Physics-Based Animation

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Chapter 2 - Articulated Figures

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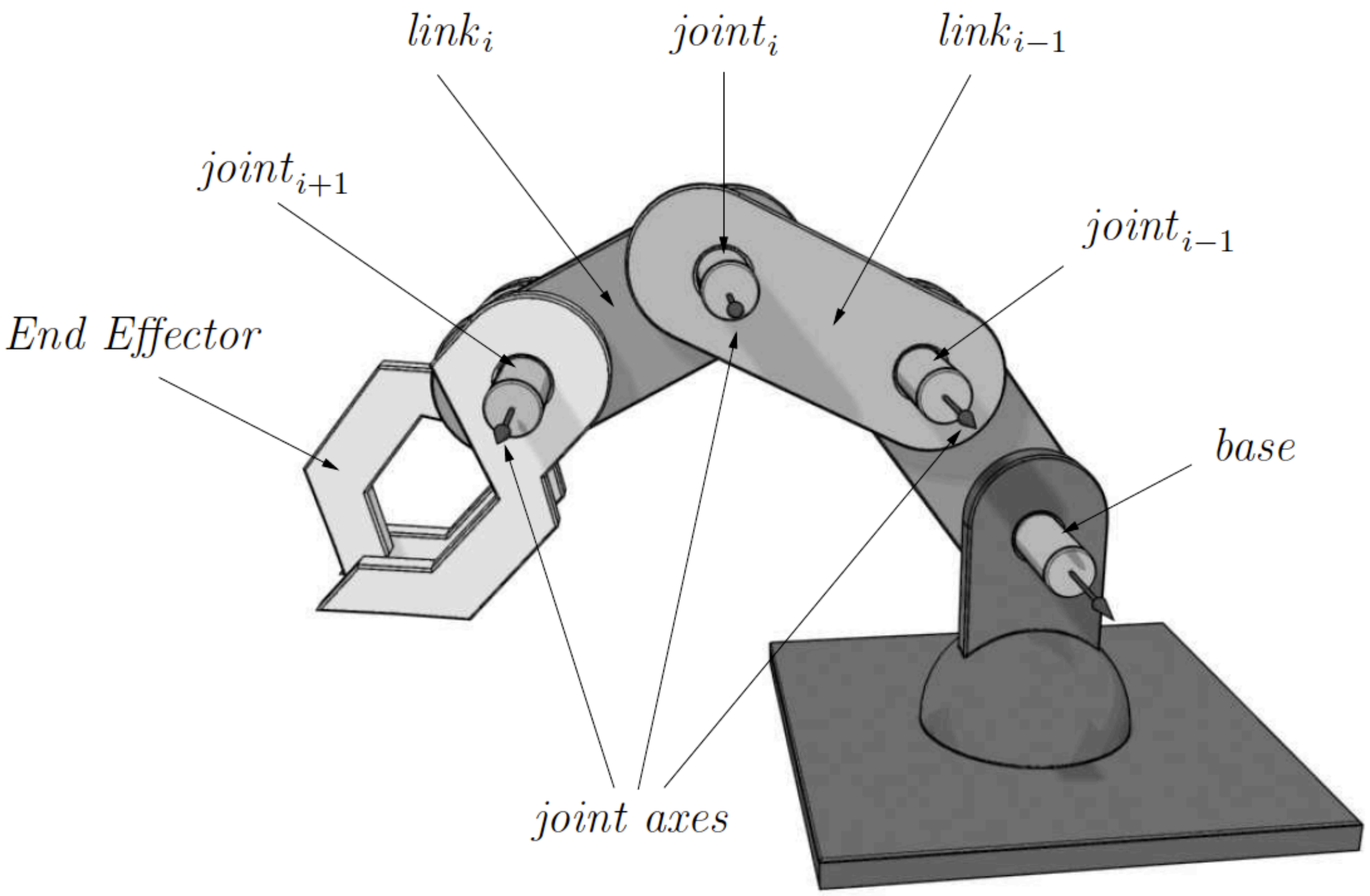
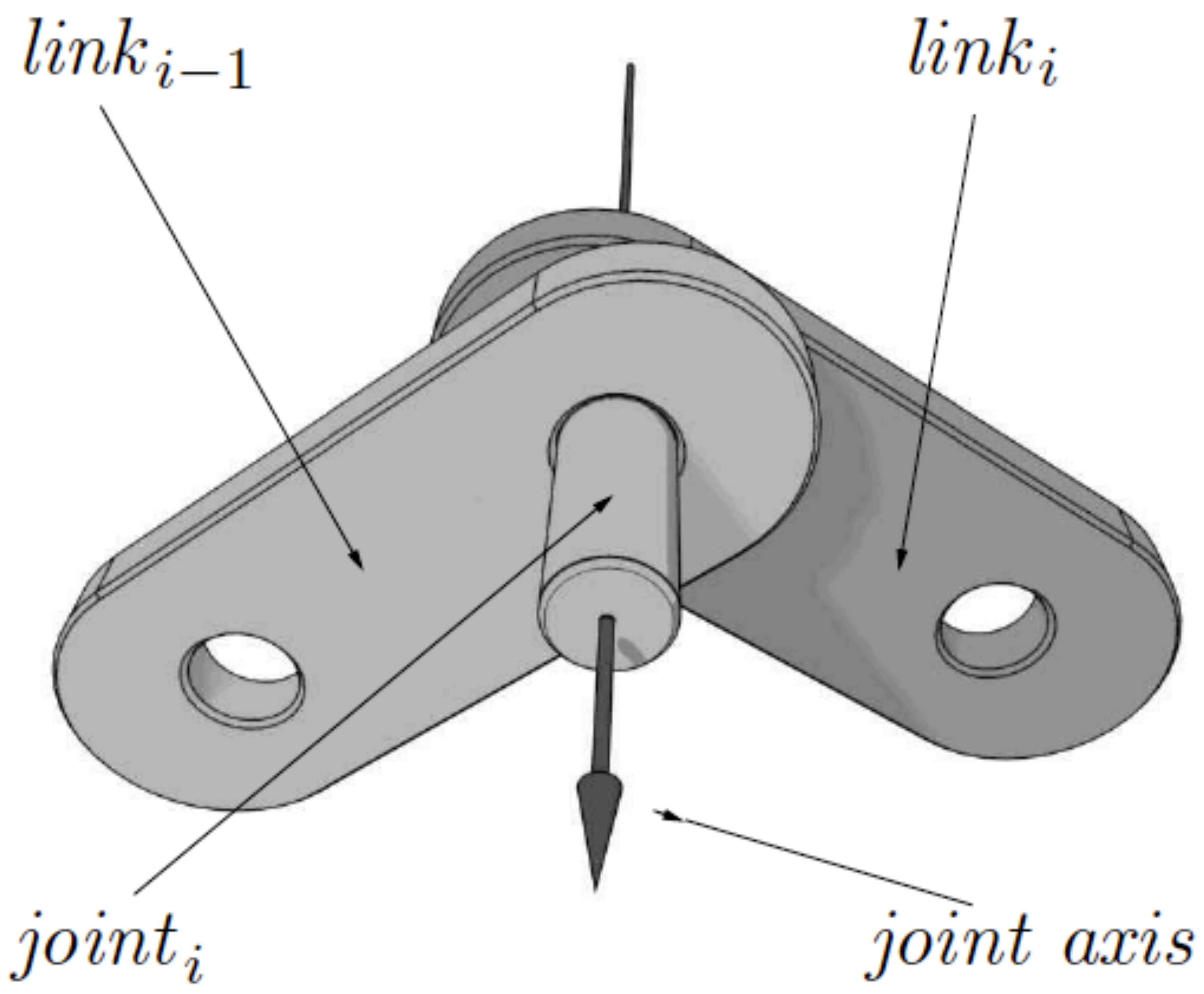
What's an articulated figure & when it's used

- An articulated figure is a construction made of links and joints.

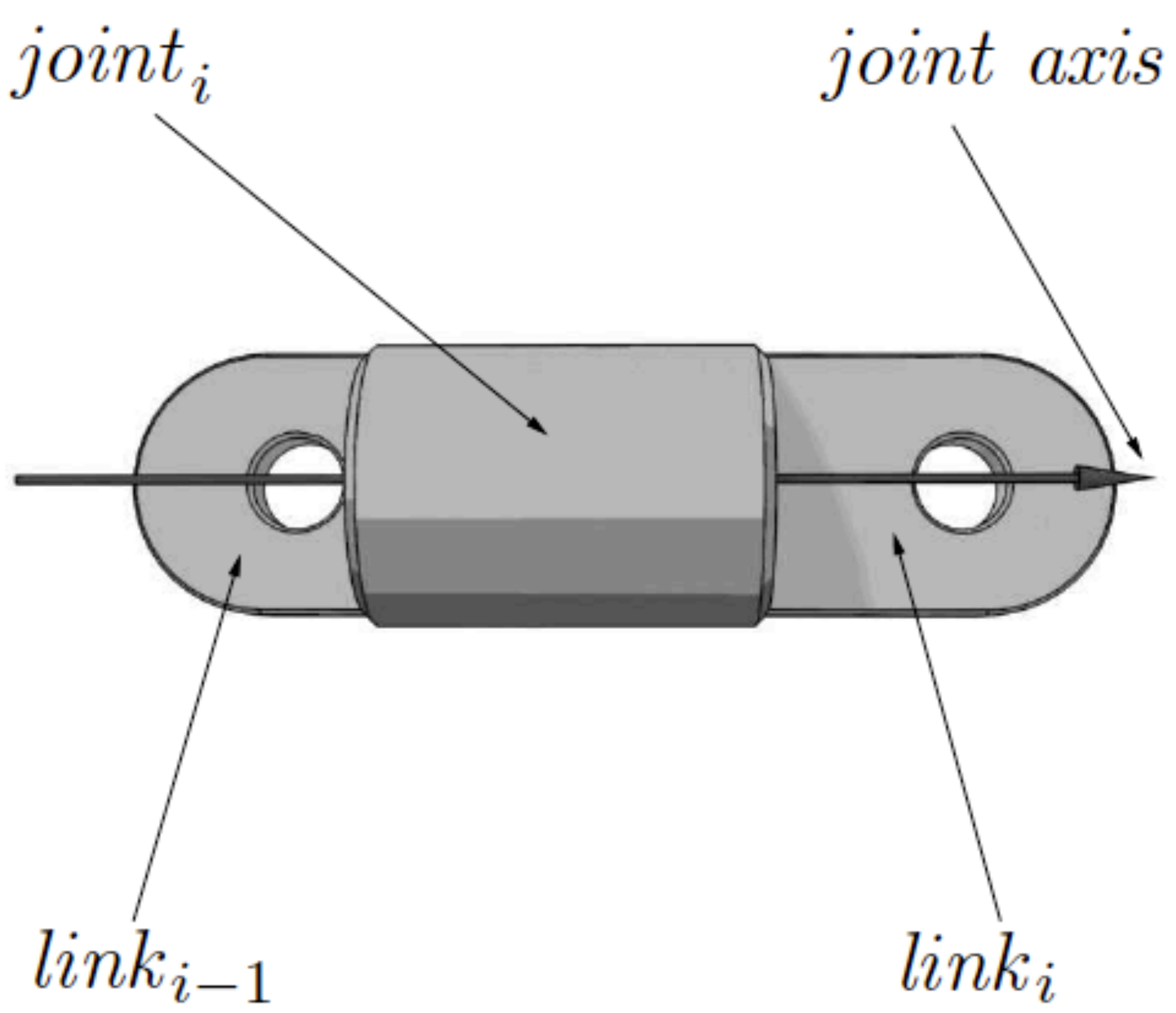
- Articulated figures are at the core of Forward and Inverse kinematics.



Joint types



Revolute (or Hinge) joint



Prismatic (or Slide) joint

How do we describe articulated figures?

- **Paired joint coordinates**

- **Denavit-Hartenberg**

- *The idea for both notations is to have a way to express points on links with respect to any link or joint frame.*

Paired joint coordinates

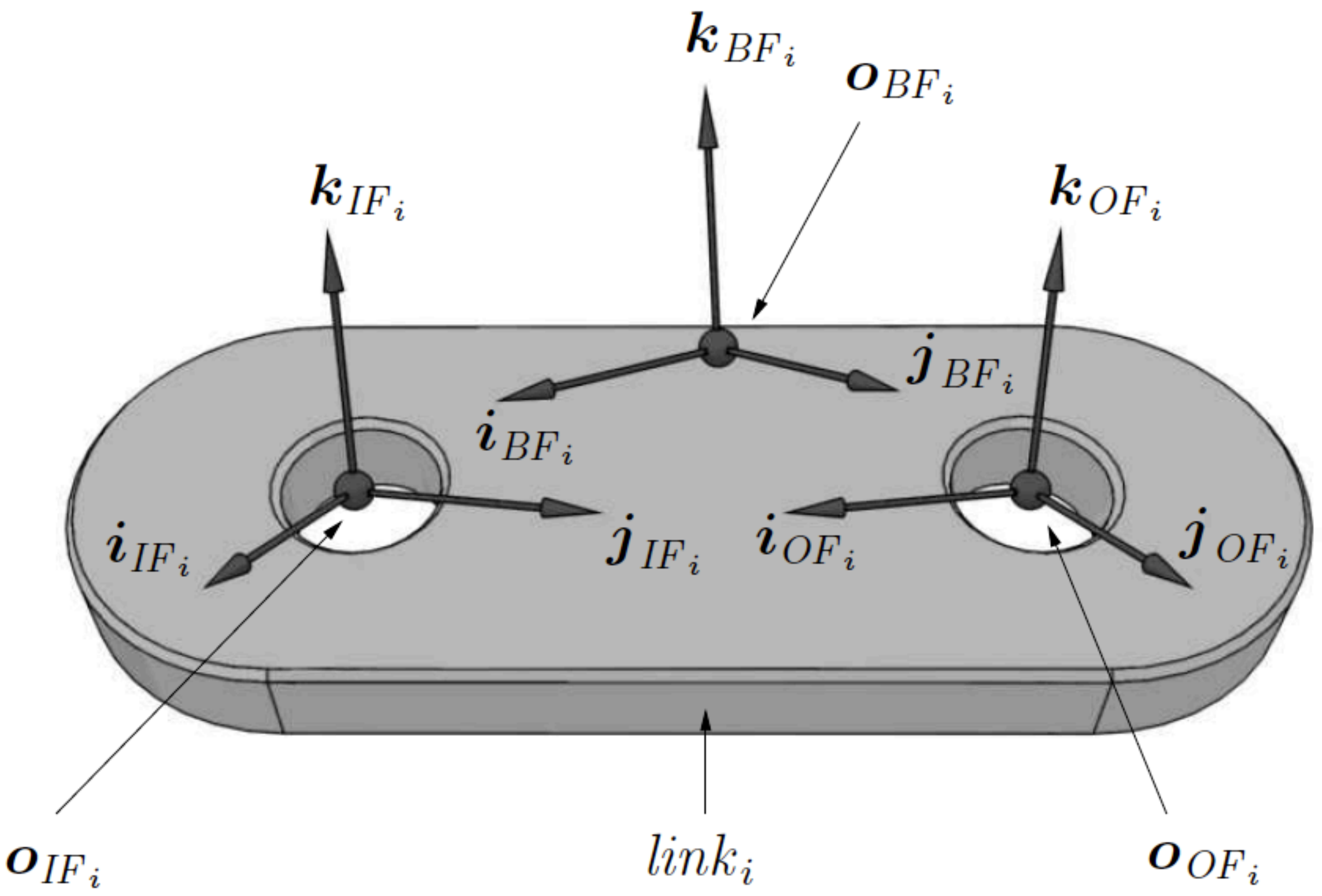
- This notation uses 3 frames per link: body frame, inner frame and outer frame.

- Body Frame (BF): associated with link i , usually the origin is set at the center of mass of the link.

- Inner frame (IF): associated with joint i , the origin is set somewhere on the joint axis.

- Outer Frame (OF): associated with joint i , the origin is set somewhere on the joint axis.

- For IF and OF the origin and axis are specified in the BF.



Now that we have these frames what?

- We can start expressing points in different coordinate systems using some transformations.

$$^{TO}\mathbf{T}_{FROM}$$

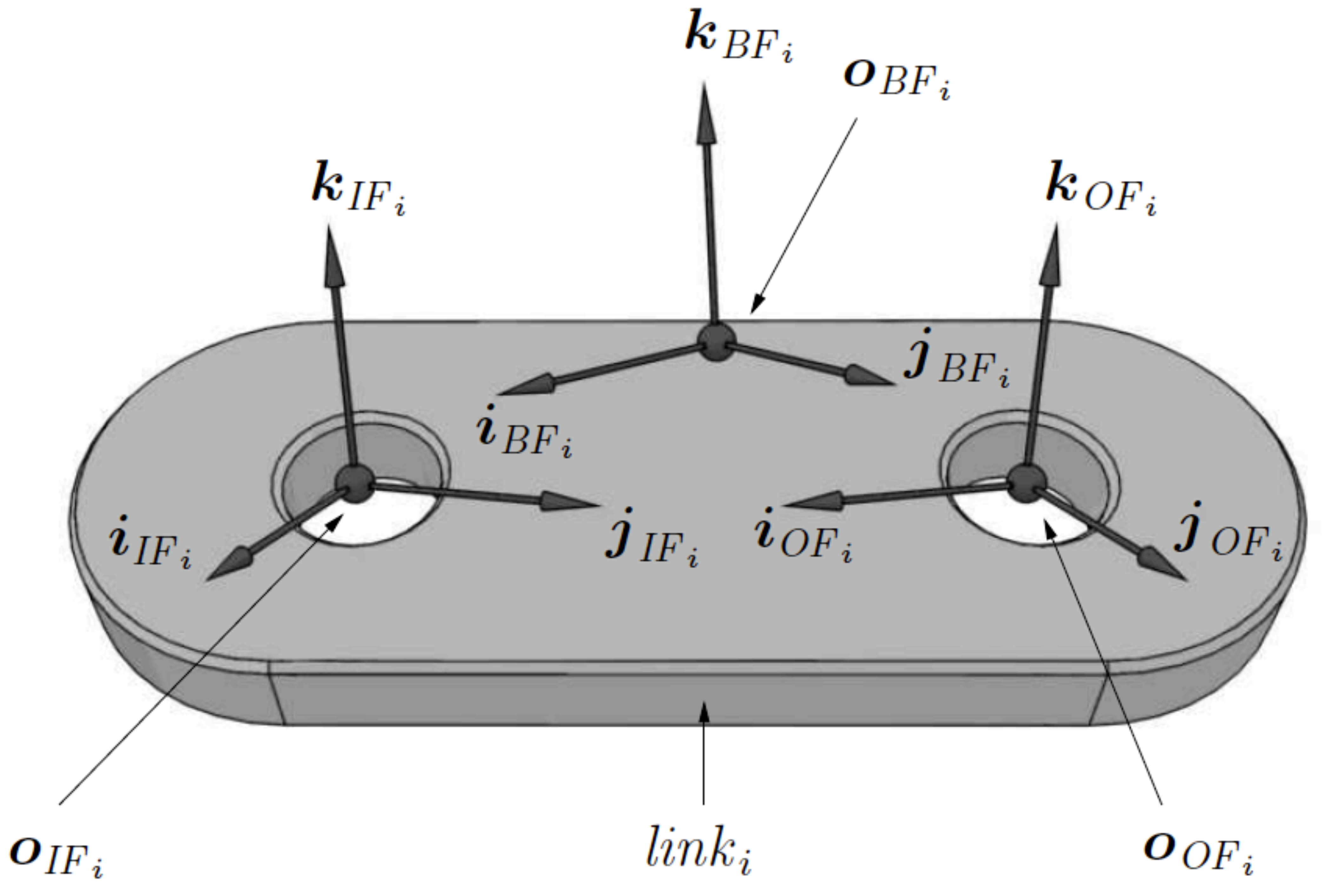
- We're gonna need 4 tranforms:

$$^{BF_i}\boldsymbol{T}_{IF_i} = \boldsymbol{T}_{IF_i}(\boldsymbol{r}_{IF_i})\boldsymbol{R}_{IF_i}(\varphi_{IF_i}, \boldsymbol{u}_{IF_i})$$

$$\boldsymbol{T}_{IF_i}(\boldsymbol{r}_{IF_i}) = \begin{bmatrix} \mathbf{1} & \boldsymbol{r}_{IF_i} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$\boldsymbol{R}_{IF_i}(\varphi_{IF_i}, \boldsymbol{u}_{IF_i}) = \begin{bmatrix} \boldsymbol{i}_{IF_i} & \boldsymbol{j}_{IF_i} & \boldsymbol{k}_{IF_i} & \mathbf{0} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For $^{BF_i}\boldsymbol{T}_{OF_i}$ is exactly the same but using the OF to compute the matrices.



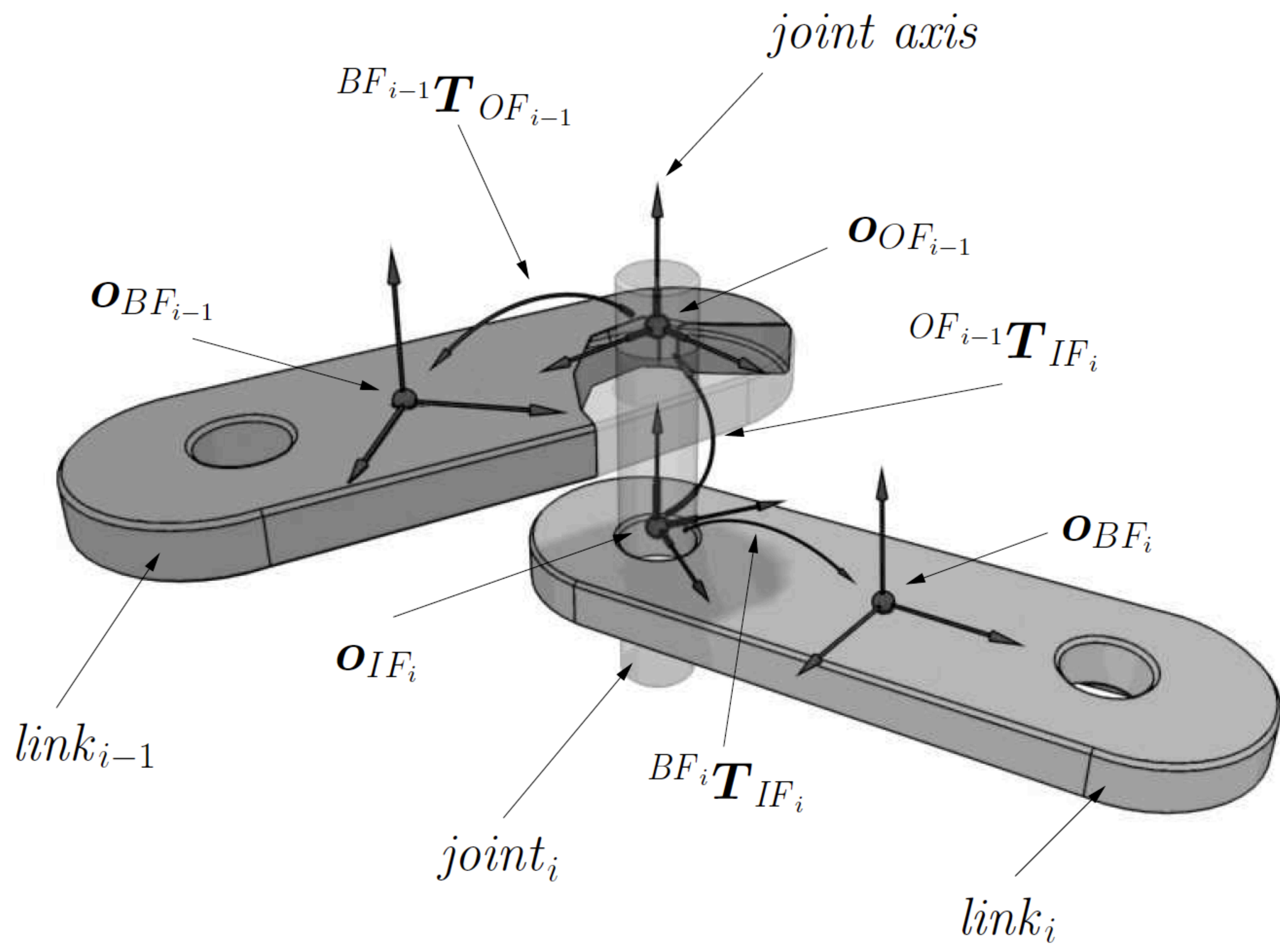
Now that we have these frames what?

- The remaining 2 transforms:

$$^{OF_{i-1}}\boldsymbol{T}_{IF_i}(\boldsymbol{d}_i, \varphi_i, \boldsymbol{u}_i) = \boldsymbol{T}_i(\boldsymbol{d}_i)\boldsymbol{R}_i(\varphi_i, \boldsymbol{u}_i)$$

$$\boldsymbol{T}(\boldsymbol{d_i}) = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{R}(\varphi_i, \boldsymbol{u}_i) = \begin{bmatrix} u_x^2 + (1-u_x^1)c\varphi & u_z u_y(1-c\varphi) - u_z s\varphi & u_x u_z(1-c\varphi) + u_y s\varphi & 0 \\ u_x u_y(1-c\varphi) + u_z s\varphi & u_y^2 + (1-u_y^2)c\varphi & u_y u_z(1-c\varphi) - u_x s\varphi & 0 \\ u_x u_z(1-c\varphi) - u_y s\varphi & u_y u_z(1-c\varphi) + u_x s\varphi & u_z^2 + (1-u_z^2)c\varphi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- For the last one $^{(i-1)}\boldsymbol{T}_i(\boldsymbol{d}_i, \varphi_i, \boldsymbol{u}_i)$
we just cook everything together:

$$\left(^{BF_{i-1}}\boldsymbol{T}_{OF_{i-1}}\right)\left(^{OF_{i-1}}\boldsymbol{T}_{IF_i}(\boldsymbol{d}_i, \varphi_i, \boldsymbol{u}_i)\right)\left(^{BF_i}\boldsymbol{T}_{IF_i}\right)^{-1}$$

Denavit-Hartenberg notation

- 4 parameters:

link length	a_i	The perpendicular distance between the axes of $joint_i$ and $joint_{i+1}$.
link twist	α_i	The angle between the axes of $joint_i$ and $joint_{i+1}$. The angle α_i is measured around the $\boldsymbol{x}_i$-axis. Positive angles are measured counterclockwise when looking from the tip of vector $\boldsymbol{x}_i$ toward its foot.
link offset	d_i	The distance between the origins of the coordinate frames attached to joint $joint_{i-1}$ and $joint_i$ measured along the axis of $joint_i$. For a prismatic joint this is a joint parameter.
joint angle	φ_i	The angle between the link lengths $\boldsymbol{a}_{i-1}$ and $\boldsymbol{a}_i$. The angle φ_i is measured around the $\boldsymbol{z}_i$-axis. Positive angles are measured counterclockwise when looking from the tip of vector $\boldsymbol{z}_i$ toward its foot. For a revolute joint this is a joint parameter.

- A bit redundant since every joint has max 1 DOF, if we want more we need to stack joints and connect them with links of length 0.

Link length

- 3 methods to compute it: pseudo-naive, geometric and analytic.

Parametrise

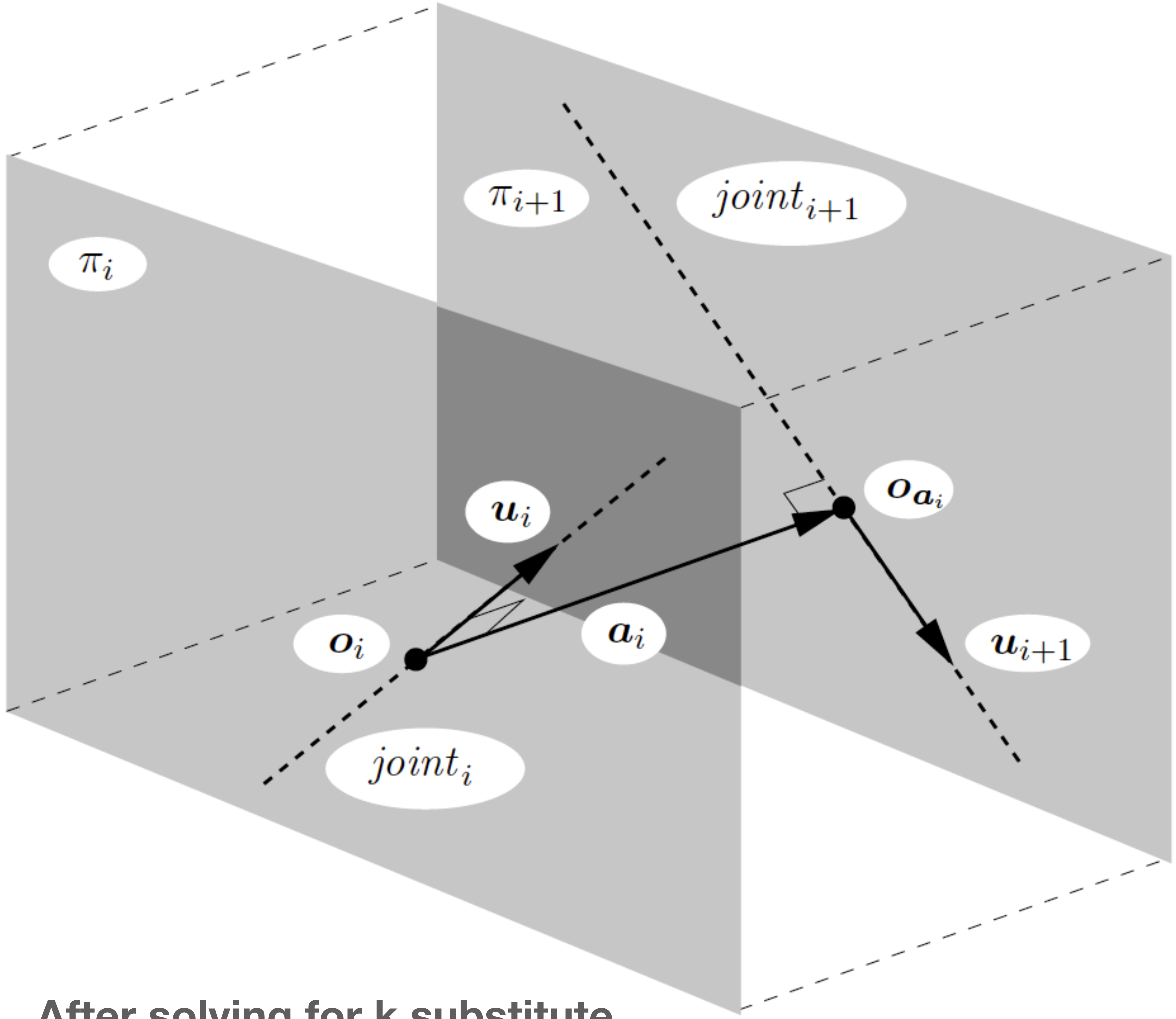
$$\boldsymbol{l}_i(s) = \boldsymbol{p}_i + s\boldsymbol{u}_i$$
$$\boldsymbol{l}_{i+1}(t) = \boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1}$$

$$\boldsymbol{p}_i + s\boldsymbol{u}_i + k\boldsymbol{u}_i \times \boldsymbol{u}_{i+1} = \boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1}$$

Solve for s, k, t

$$a_i = \frac{(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \cdot (\boldsymbol{u}_i \times \boldsymbol{u}_{i+1})}{\left\|\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}\right\|_2}$$

NOTE: k is what we're looking for



$$\boldsymbol{o}_i = \boldsymbol{l}_i(s) = \boldsymbol{p}_i + \frac{(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \cdot (\boldsymbol{u}_i \left\|\boldsymbol{u}_{i+1}\right\|_2^2 - \boldsymbol{u}_{i+1}(\boldsymbol{u}_i \cdot \boldsymbol{u}_{i+1}))}{\left\|\boldsymbol{u}_i\right\|_2^2 \left\|\boldsymbol{u}_{i+1}\right\|_2^2 - (\boldsymbol{u}_i \cdot \boldsymbol{u}_{i+1})^2}\boldsymbol{u}_i$$

$$\boldsymbol{o}_{\boldsymbol{a}_i} = \boldsymbol{l}_{i+1}(t) = \boldsymbol{p}_{i+1} + \frac{(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \cdot (\boldsymbol{u}_i(\boldsymbol{u}_i \cdot \boldsymbol{u}_{i+1}) - \boldsymbol{u}_{i+1} \left\|\boldsymbol{u}_i\right\|_2^2)}{\left\|\boldsymbol{u}_i\right\|_2^2 \left\|\boldsymbol{u}_{i+1}\right\|_2^2 - (\boldsymbol{u}_i \cdot \boldsymbol{u}_{i+1})^2}\boldsymbol{u}_{i+1}$$

After solving for k substitute

Link length - analytic approach

Parametrise

$$\boldsymbol{l}_i(s) = \boldsymbol{p}_i + s\boldsymbol{u}_i$$
$$\boldsymbol{l}_{i+1}(t) = \boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1}$$

Treats the problem
like an optimisation problem

$$a_i = \min d(s,t) = \min \sqrt{\left(\boldsymbol{l}_{i+1}(t) - \boldsymbol{l}_i(s)\right) \cdot \left(\boldsymbol{l}_{i+1}(t) - \boldsymbol{l}_i(s)\right)}$$
$$= \min \sqrt{\left(\boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1} - \boldsymbol{p}_i - s\boldsymbol{u}_i\right) \cdot \left(\boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1} - \boldsymbol{p}_i - s\boldsymbol{u}_i\right)}$$

The trick is to
find the stationary points.
For this to be true we can set just
just the numerator = 0 and solve
for s and t finding the two origin points.

$$\frac{\partial d(s,t)}{\partial s} = \frac{\left(\boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1} - \boldsymbol{p}_i - s\boldsymbol{u}_i\right) \cdot \boldsymbol{u}_i}{\sqrt{\left(\boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1} - \boldsymbol{p}_i - s\boldsymbol{u}_i\right) \cdot \left(\boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1} - \boldsymbol{p}_i - s\boldsymbol{u}_i\right)}} = 0$$
$$\frac{\partial d(s,t)}{\partial t} = \frac{\left(\boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1} - \boldsymbol{p}_i - s\boldsymbol{u}_i\right) \cdot \boldsymbol{u}_{i+1}}{\sqrt{\left(\boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1} - \boldsymbol{p}_i - s\boldsymbol{u}_i\right) \cdot \left(\boldsymbol{p}_{i+1} + t\boldsymbol{u}_{i+1} - \boldsymbol{p}_i - s\boldsymbol{u}_i\right)}} = 0$$

$$\boldsymbol{a}_i = \boldsymbol{o}_{\boldsymbol{a}_i} - \boldsymbol{o}_i$$
$$a_i = \left\|\boldsymbol{a}_i\right\|_2 = \left\|\boldsymbol{o}_{\boldsymbol{a}_i} - \boldsymbol{o}_i\right\|_2$$

It can be shown that the vector is perpendicular to both joint axes

Special cases to consider when computing link length

- The joint axes intersect

$$\boldsymbol{a}_i = \frac{\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}}{\left\|\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}\right\|_2}$$
$$a_i = 0$$

- The joint axes are parallel

$$\boldsymbol{a}_i = (\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) - \left((\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \cdot \frac{\boldsymbol{u}_i}{\left\|\boldsymbol{u}_i\right\|_2}\right) \frac{\boldsymbol{u}_i}{\left\|\boldsymbol{u}_i\right\|_2}$$
$$a_i = \left\|\boldsymbol{a}_i\right\|_2$$

- The first joint: we can choose the coordinate frame arbitrarily, but we can be clever and choose it to be the exact same frame as link 1.

- The last joint: we only need to specify the axis of the last joint. So we can choose the remaining parameters so that the others are 0.

Coordinate frame attachment

1. **The Origin:** Let the origin of the i^{th} link frame be at the point $\boldsymbol{o}_i$ on the axis of $joint_i$.

2. **The $\boldsymbol{z}_i$-axis:** Let the $\boldsymbol{z}_i$-axis be along the i^{th} joint axis. That is, let the $\boldsymbol{z}_i$ be parallel to vector $\boldsymbol{u}_i$ from (2.41)

$$\boldsymbol{z}_i = \frac{\boldsymbol{u}_i}{\left\|\boldsymbol{u}_i\right\|_2} \tag{2.42}$$

3. **The $\boldsymbol{x}_i$-axis:** Let the $\boldsymbol{x}_i$-axis be along the link vector $\boldsymbol{a}_i$ from (2.32a)

$$\boldsymbol{x}_i = \frac{\boldsymbol{a}_i}{\left\|\boldsymbol{a}_i\right\|_2} \tag{2.43}$$

4. **The $\boldsymbol{y}_i$-axis:** Let the $\boldsymbol{x}_i$-axis be such that the vectors $\boldsymbol{x}_i, \boldsymbol{y}_i, \boldsymbol{z}_i$ form a right-handed orthogonal coordinate system. That is, let $\boldsymbol{y}_i$ be given as

$$\boldsymbol{y}_i = \frac{\boldsymbol{z}_i \times \boldsymbol{x}_i}{\left\|\boldsymbol{z}_i \times \boldsymbol{x}_i\right\|_2} \tag{2.44}$$

The link twist

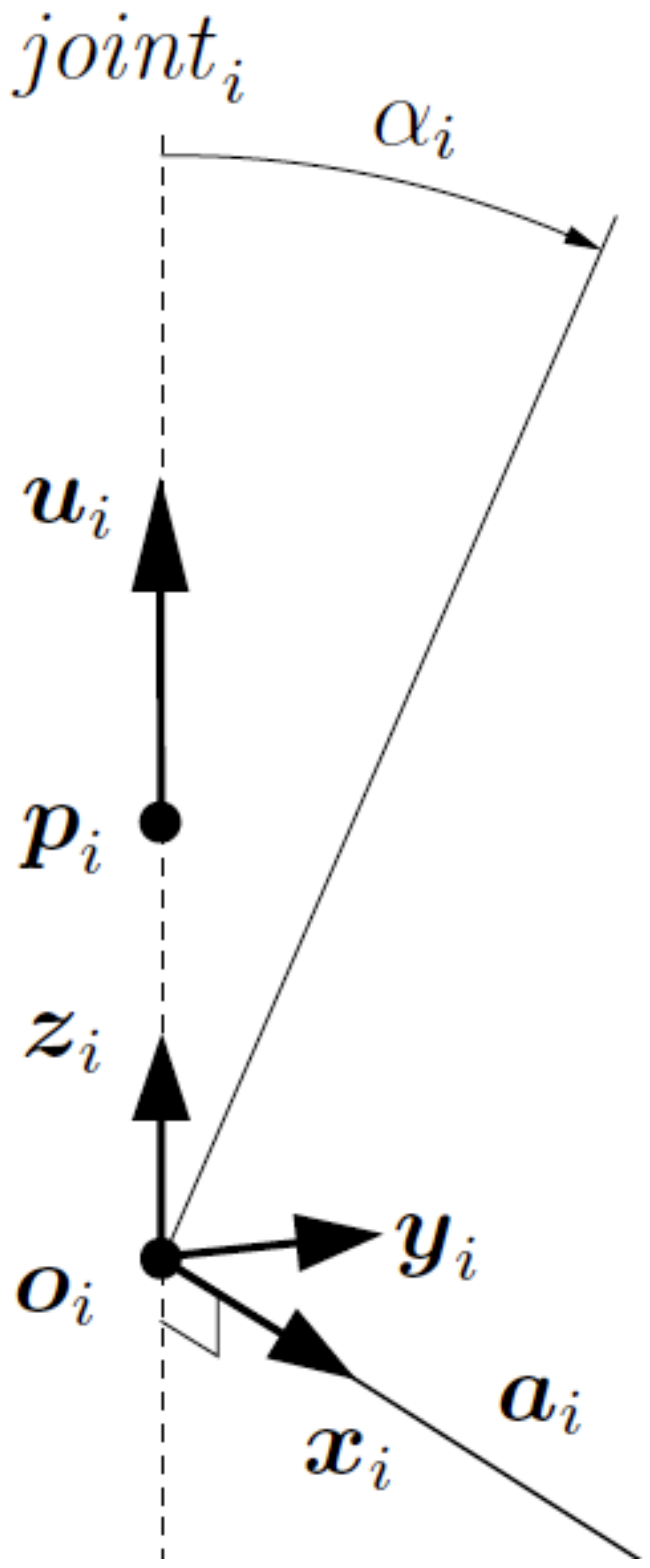
Done	link length	a_i	The perpendicular distance between the axes of $joint_i$ and $joint_{i+1}$.
-->	link twist	α_i	The angle between the axes of $joint_i$ and $joint_{i+1}$. The angle α_i is measured around the $\boldsymbol{x}_i$-axis. Positive angles are measured counterclockwise when looking from the tip of vector $\boldsymbol{x}_i$ toward its foot.
	link offset	d_i	The distance between the origins of the coordinate frames attached to joint $joint_{i-1}$ and $joint_i$ measured along the axis of $joint_i$. For a prismatic joint this is a joint parameter.
	joint angle	φ_i	The angle between the link lengths $\boldsymbol{a}_{i-1}$ and $\boldsymbol{a}_i$. The angle φ_i is measured around the $\boldsymbol{z}_i$-axis. Positive angles are measured counterclockwise when looking from the tip of vector $\boldsymbol{z}_i$ toward its foot. For a revolute joint this is a joint parameter.

The link twist

$$\arctan2(n,d) = \begin{cases} \arctan\left(\frac{n}{d}\right) & \text{if} \quad n > 0 \wedge d > 0 \\ \arctan\left(\frac{n}{d}\right) & \text{if} \quad n < 0 \wedge d > 0 \\ \arctan\left(\frac{n}{d}\right) + \pi & \text{if} \quad n > 0 \wedge d < 0 \\ \arctan\left(\frac{n}{d}\right) - \pi & \text{if} \quad n < 0 \wedge d < 0 \end{cases}$$

- If we just do arctan2(cos(alpha), sin(alpha)) isn't gonna work because of the range of sin.

$$\alpha_i = \begin{cases} +\arctan2\left(\frac{\|\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}\|_2}{\|\boldsymbol{u}_i\|_2 \|\boldsymbol{u}_{i+1}\|_2}, \frac{\boldsymbol{u}_i \cdot \boldsymbol{u}_{i+1}}{\|\boldsymbol{u}_i\|_2 \|\boldsymbol{u}_{i+1}\|_2}\right) & \text{if} \quad (\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}) \cdot \boldsymbol{a}_i \geq 0 \\ -\arctan2\left(\frac{\|\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}\|_2}{\|\boldsymbol{u}_i\|_2 \|\boldsymbol{u}_{i+1}\|_2}, \frac{\boldsymbol{u}_i \cdot \boldsymbol{u}_{i+1}}{\|\boldsymbol{u}_i\|_2 \|\boldsymbol{u}_{i+1}\|_2}\right) & \text{if} \quad (\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}) \cdot \boldsymbol{a}_i < 0 \end{cases}$$



The link offset

Done	link length	a_i	The perpendicular distance between the axes of $joint_i$ and $joint_{i+1}$.
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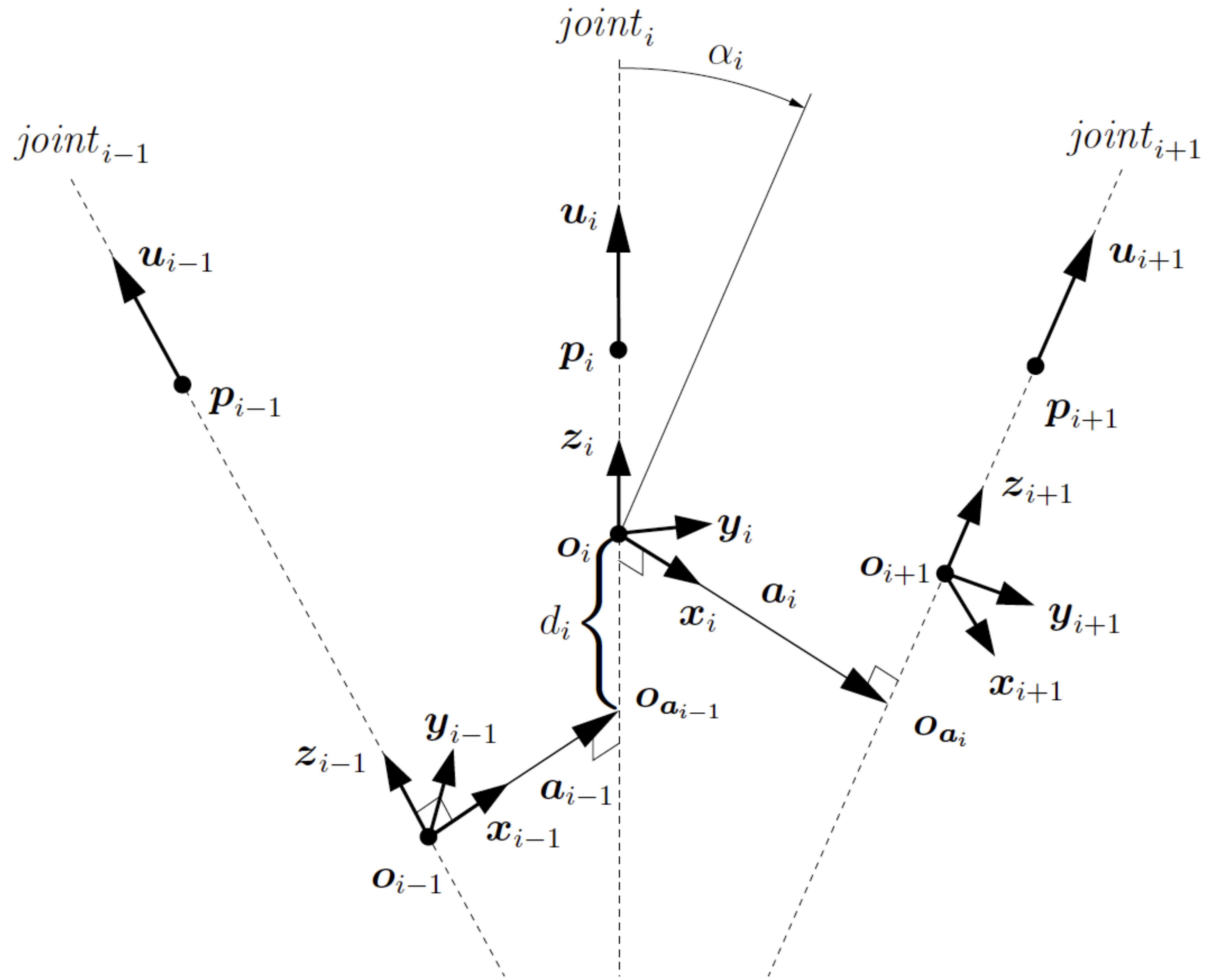
The link offset

$$d_i = \begin{cases} +\left\|\boldsymbol{o}_i - \boldsymbol{o}_{\boldsymbol{a}_{i-1}}\right\|_2 & \text{if} \quad (\boldsymbol{o}_i - \boldsymbol{o}_{\boldsymbol{a}_{i-1}}) \cdot \boldsymbol{u}_i \geq 0 \\ -\left\|\boldsymbol{o}_i - \boldsymbol{o}_{\boldsymbol{a}_{i-1}}\right\|_2 & \text{if} \quad (\boldsymbol{o}_i - \boldsymbol{o}_{\boldsymbol{a}_{i-1}}) \cdot \boldsymbol{u}_i < 0 \end{cases}$$



The joint angle

Done	link length	a_i	The perpendicular distance between the axes of $joint_i$ and $joint_{i+1}$.
Done	link twist	α_i	The angle between the axes of $joint_i$ and $joint_{i+1}$. The angle α_i is measured around the $\boldsymbol{x}_i$-axis. Positive angles are measured counterclockwise when looking from the tip of vector $\boldsymbol{x}_i$ toward its foot.
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The joint angle

- Very similar to link twist but with different axes. Recall:

$$\alpha_i = \begin{cases} +\operatorname{arctan2}\left(\frac{\left\|\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}\right\|_2}{\left\|\boldsymbol{u}_i\right\|_2 \left\|\boldsymbol{u}_{i+1}\right\|_2}, \frac{\boldsymbol{u}_i \cdot \boldsymbol{u}_{i+1}}{\left\|\boldsymbol{u}_i\right\|_2 \left\|\boldsymbol{u}_{i+1}\right\|_2}\right) & \text{if} \quad (\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}) \cdot \boldsymbol{a}_i \geq 0 \\ -\operatorname{arctan2}\left(\frac{\left\|\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}\right\|_2}{\left\|\boldsymbol{u}_i\right\|_2 \left\|\boldsymbol{u}_{i+1}\right\|_2}, \frac{\boldsymbol{u}_i \cdot \boldsymbol{u}_{i+1}}{\left\|\boldsymbol{u}_i\right\|_2 \left\|\boldsymbol{u}_{i+1}\right\|_2}\right) & \text{if} \quad (\boldsymbol{u}_i \times \boldsymbol{u}_{i+1}) \cdot \boldsymbol{a}_i < 0 \end{cases}$$

- In this case:

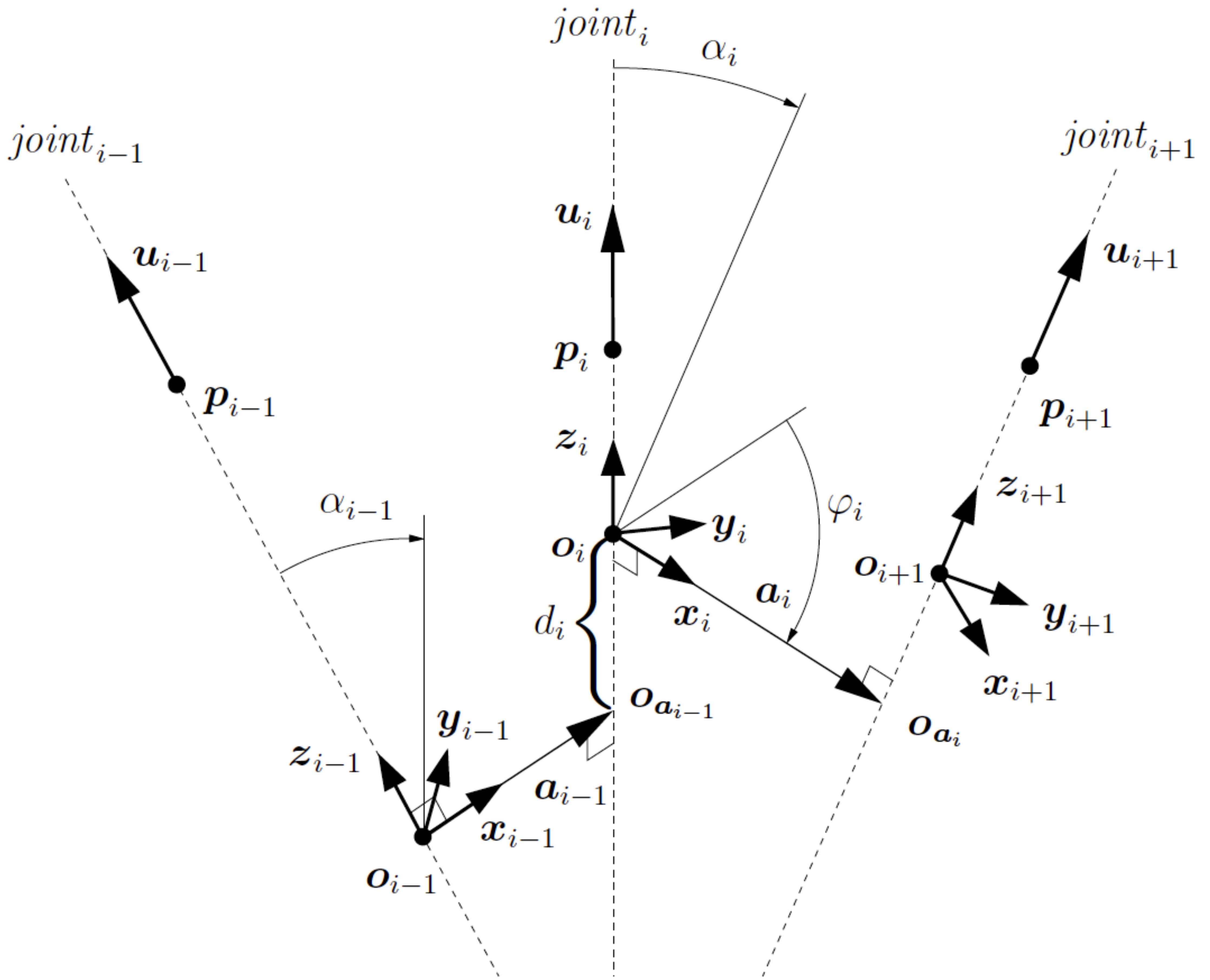
$$\varphi_i = \begin{cases} +\operatorname{arctan2}\left(\frac{\left\|\boldsymbol{a}_{i-1} \times \boldsymbol{a}_i\right\|_2}{\left\|\boldsymbol{a}_{i-1}\right\|_2 \left\|\boldsymbol{a}_i\right\|_2}, \frac{\boldsymbol{a}_{i-1} \cdot \boldsymbol{a}_i}{\left\|\boldsymbol{a}_{i-1}\right\|_2 \left\|\boldsymbol{a}_i\right\|_2}\right) & \text{if} \quad (\boldsymbol{a}_{i-1} \times \boldsymbol{a}_i) \cdot \boldsymbol{z}_i \geq 0 \\ -\operatorname{arctan2}\left(\frac{\left\|\boldsymbol{a}_{i-1} \times \boldsymbol{a}_i\right\|_2}{\left\|\boldsymbol{a}_{i-1}\right\|_2 \left\|\boldsymbol{a}_i\right\|_2}, \frac{\boldsymbol{a}_{i-1} \cdot \boldsymbol{a}_i}{\left\|\boldsymbol{a}_{i-1}\right\|_2 \left\|\boldsymbol{a}_i\right\|_2}\right) & \text{if} \quad (\boldsymbol{a}_{i-1} \times \boldsymbol{a}_i) \cdot \boldsymbol{z}_i < 0 \end{cases}$$

One last thing!

$$^{(i-1)}\boldsymbol{T}_i(d_i, \varphi_i, a_{i-1}, \alpha_{i-1}) = \boldsymbol{R}_{\boldsymbol{x}_i}(\alpha_{i-1})\boldsymbol{T}_{\boldsymbol{x}_i}(a_{i-1})\boldsymbol{T}_{\boldsymbol{z}_i}(d_i)\boldsymbol{R}_{\boldsymbol{z}_i}(\varphi_i)$$

OR

$$^{(i-1)}\boldsymbol{T}_i(d_i, \varphi_i)$$



Thank you!