Lecture 02

Paul Scemama

OCW: *Here we introduce the method of elimination, an essential tool for working with matrices. The method follows a simple algorithm. To help make sense of material presented later, we describe this algorithm in terms of matrix multiplication.*

1 Setup

We will talk about the following 4 topics:

- Elimination and when it succeeds/fails
- Back-substitution
- Elimination matrices
- Matrix multiplication
- Permutation matrices (intro)
- Inverses (touched on)

Let's setup an example. We have a system of equations,

$$x + 2y + z = 2 \tag{1}$$
$$3x + 8y + z = 12 \tag{2}$$
$$4y + z = 2 \tag{3}$$

So we have,

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \tag{4}$$

2 Elimination

Our first goal is to *knockout (reduce, eliminate)* x in equation 2.

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}$$

where we call $\boxed{1}$ the first *pivot*. The reason being is we will use this element to *knockout* x in equation 2. We notice that if we subtract from row 2, $3\times$ row 1, we can eliminate x in equation 2.

$$\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} \overset{r_2-3r_1}{\rightarrow} \begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

where $\boxed{2}$ is now the second *pivot*. Now we want to eliminate y in equation 3. We now use this second pivot to knockout y in equation 3 by noticing that if we subtract from row 3, $2\times$ row 2, we can eliminate y in equation 3.

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 4 & 1 \end{bmatrix} \overset{r_3-2r_2}{\rightarrow} \begin{bmatrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{bmatrix} = U \text{ for "upper-triangular"}$$

where $\boxed{\cdot}$ denotes all three pivots. *Note* that pivots cannot be 0.

How could this have failed (failed to come up with* 3 *pivots)?

1. 0 in the pivot position; try to exchange rows.
   - Can get out of trouble if nonzero below.
2. Can't exchange to get out of trouble...**failure**
   - Matrix is *not invertible.*

Above, we've ignored b, but we need to manipulate b with the same operations. Here we do this,

$$\overbrace{\begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}}^{b} \overset{r_2-3r_1}{\rightarrow} \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} \overset{r_3-2r_2}{\rightarrow} \overbrace{\begin{bmatrix} 2 \\ 6 \\ -10 \end{bmatrix}}^{c}$$

where we call the resultant vector (after manipulation) c by convention.

3 Back-substitution

Back-substitution is the name for solving the equivalent system $Ux = c$ that results from *elimination.* The reason for the name "back-substitution" comes from the fact that we first use the last equation (which is $z = -10$) and plug it into the equation above to solve for y, and then using y (and z) to solve the final and first equation for x. We are both substitution and going backwards.

Let us do it for our example; we can express $Ux = c$ as

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 6 \\ 0 & 0 & 5 & -10 \end{array}\right] \tag{5}$$

The equations for this system are

$$
\begin{aligned}
x + 2y + z &= 2 \\
2y - 2z &= 6 \\
5z &= -10
\end{aligned}
$$

And to perform back-substitution we do

$$
\begin{array}{lll}
\text{In (3), solve for } z\text{:} & 5z = -10 & \rightarrow\ z = -2 \\
\text{In (2), solve for } y \text{ using } z = -2\text{:} & 2y - 2(-2) = 6 & \rightarrow\ y = 1 \\
\text{In (1), solve for } x \text{ using } z = -2 \text{ and } y = 1\text{:} & x + 2(1) + (-2) = 2 & \rightarrow\ x = 2
\end{array}
$$

This is the end of what you could call *part* 1 of the lecture; where we've discussed two topics that come hand in hand - elimination and back-substitution. We now move onto *elimination matrices* in which we view elimination as multiplying A by matrices to get to U.

4 Elimination matrices

We now want to express elimination *steps* with matrices. But first, we make a special note of how we can think about a $A \cdot x$ versus $x \cdot A$.

4.1 Digression

The first thing to notice is that

$$
Matrix \times Column = Column
$$

$$
\begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \text{a combination of the columns of } A = 3col_1 + 4col_2 + 5col_3
$$

Now there is a parallel for rows.

$$
Row \times Matrix = Row
$$

$$
\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} - & - & - & - \\ - & - & - & - \\ - & - & - & - \end{bmatrix} = \text{a combination of the rows of } A = 1row_1 + 2row_2 + 7row_3
$$

4.2 Digression end

Let us start with the same A as we started with in *part one* of the lecture.

$$
A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}
$$

4.3 Step 1

Recall the first step of elimination for A - we subtracted $3 \times row_1$ from row_2. How can we express this as a matrix E_{21} such that when it is multiplied to A, the result is as if we subtracted $3 \times row_1$ from row_2? (We call this $E_{ij} = E_{21}$ because we will see that $E_{21} \cdot A$ will result in the $2,1$ position of A to become 0).

We learned in the *digression* that $row \times matrix = row$. Let's go row by row. The first row of A we do not want to alter. More precisely, $row \times matrix =$ a combination of the *matrix's* rows.

As an example, let $\{r_1, r_2, r_3\}$ denote the rows of a matrix A, and suppose that we wanted to multiply A by something so that the product is a row itself and a combination of the rows of A, namely $3r_1 + 4r_2 + 283r_3$. How would we do this? Well of course we would multiply A by a row vector, and this row vector would contain the coefficients used in creating the combination of $\{r_1, r_2, r_3\}$, so this row vector would be

$$\begin{bmatrix} 3 & 4 & 283 \end{bmatrix}$$

Using this knowledge, we can find E_{21} such that when it is multiplied to A, the result is as if we subtracted $3 \times row_1$ from row_2.

$$\overbrace{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}}^{E_{21}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

We don't want to change the first row of A when we do this multiplication by E_{21}, so we want

$$\overbrace{\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}}^{E_{21}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

The first row of E_{21} should then be,

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ - & - & - \\ - & - & - \end{bmatrix}}^{E_{21}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & 1 \\ - & - & - \\ - & - & - \end{bmatrix}$$

We want the second row of A to be subtracted by 3 times the first row. In other words we want the second row of the output to equal the second row of A subtracted by 3 times the first row of A,

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ - & - & - \\ - & - & - \end{bmatrix}}^{E_{21}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & 1 \\ & \begin{bmatrix} -3r_1 + r_2 + 0r_3 \end{bmatrix} & \\ - & - & - \end{bmatrix}$$

From our knowledge attained in the *digression*, the second row of E_{21} should then be,

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ - & - & - \end{bmatrix}}^{E_{21}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & 1 \\ & [-3r_1 + r_2 + 0r_3] & \\ - & - & - \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ - & - & - \end{bmatrix}$$

And finally we don't want to alter the last row,

$$\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{E_{21}} \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

4.4 Step 2

The second step we made was subtract 2 times the second row from the third row. So we multiply the result from above by E_{32},

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix} = U$$

The steps together can be expressed as,

$$E_{32}(E_{21}A) = U$$

Because of the associate property of matrices, we can write

$$(E_{32} \cdot E_{21})A = U$$

5 Permutation matrices (intro)

Another type of *elementary* matrix is a *permutation matrix* which exchanges rows. For example, suppose we have the following matrix,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

What matrix would we multiply to it to exchange its rows?

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

And what about switching columns?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

6 Inverses (touched on)

There's a better way to do matrix elimination that won't be apparent until a bit later.

- Instead of saying: *how do I get from $A \to U$*
- Rather, *how do I get from $U \to A$*

This introduces the notion of inverses. Think of a matrix M_2 that we apply to a matrix A that does some operation on A and outputs a resultant matrix.

$$M_2 A = A^{'}$$

We want to multiply $A^{'}$ by some other matrix M_1 that *reverses* what M_2 did. We want

$$M_1 A^{'} = A$$

Because $A^{'} = M_2 A$,

$$M_1 A^{'} = M_1(M_2 A) = (M_1 M_2)A \quad \text{which we } \textit{want} \text{ to} = A$$

So we want

$$(M_1 M_2)A = A$$

So it is clear that we want M_1 such that

$$(M_1 M_2) = I$$

since $IA = A$

In short, to find the inverse of a matrix M_2 we find an M_1 such that $M_1 M_2 = I$.