Lecture 07

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OCW: *We apply the method of elimination to all matrices, invertible or not. Counting the pivots gives us the rank of the matrix. Further simplifying the matrix puts it in reduced row echelon form R and improves our description of the nullspace.*

1 Outline

- Computing the nullspace ($A\boldsymbol{x} = 0$).
- Pivot variables & free variables.
- Special solutions: $rref(A) = R$.

2 Computing the nullspace

As is often common, let's introduce this section with an example matrix A,

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$$

Notice that the rows and the columns both contain vectors that are a *linear combination* others. We call a collection of vectors in which this is the case a *dependent* set of vectors. When we do something like *elimination* on A, we would expect it to uncover this dependence (or lack thereof) in the columns or rows.

Something to note is that when we do elimination, we *do not* change the nullspace or the solutions to the system; but we *do* actually change the column space. However, *elimination* uncovers both properties of the column space and of the nullspace.

2.1 Uncovering pivots and dependency in A

We're now going to go through elimination to uncover some connections between A and its nullspace and column space. We begin by identifying the first pivot and *eliminating* the coefficients beneath it.

$$A = \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

In knocking out the coefficients beneath the first pivot, we've also knocked out the coefficient on where a pivot *would be located* in the next column (denoted by the circled 0). Therefore, we have no pivot in the second column. In this situation, we continue onto the next column to see if there is a pivot; however the fact that the second column had no pivot tells us that the second column is dependent on the columns previous. I.e. it is a linear combination of the previous columns (in this case $col2 = 2col1$).

Let's continue doing elimination and see what else we find.

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & \boxed{2} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U = \text{Echelon Form} \tag{1}$$

Again, we see that in knocking out the coefficients below the pivot (denoted by a surrounding box) in the third column, we also knockout the *would be* pivot in the fourth column. This indicates that the fourth column is a linear combination of the first three columns.

We have now discovered that there are two pivots in A. We will call this the *rank* of A. This is a very crucial number in all of linear algebra as it tells us a lot about the matrix A. For one, it tells us how *"large"* the column space is in some sense. We will discuss more later on.

2.2 Writing solutions for $Ax = 0$

Remark: When considering a system $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b} \neq \mathbf{0}$, the relation $U\boldsymbol{x} = \boldsymbol{b}'$ was equivalent, where $\boldsymbol{b'}$ followed the same elimination steps as from A to U. Therefore, we could not say a solution to $A\boldsymbol{x} = \boldsymbol{b}$ was a solution to $U\boldsymbol{x} = \boldsymbol{b}$. However, in the case of $A\boldsymbol{x} = \mathbf{0}$, any elimination steps to take A to U as applied to $\mathbf{0}$ would result in $\mathbf{0}$ again. Therefore, we can say that a solution to $A\boldsymbol{x} = \mathbf{0}$ is the same solution to $U\boldsymbol{x} = \mathbf{0}$.

The A we've been considering is clearly not invertible–it does not have a pivot in every column (there are other ways to say the same thing). How do we write solutions for $A\boldsymbol{x} = 0$ in this case?

The first thing to do is to separate the *pivot* variables (columns) and the *free* variables (columns), where by *free* column we mean the column that didn't have a pivot. The reason for calling it *free* is because the value of the variable associated with it can be anything since its coefficient is 0. With this in mind, we can say x_2 and x_4 (corresponding to second and fourth columns of A) can be anything; and then we can solve for x_1 and x_3 (corresponding to the pivot columns). And this process will provide us with solutions to the system.

As an example, let's look at $U\boldsymbol{x} = 0$ from (1). Taking the equations out of matrix form we have:

$$x_1 + 2x_2 + 2x_3 + 2x_4 = 0 \tag{2}$$
$$2x_3 + 4x_4 = 0 \tag{3}$$

Like we said, we can choose any two values for x_2 and x_4 and then solve for x_1 and x_3 resulting in a solution to $A\boldsymbol{x} = 0$. Let's suppose we choose $x_2 = 1$ and $x_4 = 0$ (**this is just a choice, but it is a convenient one.**). So we have

$$x_1 + 2(1) + 2x_3 + 0 = x_1 + 2 + 2x_3 = 0$$
$$2x_3 + 0 = 2x_3 = 0$$

Then in solving for x_1 and x_3 we get

$$x_3 = 0;\ x_1 = -2$$

and so a solution to $A\boldsymbol{x} = 0$ is

$$\boldsymbol{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

This solution is saying -2 *times* the first column of A plus 1 *times* the second column of A equals the *zero* column. Now we have a single vector in the null space of A. What are some others? Well, because we know the null space is a vector space, any multiple of (4) is also in the null space. I.e. the following line in 4-dimensional space is in the null space of A:

$$c\boldsymbol{x} = c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

A question: is this the whole null space? No it is not. We have two free variables and we considered all the solutions to $A\boldsymbol{x} = 0$ (vectors in the null space) for different values of the free variables x_2 while holding x_4 constant at 0. So now let's do the same but with different values of x_4 while holding x_2 constant at 0. Doing so results in another set of vectors in the null space described by the line:

$$d \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} \tag{6}$$

Now that we've done this, are we done? Are the set of vectors described by the line in (5) and the line in (6) the null space of A? The answer is no; because we have only considered different values of one free variable while holding the other at 0 and vice versa. How do we get all those missing vectors that are found by co-varying the free variables? We can actually just take all linear combinations between any of the vectors in (5) and any of the vectors in (6). So we can say the null space is

$$\boldsymbol{x} = c \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{special} + d \underbrace{\begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}}_{special}$$

Additionally, we call the vectors above the *special solutions*. These are the solutions we achieve by setting one of the free variables to 1 and the others to 0...and so on. Then we can say the null space is exactly all linear combinations of the special solutions. The number of special solutions = the number of free variables.

3 *rref(A)* $= R$

There exists a more convenient (yet directly related) form than the *Echelon* form. Starting from where we left off, we have

$$U = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

It is nice to notice that $r3$ appears through elimination due it being a linear combination of $r1$ and $r2$.

The idea is to do elimination *upwards*. In other words, we want zeros **above** and **below** the pivots. Doing so looks like

$$\begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And it is also nice to have 1s in the pivot positions as well:

$$\rightarrow \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = rref(A) = R$$

3.1 Extracting *special* solutions from R

We can get the special solutions immediately (without having to set one free variable to 1 and the others to 0 and then solving...) from inspecting R.

> Recall the ***remark*** made earlier. This extends to R as well so that we can write:
>
> $$A\boldsymbol{x} = 0 \rightarrow U\boldsymbol{x} = 0 \rightarrow R\boldsymbol{x} = 0$$
>
> For example, the *special* solutions to A are the same *special* solutions to R.

The special solutions are easy for $R\boldsymbol{x} = 0$. Suppose the first r columns of A are the pivot columns. This results in R looking like this,

$$R = \begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix}$$

where the pivot columns turn into I due to elimination and we call the free columns F. The row(s) of zeros come from dependent rows. Recall that finding $\boldsymbol{x}$ in

$$R\boldsymbol{x} = 0$$

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \boldsymbol{x} = 0$$

constitutes finding the null space. Notice that

$$\begin{bmatrix} I & F \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -F \\ I \end{bmatrix} = I(-F) + FI = 0$$

We call $\begin{bmatrix} -F \\ I \end{bmatrix} = N$. The columns of N solve $R\boldsymbol{x} = 0$. N is the *null space* matrix and provides a basis for the null space. Recall that we set the free variables to 1 and the others to 0, and then solve for the pivot variables to attain each special solution. This is equivalent to saying that in each special solution, the free variables are a column of I, and then the pivot variables are a column of $-F$. Those specials solutions give the null space matrix N. Finally, the idea is still true if the pivot columns are mixed in with the free columns. Even though I and F will be mixed together, you can still see $-F$ in the solutions.