Lecture 08

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OCW: *When does $Ax = b$ have solutions x, and how can we describe those solutions? We describe all solutions to $Ax = b$ based on the free variables and special solutions encoded in the reduced form R.*

1 Outline

1. Solvability of A.
2. Complete solutions to $A\boldsymbol{x} = \boldsymbol{b}$.
   - On one hand, are there any solutions? If so, what family of solutions are there?
   - On the other hand, are there no solutions?
3. The *rank* of a matrix, and what this tells us.

2 Solvability of $A\boldsymbol{x} = \boldsymbol{b}$

2.1 Constraints on $\boldsymbol{b}$ with an example

We begin with an example for the complete solutions to $A\boldsymbol{x} = \boldsymbol{b}$. Consider the system

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 2x_4 &= b_1 \\ 2x_1 + 4x_2 + 6x_3 + 8x_4 &= b_2 \\ 3x_1 + 6x_2 + 8x_3 + 10x_4 &= b_3 \end{aligned}$$

The first thing we can notice is that $r_3 = r_1 + r_2$. We know that elimination will reveal this to us. What this also means is that $b_1 + b_2 = b_3$ is a necessary condition for $A\boldsymbol{x} = \boldsymbol{b}$ to have a solution.

We will now see how this constraint is presented in another way by means of elimination. The system above can be expressed an *augmented matrix.*

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right]$$

where $\boxed{\cdot}$ denotes the pivots once they've been revealed. Taking the elimination steps yields an augmented matrix where A is in *echelon* form:

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right] \to \left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & b_1 \\ 0 & 0 & \boxed{2} & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 - b_1 \end{array}\right] = \left[\begin{array}{c|c} U & \boldsymbol{b}' \end{array}\right] \tag{1}$$

where $\boldsymbol{b}^{'}$ is the result of elimination steps on $\boldsymbol{b}$. Note the last row of U. The left hand side of the equation is $0x_1 + 0x_2 + 0x_3 + 0x_4$. Clearly this will always equal 0. For the system to be solveable, clearly the corresponding right hand side $(b_3 - b_2 - b_1)$ must equal 0 as well – since $0 = 0$. Therefore, we know that the solvability of the system depends on $b_3 - b_2 - b_1$ being equal to 0:

$$b_3 - b_2 - b_1 = 0 \tag{2}$$

This also is a sanity check to our statement from before, since (2) can be rewritten as

$$b_3 = b_2 + b_1$$

Simply, if a function of the left hand side of a system of equations equals some number, then the same function applied to the right hand side of the equations must equal the same number. Let's consider a $\boldsymbol{b}$ where this constraint is satisfied and so we can solve the system. If $\boldsymbol{b} = \begin{bmatrix}1 & 5 & 6\end{bmatrix}^T$, then $b_1 + b_2 = b_3$. From (1) we know

$$\boldsymbol{b} \rightarrow \boldsymbol{b}^{'}$$

$$\begin{bmatrix}1\\5\\6\end{bmatrix} \rightarrow \begin{bmatrix}1\\3\\0\end{bmatrix}$$

And $b_3^{'} = 0$ satisfies (the same constraint but presented differently) $0 = 0$ in the last row of $\left[\ U \mid \boldsymbol{b}^{'}\ \right]$.

2.2 Larger picture and bringing in $C(A)$

We know that $A\boldsymbol{x} = \boldsymbol{b}$ is solveable when $\boldsymbol{b}$ is in $C(A)$ (the column space of A). We discussed in previous lectures what kind of conditions were required for this in terms of the columns of A. Now we discuss in terms of the *rows* of A. Consider the two (disjoint and complete) possible scenarios:

1. Rows are independent.
   - then row reduction will lead $U = I$, and so there will be a solution (a unique one).
2. Rows are not independent (i.e. dependent).
   - means a combination of the rows yields another row.
   - implies that a combination of the rows yields the zero row.
   - to be solveable, the same combination of components of $\boldsymbol{b}$ must yield zero.

3 Complete solution to $A\boldsymbol{x} = \boldsymbol{b}$

Just now we discussed the *solvability* of $A\boldsymbol{x} = \boldsymbol{b}$. Given the system is solveable (i.e. we can find at least one solution), what is the *complete* solution space.
Here is the procedure to *find the complete solution* to $A\boldsymbol{x} = \boldsymbol{b}$:

1. Find a particular solution: x_P
   - Set all free variables to zero (if any). We could choose any constant (e.g. 45238) but zero is convenient.
   - Solve $A\boldsymbol{x} = \boldsymbol{b}$ for the pivot variables.
   - Example:

$$\left[\begin{array}{cccc|c} \boxed{1} & 2 & 2 & 2 & 1 \\ 0 & 0 & \boxed{2} & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \qquad \text{free} : x_2, x_4 \qquad \text{pivot} : x_1, x_3$$

   By setting the free variables to 0, we only need to consider the pivot variables in the equations:

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ 2x_3 &= 3 \end{aligned}$$

   Back substituting yeilds $x_1 = -2$ and $x_3 = \frac{3}{2}$. And so a particular solution is

$$x_P = \begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}$$

2. Add the nullspace x_N
   - In adding the nullspace x_N to a particular solution x_P, we attain the *complete* solution x_C.
   - We found the nullspace of the system in the last lecture (all linear combinations of the special solutions), so x_C can be written as

$$x_C = \underbrace{\begin{bmatrix} -2 \\ 0 \\ \frac{3}{2} \\ 0 \end{bmatrix}}_{x_P} + \underbrace{c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{x_N}$$

3.1 Plot x_C

Here we plot all solutions x_C; i.e. the space described by x_C. Figure 1 shows all solutions $\boldsymbol{x}$ to

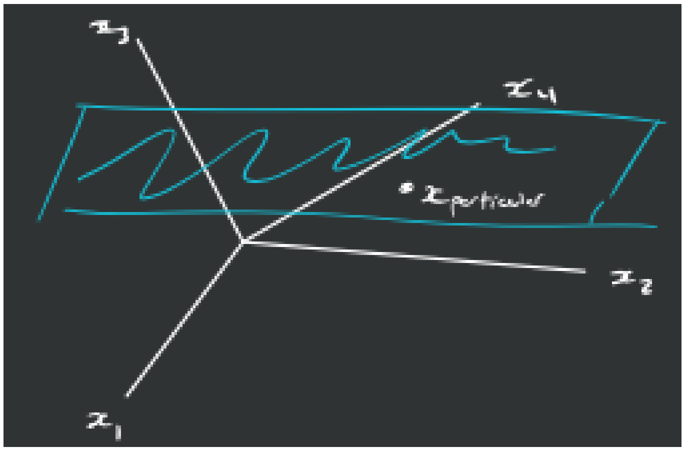

Figure 1: *A 2D plane that goes through x_P describing all solutions to the system $A\boldsymbol{x} = \boldsymbol{b}$.* The blue plane is the null space x_N.

$A\boldsymbol{x} = \boldsymbol{b}$ described earlier. It can be represented by a 2-dimensional plane embedded in $\mathbb{R}^4$ that goes through x_P. However, it does not go through the origin, and is thus not a vector space. And hence, not a subspace of $\mathbb{R}^4$. Instead, we call this an *affine space* which can informally be described as a vector space without considering the origin.

Also note that x_N is a subspace. x_N itself goes through the origin. The affine space described above places x_N *at* x_P; it is like shifting a function by a constant, but instead of a function we shift a subspace by the vector x_P.

3.2 Why $x_C = x_P + x_N$

Consider the particular solution x_P. Because it is a solution to $A\boldsymbol{x} = \boldsymbol{b}$, we can write

$$Ax_P = \boldsymbol{b} \tag{3}$$

Now consider any $x_N^* \in x_N$. By definition,

$$Ax_N^* = 0 \tag{4}$$

Putting (3) and (4) together,

$$A(x_P + x_N^*) = Ax_P + Ax_N^* = \boldsymbol{b} + 0 = \boldsymbol{b}$$

This is to say that if we have one solution to $A\boldsymbol{x} = \boldsymbol{b}$ (a particular solution), we can add anything onto it in the nullspace, because anything in the nullspace results in a zero right-hand-side, and so we still retain the correct right-hand-side of $\boldsymbol{b}$. Therefore, x_P plus any vector in the nullspace is a solution. So the space of solutions is $x_P + x_N$.

4 Generalized picture

Consider an $m \times n$ matrix A of rank r, where our current definition of *rank* is the number of pivots. We know that

- $r \leq m$ (a single row cannot contain more than one pivot) $\rightarrow$ "*Full Column Rank*"
- $r \leq n$ (a single column can't have more than one pivot) $\rightarrow$ "*Full Row Rank*"

4.1 Full Column Rank: $r = n$

We are especially interested when $r = n$; i.e. when there is a pivot in every column. This is known as A having *full rank*. This also implies that there are no free variables...that the nullspace is only the zero vector: $N(A) = \{\text{zero vector}\}$. Therefore, if a solution exists to $A\boldsymbol{x} = \boldsymbol{b}$, it is only x_P; since $x_C = x_P + 0$. Therefore, when A is of *full rank*, there is either *no solution* or there is a *single* solution.

As an example, take the following system:

$$A\boldsymbol{x} = \overbrace{\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}}^{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\boldsymbol{x}} = \overbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}}^{\boldsymbol{b}} = \boldsymbol{b}.$$

Here, the rank is 2 – column 1 and column 2 are independent. So we have an A is *full column rank*. This would yield the following row reduce echelon form ($rref$ aka R),

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

because the first two rows are independent, and the second two rows are combinations of the first two. Since we have a pivot in every column, there is nothing but the *zero vector* in the nullspace of A. Therefore, there will be *at most* one solution. What would $\boldsymbol{b}$ have to be so that there exists a solution $\boldsymbol{x}$? Well for one, a $\boldsymbol{b}$ that yields a solution $\boldsymbol{x}$ to the system could be

$$\begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix}$$

which is the sum of the columns of A. Certainly this is in the column space of A and so there exists a solution $\boldsymbol{x}$ which is equal to $\begin{bmatrix} 1 & 1 \end{bmatrix}^T$. And this is the only solution to our system below

$$A\boldsymbol{x} = \overbrace{\begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 6 & 1 \\ 5 & 1 \end{bmatrix}}^{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\boldsymbol{x}} = \overbrace{\begin{bmatrix} 4 \\ 3 \\ 7 \\ 6 \end{bmatrix}}^{\boldsymbol{b}} = \boldsymbol{b}.$$

4.2 Full Row Rank: $r = m$

We now move onto *full row rank*. This is the case where we have m pivots. For which right-hand-side $\boldsymbol{b}$ can we solve $A\boldsymbol{x} = \boldsymbol{b}$? The answer is for *every* $\boldsymbol{b}$. This is because...

- $\boldsymbol{b}$ and each column of A live in the same space: $\mathbb{R}^m$ where m is the number of rows in A.
- if there is a pivot in every row, there are m pivots. Therefore, there are m independent columns, and m independent columns span $\mathbb{R}^m$ and so they can get to every $\boldsymbol{b}$ which necessarily live in $\mathbb{R}^m$.

We've seen that if the rank is r, then we will have $n - r$ free variables. And since $r = m$ in the case of *full row rank*, we have $n - m$ free variables. For example,

$$A = \begin{bmatrix} 1 & 2 & 6 & 5 \\ 3 & 1 & 1 & 1 \end{bmatrix} \rightarrow \quad \text{rank} = 2$$
$$R = \left[\begin{array}{cc|cc} 1 & 0 & \multicolumn{2}{c}{\underbrace{\begin{matrix} - & - \\ - & - \end{matrix}}_{F}} \\ 0 & 1 & & \end{array}\right]$$

4.3 Full Rank (i.e. Invertible): $r = m = n$

A matrix A of *full rank* is a *square*, and *invertible* matrix. It will always reduce to the identity matrix in row reduced echelon form R (although the permutation of the rows/columns may differ). For example,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \rightarrow \quad \text{rank} = 2$$
$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

In this case, the nullspace is the zero vector, and there are *no* conditions on $\boldsymbol{b}$ for solvability. In other words, there will always be a solution to any $\boldsymbol{b}$, and that solution will be the only one.

5 Summary

5.1 Full Rank

- $r = m = n$
- A is square and invertible
- $R = I$
- for each $\boldsymbol{b}$...one solution...every time.

5.2 Full Column Rank

- $r = n < m$
- $R = \begin{bmatrix} I \\ 0 \end{bmatrix}$
- for each $\boldsymbol{b}$...either *one* solution, or *no* solutions.

5.3 Full Row Rank

- $r = m < n$
- $R = \begin{bmatrix} I & F \end{bmatrix}$ (with the caveat that I and F could be mixed together by the order of pivot columns).
- for each $\boldsymbol{b}$...always an infinite number of solutions

5.4 None of the above

- $r < m$ and $r < n$
- $R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$
- for each $\boldsymbol{b}$...either *zero* or *infinitely* many solutions.

In other words, *the rank tells you everything about the number of solutions you have to $A\boldsymbol{x} = \boldsymbol{b}$.* But doesn't tell you what the solutions are.