Lecture 01

Paul Scemama

OCW: *A major application of linear algebra is to solving systems of linear equations. This lecture presents three ways of thinking about these systems. The "row method" focuses on the individual equations, the "column method" focuses on combining the columns, and the "matrix method" is an even more compact and powerful way of describing systems of linear equations.*

1 Setup

Fundamental problem of Linear Algebra is to *solve a system of linear equations*. In the normal (nice) case we have n linear equations and n unknowns. Today we want to describe:

1. Row-picture
2. Column-picture
3. Matrix form

We will introduces these three concepts via examples.

2 Example 1

2 equations, and 2 unknowns.

$$\begin{aligned} 2x - y &= 0 \\ -x + 2y &= 3 \end{aligned}$$

We can *package* these equations into $A\boldsymbol{x} = \boldsymbol{b}$,

$$\underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\boldsymbol{x}} = \underbrace{\begin{bmatrix} 0 \\ 3 \end{bmatrix}}_{\boldsymbol{b}} \tag{1}$$

2.1 Row picture

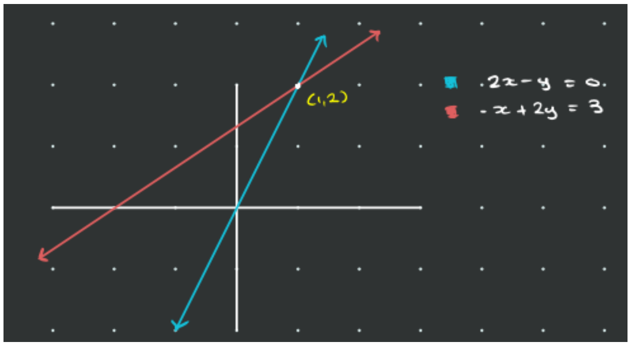

Figure 1: Row picture

We've plotted the solution to equation 1 and 2 respectively. Where they meet is where both equations are satisfied, and thus the answer to the system.

Here it is

$$x = 1$$
$$y = 2$$

2.2 Column picture

We will first expand (1),

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$x \overbrace{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}^{Col1} + y \overbrace{\begin{bmatrix} -1 \\ 2 \end{bmatrix}}^{Col2} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

What the form is asking us: how to combine $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in *the right amounts* to get $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. In other words, how to get the *right linear combination of the columns* to get $\begin{bmatrix} 0 \\ 3 \end{bmatrix}$. This can be visualized,

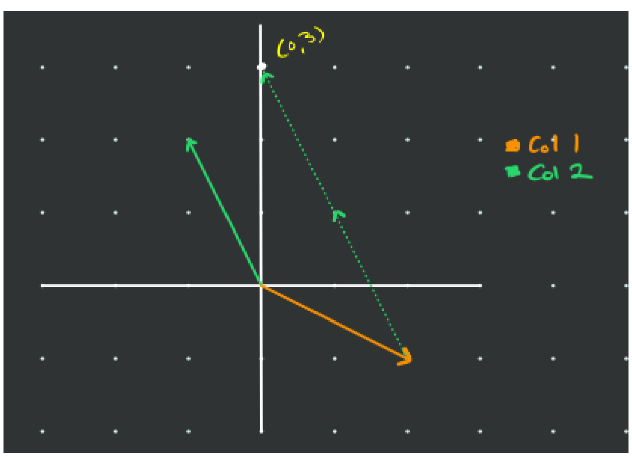

Figure 2: Column picture

If we take one of $Col1$ and then add two of $Col2$ we get to our desired answer. In other words, $1 \cdot Col1 + 2 \cdot Col2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Hence,

$$x = 1$$
$$y = 2$$

which is in agreement with our Row picture. Q: What are all the combinations? I.e., if you took all the x's and all the y's and created combinations of $Col1$ and $Col2$ with them, what would be all the results? In our example, it would be any vector in $\mathbb{R}$.

3 Example 2

3 equations, and 3 unknowns.

$$
\begin{aligned}
2x - y &= 0 \\
-x + 2y - z &= 3 \\
-3y + 4z &= 4
\end{aligned}
$$

We can again *package* these equations into $A\boldsymbol{x} = \boldsymbol{b}$, where

$$
A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ -3 & 4 & 0 \end{bmatrix} ; \quad \boldsymbol{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} ; \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}
$$

3.1 Row picture

Each row gives us a plane in 3-dimensions. These planes will meet at a point in our example, and this point is the solution $[x^*, y^*, z^*]$. We can see that it is getting a bit messy.

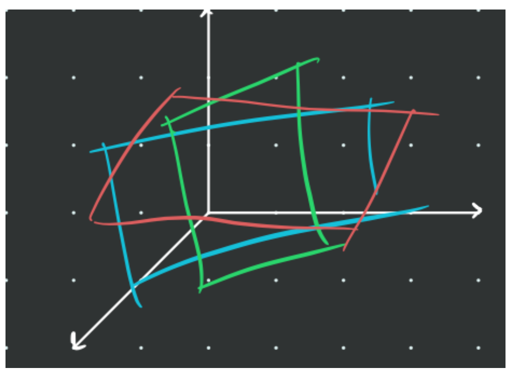

Figure 3: Row picture

3.2 Column picture

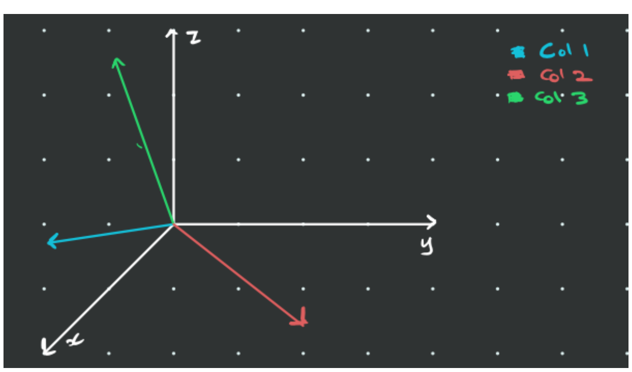

Figure 4: Column picture

We can again look at it from the column-perspective in which we treat the columns as vectors that, when combined properly with values x, y, z, arrive at $\boldsymbol{b}$.

$$
x \overbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}}^{Col1} + y \overbrace{\begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}}^{Col2} + z \overbrace{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}^{Col2} = \overbrace{\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}}^{\boldsymbol{b}}
$$

Combine $Col1, Col2, Col3$ in the **right amounts** (x, y, z) to get $\boldsymbol{b}$.

In this problem,

$$\begin{aligned} x &= 0 \\ y &= 0 \\ z &= 1 \end{aligned}$$

which is the point where the three planes met in the Row picture.

Q: Can I solve $A\boldsymbol{x} = \boldsymbol{b}$ for every b? In other words, do the linear combinations of the columns fill 3-D space?

- For this A, yes.
- A is a good matrix (nonsingular, invertible). We like these.
- There are other matrices where the answer is no.

Q: What could go wrong? When can we *not* produce any $\boldsymbol{b}$?

- If all three columns lie in the same plane, or the same line, then their combinations will all lie in that plane (or line).
- For example, if $Col1 = Col2 + Col3$.
  - In this case we could solve for some $\boldsymbol{b}$ - when $\boldsymbol{b}$ is in the plane (or line).
  - A is then singular, non-invertible.

4 9 dimensions

Think about a 9×9 matrix. Each of the 9 columns would be a vector living in 9-dimensional space, and we'd try and find the combination of these that hit the right $\boldsymbol{b}$. And we can ask, can we do this for any $\boldsymbol{b}$? Well, it, it depends on the columns. If the columns are *not* independent, then there are some $\boldsymbol{b}$'s we couldn't *reach* with combinations of our columns.

5 Formalizing a step

$A\boldsymbol{x} = \boldsymbol{b}$.

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 10 \\ 6 \end{bmatrix} = \begin{bmatrix} 12 \\ 7 \end{bmatrix}$$

So $A\boldsymbol{x}$ is a combination of the columns of A.