Lecture 05

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OCW: *To account for row exchanges in Gaussian elimination, we include a permutation matrix P in the factorization $PA = LU$. Then we learn about vector spaces and subspaces; these are central to linear algebra.*

1 Outline

- Permutations P
- Vector spaces
- Subspaces

2 Permutations P

We may need to execute row exchanges when we do elimination. Since $A = LU$ factorizes A into the eliminations steps as well as the result of elimination, we may want to ask: *what happens to* $A = LU$?

$$A = LU \quad \text{turns into} \quad PA = LU$$

where we can say that $PA = LU$ is a description of elimination *with* row exchanges for any invertible A.

2.1 A couple things about P

1. P is an identity matrix with re-ordered rows.
2. $P^{-1} = P^T$ i.e. $P^T P = I$
    - we will be interested in matrices with this property later on.

2.2 Mention of Symmetric Matrices

We call attention to a "good" type of matrix; *symmetric* matrices: $A^T = A$.

$$\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}^T = \begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 9 \\ 7 & 9 & 4 \end{bmatrix}$$

Let's just notice this family of matrices for now.

2.3 When would we get such a matrix? (symmetric)

Consider a matrix R and its transpose R^T

$$R = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} \; ; R^T = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

And now consider the product $R^T R$:

$$\begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 11 & 7 \\ 11 & 13 & 11 \\ 7 & 11 & 9 \end{bmatrix}$$

So the product of the transpose of a matrix AR^T and itself A is always symmetric.

2.4 How do we know $R^T R$ is symmetric?

If $R^T R$ is symmetric, then we know that $(R^T R)^T = R^T R$.

$$(R^T R)^T = R^T R^{TT} = R^T R \; \rightarrow \; \text{symmetric}$$

where in the first equality we know the order gets reversed (like inverses).

3 Vector spaces

What do we do with vectors?

- add them
- multiply them with scalars

A *space* means that we have a *bunch* of vectors. But not just any bunch; the space needs to allow us to add vectors and multiply them by scalars (aka take linear combinations).

3.1 Examples

$$\mathbb{R}^2 = \text{all 2-dimensional real vectors} = \text{the "xy plane"}$$

Instances of this space includes

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} \pi \\ e \end{bmatrix}, ...etc.$$

$\mathbb{R}^2$ is a vector space because it contains all two dimensional real vectors and *any of their linear combinations.*

Other examples include

$$\mathbb{R}^3 = \text{all 3-dimensional real vectors}$$

$$\mathbb{R}^n = \text{all n-dimensional real vectors}$$

Again, *the most important thing* is that we can add any two and live in the same space, and we can multiply any by a scalar and stay in the same space.

3.2 Non-examples

Let's look at an example of a space that is *not* a vector space.

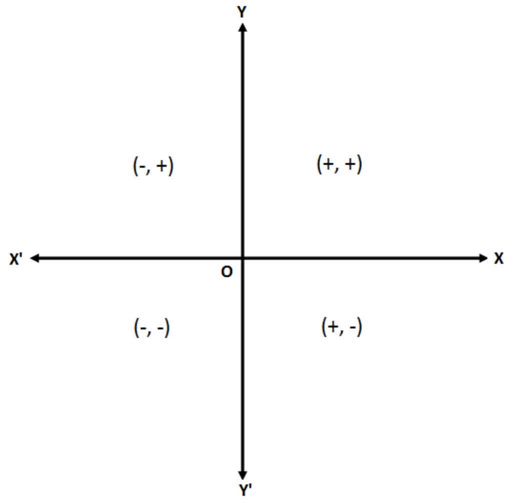

Figure 1: *Quadrants of xy-plane*

The first quadrant $(+,+)$ is *not* a vector space. Why?

- Is it "closed under addition". I.e. can we add vectors in there safely? *Yes.*
- Is it "closed under multiplication of a scalar". I.e. can we multiply a vector by a scalar safely? *No.*

It is not closed under multiplication (by a scalar). For example we can do $-3\begin{bmatrix}1\\2\end{bmatrix}$ and that is not in quadrant one $(+,+)$ anymore.

4 Subspaces

We will be interested in vector spaces **inside** $\mathbb{R}^n$. That is, spaces that follow the rules, but we don't need *all* of $\mathbb{R}^n$ to follow the rules.

For example let's look at a subspace of $\mathbb{R}^2$. In other words, a *vector space* inside of $\mathbb{R}^2$. Figure 2 illustrates both an example of a subspace of $\mathbb{R}^2$ as well as a *non*-example. Recall the two conditions a space needs to satisfy in order to be a vector space: *for every element in the space...*

1. need to be able to multiply by any scalar and remain in the space.
2. need to be able to do add elements together and remain in the space.

If we add any vector on the blue line with another vector on the blue line, we will remain on the blue line. Additionally if we multiply any vector on the blue line with a scalar, we will remain on the blue line.

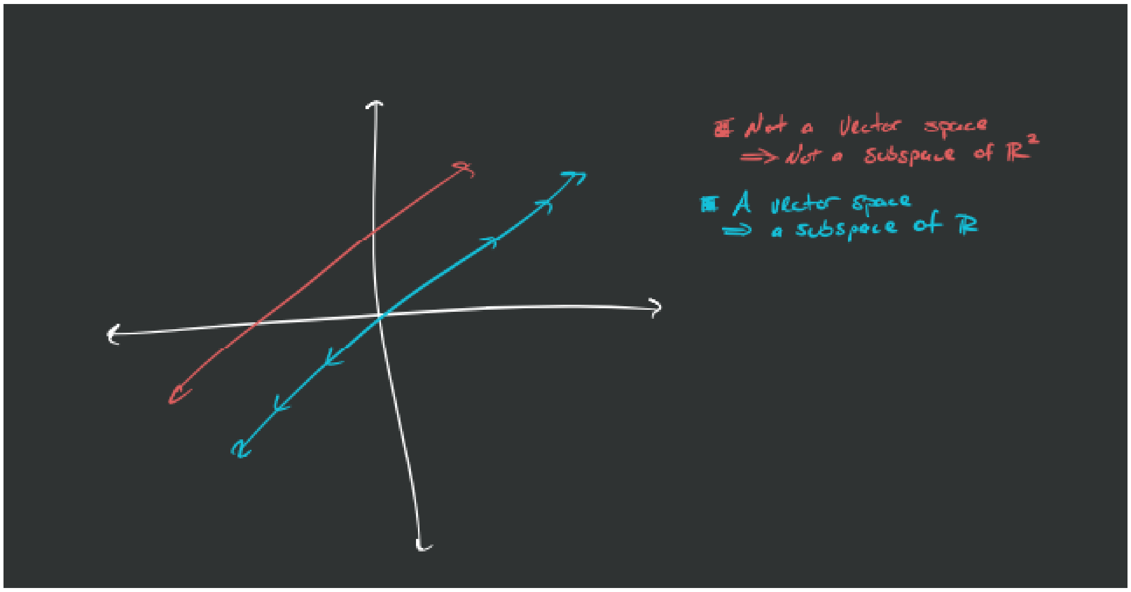

Figure 2: *An example of a subspace of* $\mathbb{R}^2$ *(blue) and a non-example of a subspace (red).*

If we add any vector on the red line with another vector on the red line, we will remain on the red line. *However*, if we multiply any vector on the red line with a scalar, we will *not always* remain on the red line. E.g. consider the scalar 0; when added to a vector on the red line it will yield the zero vector which *is not* on the red line. Therefore, the origin (or zero-vector) is always needed to be considered a vector space.

4.1 All possible subspaces of $\mathbb{R}^2$

Let's think of all the possible subspaces of $\mathbb{R}^2$.

1. All of $\mathbb{R}^2$.
2. Any line through $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
    - *not the same as* $\mathbb{R}^1$; the vectors still have two components.
3. $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. I.e. the zero vector only.

4.2 All possible subspaces of $\mathbb{R}^3$

1. $\mathbb{R}^3$ itself.
2. Any plane through the origin.
3. Any line through the origin.
4. The zero vector.

4.3 How do these subspaces come out of matrices?

Our goal is to create subspaces out of a matrix. For example consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 4 & 1 \end{bmatrix}$$

One subspace comes from the *columns*.

- The columns in A *live in* $\mathbb{R}^3$.
- We want the columns in A to be in this subspace of $\mathbb{R}^3$.
- If we take *all linear combinations of the columns* this forms a subspace of $\mathbb{R}^3$.
  - By the definition of taking all linear combinations, this will result in a vector space. And thus, since we're living in $\mathbb{R}^3$, this vector space will be a subspace of $\mathbb{R}^3$.

We call this subspace the *column space of* A, denoted $C(A)$.

There's an important idea here. We got a few vectors that live in a space. These happen to live in $\mathbb{R}^3$ so our space will contain vectors in $\mathbb{R}^3$. We then took all linear combinations of these *few vectors* to define a new space...a *subspace* of $\mathbb{R}^3$.

4.3.1 Visual of column space

Let's take a look at the column space of A. We have two column vectors in $\mathbb{R}^3$:

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} ; \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$$

We take all linear combinations and we get a plane, as shown in Figure 3.

4.4 Discussion extending to 10 dimensions

Suppose we have 5 10−dimensional vectors:

- Taking all linear combinations of these 5 vectors result in (**at most**) some sort of 5-dimensional space that passes through the origin.
  - *NOT* $\mathbb{R}^5$ since our vectors have 10 components.
  - Instead, it will be some sort of 5-dimensional space *embedded* into a 10-dimensional one.
- But if these 5 vectors were all on the same line, then we'd only get a line as the column space – a sort of 1-dimensional space *embedded* in the 10-dimensional world the vectors live.
- And if these 5 vectors were all on the same plane, then we'd only get a plane as the column space – a sort of 2-dimensional space *embedded* in the 10-dimensional world the vectors live.
- ...and so on.

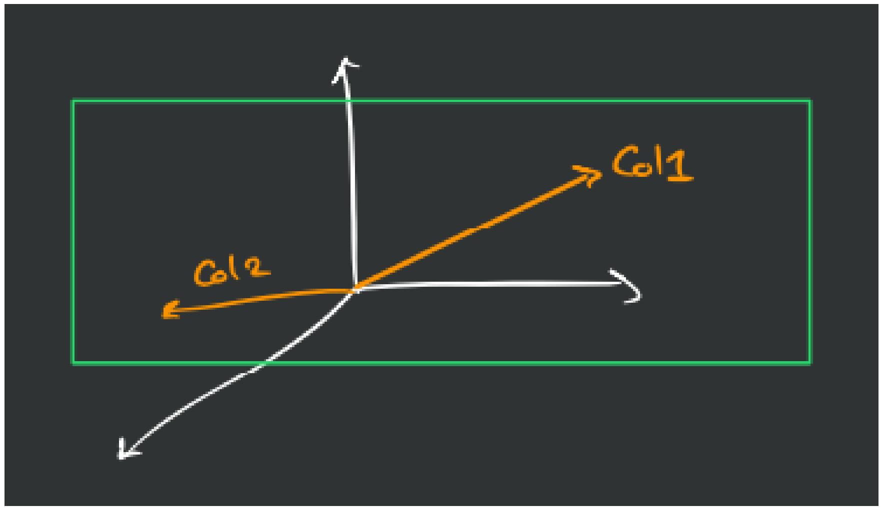

Figure 3: *The column space of A defined by taking all linear combinations of the columns of A. These two column vectors don't live on the same line, and so they define a plane in $\mathbb{R}^3$.*

4.4.1 Remarks

The column space is a great example of getting a subspace from a matrix. We will encounter other subspaces derived from a matrix soon. To get the column space we

- Take the column vectors.
- Take all linear combinations.
- ...we get a subspace of the space that the column vectors live in.

Next we will look at $A\boldsymbol{x} = \boldsymbol{b}$ and ask ourselves: *how do we understand $A\boldsymbol{x} = \boldsymbol{b}$ in the new language of column space and subspaces?*