How Linear complementarity problems (LCPs) are used in physics simulation

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How to handle contacts in physics simulation

- Penalty based methods
- Impulse based methods
- Constraint based methods

Nomenclature

- **q** = (**x**, Q) $\mathbf{u} = [\mathbf{v}^T \ \omega^T]^T \in \mathbb{R}^6$
- $\dot{\mathbf{q}} = \mathbf{H}\mathbf{u}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{1}_{3\times 3} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix} \qquad \mathbf{G} = \frac{1}{2}\begin{bmatrix} -Q_x & -Q_y & -Q_z \\ Q_s & Q_z & -Q_y \\ -Q_z & Q_s & Q_x \\ Q_y & -Q_x & Q_s \end{bmatrix}$
- Constraints

$C(\mathbf{q}_1, \mathbf{q}_2, \mathbf{u}_1, \mathbf{u}_2, \dot{\mathbf{u}}_1, \dot{\mathbf{u}}_2, \ldots, t) = 0,$ OR $C(\mathbf{q}_1, \mathbf{q}_2, \mathbf{u}_1, \mathbf{u}_2, \dot{\mathbf{u}}_1, \dot{\mathbf{u}}_2, \ldots, t) \geq 0,$

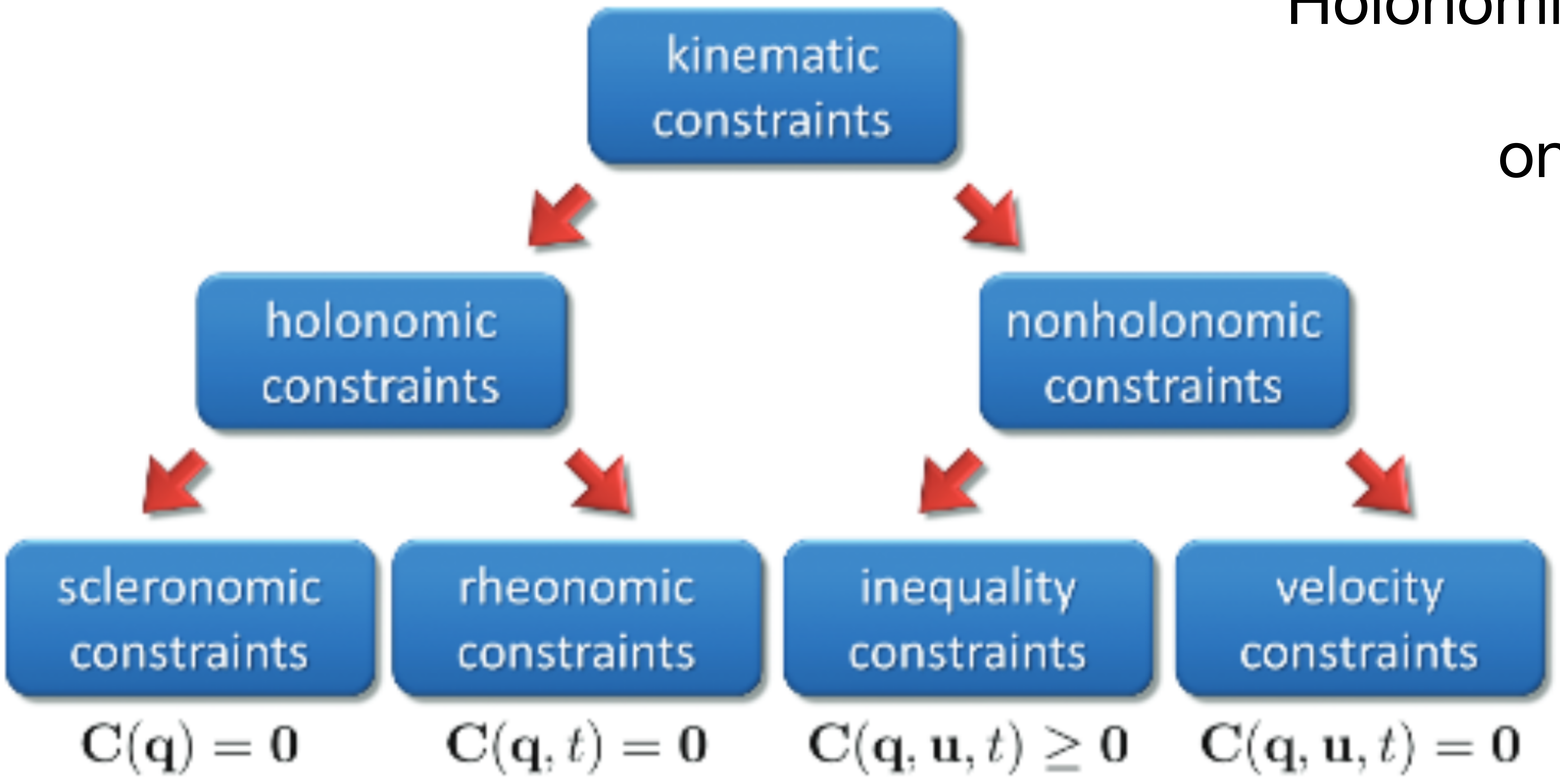
Bilateral Unilateral

For example, think of 2 spheres centered at x1 and x2 with a radius of r1 and r2 then:

$$C(x_1, x_2) = ||x_1 - x_2|| - (r_1 + r_2), = 0$$

How do we classify constraints?

Holonomic -> constraints that can be expressed using
only positions and time



the previous example

Trajectory to follow

Contacts can be split in 2 sets

- Bilateral and Unilateral, the union of both sets comprises all the contact in the scene

$$\mathcal{B} = \{i \; : \; \text{contact } i \text{ is a joint}\},$$

$$\mathcal{U} = \{i \; : \; \text{contact } i \text{ is a point contact}\},$$

Model components

- Newton-Euler equation

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} = \mathbf{g}(\mathbf{q}, \mathbf{u}, t), \quad \mathbf{M} = \begin{bmatrix} m\mathbf{1}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \mathbf{g} = \begin{bmatrix} \mathbf{f} \\ \tau - \omega \times \mathbf{I}\omega \end{bmatrix} \quad \mathbf{u} = [\mathbf{v}^T \ \omega^T]^T$$

- Kinematic map

$$\dot{\mathbf{q}} = \mathbf{H}(\mathbf{q})\mathbf{u}, \quad \mathbf{H} = \begin{bmatrix} \mathbf{1}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{G} \end{bmatrix} \quad \mathbf{G} = \frac{1}{2}\begin{bmatrix} -Q_x & -Q_y & -Q_z \\ Q_s & Q_z & -Q_y \\ -Q_z & Q_s & Q_x \\ Q_y & -Q_x & Q_s \end{bmatrix}$$

- Joint constraints, handled by unconstrained forces

$${}^{b}\mathbf{C}_n(\mathbf{q}, t) = \mathbf{0},$$

- Normal constraints (contacts as we know them), handled by constrained forces. If C>0 then f=0, if C=0 then f>0

$${}^{u}\mathbf{C}_n(\mathbf{q}, t) \geq \mathbf{0},$$

$${}^{u}\mathbf{C}_n(\mathbf{q}, t) \cdot {}^{u}\mathbf{f}_n = 0,$$

Complementarity problem 1: non linear CP

Definition 1. *(NCP):Given an unknown vector* $\mathbf{x} \in \mathbb{R}^m$ *and a known vector function* $\mathbf{y}(\mathbf{x}) : \mathbb{R}^m \rightarrow \mathbb{R}^m$*, determine* $\mathbf{x}$ *such that*

$$\mathbf{0} \leq \mathbf{y}(\mathbf{x}) \perp \mathbf{x} \geq \mathbf{0}, \tag{22}$$

where $\perp$ *implies orthogonality (i.e.* $\mathbf{y}(\mathbf{x}) \cdot \mathbf{x} = 0$*).*

Complementarity problem 2: linear CP

Definition 2. *(LCP): Given an unknown vector* $\mathbf{x} \in \mathbb{R}^m$*, a known fixed matrix* $\mathbf{A} \in \mathbb{R}^{m \times m}$*, and a known fixed vector* $\mathbf{b} \in \mathbb{R}^m$*, determine* $\mathbf{x}$ *such that*

$$\mathbf{0} \leq \mathbf{A}\mathbf{x} + \mathbf{b} \perp \mathbf{x} \geq \mathbf{0}. \tag{23}$$

For LCPs, we adopt the shorthand notation, LCP($\mathbf{A}, \mathbf{b}$*).*

Complementarity problem 3

Definition 3. *Mixed Nonlinear Complementarity Problem (mNCP): Given unknown vectors* $\mathbf{x} \in \mathbb{R}^m$ *and* $\mathbf{w} \in \mathbb{R}^n$, *and known vector functions* $\mathbf{y}(\mathbf{x}, \mathbf{w}) : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^m$ *and* $\mathbf{z}(\mathbf{x}, \mathbf{w}) : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^n$, *find* $\mathbf{x}$ *and* $\mathbf{w}$ *such that*

$$\mathbf{z}(\mathbf{x}, \mathbf{w}) = \mathbf{0}. \tag{24}$$

$$\mathbf{0} \leq \mathbf{y}(\mathbf{x}, \mathbf{w}) \perp \mathbf{x} \geq \mathbf{0}. \tag{25}$$

Complementarity problem 4

Definition 4. *Mixed Linear Complementarity Problem (mLCP): Given unknown vectors* $\mathbf{x} \in \mathbb{R}^m$ *and* $\mathbf{w} \in \mathbb{R}^n$*, known fixed square matrices* $\mathbf{F} \in \mathbb{R}^{m\times m}$ *and* $\mathbf{D} \in \mathbb{R}^{n\times n}$*, known fixed rectangular matrices* $\mathbf{B} \in \mathbb{R}^{m\times n}$ *and* $\mathbf{C} \in \mathbb{R}^{n\times m}$*, and known fixed vectors* $\mathbf{a} \in \mathbb{R}^m$ *and* $\mathbf{r} \in \mathbb{R}^n$*, determine* $\mathbf{x}$ *and* $\mathbf{w}$ *such tha:*

$$\mathbf{0} \leq \mathbf{F}\mathbf{x} + \mathbf{B}\mathbf{w} + \mathbf{a} \perp \mathbf{x} \geq \mathbf{0}, \tag{26}$$

$$\mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{w} + \mathbf{r} = \mathbf{0}. \tag{27}$$

Contact constraints in terms of acceleration

$$^{\kappa}C_{i\sigma}(q,t);\ \ \kappa \in \{b,\ u\};\ \ \sigma \in \{n,\ f\} \qquad \tilde{q} = q + \Delta q \qquad \tilde{t} = t + \Delta t$$

$$\begin{aligned}\widehat{^{\kappa}C}_{i\sigma}(\tilde{\mathbf{q}},\tilde{t}) = {}& {}^{\kappa}C_{i\sigma}(\mathbf{q},t) \\ & + \frac{\partial^{\kappa}C_{i\sigma}}{\partial \mathbf{q}}\Delta\mathbf{q} + \frac{\partial^{\kappa}C_{i\sigma}}{\partial t}\Delta t \\ & + \frac{1}{2}\left((\Delta\mathbf{q})^{T}\frac{\partial^{2\kappa}C_{i\sigma}}{\partial \mathbf{q}^{2}}\Delta\mathbf{q} + 2\frac{\partial^{2\kappa}C_{i\sigma}}{\partial\mathbf{q}\partial t}\Delta\mathbf{q}\Delta t + \frac{\partial^{2\kappa}C_{i\sigma}}{\partial t^{2}}\Delta t^{2}\right)\end{aligned}$$

$$^{\kappa}\mathbf{a}_{i\sigma} = {}^{\kappa}\mathbf{J}_{i\sigma}\dot{\mathbf{u}} + {}^{\kappa}\mathbf{k}_{i\sigma}(\mathbf{q},\mathbf{u},t)$$

$$\begin{aligned}^{\kappa}\mathbf{J}_{i\sigma} &= \frac{\partial(^{\kappa}\mathbf{C}_{i\sigma})}{\partial\mathbf{q}}\mathbf{H} \\ ^{\kappa}\mathbf{k}_{i\sigma}(\mathbf{q},\mathbf{u},t) &= \frac{\partial(^{\kappa}\mathbf{C}_{i\sigma})}{\partial\mathbf{q}}\frac{\partial\mathbf{H}}{\partial t}\mathbf{u} + \frac{\partial^{2}(^{\kappa}\mathbf{C}_{i\sigma})}{\partial\mathbf{q}\partial t}\mathbf{H}\mathbf{u} + \frac{\partial^{2}(^{\kappa}\mathbf{C}_{i\sigma})}{\partial t^{2}},\end{aligned}$$

Contact constraints in terms of acceleration

$$^{b}\mathbf{a}_n = \mathbf{0}.$$

$$\mathbf{0} \leq {}^{u}\mathbf{f}_n \perp {}^{u}\mathbf{a}_n \geq \mathbf{0}.$$

$$\mathbf{M}(\mathbf{q})\dot{\mathbf{u}} = \mathbf{g}(\mathbf{q}, \mathbf{u}, t), \quad \mathbf{M} = \begin{bmatrix} m\mathbf{1}_{3\times3} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \quad \mathbf{g} = \begin{bmatrix} \mathbf{f} \\ \tau - \omega \times \mathbf{I}\omega \end{bmatrix} \quad \mathbf{u} = [\mathbf{v}^T \ \omega^T]^T \in$$

$$\begin{aligned} \mathbf{M}(\mathbf{q})\dot{\mathbf{u}} = {}& {}^{u}\mathbf{J}_n(\mathbf{q})^T\, {}^{u}\mathbf{f}_n + {}^{u}\mathbf{J}_f(\mathbf{q})^T\, {}^{u}\mathbf{f}_f \\ & + {}^{b}\mathbf{J}_n(\mathbf{q})^T\, {}^{b}\mathbf{f}_n + {}^{b}\mathbf{J}_f(\mathbf{q})^T\, {}^{b}\mathbf{f}_f + \mathbf{g}_{\text{ext}}(\mathbf{q}, \mathbf{u}, t) \end{aligned}$$

Stuff to do next

- Discretize the equations
- Integrate in time
- Solve the LCP (different methods as Kasra’s presentation showed with different pros and cons)

References

- https://foswiki.cs.rpi.edu/foswiki/pub/RoboticsWeb/LabPublications/BETCstar_part1.pdf
- https://siggraphcontact.github.io/assets/files/SIGGRAPH21_friction_contact_notes.pdf
- https://arxiv.org/pdf/0807.1249.pdf
- http://image.diku.dk/kenny/download/erleben.13.siggraph.course.notes.pdf

Open Sourced Engines

- https://github.com/DanielChappuis/reactphysics3d/tree/develop
- https://github.com/bulletphysics/bullet3
- https://github.com/dartsim/dart
- https://github.com/nimblephysics/nimblephysics