Chapter 1.4 - The Curse of Dimensionality

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1 A first example

For practical applications of pattern recognition, we will have to deal with inputs of high-dimensionality.

- Poses serious challenges.
- Heavily influences the design of techniques.

We introduce the problem with an example.

- We have a bunch of data points we want to use in order to classify a new point into one of the classes.

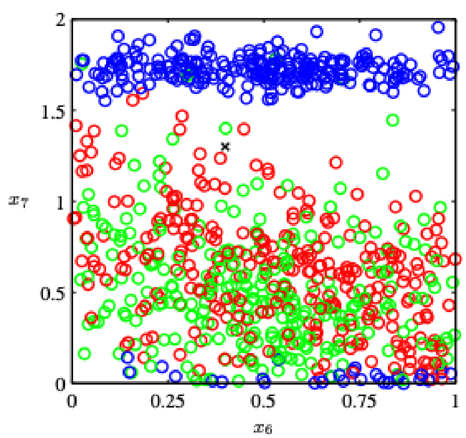

Figure 1: *Input space.*

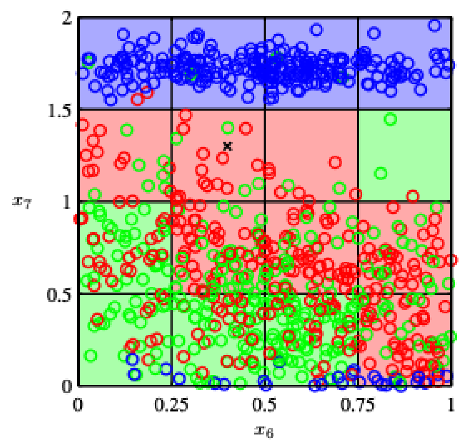

Figure 2: *Input space divided into cells.*

A simple approach would be to divide the *input space* (Figure 1 & 2) into regular cells. When we are given a new point, the identity is determined as being the same as the class having the largest number of training points in the same cell as the test point.

- Numerous problems with this approach, but the most severe is apparent when we extend the problem to inputs with higher dimensionality.
- The origin of the problem: *if we divide a region of space into regular cells, then the number of cells grows exponentially with the dimensionality of the space* (see Figure 3).
  - We would need an exponentially large number of training data to ensure all cells are nonempty.

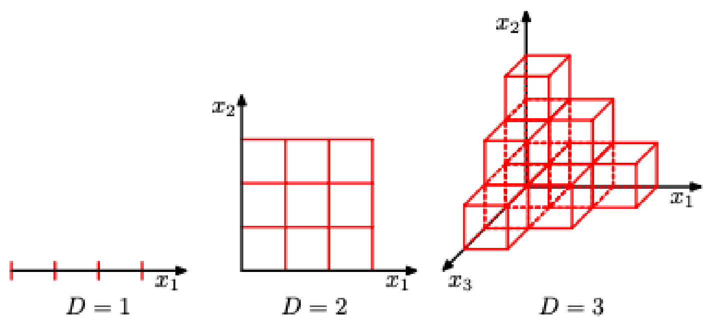

Figure 3:

2 A second example

We can gain further insight by returning to the curve fitting example.

- Consider extending it to deal with input spaces having several dimensions (several covariates).
- If each input has D variables (dimensions), then the general polynomial with $M = 3$ would take the form

$$y(\boldsymbol{x}, \boldsymbol{y}) = w_0 + \sum_{i=1}^{D} w_i x_i + \sum_{i=1}^{D}\sum_{j=1}^{D} w_{ij} x_i x_j + \sum_{i=1}^{D}\sum_{j=1}^{D}\sum_{k=1}^{D} w_i w_j w_k x_i x_j x_k$$

- As D increases, so do the number of independent coefficients proportional to $D^{3=M}$

3 Geometrical intuitions failing at higher dimensions

We live in a $3D$ world and this can cause our intuition of higher dimensions to fail. A simple example is the case where we consider a sphere with radius $r = 1$ in a space of D dimensions. What is the fraction of the volume of the sphere that lies between $r = 1 - \epsilon$ and $r = 1$?

- We can evaluate this fraction be noting the volume of a sphere with radius r in D dimensions must scale as r^D, and so

$$V_D(r) = K_D r^D \qquad \text{(where } K_D \text{ only depends on } D\text{)}$$

- The required fraction is given then by

$$\frac{V_D(1) - V_D(1-\epsilon)}{V_D(1)} = 1 - (1-\epsilon)^D$$

We see that for large D, this fraction tends to 1 even for small ϵ. Thus, in spaces of high dimension, most of the volume of a sphere is concentrated in a thing shell near the surface!

4 Discussion

Severe difficult arising in spaces of high-dimensions is sometimes referred to as the "Curse of Dimensionality".

- Although the curse of dimensionality arises important issues for pattern recognition applications, it does not prevent us from finding effect techniques applicable to high-dimensional spaces.

- Reasons being two-fold
  - Real data will often be confined to a region of space having lower effective dimensionality, and in particular the directions over which important variations in the target variables occur may be so confined.
  - Real data will typically exhibit some smoothness properties (at least locally) so that small δ applied to the input will produce δ changes in the output, so we can exploit local interpolation-like techniques to make predictions.
- Successful pattern recognition techniques exploit one or both of the above properties.

Example: an application in which images are captured of identical planar objects on a conveyor belt, where the goal is to determine their orientation.

- Each image is a point in high-dimensional space, whose dimension is determined by the number of pixels.
- Because objects can occur at different positions within the image and in different orientations, there are 3 degrees of freedom of variability between images, and a set of images will live on a $3D$ manifold embedded within the high-dimensional space.
- Due to complex relationships between object position or orientation with pixel intensities, this manifold will be highly nonlinear.
- If the goal is to output orientation irrespective of position, then there is only one degree of freedom of variability within the manifold that is significant.

For more on manifolds, see: `http://colah.github.io/posts/2014-03-NN-Manifolds-Topology/`