Lecture 06

Paul Scemama

OCW: *This lecture discusses column space and nullspace. The column space of a matrix A tells us when the equation $A\boldsymbol{x} = \boldsymbol{b}$ will have a solution $\boldsymbol{x}$. The nullspace of A tells us which values of $\boldsymbol{x}$ solve the equation $A\boldsymbol{x} = 0$.*

1 Outline

- Review.
- Column space of a matrix A.
- Null space of a matrix A.
- Some remarks.

2 Review

A *vector space* is a bunch of vectors where you can multiply any by a scalar and add them together and the result stays in the space. In a more concise mathematical form,

$v + w$ and cv are in the space. I.e. all combinations $cv + dw$ are in the space.

Examples include:

- $\mathbb{R}^3$ shown in Figure 1.

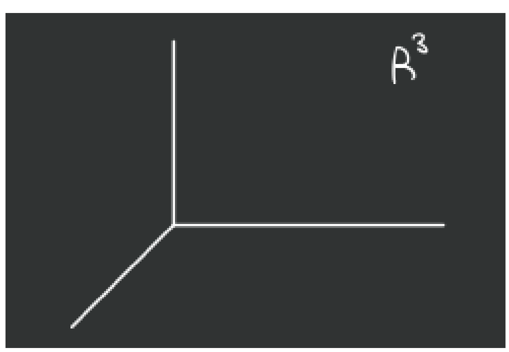

Figure 1: $\mathbb{R}^3$*; a vector space.*

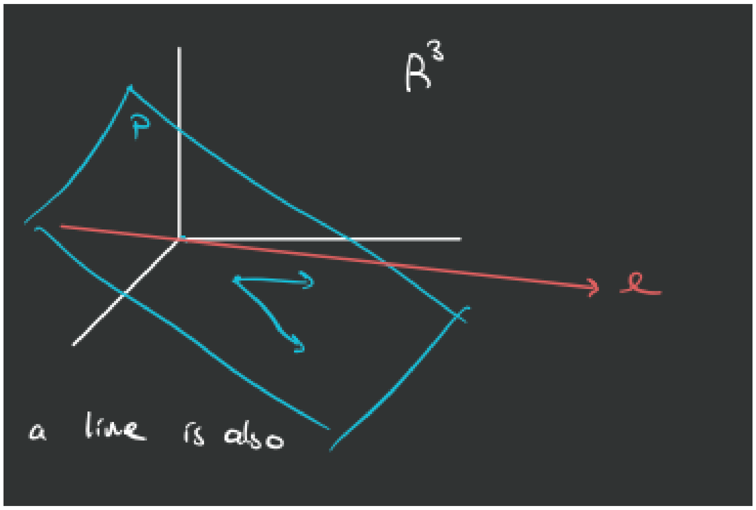

Figure 2: *A plane through the origin is a subspace of* $\mathbb{R}^3$. *The same with a line. Note that* $\mathbb{R}^2$ *is not a subspace of* $\mathbb{R}^3$ *because the vectors that live in* $\mathbb{R}^2$ *do not live in* $\mathbb{R}^3$ (they only have two entries).

- A subspace: a vector space *inside* another vector space.
- Simplest example: a plane that goes through the origin is a subspace of $\mathbb{R}^3$ shown in Figure 2.

Let's take a look at Figure 2 again. There are 2 subspaces: p and ℓ. *Assume* ℓ does not lie on the plane p.

- Consider $p \cup \ell$ = all vectors in p or ℓ or both. Is this a subspace (of $\mathbb{R}^3$)?
  - NO $\rightarrow$ we cannot *add* two vectors and remain in the space.
- Consider $p \cap \ell$ = all vectors in p and ℓ. Is this a subspace (of $\mathbb{R}^3$)?
  - YES $\rightarrow$ this space contains only the zero vector and the zero vector is a subspace. In fact, for any two subspaces S and T, their intersection $S \cap T$ *is a subspace.*

3 Column space of a matrix A

We will explore the column space of a matrix A by way of example. Consider

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

What's in $C(A)$?

- All linear combinations of the columns of A. This gives me a subspace of $\mathbb{R}^4$.
  - A subspace...since by definition taking all linear combinations of any number of vectors results a set of vectors *closed* under addition and scalar multiplication $\rightarrow$ a vector space.
  - Of $\mathbb{R}^4$...because the column vectors live in $\mathbb{R}^4$!
- We're going to be interested in this space.

Critical connection with $A\boldsymbol{x} = \boldsymbol{b}$.

- Does $A\boldsymbol{x} = \boldsymbol{b}$ have a solution for every $\boldsymbol{b}$? *NO.*
  - We have 4 equations and 3 unknowns.
  - We can't get to every vector in $\mathbb{R}^4$ by only taking linear combinations of 3 vectors (columns of A)!
- You usually can't solve 4 equations with 3 unknowns. But sometimes you can. Which $\boldsymbol{b}$'s allows this system to be solved?
  - One $\boldsymbol{b}$ would be $\begin{bmatrix}0 & 0 & 0 & 0\end{bmatrix}^T$, where we right a column vector as a transposed row vector to reduce clutter. Others include $\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}^T$, $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, and $\begin{bmatrix}2 & 3 & 4 & 5\end{bmatrix}^T$. A strategy could be to think of a $\boldsymbol{x}$, do $A\boldsymbol{x}$, and then what have I got? If we look at every possible $\boldsymbol{x}$ and do $A\boldsymbol{x}$, we get all linear combinations of the columns of A – the *column space* of A.
  - *We can solve $A\boldsymbol{x} = \boldsymbol{b}$ when $\boldsymbol{b}$ is in the column space of A.* Because by definition, the column space consists of all vectors that are $A\boldsymbol{x}$. If $\boldsymbol{b}$ is an $A\boldsymbol{x}$, then we can find $\boldsymbol{x}$. Otherwise, we cannot solve for that $\boldsymbol{b}$.
- Do all columns contribute something new? Are they *independent*?
  - No since $col1 + col2 = col3$. And furthermore we'd describe $C(A)$ as a $2D$ subspace of $\mathbb{R}^4$ since only two columns contribute something *original* to the column space (more on this in later lectures).

4 Null space of a matrix A

First things first – this is a *totally different* subspace than the column space of a matrix A.

Let's again introduce the null space by way of example. Consider the same A as the last section,

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

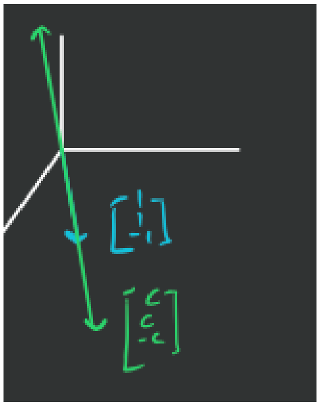

Figure 3: *A plot of* $\begin{bmatrix} c & c & -c \end{bmatrix}^T$

The null space of A, denoted $N(A)$, consists of all the $\boldsymbol{x}$'s that solve $A\boldsymbol{x} = 0$. The null space of A is also a subspace of $\mathbb{R}^3$ ($\boldsymbol{x} \in \mathbb{R}^3$). Recall that $C(A)$ is a subspace of $\mathbb{R}^4$. We will continue the same way as last section by first considering some $\boldsymbol{x}'s$ that solve $A\boldsymbol{x} = 0$.

- $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$.
- Since $col1 + col2 = col3$, $col1 + col2 - col3 = 0$. And so $\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$ is in $N(A)$. In fact, any $\begin{bmatrix} c & c & -c \end{bmatrix}^T$ is in $N(A)$, and this defines a line as shown in Figure 3.

Is this a subspace? Yes it is. Is $N(A)$ a subspace in general? Yes it is, and we will show that now. Namely, we will show that solutions to $A\boldsymbol{x} = 0$ always give a *subspace*.

Proof. *Claim* 1: if $A\boldsymbol{v} = 0$ and $A\boldsymbol{w} = 0$, then $A(\boldsymbol{v} + \boldsymbol{w})$ must be 0. This says that if $\boldsymbol{v}$ is in the null space and $\boldsymbol{w}$ is in the null space, then their sum is in the null space as well.

$$A(\boldsymbol{v} + \boldsymbol{w}) = A\boldsymbol{v} + A\boldsymbol{w} \tag{1}$$

$$A\boldsymbol{v} = 0; \;\; A\boldsymbol{w} = 0 \tag{2}$$

$$\text{thus} \;\; A(\boldsymbol{v} + \boldsymbol{w}) = A\boldsymbol{v} + A\boldsymbol{w} = 0 \tag{3}$$

where in (1) we've used the distributive property of matrix multiplication.

Claim 2: *if* $A\boldsymbol{v} = 0$*, then* $A(d\boldsymbol{v}) = 0$*.* This is saying that if $\boldsymbol{v}$ is in the null space and we multiply it by a scalar d, their product is in the null space as well.

$$A(d\boldsymbol{v}) = dA\boldsymbol{v} = d \cdot 0 = 0 \tag{4}$$

where we've used the associative property of scalar multiplication. □

5 Some remarks

Let's take the following example:

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

We'd like to know the solution to this system; that is, these 4 equations. We know that, for example, $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ is a solution. Are there any others? And is the set of all solutions a vector space? I.e. do they form a subspace of the *world* they live in. Well the answer is no, the solutions do *not* form a vector space since the zero vector is not a solution and this is a requirement. So we collect a bunch of solution vectors (all of them actually) and we know it is not a subspace. In fact, it is potentially a plane or a line that doesn't go through the origin.

6 Summary

We've introduced two different subspaces that are derived from a matrix A.

- The column space: " tell us a few columns and we will build up the space by taking all their linear combinations ".
- The null space: " have to figure out what's in it (right now) ".

They are different ways of getting a subspace and they define two different things. In the column space – give us vectors and we take all linear combinations. In the null space – give us a system that $\boldsymbol{x}$'s have to satisfy (in particular where $\boldsymbol{b} = 0$) and the $\boldsymbol{x}'s$ create a subspace (not true of $\boldsymbol{b} \neq 0$). Next time we will look at how we get a hold of the null space. Not just by eyeing like we did in this lecture.