Frictional contact course

Introduction

Overview

- Equations of Motion
- Time Integration
- Constraints (part 1)

Equations of Motion

- Newton-Euler EoM:

$$\mathbf{M}(t)\dot{\mathbf{u}}(t) = \mathbf{f}(\mathbf{q}(t), \mathbf{u}(t), t)$$
$$\mathbf{M}\dot{\mathbf{u}} = \mathbf{f}$$

- M = mass/inertia properties
- q = generalised positions
- u = generalised velocities
- f = generalised forces

Time Integration

Implicit/explict

- Explicit:

$$\mathbf{f}(\mathbf{q}^-, \mathbf{u}^-)$$

- Implicit:

$$\mathbf{f}(\mathbf{q}^+, \mathbf{u}^+)$$

- Implicit integrators have better numerical stability

Time Integration

Implicit Euler step

$$\mathbf{Mu}^+ = \mathbf{Mu} + h\mathbf{f}$$

- First order Taylor expansion: $\mathbf{u}^+ \approx \mathbf{u} + h\dot{\mathbf{u}}$
- h = time step
- Euler integration: $\mathbf{u}^+ = \mathbf{u} + h\dot{\mathbf{u}}$
- Isolate the velocities: $(\mathbf{u}^+ - \mathbf{u})/h = \dot{\mathbf{u}}$
- Substitute into EoM: $\mathbf{M}(\mathbf{u}^+ - \mathbf{u})/h = \mathbf{f}$
- Reorder: $\mathbf{Mu}^+ = \mathbf{Mu} + h\mathbf{f}$

Time Integration

Implicit Euler step

$$\mathbf{M}\mathbf{u}^{+} = \mathbf{M}\mathbf{u} + h\mathbf{f}$$

- For an n-degree of freedom system:
- Mass matrix: $\mathbf{M} \in R^{n \times n}$
- Momentum terms: $\mathbf{M}\mathbf{u} \in R^{n}$
- Applied forces: $\mathbf{f} \in R^{n}$

Time Integration
Generalised positions

$$\mathbf{q}^{+} = \mathbf{q} + \mathrm{h}\mathbf{H}(\mathbf{u}^{+})$$

- H is a kinematic map
- Example:

$$\mathbf{H}(\omega) = \frac{1}{2}\begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}\mathbf{q} = \dot{\mathbf{q}}$$

Constraints

Unilateral/Bilateral

- Recap, two main types of constraints:
  - Unilateral:

$$\phi(\mathbf{q}) \geq 0$$

  - Bilateral (fx. joint constraints):

$$\phi(\mathbf{q}) = 0$$

Constraints

Velocity constraints

- Differentiation the position constraints:
    - Unilateral:

$$\mathbf{J}\mathbf{u} \geq 0$$

    - Bilateral:

$$\mathbf{J}\mathbf{u} = 0$$

    - With:

$$\mathbf{J} = \frac{\partial \phi}{\partial \mathbf{q}}$$

Constraints

Constraint forces

- Given by: $\mathbf{f_c} = \mathbf{J^T}\lambda$

- Impulse forces: $h\mathbf{f_c}$

$$\mathbf{Mu}^{+} - \mathbf{J}^{\top}\lambda_{\mathbf{I}}^{+} = \mathbf{Mu} + \mathrm{h}\mathbf{f}$$

- Constraint impulse magnitudes: $\lambda_I \equiv h\lambda$

Constraints

Schur complement

- Constraint system:

$$\begin{bmatrix} \mathbf{M} & -\mathbf{J}^\top \\ \mathbf{J} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^+ \\ \lambda_i^+ \end{bmatrix} = \begin{bmatrix} \mathbf{Mu} + h\mathbf{f} \\ \mathbf{0} \end{bmatrix}$$

- Schur complement:

$$\underbrace{\mathbf{J}\mathbf{M}^{-1}\mathbf{J}^\top}_{\text{A}} \lambda_I^+ + \underbrace{\mathbf{J}\mathbf{M}^{-1}(\mathbf{Mu} + h\mathbf{f})}_{\text{b}} = 0$$

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Next time?

- Non-interpenetration Contact Constraint
- The Coulomb Friction Law
- The Linear Complementarity Problem Model
- The Boxed Linear Complementarity Problem Model
- (The Cone Complementarity Problem Model)