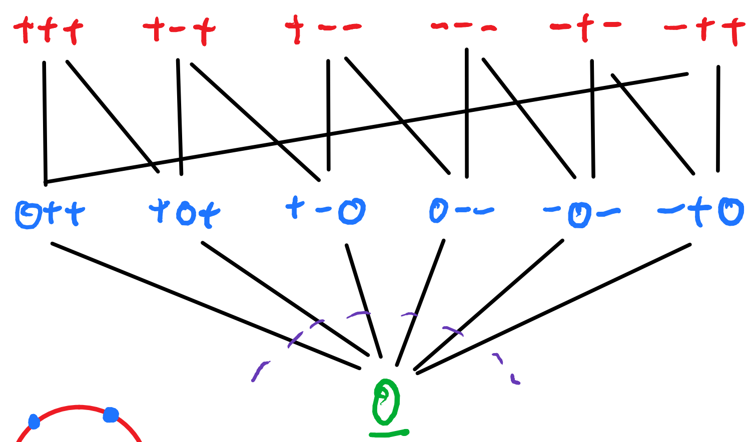
  with partial order induced by $\begin{smallmatrix} + & & - \\ & 0 & \end{smallmatrix}$ $\rightsquigarrow$ product order on $\{+,-,0\}^{\mathcal{A}}$

  poset of covectors of $\mathcal{A}$

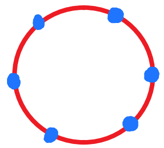
Ex.: $\mathcal{A}$:



$\mathcal{L}(\mathcal{A})$:



Think of regular cell decomposition of the sphere:



Combinatorial abstraction of these sign-posets = oriented matroids

"Def": An oriented matroid $\mathcal{M} = (E, \mathcal{L})$ consists of

- E a finite set
- $\mathcal{L} \subseteq \{+,-,0\}^E$ called covectors of $\mathcal{M}$

subject to some axioms ... [.., not here].

1.3. Back to the Topology of Arrangement complements

Thm [Deligne 1972]
If $\mathcal{A}$ is a real simplicial arrangement ($:\Leftrightarrow$ $\mathcal{L}(\mathcal{A})$ is a simplicial complex),
then $M(\mathcal{A}\otimes\mathbb{C})$ is a $K(\pi,1)$-space.

- $\sigma,\tau \in \mathcal{L}$ $\rightsquigarrow$ $\sigma\circ\tau$ is defined by : $(\sigma\circ\tau)_e := \begin{cases} \sigma_e & \text{if } \sigma_e \neq 0 \\ \tau_e & \text{else} \end{cases}$. $\forall e\in E$
- $\mathcal{T} :=$ maximal Elements in $\mathcal{L}$, called Topes

Def [Salvetti 1987, Gel'fand-Rybnikov 1990]
The Salvetti poset $S(\mathcal{A})$ of a real arrangement $\mathcal{A}$ / oriented matroid $\mathcal{M}$ is :

$$S(\mathcal{A}) / S(\mathcal{M}) := \{ (\sigma, T) \mid T \in \mathcal{T},\ \sigma \in \mathcal{L}_{\leq T} \}$$

with partial order :

$$(\sigma, T) \leq (\tau, R) \;:\Leftrightarrow\; \tau \leq_{\mathcal{L}} \sigma \text{ and } \sigma\circ R = T$$

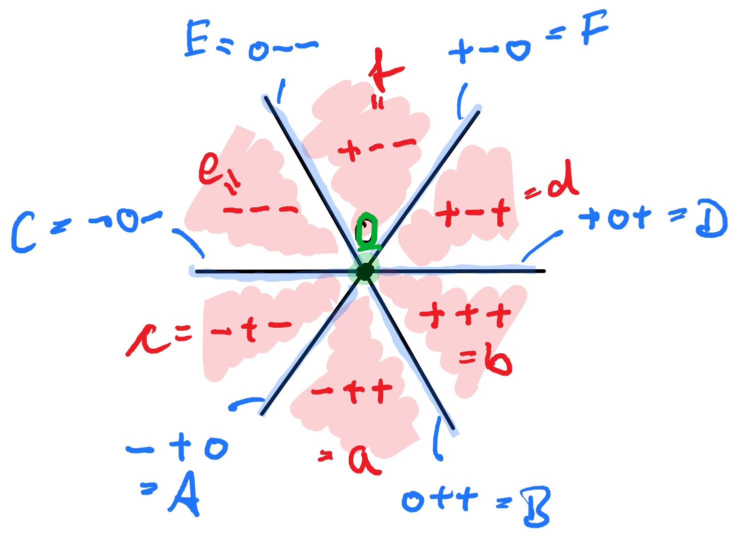
Thm [Salvetti 1987]
$S(\mathcal{A})$ is the face poset of a regular CW complex homotopy equivalent to $M(\mathcal{A}\otimes\mathbb{C})$.

[All regular CW cpxes are regarded as posets]

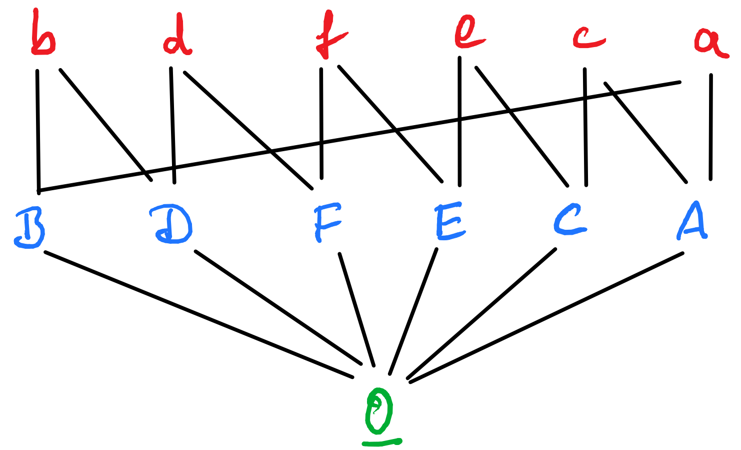
Corollary : $S(\mathcal{A})$ is a finite $K(\pi,1)$ complex if $\mathcal{A}$ is simplicial.

An example of $\mathcal{S}(\mathcal{A})$

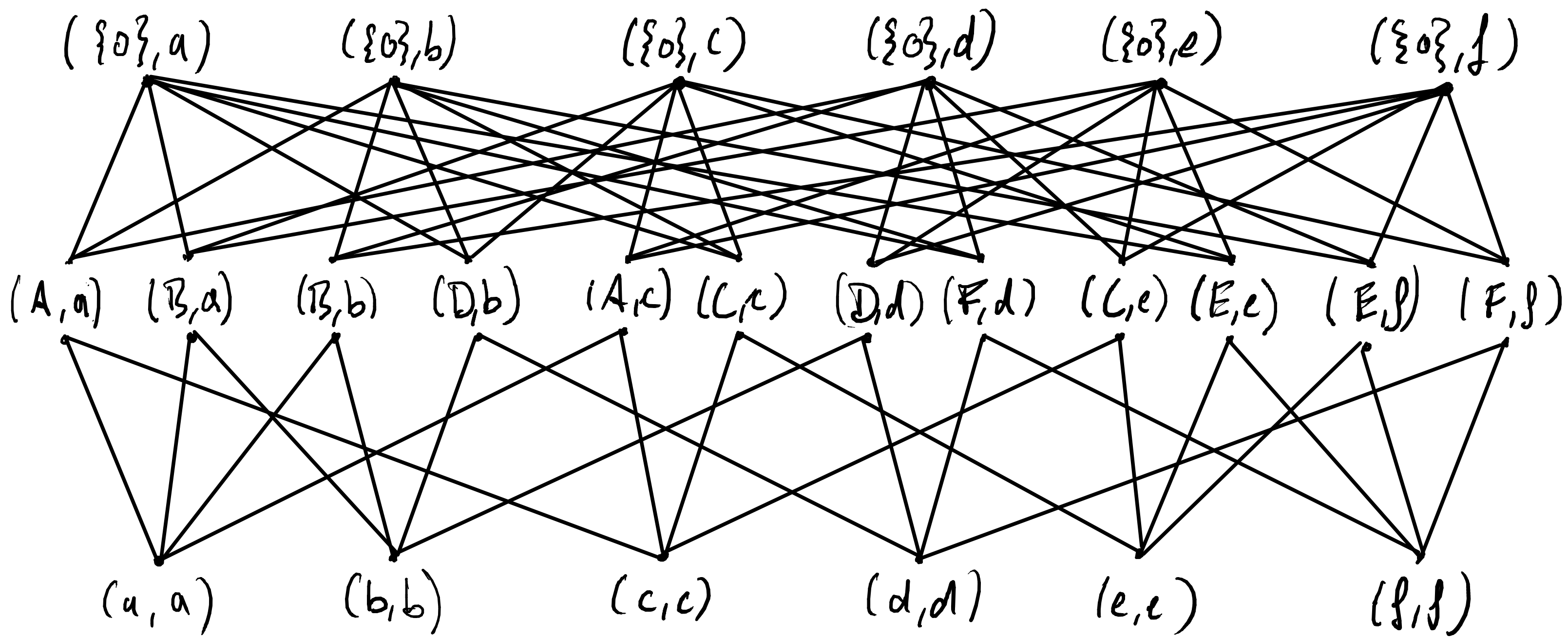
$\mathcal{A}$:



$\mathcal{L}(\mathcal{A})$:



$\mathcal{S}(\mathcal{A})$:



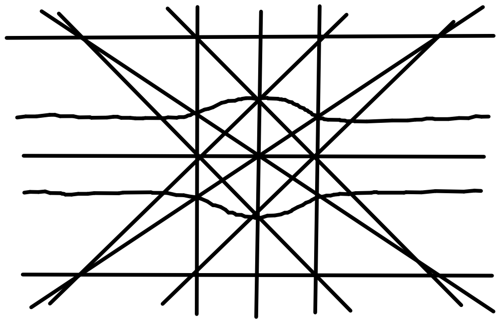
For $rk(M) = 2$ already quite large!

1.4 Topology of oriented matroids

Thm [Salvetti 1993, Cordovil 1994]
If $\mathcal{M}=(E,\mathcal{L})$ is an oriented matroid such that $\mathcal{L}$ is simplicial, then $\mathcal{S}(\mathcal{A})$ is a finite $K(\pi,1)$ complex.

Rmk: This is a proper extension of Deligne's Thm: there are infinitely many non-realizable cases!

e.g.:

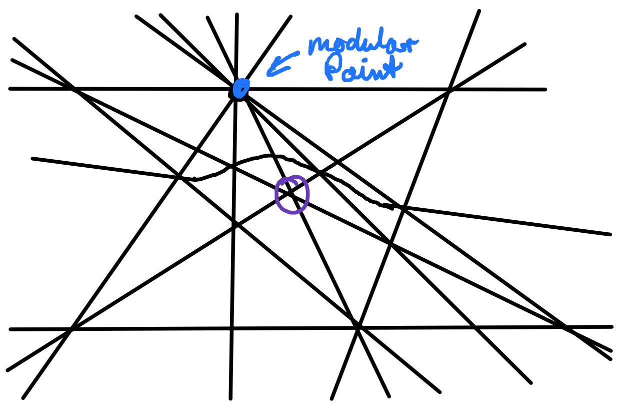


[projective picture]

Q: Does Falk, Randell and Terao's Theorem also extend to OMs in general?

i.e. $L(\mathcal{M})=\{z(\sigma)\mid \sigma\in\mathcal{L}\}$ supersolvable $\Rightarrow$ $\mathcal{S}(\mathcal{M})$ is aspherical?

e.g.:



[Non-Pappus $\subseteq \mathcal{M}$]

[• $z(\sigma)=\{e\in E\mid \sigma_e=0\}$
• $L(\mathcal{M})$ geometric lattice]

(2) Main Result

<u>Thm</u> [Quillen's Theorem B (for posets) 1973]

Let $f : P \longrightarrow Q$ be a poset map. If for all $a \leq b$ $(a, b \in Q)$ the inclusion $f^{-1}(Q_{\leq a}) =: (f \downarrow a) \hookrightarrow (f \downarrow b) := f^{-1}(Q_{\leq b})$ is a homotopy equivalence, then for $x \in P$ with $f(x) = a$ the homotopy fiber $\mathcal{F}(|\Delta(f)|, a)$ is homotopy equivalent to $|\Delta(f \downarrow a)|$.

[$|\Delta(-)| : \mathrm{Pos} \to \mathrm{Top}$ simplicial realization functor]

$\Rightarrow$ l.e.s.

$$\cdots \to \pi_{i+1}(|\Delta(Q)|, a) \longrightarrow \pi_i(|\Delta(f \downarrow a)|, x)$$
$$\to \pi_i(|\Delta(P)|, x) \longrightarrow \pi_i(|\Delta(Q)|, a) \longrightarrow \cdots$$

<u>Def.</u>: $f : P \to Q$ poset map is <u>poset quasi-fibration</u> $:\Leftrightarrow$ $(f \downarrow a) \hookrightarrow (f \downarrow b)$ is homotopy equivalence $\forall a \leq b \in Q$.

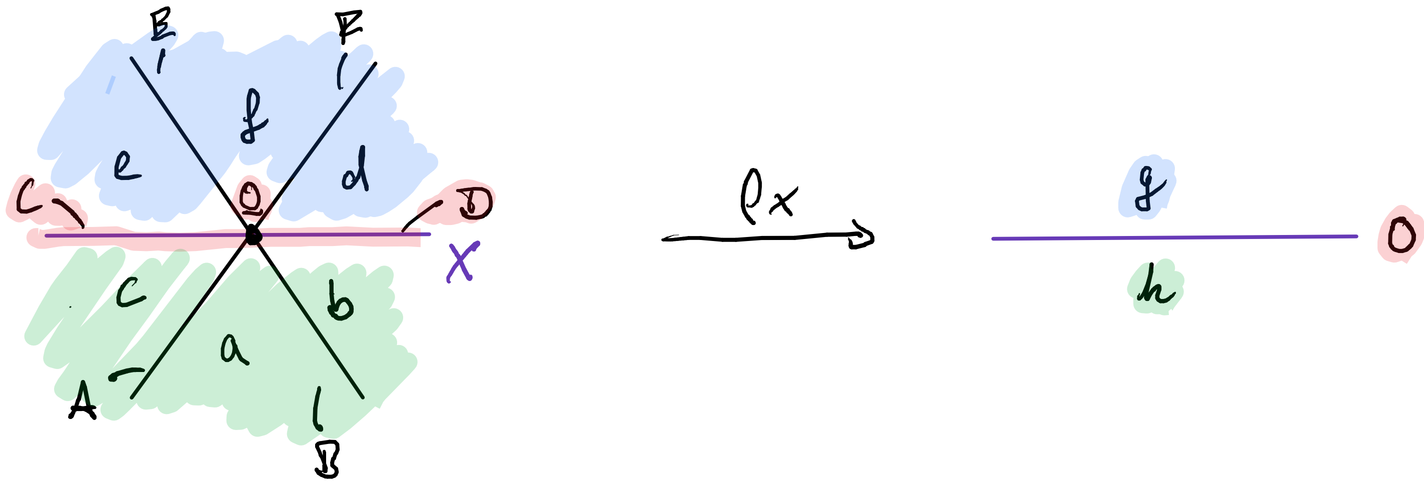
<u>Note</u> that for a regular CW cpx Σ (recall: identified with its face poset) we have: $|\Delta(\Sigma)| \cong \Sigma$

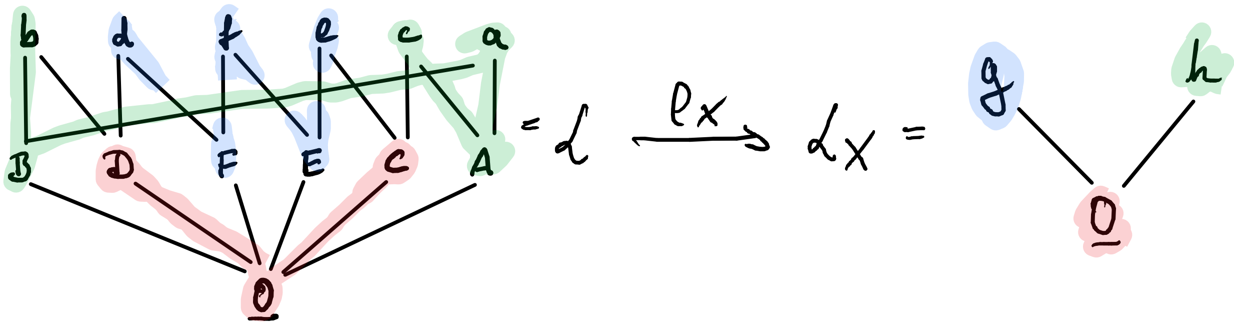
homeomorphic

② Main Result

- $X \in L(\mathcal{M}) \subseteq 2^E \rightsquigarrow \mathcal{L}_X := \{ \sigma|_X \mid \sigma \in \mathcal{L} \}$ localization of $\mathcal{L}/\mathcal{M}$
- $\rho_X : \mathcal{L} \longrightarrow \mathcal{L}_X, \ \sigma \longmapsto \sigma|_X$ localization map.

Ex.:





Fact: $\rho_X(\sigma \circ \tau) = \rho_X(\sigma) \circ \rho_X(\tau)$

$\rightsquigarrow \tilde{\rho}_X : S(\mathcal{M}) = \mathcal{S} \longrightarrow S_X := S(\mathcal{M}_X)$ induced poset map.

② Main Result

Thm [M.]
If $X \in L(\mathcal{M})$ is modular of corank 1, then

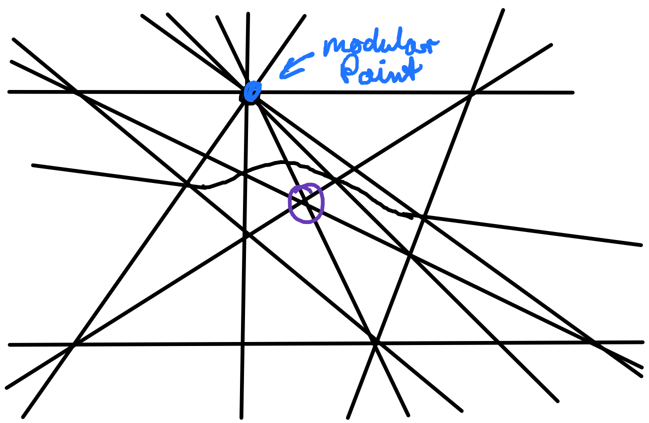
$$\tilde{p}_X : \mathcal{S} \longrightarrow \mathcal{S}_X$$

is a poset quasi-fibration with poset fiber $(f \downarrow a) \cong \mathcal{S}_{aff}(\mathcal{N})$ where $\mathcal{N}$ is an OM of rank 2.

Corollary
If $\mathcal{M}$ is supersolvable, then $\mathcal{S}(\mathcal{M})$ is a finite $K(\pi,1)$ complex.

e.g. $\mathcal{S}(\mathcal{M})$ is a finite $K(\pi,1)$ complex for

$\mathcal{M}$:



③ About the proof

Basic Ingredients:

- Shellability of $\mathcal{L}(\mathcal{M})$ [Lawrence / Björner et. al.]
- Discrete Morse Theory [Forman]
  - $\rightsquigarrow$ Patchwork Theorem [Kozlov]
  - $\rightsquigarrow$ Shellable Balls have discrete Morse functions with only one critical cell a vertex, i.e. they are collapsible [Chari]
- A nice stratification of $(\tilde{\rho}_X \downarrow (0,B')) := \tilde{\rho}_X^{-1}((\mathcal{S}_X)_{\leq (0,B')})$ for all $B' \in \mathcal{T}_X$ [Delucchi, M.]
- For $X \in L(\mathcal{M})$ modular, an isomorphism of posets

$$\mathcal{L}^{Y}_{X \vee Y} \xrightarrow{\cong} \mathcal{L}^{Y \wedge X}_{X} \quad \forall Y \in L(\mathcal{M}). \quad [M.]$$

[similar to the (trivial) iso. $[Y, X \vee Y]_L \cong [Y \wedge X, X]_L$ between intervals in L.]

3.1 Shellability of $\mathcal{L}(\mathcal{M})$

- Σ a regular cell complex, $\sigma \in \Sigma$
  $\rightsquigarrow$ $\quad \partial\sigma := \{\tau \in \Sigma \mid \tau < \sigma\} \subseteq \Sigma$
  the subcomplex of all proper faces of σ

<u>Def.</u> Let Σ be a pure d-dim'l regular cell cpx.
A linear ordering $\sigma_1, \dots, \sigma_t$ of its maximal cells is called a *shelling* if either $d=0$, or if $d \geq 1$ and:

(i) $\partial\sigma_j \cap \left(\bigcup_{i=1}^{j-1} \partial\sigma_i\right)$ is pure of dim. $d-1$ for $2 \leq j \leq t$,

(ii) $\partial\sigma_j$ has a shelling in which the $(d-1)$ cells of $\partial\sigma_j \cap \left(\bigcup_{i=1}^{j-1} \partial\sigma_i\right)$ come first for $2 \leq j \leq t$,

(iii) $\partial\sigma_1$ has a shelling.

Σ is called *shellable* if it has a shelling.

- $B \in \mathcal{T} \to \mathcal{T}(\mathcal{M}, B)$ *tope poset* with $R \leq T :\Longleftrightarrow S(B,R) \subseteq S(B,T)$
  where $S(B,T) := \{e \in E \mid B_e \cdot T_e = -\}$

3.1 Shellability of $\mathcal{L}$

Thm [Björner et al 1999]

$\mathcal{L}$ is a shellable regular cell decomposition of the $(\mathrm{rk}(\mathcal{M})-1)$-sphere.
Each linear extension of the tope poset $\mathcal{T}(\mathcal{M}, B)$ is a shelling of $\mathcal{L}$.

- Recall the map $\rho_X : \mathcal{L}^{(V)} \longrightarrow \mathcal{L}_X^{(V)}$, $\sigma \longmapsto \sigma|_X$.

The Thm leads to

Lemma 1 [M.]
Let $\sigma \in \mathcal{L}_X$. Then $\mathcal{L} \setminus \rho_X^{-1}((\mathcal{L}_X)_{\geq \sigma})$ is shellable.

Proof: $\mathcal{T}(\rho_X^{-1}((\mathcal{L}_X)_{\geq\sigma}) =: Q \subseteq \mathcal{T}$ is convex,

i.e. $\forall B \in Q$: Q is an order filter in $\mathcal{T}(\mathcal{M}, -B)$

$\Rightarrow \exists$ lin. ext. $\vdash$ of $\mathcal{T}(\mathcal{M}, -B)$
s.t. all topes $T \in Q$ come last. w.r.t. $\vdash$

$\Rightarrow$ Statement (look at the def. of a shelling). $\boxtimes$

3.2 Discrete Morse theory

Def. (Acyclic matchings)

Let $P = (P, \leq)$ be a (finite) poset.

Define a directed graph $G(P) := (V = P, \mathcal{E})$ by

$$\mathcal{E} := \{ (a,b) \mid a, b \in P \text{ with } a \lessdot b \}$$

i.e. $G(P)$ = Hasse diagram of P.

- $\underline{M} \subseteq \mathcal{E}$ is called a *matching* on P
  $:\Leftrightarrow$ each a is contained in <u>at most</u> one edge of $\underline{M}$
- Define a new graph $G(P, \underline{M}) := (V, \mathcal{E}')$ where

$$\mathcal{E}' := \mathcal{E} \setminus \underline{M} \cup \{ (b,a) \mid (a,b) \in \underline{M} \}$$

- If $G(P, \underline{M})$ does not contain any directed cycles,
  then $\underline{M}$ is called an *acyclic matching*.
- *Critical elements* $C(\underline{M}) := \{ a \in P \mid a \notin e \ \forall e \in \underline{M} \}$ $\boxed{E}$

Recall: Σ regular cell complex can be identified with its face poset.

3.2 Discrete Morse theory

<u>Thm</u> [Main Theorem of discrete Morse theory, Forman 1998] – a special case of

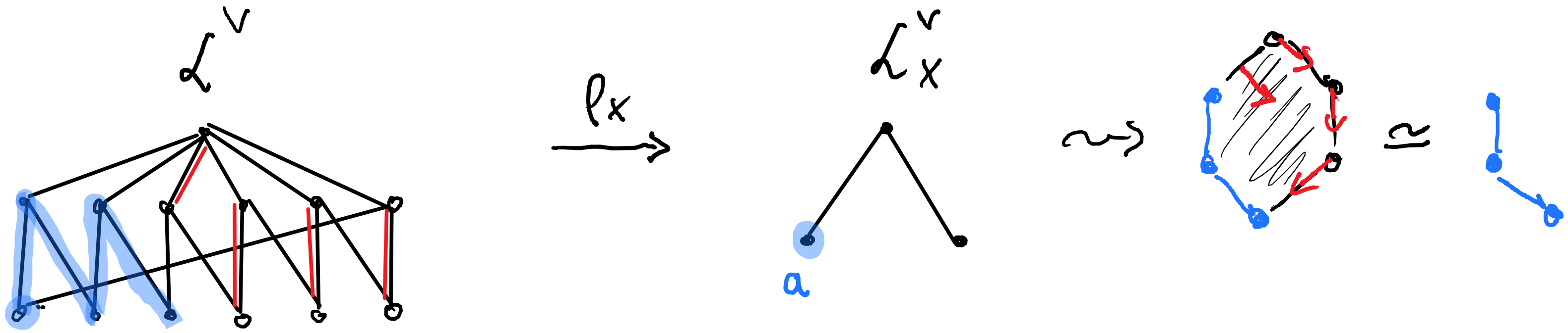
Let Σ be a regular cell complex and $\Gamma \subseteq \Sigma$ a subcomplex.

If $\underline{M}$ is an acyclic matching on Σ with $C(\underline{M}) = \Gamma$,

then Γ is a strong deformation retract of Σ.

In particular, $\Gamma \hookrightarrow \Sigma$ is a homotopy equivalence.

<u>Ex.</u>:



$\underline{M}$ acyclic matching with $C(\underline{M}) = (\rho_x \downarrow a) := \rho_x^{-1}((\mathcal{L}_x)_{\leq a})$ $\Rightarrow$ $(\rho_x \downarrow a) \xhookrightarrow{\simeq} \mathcal{L}^\vee$ homotopy equivalence.

<u>Proposition 2</u> [M.]

$\forall\, a \in \mathcal{L}_x^\vee$: $\exists$ $\underline{M}$ matching on $\mathcal{L}^\vee$ with $C(\underline{M}) = (\tilde{\rho}_x \downarrow a)$.

<u>Proof</u>: Use Lemma 1 + shellable Ball [Chari] $\Rightarrow$ Perfect Matching. ∎

3.3 Combinatorics of modular flats of $\mathcal{M}$

- $Y \in L(\mathcal{M}) \rightsquigarrow \mathcal{L}^Y := \{ \sigma|_{E \setminus Y} \mid \sigma \in \mathcal{L} \text{ with } Y \subseteq z(\sigma) \}$
  $\cong z^{-1}(L^Y)$ where $L^Y := \{ z \in L \mid Y \subseteq z \}$

  $\cap\!|$

  2^E

Lemma 3 [M.]

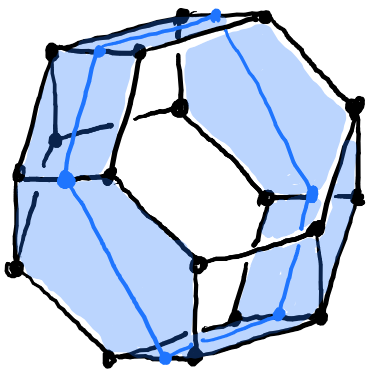
Let $X \in L(\mathcal{M})$ be a modular flat and $Y \in L(\mathcal{M})$.

Then $\bar{\rho}_X : \mathcal{L}^Y_{X \vee Y} \longrightarrow \mathcal{L}^{X \wedge Y}_X$

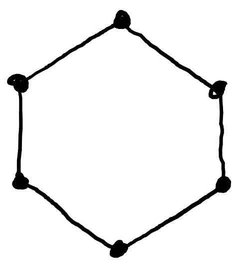
is an isomorphism of posets.

Proof: Technical.

Considerably harder than the trivial argument for L. ☒



$$\left(\begin{array}{ccc} \mathcal{L} & \xrightarrow{\rho_X} & \mathcal{L}_X \\ \uparrow j & & \| \\ \mathcal{L}^Y & \xrightarrow[\cong]{\rho_X|} & \mathcal{L}_X \end{array} \right)^{\vee}$$



3.4 A nice stratification

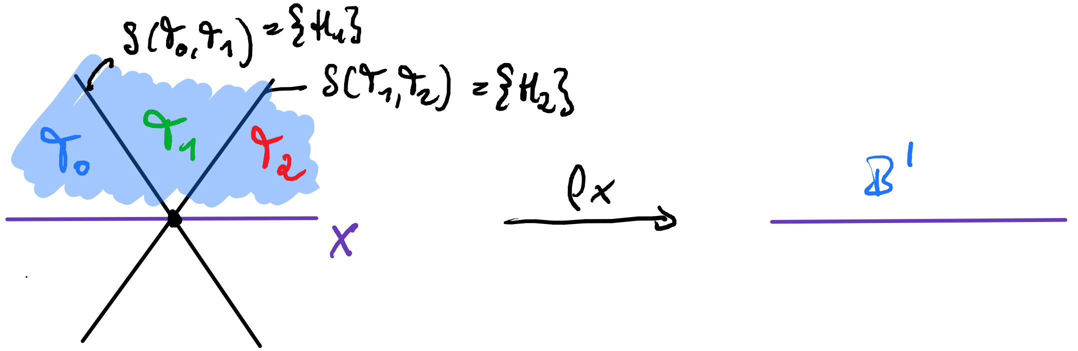
Thm 4 [Delucchi 2008, M.]

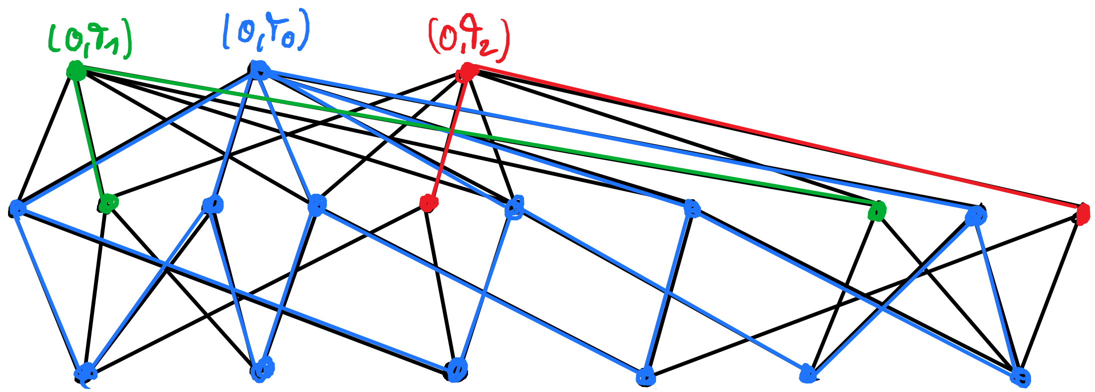
Let $X \in L$ be modular of corank 1 and $B' \in \mathcal{T}_X$.

Then there is a linear order $\mathcal{T}(\rho_X \downarrow B') = \{T_0 < T_1 < \ldots < T_m\}$ such that:

$$(\tilde{\rho}_X \downarrow (0, B')) = \bigsqcup_{i=0}^{m} N_i ,$$

where $N_i := S_{\leq (0,T_i)} \setminus \left(\bigcup_{j=0}^{i-1} S_{\leq (0,T_j)} \right)$ and $N_i \cong \begin{cases} \mathcal{L}, & i=0 \\ \mathcal{L}^{S(T_{i-1},T_i)}, & 1 \leq i \leq m \end{cases}$





$(\tilde{\rho}_X \downarrow B') = N_0 \sqcup N_1 \sqcup N_2$

$N_0 \cong \mathcal{L}$, $N_1 \cong \mathcal{L}^{H_1}$, $N_2 \cong \mathcal{L}^{H_2}$

3.5 Conclusion

- $$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\tilde{\rho}_X} & \mathcal{S}_X \\ \cup\!\!\mid & & \cup\!\!\mid \\ N_i & \xrightarrow{\tilde{\rho}_X|} & (\mathcal{S}_X)_{\leq (0,B')} \\ \cong & & \cong \\ (\mathcal{L}^{H_i})^\vee & \xrightarrow[\substack{= \rho_X \text{ or } \cong \\ \text{by Lemma 3.}}]{\rho_X|} & \mathcal{L}_X^\vee \end{array}$$ for $X \in L$ modular, $\mathrm{codk}(X) = 1$

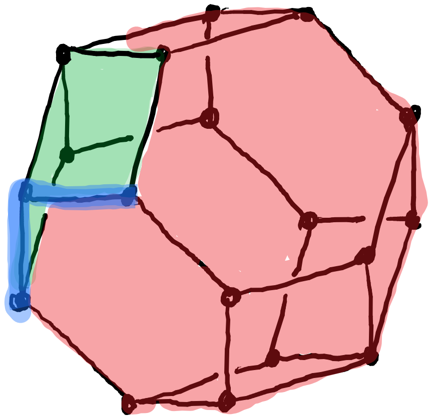
- Take $a \leq (0, B') \in \mathcal{S}_X$ with $a = (\sigma, \tau)$

- Thm 4 $\Rightarrow (\tilde{\rho}_X \downarrow (0,B')) = \bigsqcup_i \underset{\substack{\cong \\ \mathcal{L}, \mathcal{L}_X}}{N_i} \supseteq \bigsqcup_i \underbrace{(\tilde{\rho}_X \downarrow a) \cap N_i}_{\cong (\rho_X| \downarrow \sigma)}$.

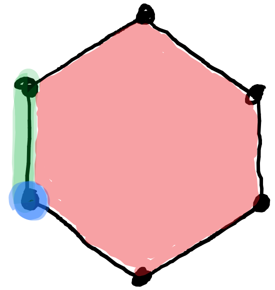
$\rightsquigarrow$ for each i, Lemma 2 gives Matching $\underline{M}_i$ on N_i with $C(\underline{M}_i) = (\rho_X| \downarrow \sigma) \cong (\tilde{\rho}_X \downarrow a) \cap N_i$

- __Patchwork Thm [Kozlov]__ : $\dot{\bigcup_i} \underline{M}_i =: \underline{M}$ is acyclic matching on $(\tilde{\rho}_X \downarrow (0,B'))$ with $C(\underline{M}) = \bigcup C(\underline{M}_i) = (\tilde{\rho}_X \downarrow a)$.

3.5 Conclusion



$$\mathcal{L}^\vee \xrightarrow{\rho_X} \mathcal{L}_X^\vee$$



$$\begin{tikzcd}
 & (\hat{\rho}_X \downarrow (0,\mathbb{1}')) & \\
(\hat{\rho}_X \downarrow a) \arrow[ur, hook, "\simeq"] \arrow[rr, hook, "\simeq"] & & (\hat{\rho}_X \downarrow b) \arrow[ul, hook', "\simeq"']
\end{tikzcd}$$

$\forall\, a \leq b \in \mathcal{S}_X$.

$\Rightarrow$ $\hat{\rho}_X$ is poset quasi-fibration.

∎