Physics Based Animation

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Impulse based multi-body animation

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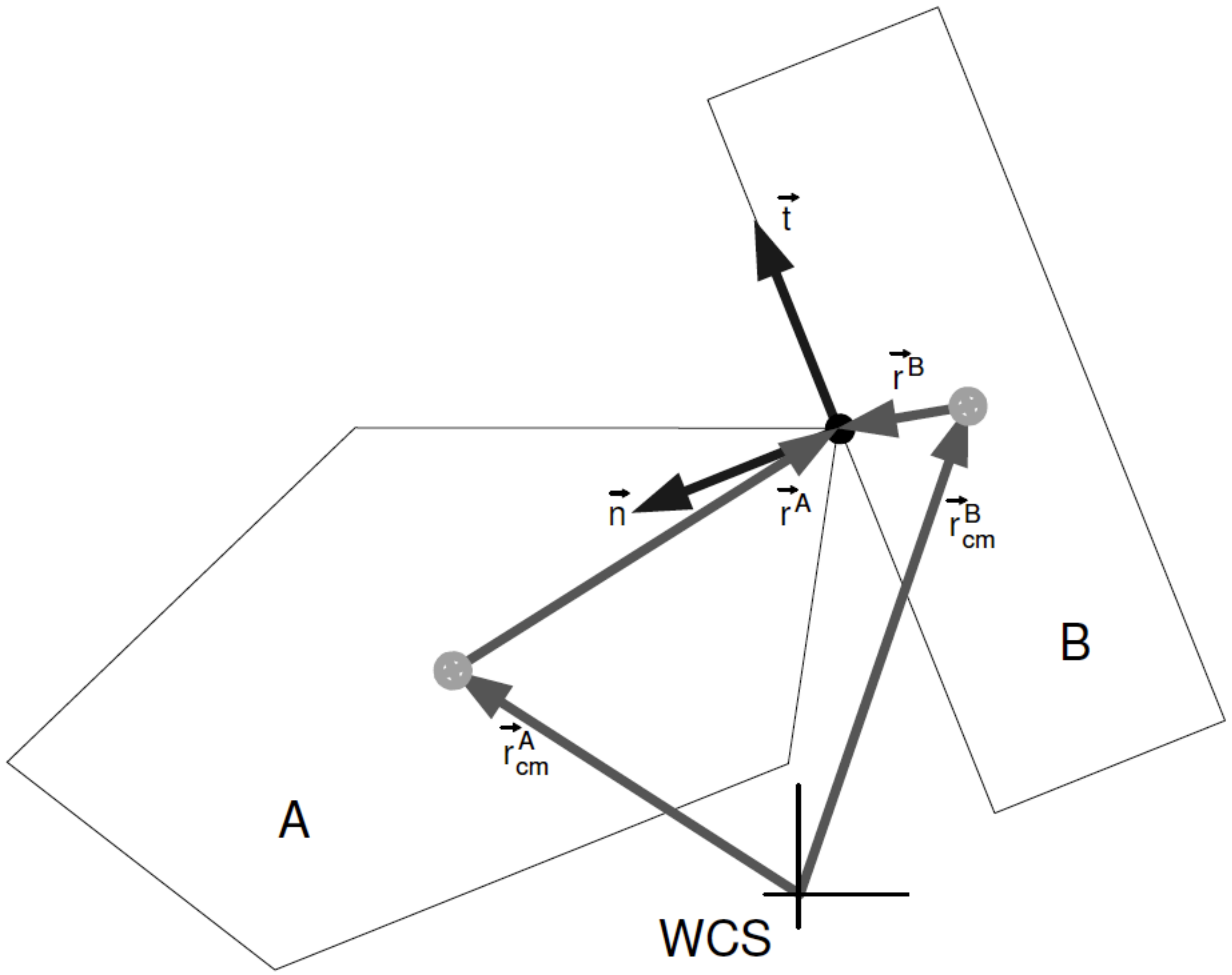
What’s an impulse-based contact?

- We simulate contacts using collision impulses.
- The impulse is the integral of the force F over time. Since force is a vector quantity, impulse is also a vector quantity. Impulse applied to an object produces an equivalent vector change in its linear momentum (mass*velocity), also in the resultant direction.
- Why do we like impulse-based contacts? Because it’s fast!

Single point contact

- Single point contacts are not very realistic since collisions usually happens over a surface.
- We will assume we can comput the normal at collision point p.
- The normal by convention is always pointing from object B to A.
- We can always reach the collision point from either object’s center of mass.

$$
\begin{aligned}
\boldsymbol{p}^{A} &= \boldsymbol{r}^{A} + \boldsymbol{r}^{A}_{\mathrm{cm}}, \\
\boldsymbol{p}^{B} &= \boldsymbol{r}^{B} + \boldsymbol{r}^{B}_{\mathrm{cm}}, \\
\boldsymbol{p}^{A} &= \boldsymbol{p}^{B}.
\end{aligned}
$$

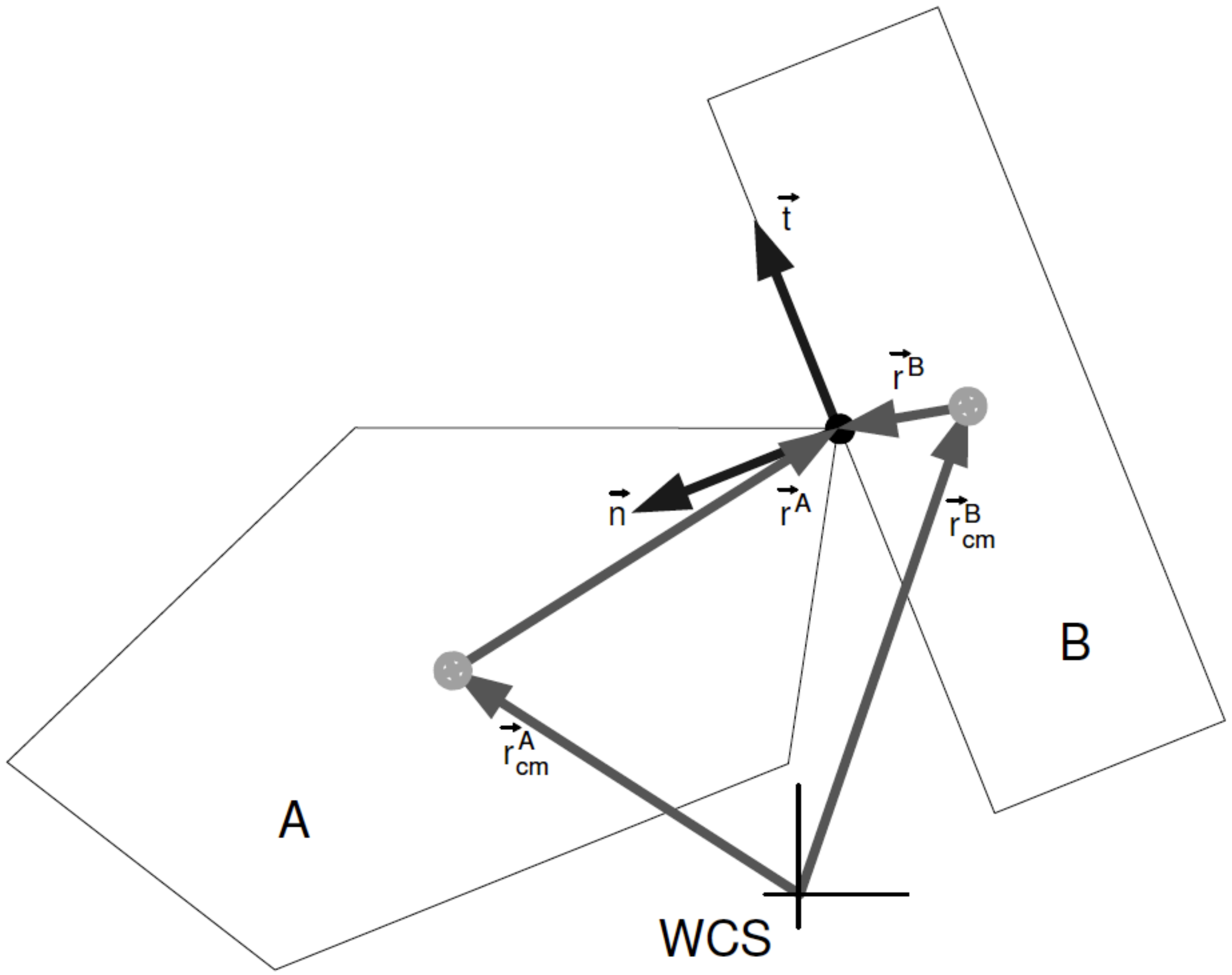


Single point contact

- In 3D the contact point and contact normal define a contact plane. We can define the plane in 2 ways: explicitely for every point or using two orthogonal vectors w.r.t normal lying on the plane.

$$\forall \boldsymbol{x} \in \mathbb{R}^3 \qquad \text{where} \qquad \boldsymbol{n} \cdot \boldsymbol{x} - \boldsymbol{n} \cdot \boldsymbol{p} = 0.$$

$$\boldsymbol{n} = \boldsymbol{t}_1 \times \boldsymbol{t}_2.$$



Single point contact - relative contact velocity

Recall, for a generic object X $\boldsymbol{p}^{X} = \boldsymbol{r}^{X} + \boldsymbol{r}_{\mathrm{cm}}^{X}$, We can use Body Frame notation $\boldsymbol{r}^{X} = \boldsymbol{R}^{X} \left[\boldsymbol{r}^{X}\right]_{\mathrm{BF}_X}$,

Obtaining $\boldsymbol{p}^{X} = \boldsymbol{R}^{X} \left[\boldsymbol{r}^{X}\right]_{\mathrm{BF}_X} + \boldsymbol{r}_{\mathrm{cm}}^{X}$. If we take the derivative we get:

$$\dot{\boldsymbol{p}}^{X} = \dot{\boldsymbol{R}}^{X} \left[\boldsymbol{r}^{X}\right]_{\mathrm{BF}_X} + \boldsymbol{v}_{\mathrm{cm}}^{X},$$
$$\dot{\boldsymbol{p}}^{X} = \left[\ \boldsymbol{\omega}^{X} \times \boldsymbol{R}_x^{X} \,\middle|\, \boldsymbol{\omega}^{X} \times \boldsymbol{R}_y^{X} \,\middle|\, \boldsymbol{\omega}^{X} \times \boldsymbol{R}_z^{X}\ \right] \left[\boldsymbol{r}^{X}\right]_{\mathrm{BF}_X} + \boldsymbol{v}_{\mathrm{cm}}^{X},$$
$$\dot{\boldsymbol{p}}^{X} = \boldsymbol{\omega}^{X} \times \left(\boldsymbol{R}^{X} \left[\boldsymbol{r}^{X}\right]_{\mathrm{BF}_X}\right) + \boldsymbol{v}_{\mathrm{cm}}^{X},$$

There's a trick! If we add zero $$\dot{\boldsymbol{p}}^{X} = \boldsymbol{\omega}^{X} \times \left(\underbrace{\boldsymbol{R}^{X} \left[\boldsymbol{r}^{X}\right]_{\mathrm{BF}_X} + \boldsymbol{r}_{\mathrm{cm}}^{X}}_{\boldsymbol{p}^{X}} - \boldsymbol{r}_{\mathrm{cm}}^{X}\right) + \boldsymbol{v}_{\mathrm{cm}}^{X}.$$ Which is equal to:

$$\dot{\boldsymbol{p}}^{X} = \boldsymbol{\omega}^{X} \times \left(\boldsymbol{p}^{X} - \boldsymbol{r}_{\mathrm{cm}}^{X}\right) + \boldsymbol{v}_{\mathrm{cm}},$$

Now we can compute the relative contact velocity $\boldsymbol{u} = \dot{\boldsymbol{p}}^{A} - \dot{\boldsymbol{p}}^{B}$.

And relative normal contact velocity $$u_n = \boldsymbol{n}^{T}\left(\dot{\boldsymbol{p}}^{A} - \dot{\boldsymbol{p}}^{B}\right),$$

Why the relative velocity is important?

- Because now we can tell if the 2 objects are separating, resting or colliding.

$$\text{relative contact} = \begin{cases} \text{Separating} & \text{if } u_n > 0 \\ \text{Resting} & \text{if } u_n = 0 \\ \text{Colliding} & \text{if } u_n < 0 \end{cases} .$$

- And we can also derive our collision test:

$$u_n = \boldsymbol{n}^T \left(\left(\boldsymbol{\omega}^{\,A} \times \boldsymbol{r}^{\,A} + \boldsymbol{v}_{cm}^{A} \right) - \left(\boldsymbol{\omega}^{\,B} \times \boldsymbol{r}^{\,B} + \boldsymbol{v}_{cm}^{B} \right) \right) < 0.$$

How contacts work in real life (sort of)

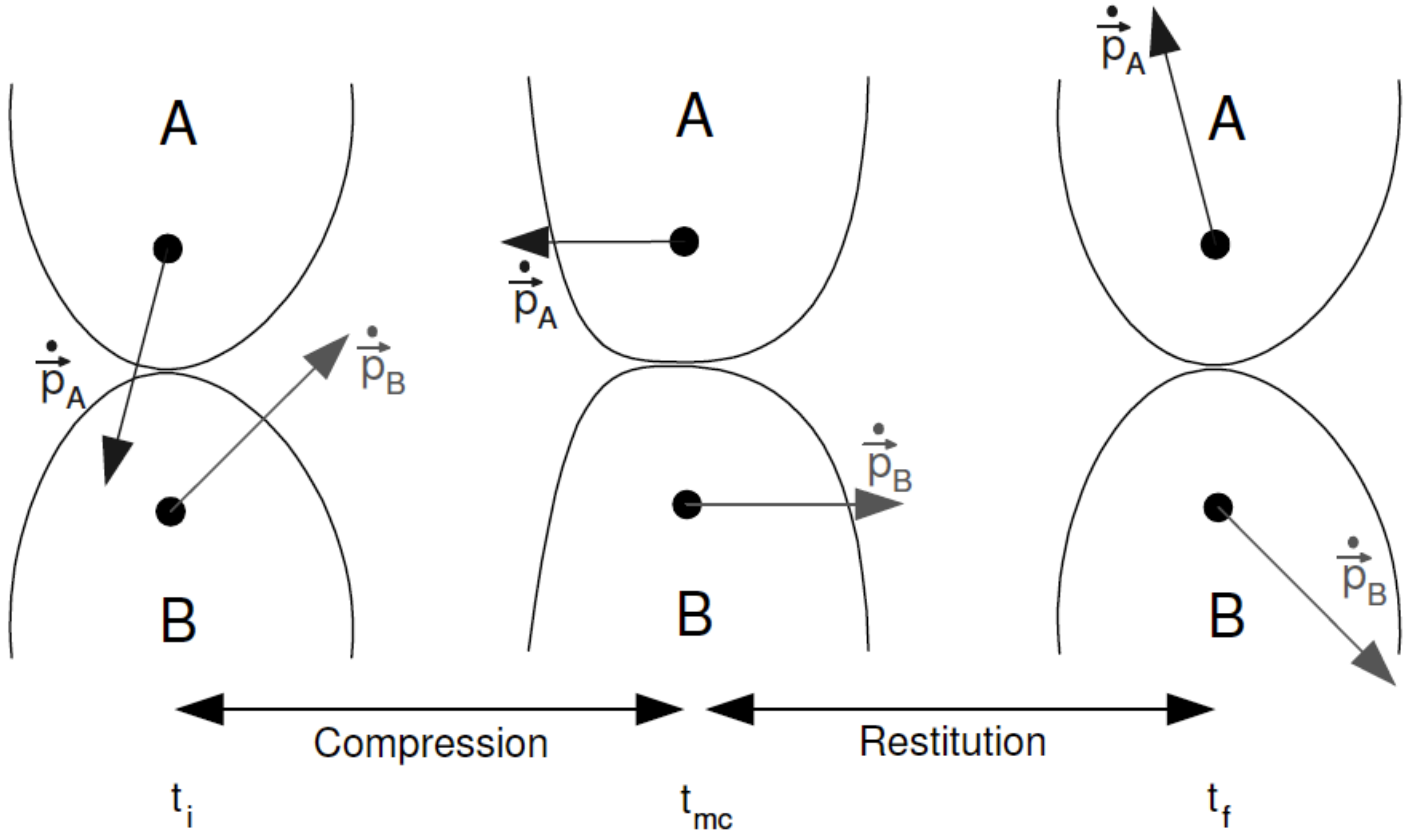
Objects move away from each other

Kinetic Energy

Impact

Elastic Energy
(objects deform)

Dissipated in Heat



Rigid bodies

Definition 6.1 (The Rigid Body Assumptions)
During a collision, the following four assumptions apply to rigid bodies:

- *The duration of the collision is zero, that is* $t_f - t_i \rightarrow 0$.
- *Only impulsive forces must be used to avoid penetration.*
- *Positions and orientations are the same before and after the collision.*
- *Non impulsive forces have no effect during the collision.*

Impulses

$$\boldsymbol{J} = \int_{t_i}^{t_f} \boldsymbol{F} dt.$$

Which can also be seen as the difference in momentum $\boldsymbol{P} = m\boldsymbol{v}_{\mathrm{cm}}$:

$$\boldsymbol{F} = \frac{d\boldsymbol{P}}{dt},$$

$$\int_{t_i}^{t_f} \boldsymbol{F} dt = \int_{t_i}^{t_f} d\boldsymbol{P},$$

$$\boldsymbol{J} = \boldsymbol{P}(t_f) - \boldsymbol{P}(t_i),$$

$$\boldsymbol{J} = \Delta \boldsymbol{P}.$$

Since $t_f - t_i \rightarrow 0.$ We need to reparametrize w.r.t time

Time always increases $\frac{d\gamma}{dt} > 0.$ So does the impulse in the normal direction $\frac{dJ_n}{dt} > 0.$

This has to hold: $\frac{d\gamma}{dJ_n} > 0.$

Collision laws

- *Algebraic Collision Laws*
- *Incremental Collision Laws*
- *Full Deformation Collision Laws*
- *Compliant Contact Model Collision Laws*

$$\textit{Collision Law} \Leftarrow \textit{Physical Law} + \textit{Contact Model}$$

Physical Law — Impact

Newton's impact law

$$u_n(\gamma_f) = -e u_n(\gamma_i), \qquad 0 \le e \le 1.$$

Poisson's hypothesis

$$J_n(\gamma_f) - J_n(\gamma_{\mathrm{mc}}) = e J_n(\gamma_{\mathrm{mc}}), \qquad 0 \le e \le 1.$$

Stronge's hypothesis

$$W_n(\gamma_f) - W_n(\gamma_{mc}) = -e^2 W_n(\gamma_{mc}), \qquad 0 \le e \le 1.$$

$$W = \int_{t_i}^{t_f} \boldsymbol{F}(t) \cdot \dot{\boldsymbol{x}}(t) dt.$$ Reparametrize $$W = \int_{\gamma_i}^{\gamma_f} \boldsymbol{F}(\gamma) \cdot \dot{\boldsymbol{x}}(\gamma) \left(\frac{dt}{d\gamma}\right) dt.$$

$$\boldsymbol{F}(\gamma) = \tfrac{d}{dt} \boldsymbol{J}(\gamma), \qquad \dot{\boldsymbol{x}}(\gamma) = \boldsymbol{u}_A(\gamma) \qquad W_A = \int_0^{\gamma_f} \boldsymbol{u}_A(\gamma) \cdot \frac{d}{d\gamma} \boldsymbol{J}(\gamma) d\gamma.$$

Physical Law — Friction

Coulomb's friction law

$$\boldsymbol{F}_t = -\mu \frac{\boldsymbol{u}_t}{\|\boldsymbol{u}_t\|} \|\boldsymbol{F}_n\|,$$

$$\dot{\boldsymbol{u}}_t = 0, \qquad \|\boldsymbol{F}_t\| \leq \mu \|\boldsymbol{F}_n\|.$$

$$\dot{\boldsymbol{u}}_t \neq 0, \qquad \|\boldsymbol{F}_t\| = \mu \|\boldsymbol{F}_n\| \quad \text{and} \quad \boldsymbol{F}_t \cdot \dot{\boldsymbol{u}}_t \leq 0.$$

Contact model — how the velocity changes

Newton's 2nd law $$\boldsymbol{F}^{\,A}(\gamma) = m_A \boldsymbol{a}_{\text{cm}}^{A}(\gamma),$$

Euler's equation of motion $$\boldsymbol{\tau}^{\,A}(\gamma) = \boldsymbol{r}^{\,A} \times \boldsymbol{F}^{\,A}(\gamma) = \boldsymbol{I}_A \boldsymbol{\alpha}^{\,A}(\gamma) + \cancel{\boldsymbol{\omega}^{\,A}(\gamma) \times \boldsymbol{I}_A \boldsymbol{\omega}^{\,A}(\gamma).}$$

Alpha is angular acceleration

Integrating:
$$\int_{\gamma_i}^{\gamma} \boldsymbol{F}^{\,A}(\gamma) d\gamma = \int_{\gamma_i}^{\gamma} m_A \boldsymbol{a}_{\text{cm}}^{A}(\gamma) d\gamma,$$
$$\int_{\gamma_i}^{\gamma} \boldsymbol{r}^{\,A} \times \boldsymbol{F}^{\,A}(\gamma) d\gamma = \int_{\gamma_i}^{\gamma} \boldsymbol{I}_A \boldsymbol{\alpha}^{\,A}(\gamma) d\gamma.$$

$$\boldsymbol{J}^{\,A}(\gamma) = m_A \underbrace{\left(\boldsymbol{v}_{\text{cm}}^{A}(\gamma) - \boldsymbol{v}_{\text{cm}}^{A}(\gamma_i)\right)}_{\Delta \boldsymbol{v}_{\text{cm}}^{A}(\gamma)},$$
$$\boldsymbol{r}^{\,A} \times \boldsymbol{J}^{\,A}(\gamma) = \boldsymbol{I}_A \underbrace{\left(\boldsymbol{\omega}^{\,A}(\gamma) - \boldsymbol{\omega}^{\,A}(\gamma_i)\right)}_{\Delta \boldsymbol{\omega}^{\,A}(\gamma)}.$$

Now we have a relationship between impulses and change in velocities

$$\boldsymbol{J}^{\,A}(\gamma) = m_A \Delta \boldsymbol{v}_{\text{cm}}^{A}(\gamma),$$
$$\boldsymbol{r}^{\,A} \times \boldsymbol{J}^{\,A}(\gamma) = \boldsymbol{I}_A \Delta \boldsymbol{\omega}^{\,A}(\gamma).$$

Contact model — how the momentum changes

$$\boldsymbol{J}^{\,A}(\gamma) = m_A \Delta \boldsymbol{v}_{\text{cm}}^{A}(\gamma),$$
$$\boldsymbol{r}^{\,A} \times \boldsymbol{J}^{\,A}(\gamma) = \boldsymbol{I}_A \Delta \boldsymbol{\omega}^{\,A}(\gamma).$$

$$\longrightarrow$$

$$\Delta \boldsymbol{v} = \frac{\boldsymbol{J}}{m},$$
$$\Delta \boldsymbol{\omega} = \boldsymbol{I}^{-1}\left(\boldsymbol{r} \times \boldsymbol{J}\right).$$

$$\dot{\boldsymbol{p}}^{\,A}(\gamma) = \boldsymbol{\omega}^{\,A}(\gamma) \times \boldsymbol{r}^{\,A} + \boldsymbol{v}_{\text{cm}}^{A}(\gamma).$$

$$\Delta \dot{\boldsymbol{p}}^{\,A}(\gamma) = \Delta \boldsymbol{\omega}^{\,A}(\gamma) \times \boldsymbol{r}^{\,A} + \Delta \boldsymbol{v}_{\text{cm}}^{A}(\gamma),$$

By substituting and doing some algebra we get:

$$\Delta \boldsymbol{u}(\gamma) = \underbrace{\left(\left(\frac{1}{m_A} + \frac{1}{m_B}\right)\mathbf{1} - \left(\left(\boldsymbol{r}^{\boldsymbol{A}}\right)^{\times} \boldsymbol{I}_A^{-1} \left(\boldsymbol{r}^{\boldsymbol{A}}\right)^{\times} + \left(\boldsymbol{r}^{\boldsymbol{B}}\right)^{\times} \boldsymbol{I}_B^{-1} \left(\boldsymbol{r}^{\boldsymbol{B}}\right)^{\times}\right)\right)}_{\boldsymbol{K}} \boldsymbol{J}^{\,A}(\gamma).$$

Properties of the collision matrix

The collision matrix $\boldsymbol{K}$ *is*

- *constant*
- *symmetric*
- *positive definite*
- *invertible*

$$\Delta \boldsymbol{u}(\gamma) = \underbrace{\left(\left(\frac{1}{m_A} + \frac{1}{m_B}\right)\mathbf{1} - \left(\left(\boldsymbol{r}^{\boldsymbol{A}}\right)^{\times} \boldsymbol{I}_A^{-1} \left(\boldsymbol{r}^{\boldsymbol{A}}\right)^{\times} + \left(\boldsymbol{r}^{\boldsymbol{B}}\right)^{\times} \boldsymbol{I}_B^{-1} \left(\boldsymbol{r}^{\boldsymbol{B}}\right)^{\times}\right)\right)}_{\boldsymbol{K}} \boldsymbol{J}^{\,A}(\gamma).$$

Differential Form of Impulse-Momentum Relation

$$\frac{d}{d\gamma}\boldsymbol{u}(\gamma) = \boldsymbol{K}\frac{d}{d\gamma}\boldsymbol{J}(\gamma),$$
$$\frac{d}{d\gamma}\boldsymbol{J}(\gamma) = \boldsymbol{K}^{-1}\frac{d}{d\gamma}\boldsymbol{u}(\gamma).$$

This notation comes handy in the Stronge's hypothesis in the contact model.

$$W = \int_0^{\gamma_f} \boldsymbol{u}(\gamma) \cdot \boldsymbol{K}^{-1}\frac{d}{d\gamma}\boldsymbol{u}(\gamma)d\gamma,$$