# Lecture Notes in Finance 1 (MiQE/F, MSc course at UNISG)

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5 February 2025

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These lecture notes are for a first M.A. course in finance. The goal is to present financial theory in a way so that it can be used directly in quantitative/empirical projects that require numerical estimations and computations. The approach is therefore formal, but the mathematics is relatively easy, for instance, linear algebra is used, but stochastic calculus is not. Also, in terms of scope, the focus is on classical concept, for instance, the tangency portfolio plays an important role, but pricing kernels do not. Optional (often more advanced) material is denoted by a star (*).

In implementing numerical computations based on these notes, my students have typically used Julia, Matlab, Python or R. Julia notebooks with numerical examples for each chapter are found at Paul Söderlind's Github page: https://github.com/PaulSoderlind/FinancialTheoryMSc

When I first set up this course many years ago, I was inspired by the texts of Bodie, Danthine&Donaldson, Elton&Gruber, and Hull. Most likely that still shows.

My students at the MiQEF program at the University of St. Gallen have asked many good questions and pointed out mistakes. Also my teaching assistants did the same. Without that, these notes would have been worse.

### Data Sources

The data used in these lecture notes are from the following sources:

1. The website of Kenneth French,
   http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

2. Bloomberg

3. Datastream

4. Federal Reserve Bank of St. Louis (FRED), http://research.stlouisfed.org/fred2/

5. The website of Robert Shiller, http://www.econ.yale.edu/~shiller/data.htm

6. yahoo! finance, http://finance.yahoo.com/

7. OlsenData, http://www.olsendata.com

# Contents

# Chapter 1

# The Basics of Return Calculations

This chapter first defines *returns*, demonstrates how to summarise their statistical properties (descriptive statistics), and discusses how to accumulate them. It then shows how *portfolio returns* depend on the returns of the assets in the portfolio. Later sections summarise the basic statistical properties of important U.S. asset classes and some key markets and trading concepts.

## 1.1 Asset Returns

### 1.1.1 Definition of a Return

The *net (rate of) return* on an asset in period $t$ is

$$R_t = \frac{V_t - V_{t-1}}{V_{t-1}} = \frac{V_t}{V_{t-1}} - 1, \tag{1.1}$$

where $V_t$ is the value of the asset in period $t$.

**Remark 1.1** *(On notation) A precise notation (which time, investment horizon, asset, units. ...) can be cumbersome. When needed, we will use $R_{it}$ (but $R_{i,t-1}$) to indicate the return of asset $i$ in period $t$ ($t-1$). However, when dealing with a single asset where the time dimension is important, then we just keep the $t$ subscript. Instead, when dealing with several assets but where the time dimension is less important, then we just keep the $i$ subscript. Sometimes we drop all subscripts. The meaning should be clear from the context.*

The gross return is

$$1 + R_t = \frac{V_t}{V_{t-1}}. \tag{1.2}$$

**Example 1.2** *(Returns)*

$$R = \frac{110 - 100}{100} = 0.1 \text{ (or } 10\%)$$
$$1 + R = \frac{110}{100} = 1.1$$

**Remark 1.3** *(% and bp) Recall that* $6\%$ *means* $6/100 = 0.06$*, and* 400 *bp (basis points) means* $400/10000 = 0.04$*. Warning: if you just drop the % symbol and thus effectively work with* $100R$ *(in this case getting 6), then you have to be careful, in particular, when accumulating returns over time and when calculating variances.*

In many cases, the values are

$$\begin{aligned} V_{t-1} &= P_{t-1} \text{ (price yesterday)} \\ V_t &= D_t + P_t \text{ (dividend + price today)}, \end{aligned} \tag{1.3}$$

so the return can be written

$$\begin{aligned} R_t &= \frac{D_t + P_t - P_{t-1}}{P_{t-1}} \\ &= \underbrace{\frac{D_t}{P_{t-1}}}_{\text{dividend yield}} + \underbrace{\frac{P_t - P_{t-1}}{P_{t-1}}}_{\text{capital gain yield}} \end{aligned} \tag{1.4}$$

**Example 1.4** *(Dividend yield ad capital gain yield)*

$$R = \frac{2}{100} + \frac{108 - 100}{100} = 0.1$$

The *excess return* of an asset (compared to the risk-free rate $R_f$) is

$$R_t^e = R_t - R_{ft}. \tag{1.5}$$

In most cases the reference return is the risk-free return, but it could also be the return of some other asset.

**Example 1.5** *(Excess return) If* $R_t = 0.08$ *and* $R_{ft} = 0.01$*, then the excess return is* $R_t^e = 0.07$ *(7%).*

**Remark 1.6** *(Approximating the risk-free return*) Suppose you have monthly equity returns and want to calculate excess returns. Do as follows. First, find a representative* *money market instrument (for instance, a T-bill or an interbank contract) with approximately one month to maturity. Second, use $R_{ft} \approx y_{t-1\,month}/12$ where $y_{t-1\,month}$ is the quoted interest rate one month earlier (because this is the rate you earn/pay on a loan between $t-1$ month and $t$). The result is an approximation since interest rates are quoted in different ways (simple, effective,...) and because the maturity may not be an exact match with the investment horizon.*

### 1.1.2 Logarithmic Returns*

It is sometimes better to work with *log returns*, especially when we compare different investment horizons for the same asset. In contrast, log returns are inconvenient when the focus is on choosing the portfolio weights: the log return of a portfolio is *not* a weighted average of the log returns of the assets in the portfolio.

Anyhow, a log return is defined as

$$r_t = \ln(1 + R_t), \tag{1.6}$$

which clearly equals $\ln(V_t/V_{t-1})$. To convert from log returns to net returns, use $R_t = \exp(r_t) - 1$.

The corresponding excess log return is

$$r_t^e = \ln(1 + R_t) - \ln(1 + R_{f,t}). \tag{1.7}$$

Assuming we invest an equal amount in both instruments in $t - 1$ ($V_{t-1} = V_{f,t-1}$), the excess log return equals $\ln(V_t/V_{f,t})$. Notice that excess log return is *not* the log of the excess return. Rather, it is the log of $(1 + R_t)/(1 + R_{f,t})$. Figure 1.1 illustrates that the difference between $r^e$ and the possible approximation $\ln(1 + R^e)$ can be substantial.

**Example 1.7** *(Excess log return) If $R_t = 0.08$ and $R_{f,t} = 0.01$, then the excess log return in 0.067.*

### 1.1.3 Inflation and Real Returns

In most portfolio choice models, it is the *real* return (measured in units of "goods") that matters, not the *nominal* return (measured in currency units). The reason is straightforward: utility depends on real goods and services, not on nominal price levels.

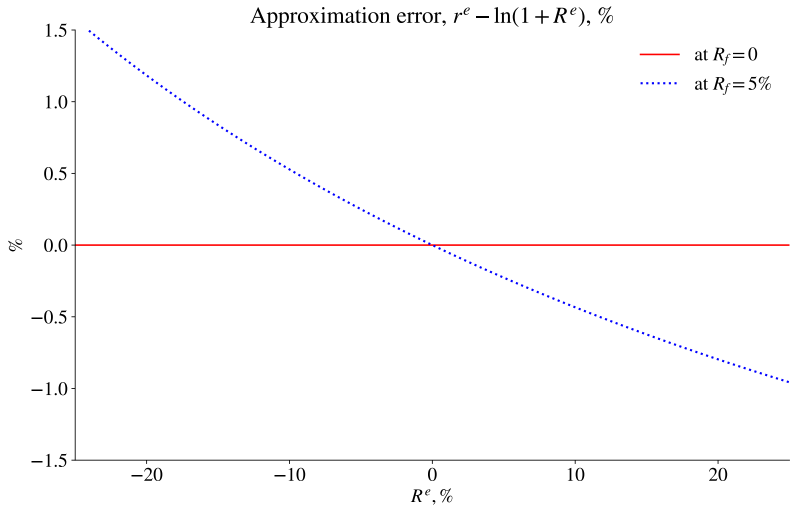



Figure 1.1: Approximation error from using $\ln(1 + R^e)$ instead of $r^e$

To see the link between real and nominal returns, let $\Gamma_t$ be the nominal price level (price of the consumption basket of an investor, measured in currency units). If $V$ in (1.1) is a nominal value (measured in currency units), then the real value is $V/\Gamma$.

**Example 1.8** *(Nominal and real prices) If $V = 110$ is the nominal value of an asset and $\Gamma = 5$ is the nominal price of the consumption basket, then the real value is $V/\Gamma = 22$. This represents the number of consumption baskets required to match the asset's value.*

The real return (corresponding to (1.1)) is

$$\begin{aligned} \tilde{R}_t &= \frac{\Gamma_{t-1}}{\Gamma_t} \frac{V_t}{V_{t-1}} - 1 \\ &= \frac{1 + R_t}{1 + \pi_t} - 1, \end{aligned} \tag{1.8}$$

where $\pi_t = \Gamma_t / \Gamma_{t-1} - 1$ is the inflation rate. We get a similar expression for the risk-free rate, so the excess real return (cf. (1.5)) is

$$\tilde{R}_t^e = \frac{R_t^e}{1 + \pi_t}. \tag{1.9}$$

**Example 1.9** *(Real returns) With $R_t = 0.08$ and $\pi_t = 0.05$, the real return is $1.08/1.05 - 1 \approx 0.029$. Also, with $R_t^e = 0.07$, the excess real return is $0.07/1.05 \approx 0.067$.*

It is clear that the real excess return (1.9) is less affected by inflation than the real net return (1.8). (Actually, for log returns the real excess log return is unaffected by inflation.)

The reason is straightforward: while inflation reduces the real value of the long position, it also reduces the real value of the short position (that is, how many consumption bundles that need to be paid back).

### 1.1.4 Descriptive Statistics of Asset Returns

The properties of returns in a sample are often summarised by the mean, standard deviation, the Sharpe ratio (mean/std of excess returns) and the coefficients from a linear regression (see below). Typically, these statistics are annualised.

**Remark 1.10** *(On notation) Mean returns are denoted* $\mathrm{E}\,R$ *or* $\mu$. *(Subscripts to indicate the asset are used when needed.) An expression like* $\mathrm{E}\,x^2$ *means the expected value of* $x^2$ *and* $\mathrm{E}\,xy$ *is the expectation of the product* $xy$. *Variances are denoted* $\sigma^2$ *or* $\mathrm{Var}(R)$ *and the standard deviations* $\sigma$ *or* $\mathrm{Std}(R)$. *Covariances are denoted* $\sigma_{ij}$ *or* $\mathrm{Cov}(R_i, R_i)$. *Clearly,* $\sigma_{ii}$ *is the same as the variance.*

The *scaling of returns* (for instance, in percentages) can often cause confusion. Let $R_{it}$ be the net return with mean $\mu$, standard deviation $\sigma$ and covariance with asset $j$ $\sigma_{ij}$. When you work with percentage returns, $100R_{it}$, then

$$100R_{it} \text{ has the } \begin{array}{ll} \text{mean:} & 100\mu \\ \text{variance:} & 100^2\sigma^2 \\ \text{standard deviation} & 100\sigma \\ \text{covariance with } 100R_{jt} & 100^2\sigma_{ij} \end{array} \tag{1.10}$$

Notice that the mean and standard deviation are scaled by 100, but the variance and covariance are scaled by $10,000$. This can easily cause problems when trading off means and variances. However, it works well when comparing means and standard deviations (for instance, the Sharpe ratio is a mean divided by a standard deviation). Also, in a regression, $\tilde{R}_{it} = \alpha + \beta\tilde{R}_{jt} + \varepsilon_{it}$, the slope is unaffected, but the intercept is scaled by 100.

It is a common convention to *annualise return statistics* when reporting the (final) results. If the return data is for a $1/k$-year horizon (for instance, $k = 12$ for monthly data), then we typically annualise as

$$\begin{array}{ll} \text{mean:} & k\mu \\ \text{variance:} & k\sigma^2 \\ \text{standard deviation} & \sqrt{k}\sigma \\ \text{covariance with } R_j & k\sigma_{ij}. \end{array} \tag{1.11}$$

For daily data use $k = 252$ (the approximate number of trading days per year) and for weekly data $k = 52$. Also, the results from a linear regression are annualised by multiplying the intercept by $k$ (since it is a mean), but not changing the slope coefficient (since it is a covariance divided by a variance). The convention in (1.11) is based on the idea that returns are almost iid (see below for details). It is probably advisable to annualise only at the very last stage of the computations.

**Example 1.11** *(Annualisation) If the monthly average return is 0.67% and the monthly standard deviation is 2.89%, then the annualised values are 8% and 10%, respectively.*

The expected excess return, $\mathrm{E}\, R_i^e$ or $\mu_i^e$, is often called a *risk premium* since it measures the expected return of taking risk (of holding asset $i$) minus the return of a risk-free asset. The *Sharpe ratio* is

$$SR = \mu^e/\sigma, \tag{1.12}$$

where $(\mu^e, \sigma)$ indicate the mean and standard deviations of the excess returns. The SR can be interpreted as a reward/risk ratio. Typically, a high Sharpe ratio is considered favourable.

**Example 1.12** *(Risk premium and Sharpe ratio) If* $(\mu^e, \sigma) = (0.1, 0.5)$*, then Sharpe ratio is 0.2.*

The so-called "market model" is a regression an asset's excess return on the excess return on the market index, $R_{mt}^e$,

$$R_t^e = \alpha + \beta R_{mt}^e + u_t. \tag{1.13}$$

A slope coefficient $\beta > 1$ indicates that the asset is strongly pro-cyclical (moves more than proportionally with the market), whereas $0 < \beta < 1$ indicates a weaker pro-cyclicality. $\beta < 0$ indicates counter-cyclicality, but such assets are rare. The $\alpha$ is often interpreted as an abnormal excess return (see the capital asset pricing model for a detailed discussion). See Table 1.1 for an example.

| | Small growth | Small value | Large growth | Large value | Equity market |
|---|---|---|---|---|---|
| mean (ann.) | 7.17 | 11.27 | 10.00 | 9.55 | 8.94 |
| std (ann.) | 23.44 | 19.96 | 16.10 | 18.53 | 15.72 |
| SR (ann.) | 0.31 | 0.56 | 0.62 | 0.52 | 0.57 |
| $\alpha$ (ann.) | −4.27 | 1.84 | 1.08 | 0.54 | −0.00 |
| $\beta$ | 1.28 | 1.05 | 1.00 | 1.01 | 1.00 |

Table 1.1: Means and std of asset class returns, US, monthly excess returns (%), 1985:01-2023:12. The mean and $\alpha$ are annualised by 12, the standard deviation by $\sqrt{12}$, and the Sharpe ratio is the ratio of the annualised mean and standard deviation.

**Remark 1.13** *(Motivation of the convention in (1.11)*) Suppose we have semi-annual data. Notice that an annual return would be* $P_t/P_{t-2}-1 \approx R_t+R_{t-1}$*.If returns are iid (in particular, the same mean and variance across time and also uncorrelated across time), then to a first approximation, the expected (or mean) annual return is* $\mathrm{E}(R_t + R_{t-1}) = 2\,\mathrm{E}\,R_t$*.Similarly, the variance and standard deviation is* $\mathrm{Var}(R_t + R_{t-1}) = 2\,\mathrm{Var}(R_t)$*, which implies* $\mathrm{Std}(R_t + R_{t-1}) = \sqrt{2}\,\mathrm{Std}(R_t)$*. Similarly, if* $\mathrm{Cov}(R_{it}, R_{j,t-1}) = 0$*, then* $\mathrm{Cov}(R_{it} + R_{i,t-1}, R_{jt} + R_{j,t-1}) = 2\sigma_{ij}$*.*

### 1.1.5 Cumulating Returns

If an investment in period $t = 0$ equals $V_0$, then its value in $t$ is

$$V_t = V_0(1 + R_1)(1 + R_2)\dots(1 + R_t), \tag{1.14}$$

where all subscripts refer to time periods and $R_\tau$ is the return on your portfolio in period $\tau$. This expression assumes that all dividends have been reinvested, making $V_t$ a *total return index*. We can clearly write this on recursive form as

$$V_t = V_{t-1}(1 + R_t). \tag{1.15}$$

**Empirical Example 1.14** *Figure 1.2 shows the cumulated return of a U.S. equity market index.*

**Example 1.15** *With net returns for three time periods* $(R_1, R_2, R_3) = (0.2, -0.35, 0.25)$*, we get portfolio values* $(1.2, 0.78, 0.975)$ *for period* $1-3$ *(assuming* $V_0 = 1$*).*

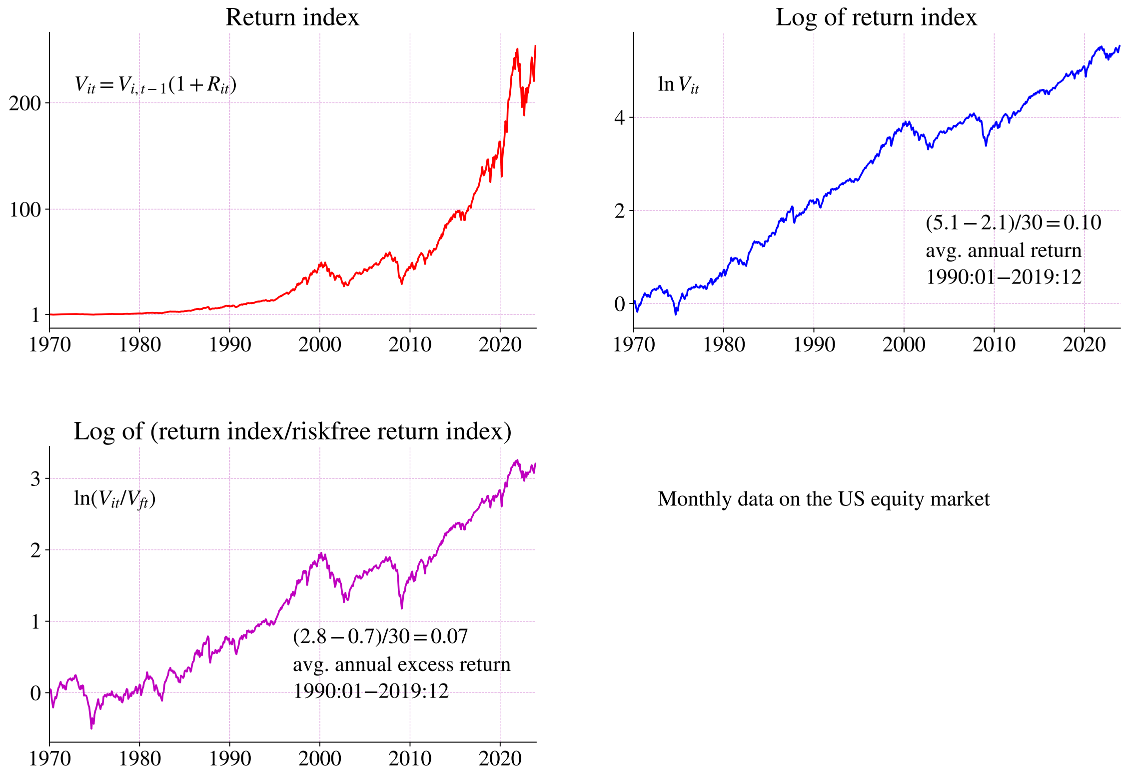



Figure 1.2: Cumulating returns

**Remark 1.16** *(Adjusted closing price) The adjusted closing price of an asset is an index calculated as (1.15) where $R_t$ is the return (including dividends, splits, etc) of holding the asset from $t-1$ to $t$. This means that it is a total return index. If you have such an index, then the returns can be calculated as $R_t = V_t/V_{t-1} - 1$, without having to handle dividend payments separately.*

Unfortunately, excess returns cannot be cumulated directly. Instead, you need to cumulate the net return $R_t$ and and the risk-free return $R_{ft}$ separately (as in (1.15)) and then form the difference

$$V_t^e = V_t - V_{ft}. \tag{1.16}$$

Sometimes the ratio $V_t/V_{ft}$ is a preferred way of illustrating the performance of the two assets.

### 1.1.6 Cumulating Logarithmic Returns*

Similarly, the log value can be calculated as

$$\ln V_t = \ln V_0 + r_1 + r_2 \ldots + r_t \text{, so} \tag{1.17}$$
$$= \ln V_{t-1} + r_t. \tag{1.18}$$

You *can* cumulate excess log returns (because it is just summing). Since the initial positions are equal ($V_0 = V_{f,0}$) we have

$$\ln(V_t / V_{ft}) = (r_1 + r_2 + \ldots + r_t) - \left(r_{f1} + r_{f2} \ldots + r_{ft}\right) \tag{1.19}$$
$$= r_1^e + r_2^e + \ldots + r_t^e \text{, so} \tag{1.20}$$
$$= \ln(V_{t-1} / V_{f,t-1}) + r_t^e, \tag{1.21}$$

starting from $\ln(V_0 / V_{f0}) = 0$. Notice that the exponential function of this gives the ratio $V_t / V_{ft}$ (not the difference).

Again, see Figure 1.2 for an illustration

## 1.2 Portfolio Returns

**Remark 1.17** *(On notation) These notes use $\Sigma_{i=1}^n x_i$ to denote the sum $x_1 + \ldots + x_n$. (In the running text, it might happen that this is sometimes written as just $\Sigma_i x_i$.) Note: $\Sigma$ may also denote a variance-covariance matrix. The distinction should be clear from the context.*

### 1.2.1 Portfolio Return: Definition

Let $R_i$ represent the return on asset $i$ over a given time period (the time subscript is omitted for convenience). The return on a portfolio ($R_p$) with the portfolio weights $w_1, w_2, \ldots, w_n$ is

$$R_p = \sum_{i=1}^{n} w_i R_i \text{, with } \sum_{i=1}^{n} w_i = 1. \tag{1.22}$$

Using vectors, this can also be written

$$R_p = w' R, \tag{1.23}$$

where $w$ is an $n$-vector of weights and $R$ an $n$-vector of asset returns.

Clearly, one of the assets in (1.22)–(1.23) could be risk-free with return $R_f$. However, in this case we will typically choose to consider $n$ risky assets and the risk-free (in total, $n + 1$) and write the portfolio return as

$$R_p = v'R + (1 - \mathbf{1}'v)R_f \tag{1.24}$$

$$= v'R^e + R_f, \tag{1.25}$$

where $v$ are the weights on the risky assets and $1 - \mathbf{1}'v$, that is, $1 - \Sigma_{i=1}^n v_i$, the weight on the risk-free asset. This automatically imposes the condition that the weights on *all* assets sum to one.

**Example 1.18** *(Portfolio return) With the portfolio weights 0.8 and 0.2 for two assets and the returns 0.1 and 0.05 for the same assets, the portfolio has the return*

$$R_p = 0.8 \times 0.10 + 0.2 \times 0.05 = 0.09,$$

*that is,* 9%*.*

**Example 1.19** *(Number of assets and portfolio returns*) For asset* 1 *we have* $P_{1,t-1} = 10$, $P_{1,t} = 11$ *and for asset* 2 *we have* $P_{2,t-1} = 8$, $P_{2,t} = 8.4$*. Assume no dividends. Yesterday you bought* 16 *of asset* 1 *and* 5 *of asset* 2*:* $16 \times 10 + 5 \times 8 = 200$*. Today your portfolio is worth* $16\times11+5\times8.4 = 218$*, so* $R_p = \frac{218-200}{200} = 0.09$*. This is the same as in Example 1.18 since the two returns are* 0.1 *(*$11/10-1$*) and* 0.05 *(*$8.4/8-1$*) respectively, and the portfolio weights are* 0.8 *(*$16 \times 10/200$*) and* 0.2 *(*$5 \times 8/200$*) respectively.*

### 1.2.2 Portfolio Return with Short Positions

The portfolio weights in (1.22) should sum to unity ($\Sigma_{i=1}^n w_i = 1$), but some weights could potentially be negative: *"short" positions*. Notice that a short position pays off if the asset price decreases. Clearly, some investors have very strict limits on their positions. For instance, mutual funds can typically not shorten assets and not put more that 10% in a particular asset. In contrast, hedge funds have very few limits.

**Remark 1.20** *(Short selling) How can we short sell an asset? Borrow the asset (for a fee) and sell it. A short position is profitable if the asset price decreases since then we can buy it back (to return it to the asset lender) for less than for what we sold it. If there are derivatives on the asset, then we don not need to borrow it: just issue a futures/option.*

**Example 1.21** *(Return on a short position) Suppose you borrow an asset (for one month, at a fee of* 0.5*) and sell it for* 100*. One month later, you buy the asset on the market for* 90*. Your profit is thus* $100 - 90 - 0.5 = 9.5$*. Expressed in terms of the initial value of the asset, this is a return of* 9.5%*. (If you can invest the 100, this may be even higher.) In practice, you typically have to provide collateral when borrowing assets.*

### 1.2.3 Zero-Cost Portfolios*

A *zero-cost "portfolio"* (also called an arbitrage portfolio) is an extreme case of short positions where the initial investment is zero. This means that the investor shortens some assets (perhaps borrows) in order to invest in other (perhaps risky) assets. The return on such a portfolio is not well defined (dividing by zero...), but we can define an excess return as follows. Split up the portfolio in a "long" portfolio and denote the weights by $w_i^L$, and and a "short" portfolio with weights $w_i^S$. Clearly, $w_i^L \geq 0$ and $w_i^S \geq 0$ and when one of them is positive then the other is zero.

**Example 1.22** *(Zero-cost portfolio) Suppose you invest 40 in asset 1, 60 in asset 2 and -100 in asset 3. The total investment is zero. We then have* $w^L = (0.4, 0.6, 0)$ *and* $w^S = (0, 0, 1)$*.*

Define the returns on the long and short portfolios as

$$R_p^L = \textstyle\sum_{i=1}^n w_i^L R_i \tag{1.26}$$

$$R_p^S = \textstyle\sum_{i=1}^n w_i^S R_i, \tag{1.27}$$

where all subscripts refer to different assets (and the subscripts for time are suppressed).

We can then consider an "excess return" of the total portfolio as

$$R_p^e = R_p^L - R_p^S = \textstyle\sum_{i=1}^n (w_i^L - w_i^S) R_i. \tag{1.28}$$

Conversely, the traditional excess return of an asset (1.5) is the return of a zero cost portfolio: a long position in the asset and a short position in the risk-free asset.

**Example 1.23** *(Excess return of a zero-cost portfolio) If the returns of the assets in Example 1.22 are* $R_1 = 10\%$*,* $R_2 = -1\%$ *and* $R_3 = 2\%$*, then the excess return is*

$$0.4 \times 0.1 + 0.6 \times (-0.01) - 0.02 = 0.014.$$

**Remark 1.24** *(A broader definition of excess returns*) The definition of the excess return of a zero-cost portfolio discussed above uses portfolio weights that sum to unity ($\Sigma w_i^L = 1$ and $\Sigma w_i^S = 1$), which is often a natural choice. However, another convention is used in some cases: the "excess return" of a zero cost portfolio is just its payoff (profit).*

## 1.3 Asset Classes

Many investors and asset managers choose to focus on asset classes, rather than on individual assets. This approach helps average out idiosyncratic (for instance, firm specific) noise and focuses attention to the macroeconomic perspective.

**Empirical Example 1.25** *Table 1.2 illustrates the return distributions for different U.S. asset classes. There are distinct differences between small and large firms (the former have higher, but more volatile returns) and between growth and value firms (the latter typically have higher returns). However, the most pronounced difference is between equity and bonds (the latter have much less volatility and often lower returns). Figures 1.3 –1.4 illustrate the dynamics behind the figures for the entire sample in Table 1.2. Table 1.3 gives the annual ranking of the asset classes (for a shorter sample). Much of portfolio management is about trying to time these changes. The changes of the ranking—and in the returns—highlight both the opportunities (if you time it right) and risks (if you don't) with such an approach.*

| | Small growth | Small value | Large growth | Large value | Bonds | T-bills | Equity market |
|---|---|---|---|---|---|---|---|
| mean | 0.85 | 1.20 | 1.09 | 1.05 | 0.46 | 0.26 | 1.00 |
| std | 6.76 | 5.75 | 4.65 | 5.35 | 1.40 | 0.21 | 4.54 |
| min | −32.48 | −28.13 | −23.22 | −27.23 | −4.39 | 0.00 | −22.64 |
| max | 28.09 | 19.58 | 14.47 | 18.21 | 5.31 | 0.79 | 13.65 |
| corr with market | 0.86 | 0.83 | 0.97 | 0.85 | −0.01 | 0.02 | 1.00 |
| beta against market | 1.28 | 1.05 | 1.00 | 1.01 | −0.00 | 0.00 | 1.00 |

Table 1.2: Descriptive statistics of asset classes, US, monthly returns (%), 1985:01-2023:12. The beta is the slope coefficient from regressing the asset on the market return.

| | 6th | | 5th | | 4th | | 3rd | | 2nd | | 1st | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2004 | TB | 1.2 | B | 3.5 | LG | 8.2 | SG | 15.7 | SV | 18.8 | LV | 20.4 |
| 2005 | SG | 0.3 | B | 2.8 | TB | 3.0 | LG | 4.5 | SV | 9.5 | LV | 14.1 |
| 2006 | B | 3.1 | TB | 4.8 | SG | 8.9 | LG | 11.4 | LV | 21.9 | SV | 22.2 |
| 2007 | SV | -13.6 | LV | -2.0 | TB | 4.7 | SG | 5.9 | B | 9.0 | LG | 12.8 |
| 2008 | SG | -40.9 | LV | -39.0 | LG | -34.1 | SV | -33.8 | TB | 1.6 | B | 13.7 |
| 2009 | B | -3.6 | TB | 0.1 | LV | 17.7 | SV | 30.4 | LG | 30.6 | SG | 36.7 |
| 2010 | TB | 0.1 | B | 5.9 | LV | 7.0 | LG | 15.3 | SV | 26.8 | SG | 28.8 |
| 2011 | LV | -10.8 | SV | -7.9 | SG | -6.1 | TB | 0.0 | LG | 4.2 | B | 9.8 |
| 2012 | TB | 0.1 | B | 2.0 | SG | 14.6 | LG | 15.3 | SV | 21.2 | LV | 28.7 |
| 2013 | B | -2.7 | TB | 0.0 | LG | 33.0 | LV | 40.3 | SV | 42.6 | SG | 44.6 |
| 2014 | TB | 0.0 | SV | 3.8 | SG | 4.7 | B | 5.1 | LV | 11.6 | LG | 13.6 |
| 2015 | SV | -10.0 | LV | -7.9 | SG | -2.9 | TB | 0.0 | B | 0.8 | LG | 4.3 |
| 2016 | TB | 0.2 | B | 1.0 | SG | 8.1 | LG | 9.0 | LV | 25.9 | SV | 36.7 |
| 2017 | TB | 0.8 | B | 2.3 | SV | 9.2 | LV | 18.2 | SG | 25.4 | LG | 29.2 |
| 2018 | LV | -14.5 | SV | -12.5 | SG | -7.6 | LG | -0.1 | B | 0.9 | TB | 1.8 |
| 2019 | TB | 2.2 | B | 6.9 | SV | 14.7 | LV | 27.8 | SG | 29.6 | LG | 33.8 |
| 2020 | LV | -3.2 | TB | 0.5 | SV | 3.9 | B | 8.0 | LG | 36.3 | SG | 57.8 |
| 2021 | B | -2.3 | TB | 0.0 | SG | 2.7 | LG | 25.3 | LV | 36.9 | SV | 42.2 |
| 2022 | SG | -27.7 | LG | -25.7 | B | -12.5 | SV | -5.9 | TB | 1.4 | LV | 4.1 |
| 2023 | B | 4.1 | TB | 4.9 | SV | 13.2 | LV | 13.8 | SG | 16.4 | LG | 37.8 |

Table 1.3: Ranking and return (in %) of asset classes, US. SG: small growth firms, SV: small value, LG: large growth, LV: large value, B: T-bonds, TB: T-bills.

## 1.4 Markets, Instruments and Some Key Terms

The initial issuance of an asset (for instance, an IPO) takes place at the *primary market* while the subsequent trading takes place on the *secondary market*. Trading in the secondary market can be done on an *exchange* (NYSE, Tokyo, EuroNext, Nasdaq, London, Shanghai, HK, CBOE, CME, etc), an *electronic platform* (EBS, Reuters), or *over the counter* (OTC).

Different asset classes are typically traded on distinct exchanges or platforms. This motives terms like the “money market”, “bond market”, “currency market”, “stock market”, and “derivative markets”.

Alternative asset classes (for instance, hedge funds, infrastructure, private equity) have gained interest over the last decade, especially among institutional long-run investors (wealthy individuals, endowments, some pension funds).

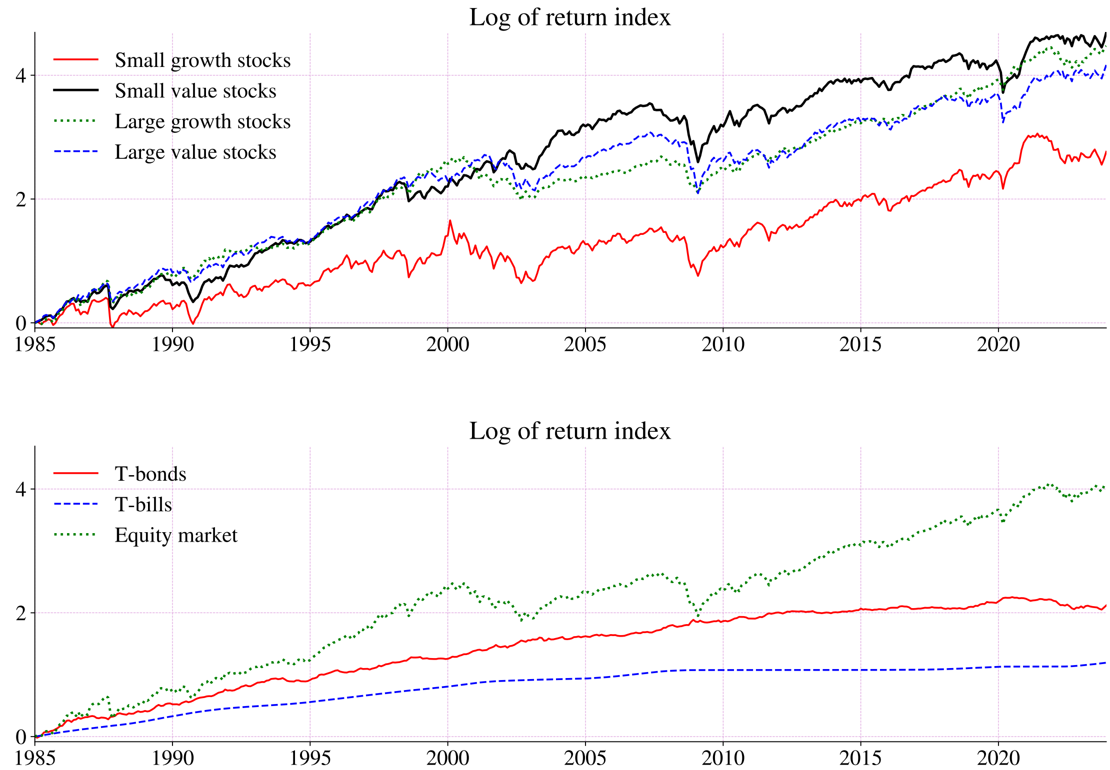



Figure 1.3: Performance of US equity and fixed income

**Remark 1.26** *(Useful terms) The following stock market terms are useful*

- Market capitalization*: value of all shares*
- Float*: number of not closely held shares*
- Volume*: number of traded shares*
- Short interest*: number of shortened shares*
- Consensus estimate*: the average forecast (of eg. earnings) across analysts*
- ROE*: net income/book value of equity*
- ROI*: (net income + interest rates)/book value of (equity + debt)*

**Remark 1.27** *(Trading costs*) As a an investor you typically pay a* commission *(eg. \$25 or \$0.025 per share, whichever is greater) to the broker. In addition, the price depends*

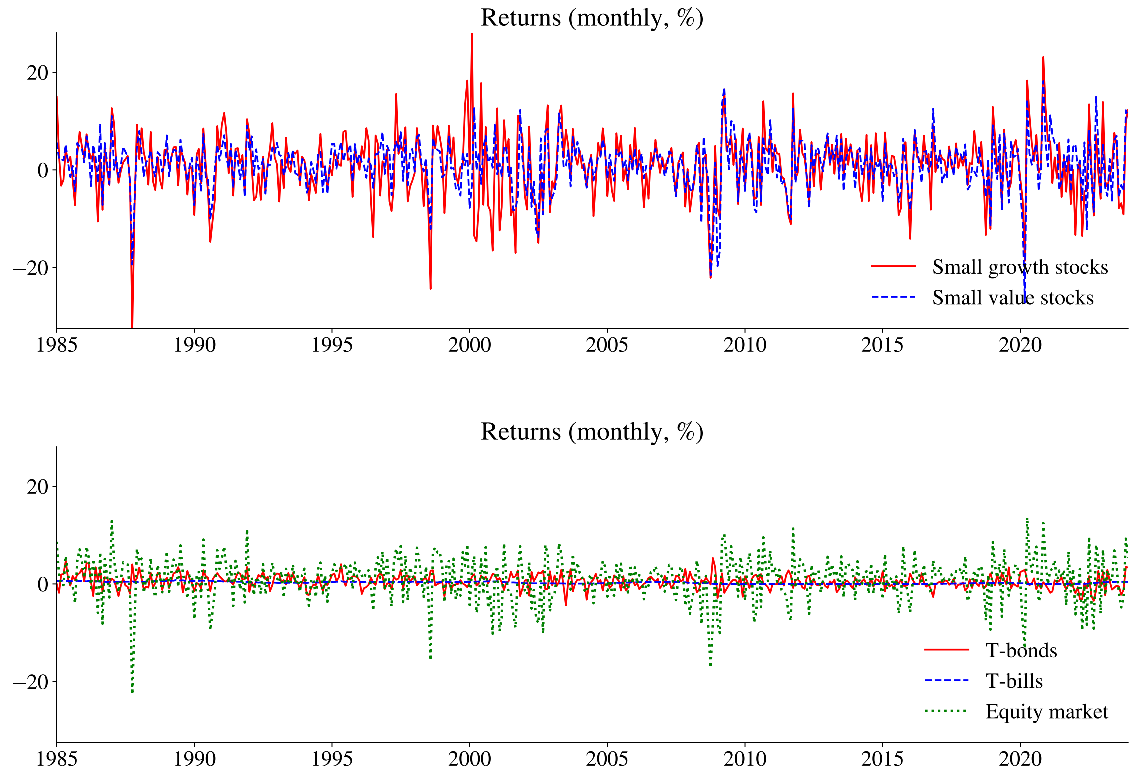



Figure 1.4: Performance of US equity and fixed income

*on whether you are buying (high price) or selling (low price).* Bid and ask *prices are:*

| | *Definition* | *Example* |
|---|---|---|
| *Ask price* | *lowest price at which someone will sell* | 90.05 |
| *Bid price* | *highest price at which someone will buy* | 90.00 |
| *Bid-ask spread* | | 0.05 |

*If you want to* buy immediately*: you submit a* market buy order *(buy at best available price) and you need to pay ask price (90.05). Instead, if you want to* sell immediately, *you submit a* market sell order *and get the bid price (90.00). A round-trip (first buy, then sell) costs* $90.05 - 90.00 = 0.05$ *(the bid-ask spread). Alternatively, you can (at least on some markets) submit a* limit buy order *at a higher bid price (eg. 90.01) or a* limit sell order *at a lower ask price (eg. 90.04). With some luck someone hits that order.*

## 1.8 Appendix – A Primer in Matrix Algebra*

This appendix introduces fundamental concepts of matrix algebra.

### 1.8.1 Matrix and Scalar Addition and Multiplication

Let $c$ be a scalar and define the matrices

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{ and } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

*Multiplying a matrix by a scalar* means multiplying each element by the scalar

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} c = \begin{bmatrix} A_{11}c & A_{12}c \\ A_{21}c & A_{22}c \end{bmatrix}.$$

**Example 1.28** *(Matrix × scalar)*

$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} 10 = \begin{bmatrix} 10 & 30 \\ 30 & 40 \end{bmatrix}.$$

*Adding/subtracting a scalar to each element of a matrix* is done by

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + cJ = \begin{bmatrix} A_{11}+c & A_{12}+c \\ A_{21}+c & A_{22}+c \end{bmatrix},$$

where $J$ is a matrix (of the same size as $A$) filled with ones. This is sometimes written $A + c$, although that notation is not universally liked. In some applications, $\mathbf{1}_n$ (or just $\mathbf{1}$) is used to represent a vector of $n$ ones.

**Example 1.29** *(Matrix ± scalar)*

$$\begin{bmatrix} 10 \\ 11 \end{bmatrix} - 10 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} + 10 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 13 \\ 13 & 14 \end{bmatrix}.$$

### 1.8.2 Adding and Multiplying: Two Matrices

Matrix *addition* (or subtraction) of matrices of the same size is element by element

$$A + B = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix}.$$

**Example 1.30** *(Matrix addition and subtraction)*

$$\begin{bmatrix} 10 \\ 11 \end{bmatrix} - \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 6 & 2 \end{bmatrix}$$

Matrix *multiplication* requires the two matrices to be conformable: the first matrix has as many columns as the second matrix has rows. Element $ij$ of the result is the multiplication of the $i$th row of the first matrix with the $j$th column of the second matrix

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}.$$

Multiplying a square matrix $A$ with a column vector $z$ gives a column vector

$$Az = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} A_{11}z_1 + A_{12}z_2 \\ A_{21}z_1 + A_{22}z_2 \end{bmatrix}.$$

**Example 1.31** *(Matrix multiplication)*

$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 10 & -4 \\ 15 & -2 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 17 \\ 26 \end{bmatrix}$$

### 1.8.3 Transpose

Transposing a column vector gives a row vector. Similarly, transposing a matrix is like flipping it around the main diagonal

$$A' = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}' = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix}.$$

**Example 1.32** *(Matrix transpose)*

$$\begin{bmatrix} 10 \\ 11 \end{bmatrix}' = \begin{bmatrix} 10 & 11 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

### 1.8.4 Inner and Outer Products, Quadratic Forms

For two column vectors $x$ and $z$, the product $x'z$ is called the *inner product* (a scalar)

$$x'z = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = x_1 z_1 + x_2 z_2,$$

and $xz'$ the *outer product* (a matrix)

$$xz' = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} z_1 & z_2 \end{bmatrix} = \begin{bmatrix} x_1 z_1 & x_1 z_2 \\ x_2 z_1 & x_2 z_2 \end{bmatrix}.$$

(Notice that $xz$ does not work for two vectors.)

**Example 1.33** *(Inner and outer products)*

$$\begin{bmatrix} 10 \\ 11 \end{bmatrix}' \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 & 11 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 75$$
$$\begin{bmatrix} 10 \\ 11 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}' = \begin{bmatrix} 10 \\ 11 \end{bmatrix} \begin{bmatrix} 2 & 5 \end{bmatrix} = \begin{bmatrix} 20 & 50 \\ 22 & 55 \end{bmatrix}$$

*If $x$ is a column vector and $A$ a square matrix, then the product $x'Ax$ is a* quadratic form *(a scalar).*

**Example 1.34** *(Quadratic form)*

$$\begin{bmatrix} 10 \\ 11 \end{bmatrix}' \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 10 \\ 11 \end{bmatrix} = 1244$$

### 1.8.5 Kronecker Product

Let $\otimes$ represent the Kronecker product, that is, if $A$ and $B$ are matrices, then

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}.$$

**Example 1.35** *(Kronecker product)*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 10 & 11 \end{bmatrix}, \text{ we get } A \otimes B = \begin{bmatrix} 10 & 11 & 30 & 33 \\ 20 & 22 & 40 & 44 \end{bmatrix}.$$

### 1.8.6 Matrix Inverse

A matrix *inverse* is the closest we get to "dividing" by a square matrix. The inverse of a matrix $A$, denoted $A^{-1}$, is such that

$$AA^{-1} = I \text{ and } A^{-1}A = I,$$

where $I$ is the *identity matrix* (ones along the diagonal, and zeros elsewhere). The matrix inverse is useful for solving systems of linear equations, $y = Ax$ as $x = A^{-1}y$.

For a $2 \times 2$ matrix we have

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^{-1} = \frac{1}{A_{11}A_{22} - A_{12}A_{21}} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}.$$

**Example 1.36** *(Matrix inverse) We have*

$$\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & -1/5 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ so}$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & -1/5 \end{bmatrix}.$$

### 1.8.7 Solving Systems of Linear Equations

If $A$ is $n \times n$ and invertible and $b$ and $y$ are $n \times 1$ vectors, then we can solve

$$Ab = y \text{ as } b = A^{-1}y.$$

This solution is unique.

$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}, \text{ gives}$$

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 11 \end{bmatrix} = \begin{bmatrix} -1.4 \\ 3.8 \end{bmatrix}.$$

### 1.8.8 OLS Notation: $X'X$ or $\sum_{t=1}^{T} x_t x_t'$?

Let $x_t$ be a $K \times 1$ vector of (of data in period $t$). We can the calculate the outer product $(K \times K)$ as $x_t x_t'$ and summing each element across $T$ observations gives the $K \times K$ matrix $S_{xx} = \sum_{t=1}^{T} x_t x_t'$.

Alternatively, let $X$ be a $T \times K$ matrix with $x_t'$ in row $t$. Then we can also calculate $S_{xx}$ as $X'X$.

**Example 1.37** *(Sum of outer product, $\sum_{t=1}^{T} x_t x_t'$)*

$$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \textit{and } x_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

*We then have*

$$\begin{aligned} \sum_{t=1}^{T} x_t x_t' &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

*In this example, the matrix happens to be diagonal, but that is not a general result. However, it will always be symmetric.*

**Example 1.38** *(Sum of outer product, $X'X$) Define*

$$X = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

*It is straightforward to calculate that* $X'X = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

Also, if $y_t$ is a scalar, then $S_{xy} = \sum_{t=1}^{T} x_t y_t$ is a $K \times 1$ vector which is calculated as $X'Y$ where $Y$ is a $T \times 1$ vector.

**Example 1.39** *With* $(y_1, y_2, y_3) = (-1.5, -0.6, 2.1)$

$$\sum_{t=1}^{T} x_t y_t = \begin{bmatrix} 1 \\ -1 \end{bmatrix}(-1.5) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}(-0.6) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} 2.1 = \begin{bmatrix} 0 \\ 3.6 \end{bmatrix}$$

### 1.8.9 Derivatives of Matrix Expressions

Let $z$ and $x$ be $n \times 1$ vectors. The *derivative of the inner product* is $\partial(x'z)/\partial x = z$.

**Example 1.40** *(Derivative of an inner product) With* $n = 2$

$$x'z = x_1 z_1 + x_2 z_2, \text{ so } \partial(x'z)/\partial x = \frac{\partial(z_1 x_1 + z_2 x_2)}{\begin{bmatrix} \partial x_1 \\ \partial x_2 \end{bmatrix}} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}.$$

Let $x$ be $n \times 1$ and $A$ a symmetric $n \times n$ matrix. The *derivative of the quadratic form* is $\partial(x'Ax)/\partial x = 2Ax$.

**Example 1.41** *(Derivative of a quadratic form) With* $n = 2$*, the quadratic form is*

$$x'Ax = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 A_{11} + x_2^2 A_{22} + 2x_1 x_2 A_{12}.$$

*The derivatives with respect to* $x_1$ *and* $x_2$ *are*

$$\partial(x'Ax)/\partial x_1 = 2x_1 A_{11} + 2x_2 A_{12} \text{ and } \partial(x'Ax)/\partial x_2 = 2x_2 A_{22} + 2x_1 A_{12}, \text{ or}$$

$$\partial(x'Ax)/\partial x = \begin{bmatrix} \partial(x'Ax)/\partial x_1 \\ \partial(x'Ax)/\partial x_2 \end{bmatrix} = 2 \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

# Chapter 2

# The Basics of Portfolio Choice

There are two key elements to portfolio choice: *(1)* how to mix the risky assets with a risk-free asset (leverage) to handle the overall risk level; and *(2)* how to mix various (risky) assets to average out volatility (diversification). This chapter introduces each of these topics. Later chapters will put them together in unified framework.

## 2.1 Expected Portfolio Return and Variance

This technical section summarizes how information about the expected returns of the investable assets and their variance-covariance matrix can be combined with portfolio weights to calculate the implications for the portfolio. In later sections, this will done for many different possible portfolio weights—to eventually find the optimal choice.

**Remark 2.1** *(Expected value and variance of a linear combination) Recall that if $w_1$ and $w_2$ are two constants, while the returns $R_1$ and $R_2$ are random variables, then*

$$\begin{aligned}\mathrm{E}(w_1 R_1 + w_2 R_2) &= w_1\mu_1 + w_2\mu_2, \text{ and} \\ \mathrm{Var}(w_1 R_1 + w_2 R_2) &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2\sigma_{12},\end{aligned}$$

*where $\mu_i = \mathrm{E}\, R_i$, $\sigma_{ij} = \mathrm{Cov}(R_i, R_j)$, and $\sigma_i^2 = \mathrm{Var}(R_i)$.*

The expected return on the portfolio is (time subscripts are suppressed)

$$\mathrm{E}\, R_p = \textstyle\sum_{i=1}^{n} w_i\mu_i \tag{2.1}$$

$$= w'\mu, \tag{2.2}$$

where $w$ is the $n$-vector of portfolio weights and $\mu$ is a corresponding vector of expected asset returns.

The variance of a portfolio return is

$$\text{Var}(R_p) = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} w_i w_j \sigma_{ij} \tag{2.3}$$
$$= w' \Sigma w, \tag{2.4}$$

where $\Sigma$ is the $n \times n$ variance-covariance matrix of the returns.

**Remark 2.2** *($n = 2$) With two assets,* $\mathrm{E}\, R_p = w_1\mu_1 + w_2\mu_2$, $\text{Var}(R_p) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}$ *and* $\text{Cov}(R_q, R_p) = v_1w_1\sigma_1^2 + v_2w_2\sigma_2^2 + (v_1w_2 + v_2w_1)\sigma_{12}$.

**Example 2.3** *(Expected value and variance of portfolio return) Let the portfolio weights be* $w = [0.8, 0.2]$. *Assume the following the expected values and covariance matrix for the returns:* $\mu = \begin{bmatrix} 9 \\ 6 \end{bmatrix} /100$ *and* $\Sigma = \begin{bmatrix} 256 & 96 \\ 96 & 144 \end{bmatrix} /100^2$, *we have*

$$\mathrm{E}\, R_p = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \end{bmatrix} \frac{1}{100} = 0.084,$$
$$\text{Var}(R_p) = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix} \begin{bmatrix} 256 & 96 \\ 96 & 144 \end{bmatrix} \frac{1}{100^2} \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} \approx 0.020, \textit{ and}$$
$$\text{Std}(R_p) \approx 0.142.$$

More details and examples are found in the statistics appendix.

## 2.2 Leverage

### 2.2.1 A Portfolio of a Single Risky Asset and a risk-free Asset

Suppose you can only invest in a risky asset (with return $R_i$) and a risk-free (with return $R_f$). The risky asset could represent the (equity) market portfolio. To observe the effect of the portfolio choice on the mean and the volatility, notice that

$$R_p = vR_i + (1 - v)R_f, \text{ so} \tag{2.5}$$
$$\mathrm{E}\, R_p = v\mu + (1 - v)R_f \text{ and} \tag{2.6}$$
$$\text{Std}(R_p) = |v|\sigma, \tag{2.7}$$

where we use ($\mu$ and $\sigma$) as short hand notation for the mean and standard deviation of the risky asset ($i$).

The expected value follows from $\mathrm{E}\, R_f = R_f$ as the risk-free rate is known. Similarly, the standard deviation follows from $\mathrm{Var}(R_p) = v^2\sigma^2$, since $\mathrm{Var}(R_f) = 0$ (the risk-free rate over the investment horizon is known when the portfolio is formed) and hence also the covariance is zero. If we use an interest rate to represent the risk-free rate, then we should typically use a maturity that corresponds to the investment horizon.

How much to put in the risky asset is a matter of *leverage*, and $v$ is often called the *leverage ratio*. This equals the investment in risky assets divided by our total capital.

**Example 2.4** *(Leveraged portfolios) Portfolio weights for three different portfolios*

| | *Portfolio A* | *Portfolio B* | *Portfolio C* | *Portfolio D* |
|---|---|---|---|---|
| $v$ *(in risky assets)* | 0.5 | 1 | 2 | $-1$ |
| $1 - v$ *(in risk-free)* | 0.5 | 0 | $-1$ | 2 |
| *Sum* | 1 | 1 | 1 | 1 |

Portfolio A*: your capital is* 200*, invest* 100 *in risky assets and* 100 *in risk-free;* Portfolio B*: your capital is* 200*, invest* 200 *in risky assets and* 0 *in risk-free;* Portfolio C*: your capital is* 200*, invest* 400 *in risky assets and* $-200$ *in risk-free (borrow 200 = short position in risk-free).*

**Example 2.5** *(Short-selling) We also consider a portfolio D which has* $v = -1$ *and* $1 - v = 2$*. This means that we short-sell the risky asset and put all the money on a bank account.*

**Remark 2.6** *(Assuming that* $R_i$ *and* $R_f$ *do not depend on* $v$*) These notes assume that the portfolio choice (here* $v$*) does not affect the returns. This means that we assume that the investor is small compared to the overall market. It also means that we effectively assume that lending (*$1 - v > 0$*) and borrowing (*$1 - v < 0$*) can be done at the same rate. This is a reasonable approximation for a large financial institution and simplifies the analysis considerably.*

The mean and the standard deviation are both scaled by the leverage ratio ($v$). Notice that taking on leverage (borrowing to invest in the risky asset) typically is a way to increase the expected return of the portfolio, but at the cost of increasing the risk. This shifts the mean up, while also increasing volatility.

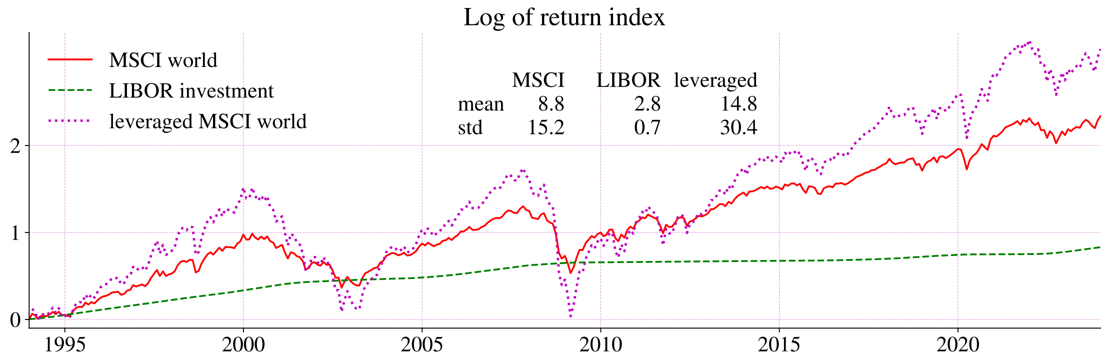



Figure 2.1: The effect of leverage on the portfolio performance

**Empirical Example 2.7** *Figure 2.1 shows the effect on the log cumulated excess returns from holding a leveraged position in equity. The LIBOR rate (London Interbank Offered Rate) is, of course, not entirely without variation in this figure, thus, the result in (2.7) applies only approximately. However, for each separate 1-month investment horizon, the LIBOR rate is known in advance and thus risk-free.*

**Example 2.8** *With* $R_i \sim N(0.095, 0.08^2)$ *(mean of 9.5%, standard deviation of 8%) and* $R_f = 0.03$, *we get (in %)*

| | *Portfolio A* | *Portfolio B* | *Portfolio C* |
|---|---|---|---|
| *Mean* | 6.25 | 9.5 | 16 |
| *Std* | 4 | 8 | 16 |

As long as the leverage ratio is positive ($v > 0$), we can combine these equations to get

$$\mathrm{E}\, R_p = R_f + SR \times \mathrm{Std}(R_p), \tag{2.8}$$

where $SR = \mu^e/\sigma$ is the *Sharpe ratio* of the risky asset. This shows that the average portfolio return is linearly related to its standard deviation. Figure 2.2 illustrates the calculations in (2.6)–(2.8).

## 2.3 Diversification

This section demonstrates that the portfolio variance can be reduced by forming a portfolio by mixing *(a)* assets that are only weakly correlated and *(b)* many assets. These diversification benefits can often be achieved without hurting the expected returns.

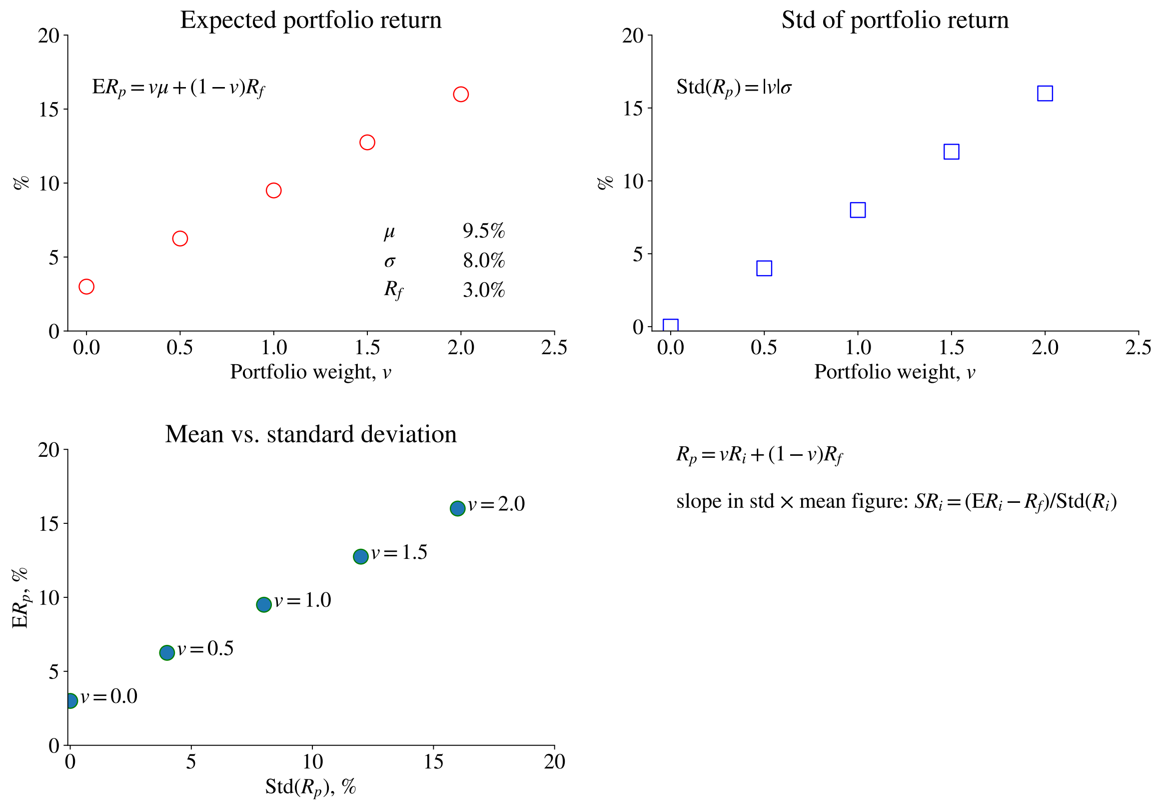



Figure 2.2: The effect of leverage on the mean and volatility of the portfolio return

Recall that the variance of a portfolio return is

$$\text{Var}(R_p) = w'\Sigma w, \tag{2.9}$$

where $w$ is the vector of portfolio weights and $\Sigma$ the variance-covariance matrix. For instance, with two assets we have

$$\text{Var}(R_p) = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12}, \tag{2.10}$$

where $w_i$ is the vector of portfolio weight on asset $i$, $\sigma_i^2$ is the variance of asset $i$ and $\sigma_{ij}$ is the covariance of assets $i$ and $j$.

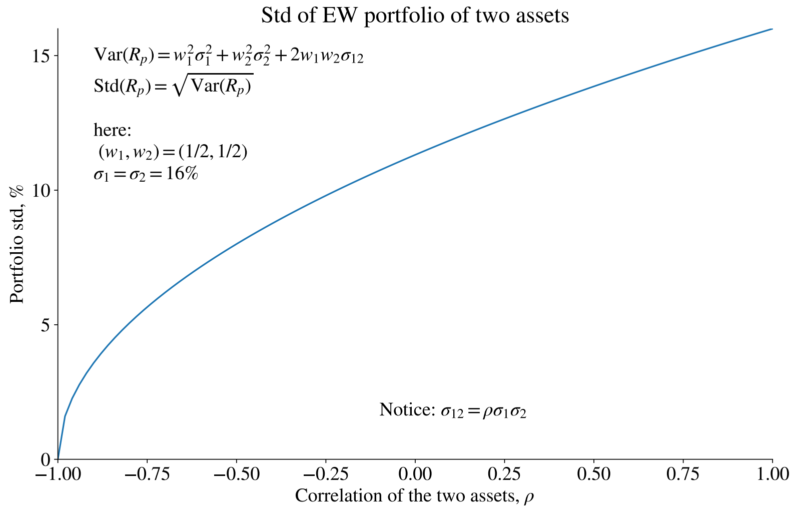



Figure 2.3: Effect of correlation on the diversification benefits

### 2.3.1 Diversification: The Correlations

As a simple example, consider an *equally weighted (EW) portfolio* of two risky assets (use $w_1 = w_2 = 1/2$ in (2.10)). Denote the correlation by $\rho$ and write as (since $\sigma_{12} = \rho\sigma_1\sigma_2$)

$$\begin{aligned}\mathrm{Var}(R_p) &= \frac{1}{4}\sigma_1^2 + \frac{1}{4}\sigma_2^2 + \frac{1}{2}\rho\sigma_1\sigma_2 \\ &= \frac{1}{2}\sigma^2(1+\rho) \text{ if } \sigma_1 = \sigma_2 = \sigma, \end{aligned} \tag{2.11}$$

where the second equality assumes that both assets have the same standard deviation.

If the assets are uncorrelated ($\rho = 0$), then the variance of this portfolio is half that of the asset—which demonstrates the importance of diversification. This effect is even stronger when the correlation is negative: with $\rho = -1$ the portfolio variance is actually zero, which we call *hedging*. In contrast, with a high correlation, the benefit from diversification is much smaller (and zero when the correlation is perfect, $\rho = 1$). See Figure 2.3 for an illustration.

**Example 2.9** *(Diversification) If* $\sigma = 16\%$ *(so* $\sigma^2 = 256/100^2$*) and* $\rho = 0.5$*, then (2.11) gives* $\mathrm{Var}(R_p) = 192/100^2$ *and thus* $\mathrm{Std}(R_p) \approx 13.9\%$*.*

**Empirical Example 2.10** *Table 2.1 provides empirical examples of the correlations between major asset classes.*

| | Small growth | Small value | Large growth | Large value | Bonds | T-bills | Equity market |
|---|---|---|---|---|---|---|---|
| Small growth | 1.00 | 0.86 | 0.81 | 0.68 | −0.10 | −0.03 | 0.86 |
| Small value | 0.86 | 1.00 | 0.72 | 0.85 | −0.12 | −0.02 | 0.83 |
| Large growth | 0.81 | 0.72 | 1.00 | 0.76 | 0.04 | 0.04 | 0.97 |
| Large value | 0.68 | 0.85 | 0.76 | 1.00 | −0.07 | 0.03 | 0.85 |
| Bonds | −0.10 | −0.12 | 0.04 | −0.07 | 1.00 | 0.21 | −0.01 |
| T-bills | −0.03 | −0.02 | 0.04 | 0.03 | 0.21 | 1.00 | 0.02 |
| Equity market | 0.86 | 0.83 | 0.97 | 0.85 | −0.01 | 0.02 | 1.00 |

Table 2.1: Correlations of asset class returns, US, monthly returns, 1985:01-2023:12

### 2.3.2 Diversification: The Number of Assets

In order to see the importance of mixing many assets in the portfolio, we will consider equally weighted portfolios of $n$ assets ($w_i = 1/n$), to focus on the basic idea. Clearly, there are other (not equally weighted) portfolios with lower variance (and the same expected return).

The variance of an equally weighted ($w_i = 1/n$ so $w_i^2 = 1/n^2$) portfolio is

$$\operatorname{Var}(R_p) = \frac{1}{n}(\overline{\sigma^2} - \overline{\sigma}_{ij}) + \overline{\sigma}_{ij}, \tag{2.12}$$

where $\overline{\sigma^2}$ is the average variance (average across the $n$ assets) and $\overline{\sigma}_{ij}$ is the average covariance of two returns (which can be treated as a constant if we pick assets randomly). In case the assets are uncorrelated, (2.12) shows that the portfolio variance goes to zero as the number of assets (included in the portfolio) goes to infinity. More realistically, $\overline{\sigma}_{ij}$ is positive. When the portfolio includes many assets, then the average covariance dominates. In the limit (as $n$ goes to infinity), only this non-diversifiable risk matters, $\overline{\sigma}_{ij}$. See Elton, Gruber, Brown, and Goetzmann (2014) 4 for a more detailed discussion.

**Example 2.11** *(Variance of portfolio return) With* $\overline{\sigma^2} = 256/100^2$ *and* $\overline{\sigma}_{ij} = 128/100^2$*, we get a portfolio variance of* $(256, 192, 170.7)/100^2$ *for* $n = (1, 2, 3)$*, and thus portfolio standard deviations of* $(16\%, 13.9\%, 13.1\%)$.

**Empirical Example 2.12** *Figure 2.4 shows an empirical example of what diversification implies. Also, Figure 2.5 shows the contributions (according to (2.12)) of the variances and the covariances to the portfolio variance. Clearly, the covariances start to dominate as the number of assets in the portfolio increases—and the portfolio variance goes to-*

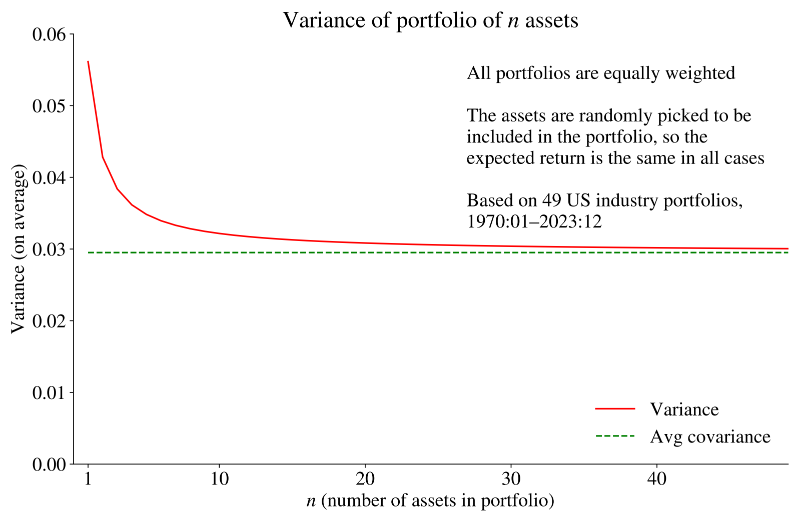



Figure 2.4: Effect of diversification

*wards the average covariance. Figure 2.6 suggests that the diversification benefits are not constant across time.*

*Proof* of (2.12). Note that $\text{Var}(R_p) = (\mathbf{1}/n)'\Sigma(\mathbf{1}/n)$, where $\mathbf{1}$ is a vector of ones. This is just summing the elements in $\Sigma$ and dividing by $n^2$. In this sum, there are $n$ variances and $n(n-1)$ covariances. We can thus write the variance as

$$\begin{aligned}\text{Var}(R_p) &= \frac{1}{n}\sum_{i=1}^{n}\frac{\sigma_i^2}{n} + \frac{n-1}{n}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\frac{\sigma_{ij}}{n(n-1)}\\ &= \frac{1}{n}\overline{\sigma^2} + \frac{n-1}{n}\overline{\sigma}_{ij},\end{aligned}$$

which can be rearranged as (2.12). □

**Remark 2.13** *(On negative covariances in (2.12)*) Formally, it can be shown that $\overline{\sigma}_{ij}$ must be non-negative as $n \to \infty$. It is simply not possible to construct a very large number of random variables (asset returns or whatever other random variable) that are, on average, negatively correlated with each other. In (2.12) this manifests itself in that $\overline{\sigma}_{ij} < 0$ would give a negative portfolio variance as $n$ increases.*

## 2.4 Covariances Do Matter

The discussion of diversification (above) was based on the assumption of an equally weighted portfolio. This section will (once again) illustrate the importance of covariances

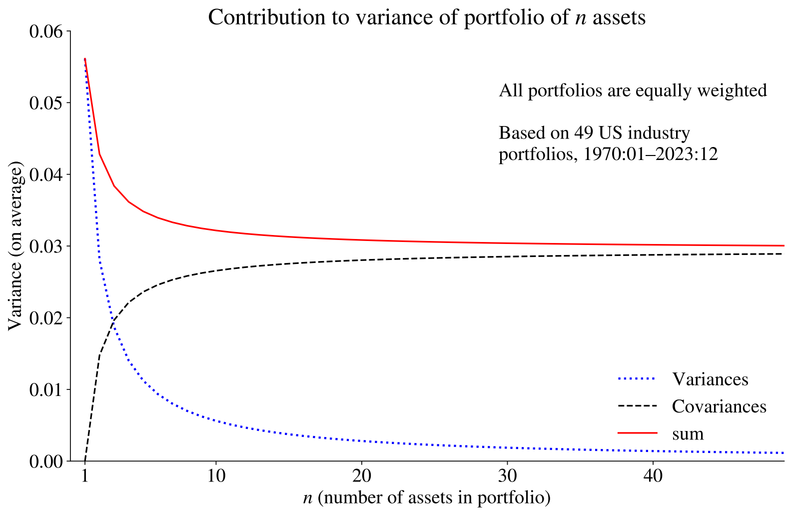



Figure 2.5: Contributions of variances and covariances to the portfolio variance

for the portfolio variance—without making that simplifying assumption.

Suppose we are initially invested in a portfolio $p$ (with portfolio weights of the risky assets in the vector $v$ risky assets and $1 - v'\mathbf{1}$ in the risk-free). We now consider a small increase ($\delta$) of the portfolio weight of asset $i$ financed by borrowing at the risk-free rate. The portfolio return of the new portfolio ($q$) would then be

$$R_q = R_p + \delta R_i^e, \tag{2.13}$$

The *incremental return*, $\delta R_i^e$, is just the excess return on the asset $i$. This is straightforward since we have increased the exposure to asset $i$ and financed it with borrowing at the risk-free rate.

The portfolio variance is

$$\sigma_q^2 = \sigma_p^2 + \underbrace{\delta^2\sigma_i^2 + 2\delta\sigma_{ip}}_{\text{incremental variance}}, \tag{2.14}$$

where $\sigma_{ip}$ is the covariance of our portfolio $p$ with asset $i$.

The *incremental variance* is $\delta^2\sigma_i^2 + 2\delta\sigma_{ip}$, so it depends on the variance of asset $i$ and on how it correlates with our current portfolio $p$. For small values of $\delta$ (say, $\delta = 5\%$) the *covariance effect might dominate* (since $\delta^2$ decreases very quickly). Conversely, adding a small amount of an uncorrelated asset ($\sigma_{ip} = 0$) to your portfolio does not change the portfolio variance much at all. See Figure 2.7 for an illustration.

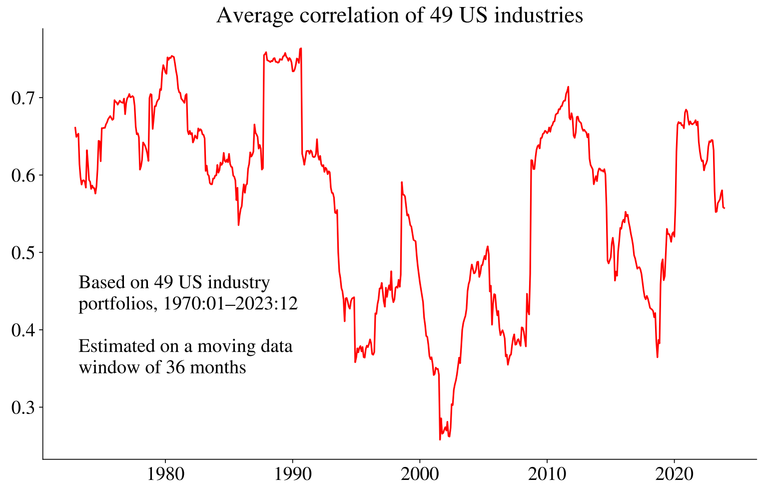



Figure 2.6: Time-varying correlations

**Example 2.14** *(of (2.14)) The easiest case is when $\sigma_p$ and $\sigma_i$ both equal 1, so $\sigma_{ip}$ equals the correlation $\rho$. Then, the incremental variance is $\delta^2 + 2\delta\rho$. For $\delta = 0.05$ we have $0.0025 + 0.04$ when $\rho = 0.8$ so the covariance effect is 16 times larger than the effect of $\delta^2\sigma_i^2$. Conversely, with $\rho = 0$ the covariance effect is zero. See also Figure 2.7.*

## 2.5 Appendix – A Primer in Statistics*

This appendix first summarizes some mathematical statistics required for the financial models discussed in these notes. Towards the end, it briefly addresses topics in estimation and testing.

### 2.5.1 The Distribution of a Random Variable

The distribution of a random variable $x$ represents the probabilities of its possible values. See Figure 2.8 for illustrations of the (discrete) distribution of a binomial variable and of several different (continuous) normal distributions, often denoted $N(\mu, \sigma^2)$ to indicate the mean and variance.

The probability that $x \leq B$ is given by the *cumulative distribution function*, cdf$(B)$. For instance, if $x$ has a $N(0, 1)$ distribution, then $\Pr(x \leq -1.645) = 0.05$ and $\Pr(x \leq 0) = 0.5$. See Figure 2.9 for an illustration.

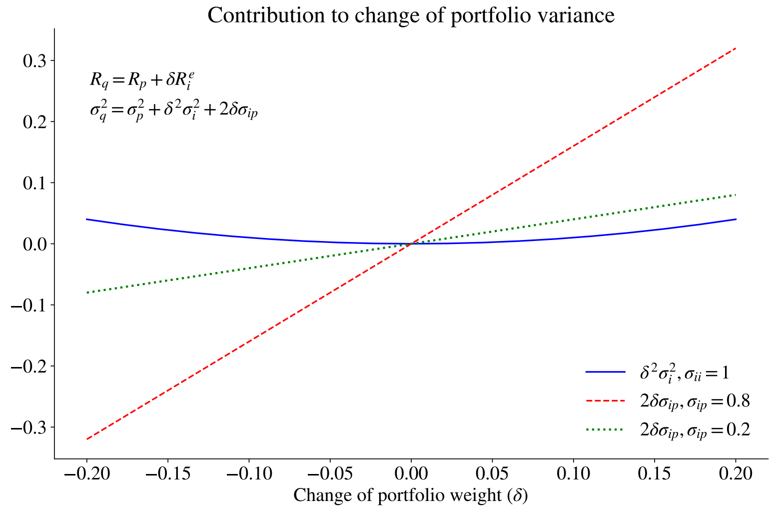



Figure 2.7: The effect of a portfolio change on the variance

If we invert the cdf, then we get the *quantiles* of the random variable. For instance, the 0.05th quantile of a $N(0, 1)$ variable is $-1.645$, while the 0.5th quantile (also called the median) is 0.

### 2.5.2 Expected Value and the Variance of a Random Variable

The expected value (or mean) of a distribution is defined as

$$\mathrm{E}\,x = \textstyle\sum_{s=1}^{S} \pi_s x_s \text{ or } \int f(x) x dx,$$

for a discrete and continuos random variable, respectively. For the former, $\pi_s$ denotes the probability of outcome $x_s$, and for the latter $f(x)$ represents the probability density function (pdf). The probabilities must sum to unity; therefore $\Sigma_{s=1}^{S} \pi_s = 1$ and $\int f(x) dx = 1$. Again, see Figure 2.8. The expected value is sometimes denoted $\mu$.

The expectation can be extended to a function $g(x)$ of the random variable as

$$\mathrm{E}\,g(x) = \textstyle\sum_{s=1}^{S} \pi_s g(x_s) \text{ or } \int f(x) g(x) dx.$$

A typical case is $g(x) = (x - \mu)^2$, which gives the variance

$$\mathrm{Var}(x) = \textstyle\sum_{s=1}^{S} \pi_s (x_s - \mu)^2 \text{ or } \int f(x)(x - \mu)^2 dx.$$

We often use $\sigma^2$ to denote the variance. The standard deviation is the square root of the

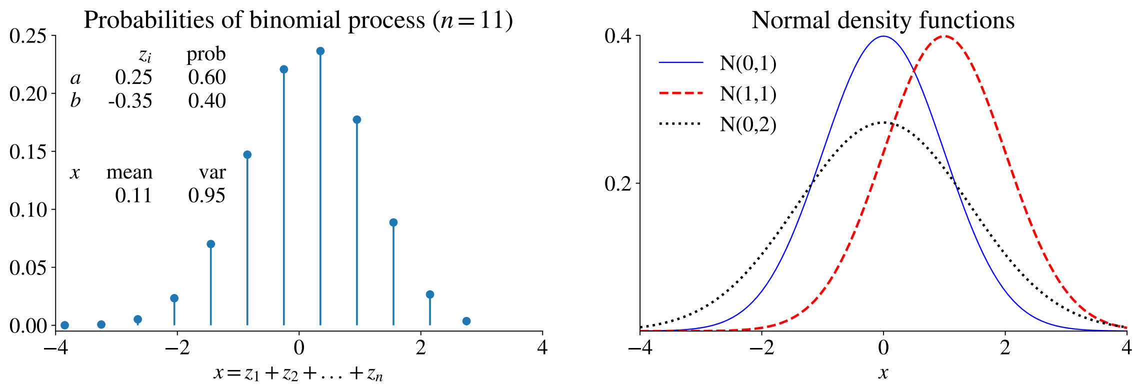



Figure 2.8: Density functions for a binomial and several normal distributions

variance, $\text{Std}(x) = \text{Var}(x)^{1/2}$.

In most of the portfolio analysis, these concepts refer to the expectations of investors as of the time of investment. This implies that they may change over time and also differ from the historical average returns.

If $a$ and $b$ are two constants, then the previous expressions directly show that

$$\begin{aligned} \mathrm{E}(a + bx) &= a + b\,\mathrm{E}\,x \\ \text{Var}(a + bx) &= b^2\,\text{Var}(x). \end{aligned}$$

Again, consider $\mathrm{E}\,g(x)$ and suppose $x$ depends on a choice variable $v$, for instance, when $x$ is the return of a portfolio of two assets, $vR_1 + (1 - v)R_2$. The derivative of $\mathrm{E}\,g(x)$ is then the expected value of the derivative, so we can interchange the order of E and the derivative

$$\frac{d\,\mathrm{E}\,g(x)}{dv} = \textstyle\sum_{s=1}^{S} \pi_s \displaystyle\frac{dg(x_s)}{dx}\frac{dx_s}{dv} = \mathrm{E}\,\frac{dg(x)}{dv}.$$

A similar expression holds for a continuous distribution.

### 2.5.3 Expected Value and the Variance of a Vector of Random Variables

There are straightforward extensions to vectors of random variables. For instance, if $x = [x_1, x_2]$ is a vector of the two random variables (returns?) $x_1$ and $x_2$ (the subscripts here indicate different variables, not time periods), then the mean of $x$ is a vector of the

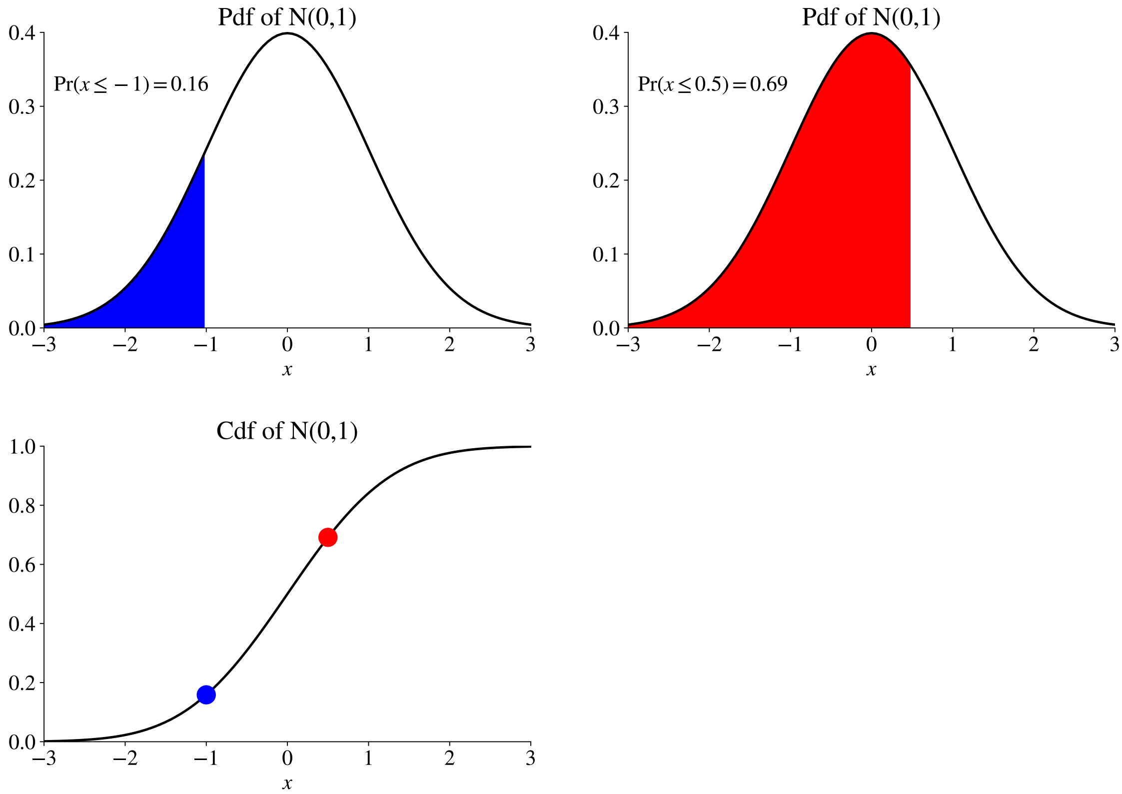



Figure 2.9: Pdf and cdf of N(0,1)

means of the two individual returns

$$\mathrm{E}\,x = \begin{bmatrix} \mathrm{E}\,x_1 \\ \mathrm{E}\,x_2 \end{bmatrix}.$$

Also, the $(2 \times 2)$ variance-covariance matrix of $x$ is

$$\mathrm{Var}(x) = \begin{bmatrix} \mathrm{Var}(x_1) & \mathrm{Cov}(x_1, x_2) \\ \mathrm{Cov}(x_2, x_1) & \mathrm{Var}(x_2) \end{bmatrix}.$$

Clearly, the matrix is symmetric (since two covariances are the same). The *correlation* of $x_1$ and $x_2$ is $\rho_{12} = \mathrm{Cov}(x_1, x_2)/[\mathrm{Std}(x_1)\,\mathrm{Std}(x_2)]$. See Figure 2.10 for an example of a bivariate distribution.

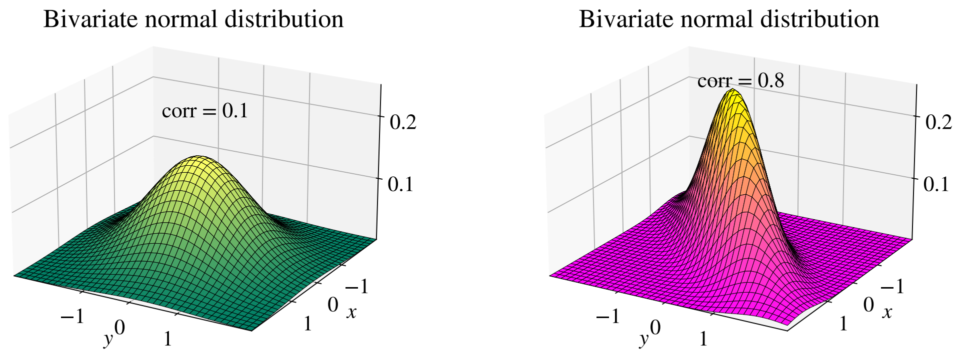



Figure 2.10: Density functions of bivariate normal distributions

### 2.5.4 Expected Value and the Variance of a Linear Combination of Random Variables

Consider a linear combination of the random variables $x_1, ..., x_n$

$$y = \sum_{i=1}^{n} w_i x_i = w'x.$$

For instance, $x$ could be a vector of portfolio returns and $w$ a vector of portfolio weights.

The expected value and the variance are

$$\begin{aligned} \mathrm{E}\, R_p &= w'\mu \\ \mathrm{Var}(R_p) &= w'\Sigma w, \end{aligned}$$

where $\mu$ is a vector of average returns and $\Sigma$ is the $n \times n$ variance-covariance matrix of $x$.

Also, consider another linear combination, $z = v'x$. Then, the covariance

$$\mathrm{Cov}(z, y) = v'\Sigma w.$$

This could, for instance, be two different portfolios.

**Remark 2.15** *(Details on the matrix form) With two assets, we have the following:*

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \text{ and } \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix},$$

*where we use $\sigma_{ii}$ to indicate $\sigma_i^2$ (this helps reading the matrices).*

$$\begin{aligned}
\mathrm{E}\, y &= w'\mu \\
&= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \\
&= w_1\mu_1 + w_2\mu_2.
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(y) &= w'\Sigma w \\
&= \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&= \begin{bmatrix} w_1\sigma_{11} + w_2\sigma_{12} & w_1\sigma_{12} + w_2\sigma_{22} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \\
&= w_1^2\sigma_{11} + w_2 w_1\sigma_{12} + w_1 w_2\sigma_{12} + w_2^2\sigma_{22}.
\end{aligned}$$

### 2.5.5 Conditional Moments

The portfolio choice analysis will be based on expected future returns, variance and covariances. In general, these represent the beliefs of the investor at the time of investment. Clearly, this means that they may change over time and differ from the properties of historical data. Also, they are to be considered *conditional* in the sense that they refer to the current information/situation—and may therefore differ from *unconditional* moments.

As an example, suppose a random variable (return?) follows an AR(1) process

$$x_{t+1} = (1-\rho)\mu + \rho x_t + u_{t+1},$$

where $u_{t+1}$ is an iid term (innovation). In this case, the *conditional* expectation and variance are

$$\begin{aligned}
\mathrm{E}_t\, x_{t+1} &= (1-\rho)\mu + \rho x_t, \text{ and} \\
\mathrm{Var}_t(x_{t+1}) &= \mathrm{Var}(u_{t+1}).
\end{aligned}$$

This differ from the *unconditional*/long-run values which do not take into consideration the current state and are

$$\begin{aligned}
\mathrm{E}\, x_{t+1} &= \mu, \text{ and} \\
\mathrm{Var}(x_{t+1}) &= \mathrm{Var}(u_{t+1})/(1-\rho^2).
\end{aligned}$$

Note that there is no difference between conditional and unconditional moments when $x$ is *iid* (independently and identically distributed), which here means $\rho = 0$. Notice that iid implies, among other things, that $x$ is unpredictable and that the variance is constant over time.

### 2.5.6 Linear Regressions

Consider the linear model

$$\begin{aligned}
y_t &= x_{1t}\beta_1 + x_{2t}\beta_2 + \cdots + x_{kt}\beta_k + u_t \\
&= x_t'\beta + u_t,
\end{aligned}$$

where $y_t$ and $u_t$ are scalars, $x_t$ a $k \times 1$ vector, and $\beta$ is a $k \times 1$ vector of the true coefficients. In this expression, one of the elements of $x_t$ is typically a constant equal to one (and the intercept is its coefficient).

Least squares minimizes the sum of the squared fitted residuals and gives

$$\hat{\beta} = S_{xx}^{-1} \textstyle\sum_{t=1}^{T} x_t y, \text{ where } S_{xx} = \textstyle\sum_{t=1}^{T} x_t x_t'.$$

Clearly, $S_{xx}$ is an $k \times k$ matrix (and is often calculated as $X'X$ if row $t$ of $X$ contains $x_t'$).

If the residuals are iid, then in large samples, we can approximate the distribution of $\hat{\beta}$ as

$$\hat{\beta} \sim N(\beta, S_{xx}^{-1}\sigma^2),$$

where $\beta$ are the true values $\sigma^2 = \mathrm{Var}(u_t)$ denotes the variance of the residuals. In contrast, with autocorrelated residuals or time-varying variance of the residuals, then we have to apply Newey-West's or White's method for approximating the variance-covariance matrix. Based on this distribution, it is straightforward to test if a single coefficient equals a particular value by a $t$-test (see below, and replace $\mu$ by the particular value to test) or if a vector of coefficients equal a particular vector of value by a $\chi^2$-test (again, see below).

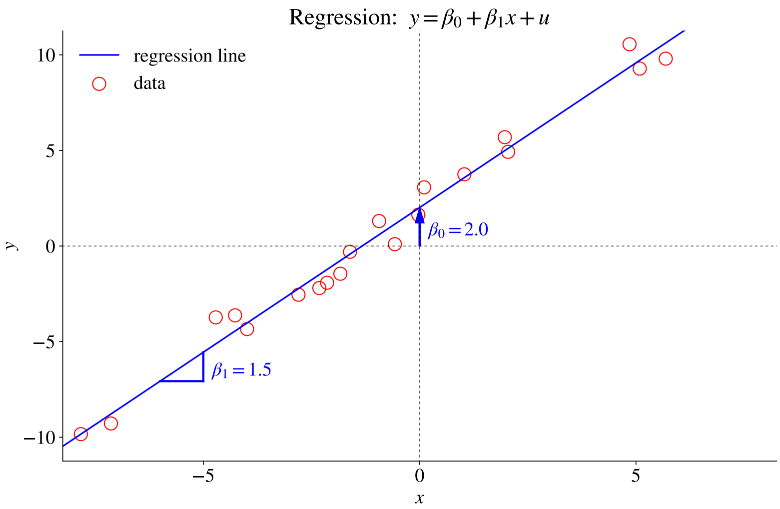



Figure 2.11: Example of OLS

### 2.5.7 Distributions Commonly Used in Tests

Suppose the random variable $x$ has a $N(\mu, \sigma^2)$ distribution. Then, the *standardized variable* $(x - \mu)/\sigma$ has a standard normal distribution

$$t = \frac{x - \mu}{\sigma} \sim N(0, 1).$$

To see this, notice that $x - \mu$ has a mean of zero and that $x/\sigma$ has a standard deviation of unity.

A $t$-distribution is sometimes used instead, since $\sigma$ has to be estimated. However, with 30 or more data points, the $t$-distribution and the $N(0, 1)$ are almost indistinguishable.

If $z \sim N(0, 1)$, then $z^2 \sim \chi_1^2$, that is, $z^2$ has a chi-square distribution with one degree of freedom. This can be generalized in several ways. For instance, if $x \sim N(\mu_x, \sigma_{xx})$ and $y \sim N(\mu_y, \sigma_{yy})$ and they are uncorrelated, then $[(x - \mu_x)/\sigma_x]^2 + [(y - \mu_y)/\sigma_y]^2 \sim \chi_2^2$.

More generally, we have

$$v'\Sigma^{-1}v \sim \chi_n^2, \text{ if the } n \times 1 \text{ vector } v \sim N(0, \Sigma).$$

See Figure 2.12 for an illustration of $\chi_n^2$ distributions.

**Example 2.16** *($\chi_2^2$ distribution) Suppose $x$ is a $2 \times 1$ vector*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N\left(\begin{bmatrix} 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}\right).$$

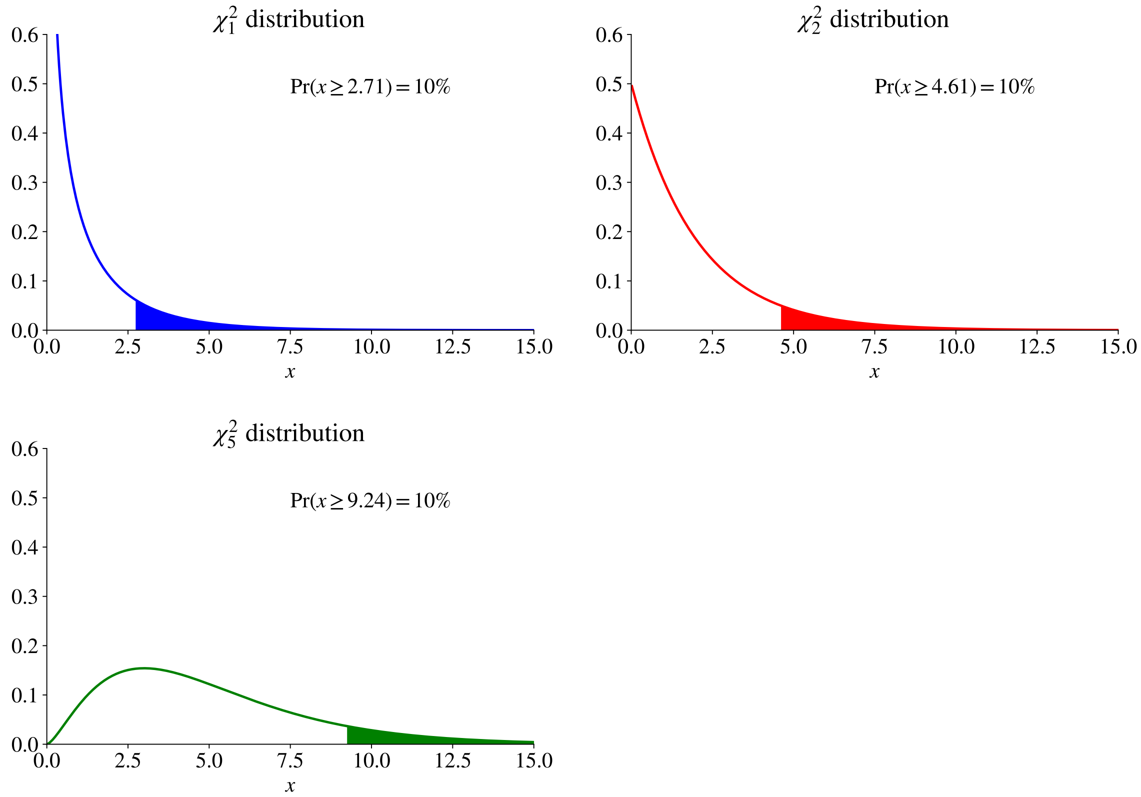



Figure 2.12: Density functions of $\chi_n^2$ distributions

*If $x_1 = 3$ and $x_2 = 5$, then*

$$\begin{bmatrix} 3-4 \\ 5-2 \end{bmatrix}' \begin{bmatrix} 5 & 3 \\ 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 3-4 \\ 5-2 \end{bmatrix} \approx 6.1$$

*has a $\chi_2^2$ distribution.*

# Chapter 3

# The Mean-Variance Frontier

## 3.1 The Mean-Variance Frontier of Risky Assets

The mean-variance frontier is based on the idea that the investor seeks high average portfolio returns but dislikes portfolio return variance.

| | $\mu, \%$ | $\Sigma$, bp | | |
|---|---|---|---|---|
| | | A | B | C |
| A | 11.5 | 166 | 34 | 58 |
| B | 9.5 | 34 | 64 | 4 |
| C | 6.0 | 58 | 4 | 100 |

Table 3.1: Characteristics of the assets in the MV examples. Notice that $\mu, \%$ is the expected return in % (that is, $\times 100$) and $\Sigma$, bp is the covariance matrix in basis points (that is, $\times 100^2$).

**Example 3.1** *(Mean and Std of a portfolio) Table 3.1 illustrates a case with three investable assets ($A$, $B$ and $C$). The mean returns are given in percentages; thus, 6% should be read as 0.06. In contrast, the covariance matrix is given in terms of basis points (bp, where 1bp* $= 1/10000$*); thus, 64bp. should be read as 0.0064. The square root of a variance is the standard deviation, so* $\sqrt{0.0064} = 0.08$*, that is, 8%. Figure 3.1 also illustrates some portfolios* $(1, 2, 3)$ *based on the three investable assets.*

To calculate a point on the mean-variance frontier, we have to find the portfolio that minimizes the portfolio variance, $\text{Var}(R_p)$, for a given expected return, $\mu^*$. The problem

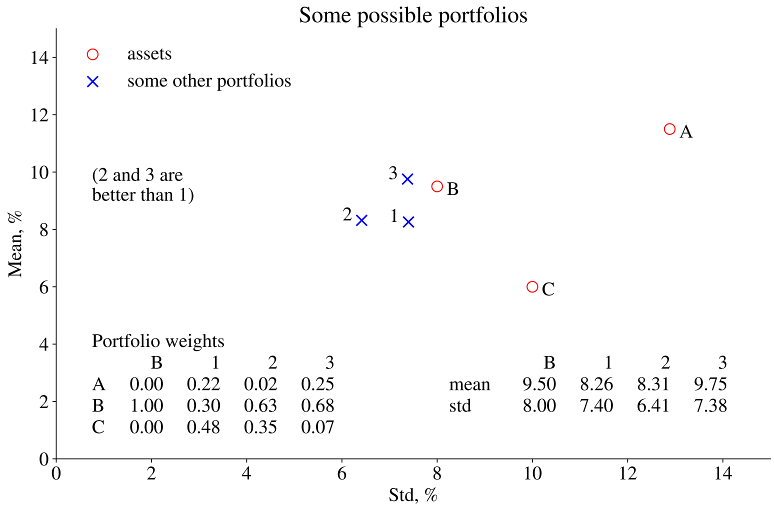



Figure 3.1: Mean vs standard deviation. The properties of the investable assets (A, B, and C) are shown in Table 3.1.

is thus

$$\begin{aligned} &\min_{w_i} \operatorname{Var}(R_p) \text{ subject to} \\ &\operatorname{E} R_p = \mu^* \text{ and } \textstyle\sum_{i=1}^n w_i = 1. \end{aligned} \tag{3.1}$$

**Remark 3.2** *(Portfolio average and variance) Let $\mu$ be the $n \times 1$ vector of average returns of all $n$ investable assets, $\Sigma$ the $n \times n$ covariance matrix of the returns and $w$ the $n \times 1$ vector of portfolio weights. The portfolio mean and variance can then be calculated as $\operatorname{E} R_p = w'\mu$ and $\operatorname{Var}(R_p) = w'\Sigma w$. All moments (means and the variance-covariance matrix) should be interpreted as the investor's beliefs, conditional on the information available at the time of the investment.*

The whole mean-variance frontier is generated by solving this problem for different values of the expected return, $\mu^*$. The results are typically shown in a figure with the *standard deviation* on the horizontal axis and the *expected return* on the vertical axis. The *efficient frontier* is the upper leg of the curve. Reasonably, a portfolio on the lower leg is dominated by one on the upper leg at the same volatility (since it has a higher expected return). Notice that there are no portfolios (based on the given investable assets and their assumed properties $\mu$ and $\Sigma$) above or to the left of the efficient frontier. See Figure 3.2 for an example.

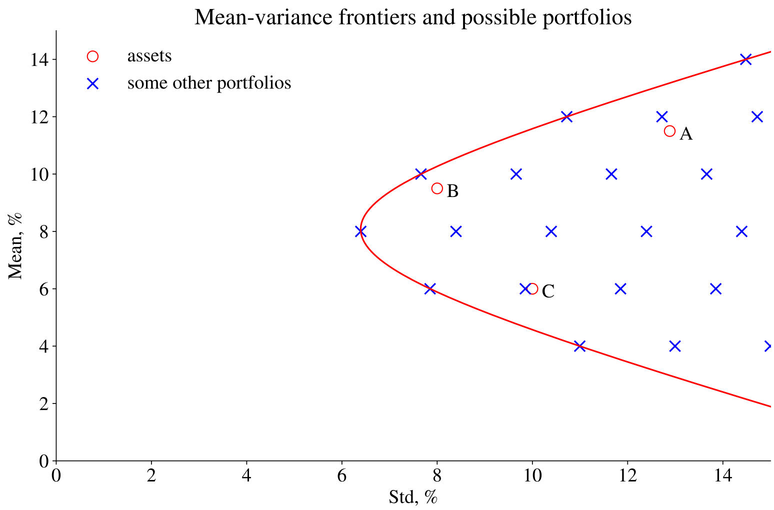



Figure 3.2: Mean-variance frontiers

**Remark 3.3** *(How many different portfolios are there with* $\mathrm{E}\,R_p = \mu^*$*?) With two assets, we require* $w\mu_1 + (1-w)\mu_2 = \mu^*$ *and there is only one choice of* $w$ *that satisfies this (assuming* $\mu_1 \neq \mu_2$*). Instead with three assets, we require* $w_1\mu_1 + w_2\mu_2 + (1-w_1-w_2)\mu_3 = \mu^*$ *which can hold for a continuum of* $(w_1, w_2)$ *values.*

### 3.1.1 The Mean Variance Frontier with Portfolio Restrictions

There are sometimes *additional restrictions*, for instance, of no short sales

$$\text{no short sales: } w_i \geq 0, \tag{3.2}$$

In other cases, there are both lower and upper bounds on the weights

$$L_i \leq w_i \leq U_i. \tag{3.3}$$

For instance, mutual funds often have to obey $L_i = 0$ and $U_i = 0.1$.

Funds may also impose restrictions on themselves; for instance, they may allow limited short sales

$$\text{limited total short sales: } \textstyle\sum_{i=1}^{n} \min(w_i, 0) \geq Q. \tag{3.4}$$

We then have to apply some explicit numerical minimization algorithm to find portfolio weights. (In contrast, with no restrictions, the solution can be found with linear

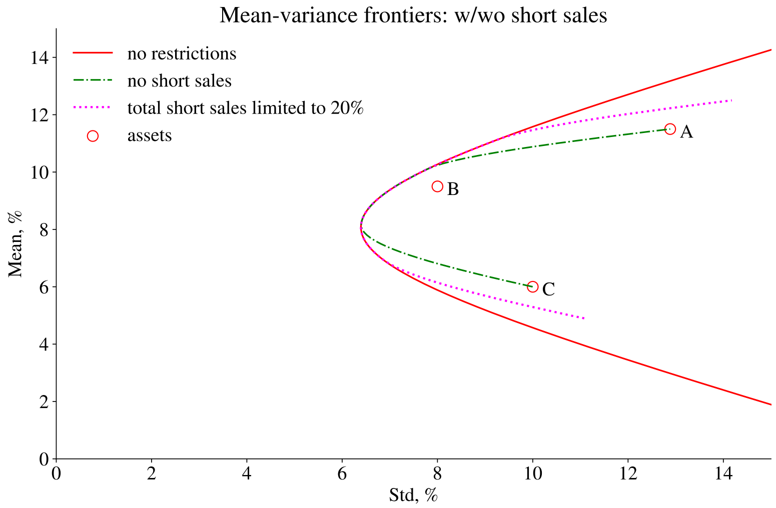



Figure 3.3: Mean-variance frontiers

algebra, as shown below.) Algorithms that solve quadratic problems are best suited. See Figures 3.3–3.4 for an example.

We can also introduce different lending and borrowing rates by defining a "lending asset" (with $w_i \geq 0$) and a "borrowing asset" (with $w_j \leq 0$).

### 3.1.2 The Mean Variance Frontier with Two Risky Assets

In the case of only two investable assets, the MV frontier can be calculated by simply calculating the mean and variance

$$\mathrm{E}\, R_p = w\mu_1 + (1-w)\mu_2 \tag{3.5}$$

$$\mathrm{Var}(R_p) = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}. \tag{3.6}$$

at a set of different portfolio weights, for instance, $w = (-1, -0.5, 0, 0.5, 1)$. The reason is that with only two assets, all portfolios of them are on the MV frontier (cf. Remark 3.3). For that reason no explicit minimization is needed. See Figure 3.5 for an example.

### 3.1.3 The Shape of the MV Frontier of Risky Assets

Consider what happens when we *add assets to the investment opportunity set*. The old mean-variance frontier is, of course, still obtainable: we can always put zero weights on the new assets. In most cases, we can do better than that: the mean-variance frontier

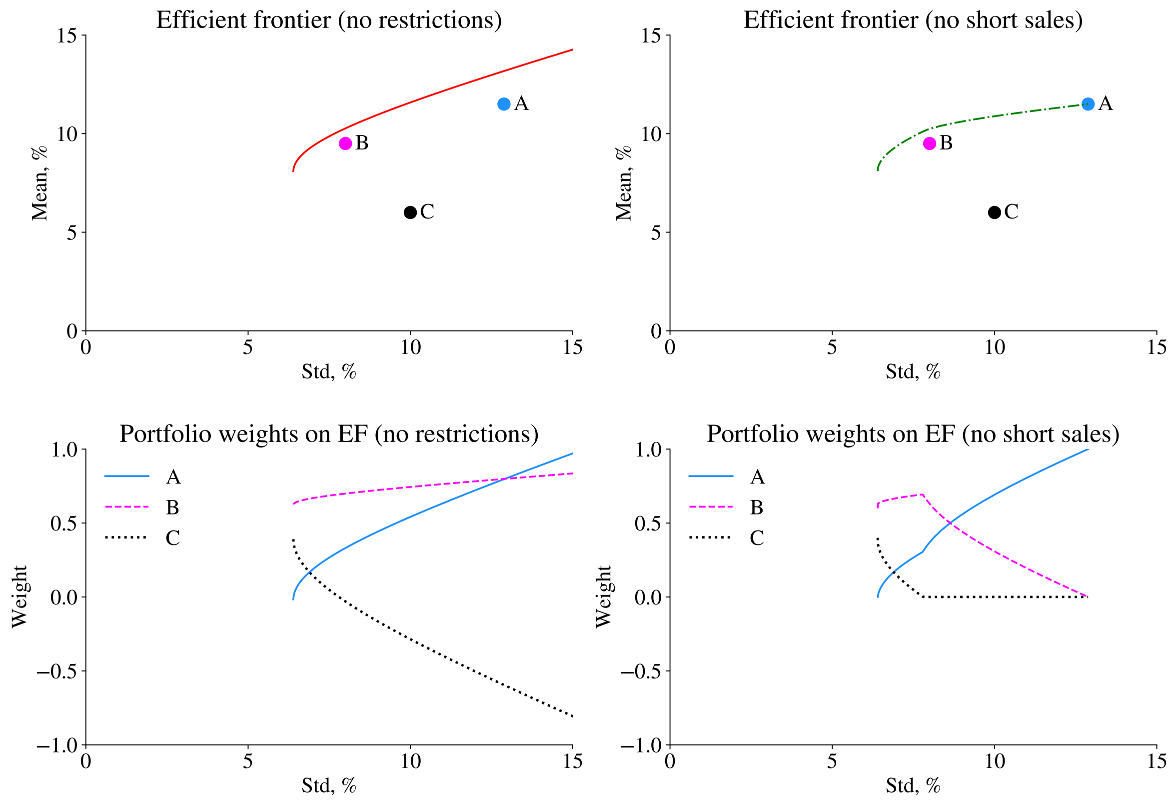



Figure 3.4: Portfolio weights on the efficient frontier

is shifted to the left (lower volatility at the same expected return). See Figure 3.6 for an example. In this case the new asset is not very attractive (low average returns, high volatility), but it may be useful in a portfolio (diversification, or for shortening).

*With intermediate correlations* ($-1 < \rho < 1$), the mean-variance frontier is a hyperbola (see Figure 3.7). Notice that the mean–volatility trade-off improves as the correlation decreases: a lower correlation means that we get a lower portfolio standard deviation at the same expected return—at least for the efficient frontier (above the bend).

**Empirical Example 3.4** *Figure 3.8 shows the MV frontier implied by the sample means and variance-covariance matrix of 10 U.S. industry portfolios. It is therefore an ex post construction.*

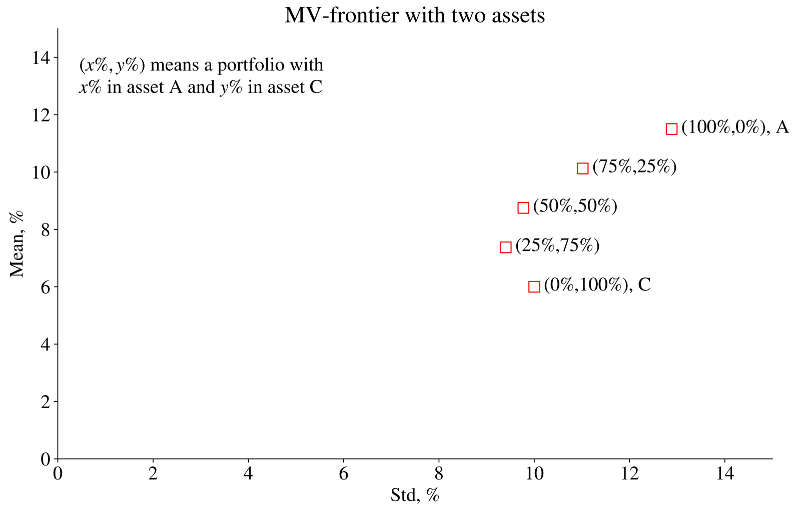



Figure 3.5: Mean-variance frontiers for two risky assets

### 3.1.4 Calculating the MV Frontier of Risky Assets

When there are no restrictions on the portfolio weights, then there are two ways of finding a point on the mean-variance frontier: let a numerical optimization routine do the job or use some simple matrix algebra. The section demonstrates the second approach.

The minimization problem (3.1) can be written

$$\begin{aligned} &\min_w w'\Sigma w \text{ subject to} \\ &w'\mu = \mu^* \text{ and } w'\mathbf{1} = 1, \end{aligned} \tag{3.7}$$

where $\mathbf{1}$ is a vector of $n$ ones (as many as there are assets). Again, $\mu$ and $\Sigma$ summarise the beliefs of the investor, conditional on the information available at the time of the investment.

**Remark 3.5** *(First order condition for optimising a differentiable function). We want to find the value of $b$ in the interval $b_{low} \leq b \leq b_{high}$, which makes the value of the differentiable function $f(b)$ as small (or large) as possible. The answer is $b_{low}$, $b_{high}$, or a value of $b$ where $df(b)/db = 0$.*

The first order conditions are

$$\begin{bmatrix} \Sigma & \mu & \mathbf{1} \\ \mu' & 0 & 0 \\ \mathbf{1}' & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mu^* \\ 1 \end{bmatrix}, \tag{3.8}$$

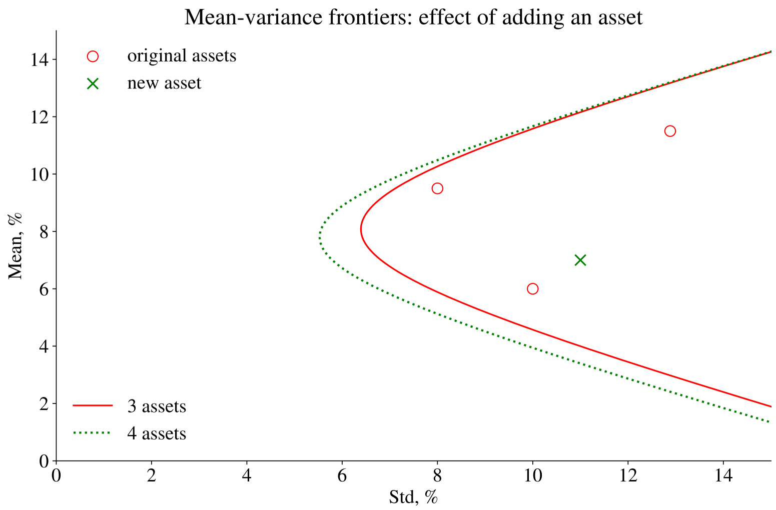



Figure 3.6: Mean-variance frontiers

where $\mathbf{0}$ is a vector of $n$ zeros. Solve for the vector $(w, \lambda, \delta)$ and extract the $w$ vector.

Using the solution in $\sqrt{w'\Sigma w}$ gives the standard deviation of a portfolio with expected return $\mu^*$ (which should equal $w'\mu$). We can trace out the entire MV frontier, by repeating this calculations for different values of the required return $\mu^*$ and then connecting the dots. In the std×mean space, the efficient frontier (the upper part) is *concave*.

**Remark 3.6** *(Alternative expression for the portfolio weights) Define the scalars $a$, $b$ and $c$ as $a = \mu'\Sigma^{-1}\mu, b = \mu'\Sigma^{-1}\mathbf{1}$, and $c = \mathbf{1}'\Sigma^{-1}\mathbf{1}$. Then, calculate the scalars (for a given required return $\mu^*$)*

$$\tilde{\lambda} = \frac{c\mu^* - b}{ac - b^2} \text{ and } \tilde{\delta} = \frac{a - b\mu^*}{ac - b^2}.$$

*The weights for a portfolio on the MV frontier of risky assets (at a given required return $\mu^*$) are then*

$$w = \Sigma^{-1}(\mu\tilde{\lambda} + \mathbf{1}\tilde{\delta}).$$

Another way to construct the MV frontier of risky assets is to retrace it by *combining any two portfolios (that are know to be) on the frontier*. For instance, we can use

$$\begin{aligned} w_\kappa &= \kappa w_g + (1-\kappa)\, w_T, \text{ where} \\ w_g &= \Sigma^{-1}\mathbf{1}/\mathbf{1}'\Sigma^{-1}\mathbf{1} \text{ and} \\ w_T &= \Sigma^{-1}\mu^e/\mathbf{1}'\Sigma^{-1}\mu^e. \end{aligned} \tag{3.9}$$

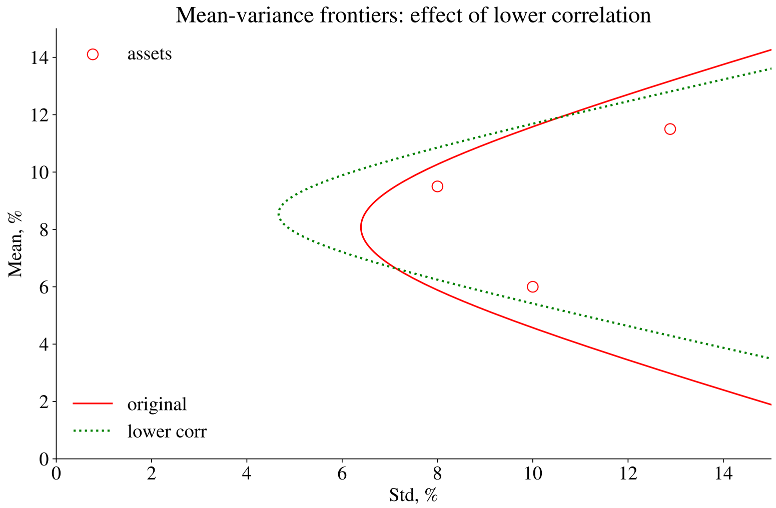



Figure 3.7: Mean-variance frontiers

The first line defines a portfolio in terms of two portfolios ($w_g$ and $w_T$) that are known to be on the MV frontier. The first ($w_g$) is the global minimum variance portfolio (lowest possible variance) and the second ($w_T$) is the tangency portfolio (to be discussed later on).

*Proof* of (3.8). We set up this as a Lagrangian problem

$$L = (w_1^2\sigma_{11} + w_2^2\sigma_{22} + 2w_1w_2\sigma_{12})/2 + \lambda(w_1\mu_1 + w_2\mu_2 - \mu^*) + \delta(w_1 + w_2 - 1).$$

Dividing the variance by 2 does not affect the solution. (Variances are denoted $\sigma_{ii}$ in order to facilitate comparison with the matrix expressions.) The first order conditions with respect to the portfolio weights are

$$\text{for } w_1 : w_1\sigma_{11} + w_2\sigma_{12} + \lambda\mu_1 + \delta = 0,$$
$$\text{for } w_2 : w_1\sigma_{12} + w_2\sigma_{22} + \lambda\mu_2 + \delta = 0.$$

Similarly, the first order conditions with respect to the Lagrange multipliers are

$$\text{for } \lambda : w_1\mu_1 + w_2\mu_2 = \mu^*,$$
$$\text{for } \delta : w_1 + w_2 = 1.$$

In matrix notation these first order conditions are

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \mu_1 & 1 \\ \sigma_{12} & \sigma_{22} & \mu_2 & 1 \\ \mu_1 & \mu_2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \lambda \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu^* \\ 1 \end{bmatrix},$$

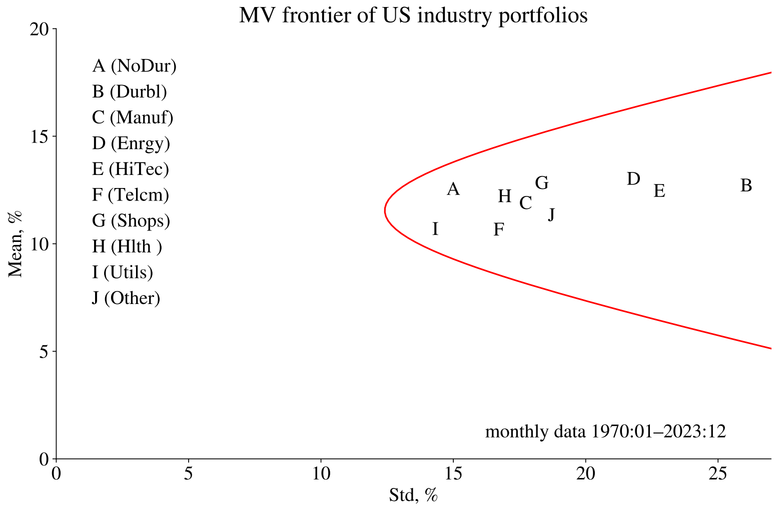



Figure 3.8: MV frontier from US industry indices

which is (3.8). $\square$

## 3.2 The Mean-Variance Frontier of Risk-free and Risky Assets

We now add a risk-free asset with return $R_f$ and notice that the restriction that $\mathrm{E}\, R_p = \mu^*$ can be written as

$$w'\mu + (1 - w'\mathbf{1})R_f = w'\mu^e + R_f = \mu^*, \tag{3.10}$$

where $\mu^e$ the vector of mean excess returns $(\mu - R_f)$. Here we use $w$ to denote the vector of portfolio weights on the risky assets only, with $1 - w'\mathbf{1}$ (that is, $1 - \Sigma_{i=1}^{n} w_i$) as the weight on the risk-free asset. This means that the requirement that all portfolio weights sum to 1 is automatically satisfied.

The minimization problem (3.1) can now be written

$$\begin{aligned} &\min_w w'\Sigma w \text{ subject to} \\ &w'\mu^e + R_f = \mu^*. \end{aligned} \tag{3.11}$$

When there are no additional constraints, then we can find an explicit solution. In other cases we need to apply an explicit numerical minimization algorithm. The weights of the risky assets for a portfolio on the MV frontier, at a given required return $\mu^*$, are

$$w = \frac{\mu^* - R_f}{\mu^{e\prime}\Sigma^{-1}\mu^e}\Sigma^{-1}\mu^e. \tag{3.12}$$

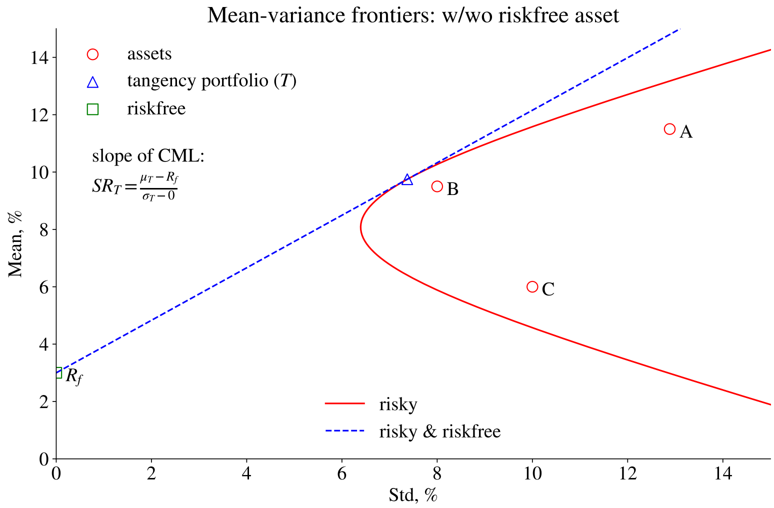



Figure 3.9: Mean-variance frontiers, w/wo risk-free asset

As mentioned before, the weight on the risk-free asset is $1 - w'\mathbf{1}$.

Repeating the calculation for different expected return, $\mu^*$, allows us to trace out the entire MV frontier. In the std×mean space, the efficient frontier (the upper part) is just a *line*, called the *Capital Market Line* (CML). See Figure 3.9 for an illustration and Figure 3.10 for an empirical example.

**Remark 3.7** *(*Alternative way to calculate $w$) The proof of (3.12) shows that we calculate $w$ by solving the following system of equations*

$$\begin{bmatrix} \Sigma & \mu^e \\ \mu^{e\prime} & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mu^* - R_f \end{bmatrix}.$$

*Proof* of (3.12). Define the Lagrangian problem

$$L = (w_1^2\sigma_{11} + w_2^2\sigma_{22} + 2w_1w_2\sigma_{12})/2 + \lambda(w_1\mu_1^e + w_2\mu_2^e + R_f - \mu^*).$$

(Variances are denoted $\sigma_{ii}$ in order to facilitate comparison with the matrix expressions.) The first order condition with respect to the portfolio weights are

$$\text{for } w_1 : w_1\sigma_{11} + w_2\sigma_{12} + \lambda\mu_1^e = 0,$$
$$\text{for } w_2 : w_1\sigma_{12} + w_2\sigma_{22} + \lambda\mu_2^e = 0.$$

Similarly, the first order condition with respect to the Lagrange multiplier is

$$w_1\mu_1^e + w_2\mu_2^e + R_f = \mu^*.$$

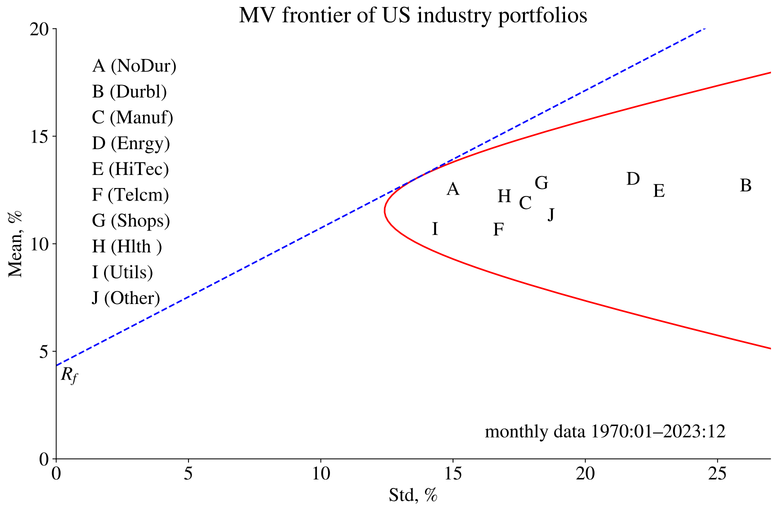



Figure 3.10: M-V frontier from US industry indices

Combine as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \mu_1^e \\ \sigma_{12} & \sigma_{22} & \mu_2^e \\ \mu_1^e & \mu_2^e & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \mu^* - R_f \end{bmatrix},$$

which can be written

$$\begin{bmatrix} \Sigma & \mu^e \\ \mu^{e\prime} & 0 \end{bmatrix} \begin{bmatrix} w \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mu^* - R_f \end{bmatrix}.$$

To simplify further, notice that the first set of equations is $\Sigma w = -\lambda \mu^e$, which can be (partially) solved as $w = -\Sigma^{-1}\lambda\mu^e$. The second set of equations is $\mu^{e\prime} w = \mu^* - R_f$. Use the (partial solution) of $w$ to write this as $-\mu^{e\prime}\Sigma^{-1}\lambda\mu^e = \mu^* - R_f$, which can be solved for as $\lambda = -(\mu^* - R_f)/\mu^{e\prime}\Sigma^{-1}\mu^e$. Finally, using this in the partial solution of $w$ gives (3.12). □

**Remark 3.8** *(Minimizing the standard deviation) It can be shown that the solution (3.12) also solves the problem* $\min \operatorname{Std}(R_p)$ *st* $\operatorname{E} R_p = \mu^*$ *and* $\Sigma_{i=1}^n w_i = 1$.

## 3.3 The Tangency Portfolio

The MV frontier for risky assets only and the frontier for risky assets plus the risk-free asset are tangent at one point—called the *tangency portfolio*: see Figure 3.9. In this case the portfolio weights from (3.8) and (3.12) coincide. Therefore, the portfolio weights of the risky assets (3.12) must sum to unity (so the weight on the risk-free asset is zero) at

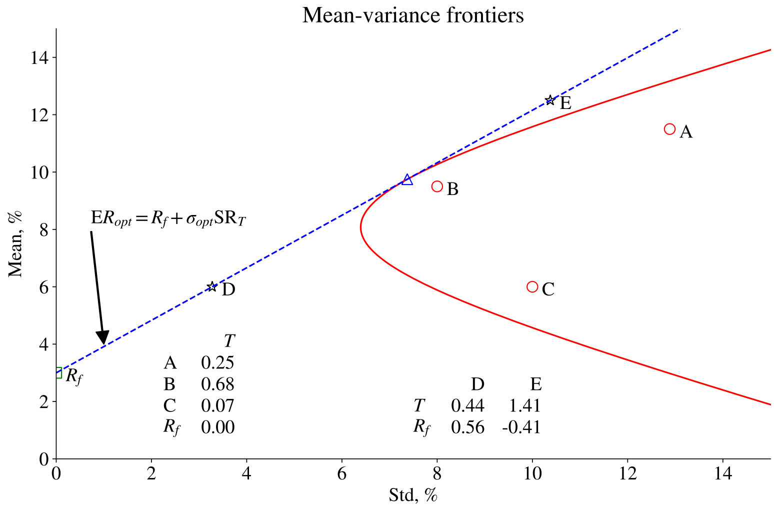



Figure 3.11: Mean-variance frontiers, creating portfolios by combining the tangency portfolio and the risk-free

this value of the required return, $\mu^*$. This gives the portfolio weights of the tangency portfolio

$$w_T = \frac{\Sigma^{-1}\mu^e}{\mathbf{1}'\Sigma^{-1}\mu^e}. \tag{3.13}$$

*Proof* of (3.13). Put the sum of the portfolio weights in (3.12) equal to one and solve for the $\mu^*$ value where that holds. Use in (3.12). □

Notice that Capital Market Line (CML) starts at the location $(\sigma, \mu) = (0, R_f)$ and goes through the point $(\mu_T, \sigma_T)$ where the latter are the mean and standard deviation of the return on the tangency portfolio. It is then clear that the slope of the CML, $(\mu_T - R_f)/(\sigma_T - 0)$, represents the *Sharpe ratio of the tangency portfolio*. The line is thus

$$\mathrm{E}\, R_{opt} = R_f + \sigma_{opt} SR_T. \tag{3.14}$$

Interestingly, the tangency portfolio has the *highest Sharpe ratio of any portfolio* that can be created from the investable assets. See Figure 3.11.

It follows that every portfolio on the CML is a combination of the tangency portfolio and the risk-free asset

$$R_{opt} = vR_T + (1 - v)R_f \tag{3.15}$$

where $R_T$ is the return on the tangency portfolio. See Figure 3.11.

**Remark 3.9** *(*Maximising the Sharpe ratio directly.) Maximizing* $v'\mu^e/\sqrt{v'\Sigma v}$ *gives* *the following n first order conditions*

$$\mu^e = \frac{v'\mu^e}{v'\Sigma v}\Sigma v.$$

*Setting $v$ equal to the tangency portfolio in (3.13) satisfy those first order conditions. (In particular, it helps to notice that $w_T'\mu^e/w_T'\Sigma w_T = \mathbf{1}'\Sigma^{-1}\mu^e$.) To be precise, any proportional scaling of the tangency portfolio ($v = \delta w_T$ where $\delta \neq 0$ is a scalar) will satisfy those first order conditions. This means any point on the capital market line. To find a unique solution, we therefore have to impose at least one restriction, for instance, that the portfolio weights $v$ on the risky assets sum to 1.*

**Remark 3.10** *(Properties of tangency portfolio*) The expected excess return and the variance of the tangency portfolio are $\mu_T^e = \mu^{e\prime}\Sigma^{-1}\mu^e/\mathbf{1}'\Sigma^{-1}\mu^e$ and $\operatorname{Var}(R_T^e) = \mu^{e\prime}\Sigma^{-1}\mu^e/\left(\mathbf{1}'\Sigma^{-1}\mu^e\right)^2$. It follows that $\mu_T^e/\operatorname{Var}(R_T^e) = \mathbf{1}'\Sigma^{-1}\mu^e$ and that the squared Sharpe ratio is $(\mu_T^e)^2/\operatorname{Var}(R_T^e) = \mu^{e\prime}\Sigma^{-1}\mu^e$.*

### 3.3.1 Examples of Tangency Portfolios*

Consider the simple case with two risky assets which are uncorrelated ($\sigma_{12} = 0$). The tangency portfolio (3.13) is then

$$\begin{bmatrix} w_{T,1} \\ w_{T,2} \end{bmatrix} = \frac{1}{\sigma_2^2\mu_1^e + \sigma_1^2\mu_2^e}\begin{bmatrix} \sigma_2^2\mu_1^e \\ \sigma_1^2\mu_2^e \end{bmatrix}. \tag{3.16}$$

This shows that if both excess returns are positive, then *(i)* the weight on asset $i$ increases when $\mu_i^e$ increases and when $\sigma_{ii}$ decreases; *(ii)* both weights are positive.

**Example 3.11** *(Tangency portfolio, numerical) When $(\mu_1^e, \mu_2^e) = (8, 5)$, the correlation is zero, and $(\sigma_1^2, \sigma_2^2) = (256, 144)$, then (3.16) gives*

$$\begin{bmatrix} w_{T,1} \\ w_{T,2} \end{bmatrix} = \begin{bmatrix} 0.47 \\ 0.53 \end{bmatrix}.$$

*When $\mu_1^e$ increases from 8 to 12, then we get*

$$\begin{bmatrix} w_{T,1} \\ w_{T,2} \end{bmatrix} = \begin{bmatrix} 0.57 \\ 0.43 \end{bmatrix}.$$

Now, consider another simple case, where both variances are the same, but the correlation is non-zero ($\sigma_1 = \sigma_2 = 1$ as a normalization, $\sigma_{12} = \rho$). Then (3.16) becomes

$$\begin{bmatrix} w_{T,1} \\ w_{T,2} \end{bmatrix} = \frac{1}{(\mu_1^e + \mu_2^e)(1-\rho)} \begin{bmatrix} \mu_1^e - \rho\mu_2^e \\ \mu_2^e - \rho\mu_1^e \end{bmatrix}. \tag{3.17}$$

Results: *(i)* both weights are positive if the returns are negatively correlated ($\rho < 0$) and both excess returns are positive; *(ii)* $w_{T,2} < 0$ if $\rho > 0$ and $\mu_1^e$ is considerably higher than $\mu_2^e$ (so $\mu_2^e < \rho\mu_1^e$). The intuition for the first result is that a negative correlation means that the assets "hedge" each other (even better than diversification), so the investor would like to hold both of them to reduce the overall risk. (Unfortunately, most assets tend to be positively correlated.) The intuition for the second result is that a positive correlation reduces the gain from holding both assets (they don't hedge each other, and there is relatively little diversification to be gained if the correlation is high). On top of this, asset 1 gives a higher expected return, so it is optimal to sell asset 2 short (essentially a risky "loan" which allows the investor to buy more of asset 1).

**Example 3.12** *(Tangency portfolio, numerical) When* $(\mu_1^e, \mu_2^e) = (8, 5)$*, and* $\rho = -0.8$ *we get*

$$\begin{bmatrix} w_{T,1} \\ w_{T,2} \end{bmatrix} = \begin{bmatrix} 0.51 \\ 0.49 \end{bmatrix}.$$

*If, instead,* $\rho = 0.8$*, then we get*

$$\begin{bmatrix} w_{T,1} \\ w_{T,2} \end{bmatrix} = \begin{bmatrix} 1.54 \\ -0.54 \end{bmatrix}.$$

## 3.4 Appendix – A Primer in Calculus*

The following derivatives (with respect to $x$) are often used in these chapters

$$\frac{d}{dx}(ax^k + bx) = akx^{k-1} + b$$
$$\frac{d}{dx}\ln x = 1/x$$
$$\frac{d}{dx}e^x = e^x.$$

Derivatives typically depend on at which $x$ value we evaluate them at ($x = 1$ or $x = 2$, say), so the derivatives are themselves functions. The first expression embeds the *sum rule* (the derivative of a sum is the sum of the derivatives).

**Example 3.13** *(Derivative of power function)* $3x^2 + 7x$ *has the derivative* $6x + 7$ *which is* $-5$ *at* $x = -2$ *and* $13$ *at* $x = 1$.

The *chain rule* says that if $g()$ and $f()$ are two functions, then the derivative of the composite function $g(f(x))$ is

$$\frac{d}{dx}g(f(x)) = g'(u)f'(x), \text{ where } u = f(x),$$

and where $g'(u)$ is short hand (Lagrange's) notation for $\frac{d}{du}g(u)$, and similarly for $f'(x)$. The derivative $g'(u)$ is often referred to as the outer derivative and $f'(x)$ as the inner derivative.

**Example 3.14** *(Chain rule) Let* $g(u) = u^2$ *and* $u = f(x) = 2-3x$*, so we are considering the composite function* $(2-3x)^2$*. We then get*

$$\frac{d}{dx}(2-3x)^2 = \underbrace{2(2-3x)}_{g'(u)}\underbrace{(-3)}_{f'(x)} = 18x - 12.$$

*This derivative is* $-12$ *at* $x = 0$ *and* $6$ *at* $x = 1$.

Consider a function of two variables, $f(x, z)$. The *partial derivative* with respect to $x$ is just a standard derivative, treating $z$ as fixed. For instance,

$$\frac{\partial}{\partial x}ax^k bz = akx^{k-1}bz$$
$$\frac{\partial}{\partial z}ax^k bz = ax^k b.$$

Suppose the function $f(x)$ gives a scalar output, but $x$ is a $n-$vector of inputs (with elements $x_1, x_2, ...x_n$). The *gradient* is then

$$\partial f(x)/\partial x = \begin{bmatrix} \partial f(x)/\partial x_1 \\ \vdots \\ \partial f(x)/\partial x_n. \end{bmatrix}$$

**Example 3.15** *(Gradient) For the function* $f(x) = (x_1 - 2)^2 + (4x_2 + 3)^2$*, the gradient is*

$$\partial f(x)/\partial x = \begin{bmatrix} 2(x_1 - 2) \\ 8(4x_2 + 3) \end{bmatrix}.$$

The *Hessian* is the $n \times n$ matrix of second derivatives

$$\partial^2 f(x)/\partial x \partial x' = \begin{bmatrix} \partial^2 f(x)/\partial x_1^2 & \cdots & \partial^2 f(x)/\partial x_1 \partial x_n \\ & \ddots & \\ \partial^2 f(x)/\partial x_n \partial x_1 & & \partial^2 f(x)/\partial x_n^2 \end{bmatrix}.$$

(In case the derivatives are continuous, then this matrix is symmetric.)

**Example 3.16** *(Hessian) Using the same function as in Example 3.15, we get*

$$\partial^2 f(x)/\partial x \partial x' = \begin{bmatrix} 2 & 0 \\ 0 & 32 \end{bmatrix}.$$

## 3.5 Appendix – A Primer in Optimization*

**Remark 3.17** *(First order condition for optimising a differentiable function). We want to find the value of $b$ in the interval $b_{low} \leq b \leq b_{high}$, which makes the value of the differentiable function $f(b)$ as small as possible (a minimization problem). The answer is $b_{low}$, $b_{high}$, or a value of $b$ where $df(b)/db = 0$. The latter is a necessary and sufficient condition for an unconstrained problem where $f(b)$ is convex. (If the function is twice differentiable, then convexity means that $f''(b) \geq 0$.) A maximization problem, except that we rather want $f(b)$ to be concave ($f''(b) \leq 0$).*

When the goal is to determine $x$ and $y$ that *minimize*

$$L = (x-2)^2 + (4y+3)^2,$$

then we have to find the values of $x$ and $y$ that satisfy the *first order conditions* $\partial L/\partial x = 0$, $\partial L/\partial y = 0$. For $L$ function above, these are

$$0 = \partial L/\partial x = 2(x-2)$$
$$0 = \partial L/\partial y = 8(4y+3),$$

which clearly requires $x = 2$ and $y = -3/4$. In this particular case, the first order condition with respect to $x$ does not depend on $y$, but that is not a general property. See Figure 3.12 for the surface of the loss function and the contours.

Also, in this case, there is a unique solution—but in more complicated problems, the first order conditions could be satisfied at different values of $x$ and $y$.

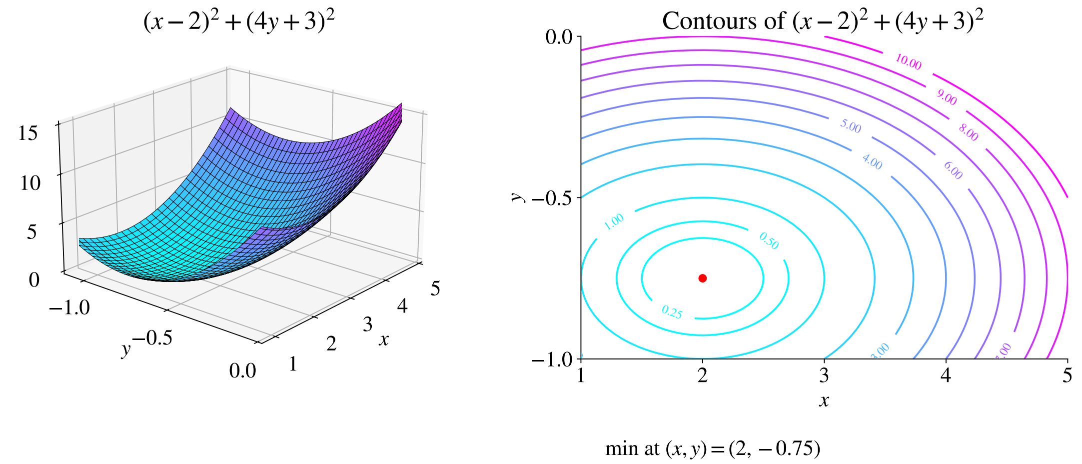



Figure 3.12: Minimization problem

A *maximization problem* has the same type of first order conditions.

If you want to add a *restriction* to the minimization problem, say

$$x + 2y = 3,$$

then we can proceed in two ways. The first is to simply substitute for $x = 3 - 2y$ in $L$ to get

$$L = (1 - 2y)^2 + (4y + 3)^2,$$

with first order condition

$$0 = \partial L/\partial y = -4(1 - 2y) + 8(4y + 3) = 40y + 20,$$

which requires $y = -1/2$, which by implies $x = 4$. (We could equally well have substituted for $y$). This is also the unique solution. See Figure 3.13. This is an easy way to eliminate an equality restriction.

The second method is to use a *Lagrangian*. The problem is then to choose $x$, $y$, *and* $\lambda$ to minimize

$$L = (x - 2)^2 + (4y + 3)^2 + \lambda\,(x + 2y - 3)\,.$$

(If you instead use $-\lambda()$ or write the restriction as $-x - 2y + 3$, you should get the same result. The interpretation of $\lambda$ differs, though.)

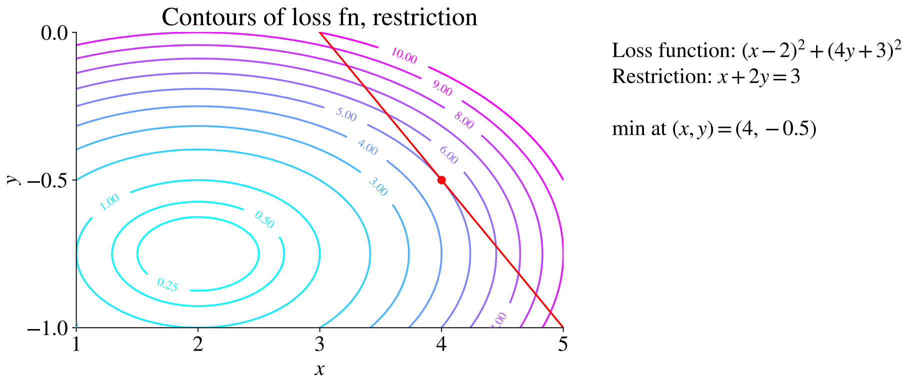



Figure 3.13: Minimization problem with restriction

The term multiplying $\lambda$ is the restriction. The first order conditions are now

$$
\begin{aligned}
0 &= \partial L/\partial x = 2(x-2) + \lambda \\
0 &= \partial L/\partial y = 8(4y+3) + 2\lambda \\
0 &= \partial L/\partial \lambda = x + 2y - 3.
\end{aligned}
$$

These are 3 equations in 3 unknowns $(x, y, \lambda)$ which we have to solve. One way is as follows. The first two conditions say

$$
\begin{aligned}
x &= 2 - \lambda/2 \\
y &= -3/4 - \lambda/16,
\end{aligned}
$$

so we need to find $\lambda$. To do that, use these latest expressions for $x$ and $y$ in the third first order condition (to substitute for $x$ and $y$)

$$
\begin{aligned}
&3 = 2 - \lambda/2 - 3/2 - \lambda/8 = 1/2 - \lambda 5/8, \text{ so} \\
&\lambda = -4.
\end{aligned}
$$

Finally, use this to calculate $x$ and $y$ as

$$x = 4 \text{ and } y = -1/2.$$

Notice that this is the same solution as before $(y = -1/2)$ and that the restriction holds $(4 + 2(-1/2) = 3)$. This second method is clearly a lot clumsier in my example, but it pays off when there are several restrictions and/or when the restriction(s) become complicated.

# Chapter 4

# The Inputs to Mean-Variance Calculations

Mean-variance analysis and portfolio choice depend on assumptions regarding average returns and the variance-covariance matrix. This means that the moments (means, variances and covariances) are *conditional* on the information available at the time of portfolio formation, depend on the *investment horizon* and that they may *change over time*.

This chapter discusses how estimates on historical data can help form such assumptions, although judgemental adjustments are likely to be made.

The *conditional* nature of the moments distinguish them different from traditional sample estimates. To illustrate, consider the definition

$$R_{t+1} = \mathrm{E}_t\, R_{t+1} + \varepsilon_{t+1}, \tag{4.1}$$

where $R_{t+1}$ is the return over the investment horizon, $\mathrm{E}_t\, R_{t+1}$ the forecast based on information when then portfolio is formed in period $t$, and $\varepsilon_{t+1}$ is the unforecasted part of the return (news, surprise).

A traditional sample estimate of the variance measures the historical variance in $R$. In contrast, the portfolio formation is based on the variance of the forecast error, $\varepsilon$. For most assets, returns are difficult to forecast; hence, the difference between the two measures of variance is small. For instance, if the forecasting model has a coefficient of determination ("$R^2$") of 0.05, which seems to be close to the upper limits of most return forecasting models, then the variance of $\varepsilon$ is 0.95 times the variance of $R$. In this situation, the sample variance of $R$ might be a good approximation. In contrast, for the risk-free rate, the conditional variance is zero, while sample variance is not (albeit small).

It is also important to consider *time-variation* in the moments. In particular, variances and covariances have considerable (predictable) movements, which motivates some kind of time-series method for estimating. In addition, *sample estimates can be noisy*, espe-

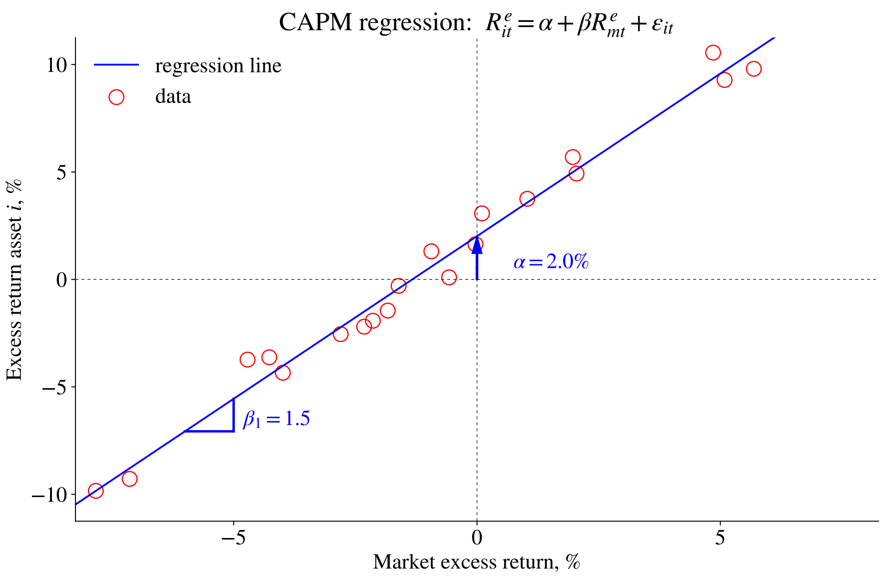



Figure 4.1: CAPM regression

cially when the sample is small, which may motivate forming a compromise between the sample estimates and a priori information ("shrinkage").

## 4.1 The Market Model: Betas

The beta (slope coefficient) from the *market model* is often used to describe the cyclicality of an asset. It is useful as a statistical description of the returns and aids in improving variance-covariance matrix estimates (more about that later). The regression is

$$\begin{aligned} R^e_{it} &= \alpha_i + \beta_i R^e_{mt} + \varepsilon_{it}, \text{ where} \\ \mathrm{E}\,\varepsilon_{it} &= 0, \ \mathrm{Cov}(\varepsilon_{it}, R^e_{mt}) = 0. \end{aligned} \tag{4.2}$$

Here, $R^e_{it}$ is the excess return on asset $i$ in period $t$, while $R^e_{mt}$ is the market excess return in the same period. The regression is done on time series ($R^e_{it}$ and $R^e_{mt}$ for $t = 1, 2, ..., T$). As usual, the regression slope is $\beta_i = \sigma_{im}/\sigma^2_m$. This regression may use the excess returns as indicated above, or the net returns. The two assumptions (the residual has a zero mean and is uncorrelated with the regressor) are standard in regression analysis. See Figures 4.1 for an illustration.

**Empirical Example 4.1** *See Figures 4.2–4.3 for results on U.S. industry portfolios. See*

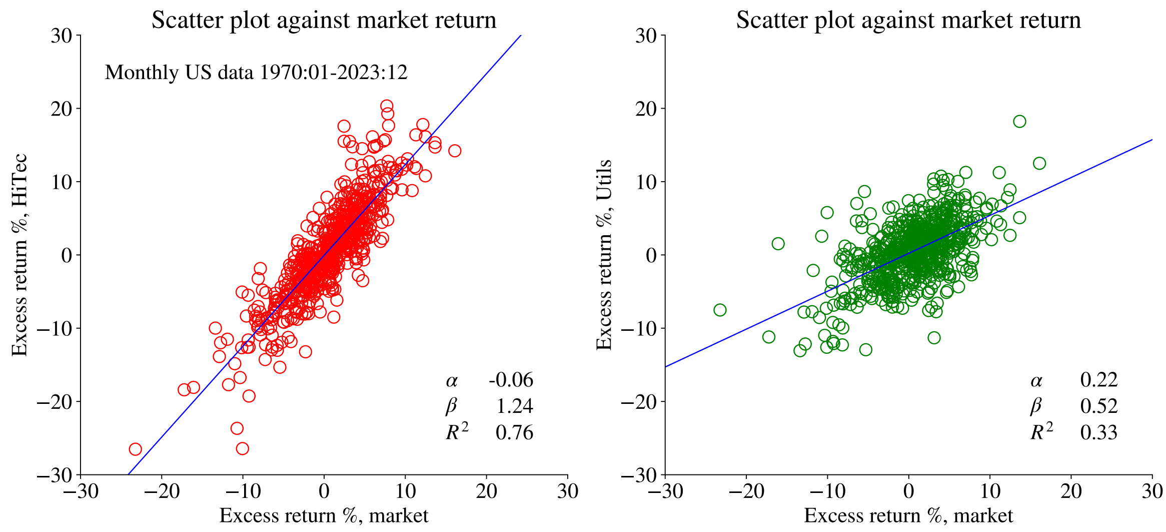



Figure 4.2: Scatter plot against market return

*also Figure 4.4 for some alternative assets.*

The beta of a portfolio is the portfolio of betas. That is, for a portfolio of assets, the beta is

$$\beta_p = \textstyle\sum_{i=1}^{n} w_i \beta_i. \tag{4.3}$$

(This follows from the fact that $\mathrm{Cov}(w_i R_i + w_j R_j, R_m) = w_i \sigma_{im} + w_j \sigma_{jm}$.)

We will later discuss how the market model can help in estimating the variance-covariance matrix of the assets. It is also clear that the $\beta$ values can be useful in *portfolio formation*. For instance, suppose we want to combine assets 1 and 2 in such a way that our overall position is market neutral ($\beta_p = 0$). This can be done by choosing the portfolio weights $(w_1, w_2) = (1, -\beta_1/\beta_2)$ or any scaling of that. Also, if the investor wants a portfolio with a particular beta ($\beta_q$), then this can be achieved by investing $\beta_q$ in the market portfolio and $1 - \beta_q$ in the risk-free asset.

The result in (4.3) also shows that the (value weighted) $\beta$ of all assets must equal 1. (This follows from the fact that the value weighted portfolio of all assets equals the market portfolio—and regressing the market on itself must give a slope of 1.)

**Remark 4.2** *(Market indices I) A market index $I_t$ is calculated as*

$$I_t = (1 + R_{mt}) I_{t-1}, \text{ where } R_{mt} = \textstyle\sum_{i=1}^{n} w_{it} R_{it},$$

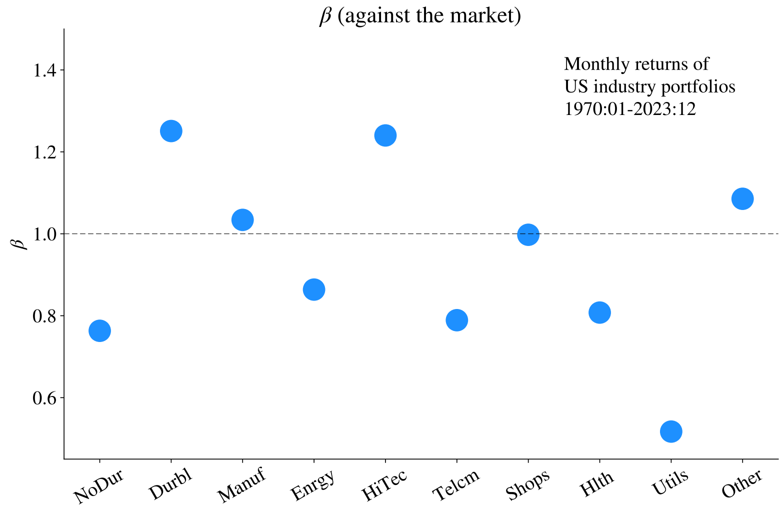



Figure 4.3: $\beta$s of US industry portfolios

*where $i$ denotes the $n$ different components/assets (for instance, stocks) of the index. This is a capital weighted return index if (a) $R_{it}$ is the net return on holding asset $i$ between $t-1$ and $t$; and (b) $w_{it}$ is the market capitalization of asset $i$ relative to the total market capitalization of all $n$ assets—measured at the end of period $t-1$. Most of the important indices are of this sort. Instead, if $R_{it}$ only includes the capital gain of holding asset $i$, then the index is a price index. In other cases, the weights may reflect the market capitalization of the floats (those shares that are actively traded). In yet other cases the weights are the same across the assets (an equally weighted index).*

**Remark 4.3** *(Market indices II*) Dow Jones Industrial Average and Nikkei 225 have very special weights. In practice, these two indices are just the average prices of all (30 or 225) stocks in the index. This means that the portfolio weights are proportional to the stock price.*

**Remark 4.4** *(Market indices III*) More recently, a large number of alternative indices have been introduced, for instance of (a) "sustainable" companies (DJSI); (b) fundamentally weighted indices (weights based on sales, earnings or dividends); (c) 1/volatility based indices; (d) performance based indices (large weights on recent winners).*

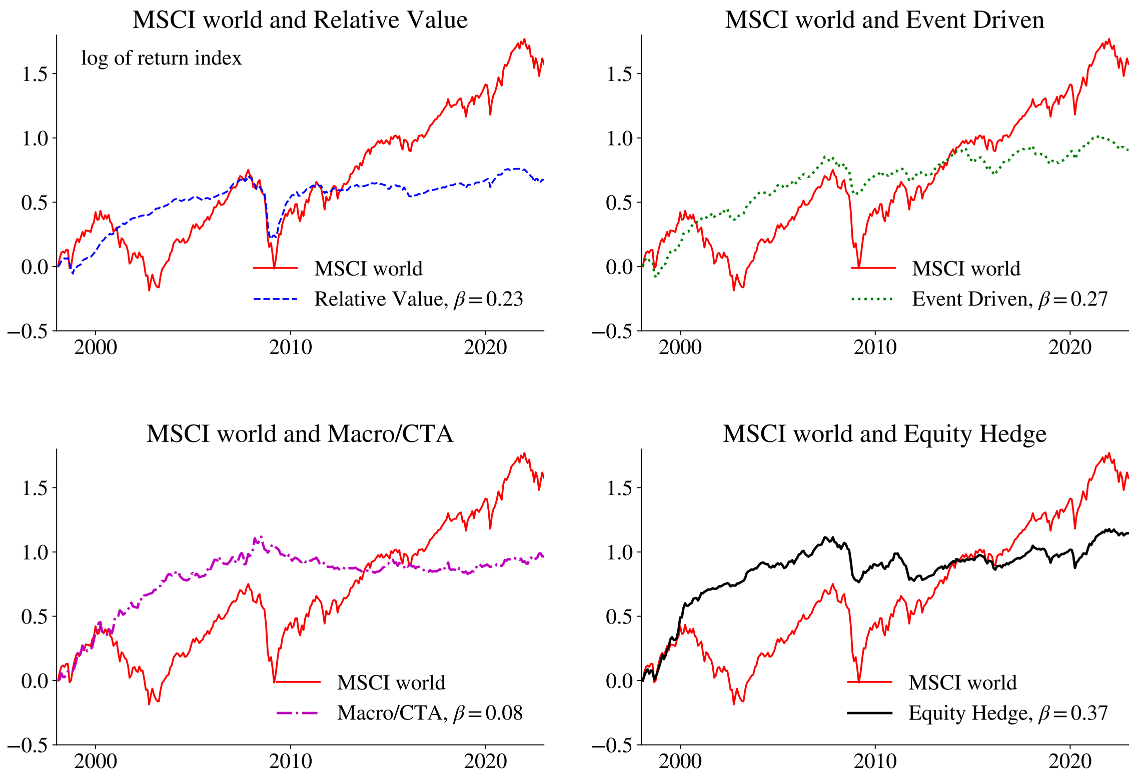



Figure 4.4: Comparing hedge fund indices with the MSCI world (equity) index

### 4.1.1 Estimating Historical Beta: OLS and Other Approaches

It is sometimes argued that the OLS estimate of beta on a historical sample may not be the best forecast of the beta for a future time periods (see, for instance, Blume (1971)). As a potential solution, we could apply a shrinkage towards the average beta (which is 1)

$$\beta = \eta\hat{\beta}_{OLS} + (1 - \eta)1. \tag{4.4}$$

This could be motivated by empirical findings or by a Bayesian principle (see Greene (2018) 16). In the latter case, $\eta$ would be higher if the sample is long and when the fit is good.

To capture time-variation in betas, we could either estimate on a moving data window or apply an exponentially weighted moving average estimate (EWMA). The latter is a weighted OLS where an observation $s$ periods ago gets the weight $\lambda^s$ where $\lambda$ is close to one (for instance, 0.95).

**Empirical Example 4.5** *See Table 4.1 for an evaluation of several methods: most are of* *the form (4.4), but an EWMA approach is also considered. A negative number indicate that the method is better than OLS.*

| OLS adj $0.67\hat{b} + 0.33$ | OLS adj $0.5\hat{b} + 0.5$ | OLS adj $0.33\hat{b} + 0.67$ | 1 1 | EWMA $0.5\hat{b} + 0.5$ |
|---|---|---|---|---|
| $-6.9$ | $-6.4$ | $-3.0$ | 10.0 | $-11.0$ |

Table 4.1: Absolute forecast errors of future betas, as a that is, the average |next 2 year $\beta -$ predicted $\beta$| compared to the results from OLS. A negative number is better performance than OLS. The models are estimated on moving 10-year windows and EWMA uses $\lambda = 0.95$. 49 industry portfolios, monthly data for 1970:01-2023:12.

### 4.1.2 Fundamental Betas

Another way to improve the beta forecasts is to incorporate information about fundamental firm variables. This is particularly useful when there is little historical data on returns (for instance, because the asset was not traded before).

It is often found that betas are related to fundamental variables as follows (with signs in parentheses indicating the effect on the beta): dividend payout (-), asset growth (+), leverage (+), liquidity (-), asset size (-), earning variability (+), earnings Beta (slope in earnings regressed on economy wide earnings) (+). Such relations can be used to make an educated guess about the beta of an asset without historical data on the returns—but with data on (at least some) of these fundamental variables.

## 4.2 Estimation of the Covariance Matrix of the Asset Returns

There are several issues with estimating variance-covariance matrices: *(1)* the number of parameters increase very quickly as the number of assets increases ($n(n+1)/2$ with $n$ assets, for instance 5,050 for 100 assets); *(2)* there may not be relevant historical data; *(3)* historical estimates have proven somewhat unreliable for future periods due to small sample issues and time-variation of the parameters.

**Remark 4.6** *(Fama-French portfolios) The 25 FFF portfolios (used in the examples below) are calculated by annual rebalancing (June/July). The US stock market is divided into $5 \times 5$ portfolios as follows. First, split up the stock market into 5 groups based on the book value/market value: put the lowest 20% in the first group, the next 20% in the* *second group etc. Second, split up the stock market into 5 groups based on size: put the smallest 20% in the first group etc. Then, form portfolios based on the intersections of these groups (also called double sorting). For instance, in Table 4.2 the portfolio in row 2, column 3 (portfolio 8) belong to the 20%-40% largest firms and the 40%-60% firms with the highest book value/market value.*

| | Book value/Market value | | | | |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 |
| Size 1 | 1 | 2 | 3 | 4 | 5 |
| 2 | 6 | 7 | 8 | 9 | 10 |
| 3 | 11 | 12 | 13 | 14 | 15 |
| 4 | 16 | 17 | 18 | 19 | 20 |
| 5 | 21 | 22 | 23 | 24 | 25 |

Table 4.2: Numbering of the FF portfolios.

## 4.3 Covariance Matrix with Time-Varying Parameters

To handle the time-variation, we could apply a simple EWMA estimator (cf. the Risk-Metrics approach of JP Morgan (1996))

$$\mu_{it} = \lambda \mu_{i,t-1} + (1-\lambda) x_{i,t-1}, \tag{4.5}$$

$$\sigma_{ij,t} = \lambda \sigma_{ij,t-1} + (1-\lambda)(x_{i,t-1} - \mu_{i,t-1})(x_{j,t-1} - \mu_{j,t-1}), \tag{4.6}$$

with $0 \leq \lambda \leq 1$ and where $x_{it}$ is element $i$ of the vector $x_t$. The first equation provides a time-varying estimate of the mean and the second equation of the covariance between $x_{it}$ and $x_{jt}$ (set $i = j$) for the variance. In many application on daily data a value of $\lambda \approx 0.94$ is common. For monthly data, often slightly lower values.

## 4.4 Covariance Matrix with Average Correlations

A commonly used method to address the instability of the variance-covariance matrix, caused by excessive parameters or insufficient data, is to replace the historical correlation with an average historical correlation. To do that, estimate $\rho_{ij}$ on historical data, but use the average estimate $\bar{\rho}$ as the "forecast" of all correlations:

$$\sigma_{ij} = \bar{\rho}\sqrt{\sigma_i^2 \sigma_j^2}. \tag{4.7}$$

Notice that $\bar{\rho}$ is the average of the $n(n-1)/2$ elements below (or above) the main diagonal of the correlation matrix.

## 4.5 Covariance Matrix from a Single-Index Model

The single-index model is another way to reduce the number of parameters that we need to estimate in order to construct the covariance matrix of assets. The model assumes that the co-movement between assets is due to a single common influence (here denoted $R_{mt}$). This means that we add one assumption to (4.2)

$$\text{Cov}(\varepsilon_{it}, \varepsilon_{jt}) = 0, \tag{4.8}$$

which says that the residuals for different assets are uncorrelated. This means that all co-movements of two assets ($R_i$ and $R_j$, say) are due to movements in the common "index" $R_m$. For instance, when there is too little (relevant) historical data to estimate covariances, then we might still be able to construct one by first assessing the beta of the asset (for instance, based on firm characteristics) and then drawing conclusions about that implies for the covariance with other assets.

If (4.2) and (4.8) are true, then the variance of asset $i$ and the covariance of assets $i$ and $j$ are

$$\sigma_i^2 = \beta_i^2 \sigma_{mm} + \text{Var}(\varepsilon_{it}) \tag{4.9}$$

$$\sigma_{ij} = \beta_i \beta_j \sigma_{mm} \text{ when } i \neq j, \tag{4.10}$$

where $\sigma_{mm}$ is the variance of $R_m$. Together, these equations show that we can calculate the whole covariance matrix by having just the variance of the index (to get $\sigma_{mm}$) and the output from $n$ regressions (to get $\beta_i$ and Var $(\varepsilon_i)$ for each asset). See Elton, Gruber, Brown, and Goetzmann (2014) 7–8 for more details on index models.

**Example 4.7** *(Two assets) Let* $[\beta_1, \beta_2] = [0.9, 1.1]$, $[\text{Var}(\varepsilon_{1t}), \text{Var}(\varepsilon_{2t})] = [100, 25]/100^2$, *and* $\sigma_{mm} = 225/100^2$. *Then*

$$\text{Cov}(R_t) \approx \begin{bmatrix} 282.25 & 222.75 \\ 222.75 & 297.25 \end{bmatrix} /100^2.$$

*Proof* of (4.9)–(4.10. By using (4.2) and (4.8) and recalling that $\text{Cov}(R_m, \varepsilon_i) = 0$ direct calculations give that the covariance of assets $i$ and $j$ ($i \neq j$) is (recalling also that $\text{Cov}\left(\varepsilon_i, \varepsilon_j\right) = 0$)

$$\begin{aligned}
\sigma_{ij} &= \text{Cov}(R_i, R_j) \\
&= \text{Cov}(\alpha_i + \beta_i R_m + \varepsilon_i, \alpha_j + \beta_j R_m + \varepsilon_j) \\
&= \beta_i \beta_j \sigma_{mm} + 0.
\end{aligned}$$

□

## 4.6 Covariance Matrix from a Multi-Index Model

The multi-index model is just a multivariate extension of the single-index model

$$R^e_{it} = a_i + b_i' I_t + \varepsilon_{it}, \text{ where} \tag{4.11}$$
$$\mathrm{E}\,\varepsilon_{it} = 0,\ \text{Cov}(\varepsilon_{it}, I_t) = \mathbf{0}, \text{ and } \text{Cov}(\varepsilon_{it}, \varepsilon_{jt}) = 0.$$

As an example, there could be two indices: the stock market return and an interest rate. An ad-hoc approach is to first try a single-index model and then test if the residuals are approximately uncorrelated. If not, then adding a second index might improve the model.

It is often found that it takes several indices to get a reasonable approximation—but that a single-index model is equally good (or better) at "forecasting" the covariance over a future period. This is much like the classical trade-off between in-sample fit (requires a large model) and forecasting (often better with a small model).

If $\Omega$ is the covariance matrix of the indices, then the covariance of assets $i$ and $j$ is

$$\sigma_i^2 = b_i' \Omega b_i + \text{Var}(\varepsilon_{it}), \tag{4.12}$$
$$\sigma_{ij} = b_i' \Omega b_j \text{ when } i \neq j, \tag{4.13}$$

where $b_i$ is the vector of slope coefficients obtained from regressing $R_{it}$ on the vector of factors ($I_t$ or $I_t^*$) as in (4.11).

## 4.7 Covariance Matrix From A Shrinkage Estimator

The historical sample covariance matrix, $S$, can exhibit significant noise in small samples. One way of handling that is to "shrink" the sample covariance matrix towards a target, $F$, as

$$\Sigma = \delta F + (1 - \delta)S, \text{ where } 0 \leq \delta \leq 1. \tag{4.14}$$

Ledoit and Wolf (2003) suggest an $F$ matrix from the single index model and Ledoit and Wolf (2004) instead suggest an $F$ matrix which implies the same correlations of all assets. See the previous sections for how to construct such $F$ matrices. In both cases, the diagonal elements of $F$ are the same as in $S$.

The articles develop algorithms for calculating an approximately optimal value of $\delta$, which tend to be large in small samples and with crude targets.

**Empirical Example 4.8** *Table 4.3 suggest that with the 25 FF assets $\delta$ is small for long samples, but may be non-trivial for shorter samples. With more assets, the $\delta$ value is likely to be larger. An alternative approach to choose $\delta$ is to investigate the performance on earlier samples.*

| | Full sample | 10-year samples |
|---|---|---|
| constant corr | 0.11 | 0.64 |
| single index model | 0.03 | 0.11 |

Table 4.3: Shrinkage parameter $\delta$ in Ledoit and Wolf's (2003,2004) covariance estimator $\delta F + (1 - \delta)S$ The target matrix $F$ is either a constant correlation covariance matrix or the covariance matrix from a single index model. 25 FF portfolios, monthly data for 1970:01–2023:12. The result for the 10-year samples is the average across moving 10-year data windows.

## 4.8 An Evaluation of Different Approaches

This section presents a simple assessment the different methods. However, it does not make any general claims as the conclusions depend on *(a)* which assets; *(b)* sample period and time horizons; and *(c)* various modelling parameters.

**Empirical Example 4.9** *Table 4.4 present results for the 25 FF portfolios. Several models are estimated on moving 10-year data windows (except the exponentially weighted moving average, EWMA, method which follows (4.5 )–(4.6)). We focus on the correlations, since most methods share the same approach for the standard deviations. For each method and period, the implied correlations are calculated and then compared with the* *actual (realised) correlations for the year after the estimation window. Then, the data window is moved one month and the procedure is repeated. The table show the average (across time and assets) of absolute forecast error of the correlation.*

| sample corr | average corr | 1-factor model | 3-factor model | shrink to avg corr | shrink to 1-factor | EWMA $\lambda = 0.95$ |
|---|---|---|---|---|---|---|
| 12.09 | 13.48 | 16.77 | 12.39 | 12.82 | 12.22 | 11.36 |

Table 4.4: Average absolute forecast errors (in %) of future correlations, that is, average |correlation next 2 years - predicted correlation|. All models (except EWMA) are estimated on moving 10-year windows. The shrinkage approach reports results from covariance matrix $= \delta F + (1-\delta)S$ with $\delta$ optimally chosen as in Ledoit and Wolf (2003,2004). 25 FF portfolios, monthly data for 1970:01-2022:12.

## 4.9 Estimating Expected Returns

A later chapter will discuss return predictability at some length. For now it suffices to say that even the best prediction models have limited performance, often with coefficient of determination ("$R^2$") below 5%. In particular, it turns out that it is hard to beat the historical average return as a predictor.

# Chapter 5

# Portfolio Choice

## 5.1 Portfolio Choice with Mean-Variance Preferences

This chapter discusses optimal portfolio choice when the investor has mean-variance preferences and can invest in both risky assets and a risk-free asset.

The investor chooses the portfolio weights to maximize expected utility

$$\mathrm{E}\, U(R_p) = \mathrm{E}\, R_p - \frac{k}{2}\mathrm{Var}(R_p), \text{ where} \tag{5.1}$$

$$R_p = v'R + (1 - w'\mathbf{1})R_f = v'R^e + R_f. \tag{5.2}$$

The $k$ parameter indicates the degree of risk aversion. (Dividing $k$ by 2 is made for convenience: it makes the equation for the optimal portfolio choice look a bit less involved.) The portfolio return in (5.2) assumes investment in risky assets (vector of portfolio weights $v$) and a risk-free asset (portfolio weight $1 - w'\mathbf{1}$). Notice that this expression automatically imposes the restriction that all portfolio weights sum to one. As before, the expectation and the variance summarise the beliefs of the investor, conditional on the information available at the time of the investment.

The optimisation problem is illustrated in Figure 5.1. This figure shows utility contours which are combinations of $\mathrm{E}\, R_p$ and $\mathrm{Std}(R_p)$ so that expected utility, $\mathrm{E}\, U(R_p)$, in (5.1) is constant. Contours further to the upper left have higher expected utility and are thus preferred. In contrast, the capital market line (CML) shows what is possible to achieve: points on or below the line.

The optimization problem in (5.1) is to move to a point *as far to the upper left as possible*. Clearly, this will be a point on the CML, so the optimal portfolio is a mix of the risk-free and the tangency portfolio. This is sometimes called the two-fund separation theorem.

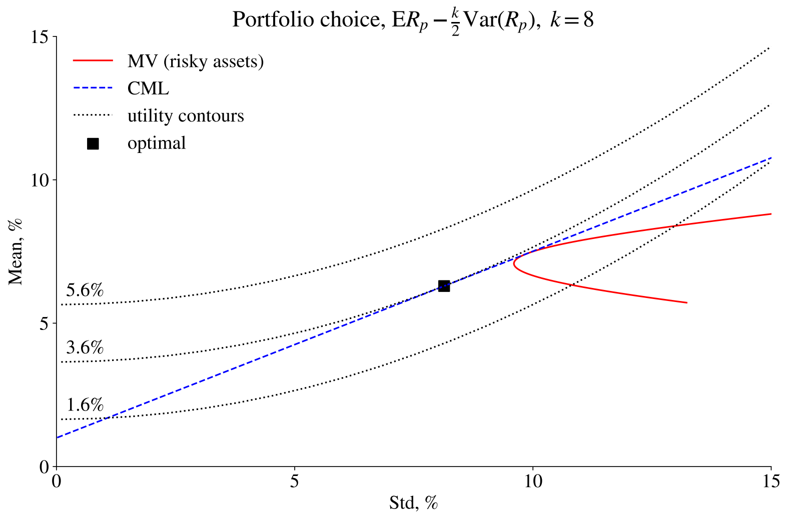



Figure 5.1: Iso-utility curves, mean-variance utility

Also, see Figure 5.2 for an illustration of how the risk aversion determines *which point on the CML* that is optimal: higher risk aversion ($k$) will lead the investor to accept more risk in exchange for a higher average return.

**Remark 5.1** *(Utility contours*) Let $u$ be a fixed level of expected utility and rewrite (5.1) as $u + \frac{k}{2}\,\mathrm{Std}(R_p)^2 = \mathrm{E}\,R_p$. For a given value of $\mathrm{Std}(R_p)$ this gives the required $\mathrm{E}\,R_p$ needed to get expected utility $u$.*

## 5.2 Optimal Portfolio of a Single Risky Asset and a risk-free Asset

Suppose initially that there is a single investable risky assset. The investor then maximizes (5.1) but where (5.2) simplifies to

$$R_p = vR_i^e + R_f. \tag{5.3}$$

**Remark 5.2** *(Real or nominal returns) The objective function (5.1) makes more sense if the returns are real, that is, nominal returns minus inflation. During periods (or over horizons) when inflation is fairly stable, the practical difference is small. However, it might matter in other periods. In particular, the risk profile of an asset will then depend on how its return covaries with inflation. For instance, equity returns are often considered to be better hedges against inflation than bond returns.*

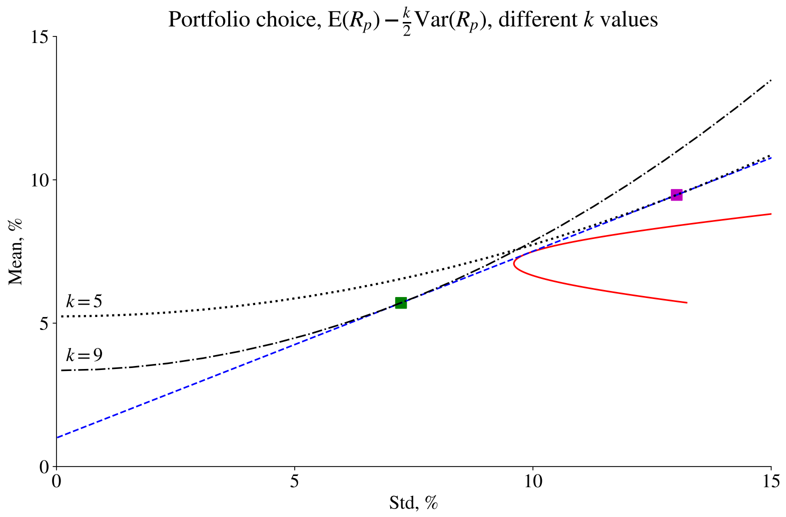



Figure 5.2: Iso-utility curves, mean-variance utility (different risk aversions)

Use the budget constraint in the objective function to get

$$\begin{aligned} \mathrm{E}\,U(R_p) &= \mathrm{E}(vR_i^e + R_f) - \frac{k}{2}\mathrm{Var}(vR_i^e + R_f) \\ &= v\mu^e + R_f - \frac{k}{2}v^2\sigma^2, \end{aligned} \tag{5.4}$$

where $(\mu^e, \sigma^2)$ denote the investor's beliefs about the mean excess return and variance of the risky asset. In the second equation we use the fact that $R_f$ is known.

The first order condition for an optimum ($d\,\mathrm{E}\,U(R_p)/dv = 0$) is

$$\mu^e - kv\sigma^2 = 0, \tag{5.5}$$

which trades off how a marginal increase of $v$ gives a higher expected return but also volatility.

Solve for the optimal portfolio weight of the risky asset as

$$v = \frac{1}{k}\frac{\mu^e}{\sigma^2}. \tag{5.6}$$

The weight on the risky asset is increasing in the expected excess return of the risky asset, but decreasing in the risk aversion and variance. See Figure 5.3, which also illustrates that the objective function is concave, meaning that the first order condition is both necessary and sufficient. (From (5.5) we also see that the 2nd-order derivative is $-k\sigma^2$, which is

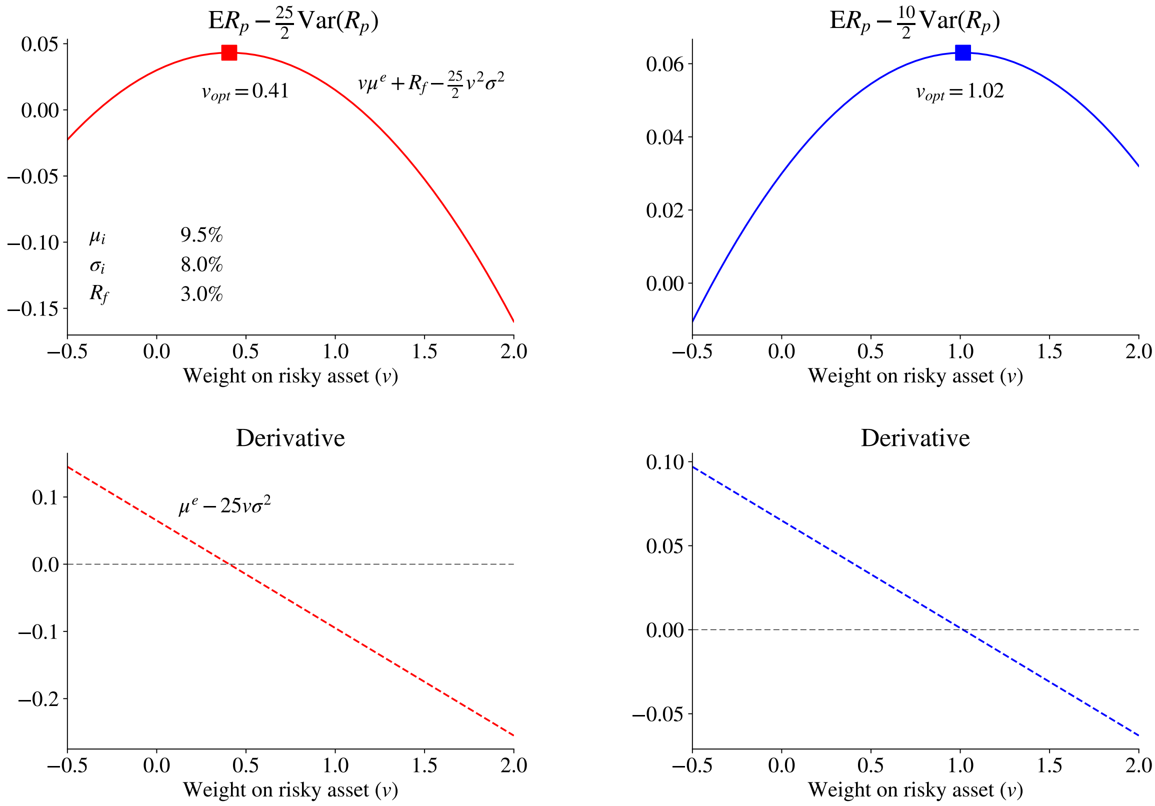



Figure 5.3: Portfolio choice

negative.)

**Example 5.3** *(Portfolio choice) If* $\mu^e = 6.5\%$, $\sigma_i = 8\%$ *and* $k = 25$, *then* $v \approx 0.41$. *Instead, with* $k = 10$, $v \approx 1.02$.

**Remark 5.4** *(*Why not use* $\mathrm{E}\, R_p - k\,\mathrm{Std}(R_p)$*?) Because it may not have a finite optimum as the objective function is not strictly concave. Too see this, consider changing (5.4) to* $v\mu^e - k\sqrt{v^2\sigma^2}$ *and suppose* $v \geq 0$ *is optimal. The objective function is then* $v(\mu^e - k\sigma)$ *where* $v = \infty$ *is optimal if* $\mu^e > k\sigma$. *The problem is that both the average returns and the standard deviation are linear in* $v$. *Instead, if we were to maximize* $v\mu^e - k(v^2\sigma^2)^{0.51}$ *(notice the* $0.51$ *instead of* $0.5$*), then the problem is well behaved.*

## 5.3 Portfolio Choice with Several Risky Assets and a Risk-free Asset

We now consider the case with $n$ investable risky assets and a risk-free asset. Combining (5.1) and (5.2) gives

$$\mathrm{E}\, U(R_p) = v'\mu^e + R_f - \frac{k}{2} v'\Sigma v, \tag{5.7}$$

where $\mu^e$ the $n$-vector of average excess returns and $\Sigma$ is the $n \times n$ covariance matrix of the returns. As before, these moments represent the beliefs of the investor, conditional on the information available at the time of the investment.

The first order conditions (for the vector $v$) are that the partial derivatives with respect to $v$ are zero

$$\mu^e - k\Sigma v = \mathbf{0}, \tag{5.8}$$

which can be solved as

$$v = \frac{1}{k}\Sigma^{-1}\mu^e. \tag{5.9}$$

Notice that the weight on the risk-free asset is $1 - v'\mathbf{1}$, where $\mathbf{1}$ is a (column) vector of ones. We will later provide an interpretation of the first order conditions.

**Remark 5.5** *For two assets, (5.9) can be written*

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{k}\frac{1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\begin{bmatrix} \sigma_{22} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11} \end{bmatrix}\begin{bmatrix} \mu_1^e \\ \mu_2^e \end{bmatrix},$$

*where we use $\sigma_{ii}$ to indicate the variance of asset $i$, since this facilitates the comparison with the matrix expressions. Notice that the denominator $\sigma_{11}\sigma_{22} - \sigma_{12}^2$ is positive since the correlation $\sigma_{12}/(\sigma_1\sigma_2)$ is between $-1$ and $1$. This means that*

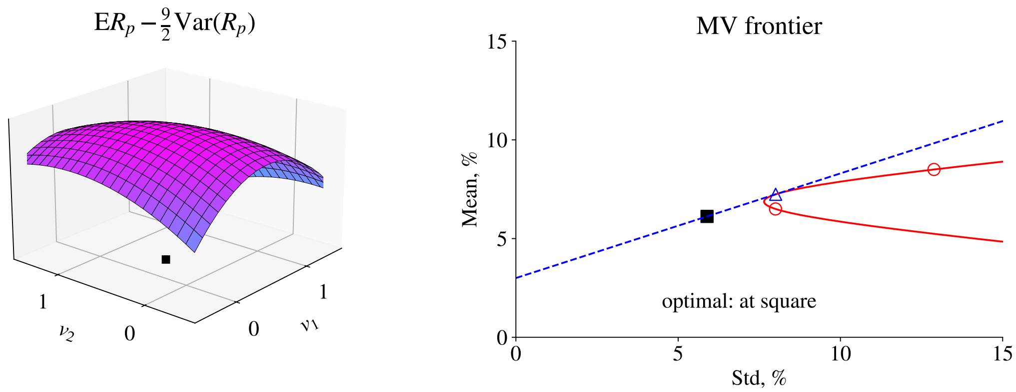
$$v_i > 0 \text{ if } SR_i > \rho SR_j,$$

*where $\rho$ is the correlation. This shows that an asset should be held in positive amounts if its Sharpe ratio exceeds the correlation times the Sharpe ratio of the other asset.*

**Example 5.6** *((5.9) with two assets) Let* $\Sigma = \begin{bmatrix} 166 & 34 \\ 34 & 64 \end{bmatrix}/100^2$, $\mu^e = \begin{bmatrix} 5.5 \\ 3.5 \end{bmatrix}/100$ *and* $k = 9$*. Then*

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \approx \begin{bmatrix} 67.6 & -35.9 \\ -35.9 & 175.3 \end{bmatrix}\begin{bmatrix} 0.055 \\ 0.035 \end{bmatrix}\frac{1}{9} \approx \begin{bmatrix} 0.27 \\ 0.46 \end{bmatrix}.$$

*The weight on the risk-free is (approximately)* $1 - 0.27 - 0.46 = 0.27$. *See also Figure 5.4.*




Riskfree, % 3.0

| $\mu$, % | $\Sigma$, bp | |
|---|---|---|
| 8.5 | 166 | 34 |
| 6.5 | 34 | 64 |

| | optimal | tangency | ratio |
|---|---|---|---|
| asset 1 | 0.27 | 0.37 | 0.74 |
| asset 2 | 0.46 | 0.63 | 0.74 |
| riskfree | 0.26 | 0.00 | |

Figure 5.4: Choice of portfolios weights

**Remark 5.7** *(Covariance of portfolios) $\Sigma v$ is the vector of covariances of each asset return, $R$, with the return on the portfolio $v'R$. Also, $\partial v'\Sigma v/\partial v = 2\Sigma v$, is the marginal contribution of each of the assets to the variance of the portfolio.*

Remark 5.7 says that $\Sigma v$ is the $n$-vector of covariances of each investable asset with the optimal portfolio. This means that row $i$ of the first order conditions (5.8) can be written

$$\mu_i^e - k\,\mathrm{Cov}(R_i, R_v) = 0, \tag{5.10}$$

The first term is the marginal increase in the portfolio excess return from increasing the weight on asset $i$ slightly—and financing it by borrowing. The second term is the risk aversion $k$ times the marginal increase in portfolio variance (divided by 2).

At at the optimum, the two terms (the "benefit" and the "cost" of increasing the position in asset $i$) are equal. See Figure 5.5 for an illustration. Off the optimum, one is larger than the other—so it is beneficial to change the portfolio. Increasing $v_i$ does not change the first term of (5.10) which is constant at $\mu_i^e$ but it will change the second term. The reason for the latter is that a higher $v_i$ value will make the portfolio return more similar to $R_i$ and thus increase the covariance.

*Proof* of (5.9). With $n = 2$ the portfolio return can be written $R_p = v_1 R_1^e + v_2 R_2^e +$

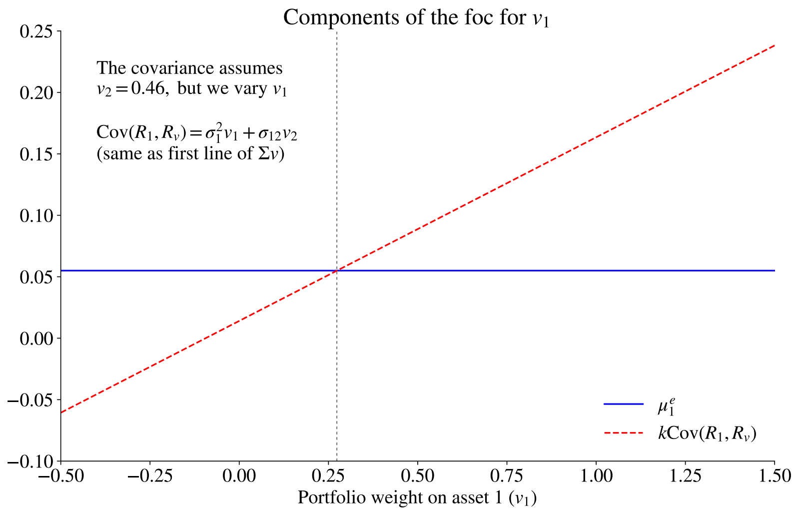



Figure 5.5: Choice of portfolios weights, first order condition. See Figure 5.4 for the parameter values.

$R_f$ ,so the objective is

$$\mathrm{E}\,U(R_p) = v_1\mu_1^e + v_2\mu_2^e + R_f - \frac{k}{2}(v_1^2\sigma_{11} + v_2^2\sigma_{22} + 2v_1v_2\sigma_{12}),$$

where $\sigma_{ii}$ denotes the variance of asset $i$ and $\sigma_{12}$ the covariance of asset 1 and 2. The first order conditions are

$$0 = \partial\,\mathrm{E}\,U(R_p)/\partial v_1 = \mu_1^e - \frac{k}{2}\,(2v_1\sigma_{11} + 2v_2\sigma_{12})$$

$$0 = \partial\,\mathrm{E}\,U(R_p)/\partial v_2 = \mu_2^e - \frac{k}{2}\,(2v_2\sigma_{22} + 2v_1\sigma_{12})\,, \text{ or}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mu_1^e \\ \mu_2^e \end{bmatrix} - k \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

This is the same as (5.8). □

**Remark 5.8** *(Several risky assets, no risk-free*) Maximizing the Lagrangian* $v'\mu - \frac{k}{2}v'\Sigma v + \theta(1 - v'\mathbf{1})$ *where* $\theta$ *is a Lagrange multiplier (for the constraint that the portfolio weights sum to one) gives the first order conditions (wrt.* $v$*)* $\mu - k\Sigma v - \theta\mathbf{1} = 0$*. Rewrite and use the restriction (*$\mathbf{1}'v{=}1$*) to write on matrix form*

$$\begin{bmatrix} k\Sigma & \mathbf{1} \\ \mathbf{1}' & 0 \end{bmatrix} \begin{bmatrix} v \\ \theta \end{bmatrix} = \begin{bmatrix} \mu \\ 1 \end{bmatrix}.$$

*Solve for* $[v; \theta]$.

## 5.4 Mean-Variance Preferences Gives a Portfolio on the Mean-Variance Frontier

It is evident that the optimal portfolio (5.9) is a scaling up/down of the *tangency portfolio* (see previous chapters)

$$w_T = \frac{\Sigma^{-1}\mu^e}{\mathbf{1}'\Sigma^{-1}\mu^e}. \tag{5.11}$$

This confirms the result previously discussed in Section 5.1 and illustrated in Figure 5.1.

To be precise, the optimal portfolio can be written

$$v = c w_T, \text{ where } c = \frac{\mathbf{1}'\Sigma^{-1}\mu^e}{k} \tag{5.12}$$

and $1 - c$ in the risk-free asset. The return on any optimal portfolio is thus a mix of the risk-free and tangency portfolio returns

$$\begin{aligned} R_{opt} &= v'R^e + R_f \\ &= cR_T^e + R_f. \end{aligned} \tag{5.13}$$

This means that the optimal portfolio is on the CML and can be constructed by combining the tangency portfolio and the risk-free asset. See Figure 5.6 for an illustration.

Equation (5.13) also shows that the beta of the optimal portfolio is

$$\beta_{opt} = c. \tag{5.14}$$

(This follows directly from the fact that regressing $cR_T^e + R_f$ on $R_T^e$ must give a slope of $c$, since $R_f$ is constant.) Equation (5.12) then shows that risk averse investors (high $k$) will chose portfolios with low $\beta$ and vice versa.

**Example 5.9** *(Portfolio choice to get a desired $\beta$) To construct a portfolio with $\beta = 1.2$ against the tangency portfolio, invest $c = 1.2$ in the tangency portfolio and $-0.2$ in the risk-free.*

**Remark 5.10** *(The mathematics of why $\max \mathrm{E}\, R_p - \frac{k}{2}\operatorname{Var}(R_p)$ gives a MV portfolio*) The efficient set solves the problem $\max \mathrm{E}\, R_p$ subject to $\operatorname{Var}(R_p) \le q$ (where we vary $q$ to trace out the efficient set). Notice that maximizing $\mathrm{E}\, R_p - \frac{k}{2}\operatorname{Var}(R_p)$ can be thought of as the Lagrangian formulation of the efficient set problem.*

|   | $\mu,\%$ | $\Sigma$, bp |   |   |
|---|---|---|---|---|
|   |   | A | B | C |
| A | 11.5 | 166 | 34 | 58 |
| B | 9.5 | 34 | 64 | 4 |
| C | 6.0 | 58 | 4 | 100 |

Table 5.1: Characteristics of the assets in the MV examples. Notice that $\mu,\%$ is the expected return in % (that is, $\times 100$) and $\Sigma$, bp is the covariance matrix in basis points (that is, $\times 100^2$).

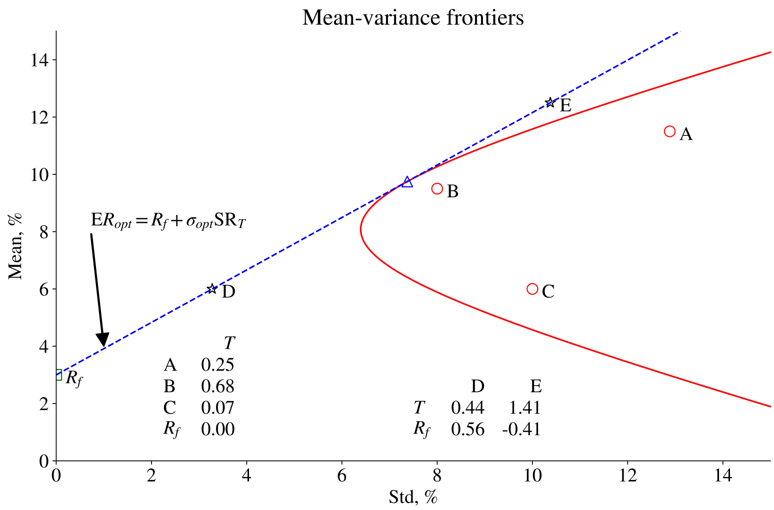



Figure 5.6: Mean-variance frontiers. The properties of the investable assets (A, B, and C) are shown in Table 5.1.

## 5.5 Portfolio Choice with Restrictions

In practice, many investors have to obey various (legal, practical och strategic) restrictions. Mutual typically have lower and upper bounds on the weights

$$L_i \leq w_i \leq U_i, \tag{5.15}$$

for instance, $L_i = 0$ and $U_i = 0.1$. Other investment funds allow some limited short sales

$$\text{limited total short sales: } \textstyle\sum_{i=1}^{n} \min(w_i, 0) \geq Q. \tag{5.16}$$

We can also introduce different lending and borrowing rates by defining a "lending asset" (with $w_i \geq 0$) and a "borrowing asset" (with $w_j \leq 0$).

Other funds with a focus on a particular investment focus may choose to track a benchmark (returns $R_b$ and portfolio weights $v_b$), but allow some tracking errors (to take bets or to reduce trading costs). This can be expressed as the following restriction

$$\text{Var}(R_p - R_b) = (w - v_b)' \Sigma (w - v_b) \leq TE, \tag{5.17}$$

where $\Sigma$ is the covariance matrix of the investable assets.

Restrictions of this type can be handled by numerical minimization algorithms.

## 5.6 Historical Estimates of the Average Returns and the Covariance Matrix

**Empirical Example 5.11** *Figure 5.7 illustrates mean returns and standard deviations, estimated by exponentially weighted moving averages (as by RiskMetrics). This means that the estimates are based on a longer and longer sample, but that old data are given lower weights. (In a sample that ends in $t$, the data in $t-s$ is given the weight $\lambda^{t-s}$, where $\lambda < 1$.) This leads to (time-varying) portfolio weights shown in Figure 5.8.*

**Empirical Example 5.12** *Figure 5.9 shows how the optimal portfolio weights (based on mean-variance preferences) change over time. It is evident that the portfolio weights change very dramatically—perhaps too much to be realistic. It also seems that changes in estimated average returns cause more dramatic movements in the portfolio weights than the changes in the estimated covariance matrix.*

**Empirical Example 5.13** *This means that, in practical application of the MV framework, we typically put restrictions on the levels and changes of the portfolio weights, see Figure 5.10 for an example.*

## 5.7 Appendix – A Primer on Using Numerical Optimization Routines*

Reference: Brandimarte (2006), Stan manual (`http://mc-stan.org/users/documentation/`), Kochenderfer and Wheeler (2019)

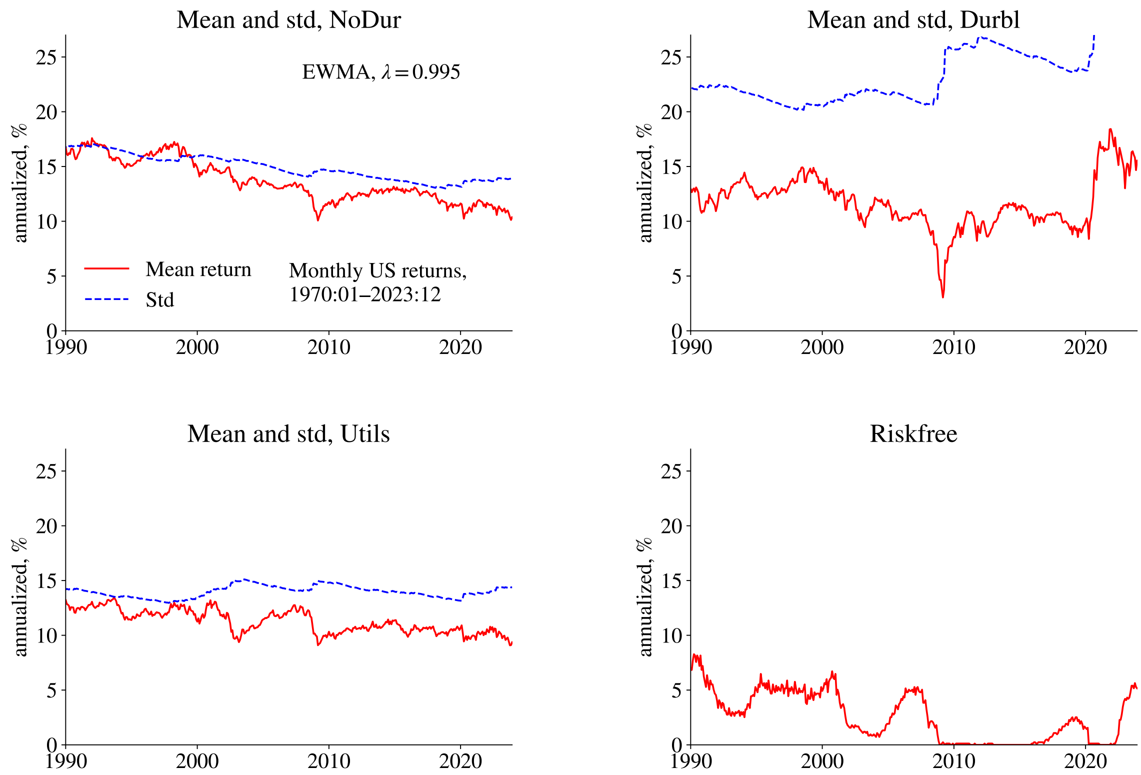



Figure 5.7: Dynamically updated estimates, 3 U.S. industries

### 5.7.1 Unconstrained Minimization

Consider the loss function

$$f(\theta) = (x - 2)^2 + (4y + 3)^2, \tag{5.18}$$

where $\theta = (x, y)$ contains the two choice variables. (Clearly, the minimum of $f(\theta)$ is at $(x, y) = (2, -3/4)$.) Since this loss function is particularly simple—quadratic and also separable in $x$ and $y$—the solutions below are straightforward. However, the methods presented can also be used with more complicated loss functions.

A numerical minimization routine searches through different values of $\theta$, typically starting from an initial guess, to find the values that makes $f(\theta)$ as small as possible. Convergence criteria, often set by the user, determine when the search will stop, for instance, when the improvement in $f(\theta)$ is smaller than a certain threshold or when the $\theta$ values stabilise. The starting guess is often important, so be sure to use reasonable values. See Figure 5.11 for an example.

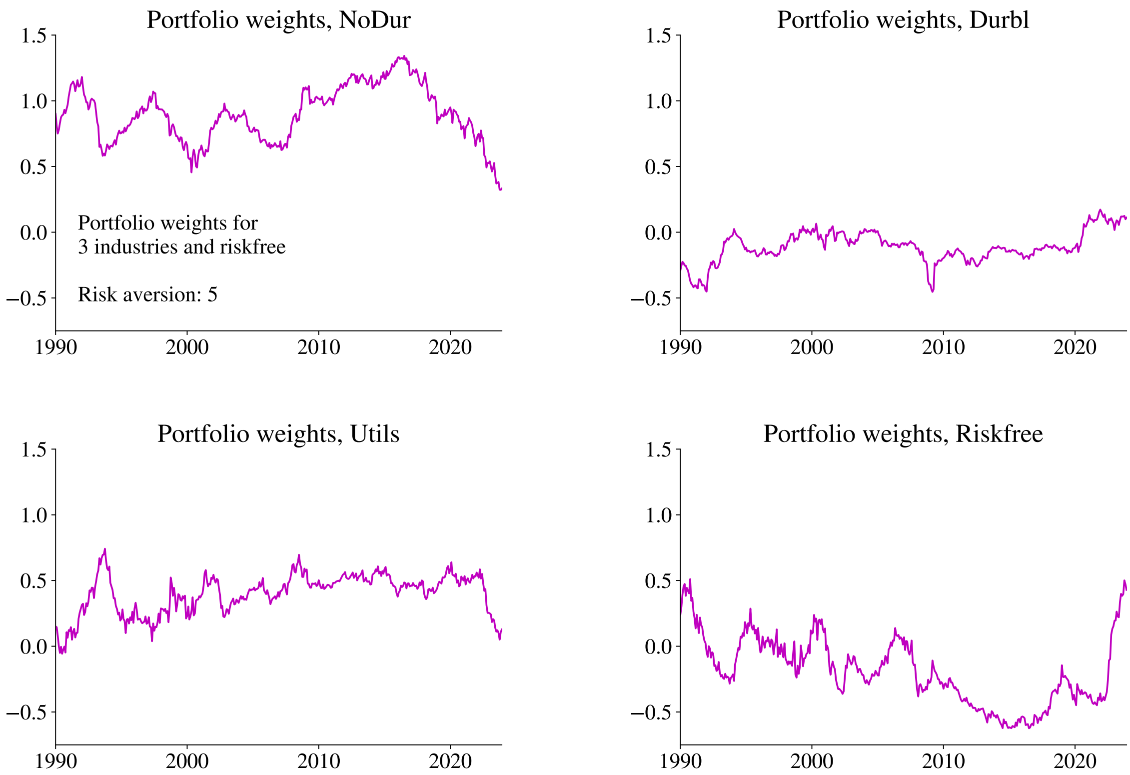



Figure 5.8: Dynamically updated portfolio weights, T-bill and 3 U.S. industries

Some algorithms use derivatives of the loss function (which may have to be coded by the user), while others do not ("derivative free"). The latter type is often slower, but sometimes more robust.

Most optimization algorithms are for minimizing a function value. In case you want to maximize, then just change the sign of the function and then minimize it. For instance, if you want to maximize $g(\theta)$, then you can do that by minimizing $-g(\theta)$.

### 5.7.2 Bounds on Variables

Many numerical optimization packages have options for setting bounds on the solution ("box minimization"). As an alternative, we could transform the variables and then apply an algorithm for unconstrained optimisation. The latter is briefly discussed below.

A simple way to handle a lower bound, such as $a \leq x$, is to let the routine optimize, without any restrictions, with respect to a transformed variable, $\tilde{x} = \ln(x - a)$. Within the loss function—and also after having obtained the minimizer—the variable can be transformed back using $x = \exp(\tilde{x}) + a$.

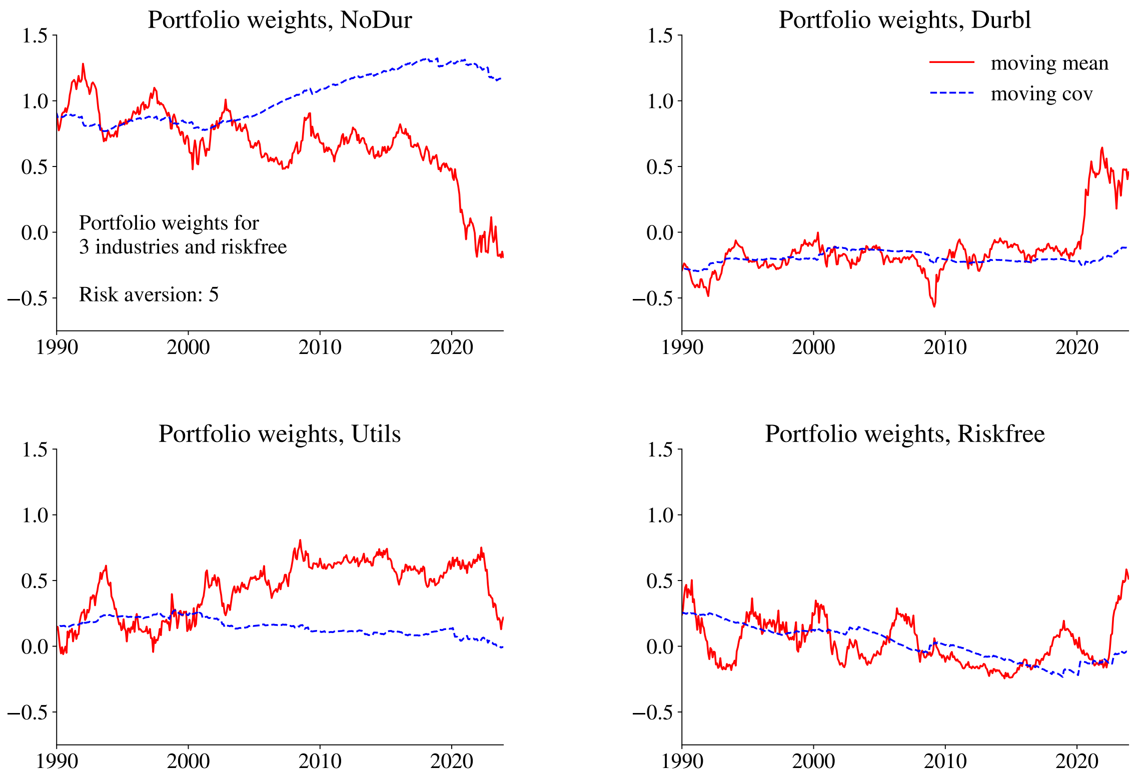



Figure 5.9: Dynamically updated portfolio weights, T-bill and 3 U.S. industries

Instead, with an upper bound, $x \leq b$, we optimize over $\tilde{x} = \ln(b - x)$ and transform back using $x = b - \exp(\tilde{x})$.

Suppose we use the same loss function (5.18) as before, but also impose the bounds

$$2.75 \leq x \text{ and } y \leq -0.3. \tag{5.19}$$

The solution is $(x, y) = (2.75, -3/4)$, so only one of the bounds is really binding. See Figure 5.12 for an illustration.

**Remark 5.14** *With both lower and upper bounds $a \leq x \leq b$, we instead work with the (unbounded) $v = \operatorname{logit}(\frac{x-a}{b-a})$, where the logit function and its inverse are defined as $\operatorname{logit}(u) = \ln(\frac{u}{1-u})$ and $\operatorname{logit}^{-1}(v) = \frac{1}{1+\exp(-v)}$. (The inverse is also called the logistic function.) We can transform back using $x = a + (b-a)\operatorname{logit}^{-1}(v)$*

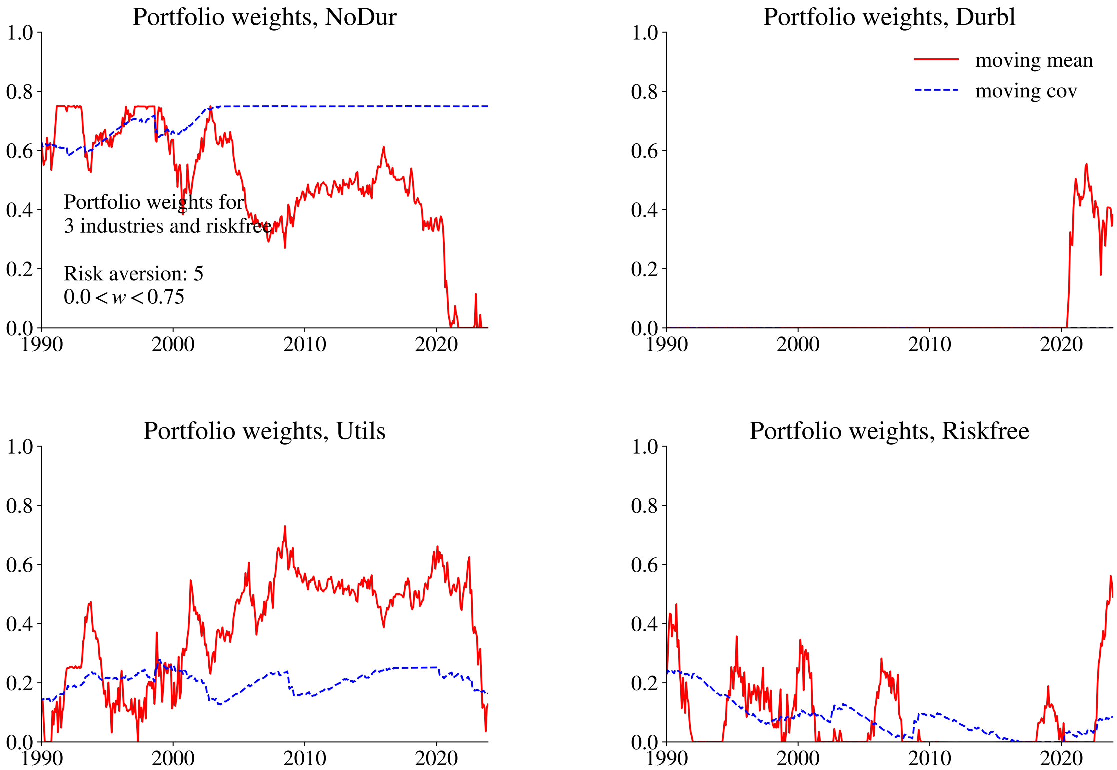



Figure 5.10: Dynamically updated portfolio weights (no short sales), T-bill and 3 U.S. industries

### 5.7.3 Equality Constraints

Suppose you want an *equality constraint* on the minimization problem, say

$$h_1(\theta) = x + 2y - 3 = 0. \tag{5.20}$$

One way to handle this is to use the constraint to rewrite the loss function (in this case, we would use $x = 3 - 2y$ to replace $x$ in (5.18)). If this is tricky, then we try to find a routine that can handle equality constraints. The short discussion below outlines how these routines work (and also suggests how we could construct such a routine ourselves).

A simple approach is to apply a penalty for deviations from the constraint, thereby modifying the overall loss function to

$$f(\theta) + \lambda \textstyle\sum_{i=1}^{p} h_i(\theta)^2, \tag{5.21}$$

where $h_i(\theta)$ is the $i$th equality constraint. In our example (5.20), there is only one re-

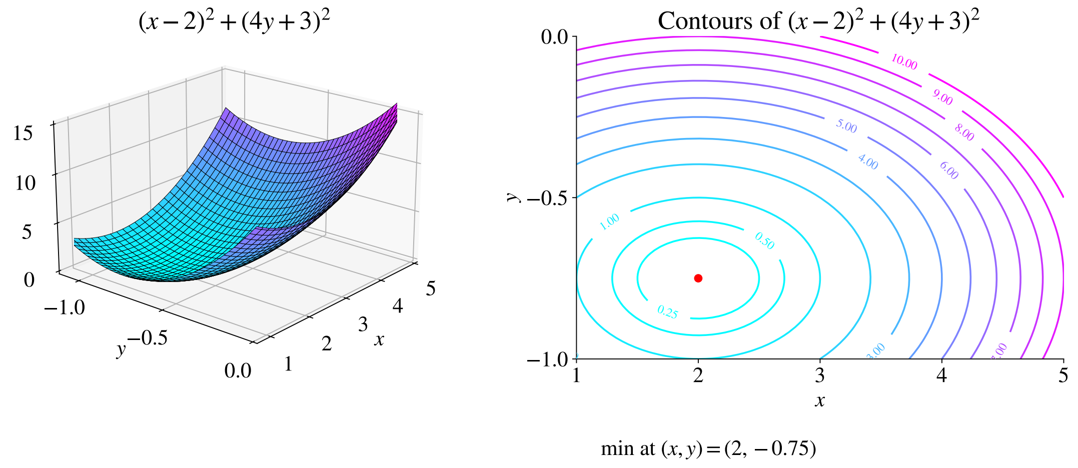



Figure 5.11: Numerical optimization, no restrictions

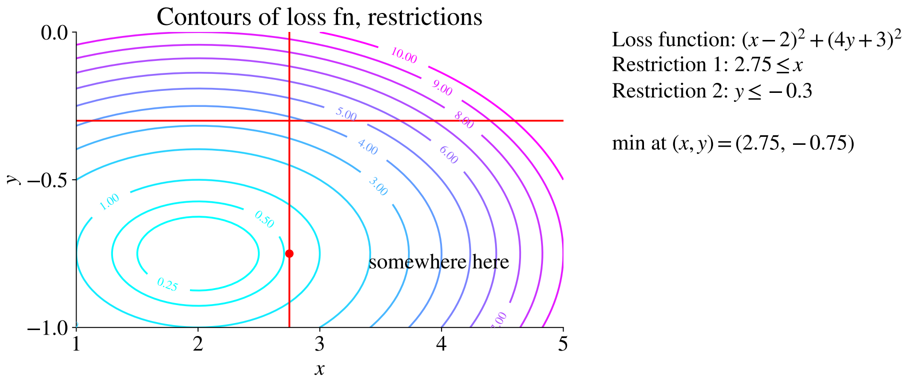



Figure 5.12: Numerical optimization with bounds on the solution

striction ($p = 1$). See Figure 5.13 for an example. (The solution should be very close to $(x, y) = (4, -1/2)$.)

Start by setting $\lambda = 0$ and find the optimal value of $\theta$, and call it $\theta_1$. This is clearly the unconstrained solution. Then, increase $\lambda$ and redo the optimization (using $\theta_1$ as the starting guess) to get the optimal value $\theta_2$. Now, increase $\lambda$ further and redo the optimization (using $\theta_2$ as the starting guess). Keep doing this (with higher and higher values of $\lambda$) until the solutions do not change much anymore. It is often worthwhile to experiment a bit with the sequence of $\lambda$ values. In general, it seems as if initially making small increases and later larger ones works well in many cases. See Figure 5.14 for an example. (Clearly, there are more systematic ways to pick the sequence of penalties.)

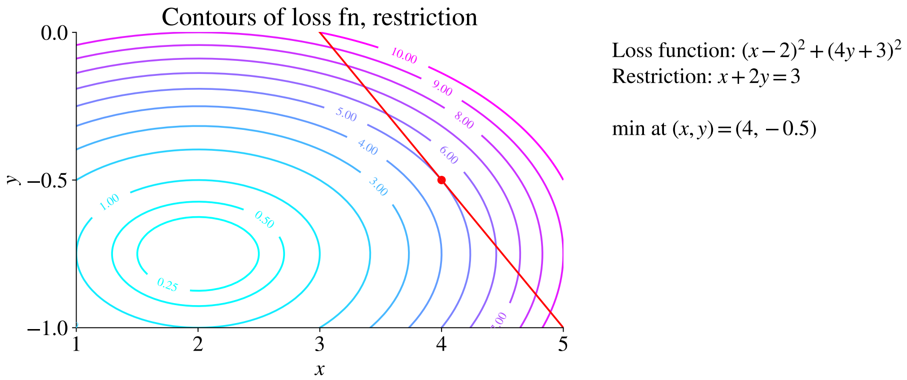



Figure 5.13: Numerical optimization with an equality restriction

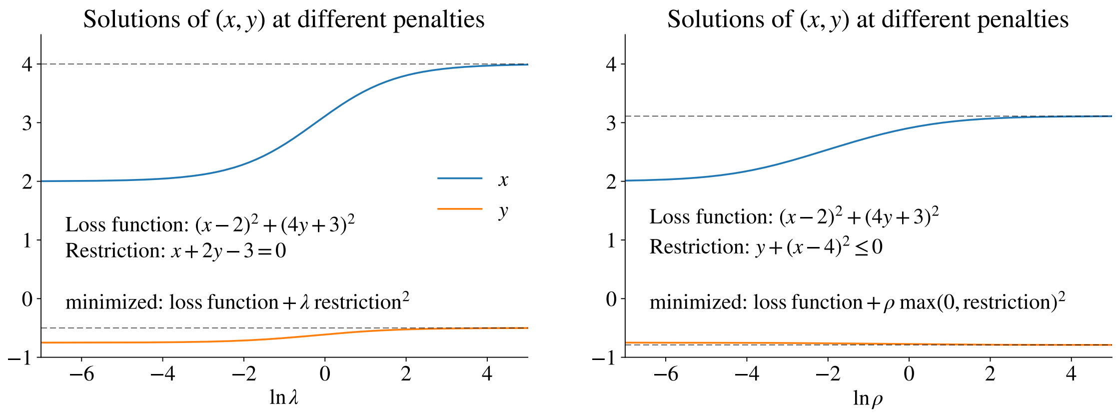



Figure 5.14: Numerical optimizations with penalty on the restriction

### 5.7.4 Inequality Constraints

Instead, we now want to minimize (5.18) under the *inequality constraint* $y \leq -(x-4)^2$.

It is convenient to rewrite all inequality constraints on a common form, and we here choose to write them all on $\leq 0$ form, which gives

$$g_1(\theta) = y + (x-4)^2 \leq 0. \tag{5.22}$$

Now, we minimise the overall loss function

$$f(\theta) + \rho \textstyle\sum_{j=1}^{q} \max[0, g_j(\theta)]^2, \tag{5.23}$$

where $g_j(\theta)$ is the $j$th inequality constraint (there is only one in our example). Notice that

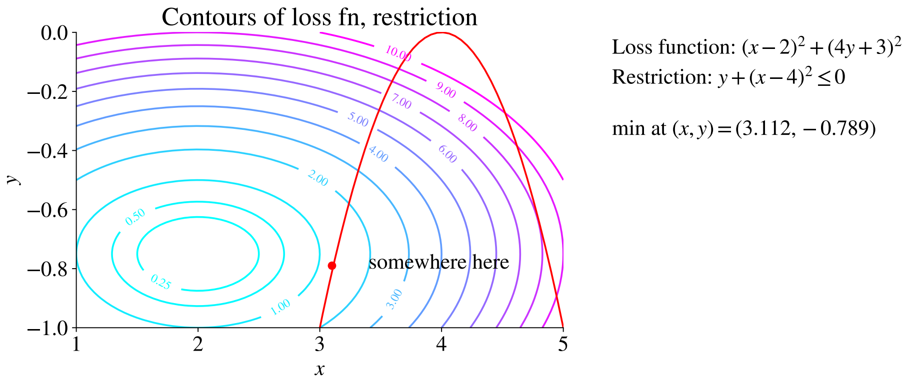



Figure 5.15: Numerical optimization with inequality restriction

$\rho$ plays the same role as $\lambda$: start by solving for $\rho = 0$, then use that solution as a starting guess for the problem with a higher $\rho$, etc. See Figure 5.15 for an example. (The solution should be close to $(x, y) = (3.1, -0.79)$.) See Figure 5.14 for an iterative approach with a larger and larger penalty.

Finally, we can combine equality and inequality constraints as

$$f(\theta) + \lambda \sum_{i=1}^{p} h_i(\theta)^2 + \rho \sum_{j=1}^{q} \max[0, g_j(\theta)]^2. \tag{5.24}$$

# Chapter 6

# CAPM

## 6.1 Beta Representation of Expected Returns

### 6.1.1 Beta Representation: Definition

The beta representation (and eventually also CAPM as developed by Sharpe (1964), Lintner (1965) and Mossin (1966)) follows from the analysis of portfolio choice based on mean-variance preferences. From an earlier chapter, we notice two things. First, the optimal portfolio weights, $v$, are proportional to the tangency portfolio, $w_T$,

$$v = c w_T, \tag{6.1}$$

where $c$ is a scalar ($c = \mathbf{1}'\Sigma^{-1}\mu^e/k$).

Second, the *first order conditions* for optimal portfolio choice are the $n$ equations in

$$\mathrm{E}\, R^e = k\Sigma v. \tag{6.2}$$

Recall that element $i$ of the vector $\Sigma v$ is the covariance between $R_i$ and $R_v$.

Combining these two observations shows that the first order conditions for optimal portfolio choice can be written

$$\mathrm{E}\, R_i^e = kc\sigma_{iT}, \text{ for } i = 1, \ldots, n, \tag{6.3}$$

where $k$ is the risk aversion and $\sigma_{iT}$ is shorthand notation for the covariance of $R_i$ and $R_T$, $\mathrm{Cov}(R_i, R_T)$. I use the notation $\mathrm{E}\, R_i^e$ for the left hand side (rather than the equivalent $\mu_i^e$) to suggest that this is a result, not "data."

We can express this as a *beta representation*. Let $\mu_T^e$ be the expected excess return on the tangency portfolio and rewrite (6.3) as

$$\mathrm{E}\, R_i^e = \beta_i \mu_T^e \text{ or} \tag{6.4}$$

$$\mathrm{E}\, R_i = R_f + \beta_i \mu_T^e, \text{ where} \tag{6.5}$$

$$\beta_i = \sigma_{iT}/\sigma_T^2. \tag{6.6}$$

(See below for a proof). It is important to acknowledge that this expression *does not say anything about causality*: it just shows how expected returns and betas relate to each other according to the first order conditions for optimal portfolio choice. Plotting $\mathrm{E}\, R_i^e$ or $\mathrm{E}\, R_i$ against $\beta_i$ gives the *security market line*, see Figure 6.1.

*Proof* of (6.4). (6.3) holds for any asset/portfolio, also the tangency portfolio so $\mathrm{E}\, R_T^e = kc\sigma_T^2$. Solve for $kc$ and use back in (6.3). (More details: in vector form, (6.3) is $\mu^e = kc\Sigma w_T$. Premultiply both sides by $w_T'$ to get $w_T'\mu^e = kcw_T'\Sigma w_T$ which is the same as $\mu_T^e = kc\sigma_T^2$.) $\square$

The $\beta_i$ is clearly the slope coefficient in a (time series) OLS regression

$$R_{it}^e = \alpha_i + \beta_i R_{Tt}^e + \varepsilon_{it}, \tag{6.7}$$

where $R_{it}^e$ is the excess return on asset $i$ in period $t$.

**Remark 6.1** *(Calculating $\beta_i$ from the covariance matrix*) The traditional way of estimating $\beta_i$ is to run a regression. However, if we know the variance-covariance matrix $\Sigma$ of the investable assets, then we can also use the fact that $\beta_i = \sigma_{iT}/\sigma_T^2$ where $\sigma_{iT} = w_i'\Sigma w_T$. Using the asset price characteristics in Table (6.1), together with the weights of the tangency portfolio gives the $\beta$ values in Figure 6.1.*

| | $\mu$, % | $\Sigma$, bp | | |
|---|---|---|---|---|
| | | A | B | C |
| A | 11.5 | 166 | 34 | 58 |
| B | 9.5 | 34 | 64 | 4 |
| C | 6.0 | 58 | 4 | 100 |

Table 6.1: Characteristics of the assets in the MV examples. Notice that $\mu$, % is the expected return in % (that is, $\times 100$) and $\Sigma$, bp is the covariance matrix in basis points (that is, $\times 100^2$).

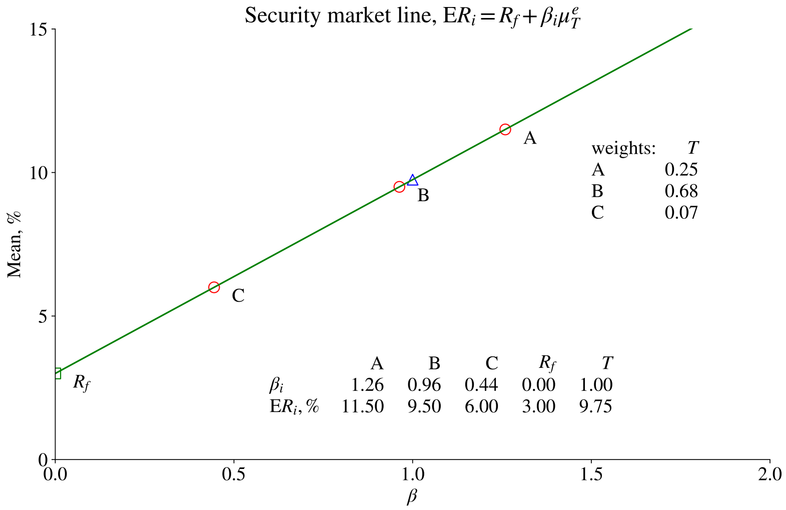



Figure 6.1: Security market line. The properties of the investable assets (A, B, and C) are shown in Table 6.1.

**Example 6.2** *($\beta_i$ vs $\mathrm{E}\,R_i$) With $R_f = 3\%$ and $\mu_T = 9.75\%$ (so $\mu_T^e = 6.75\%$) we get*

| $\beta_i$ | $\mathrm{E}\,R_i$ | *Comment* |
|---:|---:|---:|
| 0.44 | 6.0% | *low $\beta$, low avg. return* |
| 1.5 | 13.12% | *high $\beta$, high avg. return* |
| 1 | 9.75% | *same risk as market* |
| 0 | 3% | *no risk* |
| $-0.5$ | $-0.38\%$ | *the opposite of risk* |

### 6.1.2 Betas of Portfolios

Recall that the beta of any portfolio (not just the optimal one) is the weighted average (portfolio) of the betas of its components. That is, the portfolio with return

$$R_p = w'R^e + R_f \text{ has the beta} \tag{6.8}$$

$$\beta_p = w'\beta. \tag{6.9}$$

(This follows directly from $\beta_p = \mathrm{Cov}(\Sigma_{i=1}^n w_i R_i, R_T)/\sigma_T^2 = \Sigma_{i=1}^n w_i \beta_i$.)

**Example 6.3** *Let $(\beta_1, \beta_2) = (1.2, 0.8)$. The portfolio return $R_p = 0.6R_1 + 0.4R_2$ has the beta $\beta_p = 0.6 \times 1.2 + 0.4 \times 0.8 = 1.04$.*

In particular, consider the portfolios on the capital market line (CML): $R_{opt} = cR_T^e + R_f$, where $c$ is a scalar. Using the result in (6.9) and noticing that the tangency portfolio has $\beta_T = 1$ gives that

$$\beta = c \text{ for any portfolio on the CML.} \tag{6.10}$$

This implies that it is straightforward to create a portfolio with any desired $\beta$: just invest $\beta$ in the tangency portfolio and $1-\beta$ in the risk-free.

**Example 6.4** *(Creating a portfolio with $\beta_p = 0.44$) We can create a portfolio with $\beta = 0.44$ by investing 0.44 in the tangency portfolio and 0.56 in the risk-free.*

### 6.1.3 Beta Representation and the Capital Market Line

The beta representation (6.4) means that two assets with the same betas should have the same expected returns—even if they have very different volatilities.

To be precise, consider the regression (6.7) which has the usual property that the residual is uncorrelated with the regressor. We can therefore write the variance of $R_i$ as

$$\sigma_i^2 = \beta_i^2 \sigma_T^2 + \sigma_\varepsilon^2. \tag{6.11}$$

This says that the variance of $R_i$ has two components: *systematic risk* (the comovement of $R_i$ with $R_T$, $\beta_i^2\sigma_T^2$) and *idiosyncratic noise* (the variance of $\varepsilon_i$, $\sigma_\varepsilon^2$). In particular, *MV efficient portfolios have only systematic risk* ($\sigma_\varepsilon^2 = 0$) since they are formed from the tangency portfolio and risk-free ($R_{opt} = cR_T + (1-c)R_f$). All other portfolios with the same $\beta$ are to the right in the MV figure: see Figure 6.2 for an illustration.

**Example 6.5** *In Figure 6.2, we want to understand the mean return (vertical location) of asset $C$ (taking its volatility and $\beta$ as given). We notice that $C$ has the same systematic risk as the efficient portfolio $D$. According to the beta representation, $C$ must then have the same average return as $D$.*

### 6.1.4 The Tangency Portfolio is the Market Portfolio

To determine the equilibrium asset prices (and therefore expected returns) we have to equate demand (the mean variance portfolios) with supply, which we assume is exogenous. Since we assume a fixed and exogenous supply (say, 2000 shares of asset 1 and 407 shares of asset 2,...), prices, and therefore returns, are driven by demand.

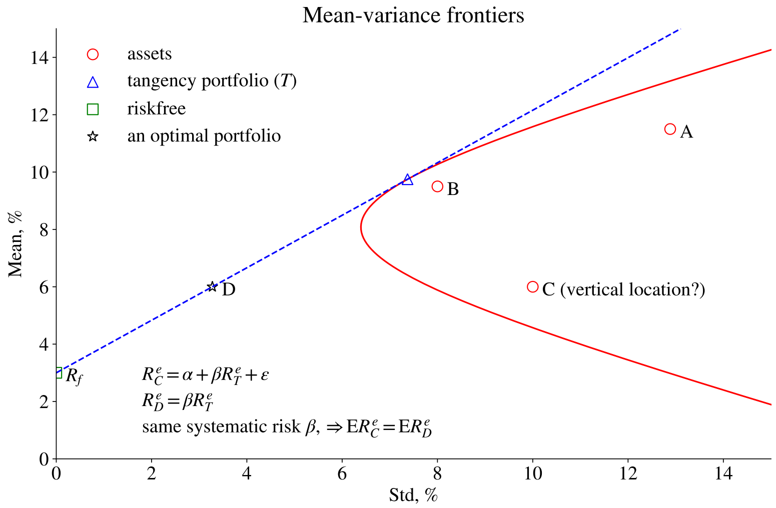



Figure 6.2: Mean-variance frontier and expected returns

Suppose all investors have the same beliefs about the asset returns (same expected returns and covariance matrix). They will then all mix the same tangency portfolio with the risk-free—but possibly in different proportions due to different risk aversions.

In equilibrium, net supply of the risk-free assets is zero (lending = borrowing), so *the average investor must hold the tangency portfolio and no risk-free assets*. Therefore, *the tangency portfolio must be the market portfolio*, so we can replace $R_T$ with $R_m$ in all expressions above. In particular, CAPM says

$$\mathrm{E}\, R_i^e = \beta_i \mu_m^e \text{ where} \tag{6.12}$$

$$\beta_i = \sigma_{im}/\sigma_m^2. \tag{6.13}$$

As discussed before, this expression is just a characterisation of the equilibrium (the first order conditions), and CAPM is silent on how that equilibrium is reached. One possible *story* is that $\beta_i$ is driven by the firm characteristics (industry, size, leverage, etc.) and that equilibrium is reached as follows: high $\beta$ assets are in low demand since they are too procyclical (pay off at the wrong time) which means that (in equilibrium) the share price will be low. For a given dividend, this means a higher dividend/price ratio, which contributes to a high average return.

Clearly, *CAPM relies on very strong assumptions*, in particular, the assumption about all investors having the same beliefs. Also, it rules out that investors face other types of financial risks (not just asset market risks). These issues will be discussed in subsequent lecture notes.

### 6.1.5 Properties of the Market Portfolio*

It is straightforward to show that the market risk premium (expected excess return) is proportional to the market volatility

$$\mathrm{E}\, R_m^e = k_m \sigma_m^2, \tag{6.14}$$

where we used the subscript $m$ to indicate that this is the market portfolio (which equals the tangency portfolio). Also, $k_m$ indicates the risk aversion for the (average) investor (who holds no risk-free assets). This says that the market risk premium increases if the risk aversion ($k_m$) or market variance does.

*Proof* of (6.14). Recall the first order conditions for optimal portfolio choice for the investor with risk aversion $k_T$, which imply holding the tangency portfolio $\mu^e = k_T \Sigma w$. Premultiply both sides by $w'$ and set $m = T$. $\square$

### 6.1.6 Summarizing MV and CAPM: CML and SML

According to MV analysis, and assuming that the market portfolio equals the tangency portfolio, average return of all optimal (effective) portfolios (denoted $opt$) obey

$$\mathrm{E}\, R_{opt} = R_f + \sigma_{opt} SR_m. \tag{6.15}$$

The plot of $\mathrm{E}\, R_{opt}$ against $\sigma_{opt}$ is called the *capital market line*. See Figure 6.3 for an example.

According to CAPM, the average return on all portfolios (optimal or not), obey the beta representation (6.5)

$$\mathrm{E}\, R_i = R_f + \beta_i \mu_m^e. \tag{6.16}$$

The plot of $\mathrm{E}\, R_i$ against $\beta_i$ (for different assets, $i$) is called the *security market line*. See Figure 6.3 for an example.

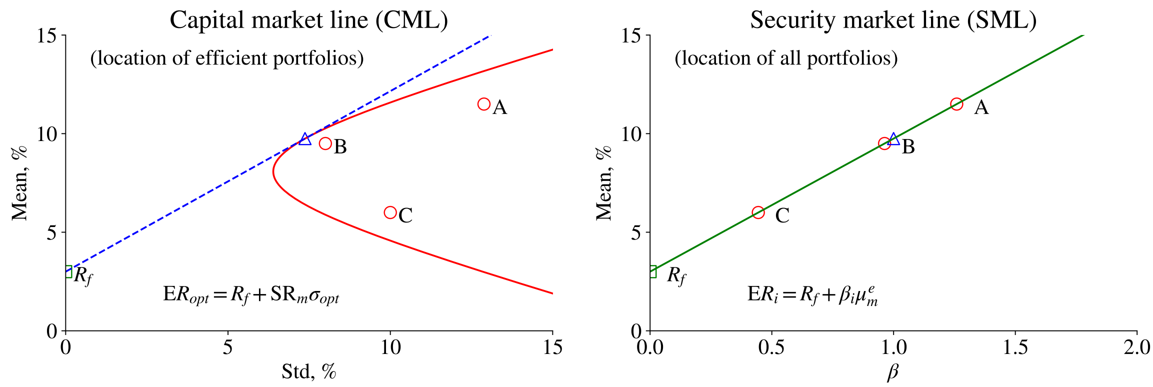



Figure 6.3: CML and SML

### 6.1.7 CAPM and Stochastic Discount Factors*

For future reference, we here notice that the CAPM expressions (6.12)–(6.13) can also be written in terms of a "stochastic discount factor" (SDF) model. This model implies

$$\mathrm{E}\, R_i^e M = 0, \text{ where} \tag{6.17}$$

$$M = a - bR_m^e, \text{ with } b > 0. \tag{6.18}$$

Many asset pricing models can be written on a similar form, as will be discussed in later chapters.

*Proof* of (6.17)–(6.18) giving (6.12)–(6.13 ). Recall that $\mathrm{Cov}(x, y) = \mathrm{E}\, xy - \mathrm{E}\, x \times \mathrm{E}\, y$, so $\mathrm{E}\, xy = 0$ can be rearranged as $\mathrm{E}\, y = -\mathrm{Cov}(x, y)/\,\mathrm{E}\, x$. Applying to (6.17)–(6.18) gives

$$\mathrm{E}\, R_i^e = b\frac{\sigma_{im}}{\mathrm{E}(a - bR_m^e)}$$

We can, of course, apply this expression to the market excess return (instead of asset $i$) to get

$$\mathrm{E}\, R_m^e = b\frac{\sigma_m^2}{\mathrm{E}(a - bR_m^e)}.$$

Solve for $b/\,\mathrm{E}(a - bR_m^e)$ and use that in the first equation to get the CAMP expression (6.12)–(6.13). □

### 6.1.8 Back to Prices (The Gordon Model)*

The gross return, $1 + R_{t+1}$, is defined as

$$1 + R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t}, \tag{6.19}$$

where $P_t$ is the asset price and $D_{t+1}$ is the dividend it gives at the beginning of the next period. If we assume that expected returns are constant across time (denoted $R$, for instance 10%) and that dividends are expected to grow at the rate $g$ (for instance, 2%), then it is straightforward to show that the asset price is

$$P_t = \mathrm{E}_t\, D_{t+1} \textstyle\sum_{s=1}^{\infty} \dfrac{(1+g)^{s-1}}{(1+R)^s} = \dfrac{\mathrm{E}_t\, D_{t+1}}{R-g}. \tag{6.20}$$

Clearly, higher (expected) dividends and/or a higher growth rate increases the asset price. In addition, a lower expected ("required") *future return* also increases *today's asset price*.

In CAPM, a lower expected return could be driven by a lower beta or by a lower risk-free rate. One way of interpreting this is as follows. If an asset (suddenly) gets a lower beta, that means that it has less systematic risk than before. It is, therefore, more useful in portfolio formation and becomes more demanded—so the price level increases. With a higher price level, the dividend yield is lower, which contributes to a lower return (recall the return is the dividend yield plus the capital gains yield).

## 6.2 Testing CAPM

### 6.2.1 Testing a Single Asset

The basic implication of CAPM is that the expected excess return of an asset ($\mathrm{E}\, R^e_{it}$) is linearly related to the expected excess return on the market portfolio ($\mathrm{E}\, R^e_{mt}$) according to (6.13). This could be tested by the regression (6.7 ), but where we use the market return to proxy for the tangency portfolio return.

In particular, take average (over time) of the regression to get

$$\bar{R}^e_i = \hat{\alpha}_i + \hat{\beta}_i \bar{R}^e_m, \tag{6.21}$$

where $\bar{R}^e_i$ is the average excess return on asset $i$ in the sample ($\bar{R}^e_i = \Sigma^T_{t=1} R^e_{it}/T$). (This follows from the fact that $\bar{\varepsilon}_i = 0$ by construction.)

The OLS estimate of $\beta_i$ is the sample analogue to the true $\beta_i$. It is then clear that *CAPM implies* that the intercept ($\alpha_i$) of the regression should be zero, which is what empirical tests of CAPM focus on.

However, this interpretation relies on a big *assumption*: that market expectations are (on average) well represented by the data for a particular sample. In principle, the CAPM equation (6.13) could be true, but for beliefs (about expected returns and betas) that we cannot estimate a historical sample.

The test of the null hypothesis that $\alpha_i = 0$ uses the fact that, under fairly mild conditions, the t-statistic has an asymptotically normal distribution, that is

$$\frac{\hat{\alpha}_i}{\text{Std}(\hat{\alpha}_i)} \xrightarrow{d} N(0,1) \text{ under } H_0 : \alpha_i = 0. \tag{6.22}$$

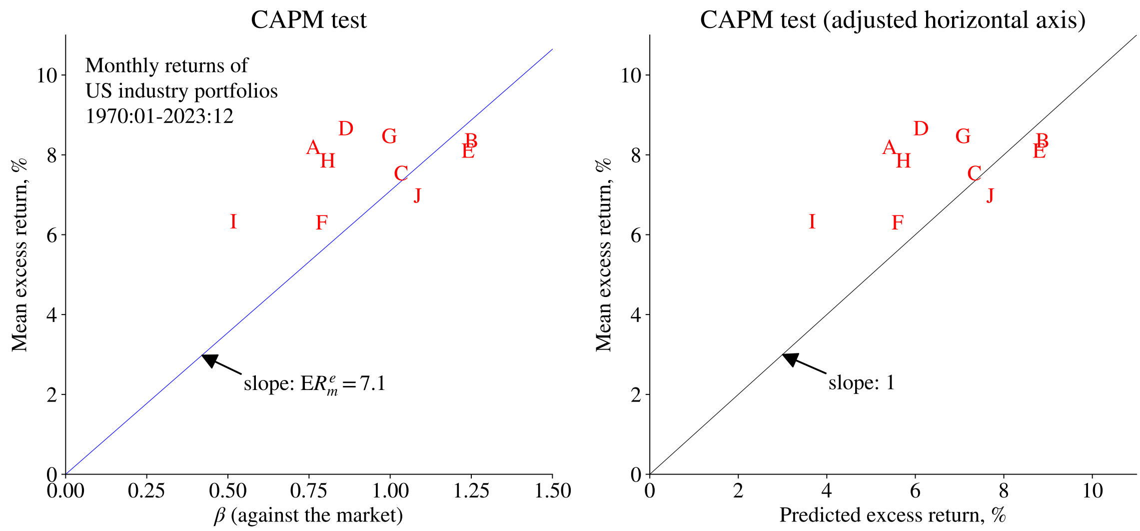
In this expression, $\hat{\alpha}_i$ is the estimate of the intercept in (6.7) and $\text{Std}(\hat{\alpha}_i)$ its standard deviation (for instance, from the usual OLS results, also see Remark 6.7). Note that this is the distribution under the null hypothesis that the true value of the intercept is zero, that is, that CAPM is correct. We typically reject the null hypothesis ($\alpha_i = 0$) when the the t-statistic is very negative or very positive (for instance, lower than $-1.64$ or higher than $1.64$ for the 10% significance level, and $-1.96/1.96$ for the 5% level).

The test assets are often portfolios of firms with similar characteristics, for instance, small size or having their main operations in the retail industry. There are two main reasons for testing the model on such portfolios: individual stocks are extremely volatile and firms can change substantially over time (so the beta changes). Moreover, it is of interest to see how the deviations from CAPM are related to firm characteristics (size, industry, etc), since that can possibly suggest how the model needs to be improved.

The empirical results from such tests vary with the test assets used. For US portfolios, CAPM seems to work reasonably well for some types of portfolios (for instance, portfolios based on firm size or industry), but much worse for other types of portfolios (for instance, portfolios based on firm dividend yield or book value/market value ratio).

**Empirical Example 6.6** *Figure 6.4 shows some results for US industry portfolios, while Table 6.2 and Figures 6.5–6.6 for US size/book-to-market portfolios. In these figures, the results are plotted in one of two different ways:*

$$\begin{array}{lll} & \underline{\textit{horizontal axis}} & \underline{\textit{vertical axis}} \\ 1: & \beta_i & \bar{R}_i^e \\ 2: & \beta_i \bar{R}_m^e & \bar{R}_i^e, \end{array} \tag{6.23}$$




| | $\alpha$ (ann.) | $t$-stat | $\sigma$ (ann.) |
|---|---|---|---|
| A (NoDur) | 2.73 | 2.29 | 8.71 |
| B (Durbl) | -0.57 | -0.25 | 16.72 |
| C (Manuf) | 0.15 | 0.17 | 6.38 |
| D (Enrgy) | 2.49 | 1.07 | 16.87 |
| E (HiTec) | -0.75 | -0.49 | 11.19 |
| F (Telcm) | 0.66 | 0.44 | 10.95 |
| G (Shops) | 1.34 | 1.08 | 9.03 |
| H (Hlth ) | 2.08 | 1.38 | 10.95 |
| I (Utils) | 2.61 | 1.63 | 11.69 |
| J (Other) | -0.77 | -0.81 | 6.97 |

CAPM: $R_i^e = \alpha_i + \beta_i R_m^e + e_i$
Predicted excess return: $\beta_i \mathrm{E} R_m^e$

10% crit. value (Bonferroni): 2.58

Test if all $\alpha_i = 0$:

| | |
|---|---|
| Wald stat | 11.20 |
| 5% crit val | 18.31 |
| p-value | 0.34 |

Figure 6.4: CAPM regressions on US industry indices

*where $\bar{R}_i^e$ indicates the (time) average excess return of asset $i$. In the first approach, CAPM says that all data points (different assets, $i$) should cluster around a straight line with a slope equal to the average market excess return, $\bar{R}_m^e$. In the second approach, CAPM says that all data points should cluster around a 45-degree line. In either case, the vertical distance to the line is $\alpha_i$ (which should be zero according to CAPM).*

### 6.2.2 Testing Several Assets

In most cases there are several ($n$) test assets, and we actually want to test if all the $\alpha_i$ (for $i = 1, 2, ..., n$) are zero. Ideally we then want to take into account the correlation of the different alphas.

While it is straightforward to construct such a test, it requires setting up a system of regression equations and test across regressions. See Remark 6.7 for details. Alternatively, we can apply a *Bonferroni adjustment* of the individual t-stats: reject CAPM at the 10% significance level only if the largest t-stat (in absolute terms) exceeds the critical value at the $0.10/n$ significance level. For instance, with $n = 25$, the critical value from a standard normal distribution would be 2.88 instead of 1.64. The motivation for this is that repeated single-asset testing (using a traditional critical value) will, by pure randomness, reject 10% of the cases—even if the null hypothesis is true. The Bonferroni adjustment takes this into account to correct the "family-wise" false rejection rate.

**Remark 6.7** *(*Variance-covariance matrix if OLS estimate of $\alpha$) The "system estimation" is actually n separate OLS regressions. Assuming residuals have no autocorrelation or heteroskedasticity (the standard Gauss-Markov assumption), it is straightforward to show that the variance-covariance matrix of the vector of the estimated $\hat{\alpha}$-values $(\hat{\alpha}_1, \hat{\alpha}_2, ..., \hat{\alpha}_n)$ is $V = \Omega(1 + SR_m^2)/T$, where $\Omega$ is the variance-covariance matrix of the residuals, $SR_m$ is the Sharpe ratio of the market and $T$ the (time) length of the sample. This holds also for a single asset as in (6.22). For monthly or longer return periods, autocorrelation is rarely a problem and a heteroskedasticity-robust $V$ is typically similar (for $\alpha$, not for $\beta$). The joint hypothesis that all alphas are zero can be tested with an $F$- or $\chi^2$-test. The latter has the test statistic $\hat{\alpha}' V^{-1} \hat{\alpha}$, which is distributed as a $\chi_n^2$ variable under the null hypothesis.*

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | −3.39 | 0.07 | 0.57 | 2.34 | 2.60 |
| 2 | −2.24 | 0.72 | 1.70 | 2.50 | 2.02 |
| 3 | −2.00 | 1.58 | 1.36 | 2.47 | 2.40 |
| 4 | −0.65 | 0.56 | 1.47 | 2.08 | 1.64 |
| 5 | 0.13 | 1.22 | 1.34 | 0.09 | 0.96 |

Table 6.2: t-stats for $\alpha$ in CAPM, 25 FF portfolios 1970:01-2023:12. NW uses 1 lag. The Bonferroni adjusted 10% and 5% critical values are 2.88 and 3.09.

A quite different approach to study a cross-section of assets is to first perform a CAPM regression and then the following cross-sectional regression

$$\bar{R}_i^e = \gamma + \lambda \hat{\beta}_i + u_i, \tag{6.24}$$

where $\bar{R}_i^e$ is the (sample) average excess return on asset $i$. Notice that the estimated betas are used as regressors and that there are as many data points as there are assets ($n$).

There are severe econometric problems with this regression equation since the regressor contains measurement errors (it is only an estimate), which typically tend to bias the slope coefficient towards zero. To get the intuition for this bias, consider an extremely noisy measurement of the regressor: it would be virtually uncorrelated with the dependent variable (noise isn't correlated with anything), so the estimated slope coefficient would be close to zero.

If we could overcome this bias (and we can by being careful), then the testable implications of CAPM is that $\gamma = 0$ and that $\lambda$ equals the average market excess return. We also want (6.24) to have a high $R^2$—since it should be unity in a very large sample (if CAPM holds).

### 6.2.3 Representative Results of the CAPM Test

One of the more interesting studies is Fama and French (1993) (see also Fama and French (1996)). They construct 25 stock portfolios according to two characteristics of the firm: the size (by market capitalization) and the book-value-to-market-value ratio (BE/ME).

They run a traditional CAPM regression on each of the 25 portfolios (monthly data 1963–1991)—and then study if the expected excess returns are related to the betas as they should according to CAPM. However, it is found that there is almost no relation between $\bar{R}_i^e$ and $\beta_i$ (there is a cloud in the $\beta_i \times \bar{R}_i^e$ space). This is due to the combination of two features of the data. First, *within a BE/ME quintile*, there is a positive relation (across size quantiles) between $\bar{R}_i^e$ and $\beta_i$—as predicted by CAPM. Second, *within a size quintile* there is a negative relation (across BE/ME quantiles) between $\hat{R}_i^e$ and $\beta_i$—in stark contrast to CAPM. See Figures 6.5–6.6 for an analysis on more recent data.

## 6.3 Appendix – Asset Value as Discounted Cash Flow*

### 6.3.1 Fundamental Asset Value

A *present value* is a sum of discounted future cash flows. A higher discount rate and longer time until the cash flow reduces the present value.

**Remark 6.8** *If the cash flow is $-150$ in $t$, 100 in $t+1$ and 130 in $t+2$, and the discount rate $R = 0.1$ then*

$$-150 + \frac{100}{1+R} + \frac{130}{(1+R)^2} \approx 48.3 \textit{ for } R = 0.1.$$

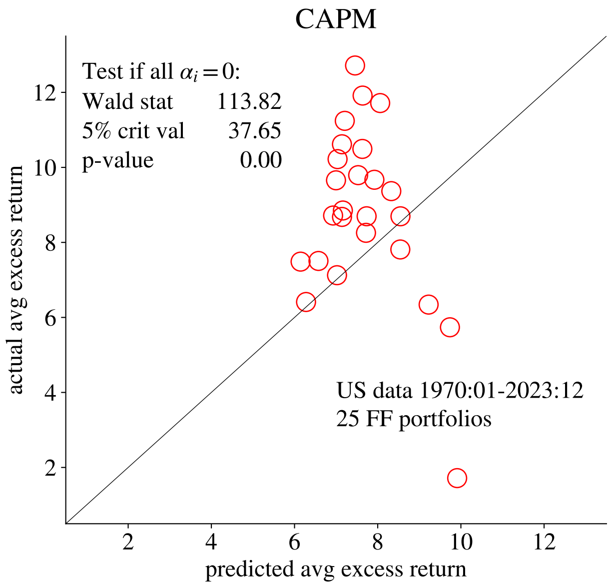



Figure 6.5: CAPM, FF portfolios

Many assets are long-lived. A fundamental valuation of the asset is that its (fair) price equals the present value of future cash flow

$$P_t = \frac{\mathrm{E}_t\, D_{t+1}}{1+R} + \frac{\mathrm{E}_t\, D_{t+2}}{(1+R)^2} + \frac{\mathrm{E}_t\, D_{t+3}}{(1+R)^3} + \dots$$
$$= \textstyle\sum_{s=1}^{\infty} \frac{\mathrm{E}_t\, D_{t+s}}{(1+R)^s}, \tag{6.25}$$

where $D_{t+s}$ are the future cash flows to the investor. In this expression subscripts refer to time periods and the fact that this refers to a particular asset ($i$) is assumed to be implicitly understood.

For shares the cash flows are the dividend payments, while for bonds they are the coupon (and face value) payments. In this section, the discount rate $R$ is given (and here assumed to be constant). In general, the discount rate depends on both the risk-free rate and the risk of the asset. (This is one of the main topics of the rest of these notes, see, for instance, the discussion of CAPM.) In project evaluation, the discount rate is often a weighted average (“WACC”) of the required return on equity and the after tax borrowing rate.

**Remark 6.9** *(What if the company cancels dividends in order to invest more?*) Suppose the investment project generates an annual return of ROE—and all earnings are paid out*

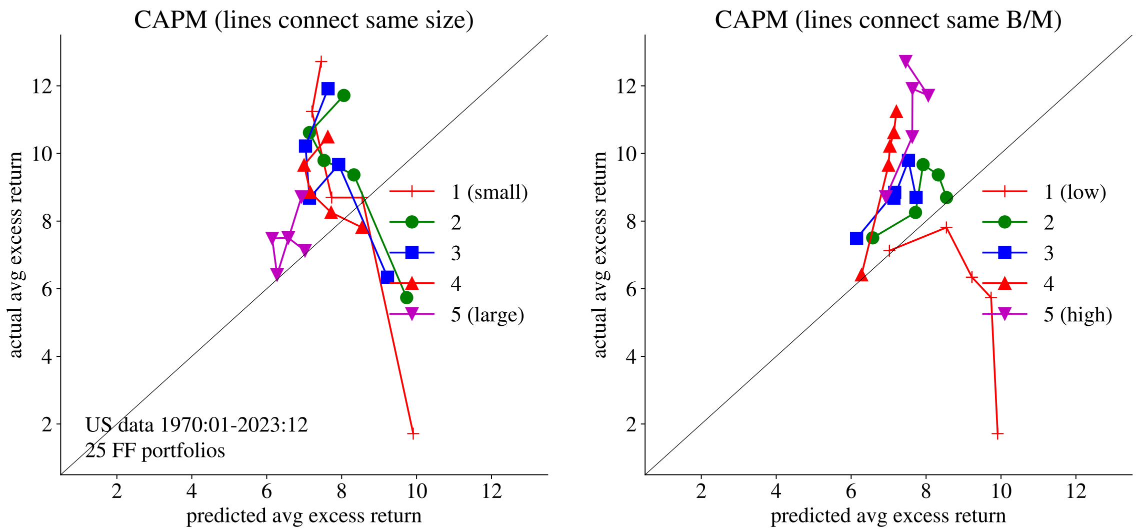



Figure 6.6: CAPM, FF portfolios

*in period 3:*

$$\textit{Old plan: } P_0 = \frac{\mathrm{E}_0 D_1}{1+R} + \frac{\mathrm{E}_0 D_2}{(1+R)^2} + \frac{\mathrm{E}_0 D_3}{(1+R)^3} + \ldots$$

$$\textit{New plan: } \tilde{P}_0 = \frac{\mathbf{0}}{1+R} + \frac{\mathrm{E}_0 D_2}{(1+R)^2} + \frac{\mathrm{E}_0 D_3 + \mathrm{E}_0 D_1(1+ROE)^2}{(1+R)^3} + \ldots$$

*Same value* ($\tilde{P}_0 = P_0$) *if* $ROE = R$.

In general, *dividends* reduce the stock price on the ex-dividend day (when the next dividend belongs to the seller, rather than the buyer of the stock) by an amount equal to the dividend. In contrast, a *stock repurchase* does not directly affect the stock price, but clearly reduces the number of outstanding (floating) shares. Both methods (if of same size) are likely to reduce the market value of the firm with the same amount.

**Remark 6.10** *(Dividends and stock repurchases*) Suppose the total value of a firm is 100, of which 90 is the present value of future earnings and 10 is cash. With 10 outstanding shares, the share price is 10* (100/10)*. If the firm distributes the cash as dividends, then the remaining total value of the firm is 90 so the share price is now 9. Overall the share holders have this period (assuming no other news) received a zero return (dividend yield + capital gain). Instead, if the firm buys back one share at the price of 10, then the total firm value becomes 90 and there are now 9 outstanding shares, so the share price* *would be unchanged at 10. Again, the return is zero. (Taxes and behavioural aspects can complicate this story.)*

**Remark 6.11** *(Valuation in terms of earnings instead of dividends**) Earnings can be spent on dividends or kept on the balance sheet as cash or some other asset (an "investment"): $E = D + I$. The firm value is*

$$P_0 = \frac{\overbrace{\mathrm{E}_0 \, D_1}^{E_1 - I_1}}{1+R} + \frac{\overbrace{\mathrm{E}_0 \, D_2}^{E_2 - I_2}}{(1+R)^2} + \frac{\overbrace{\mathrm{E}_0 \, D_3}^{E_3 - I_3}}{(1+R)^3} + \ldots$$

*This shows that the firm value equals the present value of future earnings minus the present value of new investment expenditures used to generate those earnings.*

**Remark 6.12** *(From income to cash flow**) To calculate the cash flow start with the net income (profit) before interests and taxes (EBIT) from the income statement, subtract the taxes (they are costs...), add back the depreciations (it is just an accounting item), subtract the capital expenditure (buying machines takes cash, even if it is not booked as a cost) and also subtract the change in the net working capital (current assets minus current liabilities, booked as income but you have not received it yet). All financial transactions are disregarded, so the cash flow must be used to pay all bond and equity holders.*

**Remark 6.13** *(Internal Rate of Return) The IRR is the $R$ that makes the net present value of a cash flow process zero. For instance, if the cash flow is $-150$ in $t$ (an investment), 100 in $t+1$ and 130 in $t+2$, then*

$$-150 + \frac{100}{1+R} + \frac{130}{(1+R)^2} \approx 0 \text{ for } R = 0.32.$$

*Typically we have to solve for the IRR by numerical methods. Notice that there may be more than one IRR if the cash flow process changes sign more than once.*

### 6.3.2 "Speculative" Valuation

An alternative view of the asset value is the present of the next dividend plus what you expect to resell the asset for

$$P_t = \frac{\mathrm{E}_t \, D_{t+1} + \mathrm{E}_t \, P_{t+1}}{1+R}. \tag{6.26}$$

This is the same as the fundamental valuation (6.25) *if* you expect to resell it at your (expected next period) fundamental valuation. Otherwise not.

*Proof* of fundamental = speculative asset value, if $\mathrm{E}_t\, P_{t+1}$ follows fundamental valuation. Use (6.25) to write

$$P_{t+1} = \frac{\mathrm{E}_{t+1}\, D_{t+2}}{1+R} + \frac{\mathrm{E}_{t+1}\, D_{t+3}}{(1+R)^2} + \ldots.$$

Take expectations as of period $t$ and use in (6.26)

$$P_t = \frac{\mathrm{E}_t\, D_{t+1}}{1+R} + \frac{\mathrm{E}_t\, \mathrm{E}_{t+1}\, D_{t+2}}{(1+R)^2} + \frac{\mathrm{E}_t\, \mathrm{E}_{t+1}\, D_{t+3}}{(1+R)^3} + \ldots..$$

Recall that $\mathrm{E}_t(\mathrm{E}_{t+1}\, D_{t+s})) = \mathrm{E}_t\, D_{t+s}$ (the "law of iterated expectations.") to complete the proof. □

**Remark 6.14** *(Law of iterated expectations) The law of iterated expectations implies that*

$$\mathrm{E}_t(\mathrm{E}_{t+1}\, y_{t+2}) = \mathrm{E}_t\, y_{t+2}$$

*To see why, let $y_{t+2} = \mathrm{E}_{t+1}\, y_{t+2} + \varepsilon_{t+2}$, so $\varepsilon_{t+2}$ is a surprise in $t+2$. The equation above can then be written*

$$\mathrm{E}_t(y_{t+2} - \varepsilon_{t+2}) = \mathrm{E}_t\, y_{t+2},$$

*which holds if $\mathrm{E}_t\, \varepsilon_{t+2} = 0$. That is, the surprise in $t+2$ cannot be predicted by any information in period $t$. Basically, this is the same as saying that we know more, not less, as time goes by.*

### 6.3.3 Fundamental Valuation and Returns

The return from holding the asset from $t$ to $t+1$ is

$$R_{t+1} = \frac{D_{t+1} + P_{t+1}}{P_t} - 1. \tag{6.27}$$

If the discount rate in (6.25) is constant over time, then it equals the expected return

$$\mathrm{E}_t\, R_{t+1} = R. \tag{6.28}$$

It follows that if there is no news between $t$ and $t+1$ (so expectations are unchanged, $\mathrm{E}_t\, D_{t+s} = \mathrm{E}_{t+1}\, D_{t+s}$), then

$$R_{t+1} = R \text{ (if no news)}. \tag{6.29}$$

Notice that this return does *not* depend on the level or growth rate of the dividends. Old information is in $P_t$, and does not affect $R_{t+1}$.

*Proof* of (6.29)–(6.28). Use (6.25) to write

$$P_{t+1} = \frac{\mathrm{E}_{t+1} D_{t+2}}{1+R} + \frac{\mathrm{E}_{t+1} D_{t+3}}{(1+R)^2} + \ldots.$$

Use in the realized return (6.27) and take expectations as of $t$ to get (using $\mathrm{E}_t\, \mathrm{E}_{t+1}\, D_{t+s} = \mathrm{E}_t\, D_{t+s}$)

$$\mathrm{E}_t\, R_{t+1} = \frac{\mathrm{E}_t\, D_{t+1} + \frac{\mathrm{E}_t D_{t+2}}{1+R} + \frac{\mathrm{E}_t D_{t+3}}{(1+R)^2} + \ldots}{\frac{\mathrm{E}_t D_{t+1}}{1+R} + \frac{\mathrm{E}_t D_{t+2}}{(1+R)^2} + \frac{\mathrm{E}_t D_{t+3}}{(1+R)^3} + \ldots} - 1 = R.$$

In addition, if expectations are unchanged, then $R_{t+1} = \mathrm{E}_t\, R_{t+1}$. (This can also be proved directly by substituting for $P_{t+1}$ in (6.27).) $\square$

### 6.3.4 Asset Price with constant Cash Flow Growth

With *constant dividend growth forever* (growing perpetuity), $\mathrm{E}_t\, D_{t+s+1} = (1+g)\, \mathrm{E}_t\, D_{t+s}$, so (6.25) becomes

$$P_t = \mathrm{E}_t\, D_{t+1} \textstyle\sum_{s=1}^{\infty} \frac{(1+g)^{s-1}}{(1+R)^s} = \frac{\mathrm{E}_t\, D_{t+1}}{R-g}. \tag{6.30}$$

This is the "Gordon model." The asset price (6.30) is high when: *(a)* dividends are expected to be high; *(b)* the growth rate ($g$) is believed to be high; and *(c)* when discounting ($R$) is low.

Inverting this formula to get the discount rate ("cost of equity capital")

$$R = \frac{\mathrm{E}_t\, D_{t+1}}{P_t} + g. \tag{6.31}$$

**Example 6.15** *(Asset price as sum of discounted cash flows) With $D_1 = 100$, $R = 0.1$ and $g = 2\%$,*

$$P_0 = 100/(0.1 - 0.02) = 1250$$

*Proof* of (6.30) Write the first equality of (6.30) as $P_t = \frac{\mathrm{E}_t D_{t+1}}{1+R} \sum_{s=0}^{\infty} (\frac{1+g}{1+R})^s$. Recall the fact that for a geometric series, $\sum_{s=0}^{\infty} r^s = 1/(1-r)$ if $|r| < 1$. Apply this on $r = (1+g)/(1+R)$, to get that

$$P_t = \frac{\mathrm{E}_t\, D_{t+1}}{1+R} \frac{1}{1-(1+g)/(1+R)} = \frac{\mathrm{E}_t\, D_{t+1}}{R-g}.$$

$\square$

### 6.3.5 Valuation Multiples

The *price-earnings ratio* (p/e) is

$$\text{“}p/e\text{”} = \frac{P}{e}, \tag{6.32}$$

where $e$ is short for earnings per share.

If dividends are proportional to earnings, $D_t = k \times e_t$ in each period and earnings grow at the rate $g$, $e_{t+1} = (1+g)e_t$, then

$$(p/e)_t = \frac{P_t}{e_t} = \frac{\overbrace{ke_{t+1}}^{D_{t+1}}/(R-g)}{e_t} = k\frac{1+g}{R-g},$$

where notation $(i)$ for the firm is suppressed.

**Example 6.16** *$R = 0.1$, $g = 2\%$ and $k = 1$ (a "cash cow")*

$$p/e = 1 \times \frac{1.02}{0.1 - 0.02} = 12.75$$

*Instead, with $g = 5\%$ we have $p/e = 21$. This shows that $p/e$ is very sensitive to assumptions about the growth rate.*

The *multiples approach* is to use a comparison with a peer group (in the market or recent M&A transactions) in order price an asset (here denoted $i$). It has the advantage that we do not need to specify growth or discount rate. The *equity value method* is calculate the share value of company $i$ as

$$P_{i,t} = \left(\frac{P_t}{e_t}\right)_{peers} \times e_{i,t}, \text{ so } \frac{P_{i,t}}{e_{i,t}} = \left(\frac{P_t}{e_t}\right)_{peers}. \tag{6.33}$$

As alternatives to $e$, use cash flow and book value. In general, this approach makes sense if firm $i$ and the peers have similar growth and risk, while the dividends might differ.

**Remark 6.17** *(The discounted cash flow model vs. the multiples approach) To simplify, assume $D = e$ and assume constant growth. This means that $P = (1+g)e/(R-g)$ for both $i$ and peers. To have $P_i/e_i = (P/e)_{peers}$ as in (6.33), the following must hold*

$$\frac{P_i}{e_i} = \left(\frac{1+g}{R-g}\right)_i = \left(\frac{1+g}{R-g}\right)_{peers} = \left(\frac{P}{e}\right)_{peers}.$$

*This shows that the discount and growth rates must be similar.*

# Chapter 7

# Downside Risk Measures

The mean-variance framework is often criticized for failing to distinguish between the downside of the return distribution (considered to represent risk) and upside (considered to represent potential). This chapter introduces several commonly used downside risk measures and also explores more general methods to describe the return distribution.

## 7.1 Value at Risk

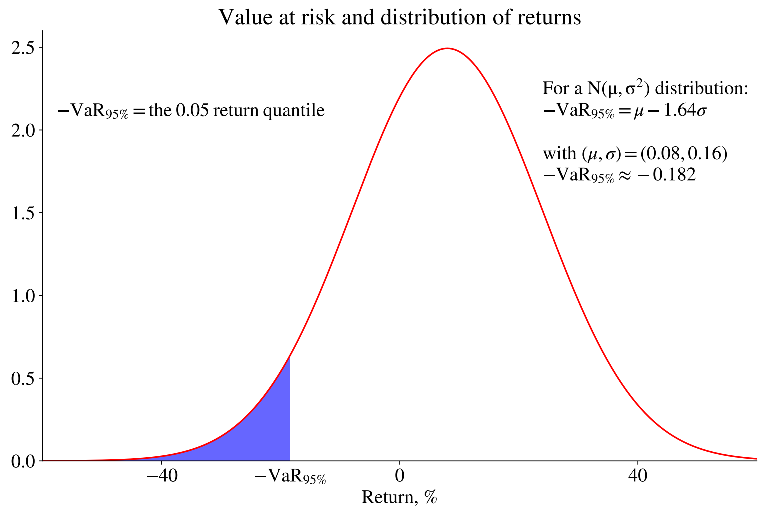



Figure 7.1: Value at risk

The Value at Risk (VaR) measures the downside risk by focusing on a quantile of the return (or loss) distribution,

**Remark 7.1** *(Quantile of a distribution) The 0.05 quantile is the value $x$ such that there is only a 5% probability of a lower number,* $\Pr(R \leq x) = 0.05$.

The 95% Value at Risk ($\text{VaR}_{95\%}$) is a number such that there is only a 5% chance that the loss as a fraction of the investment (which is the negative of the return, $-R$) is larger

$$\Pr(-R \geq \text{VaR}_{95\%}) = 5\%. \tag{7.1}$$

Here, 95% is the confidence level of the VaR. For instance, with $\text{VaR}_{95\%} = 18\%$, then we are 95% sure that we will not lose more than 18% of our investment.

To work with the return distribution, not the loss distribution, we notice that (7.1) is the same as

$$\Pr(R \leq -\text{VaR}_{95\%}) = 5\%, \tag{7.2}$$

so $-\text{VaR}_{95\%}$ is the 0.05 *quantile* of the return distribution. More generally, the VaR for confidence level $\alpha$ (instead of 0.95) is

$$\text{VaR}_\alpha = -(\text{the } 1-\alpha \text{ quantile of the } R \text{ distribution}). \tag{7.3}$$

If the return is *normally distributed*, $R \sim N(\mu, \sigma^2)$, then

$$\text{VaR}_\alpha = -(\mu + c\sigma), \tag{7.4}$$

where $c$ is the $1-\alpha$ quantile of a $N(0,1)$ distribution. For instance, $c$ is approximately $(-1.64, -1.96, -2.33)$ for the $(0.05, 0.025, 0.01)$ levels, respectively. See Figures 7.1–7.2. Since $c < 0$, the VaR is here strictly increasing the the standard deviation, which will later be important when we consider portfolio choice based on a VaR.

**Example 7.2** *(VaR and regulation of bank capital) Bank regulations have used 3 times the 99% VaR for 10-day returns as the required bank capital.*

Note that the return distribution depends on the *investment horizon*; therefore, the VaR is typically calculated for a particular return period (for instance, one day). Multi-period VaRs are calculated by either explicitly constructing the distribution of multi-period returns, or by making simplifying assumptions about the relation between returns in different periods (for instance, that they are uncorrelated). If the returns are iid, then a $q$-period return has the mean $q\mu$ and variance $q\sigma^2$, where $\mu$ and $\sigma^2$ are the mean and variance of the one-period returns respectively. If the mean is zero, then the $q$-day VaR is $\sqrt{q}$ times the one-day VaR.

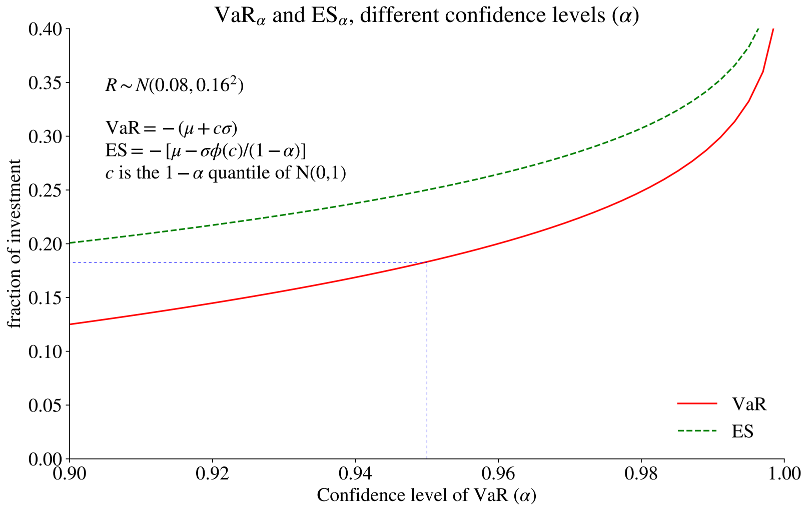



Figure 7.2: Value at risk, different probability levels

**Example 7.3** *(The London whale) The broad outline of the "London whale" (JPM) story is as follows: at the end of 2011, top management instructed the division to bring down the RWA (risk weighted asset) exposure to (various) credit derivatives. However, that would (a) have caused high execution costs and (b) the portfolio had recently performed well, so the division invented a new VaR method and pushed it through the Risk Office without the usual parallel testing. They went on to triple the positions (and lose \$719 million in 2012Q1). Interestingly, the two VaR models show divergent paths for the value at risk.*

### 7.1.1 Backtesting a VaR model

Backtesting a VaR model amounts to comparing how well the VaR model can describe the 5% quantile (say) of the ex post data. This is done by determining whether the returns fall below the VaR$_{95\%}$ approximately 5% of the times in the sample. This could be a long sample period or a set of subsamples. In particular, a model with extended periods of under- and then over-performance which average (in the full sample) to roughly 5% is unlikely to be a useful model.

**Empirical Example 7.4** *Figure 7.3 shows the distribution and VaRs (for different probability levels) for the daily S&P 500 returns for a long sample. The results indicate that the $N()$-based model has a reasonable coverage for the 95%, but perhaps not for the 99% confidence level.*

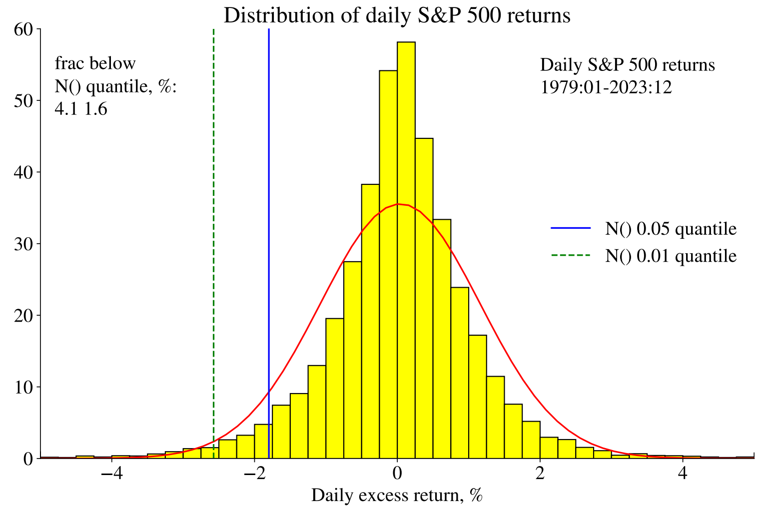



Figure 7.3: Return distribution and VaR for S&P 500

**Empirical Example 7.5** *Figure 7.4 shows backtesting on many subsamples. The results indicate that a static VaR model for S&P 500 has long cycles of under- and then over-performance.*

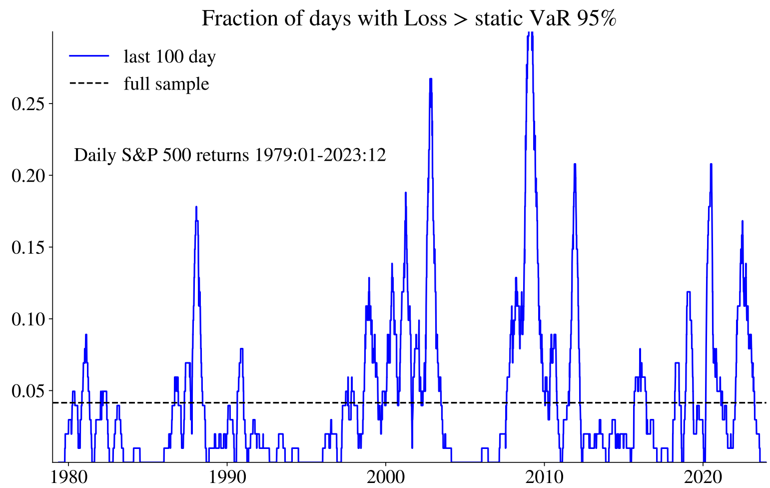



Figure 7.4: Backtesting a static VaR model on a moving data window

### 7.1.2 A Simple Dynamic VaR

It is well known that financial *volatility changes over time*, which needs taking account of in a reliable VaR model. One particularly simple approach is to estimate means and variances using the recursive formulas (cf. the RiskMetrics of JP Morgan (1996))

$$\mu_t = \lambda \mu_{t-1} + (1-\lambda) R_{t-1} \tag{7.5}$$

$$\sigma_t^2 = \lambda \sigma_{t-1}^2 + (1-\lambda)(R_{t-1} - \mu_{t-1})^2, \tag{7.6}$$

where $0 < \lambda < 1$ and often high (around $0.90 - 0.95$ for daily data). The estimate of the mean is an update of yesterday's estimate, using yesterday's return for the update. This is the same as a weighted average of past returns (actually, an exponentially weighted moving average), with recent data having higher weights than old data.

The variance is a similar, with updating using the square of yesterday's surprise.

**Empirical Example 7.6** *Figure 7.5 illustrates the VaR calculated from a time series model for daily S&P returns. In this case, the VaR changes from day to day as both the mean return (the forecast) as well as the standard error (of the forecast error) do. Figures 7.5–7.6 show results from backtesting a VaR model which assumes that one-day returns are normally distributed, but where the volatility is time varying. Clearly, this means that the VaR is also time varying: use (7.4) but allow $\sigma$ (and less importantly, $\mu$) to change from day to day. The evidence suggests that this model works relatively well at the 95% confidence level and that it is important to account for the time-varying volatility (or else there will be prolonged periods when the VaR performs poorly).*

### 7.1.3 Value at Risk of a Portfolio

The general way of calculating the VaR of a portfolio is the same as for an individual asset: first calculate (or estimate) the parameters of the distribution, then find the quantile.

However, in some special cases, it is possible to directly translate the VaR values of the individual assets into a portfolio VaR.

**Remark 7.7** *Suppose the assets in the portfolio are jointly normally distributed with zero means, so the VaR of asset $i$ is $VaR_i = 1.64\sigma_i$. (The index on VaR here indicates the asset, not a confidence level.) Let $\upsilon$ be a vector where $\upsilon_i = w_i VaR_i$, where $w_i$ is the portfolio weight. Then, $VaR_p = [\upsilon' \operatorname{Corr}(R)\upsilon]^{1/2}$, where* Corr$(R)$ *is the correlation matrix of the*

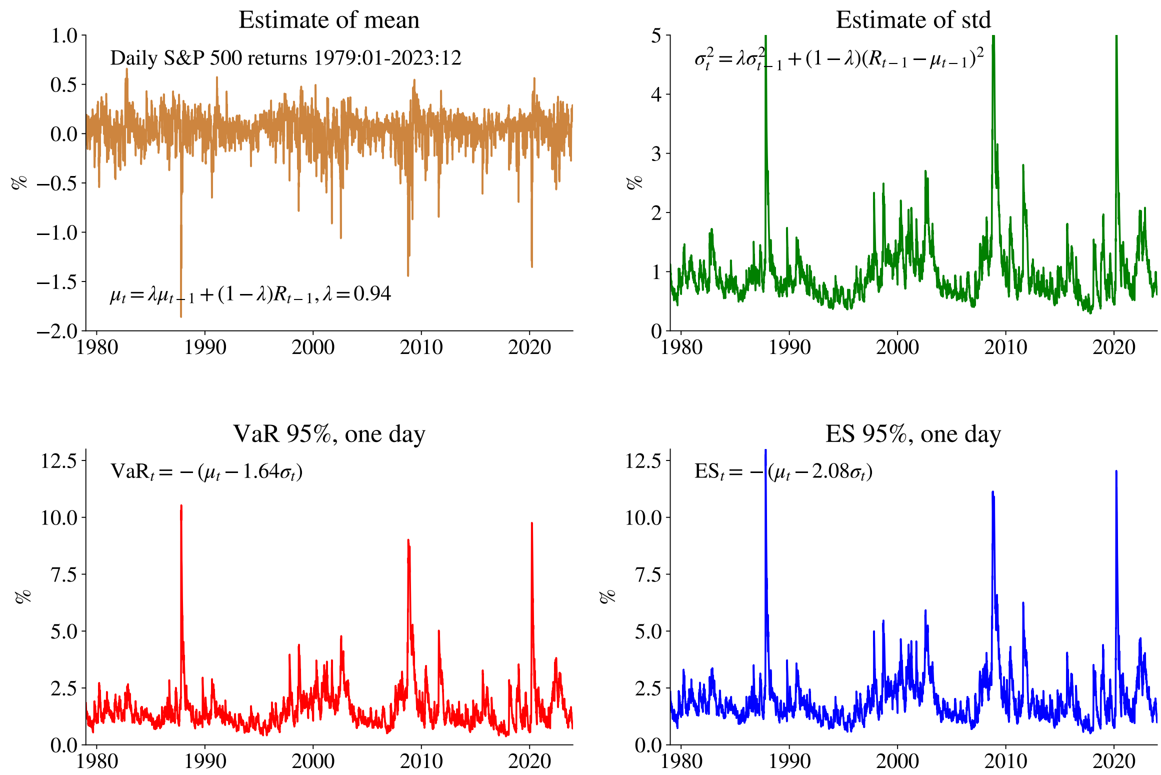



Figure 7.5: A dynamic VaR model

*assets. (To see this, recall that VaR$_p$ = $1.64\sigma_p$ and that we can calculate $\sigma_p$ from the $\sigma_i$ values and correlations.)*

### 7.1.4 Index Models for Calculating the Value at Risk

Consider a multi-index model

$$R_t = a + b' I_t + e_t, \tag{7.7}$$

where $b$ is a $k \times 1$ vector of the $b_i$ coefficients and $I$ is a $k \times 1$ vector of the $I_i$ indices. As usual, we assume $\mathrm{E}\, e_t = 0$ and $\mathrm{Cov}\,(e_t, I_{it}) = 0$. This model can be used to generate the inputs to a VaR model. For instance, the mean and standard deviation of the return are

$$\begin{aligned} \mu &= a + b' \mathrm{E}\, I_t \\ \sigma &= \sqrt{b' \mathrm{Cov}(I_t) b + \mathrm{Var}(e_t)}, \end{aligned} \tag{7.8}$$

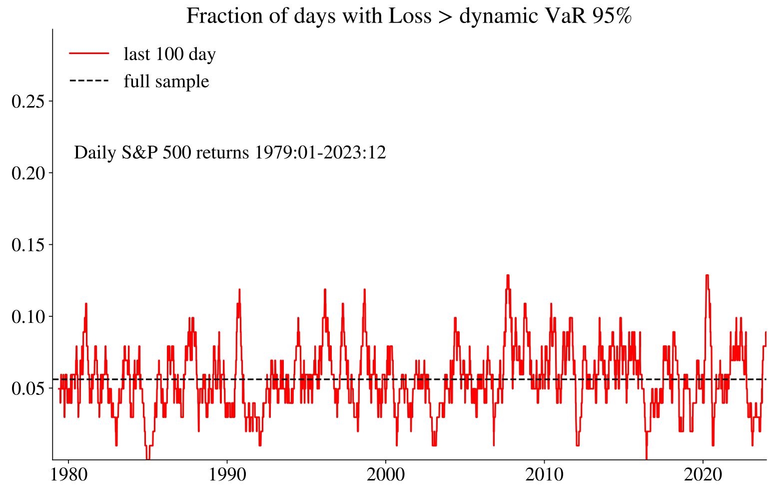



Figure 7.6: Backtesting a dynamic VaR model on a moving data window

which can be used in (7.4), that is, an assumption of a normal return distribution. If the return is of a well diversified portfolio and the indices include the key market indices, then the idiosyncratic risk Var$(e)$ is close to zero. The RiskMetrics approach is to make this assumption.

A *stand-alone VaR* assesses the contribution of different factors (indices) on the overall VaR. For instance, the indices in (7.7) could include: an equity indices, interest rates, exchange rates and perhaps also a few commodity indices. Then, an *equity VaR* is calculated by setting all elements in $b$, except those for the equity indices, to zero. Often, the intercept, $a$, is also set to zero. Similarly, an *interest rate VaR* is calculated by setting all elements in $b$, except referring to the interest rates, to zero. And so forth for an *FX VaR* and a *commodity VaR*. Clearly, these different VaRs do not add up to the total VaR, but they still give an indication of where the main risk comes from.

If an asset or a portfolio is a non-linear function of the indices, then (7.7) can be thought of as a first-order Taylor approximation where $b_i$ represents the partial derivative of the asset return with respect to index $i$. For instance, an option is a non-linear function of the underlying asset value and its volatility. This approach, when combined with the normal assumption in (7.4), is called the *delta-normal method*.

## 7.2 Expected Shortfall

While the value at risk is a useful risk measure, it has the strange property that it considers whether an outcome is in the tail of the return distribution, not how far out.

In addition, the VaR concept has been criticized for having poor aggregation properties. In particular, the VaR for a portfolio is not necessarily (weakly) lower than the portfolio of the VaRs even if the assets all have the same volatility, which contradicts the notion of diversification benefits. (To get this unfortunate property, the return distributions must be heavily skewed.) See McNeil, Frey, and Embrechts (2005) and Alexander (2008) for more detailed discussions.

The expected shortfall (ES, also called conditional VaR, average value at risk and expected tail loss) has better properties. It is the expected loss when the return actually is below the $\text{VaR}_\alpha$, that is,

$$\text{ES}_\alpha = -\,\text{E}(R|R \leq -\text{VaR}_\alpha). \tag{7.9}$$

See Figures 7.7 for an illustration.

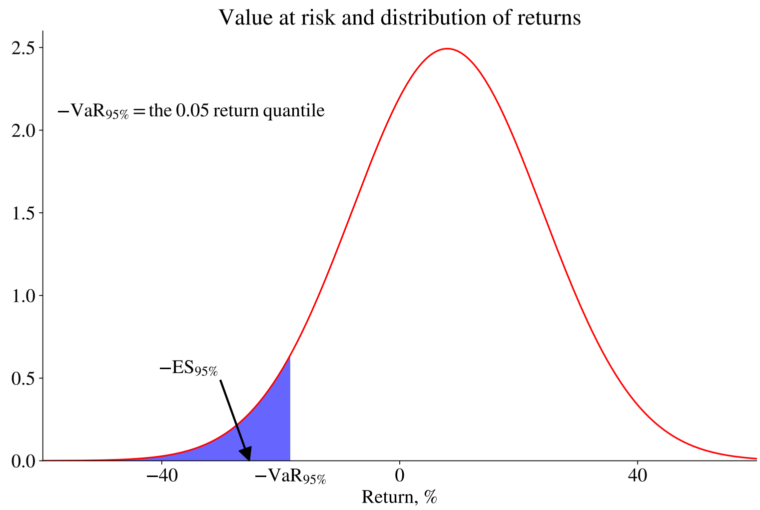



Figure 7.7: Value at risk and expected shortfall

**Empirical Example 7.8** *See Figure 7.5 for an empirical estimate of ES, based on the dynamic estimates of the mean and variance in (7.5)–(7.6).*

**Empirical Example 7.9** *See Table 7.1 for an empirical comparison of the VaR, ES and some alternative downside risk measures (discussed below) for two stock indices.*

| | Small growth | Large value |
|---|---|---|
| Std | 8.1 | 5.9 |
| VaR (95%) | 12.8 | 8.9 |
| ES (95%) | 17.9 | 13.1 |
| SemiStd | 5.7 | 3.9 |
| Drawdown | 78.4 | 63.2 |

Table 7.1: Risk measures of monthly returns of two stock indices (%), US data 1970:01-2023:12.

For a normally distributed return $R \sim N(\mu, \sigma^2)$ we have

$$\mathrm{ES}_\alpha = -\left(\mu - \frac{\phi(c)}{1-\alpha}\sigma\right), \tag{7.10}$$

where $\phi()$ is the pdf of a $N(0,1)$ variable and $c$ is the $1-\alpha$ quantile of a $N(0,1)$ distribution. In this case, the ES is strictly increasing in the standard deviation, which will later be important when we consider portfolio choice. See Figure 7.2 for an illustration.

**Example 7.10** *(ES) If $\mu = 8\%$ and $\sigma = 16\%$, the 95% expected shortfall is $ES_{95\%} = -(0.08 - 2.08 \times 0.16) \approx 0.25$ (since $\phi(-1.64)/0.05 \approx 2.08$) and the 97.5% expected shortfall is $ES_{97.5\%} = -(0.08 - 2.34 \times 0.16) \approx 0.29$ (since $\phi(-1.96)/0.025 \approx 2.34$)*

*Proof* of (7.10). If $x \sim N(\mu, \sigma^2)$, then it is well known that $\mathrm{E}(x|x \leq b) = \mu - \sigma\phi(b_0)/\Phi(b_0)$ where $b_0 = (b-\mu)/\sigma$ and where $\phi()$ and $\Phi()$ are the pdf and cdf of a $N(0,1)$ variable respectively. To apply this, use $b = -\mathrm{VaR}_\alpha = \mu + c\sigma$ so $b_0 = c$. Clearly, $\Phi(c) = 1-\alpha$, so $\mathrm{E}(R|R \leq -\mathrm{VaR}_\alpha) = \mu - \sigma\phi(c)/(1-\alpha)$. Multiply by $-1$. $\square$

To estimate the average shortfall from a sample, calculate the average $-R_t$ for observations where $R_t \leq -\mathrm{VaR}_\alpha$

$$\mathrm{ES}_\alpha = \frac{-1}{\sum_{t=1}^{T}\delta_t}\sum\nolimits_{t=1}^{T}\delta_t R_t, \text{ where } \delta_t = \begin{cases} 1 \text{ if } R_t \leq -\mathrm{VaR}_\alpha \\ 0 \text{ otherwise.} \end{cases} \tag{7.11}$$

This can be used in backtesting an ES model.

**Empirical Example 7.11** *See Figure 7.8 for a back testing of the dynamic ES model previously shown in Figure 7.5.*

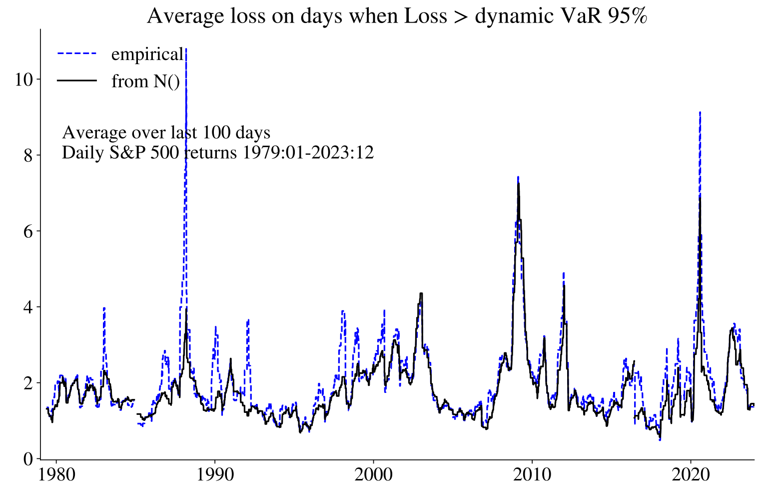



Figure 7.8: Backtesting a dynamic ES model on a moving data window

## 7.3 Target Semivariance

Reference: Bawa and Lindenberg (1977) and Nantell and Price (1979)

The target semivariance (also called the lower partial 2nd moment) is defined as

$$\lambda(h) = \mathrm{E}[\min(R - h, 0)^2], \tag{7.12}$$

where $h$ is a "target level" chosen by the investor. Also, $\sqrt{\lambda(h)}$ with $h = \mu$ is called the semi-standard deviation. In comparison with the variance, $\sigma^2 = \mathrm{E}(R - \mathrm{E}\, R)^2$, the target semivariance differs in two aspects: *(i)* it uses the target level $h$ as a reference point instead of the mean $\mu$: and *(ii)* only negative deviations from the reference point are given any weight.

For a normally distributed variable, the target semivariance $\lambda_p(h)$ is increasing in the standard deviation, see Remark 7.13, which will later be important when we consider portfolio choice. See also Figure 7.9 for an illustration.

To estimate the target semivariance from the empirical return distribution (for back-testing a model), use

$$\lambda(h) = \frac{1}{T} \sum\nolimits_{t=1}^{T} \min(R_t - h, 0)^2. \tag{7.13}$$

**Remark 7.12** *(*Alternative scaling of $\lambda(h)$) Some analysts define $\lambda(h)$ by just including those observations when $R_t \leq h$. This means multiplying $\lambda(h)$ in (7.13) by $T/$(number of periods when $R_t \leq h$). Conceptually, this is estimating* $\mathrm{E}[(R - h)^2 | R_t \leq h]$.

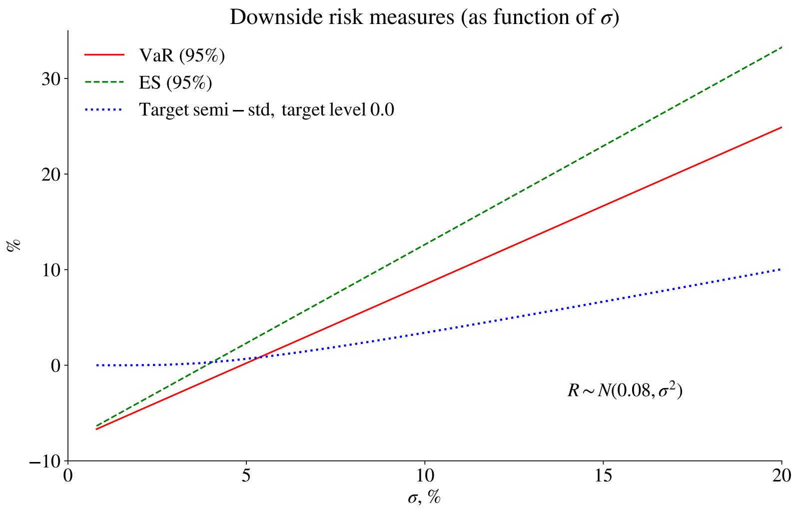



Figure 7.9: Downside risk measures as functions of the standard deviation for a N($\mu$,$\sigma^2$) variable

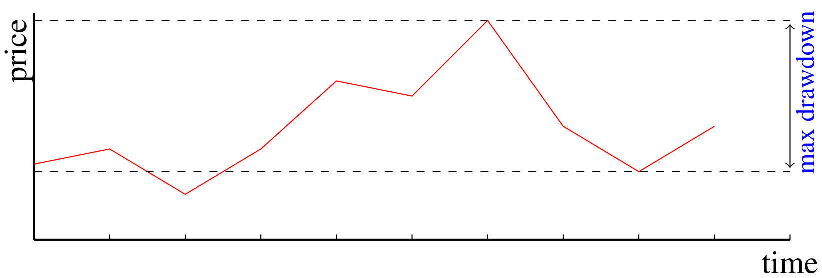



Figure 7.10: Max drawdown

**Remark 7.13** *(Target semivariance calculation for normally distributed variable*) For an $N(\mu,\sigma^2)$ variable, target semivariance around the target level $h$ is*

$$\lambda_p(h) = \sigma^2 a\phi(a) + \sigma^2(a^2+1)\Phi(a), \text{ where } a = (h-\mu)/\sigma,$$

*where $\phi()$ and $\Phi()$ are the pdf and cdf of a $N(0,1)$ variable, respectively. Notice that $\lambda_p(h) = \sigma^2/2$ for $h=\mu$. It is straightforward to show that*

$$\frac{d\lambda_p(h)}{d\sigma} = 2\sigma\Phi(a),$$

*so the target semivariance is a strictly increasing function of the standard deviation.*

**Remark 7.14** *(Sortino ratio) The Sortino ratio is an alternative to the Sharpe ratio (as a*

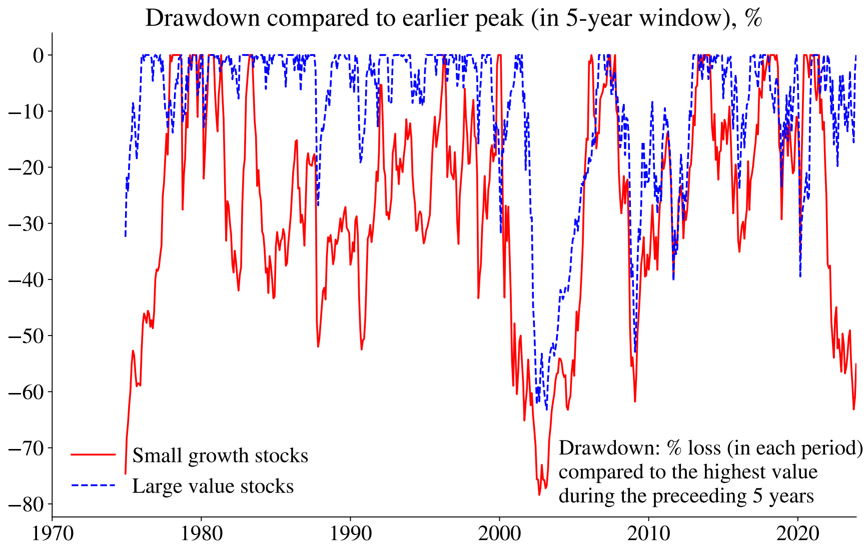



Figure 7.11: Drawdown

*measure of performance). It is* $(\mathrm{E}\, R - h)/\sqrt{\lambda(h)}$.

**Empirical Example 7.15** *See Table 7.2 for an empirical rank correlation of the different risk measures for 25 FF portfolios. Most of the risk measures have very rank correlations, meaning that they give very similar ranking of "riskiness" of these 25 assets. However, max drawdown is different, mostly likely since it is focused on the extreme left tail of the distribution.*

| | Std | VaR (95%) | ES (95%) | SemiStd | Drawdown |
|---|---|---|---|---|---|
| Std | 1.00 | 0.94 | 0.97 | 0.97 | 0.61 |
| VaR (95%) | 0.94 | 1.00 | 0.95 | 0.95 | 0.66 |
| ES (95%) | 0.97 | 0.95 | 1.00 | 0.97 | 0.64 |
| SemiStd | 0.97 | 0.95 | 0.97 | 1.00 | 0.60 |
| Drawdown | 0.61 | 0.66 | 0.64 | 0.60 | 1.00 |

Table 7.2: Correlation of rank of risk measures across the 25 FF portfolios (%), US data 1970:01-2023:12. The VaR and ES are based on the empirical return distribution. The max drawdown is calculated over a moving 5-year data window.

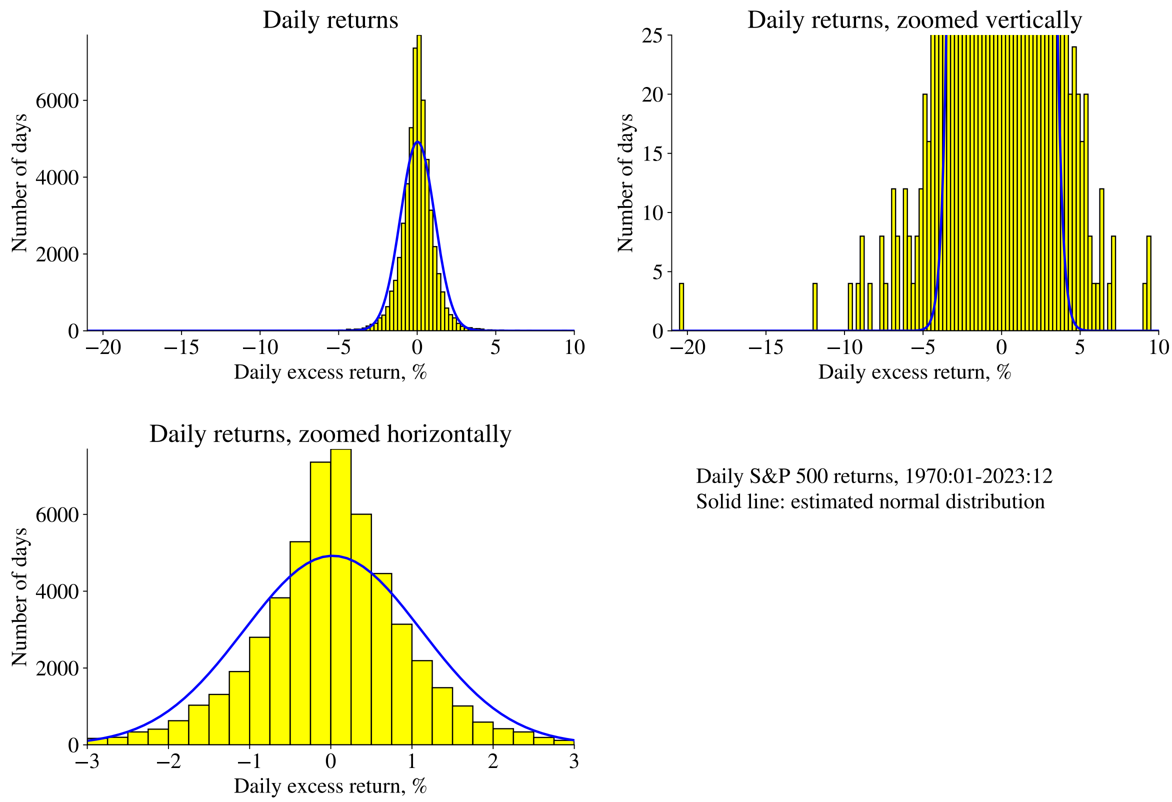



Figure 7.12: Distribution of daily S&P returns

## 7.4 Max Drawdown

An alternative measure is the (percentage) *maximum drawdown* over a given horizon, for instance, 5 years, say. This is the largest loss from peak to bottom within the given horizon–see Figure 7.10. This is a useful measure when the investor do not know exactly when he/she has to exit the investment—since it indicates the worst (peak to bottom) outcome over the sample.

**Empirical Example 7.16** *See Figure 7.11 for a comparison of the max drawdown of two return series. The results suggest that small growth stocks are considerably more risky than large value stocks.*

## 7.5 Empirical Return Distributions

Are returns normally distributed? Mostly not, but it depends on the asset type and on the data frequency. Options returns have very non-normal distributions (in particular, since

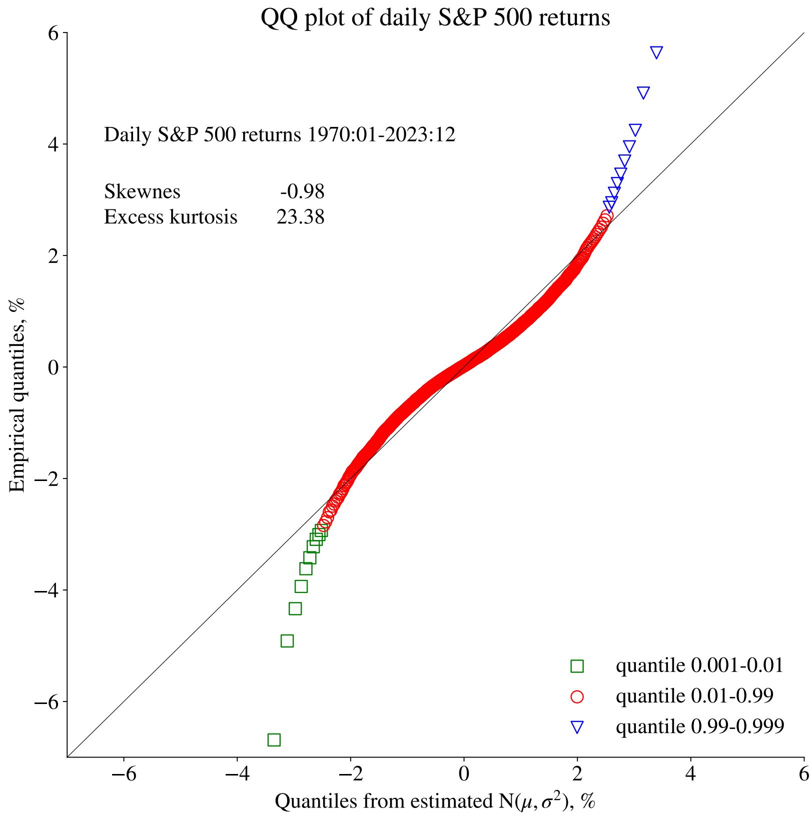



Figure 7.13: Quantiles of daily S&P returns

the return is $-100\%$ on many expiration days). Stock returns are typically non-normal at short horizons, but may appear approximately normal over longer horizons.

To assess the normality of returns, the usual econometric techniques (Bera–Jarque and Kolmogorov-Smirnov tests) are useful, but a visual inspection of the histogram and a QQ-plot also give useful clues.

**Remark 7.17** *(Reading a QQ plot) A QQ plot is a way to assess if the empirical distribution conforms reasonably well to a prespecified theoretical distribution, for instance, a normal distribution where the mean and variance have been estimated from the data. Each point in the QQ plot shows a specific percentile (quantile) according to the empirical as well as according to the theoretical distribution. For instance, if the 2th percentile (0.02 percentile) is at -10 in the empirical distribution, but at only -3 in the theoretical distribution, then this indicates that the two distributions have fairly different left tails.*

**Empirical Example 7.18** *See Figures 7.12–7.14 for empirical histograms and QQ-plots* *of S&P 500 returns. It is observed, among other findings, that empirical returns distributions exhibit more extreme negative returns than a normal distribution would suggest, and that the return distribution appears closer to a normal distribution as the return horizon increases.*

There is one caveat to this way of studying data: it only provides evidence on the unconditional distribution. For instance, nothing rules out the possibility that we could estimate a model for time-varying volatility (for instance, a GARCH model) of the returns and thus generate a description for how the VaR changes over time. However, data with time varying volatility will typically not have an unconditional normal distribution.

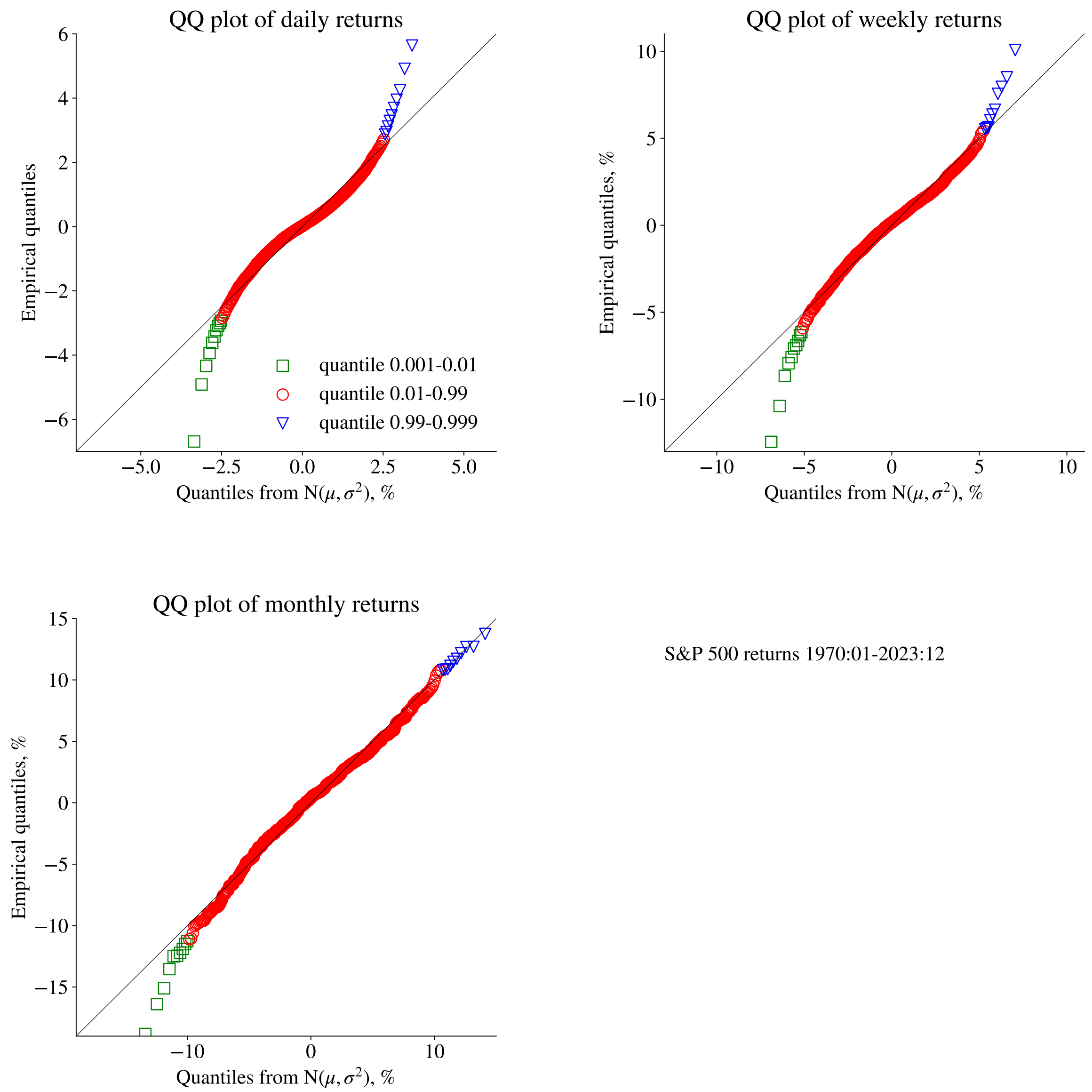



Figure 7.14: Distribution of S&P returns (different horizons)

# Chapter 8

# Utility-Based Portfolio Choice

## 8.1 Utility Functions and Risky Investments

Any model of portfolio choice must embody a notion of "what is best?" In finance, that often means a portfolio that strikes a good balance between expected return and its variance. However, in order to make sense of that idea—and to go beyond it—we must refer to basic economic utility theory.

### 8.1.1 Specification of Utility Functions

In finance, the key features of the utility functions that we use are as follows. Figure 8.1 gives an illustration. *First*, utility is a function of a scalar argument, $U(x)$. This argument $(x)$ can be end-of-period wealth, a consumption basket or the *real* (inflation adjusted) portfolio return. In one-period investment problems, this choice of $x$ is irrelevant since consumption equals wealth, which is proportional to the portfolio return.

*Second*, uncertainty is incorporated by letting investors maximize expected utility, $\mathrm{E}\,U(x)$. The reasons is that returns (and therefore wealth and consumption) are uncertain. Hence, we need a way to rank portfolios at the time of investment, before the uncertainty is resolved. For instance, if there are $S$ possible states with outcomes $x_1, x_2, ..., x_S$ and probabilities $\pi_1, \pi_2, ...\pi_S$, then expected utility is

$$\mathrm{E}\,U(x) = \textstyle\sum_{s=1}^{S} \pi_s U(x_s). \tag{8.1}$$

For instance, the outcomes could represent portfolio returns, which depend on the state *and* the portfolio weights. That is, $x_s$ is not a fixed list, but rather functions of the investor's choices.

**Example 8.1** *($\mathrm{E}\,U(W)$) Suppose there are two states of the world: $W$ (wealth) will be*

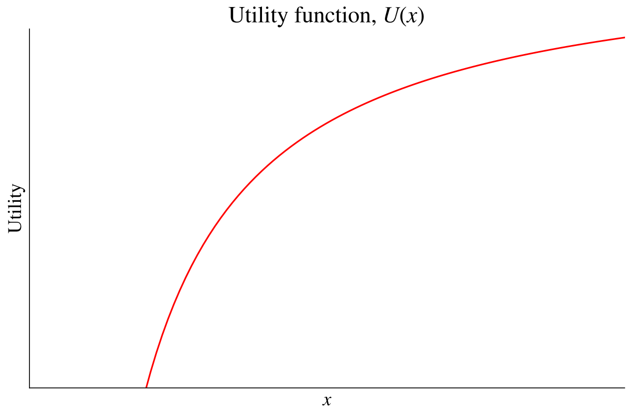



Figure 8.1: A utility function

*either* 0.85 *or* 1.15 *with probabilities* 1/3 *and* 2/3*. If* $U(W) = \ln W$*, then* $\mathrm{E}\,U(W) = 1/3 \times \ln 0.85 + 2/3 \times \ln 1.15 \approx 0.039$*. If the investor had picked another portfolio the outcomes would instead be* 0.9 *and* 1.05*, with expected utility* $1/3 \times \ln 0.9 + 2/3 \times \ln 1.05 \approx -0.003$.

*Third*, the functional form of the utility function is such that more is better (the function is increasing) and uncertainty is bad (the function is concave). The latter means that investors are risk averse.

**Remark 8.2** *(Expected utility theorem*) Expected utility,* $\mathrm{E}\,U(W)$*, is the right thing to maximize if the investors' preferences* $U(W)$ *are* (1) *complete: can rank all possible outcomes (that is, we know what we like);* (2) *transitive: if* $A$ *is better than* $B$*, and* $B$ *is better than* $C$*, then* $A$ *is better than* $C$ *(a form of consistency);* (3) *independent: if* $X$ *and* $Y$ *are equally preferred, and* $Z$ *is some other outcome, then the following gambles are equally preferred* (a) $X$ *with prob* $\pi$ *and* $Z$ *with prob* $1-\pi$ *and* (b) $Y$ *with prob* $\pi$ *and* $Z$ *with prob* $1-\pi$*,(this is the key assumption); and* (4) *such that every gamble has a certainty equivalent (a non-random outcome that gives the same utility, fairly trivial).*

### 8.1.2 Basic Properties of Utility Functions: (1) More is Better

The idea that *more is better* (non-satiation) is trivial. It means that the utility function is upward sloping. If $U(W)$ is differentiable, then this is the same as marginal utility being positive, $U'(W) > 0$.

**Example 8.3** *(Logarithmic utility)* $U(W) = \ln W$ *so* $U'(W) = 1/W > 0$ *(assuming* $W > 0$*).*

### 8.1.3 Basic Properties of Utility Functions: (2) Risk is Bad

With expected utility, *risk aversion* (uncertainty is considered to be bad) is captured by the concavity of the utility function.

In contrast, a linear utility function implies risk-neutrality, which we rule out because investors appear to care about risk. (Some may seems not to do so, but they are often not gambling with their own money.)

As an example, consider Figure 8.2. It shows a case where the portfolio (or wealth, or consumption,...) of an investor will be worth either $x^-$ or $x^+$ with equal probabilities. The utility function embodies risk aversion since the utility of getting the expected payoff for sure, $U(\mathrm{E}\,x)$, is higher than the expected utility from owning the uncertain asset

$$U(\mathrm{E}\,x) > 0.5U(x^-) + 0.5U(x^+) = \mathrm{E}\,U(x). \tag{8.2}$$

**Remark 8.4** *(*Risk aversion and "marginal utility") Rearranging (8.2) gives*

$$U(\mathrm{E}\,x) - U(x^-) > U(x^+) - U(\mathrm{E}\,x),$$

*which says that moving from a low to a mid value of x (left hand side) counts for more than moving from a mid value to a high value (right hand side). Another way of phrasing the same thing is that a poor person appreciates an extra dollar more than a rich person. This is a key property of a concave utility function.*

The lowest price, $P$, the investor is willing to sell this risky portfolio for is the certain amount that gives the same utility as $\mathrm{E}\,U(x)$, that is, the value of $P$ that solves the equation

$$U(P) = \mathrm{E}\,U(x). \tag{8.3}$$

This price ($P$), called the *certainty equivalent* of the portfolio, is less than the expected payoff

$$P < \mathrm{E}\,x = 0.5x^- + 0.5x^+. \tag{8.4}$$

(The result follows from $U(P) < U(\mathrm{E}\,x)$ and that $U()$ is an increasing function.) Again, see Figure 8.2 for an illustration.

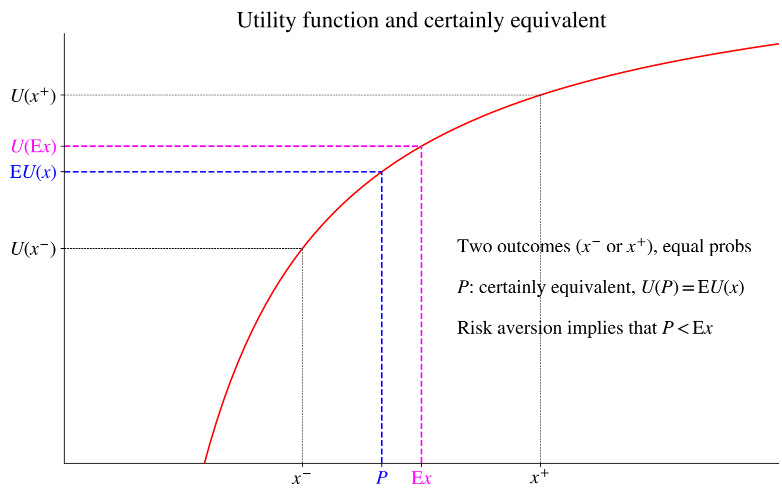



Figure 8.2: Certainty equivalent

**Example 8.5** *(Certainty equivalent) Suppose you have a CRRA utility function, $x^{1-\gamma}/(1-\gamma)$, and own an asset that gives either 0.85 or 1.15 with equal probabilities. What is the certainty equivalent (that is, the lowest price you would sell this asset for)? The answer is the $P$ that solves*

$$\frac{P^{1-\gamma}}{1-\gamma} = 0.5\frac{0.85^{1-\gamma}}{1-\gamma} + 0.5\frac{1.15^{1-\gamma}}{1-\gamma}.$$

*For instance, with $\gamma = 0, 2, 5, 10,$ and 25 we have $P \approx 1, 0.977, 0.947, 0.912,$ and $0.875$. Notice that $P$ goes from the average (1) to the lowest outcome (0.85) as risk aversion increases.*

This means that the *expected net return* on the risky portfolio that the investor demands is

$$\mathrm{E}\, R_x = \frac{\mathrm{E}\, x}{P} - 1 > 0, \tag{8.5}$$

which is greater than zero. This "required return" is higher if the investor is very risk averse (since $P$ is lower). On the other hand, it goes towards zero as the investor becomes less and less risk averse. Notice that this analysis applies to the portfolio return (or wealth, or consumption,...), that is, the argument of the utility function—not to any individual asset. To analyse an individual asset, we need to study how it changes the argument of the utility function, so the covariances with the other assets play a key role (not treated here).

**Example 8.6** *(Risk premium in a simple case) Using the $k = 2$ case in Example 8.5 we get the expected net return (8.5) $1/0.977 - 1 \approx 2.4\%$, since $\mathrm{E}\, x = 1$. Instead, with $k = 25$ we get $1/0.875 - 1 \approx 14.3\%$.*

## 8.2 Utility-Based Portfolio Choice and Mean-Variance Frontiers

### 8.2.1 Utility-Based Portfolio Choice with a Single Risky Asset

Suppose the investor maximizes expected utility from the portfolio return by choosing between a risky and a risk-free asset

$$\max_v \mathrm{E}\, U(R_p), \text{ with } R_p = vR^e + R_f, \tag{8.6}$$

where the excess return of the risky asset is denoted $R^e$.

The first order condition with respect to the weight on risky assets is

$$\begin{aligned} \frac{d\, \mathrm{E}\, U(vR^e + R_f)}{dv} &= 0 \text{ or} \\ \mathrm{E}[U'(R_p)R^e] &= 0, \end{aligned} \tag{8.7}$$

where $U'(R_p)$ is the marginal utility evaluated at $R_p = vR^e + R_f$. Notice that the order of E and $\partial$ are different in the first and second expressions. This is permissible since E defines a sum (and a derivative of a sum is the sum of derivatives, see below for a remark). Also, notice that the second expression is the expectation of the *product* of marginal utility and the excess return.

**Remark 8.7** *(Stochastic discount factor models*) Equation (8.7) is on the same form as a stochastic discount factor (SDF) to asset pricing, where* $\mathrm{E}\, MR^e = 0$ *is a key condition.*

As an example, with a CRRA utility function the first order condition (8.7) can be written

$$\mathrm{E}\, \frac{R^e}{(vR^e + R_f)^\gamma} = 0, \tag{8.8}$$

which is an expectation of a non-linear expression. Solving for $v$ can therefore be tricky.

**Remark 8.8** *(*Interchanging the order of* E *and* $\partial$*) Consider expected utility in (8.1) and let the outcomes be functions of a portfolio weight* $v$*, as in* $x_s(v)$*. Differentiating wrt.* $v$ *then gives*

$$\frac{d\, \mathrm{E}\, U(x)}{dv} = \textstyle\sum_{s=1}^{S} \pi_s \frac{dU(x_s)}{dx}\frac{dx_s}{dv} = \mathrm{E}\, \frac{dU(x)}{dv},$$

*where the last expression uses a short hand notation for how utility depends on* $v$.

Clearly, the first order condition (8.7) defines one equation in one unknown ($v$), making it possible to solve for the portfolio weight. Unfortunately, that can be complicated.

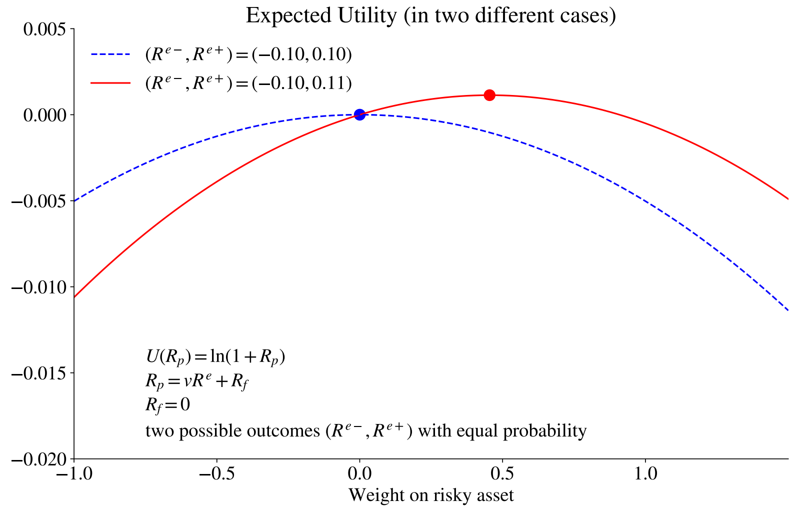



Figure 8.3: Example of portfolio choice with a log utility function

The expectation requires integration and marginal utility might be non-linear, together requiring numerical methods. Explicit solutions are only possible in a few simple cases.

**Example 8.9** *(Portfolio choice with log utility and two states) Suppose* $U(R_p) = \ln(R_p + 1)$*, and that there is one risky asset and a risk-free asset. The excess return on the risky asset* $R^e$ *is either a low value* $R^{e-}$ *(with probability* $\pi$*) or a high value* $R^{e+}$*(with probability* $1 - \pi$*). The optimization problem is*

$$\max_v \pi \ln(vR^{e-} + R_f + 1) + (1 - \pi) \ln(vR^{e+} + R_f + 1).$$

*The first order condition (*$\partial \operatorname{E} U(R_p)/\partial v = 0$*) is*

$$0 = \pi \frac{R^{e-}}{vR^{e-} + R_f + 1} + (1 - \pi) \frac{R^{e+}}{vR^{e+} + R_f + 1}, \text{ so}$$
$$v = -(1 + R_f) \frac{\pi R^{e-} + (1 - \pi) R^{e+}}{R^{e-} R^{e+}}.$$

*See Figure 8.3 for an illustration.*

**Remark 8.10** *(*When to put all investments in the risk-free asset?*) Suppose* $v = 0$ *would be an optimal decision, then the portfolio return equals the risk-free rate which is not random. The first order condition (8.7) can then be written*

$$\operatorname{E}[U'(R_f) R^e] = U'(R_f) \operatorname{E} R^e = 0$$

*which holds only if* $\mathrm{E}\, R^e = 0$*. This shows that it is optimal to make zero investment in the risky asset when its expected excess return is zero. (Why take on risk if it does not give any benefits?)*

### 8.2.2 Utility-Based Portfolio Choice with Several Risky Assets

We now consider the case with $n$ risky assets and a risk-free asset. The optimization problem is

$$\max_{v_1, v_2, \ldots} \mathrm{E}\, U(R_p), \text{ where} \tag{8.9}$$

$$R_p = \textstyle\sum_{i=1}^n v_i R_i^e + R_f. \tag{8.10}$$

The first order conditions for the portfolio weights on the risky assets are

$$\mathrm{E}[U'(R_p) R_i^e] = 0 \text{ for } i = 1, 2, \ldots, n, \tag{8.11}$$

which defines $n$ (non-linear) equations in $n$ unknowns: $v_1, v_2, \ldots, v_n$. For instance, with a CRRA utility function we get

$$\mathrm{E} \frac{R_i^e}{(\sum_{i=1}^n v_i R_i^e + R_f)^\gamma} = 0 \text{ for } i = 1, 2, \ldots, n. \tag{8.12}$$

Notice that calculating the expectation involves integrating over $n$ dimensions. See Figures 8.4 –8.5 for illustrations. The (explicit or numerical) solution is often hard to obtain—so it would be convenient if we could simplify the problem.

### 8.2.3 Is the Optimal Portfolio on the Mean-Variance Frontier?

There are important cases where we can side-step most of the problems with solving the general portfolio choice problem (8.11). In particular, sometimes we can show that the portfolio will be on the mean-variance frontier.

The optimal portfolio is on the mean-variance frontier when optimisation problem can be rewritten as a function in terms of the expected return (positive derivative) and the variance (negative derivative) only

$$\max_v V(\mathrm{E}\, R_p, \mathrm{Var}(R_p)), \tag{8.13}$$

where $\partial V()/\partial\, \mathrm{E}\, R_p > 0$ and $\partial V()/\partial\, \mathrm{Var}(R_p) < 0$. In this case, we should interpret $V()$ as incorporating the preferences, all relevant restrictions and also the features of the return

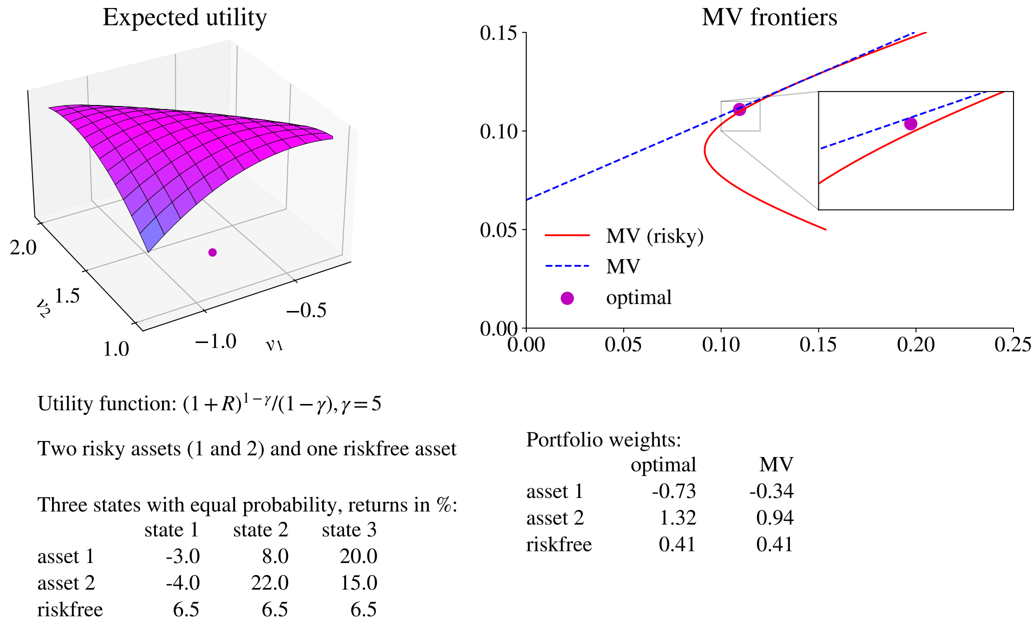



Figure 8.4: Example of when the optimal portfolio is (very slightly) off the MV frontier

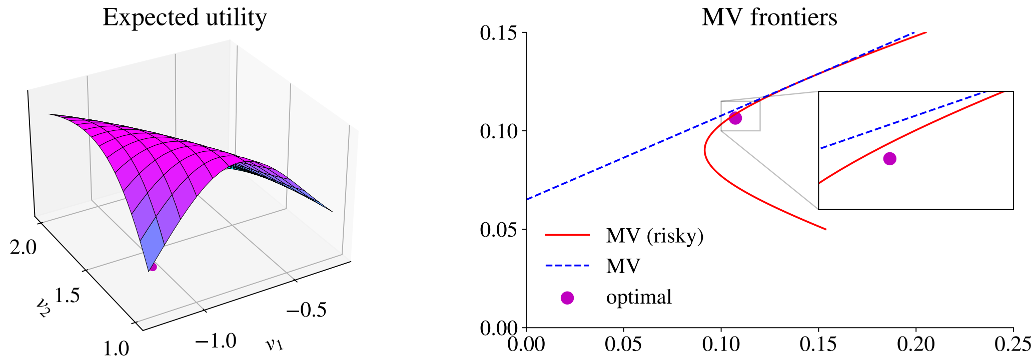
distribution. See Danthine and Donaldson (2005) 4–6 and Huang and Litzenberger (1988) 4–5 for more detailed discussions.

Figure 8.6 shows the utility contours (curves with equal utility) for some cases when (8.13) holds. The optimum is on the mean-variance frontier since the investor wants to move as far to the upper left as possible, but only portfolios on/below the mean-variance frontier are feasible.

In contrast, see Figures 8.4 and 8.5 for examples when (8.13) does not hold. For instance, the preferences may include concerns about the skewness of the portfolio returns, at the same time as the return distribution is such that the skewness is not a function of the mean and variance only. In such cases, the optimal portfolio may be off the MV frontier.

### 8.2.4 Special Cases

This section outlines special cases when the utility-based portfolio choice problem can be rewritten as in (8.13) (in terms of mean and variance only), so that the optimal portfolio is on the mean-variance frontier.




Utility function:
$E(1+R)-(k/2)\text{Var}(R)+(l/3)\text{Skew}(R)$
$k=3.6, l=0.15$

Portfolio weights:

| | optimal | MV |
|---|---|---|
| asset 1 | -1.04 | -0.18 |
| asset 2 | 1.35 | 0.48 |
| riskfree | 0.70 | 0.70 |

Figure 8.5: Example of when the optimal portfolio is (very slightly) off the MV frontier

### Case 1: Mean-Variance Utility

We already know that if the investor maximizes $\mathrm{E}\, R_p - \mathrm{Var}(R_p)k/2$, then the optimal portfolio is on the mean-variance frontier. Clearly, this is the same as assuming that the utility function is $U(R_p) = R_p - (R_p - \mathrm{E}\, R_p)^2 k/2$. (Evaluate $\mathrm{E}\, U(R_p)$ to see this.)

### Case 2: Quadratic Utility

If utility is quadratic in the return (or equivalently, in wealth)

$$U(R_p) = R_p - kR_p^2/2, \tag{8.14}$$

then expected utility can be written

$$\mathrm{E}\, U(R_p) = \mathrm{E}\, R_p - k[\mathrm{Var}(R_p) + (\mathrm{E}\, R_p)^2]/2 \tag{8.15}$$

since $\mathrm{Var}(R_p) = \mathrm{E}\, R_p^2 - (\mathrm{E}\, R_p)^2$. For $k > 0$ this function is decreasing in the variance, and increasing in the mean return as long as $k\, \mathrm{E}\, R_p < 1$. In this case, the optimal portfolio is on the mean-variance frontier.

The main drawback of this utility function is that we have to make sure that we are on the portion of the curve where expected utility is increasing in $\mathrm{E}\, R_p$ (below the so called "bliss point"). Moreover, the quadratic utility function has the strange property that the

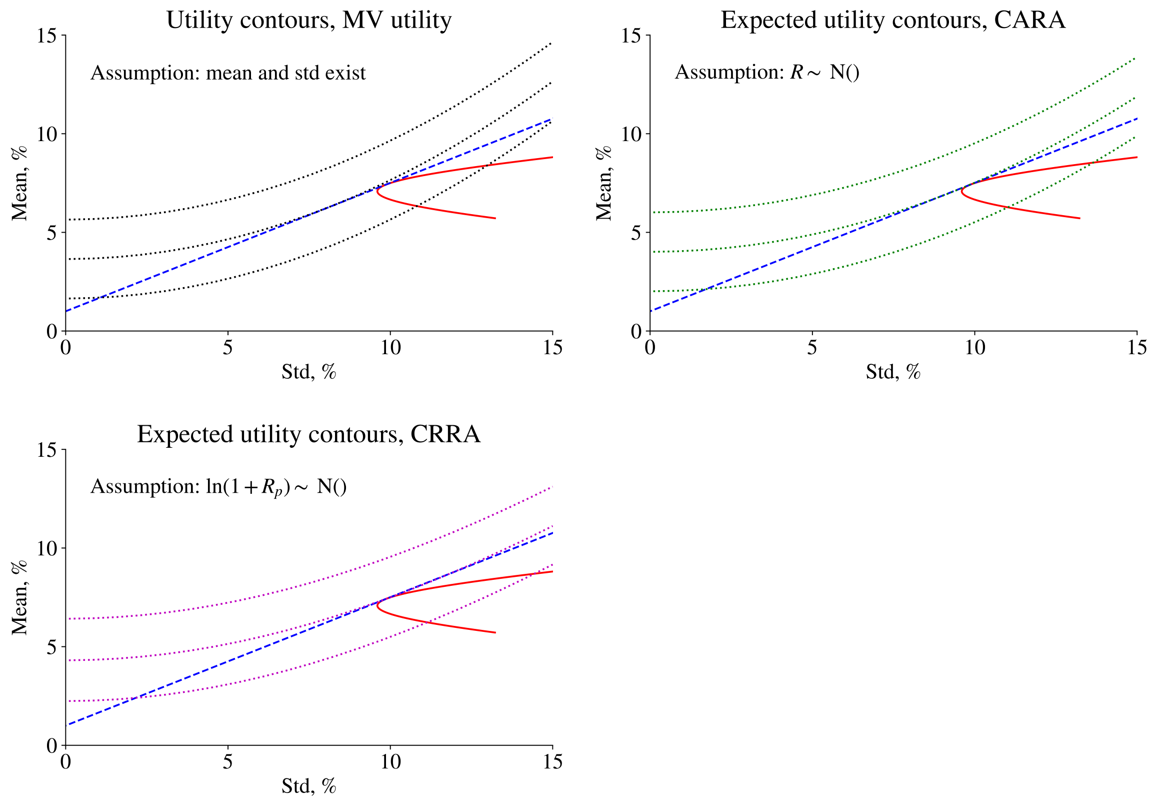



Figure 8.6: Contours with same utility level for different utility functions and return distributions. The means and standard deviations (on the axes) are for the net returns (not log returns).

amount invested in risky assets decreases as wealth increases (increasing absolute risk aversion).

### Case 3: Normally Distributed Returns

When the distribution of any *portfolio* return is fully described by the mean and variance, then maximizing $\mathrm{E}\,U(R_p)$ will result in a mean variance portfolio, at least if the utility function is strictly increasing and displays risk aversion (is concave). An appendix provides details of the mathematics.

The basic intuition is as follows. Write $\mathrm{E}\,U()$ as a (Taylor) series expansion—it will then include higher moments of the return distribution. However, in a normal distribution all odd moments (except the mean) are zero and all even moments are increasing in the variance. In short, the optimisation problem can be written on the form in (8.13).

Normally distributed returns should be considered as just an approximation for three reasons. *First*, limited liability means that the net return can never be below $-100\%$ (the asset price cannot be negative). However, such returns are possible in a normal distribution, although they may have very low probabilities. *Second*, empirical evidence suggests that most asset returns have distributions with fatter tails and more skewness than implied by a normal distribution, especially when the returns are measured over short horizons. *Third*, some assets with non-linear payoffs, like options, have return distributions that must be non-normal.

As an example of what happens when we combine a normal distribution with a valid utility function, consider the next propositions. Further examples/applications, for instance, using the Telser criterion are discussed in a separate section below.

**Proposition 8.11** *If returns are normally distributed, then maximizing the expected value a utility function with* constant absolute risk aversion $k > 0$ *(CARA)*

$$U(R_p) = -\exp(-R_p k)$$

*is the same as solving a mean-variance problem. See Figure 8.6. (The proof is in the appendix.)*

### Case 4: CRRA Utility and Lognormally Distributed Portfolio Returns

**Proposition 8.12** *Consider a CRRA utility function,* $(1 + R_p)^{1-\gamma}/(1 - \gamma)$*, and suppose all log portfolio returns,* $r_p = \ln(1 + R_p)$*, are normally distributed. The solution is then, once again, on the mean-variance frontier. See Figure 8.6. (The proof is in the appendix.)*

This result is especially useful in analysis of multi-period investments. Notice, however, that this should be thought of as an approximation since $1 + R_p = \alpha(1 + R_1) + (1 - \alpha)(1 + R_2)$ is not lognormally distributed even if both $R_1$ and $R_2$ are.

### 8.2.5 Application of Normal Returns

This section gives a few examples of how fairly non-standard preferences, combined with normally distributed portfolio returns, give optimal portfolios on the mean-variance frontier.

The down-side risk measure Value at Risk (VaR) is just a quantile of the loss distribution, while Expected Shortfall (ES) is the average loss in case the loss is beyond the

VaR. Target semivariance (TSV) is the average squared deviation around a target, but only counting the downside. Another chapter discusses the details and shows that, when returns are normally distributed, then *all three measures are increasing in the standard deviation*. Remark 8.13 summarises the key features, and details are in another chapter.

**Remark 8.13** *(VaR, ES and TSV with normally distributed returns) If the return is normally distributed, $R \sim N(\mu, \sigma^2)$, then $VaR_\alpha = -(\mu + c\sigma)$, where c is the $1-\alpha$ quantile of a N(0,1) distribution ($-1.64$ for 5%). Also, $ES_\alpha = -[\mu - \phi(c)\sigma/(1-\alpha]$, where $\phi()$ is the pdf or a $N(0,1)$ variable. Finally, it can be shown that the TSV $\lambda(h)$ is a strictly increasing function of the standard deviation, $d\lambda_p(h)/d\sigma = 2\sigma\Phi(a)$, where $\Phi()$ is the distribution function of a standard normal and $a = (h-\mu)/\sigma$.*

With normally distributed returns, the VaR, ES and TS are strictly increasing functions of the standard deviation (and the variance). In this case, the portfolio that *minimizes the VaR, ES* or *TS* at a given average return will be on the mean-variance frontier.

Another portfolio choice approach is to *use the value at risk (VaR) as a restriction*. For instance, the *Telser criterion* maximizes the expected portfolio return subject to the restriction that the value at risk does not exceed a given level $V^*$

$$\max_v \mathrm{E}\, R_p \text{ st. } \mathrm{VaR}_\alpha < V^*. \tag{8.16}$$

When returns are normally distributed, Remark 8.13 shows that the restriction can be rewritten as

$$\mathrm{E}\, R_p > -V^* - c\,\mathrm{Std}(R_p), \tag{8.17}$$

where $c$ is, for instance, $-1.64$ when the $\mathrm{VaR}_\alpha$ has a confidence level $\alpha = 95\%$.

**Example 8.14** *With a VaR confidence level of 95% and $V^* = 0.1$, then (8.17) gives* $\mathrm{E}\, R_p > -0.1 + 1.64\,\mathrm{Std}(R_p)$.

The optimization problem is illustrated in Figure 8.7. The objective is to find the portfolio with the highest expected return that satisfies the VaR restriction, which means that is has to be on or above the line defined by (8.17). Also, only points on or below the CLM (the mean-variance frontier based on both risky assets and a risk-free asset) are feasible.

The optimal portfolio is therefore where the restriction intersects the CLM: the Telser criterion applied to normally distributed returns gives a mean-variance portfolio. To be

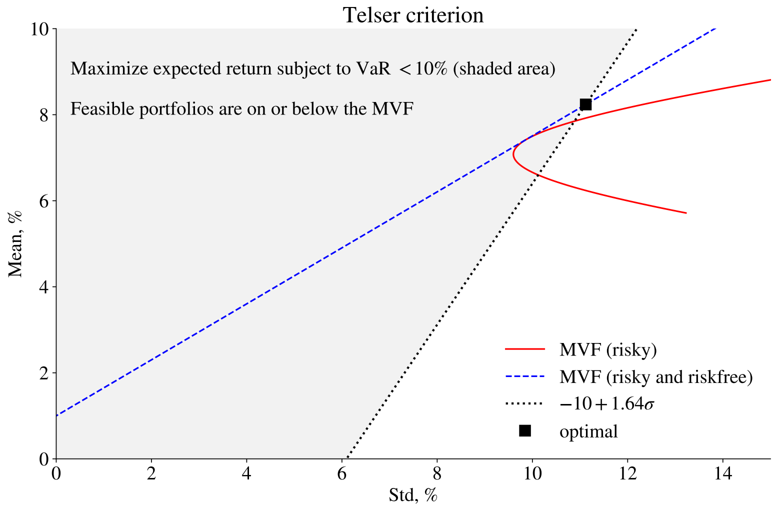



Figure 8.7: Telser criterion and VaR

precise, the optimal portfolio puts

$$w = -\frac{R_f + V^*}{\mu_T^e + c\sigma_T}. \tag{8.18}$$

in the tangency portfolio (with average excess return $\mu_T^e$ and standard deviation $\sigma_T$) and the rest $(1 - w)$ in the risk-free asset.

We could instead use a restriction on expected shortfall or target semivariance, which define areas in a MV figure similar to that in Figure 8.7.

**Example 8.15** *(Optimal portfolio. Telser) Let $\mu_T^e = 6.5\%$, $\sigma_T = 10\%$ and $R_f = 1\%$. The optimal portfolio with $V^* = 10\%$ is then*

$$w = -\frac{0.01 + 0.10}{0.065 - 1.64 \times 0.1} \approx 1.11.$$

*Instead, if the restriction is that VaR$< 4.5\%$, then the weight is $w \approx 0.55$.*

*Proof* of (8.18). As usual, the average return on a portfolio $p$ of the CML is

$$\mu_p = R_f + SR_T\sigma_p,$$

where $SR_T$ is the the Sharpe ratio of the tangency portfolio. This equals the mean return required by the VaR restriction (8.17) when

$$\sigma_p = -\frac{R_f + V^*}{SR_T + c}.$$

Since $\sigma_p = w\sigma_T$ (assuming $w \geq 0$), the optimal portfolio weight on the tangency portfolio is (8.18). $\square$

## 8.3 Behavioural Finance

Reference: Elton, Gruber, Brown, and Goetzmann (2014) 20; Forbes (2009); Shefrin (2005)

There is relatively little direct evidence on investor's preferences (utility). For obvious reasons, we can't know for sure what people really like. The evidence we do have is from two sources: "laboratory" experiments designed to elicit information about the test subject's preferences for risk, and a lot of indirect information.

### 8.3.1 Evidence on Utility Theory

The laboratory experiments are typically organized at university campuses (mostly by psychologists and economists) and involve only small compensations—so the test subjects are those students who really need the monetary compensation for taking part or those that are interested in this type of psychological experiments. The results vary quite a bit, but a main theme is that the key assumptions in utility-based portfolio choice might be reasonable. There are, however, some important systematic deviations from these assumptions.

For instance, investors seem to be unwilling to realize losses, that is, to sell off assets which they have made a loss on (often called the "disposition effect"). They also seem to treat the investment problem much more on an asset-by-asset basis than suggested by mean-variance analysis which pays a lot of attention to the covariance of assets (sometimes called mental accounting). Discounting appears to be non-linear in the sense that discounting is higher when comparing today with dates in the near future than when comparing two dates in the distant future. (Hyperbolic discount factors might be a way to model this, but lead to time-inconsistent behaviour: today we may prefer an asset that pays off in $t + 2$ to an asset than pays off in $t + 1$, but tomorrow our ranking might be reversed.) Finally, the results seem to move towards tougher play as the experiments are repeated and/or as more competition is introduced—although the experiments seldom converge to ultra tough/egoistic behaviour (as typically assumed by utility theory).

The indirect evidence is broadly in line with the implications of utility-based theory—especially now that the costs for holding well diversified portfolios have decreased (mutual funds). However, there are clearly some systematic deviations from the theoretical implications. For instance, many investors seem to be too little diversified. In particular, many investors hold assets in companies/countries that are very strongly correlated to their labour income (local bias). Moreover, diversification is often done in a naive fashion and depends on the "menu" of choices. For instance, many pension savers seem to diversify by putting the fraction $1/n$ in each of the $n$ funds offered by the firm/bank—irrespective of what kind of funds they are. There are, of course, also large chunks of wealth invested for control reasons rather than for a pure portfolio investment reason (which explains part of the so called "home bias"—the fact that many investors do not diversify internationally).

### 8.3.2 Evidence on Expectations Formation (Forecasting)

In laboratory experiments (and studies of the properties of forecasts made by analysts), several interesting results emerge on how investors seem to form expectations. First, complex situations are often approached by treating them as a simplified representative problem—even against better knowledge (often called "representativeness")—and stands in contrast to the idea of Bayesian learning where investors update and learn from their mistakes. Second (and fairly similar), difficult problems are often handled as if they were similar to some old/easy problem—and all that is required is a small modification of the logic (called "anchoring"). Third, recent events/data are given much higher weight than they typically warrant (often called "recency bias" or "availability"). Finally, most forecasters seem to be overconfident: they draw (too) strong conclusions from small data sets ("law of small numbers") and overstate the precision of their own forecasts.

Notice, however, that it is typically difficult to disentangle (distorted) beliefs from non-traditional preferences. For instance, the aversion of selling off bad investments, may equally well be driven by a belief that past losers will recover.

### 8.3.3 Prospect Theory

The *prospect theory* (developed by Kahneman and Tversky) tries to explain several of these things by postulating that the utility function is concave over some reference point (which may shift), but convex below it. This means that gains are treated in a risk-averse way, but losses in a risk-loving way. For instance, after a loss (so we are below the reference point) an asset looks less risky than after a gain—which might explain why investors hold on to losing investments. Clearly, an alternative explanation is that investors believe in mean-reversion (losing positions will recover, winning positions will fall back).

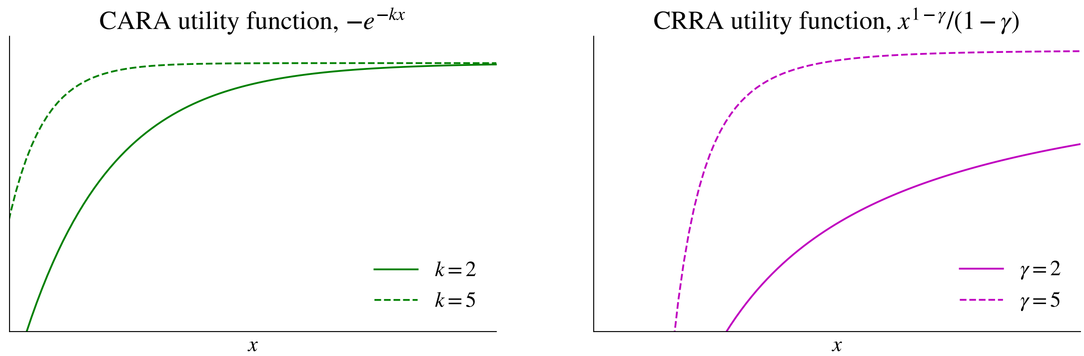



Figure 8.8: Examples of utility functions

In general, it is hard to make a clear distinction between non-classical preferences and (potentially distorted) beliefs.

## 8.4 Appendix – Is Risk Aversion Related to the Level of Wealth?*

This section discusses how risk aversion is related to the wealth level. (In contrast, when we use the portfolio return as the argument of the utility function, then this amounts to disregarding differences across wealth levels.)

First, define *absolute risk aversion* as

$$A(W) = \frac{-U''(W)}{U'(W)}, \tag{8.19}$$

where $U'(W)$ is the first derivative and $U''(W)$ the second derivative. Second, define *relative risk aversion* as

$$R(W) = WA(W) = \frac{-WU''(W)}{U'(W)}. \tag{8.20}$$

These two definitions are strongly related to the attitude towards taking risk (see below).

Figure 8.8 demonstrates two commonly used utility functions, and the following discussion outlines their main properties.

The *CARA utility function* (constant absolute risk aversion), $U(W) = -e^{-kW}$, is quite simple to use (in particular when returns are normally distributed), but has the unappealing feature that the amount invested in the risky asset (in a risky/risk-free trade-off) is constant across wealth levels. This means, of course, that wealthy investors would have a lower portfolio weight on risky assets.

**Remark 8.16** *(Risk aversion in CARA utility function) $U(W) = -e^{-kW}$ gives $U'(W) = ke^{-kW}$ and $U''(W) = -k^2e^{-kW}$, so we have $A(W) = k$. This means an increasing relative risk aversion, $R(W) = Wk$, so a poor investor typically has a larger portfolio weight on the risky asset than a rich investor.*

The *CRRA utility function* (constant relative risk aversion) is often harder to work with, but has the nice property that the portfolio weights are unaffected by the wealth level. This fits with historical data which show no trends in portfolio weights or risk premia—in spite of investors having become much richer over time.

**Remark 8.17** *(Risk aversion in CRRA utility function) $U(W) = W^{1-\gamma}/(1-\gamma)$ gives $U'(W) = W^{-\gamma}$ and $U''(W) = -\gamma W^{-\gamma-1}$, so we have $A(W) = \gamma/W$ and $R(W) = \gamma$. The absolute risk aversion decreases with the wealth level in such a way that the relative risk aversion is constant. In this case, a poor investor typically has the same portfolio weight on the risky asset as a rich investor.*

To understand the concepts of absolute and relative risk aversion, consider an investor with wealth $W$ who can choose between taking on a zero mean risk $Z$ (so $\mathrm{E}\,Z = 0$) or pay a price $P$. The investor is indifferent if

$$\mathrm{E}\,U(W+Z) = U(W-P). \tag{8.21}$$

If $Z$ is a small risk, then we can use a second order approximation and solve for the price as

$$P \approx A(W)\,\mathrm{Var}(Z)/2. \tag{8.22}$$

This says that the price the investor is willing to pay to avoid the risk $Z$ is proportional to the *absolute risk aversion* $A(W)$.

**Example 8.18** *(Willingness to pay to avoid a risk) Suppose the investor has a CARA utility function with $A(W) = 5$ and that* $\mathrm{Var}(Z) = 1$. *Then,* $P = 5 \times 1/2 = 2.5$.

*Proof* of (8.22). First, approximate as

$$\begin{aligned}\mathrm{E}\,U(W+Z) &\approx U(W) + U'(W)\,\mathrm{E}\,Z + U''(W)\,\mathrm{E}\,Z^2/2\\ &= U(W) + U''(W)\,\mathrm{Var}(Z)/2,\end{aligned}$$

since $\mathrm{E}\,Z = 0$. Second, approximate $U(W-P) \approx U(W) - U'(W)P$. Finally, make the two approximations equal to get (8.22). $\square$

If we change the setting in (8.21)–(8.22) to make the risk proportional to wealth, that is $Z = Wz$ where $z$ is the risk factor, then (8.22) directly gives

$$\begin{aligned} P &\approx A(W)W^2 \operatorname{Var}(z)/2, \text{ so} \\ P/W &\approx R(W) \operatorname{Var}(z)/2. \end{aligned} \tag{8.23}$$

This says that the fraction of wealth $(P/W)$ that the investor is willing to pay to avoid the risk $(z)$ is proportional to the *relative risk aversion* $R(W)$.

**Example 8.19** *(Willingness to pay to avoid a risk) Suppose the investor has a CRRA utility function with* $R(W) = 5$ *and that* $\operatorname{Var}(z) = 0.2$*. Then,* $P/W = 5 \times 0.2/2 = 0.5$.

These results mostly carry over to the portfolio choice: high absolute risk aversion typically implies that only small *amounts* are invested in risky assets, whereas a high relative risk aversion typically leads to small *portfolio weights* of risky assets.

## 8.5 Appendix – Extra Details on Portfolio Choice with Normally Distributed Returns*

### 8.5.1 Case 3 and 4: Proofs

*Proof* of Proposition 8.11. First, recall that if $x \sim N(\mu, \sigma^2)$, then $\operatorname{E} e^x = e^{\mu+\sigma^2/2}$. Therefore, rewrite expected utility as

$$\operatorname{E} U(R_p) = \operatorname{E}\left[-\exp\left(-R_p k\right)\right] = -\exp\left[-\operatorname{E} R_p k + \operatorname{Var}(R_p) k^2/2\right].$$

Notice that the assumption of normally distributed returns is crucial for this result. Second, recall that if $x$ maximizes $f(x)$, then it also maximizes $g[f(x)]$ if $g$ is a strictly increasing function. The function $-\ln(-z)/k$ is defined for $z < 0$ and it is increasing in $z$. We can apply this function by letting $z$ be the right hand side of the previous equation to get

$$-\ln(-z)/k = \operatorname{E} R_p - \operatorname{Var}(R_p) k/2.$$

Therefore, maximizing the expected CARA utility or MV preferences (in terms of the returns) gives the same solution. □

*Proof* of Proposition 8.12. Notice that

$$\frac{\operatorname{E}(1+R_p)^{1-\gamma}}{1-\gamma} = \frac{\operatorname{E}\exp[(1-\gamma)r_p]}{1-\gamma}, \text{ where } r_p = \ln(1+R_p).$$

Since $r_p$ is normally distributed, the expectation is (recall that if $x \sim N(\mu, \sigma^2)$, $\mathrm{E}\, e^x = e^{\mu+\sigma^2/2}$)

$$\frac{1}{1-\gamma}\,\mathrm{E}\exp[(1-\gamma)r_p] = \frac{1}{1-\gamma}\exp[(1-\gamma)\,\mathrm{E}\, r_p + (1-\gamma)^2\,\mathrm{Var}(r_p)/2].$$

Assume that $\gamma > 1$. The function $\ln\left[z(1-\gamma)\right]/(1-\gamma)$ is then defined for $z < 0$ and it is increasing in $z$. Let $z$ be the the right hand side of the previous equation and apply the transformation to get

$$\mathrm{E}\, r_p + (1-\gamma)\,\mathrm{Var}(r_p)/2,$$

which is increasing in the expected log return and decreasing in the variance of the log return (since we assumed $1-\gamma < 0$). To express this in terms of the mean and variance of the return instead of the log return we use the following fact: if $\ln y \sim N(\mu, \sigma^2)$, then $\mathrm{E}\, y = \exp(\mu + \sigma^2/2)$ and $\mathrm{Std}\,(y)\,/\,\mathrm{E}\, y = (\exp(\sigma^2) - 1)^{1/2}$. Using this fact in the previous expression gives

$$\ln(1 + \mathrm{E}\, R_p) - \gamma \ln[\mathrm{Var}(R_p)/(1 + \mathrm{E}\, R_p)^2 + 1]/2,$$

which is increasing in $\mathrm{E}\, R_p$ and decreasing in $\mathrm{Var}(R_p)$. We therefore get a mean-variance portfolio. $\square$

### 8.5.2 Case 3: Normally Distributed Returns, the Math of the General Case*

This section shows that if the returns are normally distributed and all even derivatives of the utility function are negative, then expected utility can be written in terms of the mean (increasing) and variance (decreasing).

**Remark 8.20** *(Taylor series expansion) Recall that a Taylor series expansion of a function $f(x)$ around the point $x_0$ is $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n f(x_0)}{dx^n}(x-x_0)^n$, where $d^n f(x_0)/dx^n$ is the $n$th derivative of $f()$ evaluated at $x_0$ and $n!$ is the factorial ($n! = 1 \times 2 \times \ldots \times n$ and $0! = 1$ by definition).*

**Remark 8.21** *(Higher central moments for a normal distribution) If $x$ is normally distributed, then $\mathrm{E}\,(x-\mu)^n = 0$ if $n$ is odd and related to $\sigma = \mathrm{Std}(x)$ if $n$ is even. To be precise, for even $n$, $\mathrm{E}\,(x-\mu)^n = \sigma^n \times (n-1)!!$, where $(n-1)!!$ is the product of all odd numbers up to and including $n-1$, that is, the "double factorial" $1 \times 3 \times \ldots \times (n-3) \times (n-1)$. For instance, $n = 4$ gives $3\sigma^4$ and $n = 6$ gives $15\sigma^6$.*

Do a Taylor series expansion of the utility function $U(R_p)$ around the average portfolio return ($\mathrm{E}\,R_p$) to get

$$U(R_p) = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n U(\mathrm{E}\,R_p)}{dR_p^n} \left(R_p - \mathrm{E}\,R_p\right)^n, \tag{8.24}$$

where $d^n U(\mathrm{E}\,R_p)/dR_p^n$ denotes the $n$th derivative of the utility function evaluated at the point $\mathrm{E}\,R_p$. For instance, $d^2 U(\mathrm{E}\,R_p)/dW^2$ is the same as $U''(\mathrm{E}\,R_p)$.

**Remark 8.22** *(Taylor expansion of a CRRA utility function*) For a CRRA utility function,* $(1 + R_p)^{1-\gamma}/(1-\gamma)$, *we have*

$$U'(\mathrm{E}\,R_p) = (1+\mathrm{E}\,R_p)^{-\gamma} > 0,\ U''(\mathrm{E}\,R_p) = -\gamma(1+\mathrm{E}\,R_p)^{-\gamma-1} < 0,\ U'''(\mathrm{E}\,R_p) = \gamma(1+\gamma)(1+\mu_p)^{-\gamma-2} >$$

*so high variance is bad, but high mean and skewness are considered good.*

Take expectations, but notice that $d^n U(\mathrm{E}\,R_p)/dR_p^n$ is not random, only the $(R_p - \mathrm{E}\,R_p)^n$ terms are. Recall that $\mathrm{E}(R_p - \mathrm{E}\,R_p) = 0$ and that $\mathrm{E}(R_p - \mathrm{E}\,R_p)^2 = \mathrm{Var}(R_p)$, and use Remark (8.21) to get

$$\mathrm{E}\,U(R_p) = U(\mathrm{E}\,R_p) + \sum_{n=2,4,6\ldots} \frac{d^n U(\mathrm{E}\,R_p)}{dR_p^n} \frac{\mathrm{Var}(R_p)^{n/2}}{n!!}, \tag{8.25}$$

where $n!!$ is the product of all even numbers up to $n$ ($2 \times 4 \times \ldots \times (n-2) \times n$). If all even derivatives are negative (as they would with, for instance, a CRRA or CARA utility function), then this expression is guaranteed to be decreasing in $\mathrm{Var}(R_p)$. Clearly, it could happen in other cases too.

# Chapter 9

# Multi-Factor Models

## 9.1 Factor Investment

A number of factor returns related to, for instance, firm characteristics like size and profitability, have shown good performance over long periods of time. It is therefore common to base investment strategies on those characteristics—and a large number of funds and other investment vehicles have been developed for this purpose. This approach is called *factor investing* or "smart beta." It is essentially a dynamic trading strategy since the characteristics change over time.

In studies of investment fund performance, it is often found that the abnormal performance ($\alpha$ from a CAPM regression) can be explained by a fairly small set of factors. This suggests that fund managers have indeed been able to invest in those characteristics that have historically paid off. This suggests that a multi-factor model would be empirically more appropriate than CAPM.

This chapter provides theoretical motivation of different multi-factor models and performs an empirical test of a commonly used model.

**Empirical Example 9.1** *(Fama-French factors) Figure 9.1 illustrate several of the factors discussed by Fama and French (1993) and Fama and French (2015), while Table 9.1 summarises the return patterns. It is clear that several of the factors have earned substantial risk premia (average excess returns) and have fairly reasonable volatilities. As a consequence, the Sharpe ratios are good. In addition, several of these portfolios are virtually uncorrelated with the market excess return, so the $\alpha$ values are similar to the average excess returns.*

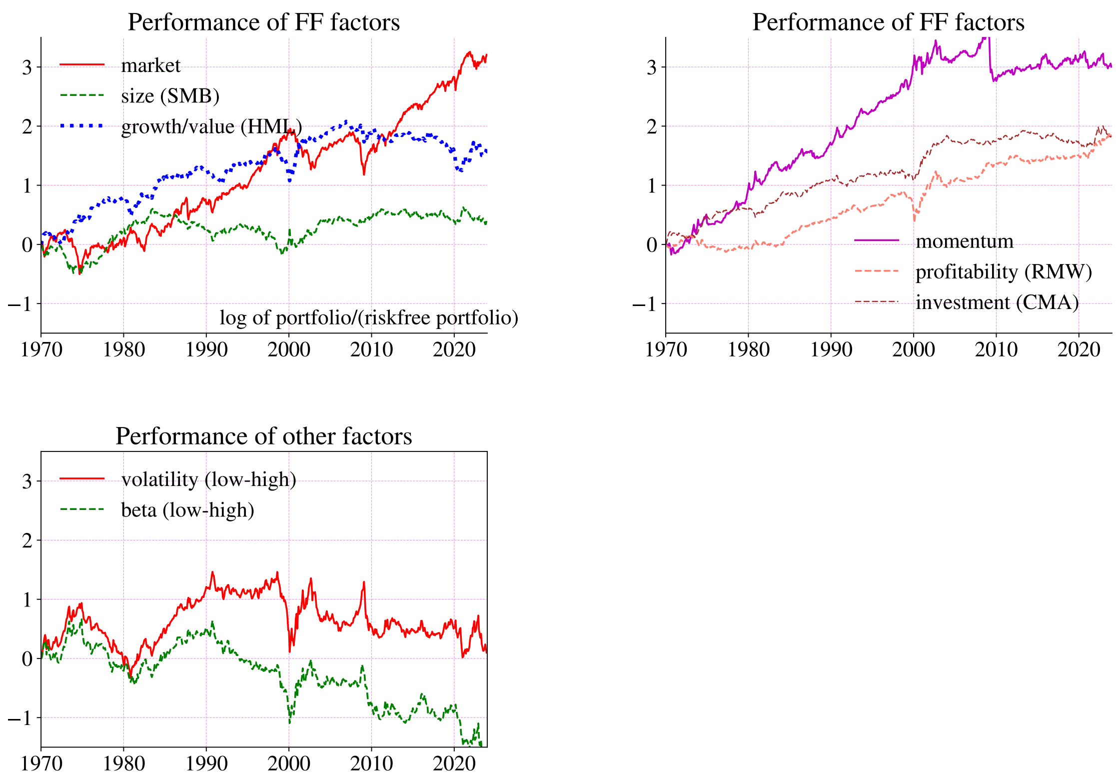



Figure 9.1: Cumulated returns of important equity factors

## 9.2 An Overview of Multi-Factor Models

This section gives a short introduction to multi-factor models. Model details and proofs are in later sections.

A multi-factor model extends the market model by allowing more factors to explain the return on an asset. For instance, a two-factor model is

$$R^e_{it} = \alpha + \beta_{im} R^e_{mt} + \beta_{io} R^e_{ct} + \varepsilon_{it}, \tag{9.1}$$

where $R^e_m$ is the excess return on the market and $R^e_c$ is the excess return on some other portfolio. As usual, we require $\mathrm{E}\,\varepsilon_{it} = 0$, and that $\varepsilon_{it}$ is uncorrelated to all regressors.

The pricing implication is a multi-beta model

$$\mathrm{E}\,R^e_i = \beta_{im}\mu^e_m + \beta_{ic}\mu^e_c. \tag{9.2}$$

Notice that there is no intercept, so $\alpha$ in (9.1) should be zero.

| | $\mu, \%$ | $\sigma, \%$ | SR | $\beta$ | $\alpha, \%$ |
|---|---|---|---|---|---|
| market | 7.10 | 15.99 | 0.44 | 1.00 | 0.00 |
| size | 1.36 | 10.64 | 0.13 | 0.19 | 0.03 |
| growth/value | 3.64 | 10.73 | 0.34 | −0.14 | 4.66 |
| momentum | 6.80 | 15.08 | 0.45 | −0.18 | 8.08 |
| profitability | 3.67 | 7.93 | 0.46 | −0.10 | 4.36 |
| investment | 3.73 | 7.20 | 0.52 | −0.16 | 4.90 |
| volatility (low-high) | 2.90 | 22.69 | 0.13 | −0.89 | 9.19 |
| beta (low-high) | −1.15 | 20.18 | −0.06 | −0.83 | 4.72 |

Table 9.1: Descriptive statistics of excess returns of different US equity portfolios (including the Fama-French factors and more), annualised. Monthly data 1970:01-2023:12.

**Remark 9.2** *(When factors are not excess returns*) Equation (9.2) assumes that the factor can be expressed as an excess return, but that is not always the case. For instance, it could be that the second factor is a macro variable like inflation surprises. Then there are two possible ways to proceed. First, find that portfolio which mimics the movements in the inflation surprises best and use the excess return of that (factor mimicking) portfolio in (9.1) and (9.2). Second, we could instead* (1) *estimate the betas* $(\beta_{im}, \beta_{ic})$ *by a time series regression of (9.1), but allowing for an intercept; and (2) estimate the factor risk premia* $(\mu^e_m, \mu^e_c)$ *by a cross-section of (9.2) where the dependent variable is the historical average returns of different assets* $(i = 1, 2, ...n)$ *and the regressors are the betas from the first step.*

We will consider several *theoretical* multi-factor models: the “CAPM with background risk” as well as a consumption-based model.

There are also several *empirically motivated* multi-factor models, that is, empirical models that have been found to work well in practice. For instance, Fama and French (1993) estimate a three-factor model (capturing the market, the difference between small and large firms and the difference between value firms and growth firms) and show that it empirically performs much better than CAPM. Also, the multi-factor model by MSCI Barra is widely used in the financial industry. It uses a set of firm characteristics as factors, for instance, size, volatility, price momentum, and industry/country (see Stefek (2002)). This model is often used to value firms without a price history (for instance, before an IPO) or to find mispriced assets.

## 9.3 Portfolio Choice with Background Risk

This section discusses the portfolio problem when there is "background risk" (see Mayers (1972)). For instance, labour income, real estate or a private business as both background income and risk. The same applies to the value of a liability stream, even if it's just a target retirement wealth or a planned future house purchase.

The existence of background risk typically affects portfolio choice and, consequently, asset prices

### 9.3.1 Portfolio Choice with Background Risk: One Risky Asset

Consider a mean-variance investor who forms a financial portfolio by choosing between a risky asset (henceforth called "equity") with return $R_i$ and a risk-free asset with return $R_f$. The investor also has a background risk—in the form of an endowment (positive or negative) of a non-traded asset (with return $R_c$). This could, for instance, be labour income or a house (positive endowment) or a combination of both. We refer to this as a non-traded asset and it accounts for the fraction $\phi$ of total wealth, while the financial portfolio accounts for $1-\phi$. When the non-traded asset is a liability, then $\phi < 0$. The "return" on the total portfolio, $R_p$, is

$$R_p = (1-\phi)R_{Fin} + \phi R_c, \text{ with} \tag{9.3}$$

$$R_{Fin} = wR_i + (1-w)R_f, \tag{9.4}$$

where $R_c$ is the "return" (change of value) of the non-traded asset.

The investor chooses $w$ to maximize

$$\mathrm{E}\, U(R_p) = \mathrm{E}\, R_p - \frac{k}{2}\operatorname{Var}(R_p), \tag{9.5}$$

and the optimal value is

$$w = \frac{\mu_i^e/k - \phi\sigma_{ic}}{(1-\phi)\sigma_i^2}, \tag{9.6}$$

where $\sigma_i^2$ is the variance of equity and $\sigma_{ic}$ is the covariance of equity and the non-traded asset.

*Proof* of (9.6). To simplify the notation, write the portfolio return as $R_p = vR_i^e + \phi R_c^e + R_f$, where $v = (1-\phi)w$. Use in the objective function to get

$$\mathrm{E}\, U(R_p) = v\mu_i^e + \phi\mu_c^e + R_f - \frac{k}{2}(v^2\sigma_i^2 + \phi^2\sigma_c^2 + 2v\phi\sigma_{ic}),$$

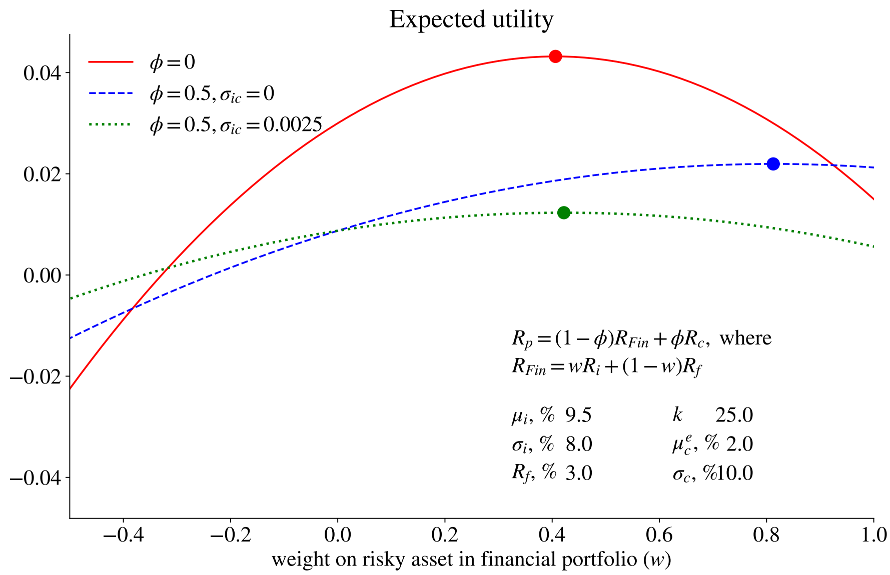



Figure 9.2: Portfolio choice with non-traded assets

The first order condition is $\mu_i^e - k(v\sigma_i^2 + \phi\sigma_{ic}) = 0$, solve for $v$ and divide by $1-\phi$ to get (9.6). $\square$

The second term in the optimal portfolio weight, *the hedging term*, depends on how important the non-traded asset is in the portfolio ($\phi$) and also on the covariance term ($\sigma_{ic}$). Clearly, if there is no non-traded asset in the total portfolio ($\phi = 0$), then we are back in a traditional MV case.

**Remark 9.3** *(Interpreting the hedging term*) The hedging term in (9.6) is related to the slope coefficient in a regression of the non-traded asset on equity,* $R_{ct}^e = \alpha + \gamma R_{it}^e + \varepsilon_t$*, since* $\gamma = \sigma_{ic}/\sigma_i^2$.

**Example 9.4** *(Details on Figure 9.2) Based on the parameters in the figure, the optimal* $w$ *is 0.41, 0.81 and 0.42 in the three cases.*

Several things can be noticed. First, *when the covariance is zero* ($\sigma_{ic} = 0$), then the equity weight is increasing in the amount of non-traded assets ($\phi$), while the opposite holds for the risk-free asset; see Figure 9.2 for an illustration and Example 9.4 for more details. The intuition is that a zero covariance means that the non-traded asset is quite similar to a bond: having an endowment of a bond-like asset in the overall portfolio means that the financial portfolio should be tilted towards equity.

Second, *when the covariance is positive* ($\sigma_{ic} > 0$) and we have a positive exposure to the non-traded asset ($\phi > 0$), then the hedging term will reduce the equity weight and increase the risk-free weight. The intuition is that the overall portfolio now includes a lot of "equity like" assets, so the financial portfolio should be tilted towards the risk-free asset. The opposite holds when the exposure to the non-traded asset is negative (a liability, $\phi < 0$) or when the non-traded asset is negatively correlated with equity ($\sigma_{ic} < 0$, assuming a positive exposure, $\phi > 0$). Again, see Figure 9.2.

**Example 9.5** *(Portfolio choice of young and old) Consider the common portfolio advice that* young investors *(with labour income) should invest relatively more in stocks than* old investors. *In this case, the non-traded asset is an endowment of "human capital," that is, the present value of future labour income—and current labour income can loosely be interpreted as its return. The analysis in the previous section suggests that a low correlation of stock returns and wages means that the young investor is endowed with a bond-like asset, so the financial portfolio should be tilted towards equity. Old investors less so.*

### 9.3.2 Portfolio Choice with Background Risk: Several Risky Assets

| | $\mu$, % | $\Sigma$, bp | | |
|---|---|---|---|---|
| | | A | B | C |
| A | 11.5 | 166 | 34 | 58 |
| B | 9.5 | 34 | 64 | 4 |
| C | 6.0 | 58 | 4 | 100 |

Table 9.2: Characteristics of the assets in the MV examples. Notice that $\mu$, % is the expected return in % (that is, $\times 100$) and $\Sigma$, bp is the covariance matrix in basis points (that is, $\times 100^2$).

With several risky assets the financial portfolio return (9.4) is

$$R_{Fin} = w'R + (1 - \mathbf{1}'w)R_f, \tag{9.7}$$

where $w$ now is a vector of portfolio weights, $R$ a vector of returns on the risky assets and $\mathbf{1}$ is a vector of ones (so $\mathbf{1}'w$ is the sum of the elements in the $w$ vector). The optimal portfolio are now

$$w = \Sigma^{-1}\frac{\mu^e/k - \phi S_c}{1 - \phi}. \tag{9.8}$$

where $\Sigma$ is the covariance matrix of all risky assets (not including the non-traded asset) and $S_c$ is a vector of covariances of the assets with the non-traded asset. The portfolio weights of the financial subportfolio will (as long as $\phi S_c \neq 0$) give a return that is *off the mean-variance frontier*—and will differ across investors if the non-traded asset do: the *two-fund separation theorem is no longer valid*. See Figure 9.3 for an illustration and Example 9.6 for more details. Note that the optimal portfolio tends to have lower weights on assets that are positive correlated with the non-traded asset and vice versa.

**Example 9.6** *(Details on Figure 9.3) With the parameters in the figure, the optimal portfolio weights on the three investable risky assets are 0.24, 0.45 and 0.49.*

*Proof* of (9.8). The portfolio return (9.7) can be written $R_p = v'R^e + \phi R_c^e + R_f$, where $v = (1-\phi)w$. The investor solves

$$\max_v v'\mu^e + \phi\mu_c^e + R_f - \frac{k}{2}(v'\Sigma v + \phi^2\sigma_c^2 + 2\phi v'S_c),$$

with first order conditions

$$\mu^e - k(\Sigma v + \phi S_c) = 0.$$

Solve for $v$ and divide by $1-\phi$ to get (9.8). $\square$

**Remark 9.7** *(Interpreting the hedging terms*) The hedging terms are related to the slope coefficients from a regression of $R_c^e$ on the vector of investable risky assets ($R^e$) $R_{ct}^e = \alpha + \gamma' R_t^e + \varepsilon_t$, since $\gamma = \Sigma^{-1} S_c$.*

**Example 9.8** *(Portfolio choice of a pharmaceutical engineer) Suppose asset 1 is an index of pharmaceutical stocks, and asset 2 is the rest of the equity market. Consider a person working as a pharmaceutical engineer: the covariance of her labour with asset 1 is likely to be high, while the covariance with asset 2 might be fairly small. This person should therefore tilt the financial portfolio away from pharmaceutical stocks: the market portfolio is not the best for everyone.*

**Remark 9.9** *(Transformed assets*) However, the optimal portfolio $w$ is on the mean-variance frontier of some transformed assets with returns $Z_i$. We can rewrite the portfolio return as*

$$R_p = w'Z + (1-\mathbf{1}'w)Z_f, \text{ where}$$
$$Z_i = (1-\phi)R_i + \phi R_c \text{ and and } Z_f = (1-\phi)R_f + \phi R_c.$$

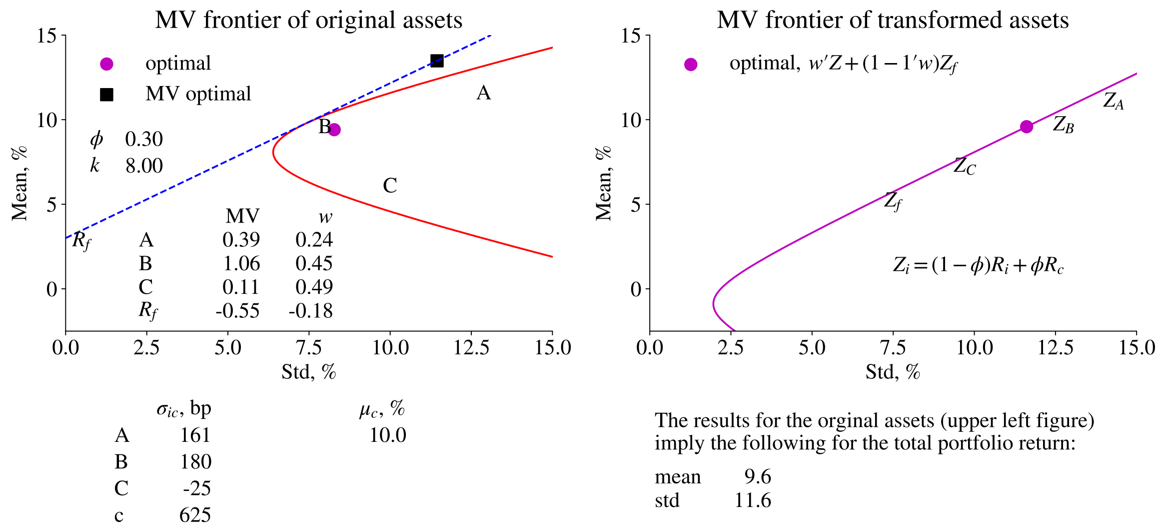



Figure 9.3: Portfolio choice with background risk. The properties of the investable assets (A, B, and C) are shown in Table 9.2.

*Notice that all these transformed assets (also $Z_f$) are risky. The optimal portfolio will be on the mean-variance frontier of $Z = (Z_i, Z_f)$. See Figure 9.3 (right panel). (The "proof" is that maximizing the objective function (9.5) subject to this new definition of the portfolio return is a traditional mean-variance problem—but in terms of the transformed assets $Z$.)*

### 9.3.3 Asset Pricing Implications of Background Risk I

If the background risk affects portfolio choice for a large fraction of the investors, then it is also likely to influence the (equilibrium) asset pricing. For instance, an asset which provides an effective hedge against background risk will be greatly demanded—and therefore generate low returns.

Under strong assumptions (MV preferences, same beliefs, similar background risk) this leads to an extension of the traditional CAPM expression for expected returns in the form of a *multi-beta model*

$$\mathrm{E}\, R_i^e = \beta_{im}\mu_m^e + \beta_{ic}\mu_c^e, \tag{9.9}$$

where $\mu_m^e$ and $\mu_c^e$ are the average excess returns on the (here two) factors. Also, $\beta_{im}$ and

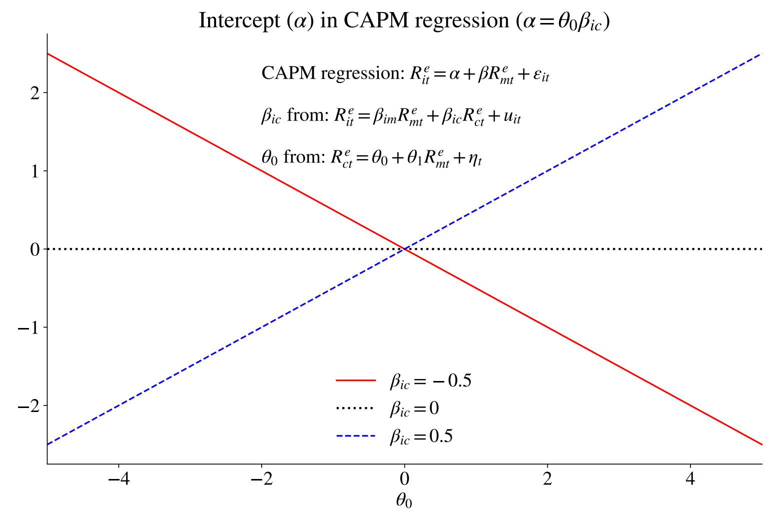



Figure 9.4: $\alpha$ in CAPM regression

$\beta_{ic}$ are the multiple regression coefficients from the *multi-factor* regression

$$R^e_{it} = a_i + \beta_{im} R^e_{mt} + \beta_{ic} R^e_{ct} + u_{it}. \tag{9.10}$$

Notice that there is no intercept in (9.9), so the testable implication is that $\alpha = 0$ in (9.10). (The proof is in the Appendix.)

In this case, the expected excess return on asset $i$ depends on how it is related to both the (financial) market and the background risk. The key implication of (9.9) is that there are two risk factors that influence the required risk premium of asset $i$: both the market and the background risk.

**Example 9.10** *(Multi-factor model) With* $(\beta_{im}, \beta_{ic}) = (0.37, 0.26)$ *and* $(\mu^e_m, \mu^e_c) = (0.08, 0.06)$*, the expected excess return of asset i should be*

$$\mathrm{E}\, R^e_i = 0.37 \times 0.08 + 0.26 \times 0.06 \approx 0.045.$$

### 9.3.4 Asset Pricing Implications of Background Risk II: Reinterpretation of CAPM Results

Consider the standard CAPM regression

$$R^e_{it} = \alpha_i + \beta_i R^e_{mt} + \varepsilon_{it}, \tag{9.11}$$

where $R^e_{mt}$ is the market excess return in period $t$. We use time series data to estimate it. The intercept $\alpha$ is likely to be non-zero if the $R_{it}$ returns are driven by a multi-factor model. That is, (9.11) suffers from an omitted variable bias.

To be precise, suppose the two-factor model (9.10) holds with a zero intercept. Then, the OLS estimate of $\alpha_i$ from (9.11) is

$$\hat{\alpha}_i = \hat{\theta}_0 \hat{\beta}_{ic}, \tag{9.12}$$

where $\hat{\beta}_{ic}$ is the beta in the two-factor model regression (9.10) and $\hat{\theta}_0$ is the estimate of the intercept in

$$R^e_{ct} = \theta_0 + \theta_1 R^e_{mt} + \eta_t. \tag{9.13}$$

(To show this, apply Remark 9.11.) Together, these two equations suggest that non-zero alphas from CAPM regression may be explained by a combination of *(1)* a missing factor ($\beta_{ic} \neq 0$); *(2)* and that factor is not "priced" by the market returns alone ($\theta_0 \neq 0$). This is illustrated in Figure 9.4.

**Remark 9.11** *(Omitted variable bias in OLS) Suppose the correct regression model is $y_t = x_t'\beta + h_t\gamma + u_t$, but we omit the $h_t$ regressor and estimate $y_t = x_t'\delta + \varepsilon_t$ by OLS. It is well known that the OLS estimate $\hat{\delta} = \hat{\beta} + \hat{\theta}\hat{\gamma}$, where $\hat{\theta}$ is from regressing $h_t = x_t'\theta + \eta_t$.*

## 9.4 Heterogeneous Investors

**To do: rewrite this section**

## 9.5 Joint Portfolio and Savings Choice

The basic *consumption-based* multi-period investment problem assumes that the investor derives utility from consumption in every period and that the utility in one period is additively separable from the utility in other periods. For instance, if the investor plans for 2 periods (labelled 1 and 2), then the task is to maximize expected utility

$$\max U(c_1) + \delta \,\mathrm{E}_1\, U(c_2), \text{ subject to} \tag{9.14}$$

$$c_1 + I_1 = W_1 \tag{9.15}$$

$$c_2 + I_2 = (1 + R_p)I_1 + y_2, \text{ where} \tag{9.16}$$

$$R_p = v' R^e + R_f. \tag{9.17}$$

In equation (9.14), $c_t$ is consumption in period $t$. The current period (when the portfolio is chosen) is period 1—so all expectations are made on the basis of the information available then. The constant $\delta$ is the time discounting, with $0 < \delta < 1$ indicating impatience. (In an equilibrium without risk, we will get a positive real interest rate if investors are impatient.)

Equation (9.15) is the budget constraint for period 1: an initial wealth (including exogenous income), $W_1$, is split between consumption, $c_1$, and investment, $I_1$. Equation (9.16) is the budget constraint for period 2: consumption plus investment must equal the wealth at the beginning of period 2 plus (exogenous) income, $y_2$. The wealth at the beginning of period 2 equals the investment in period 1, $I_1$, times the gross portfolio return—which in turn (see (9.17)) depends on the portfolio weights chosen in period 1 ($v$) as well as on the returns on the assets (from holding them from period 1 to period 2).

Obtaining closed-form solutions is typically difficult. However, we can gain some insights by studying the first order conditions. The optimization problem involves maximizing with respect to the investment level (mostly a macro topic, but summarized in a Remark below) and how to form the investment portfolio, which is the focus below.

**Remark 9.12** *(**Investment level *in period 1, $I_1$) It is clear that $I_2 = 0$ since investing in period 2 is just a waste, so we can maximise with respect to $I_1$. The first order condition for $I_1$ is that the derivative of (9.14) wrt $I_1$ is zero*

$$U'(c_1)\frac{\partial c_1}{\partial I_1} + \delta \,\mathrm{E}_1\left[U'(c_2)\frac{\partial c_2}{\partial I_1}\right] = 0$$

$$-U'(c_1) + \delta \,\mathrm{E}_1\left[U'(c_2)\left(1 + R_p\right)\right] = 0,$$

*where $U'(c_t)$ is the marginal utility in period $t$. Notice that we use the chain rule for the derivatives where $\partial c_1/\partial I_1 = -1$ and $\partial c_2/\partial I_1 = 1 + R_p$ can be calculated from the budget constraints (9.15)–(9.16). This says that consumption should be planned such that the marginal loss of utility from investing (decreasing $c_1$) equals the discounted expected marginal gain of utility from increasing $c_2$ by the gross return on the investment. For* *instance, with logarithmic utility we get* $\mathrm{E}_1\, c_2/c_1 = \delta\left(1 + R_f\right)$*, which says that when* $R_f$ *is high, then the expected (planned) consumption path is upward sloping.*

### 9.5.1 Optimal Portfolio Choice

This section studies the *portfolio choice*, that is, the portfolio weights in the vector $v$. The first order condition for $v_i$ is

$$\begin{aligned} \delta\,\mathrm{E}_1\left[U'(c_2)\frac{\partial c_2}{\partial v_i}\right] &= 0 \\ \mathrm{E}_1\left[U'(c_2) R_i^e\right] &= 0 \text{ for } i = 1, \ldots, n. \end{aligned} \tag{9.18}$$

This uses the chain rule and differentiates the budget constraint (9.16) to get $\partial c_2/\partial v_i = I_1 R_i^e$. For the second line, we divide both sides by $\delta I_1$ since it is known at the time of the portfolio choice (and not zero). The expression says that excess returns should be "orthogonal" to marginal utility.

**Remark 9.13** *(Rewriting* $\mathrm{E}\, xy = 0$*) Recall that, by definition,* $\mathrm{Cov}(x, y) = \mathrm{E}\, xy - \mathrm{E}\, x \times \mathrm{E}\, y$*.* $\mathrm{E}\, xy = 0$*, so* $\mathrm{E}\, y = -\mathrm{Cov}(x, y)/\,\mathrm{E}\, x$*.*

The first order conditions still contain some useful information. Use Remark 9.13 to write (9.18) as

$$\mathrm{E}_1\, R_i^e = \frac{-\mathrm{Cov}_1[U'(c_2), R_i^e]}{\mathrm{E}_1\, U'(c_2)}, \tag{9.19}$$

where the time subscripts on the expectation and covariance operators indicate that they are conditional on the information in period 1.

First, the denominator is positive (marginal utility always is). Second, suppose the return is procyclical, $\mathrm{Cov}(c_2, R_i^e) > 0$. This will make $\mathrm{Cov}[U'(c_2), R_i^e] < 0$, since marginal utility $U'(c_2)$ is a decreasing function of the consumption level (the utility function is concave). Together, this creates a positive risk premium, $\mathrm{E}\, R_i^e > 0$. That is, *an asset is risky if it is procyclical. (*Recall that risky assets have high risk premia since otherwise no one would like to buy those assets.)

**Remark 9.14** *(*Linearizing* $U'(c)$*) A first-order Taylor approximation of marginal utility around* $\bar{c}$ *is* $U'(C) \approx U'(\bar{c}) + U''(\bar{c})(c - \bar{c})$*. The numerator in (9.19) can thus be written* $-\mathrm{Cov}[U'(c_2), R_i^e] \approx -U''(\bar{c})\,\mathrm{Cov}(c_2, R_i^e)$*,where* $-U''(\bar{c}) > 0$ *since the utility function is concave.*

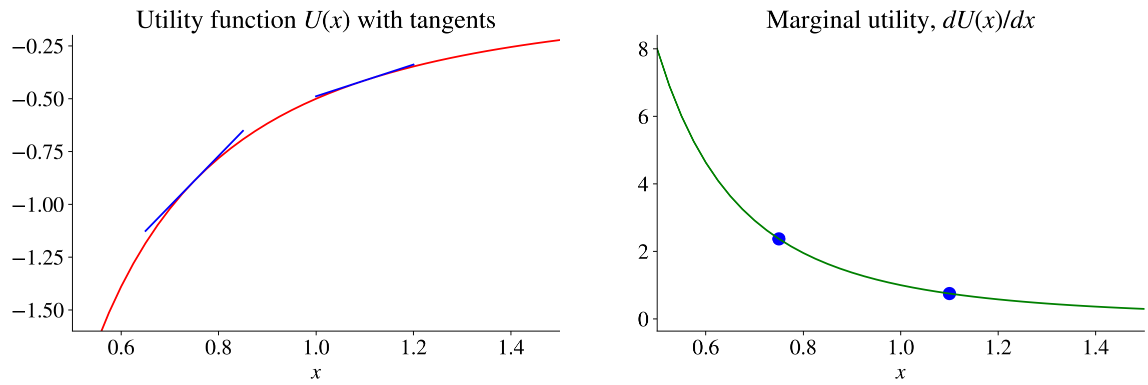



Figure 9.5: Utility function

Although these results were derived from a two-period problem, it can be shown that a problem with more periods gives the same first-order conditions. In this case, the objective function is

$$U(c_1) + \delta \operatorname{E}_1 U(c_2) + \delta^2 \operatorname{E}_1 U(c_3) + \dots \delta^{T-1} \operatorname{E}_1 U(c_T). \tag{9.20}$$

### 9.5.2 The Stochastic Discount Factor or Pricing Kernel

Define the *stochastic discount factor* (SDF) or pricing kernel

$$M = \delta \frac{U'(c_2)}{U'(c_1)}, \tag{9.21}$$

where we, for convenience drop the time subscript. Rewrite the first order condition for $v_i$ (9.18) and the expression for expected excess returns (9.19) as

$$\operatorname{E}_1(M R_i^e) = 0, \text{ and} \tag{9.22}$$

$$\operatorname{E}_1 R_i^e = -\operatorname{Cov}_1(M, R_i^e) / \operatorname{E}_1 M. \tag{9.23}$$

Notice that the *same* SDF is used for each asset $i$. Such SDFs can be derived in many ways (here we used a consumption plan approach), but they typically imply an expression like (9.22). (Similarly, the foc in Remark 9.12 can be written $\operatorname{E}_1 \left[M \left(1 + R_p\right)\right] = 1$.)

**Remark 9.15** *(Pricing with a stochastic discount factor, SDF*) Let $M_{t+1}$ be an SDF and $x_{t+1}$ the payoff of an asset in $t+1$. Most asset pricing theories imply that that the price today of the asset today ($P_t$) must satisfy (a) $P_t = \operatorname{E}_t M_{t+1} x_{t+1}$. This implies that the gross return must satisfy (b) $\operatorname{E}_t M_{t+1}(1 + R_{t+1}) = 1$ and the excess returns must satisfy* *(c)* $\mathrm{E}_t\, M_{t+1} R^e_{t+1} = 0$.

### 9.5.3 The Equity Premium Puzzle*

**Remark 9.16** *(Stein's lemma) If $x$ and $y$ have a bivariate normal distribution and $h(y)$ is a differentiable function such that* $\mathrm{E}[|h'(y)|] < \infty$*, then* $\mathrm{Cov}[x, h(y)] = \mathrm{Cov}(x, y)\,\mathrm{E}[h'(y)]$.

With CRRA utility, $c^{1-\gamma}/(1-\gamma)$, the SDF is

$$M_2 = \delta (c_2/c_1)^{-\gamma}$$

If the excess return, $R^e_t$, and consumption growth, $\Delta c_t$, have a bivariate normal distribution, then by using Stein's lemma, we can rewrite the risk premium (9.23) as

$$\mathrm{E}_1\, R^e = \mathrm{Cov}_1(R^e, \Delta c)\gamma \tag{9.24}$$

$$= \mathrm{Corr}_1(R^e, \Delta c)\, \mathrm{Std}_1(R^e)\, \mathrm{Std}_1(\Delta c)\gamma. \tag{9.25}$$

The "equity premium puzzle" is that, over a long U.S. sample of the equity market and consumption per capita, $\mathrm{E}_1\, R^e \approx 0.08$, $\mathrm{Corr}_1(R^e, \Delta c) \approx 0.15$, $\mathrm{Std}_1(R^e) \approx 0.2$ and $\mathrm{Std}_1(\Delta c) \approx 0.02$, so an implausibly high risk aversion ($\gamma \approx 133$) is required to account for the high risk premia on the equity market. Basically, consumption is not volatile enough to explain the risk premium. See Cochrane (2005) for an extensive analysis.

*Proof* of (9.24). Stein's lemma gives $\mathrm{Cov}[R^e, \exp(\ln M)] = \mathrm{Cov}(R^e, \ln M)\,\mathrm{E}\, M$. (In terms of Stein's lemma, $x = R^e$, $y = \ln M$ and $h() = \exp()$.) Finally, notice that $\mathrm{Cov}(R^e, \ln M) = -\gamma\, \mathrm{Cov}(R^e, \Delta c)$. □

### 9.5.4 From a Consumption-Based Model to CAPM

Suppose the stochastic discount factor (ratio of marginal utilities) is an affine function of the market excess return

$$M = a - bR^e_m, \text{ with } b > 0. \tag{9.26}$$

This would, for instance, be the case in a Lucas model where consumption equals the market return and the utility function is quadratic—but it could be true in other cases as well. From (9.23) some rearrangements we get

$$\mathrm{E}\, R^e_i = \beta_i\, \mathrm{E}\, R^e_m, \text{ where } \beta_i = \sigma_{im}/\sigma^2_m, \tag{9.27}$$

which is the standard CAPM expression. This means that CAPM is consistent with (some) multi-period utility based portfolio choice models.

*Proof* of (9.27). Using (9.26) in (9.23) gives $\mathrm{E}\, R_i^e = b\sigma_{im}/\,\mathrm{E}(a - bR_m^e)$. We can, of course, apply this expression to the market excess return (instead of asset $i$) to get $\mathrm{E}\, R_m^e = b\sigma_m^2/\,\mathrm{E}(a - bR_m^e)$.Solve for $b/\,\mathrm{E}(a - bR_m^e)$ and use that in the first equation to get (9.27). □

### 9.5.5 From a Consumption-Based Model to a Multi-Factor Model

The consumption-based model may not look like a factor model, but it could easily be written as one. The idea is to assume that the stochastic discount factor (ratio of marginal utilities) is a linear function of some key macroeconomic variables, for instance, output ($y$) and interest rates ($r$)

$$M = ay + br, \tag{9.28}$$

where we have dropped time subscripts.

Such a formulation makes a lot of sense in most macro models—at least as an approximation. It is then possible to write (9.23) as

$$\mathrm{E}\, R_i^e = \frac{a\sigma_{iy} + b\sigma_{ir}}{-\mathrm{E}(ay + br)} \tag{9.29}$$

$$= \beta_{iy}\mu_y^e + \beta_{ir}\mu_r^e, \tag{9.30}$$

where $(\beta_{iy}, \beta_{ir})$ are from a multiple regression of $R_{it}^e$ on excess returns on assets that are perfectly correlated with $y$ and $r$ respectively ("factor mimicking portfolios"), while $(\mu_y^e, \mu_r^e)$ are the corresponding average excess returns. (The proof is in the Appendix.) The more general insight is that when the SDF is linear in $K$ factors, then we get a $K$-beta model for average returns.

## 9.6 Testing Multi-Factors Models

Let $R_{ot}^e$ be a vector of factors *excess returns*. We can then test a multi-factor model by testing whether $\alpha = 0$ in the regression

$$R_{it}^e = \alpha + \beta' R_{ot}^e + \varepsilon_{it}. \tag{9.31}$$

(This test becomes invalid if some factors are not excess returns.)

The t-test of the null hypothesis that $\alpha_i = 0$ uses the fact that, under fairly mild conditions, the t-statistic has an asymptotically normal distribution, that is

$$\frac{\hat{\alpha}_i}{\text{Std}(\hat{\alpha}_i)} \xrightarrow{d} N(0,1) \text{ under } H_0 : \alpha_i = 0. \tag{9.32}$$

Fama and French (1993) try a multi-factor model. They find that a three-factor model fits the 25 stock portfolios fairly well (two more factors are needed to also fit the seven bond portfolios that they use). This three-factor model is rejected at traditional significance levels, but it can still capture a fair amount of the variation of expected returns.

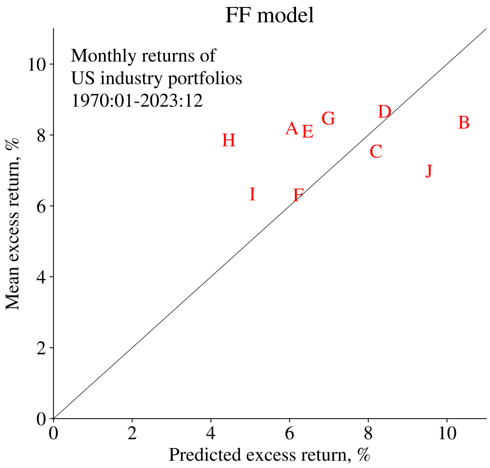
**Remark 9.17** *(Fama-French factors) Fama and French (1993) use three factors: the market excess return, the return on a portfolio of small stocks minus the return on a portfolio of big stocks (SMB), and the return on a portfolio with a high ratio of book value to market value minus the return on a portfolio with a low ratio (HML). All three are excess returns (although only the first is in excess of a risk-free return), since they are long-short portfolios. He and Ng (1994) try to relate these factors to macroeconomic series.*

Chen, Roll, and Ross (1986) use a number of macro variables as factors—along with traditional market indices. They find that industrial production and inflation surprises are priced factors, while the market index might not be. For such (non-return) factors it is common to use factor mimicking portfolios: the excess return on portfolios strongly correlated with the factors.

**Empirical Example 9.18** *Figure 9.6 shows some results for the Fama-French model on US industry portfolios and Figures 9.7–9.8 on the 25 Fama-French portfolios. The results indicate that the FF model is a considerable improvement compared to CAPM for the 25 FF portfolios, but perhaps not so much for the industry portfolios. Even for the 25 FF portfolios, strict statistical tests reject also the FF model, but the fit of the average returns is clearly better than CAPM.*

## 9.7 Appendix – Details on the Asset Pricing Implications*

**Remark 9.19** *Let portfolio $q$ have the portfolio weights $v$. Then, the covariance with portfolio $p$ (with weights $w$) is then* $\text{Cov}(R_q, R_p) = v'\Sigma w$.




| | $\alpha$ (ann.) | $t$-stat | $\sigma$ (ann.) |
|---|---|---|---|
| A (NoDur) | 2.09 | 1.79 | 8.46 |
| B (Durbl) | -2.13 | -0.95 | 16.26 |
| C (Manuf) | -0.71 | -0.84 | 6.08 |
| D (Enrgy) | 0.20 | 0.09 | 16.00 |
| E (HiTec) | 1.60 | 1.21 | 9.55 |
| F (Telcm) | 0.03 | 0.02 | 10.69 |
| G (Shops) | 1.44 | 1.16 | 8.99 |
| H (Hlth ) | 3.35 | 2.33 | 10.40 |
| I (Utils) | 1.23 | 0.81 | 10.98 |
| J (Other) | -2.62 | -3.38 | 5.61 |

Fama-French model
Factors: US market, SMB (size), and HML (book-to-market)
Predicted excess return: $\beta_m R_m^e + \beta_{SMB} R_{SMB} + \beta_{HML} R_{HML}$

10% crit. value (Bonferroni): 2.58

Test if all $\alpha_i = 0$:

| | |
|---|---|
| Wald stat | 24.03 |
| 5% crit val | 18.31 |
| p-value | 0.01 |

Figure 9.6: Fama-French regressions on US industry indices

*Proof* of (9.9). (A slow proof to give the key details.) Assume $k$ in (9.8) is such that $w$ equals the (financial) market portfolio, $w_m$ in (9.8). For any portfolio with portfolio weights $w_p$, the covariance with the market return (times $1-\phi$) is

$$\begin{aligned}(1-\phi)\sigma_{pm} &= (1-\phi)w_p'\Sigma w_m \\ &= w_p'\Sigma\Sigma^{-1}(\mu^e/k - S_c\phi) \text{ using (9.8)} \\ &= \mu_p^e/k - \phi\sigma_{pc},\end{aligned}$$

since $w_p'\mu^e = \mu_p^e$ and $w_p'S_c = \sigma_{pc}$. Put $\mu_p^e/k$ on the LHS and rewrite as

$$\mu_p^e/k = \begin{bmatrix} 1-\phi & \phi \end{bmatrix} \begin{bmatrix} \sigma_{pm} \\ \sigma_{pc} \end{bmatrix} \quad (*)$$

Invert the definition of the regression slopes

$$\begin{bmatrix} \beta_{pm} \\ \beta_{pc} \end{bmatrix} = \begin{bmatrix} \sigma_{mm} & \sigma_{mc} \\ \sigma_{mc} & \sigma_{cc} \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{pm} \\ \sigma_{pc} \end{bmatrix}$$

to substitute for the $[\sigma_{pm}, \sigma_{pc}]$ vector in (*)

$$\mu_p^e/k = \begin{bmatrix} 1-\phi & \phi \end{bmatrix} \begin{bmatrix} \sigma_{mm} & \sigma_{mc} \\ \sigma_{mc} & \sigma_{cc} \end{bmatrix} \begin{bmatrix} \beta_{pm} \\ \beta_{pc} \end{bmatrix}. \quad (**)$$

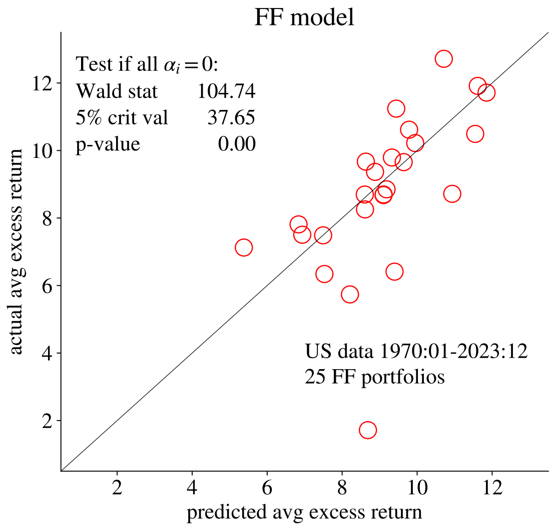



Figure 9.7: FF, FF portfolios

($\sigma_{mm}$ and $\sigma_{cc}$ are here used to denote variances.) The second line just multiplies and divides by the covariance matrix. The third line follows from the usual definition of regression coefficients, $\beta = \text{Var}(x)^{-1}\,\text{Cov}(x, y)$. The last shows the result from multiplying.

For the market return $(\beta_{mm}, \beta_{mc}) = (1, 0)$, so (**) gives

$$\mu_m^e/k = \begin{bmatrix} 1-\phi & \phi \end{bmatrix} \begin{bmatrix} \sigma_{mm} \\ \sigma_{mc} \end{bmatrix}.$$

For any asset that is perfectly correlated with $R_c$ (a "factor mimicking" portfolio) we have $(\beta_{cm}, \beta_{cc}) = (0, 1)$, so (**) gives

$$\mu_c^e/k = \begin{bmatrix} 1-\phi & \phi \end{bmatrix} \begin{bmatrix} \sigma_{mc} \\ \sigma_{cc} \end{bmatrix}.$$

Use these last two equations to substitute for the first two terms in (**) and cancel the $1/k$ factors to get

$$\mu_p^e = \begin{bmatrix} \mu_m^e & \mu_c^e \end{bmatrix} \begin{bmatrix} \beta_{pm} \\ \beta_{pc} \end{bmatrix},$$

which is (9.9). $\square$

*

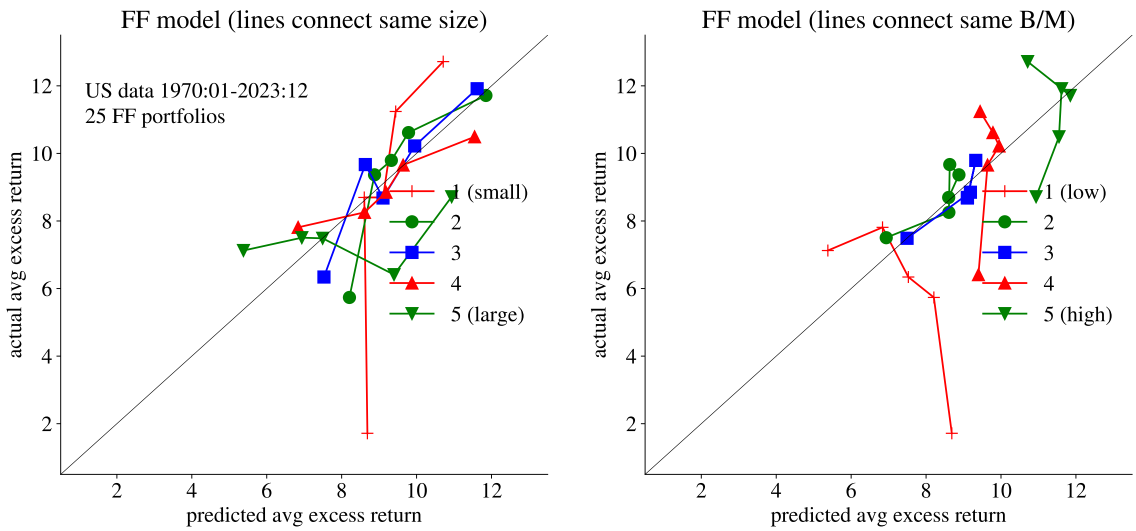



Figure 9.8: FF, FF portfolios

*Proof* of 9.30. Rewrite (9.29) as

$$\mathrm{E}\, R_i^e = \frac{1}{-\mathrm{E}(ay+br)} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \sigma_{yy} & \sigma_{yr} \\ \sigma_{yr} & \sigma_{rr} \end{bmatrix} \begin{bmatrix} \sigma_{yy} & \sigma_{yr} \\ \sigma_{yr} & \sigma_{rr} \end{bmatrix}^{-1} \begin{bmatrix} \sigma_{iy} \\ \sigma_{ir} \end{bmatrix} \qquad (+)$$

$$= \frac{1}{-\mathrm{E}(ay+br)} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \sigma_{yy} & \sigma_{yr} \\ \sigma_{yr} & \sigma_{rr} \end{bmatrix} \begin{bmatrix} \beta_{iy} \\ \beta_{ir} \end{bmatrix}.$$

Applying (+) to an asset that is perfectly correlated with $y$ gives $(\beta_{iy}, \beta_{ir}) = (1, 0)$, so

$$\mu_y^e = \frac{1}{-\mathrm{E}(ay+br)} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \sigma_{yy} \\ \sigma_{yr} \end{bmatrix}.$$

Similarly, applying (+) to an asset that is perfectly correlated with $r$ gives $(\beta_{iy}, \beta_{ir}) = (0, 1)$, so

$$\mu_r^e = \frac{1}{-\mathrm{E}(ay+br)} \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} \sigma_{yr} \\ \sigma_{rr} \end{bmatrix}$$

Use these two expressions to rewrite (+) as

$$\mathrm{E}\, R_i^e = \begin{bmatrix} \mu_y^e & \mu_r^e \end{bmatrix} \begin{bmatrix} \beta_{iy} \\ \beta_{ir} \end{bmatrix}.$$

□

# Chapter 10

# Efficient Markets

## 10.1 The Efficient Market Hypothesis

The efficient market hypothesis (EMH) says that it is very *hard to predict future asset returns*. If this is true (evidence is discussed later), then active management, such as security analysis and market timing is of limited use and incurs costs (management fees, trading costs). Instead, it is more practical to apply a passive approach that satisfies individual requirements (diversification, hedging background risk, appropriate risk level, etc). The implications are thus significant.

### 10.1.1 Different Versions of the Classical Efficient Market Hypothesis

A precise formulation of the EMH needs to specify three things. First, what type of information is used in making those forecasts? Is it price and trading volume data (referred to as the weak form of the EMH), all public information (the semi-strong form), or perhaps all public and private information (the strong form)? Most modern analysis is focused on the weak or semi-strong forms as private information is likely to have predictive power. Second, what is supposed to be unpredictable? Most modern financial theory would focus on *excess returns*, since they represent risk compensation. (Earlier finance would also study the predictability of prices and net returns.) Third, what is the link between predictability and expectations? If an excess return is almost unpredictable, then rational investors would have nearly constant expectations (close the long-run average), that is, almost constant expected risk premia, and portfolio weights are likely to be fairly stable over time. The opposite holds if excess returns are straightforward to predict.

Rejection of the EMH can have different sources: changes in risk or in risk aversion (both rational reasons) or in inefficiencies. It is typically very hard to disentangle these possible sources.

This chapter will present methods and empirical results. The first sections deal with traditional in-sample methods, initially focusing on the return history of the same asset, but later broadening the scope to bring in other types of predictors (fundamental valuation ratios, lagged returns of other assets, etc.) Later sections will instead focus on out-of-sample methods (recursive regressions, trading strategies, etc).

## 10.2 Autocorrelations and Autoregressions

Autocorrelations and autoregressions are tools for studying whether past and current returns can predict future returns (typically of the same asset).

### 10.2.1 Autocorrelation Coefficients

The autocovariances of the $R_t$ process can be estimated as

$$\hat{\gamma}_s = \frac{1}{T} \sum_{t=1+s}^{T} \left(R_t - \bar{R}\right)\left(R_{t-s} - \bar{R}\right), \text{ where} \tag{10.1}$$

$\bar{R} = \frac{1}{T}\Sigma_{t=1}^{T} R_t/T$ is the sample average estimated from the full sample. (In time series analysis we typically divide by $T$ in (10.1) even if there are only $T - s$ observations to estimate $\gamma_s$ from.) In most of the applications of this chapter, $R_t$ indicates either a return or an excess return (see the tables/figures for details).

Autocorrelations are then estimated as

$$\hat{\rho}_s = \hat{\gamma}_s/\hat{\gamma}_0. \tag{10.2}$$

The sampling properties of $\hat{\rho}_s$ are complicated, but there are several useful large sample results for Gaussian processes (these results typically carry over to processes which are similar to the Gaussian). When the true autocorrelations are all zero (not $\rho_0$, of course), then for any lag $s$ different from zero

$$\sqrt{T}\hat{\rho}_s \to^d N(0,1), \tag{10.3}$$

so $\sqrt{T}\hat{\rho}_s$ can be used as a t-stat.

**Example 10.1** *(t-test) Reject the null hypothesis that* $\rho_1 = 0$ *on the 10% significance level if* $\sqrt{T}|\hat{\rho}_1| > 1.64$.

**Empirical Example 10.2** *Figures 10.1–10.2 show autocorrelations for daily U.S., equity data. There is little autocorrelation for large caps, but more for small caps.*

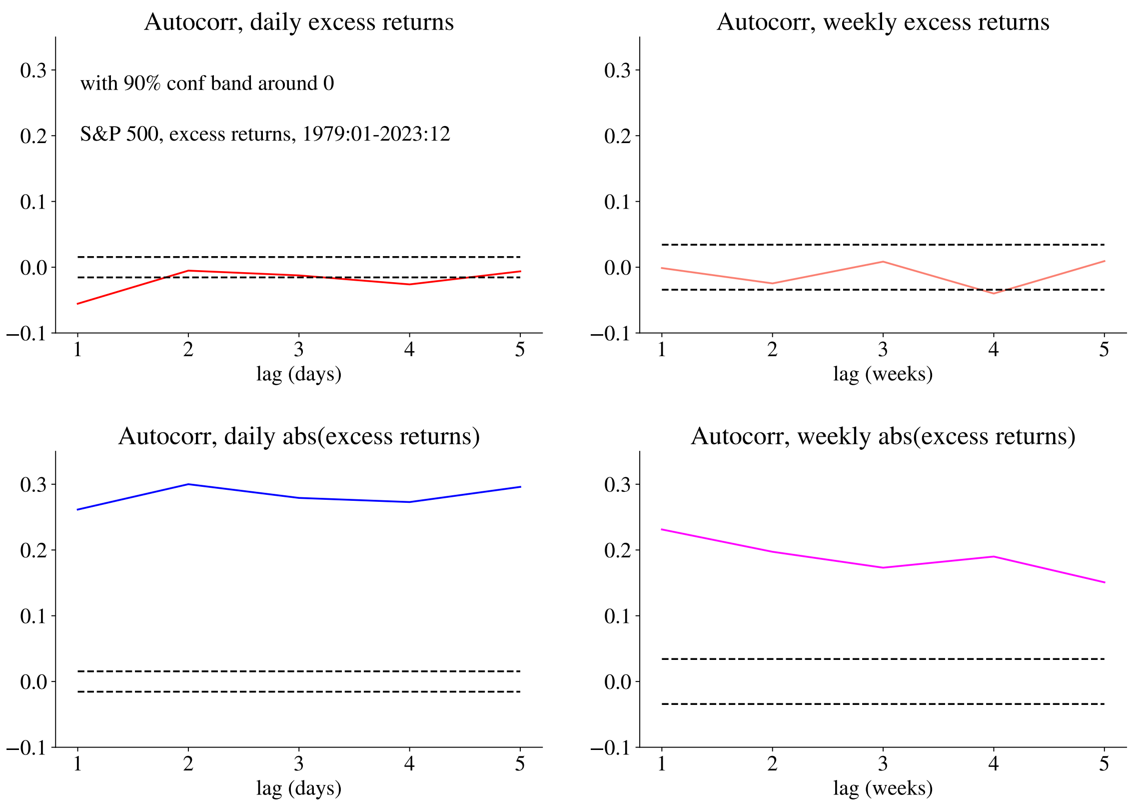



Figure 10.1: Predictability of US stock returns

### 10.2.2 Autoregressions

An alternative method for testing autocorrelations is to estimate an AR model

$$R_t = c + a_1 R_{t-1} + a_2 R_{t-2} + ... + a_p R_{t-p} + \varepsilon_t, \tag{10.4}$$

and test if all slope coefficients $(a_1, a_2, ..., a_p)$ are zero with a $\chi^2$ or $F$ test. This approach is somewhat less general than testing if all autocorrelations are zero, but is easy to implement (and the difference is not large).

**Empirical Example 10.3** *Table 10.1 shows results from estimating an AR model on daily data for S&P 500 returns.*

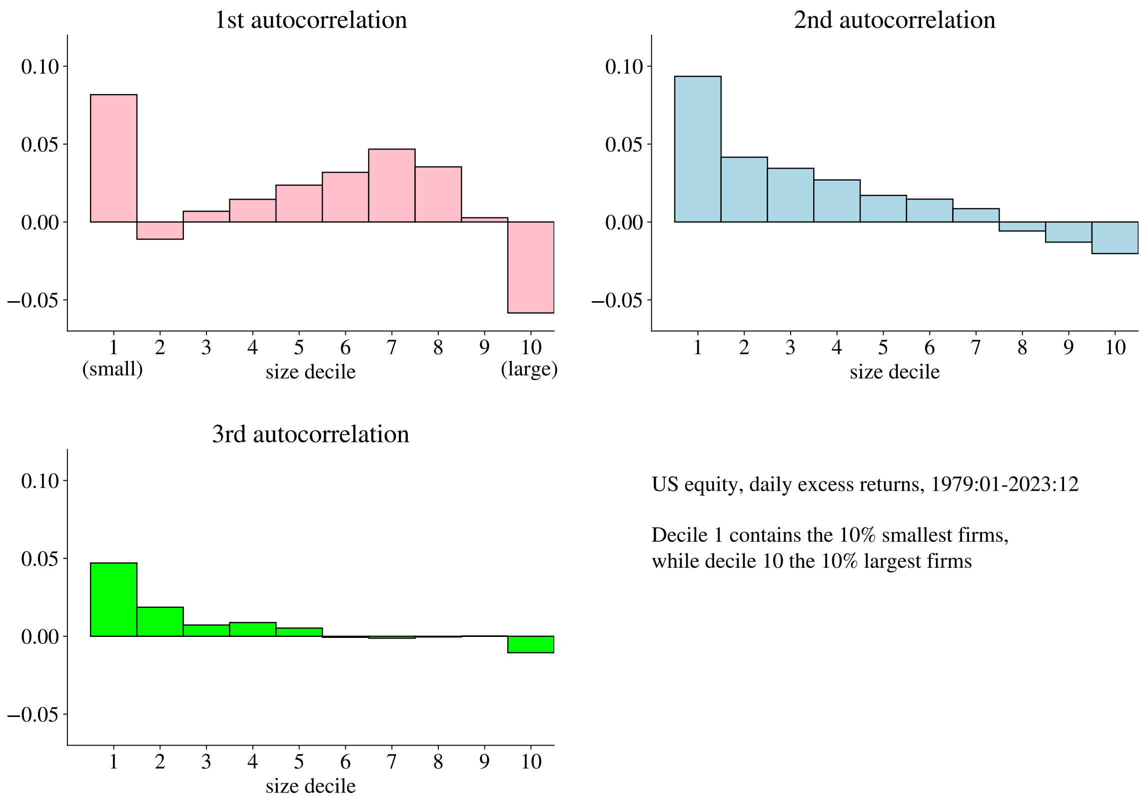



Figure 10.2: Predictability of US stock returns, size deciles

The autoregression can also allow for the coefficients to depend on the market situation. For instance, consider an AR(1), but where the autoregression coefficient may be different depending on the sign of last period's return

$$R_t = \alpha + \beta Q_{t-1} R_{t-1} + \gamma(1 - Q_{t-1}) R_{t-1} + \varepsilon_t, \text{ where} \tag{10.5}$$

$$Q_{t-1} = \begin{cases} 1 \text{ if } R_{t-1} < 0 \\ 0 \text{ else.} \end{cases}$$

**Empirical Example 10.4** *Figure 10.3 shows regression results from daily S&P 500 data. The reversal back after a negative shock is the most prominent finding.*

**Empirical Example 10.5** *Figure 10.4 shows results from autoregressions for different investment horizons. For the business cycle frequency (3-4 years), there is some evidence of negative autocorrelation, that is, reversals. However, testing long-run returns is challenging because it requires a very long sample to have enough (non-overlapping) return periods, and it is unclear if data from long ago is informative about today's economy.*

| | (1) |
|---|---|
| lag 1 | −0.06 |
| | (−2.76) |
| lag 2 | −0.02 |
| | (−0.67) |
| lag 3 | −0.02 |
| | (−0.92) |
| lag 4 | −0.03 |
| | (−1.95) |
| lag 5 | −0.01 |
| | (−0.50) |
| c | 0.03 |
| | (3.01) |
| $R^2$ | 0.01 |
| All slopes | 0.00 |
| obs | 10656 |

Table 10.1: AR(5) of daily S&P returns 1979:01-2023:12. Numbers in parentheses are t-stats, based on Newey-West with 3 lags. All slopes is the p-value for all slope coefficients being zero.

The empirical evidence reported in the section suggest little autocorrelation for daily returns for large-cap stocks (like those in S&P 500), but perhaps more for small- cap stocks. There is also some indication of non-linearity, with more autocorrelation in down markets. For longer return horizons, there appear to be some negative autocorrelation on the business cycle frequency.

## 10.3 Other Predictors and Methods

There are many other possible predictors of future stock returns. For instance, lagged short-run returns on other assets have been employed to predict short-run returns, and both the dividend-price ratio and nominal interest rates to predict long-run returns.

### 10.3.1 Lead-Lags

Stock indices have more positive autocorrelation than (most) individual stocks: there should therefore be fairly strong cross-autocorrelations among individual stocks. Indeed, this is also what is found in US data where returns of large size stocks forecast returns of

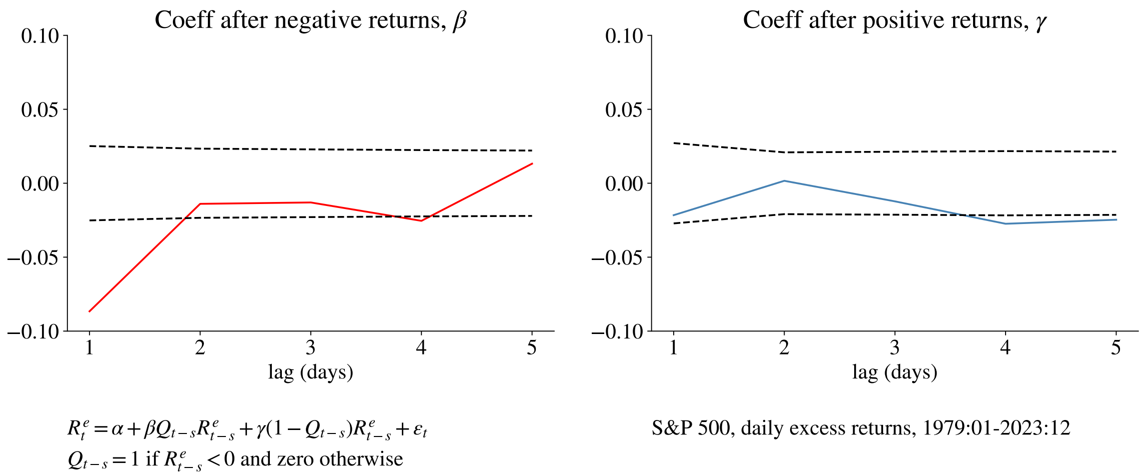



Figure 10.3: Predictability of US stock returns, results from a regression with interactive dummies

small size stocks.

**Empirical Example 10.6** *Figure 10.5 shows (for different size deciles) the regressions coefficients on the 1-day own lag and the 1-day lag of large caps. The results suggest considerable spillover from large caps to the other size deciles.*

### 10.3.2 Earnings-Price Ratio as a Predictor

One of the most effective methods to forecast long-run returns is a regression of future returns on the current earnings-price (or dividend-price) ratio

$$R^e_{s,t} = \alpha + \beta_q \ln(e_{t-s}/p_{t-s}) + \varepsilon_t, \tag{10.6}$$

where $R^e_{s,t}$ is the $s$-period excess return over the period $t-s$ to $t$.

**Empirical Example 10.7** *Figure 10.6 shows results from estimating (10.6) for different investment horizons on data for a U.S. stock market index.*

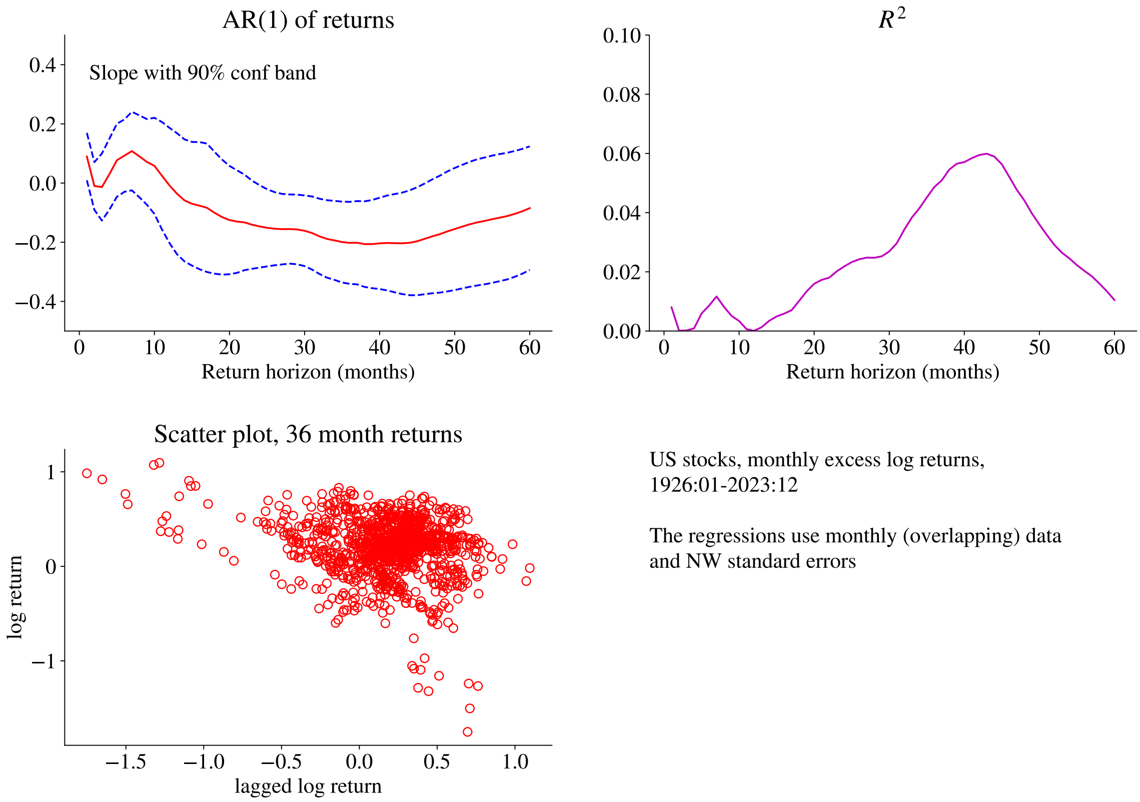



Figure 10.4: Predictability of long-run US stock returns

## 10.4 Out-of-Sample Forecasting Performance

### 10.4.1 In-Sample versus Out-of-Sample Forecasting

In-sample evidence on predictability can potentially be misleading because of *(a)* in-sample overfitting; and/or *(b)* structural breaks.

To gauge the out-of-sample predictability, estimate the prediction equation using data up to and including $t-1$ and then make a forecast for period $t$. This is then an out-of-sample forecast.

The forecasting performance is then compared with a benchmark model, for instance, using the historical average as the predictor. Notice that this benchmark model is also estimated on the same sample. Then, the estimation is redone using data up to and including $t$ and a forecast is made for $t+1$. This is called a recursive approach. An alternative is to instead use a moving data window (ending in $t-1$ and then in $t$,...) where really old data points are discarded. Yet another approach is downweight old data. See Figure 10.7.

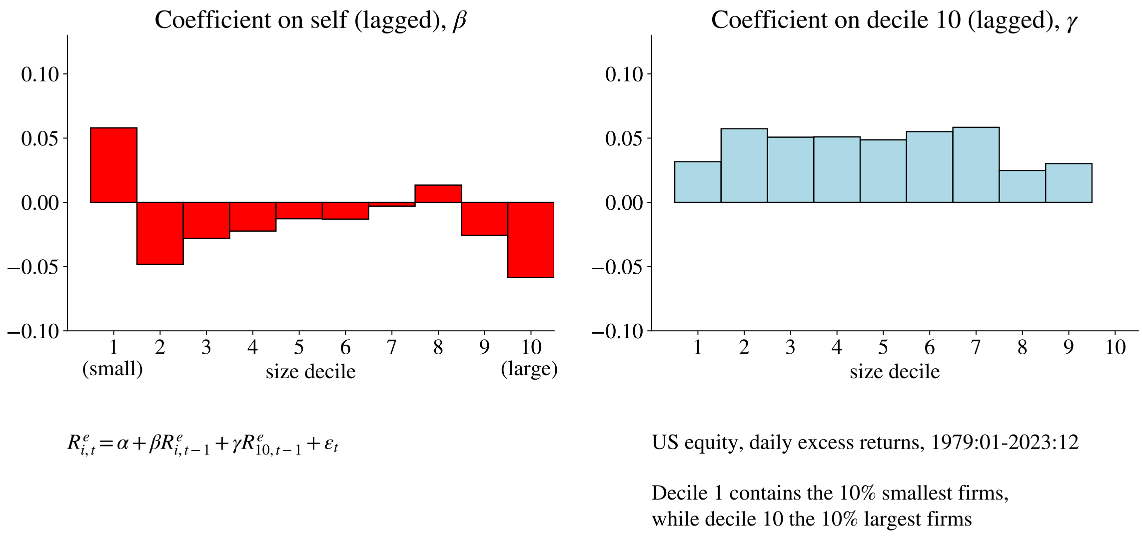



Figure 10.5: Coefficients from multiple prediction regressions

To formalise the comparison, study the root mean square error, RMSE, and the out-of-sample coefficient of determination (denoted $R^2_{OS}$)

$$R^2_{OS} = 1 - \sum\nolimits_{t=s}^{T} (R_t - \hat{R}_t)^2 / \sum\nolimits_{t=s}^{T} (R_t - \tilde{R}_t)^2, \tag{10.7}$$

where $s$ is the first period with an out-of-sample forecast, $\hat{R}_t$ is the forecast based on the prediction model (estimated on data up to and including $t-1$) and $\tilde{R}_t$ is the prediction from some benchmark model (also estimated on data up to and including $t-1$). Make sure not to confuse $R^2_{OS}$ with a return.

**Example 10.8** *($R^2_{OS}$)*

$$R^2_{OS} = 1 - \frac{0.4}{0.5} = 0.2 \text{ (your model is better)}$$
$$R^2_{OS} = 1 - \frac{0.5}{0.4} = -0.25 \text{ (your model is worse)}$$

To test the relative predictive performance, define

$$g_t = (R_t - \hat{R}_t)^2 - (R_t - \tilde{R}_t)^2, \tag{10.8}$$

and test if the sample average, $\bar{g}$, differs from zero, a method described by Diebold and Mariano (1995). Instead of squared forecast errors, we could consider absolute values or an indicator of whether the sign is right, etc. If there is little or no autocorrelation in $g_t$,

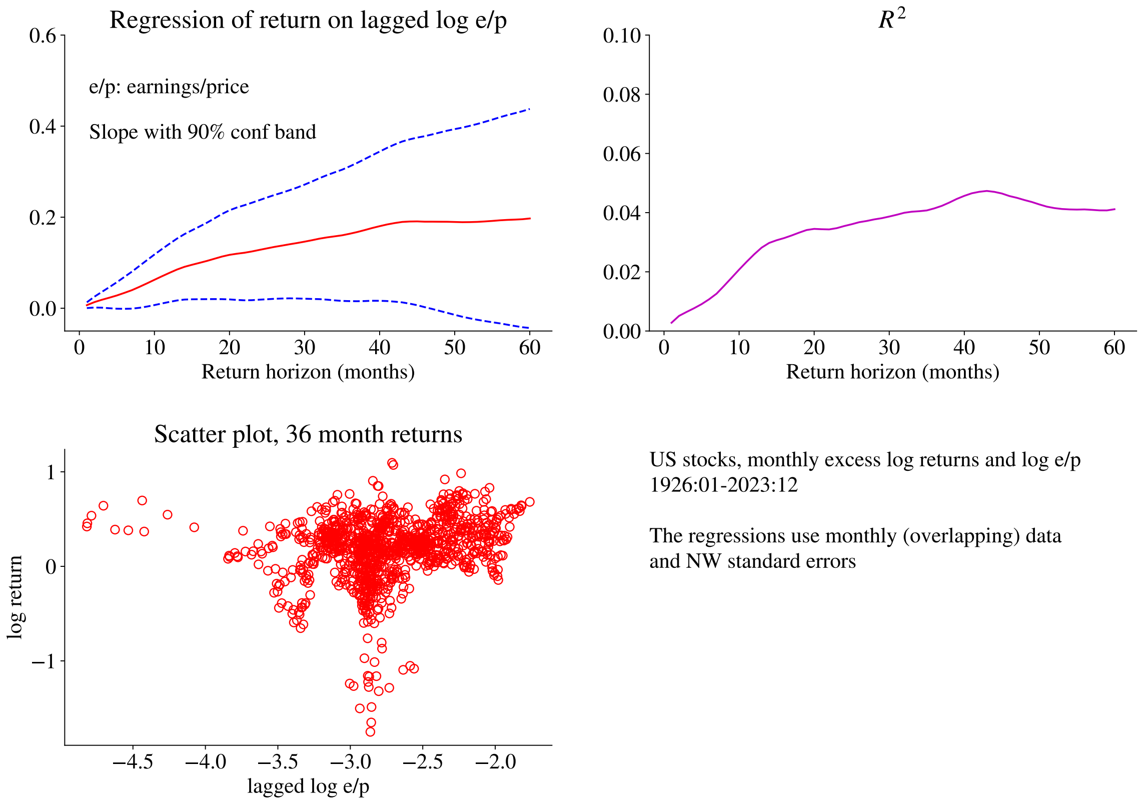



Figure 10.6: Predictability of long-run US stock returns

then $\text{Var}(\bar{g}) = \text{Var}(g_t)/T$ so the $t$-stat is

$$\frac{\bar{g}}{\text{Std}(g_t)/\sqrt{T}}, \tag{10.9}$$

which could be compared with a $N(0,1)$ distribution.

**Empirical Example 10.9** *Figure 10.8 shows results based on daily data for different size deciles. It seems as if an AR(1) model is better than the historical average for small caps, but worse for large caps.*

**Empirical Example 10.10** *Figure 10.9 shows how an e/p regression for a U.S. stock market index compares with the historical average—at different investment horizons. It seems as if it's consistently worse, which is similar to the findings of Goyal and Welch (2008).*

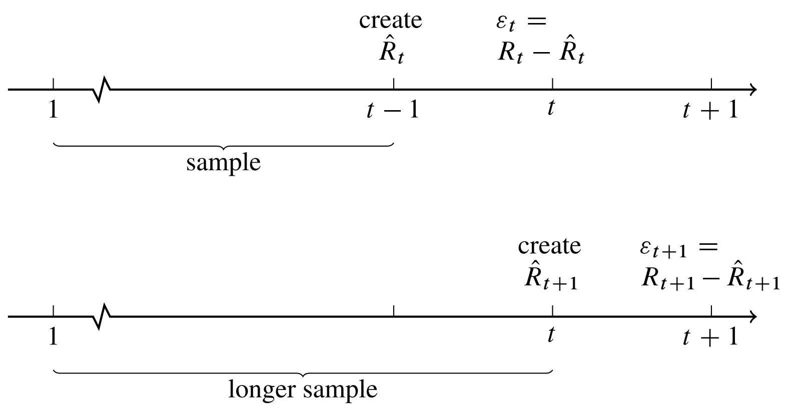



Figure 10.7: Out-of-sample forecasting

### 10.4.2 Trading Strategies

Another way to measure predictability and to illustrate its economic importance is to calculate the return of a *dynamic trading strategy*, and then measure the performance of this strategy in comparison to some benchmark portfolio return. The trading strategy should be based on the variables that are supposed to forecast returns.

A common way to measure the performance of a portfolio is its *alpha* from a regression on the market excess return. Neutral performance requires $\alpha = 0$, which can be tested with a $t$ test.

**Empirical Example 10.11** *Figure 10.10 for an empirical example based on a momentum strategy (bet on recent winners, bet against recent losers) on daily data for the 25 FF portfolios. The upper left figure shows that the strategy has high average returns and* $\alpha$*. It also shows that frequent rebalancing is important for the performance. (At least before trading costs). The lower left figure illustrates the magnitude of trading costs that the strategy can handle, while still generating a positive average excess return. The upper right figure instead investigates the importance of a formation lag: having a time gap between the sorting and portfolio formation. The result suggests a short gap, perhaps shorter than a week.*

### 10.4.3 Technical Analysis

Main reference: Neely (1997), The Economist (1993), Brock, Lakonishok, and LeBaron (1992), Lo, Mamaysky, and Wang (2000)

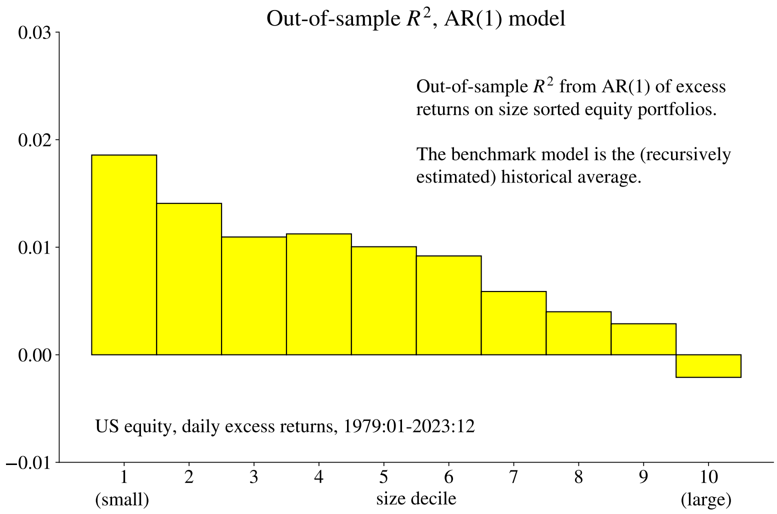



Figure 10.8: Short-run predictability of US stock returns, out-of-sample.

Technical analysis is typically a data mining exercise which looks for local trends or systematic non-linear patterns. The basic idea is that markets are not instantaneously efficient: prices exhibit delayed and predictable reaction to news. In practice, technical analysis amounts to analysing different transformations (for instance, a moving average) of prices—and to spot patterns. This section summarizes some simple trading rules that are used.

Many trading rules rely on some kind of local trend which can be thought of as positive autocorrelation in price movements (also called momentum[1]).

A *moving average rule* involves buying when a short moving average (equally weighted or exponentially weighted) exceeds a long moving average. The idea is that this signals a new upward trend. Let $S$ $(L)$ be the lag order of a short (long) moving average, with $S < L$ and let $b$ be a bandwidth (perhaps 0.01). Then, a MA rule for period $t$ could be

$$\begin{bmatrix} \text{buy in } t \text{ if} & MA_{t-1}(S) > MA_{t-1}(L)(1+b) \\ \text{sell in } t \text{ if} & MA_{t-1}(S) < MA_{t-1}(L)(1-b) \\ \text{no change} & \text{otherwise} \end{bmatrix}, \text{ where} \tag{10.10}$$

$$MA_{t-1}(S) = (p_{t-1} + \ldots + p_{t-S})/S.$$

[1] In physics, momentum equals the mass times speed.

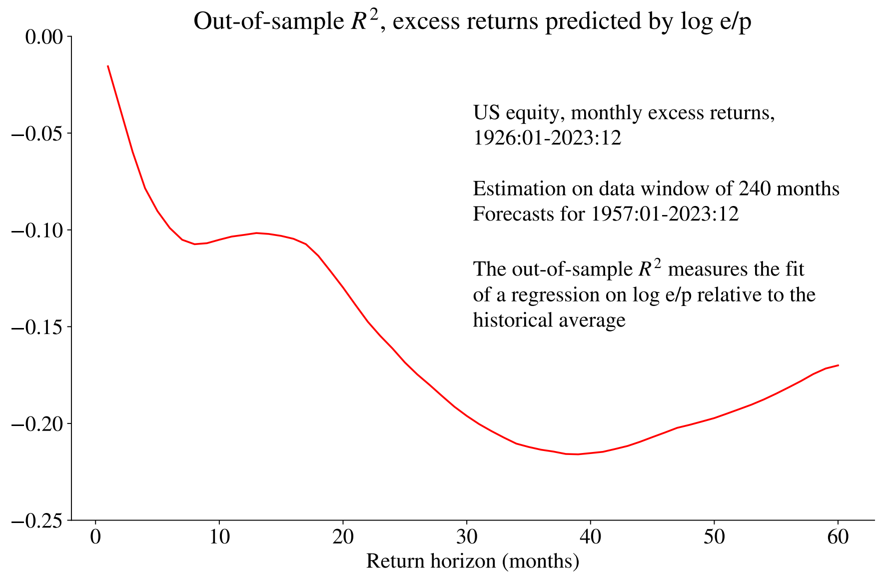



Figure 10.9: Predictability of long-run US stock returns, out-of-sample

The difference between the two moving averages is called an *oscillator*

$$\text{oscillator}_t = MA_t(S) - MA_t(L), \tag{10.11}$$

(or sometimes, moving average convergence divergence, MACD) and the sign is taken as a trading signal (this is the same as a moving average crossing, MAC). A version of the moving average oscillator is the *relative strength index*[2], which is the ratio of average price level (or returns) on "up" days to the average price (or returns) on "down" days—during the last $z$ (14 perhaps) days. Yet another version is to compare the oscillator$_t$ to a moving average of the oscillator (also called a signal line).

The *trading range break-out rule* generally involves buying when the price rises above a previous peak (local maximum). The idea is that a previous peak is a *resistance level* in the sense that some investors are willing to sell when the price reaches that value (perhaps because they believe that prices cannot pass this level; clear risk of circular reasoning or self-fulfilling prophecies; round numbers often play the role as resistance levels). Once this artificial resistance level has been broken, the price can possibly rise substantially. On the downside, a *support level* plays the same role: some investors are willing to buy when the price reaches that value. To implement this, it is common to let the resistance/support

[2]Not to be confused with relative strength, which typically refers to the ratio of two different asset prices (for instance, an equity compared to the market).

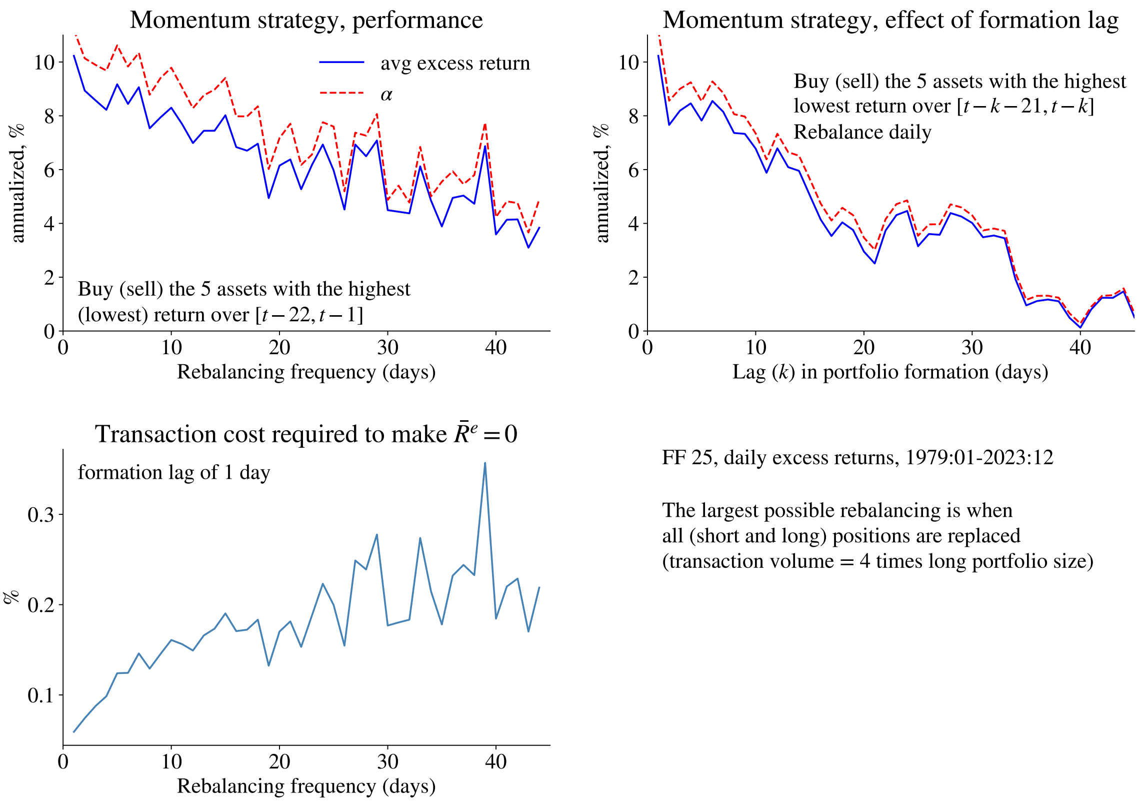



Figure 10.10: Predictability of US stock returns, momentum strategy

levels be proxied by minimum and maximum values over a data window of length $L$. With a bandwidth $b$ (perhaps 0.01), the rule for period $t$ could be

$$\begin{bmatrix} \text{buy in } t \text{ if} & P_t > M_{t-1}(1+b) \\ \text{sell in } t \text{ if} & P_t < m_{t-1}(1-b) \\ \text{no change} & \text{otherwise} \end{bmatrix}, \text{ where} \tag{10.12}$$

$$M_{t-1} = \max(p_{t-1}, \ldots, p_{t-S})$$
$$m_{t-1} = \min(p_{t-1}, \ldots, p_{t-S}).$$

When the price is already trending up, then the trading range break-out rule may be replaced by a *channel rule*, which works as follows. First, draw a *trend line* through previous lows and a *channel line* through previous peaks. Extend these lines. If the price moves above the channel (band) defined by these lines, then buy. A version of this is to define the channel by a *Bollinger band*, which is $\pm 2$ standard deviations from a moving data window around a moving average.

If we instead believe in mean reversion of the prices, then we can essentially reverse the previous trading rules: we would typically sell when the price is high.

**Empirical Example 10.12** *Figure 10.11 shows the idea of a reversal rule for S&P 500: buy when recent (a short MA) index values are outside the medium term trend (a long MA). The performance of implementing this rule over a long sample is shown in Table 10.2. The evidence suggest that the buy and sell signals do contain some information: average returns are high (low) after buy (sell) signals, but that this may come at the cost of higher uncertainty.*

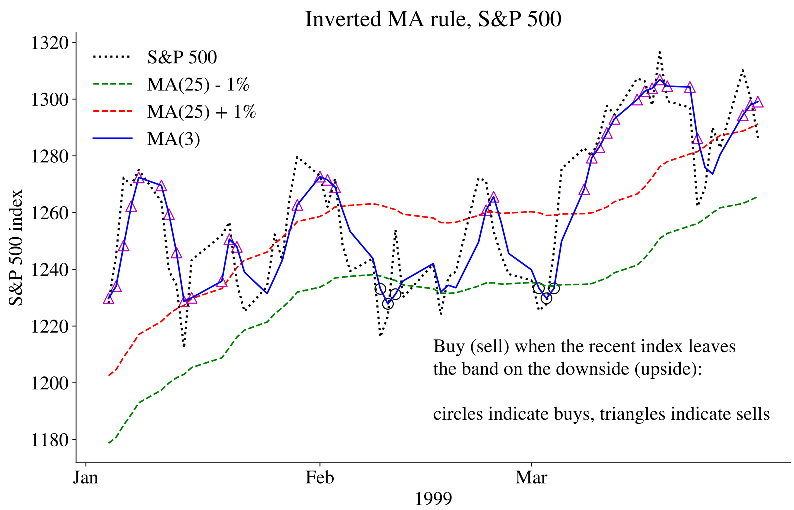



Figure 10.11: Example of a trading rule, illustration over short subsample

## 10.5 Security Analysts

Reference: Makridakis, Wheelwright, and Hyndman (1998) 10 and Elton, Gruber, Brown, and Goetzmann (2014) 27

**To do: update this section with more recent evidence**

### 10.5.1 Evidence on Analysts' Performance

Makridakis, Wheelwright, and Hyndman (1998) show that there is little evidence that the average stock analyst beats (on average) the market (or a passive index portfolio). In fact, less than half of the analysts beat the market. However, there are analysts which seem

| | Mean | Std |
|---|---|---|
| All days | 6.9 | 18.1 |
| After buy signal | 15.4 | 27.4 |
| After neutral signal | 5.0 | 14.6 |
| After sell signal | 3.9 | 13.7 |
| Strategy | 8.9 | 27.8 |
| Transaction cost | 0.1 | |

Table 10.2: Excess returns (annualized, in %) from technical trading rule (Inverted MA rule). Daily S&P 500 data 1990:01-2023:12. The trading strategy involves (a) on everay day: hold one unit of the index and short the riskfree; (b) on days after a buy signal: double the position in (a); (c) on days after a sell signal: short sell the position in (a), effectively having a zero investment. The transaction costs shows the cost (in %) of trade that the strategy can pay and still perform as well as the static holding of (a).

to outperform the market for some time, but the autocorrelation in over-performance is weak. The evidence from mutual funds is similar.

It should be remembered that many analysts also are sales persons: either of a stock (for instance, since the bank is underwriting an offering) or of trading services. It could well be that their objective function is quite different from minimizing the squared forecast errors. (The number of litigations in the US after the technology boom/bust should serve as a strong reminder of this.)

### 10.5.2 Do Security Analysts Overreact?

The paper by Bondt and Thaler (1990) compares the (semi-annual) forecasts (one- and two-year time horizons) with actual changes in earnings per share (1976-1984) for several hundred companies. The paper has regressions like

$$\text{Actual earnings change} = \alpha + \beta(\text{forecasted earnings change}) + \text{ residual},$$

and then studies the estimates of the $\alpha$ and $\beta$ coefficients. With rational expectations (and a long enough sample), we should have $\alpha = 0$ (no constant bias in forecasts) and $\beta = 1$ (proportionality, for instance no exaggeration).

The main result is that $0 < \beta < 1$, so that the forecasted change tends to be too wild in a systematic way: a forecasted change of 1% is (on average) followed by a less than 1% actual change in the same direction. This means that analysts in this sample tended to be too extreme—to exaggerate both positive and negative news.

### 10.5.3 High-Frequency Trading Based on Recommendations from Stock Analysts

Barber, Lehavy, McNichols, and Trueman (2001) give a somewhat different picture. They focus on the profitability of a trading strategy based on analyst recommendations. They use a huge data set (some 360,000 recommendations, US stocks) for the period 1985–1996. They sort stocks in to five portfolios depending on the consensus (average) recommendation—and redo the sorting every day (if a new recommendation is published). They find that such a daily trading strategy gives an annual 4% abnormal return on the portfolio of the most highly recommended stocks, and an annual -5% abnormal return on the least favourably recommended stocks.

This strategy requires a lot of trading (a turnover of 400% annually), so trading costs would typically reduce the abnormal return on the best portfolio to almost zero. A less frequent rebalancing (weekly, monthly) gives a very small abnormal return for the best stocks, but still a negative abnormal return for the worst stocks. Chance and Hemler (2001) obtain similar results when studying the investment advise by 30 professional "market timers."

### 10.5.4 Economic Experts

Several papers, for instance, Bondt (1991) and Söderlind (2010), have studied whether economic experts can predict the broad stock markets. The results suggests that they cannot. For instance, Söderlind (2010) shows that the economic experts that participate in the semi-annual Livingston survey (mostly bank economists) *(ii)* forecast the S&P worse than the historical average (recursively estimated), and that their forecasts are strongly correlated with recent market data (which in itself, cannot predict future returns).

### 10.5.5 Analysts and Industries

Boni and Womack (2006) study data on some 170,000 recommendations for a very large number of U.S. companies for the period 1996–2002. Focusing on revisions of recommendations, the papers shows that analysts are better at ranking firms within an industry than ranking industries.

### 10.5.6 Insiders

Corporate insiders *used to* earn superior returns, mostly driven by selling off stocks before negative returns. (There is little/no systematic evidence of insiders gaining by buying before high returns.) Actually, investors who followed the insider's registered transactions (in the U.S., these are made public six weeks after the reporting period), also used to earn some superior returns. It seems as if these patterns have more or less disappeared.

### 10.5.7 Mutual Funds

The general evidence on mutual funds is that they, on average, have zero alphas (or worse, after fees), and that there is little persistence in overperformance, at least among good funds(possible exceptions: hedge funds and private equity funds), while bad funds can stay bad for a long while.

# Chapter 11

# Performance Analysis

## 11.1 Performance Evaluation

### 11.1.1 The Idea behind Performance Evaluation

Traditional performance analysis seeks to answer the question: "Should we include an asset in our portfolio, assuming future returns will follow the same distribution as a historical sample." Therefore, a high return is compared with the risk of the strategy.

Most performance measures rely on mean-variance (MV) analysis; however, the full MV portfolio optimization problem is not solved from scratch in these cases. Instead, the performance measures can be seen as different approximations of the MV problem, where the issue is whether we should invest in fund $p$ or in fund $q$. (A mix of the two is not considered.) Although the analysis is based on the MV model, it does not assume that all portfolios conform to Capital Asset Pricing Model's (CAPM's) beta representation or that the market portfolio is the optimal choice for every investor. One motivation of this approach could be that the investor who is doing the performance evaluation is a MV investor, but that the market is influenced by non-MV investors. Alternatively, that we regard CAPM as a useful approximation.

Mutual fund evaluations typically find *(i)* neutral performance on average (or less due to trading costs and fees); *(ii)* poorer performance among large funds; *(iii)* better performance in less liquid and possibly less efficient markets; and *(iv)* little persistence in performance, except for very bad funds.

**Example 11.1** *(Steadman's funds*) "How can a fund be this bad?" (NYT, 1991) (the four Steadman funds rank among the six worst performers of the 244 stock funds tracked by Lipper Analytical Services for the 15 years that ended on Oct. 31. The Oceanographic*

*Fund comes in at No. 243 and Steadman American Industry Fund, No. 244); "Steadman's creature just won't die" (Forbes, 1999); "Those awful Steadman's funds returning under a new name" (Baltimore Sun, 2002).*

Several popular performance measures are related to the CAPM regression

$$R_t^e = \alpha + \beta R_{mt}^e + \varepsilon_t, \tag{11.1}$$

where $\mathrm{E}\,\varepsilon_t = 0$ and $\mathrm{Cov}(R_{mt}^e, \varepsilon_t) = 0$. In many cases, $R_{mt}^e$ represents the excess return on the market, but it could be some other benchmark return, for instance, for a segment of the market.

**Example 11.2** *(Statistics for example of performance evaluations) The examples below use the following information about portfolios $m$ (the market), $p$, and $q$*

| | $\alpha$ | $\beta$ | $\mathrm{Std}(\varepsilon)$ | $\mu^e$ | $\sigma$ |
|---|---|---|---|---|---|
| $m$ | 0.00 | 1.00 | 0.00 | 10.00 | 18.00 |
| $p$ | 1.00 | 0.90 | 14.00 | 10.00 | 21.41 |
| $q$ | 5.00 | 1.30 | 3.00 | 18.00 | 23.59 |

Table 11.1: Basic facts about the market and two other portfolios, $\alpha$, $\beta$, and $\mathrm{Std}(\varepsilon)$ are from CAPM regression: $R_{it}^e = \alpha + \beta R_{mt}^e + \varepsilon_{it}$

### 11.1.2 Alpha

The intercept ($\alpha$) from the regression (11.1) is often used as a performance measure. In CAPM, $\alpha$ measures the risk adjusted return. To see that, construct a portfolio with the weight $\beta_i$ on the market portfolio (or some other benchmark) and $1 - \beta_i$ on the risk-free asset. The excess return on this portfolio is

$$R_p^e = \beta_i R_m^e, \tag{11.2}$$

since $R_p = \beta_i R_m + (1 - \beta_i) R_f$. This portfolio has the same systematic risk (sensitivity to the market) as asset $i$. As a practical matter, it is typically straightforward to create this portfolio by investing in an index tracking fund.

The $\alpha$ is then the difference in average excess returns of two portfolios with the same systematic risk

$$\alpha_i = \mathrm{E}\,R_i^e - \mathrm{E}\,R_p^e. \tag{11.3}$$

**Empirical Example 11.3** *Table 11.2 shows the various performance measures for two large mutual funds. The Vanguard fund seems to perform best.*

| | $\alpha$ | SR | $M^2$ | AR | Treynor | $T^2$ |
|---|---|---|---|---|---|---|
| Market | 0.00 | 0.37 | 0.00 | | 6.70 | 0.00 |
| Putnam | −0.22 | 0.34 | −0.52 | −0.05 | 6.42 | −0.28 |
| Vanguard | 1.83 | 0.51 | 2.53 | 0.49 | 9.77 | 3.07 |

Table 11.2: Performance Measures of Putnam Asset Allocation: Growth A and Vanguard Wellington, weekly data 1999:01-2023:12 (annualized figures)

### 11.1.3 Sharpe Ratio and $M^2$

Suppose we want to determine whether fund $p$ is better than fund $q$ for allocating *all* our savings in. Again, we don't allow a mix of them. With MV preferences, the answer is that $p$ is better if it has a higher Sharpe ratio—defined as

$$SR = \mu^e/\sigma. \tag{11.4}$$

Intuitively, for a given level of volatility, we obtain the highest expected return.

**Example 11.4** *(Performance measure) From Example 11.2 we get the performance measures in Table 11.3.*

| | $SR$ | $M_i^2 - M_m^2$ | AR | Treynor | $T^2$ |
|---|---|---|---|---|---|
| $m$ | 0.56 | 0.00 | | 10.00 | 0.00 |
| $p$ | 0.47 | −1.59 | 0.07 | 11.11 | 1.11 |
| $q$ | 0.76 | 3.73 | 1.67 | 13.85 | 3.85 |

Table 11.3: Performance Measures

**Remark 11.5** *(Sortino ratio) The Sortino ratio is an alternative to the Sharpe ratio. It replaces $\sigma_p$ with a measure of variation on the downside (typically, the square root of a semivariance).*

The $M^2$ ("Modigliani and Modigliani") measure is

$$M^2 = R_f + SR_p\sigma_m, \tag{11.5}$$

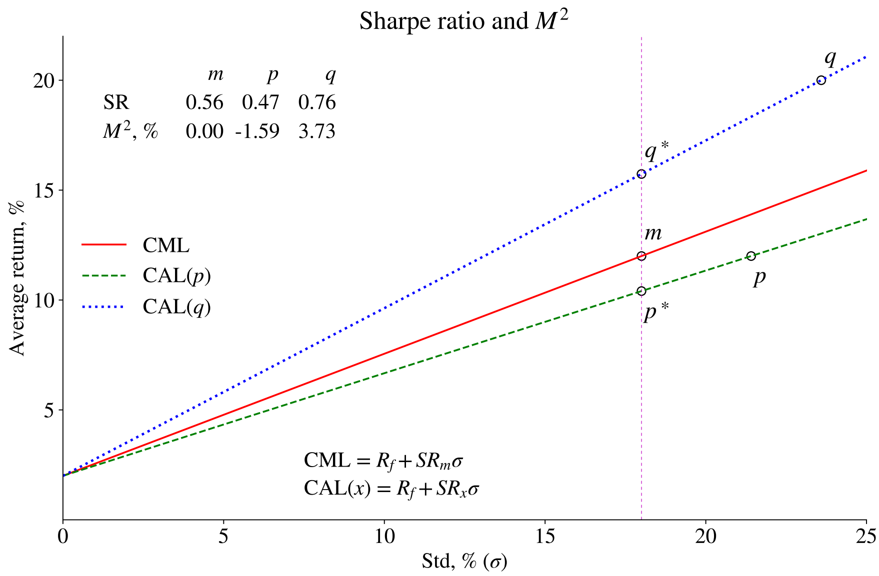



Figure 11.1: Sharpe ratio and $M^2$

which is simple transformation of the Sharpe ratio. The difference of $M^2$ for portfolio $p$ and the market (or another benchmark) $m$ can also be written as a difference of two risk-adjusted expected returns

$$\begin{aligned} M_p^2 - M_m^2 &= (SR_p - SR_m)\sigma_m \\ &= \mu_{p*}^e - \mu_m^e \end{aligned} \tag{11.6}$$

(or just $\mu_{p*} - \mu_m$). In this expression, $\mu_{p*}^e$ is the expected return on a mix of portfolio $p$ and the risk-free asset such that the volatility is the same as for the market return

$$R_{p*} = aR_p + (1-a)R_f, \text{ with } a = \sigma_m/\sigma_p. \tag{11.7}$$

The risk-adjustment here is thus to make the portfolios have the same volatility as the market. See Example 11.4 and Figure 11.1 for an illustration, which illustrate the relationship between the Sharpe ratio and $M^2$, highlighting the differences in risk-adjusted average returns.

*Proof* of (11.6). Notice that $SR_p = SR_{p*}$ and that $\sigma_{p*} = \sigma_m$. The first line of (11.6) can then be written $SR_{p*}\sigma_{p*} - SR_m\sigma_m$, which can be simplified as the 2nd line. □

### 11.1.4 Appraisal and Information Ratios

If the question is "should I *add* fund $p$ or fund $q$ to my holding of the market portfolio?," then the appraisal ratio provides an answer. The appraisal ratio is

$$AR = \alpha / \operatorname{Std}(\varepsilon_t), \tag{11.8}$$

where $\alpha$ is the intercept and $\operatorname{Std}(\varepsilon_t)$ is the standard deviation of the residual ("tracking error") of the CAPM regression (11.1). If you think of (11.2) as the benchmark return, then $AR$ is the average extra return per unit of extra standard deviation.

The motivation for $AR$ indicating the best addition to the market portfolio is that the tangency portfolio based on the market portfolio and portfolio $p$, has the following squared Sharpe ratio

$$SR_T^2 = AR^2 + SR_m^2. \tag{11.9}$$

(The proof is found below.) A higher $AR$ gives a higher Sharpe ratio of the optimal mix of portfolio $p$ and the market portfolio. See Example 11.4 for an illustration.

The *information ratio*

$$IR_p = \frac{\mathrm{E}(R_p - R_b)}{\operatorname{Std}(R_p - R_b)}, \tag{11.10}$$

where $R_b$ is some benchmark return. The information ratio is similar to both the Sharpe ratio and the appraisal ratio. The denominator in (11.10) can be thought of as the tracking error relative to the benchmark—and the numerator as the average active return (the gain from actively deviating from the benchmark). In fact, when the benchmark is as in (11.2) , then the information ratio is the same as the appraisal ratio. Instead, when $R_f$ is the benchmark, then the information ratio equals the Sharpe ratio.

*Proof* of (11.9). From the CAPM regression (11.1) we have

$$\operatorname{Cov}\left(\begin{bmatrix} R_i^e \\ R_m^e \end{bmatrix}\right) = \begin{bmatrix} \beta_i^2\sigma_m^2 + \operatorname{Var}(\varepsilon_{it}) & \beta_i\sigma_m^2 \\ \beta_i\sigma_m^2 & \sigma_m^2 \end{bmatrix}, \text{ and } \begin{bmatrix} \mu_i^e \\ \mu_m^e \end{bmatrix} = \begin{bmatrix} \alpha_i + \beta_i\mu_m^e \\ \mu_m^e \end{bmatrix}.$$

As usual, the square of the Sharpe ratio of the tangency portfolio is $\mu^{e\prime}\Sigma^{-1}\mu^e$. Combining, we get that the squared Sharpe ratio for the tangency portfolio (using both $R_{it}$ and $R_{mt}$) is

$$\left(\frac{\mu_T^e}{\sigma_T}\right)^2 = \frac{\alpha_i^2}{\operatorname{Var}(\varepsilon_{it})} + \left(\frac{\mu_m^e}{\sigma_m}\right)^2.$$

□

### 11.1.5 Treynor's Ratio and $T^2$

Suppose instead that the issue is if we should add a *small* amount of fund $p$ or fund $q$ to an already well diversified portfolio (not necessarily the market portfolio). In this case, Treynor's ratio might be useful

$$TR = \mu^e/\beta. \tag{11.11}$$

The basic intuition is that, with a *diversified portfolio* and a *small investment*, idiosyncratic risk becomes negligible, whereas only systematic risk ($\beta$) remains significant.

The $TR$ measure can be rephrased in terms of expected returns—and could then perhaps be called the $T^2$ measure

$$\begin{aligned} T^2 &= \mu_p^e/\beta_p - \mu_m^e \\ &= \mu_{p*}^e - \mu_m^e, \end{aligned} \tag{11.12}$$

In this expression, $\mu_{p*}^e$ is the expected return on a mix of portfolio $p$ and the risk-free asset such that the beta is one (the same as for the market return)

$$R_{p*} = aR_p + (1-a)R_f, \text{ with } a = 1/\beta_p, \tag{11.13}$$

so $\mu_{p*}^e = \mu_p^e/\beta_p$. The risk-adjustment here is thus to make the portfolios have the same $\beta$ as the market. See Example 11.4 and Figure 11.2 for an illustration which illustrate Treynor's Ratio and $T^2$, highlighting the differences in risk-adjusted average returns.

### 11.1.6 Relationships among the Various Performance Measures

The different measures can give different answers when comparing portfolios, but they all share one thing: they are increasing in alpha. By using the expected values from the CAPM regression (11.2), $\mu_p^e = \alpha_p + \beta_p\mu_m^e$, simple rearrangements give

$$\begin{aligned} SR_p &= \frac{\alpha_p}{\sigma_p} + \text{Corr}(R_p, R_m)SR_m \\ AR_p &= \frac{\alpha_p}{\text{Std}(\varepsilon_{pt})} \\ TR_p &= \frac{\alpha_p}{\beta_p} + \mu_m^e. \end{aligned} \tag{11.14}$$

and $M^2$ is just a scaling of the Sharpe ratio. Notice that these expressions do not assume that CAPM is the right pricing model—we just use the definition of the intercept and slope in the CAPM regression.

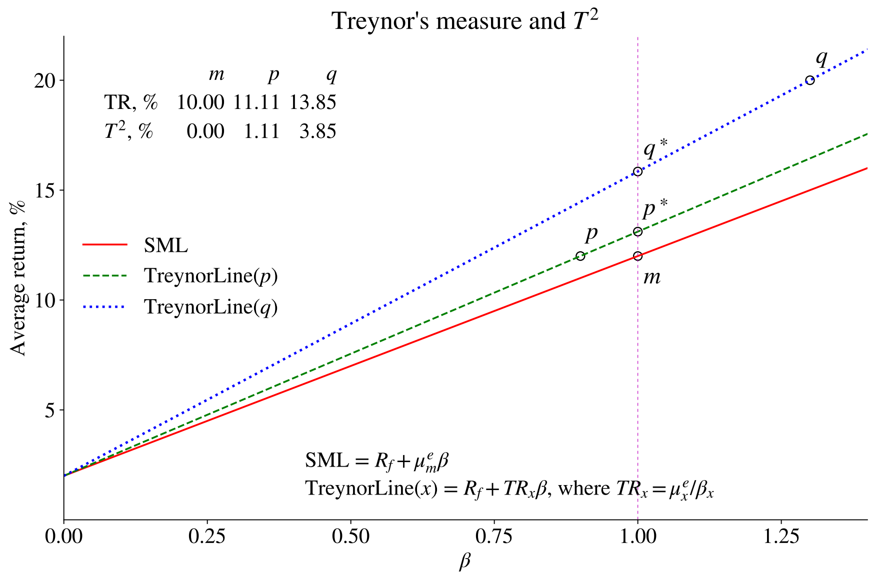



Figure 11.2: Treynor's ratio

Since alpha is the driving force in all these measurements, this further motivates it use as a performance measure in itself.

*Proof* of (11.14). Taking expectations of the CAPM regression (11.1) gives $\mu_p^e = \alpha_p + \beta_p \mu_m^e$, where $\beta_p = \text{Cov}(R_p, R_m)/\sigma_m^2$. The Sharpe ratio is therefore

$$SR_p = \frac{\mu_p^e}{\sigma_p} = \frac{\alpha_p}{\sigma_p} + \frac{\beta_p}{\sigma_p}\mu_m^e,$$

which can be written as in (11.14) since

$$\frac{\beta_p}{\sigma_p}\mu_m^e = \frac{\text{Cov}(R_p, R_m)}{\sigma_m \sigma_p}\frac{\mu_m^e}{\sigma_m}.$$

The $AR_p$ in (11.14) is just a definition. The $TR_p$ measure can be written

$$TR_p = \frac{\mu_p^e}{\beta_p} = \frac{\alpha_p}{\beta_p} + \mu_m^e,$$

where the second equality uses the expression for $\mu_p^e$ from above. □

### 11.1.7 More Sophisticated Performance Measures

This section goes beyond CAPM to consider more sophisticated performance measures.

The logic of using $\alpha$ from a CAPM regression can be extended to a *multi-factor model* where the factors are excess returns

$$R_t^e = \alpha + \beta_m R_{mt}^e + \beta_c R_{ct}^e + ... + \varepsilon_t. \tag{11.15}$$

Once again $\alpha$ can be seen as a performance measure and the rest (excluding $\varepsilon_t$) as a portfolio that could be easily replicated.

If there are predictable movements in the market excess return, then it makes sense to add a "market timing" factor to the CAPM regression. For instance, Treynor and Mazuy (1966) argue that market timing is analogous to having a beta that varies linearly with the market excess return

$$\beta = b + cR^e_{mt}. \tag{11.16}$$

Using this in a traditional market model (CAPM) regression, $R^e_t = a_i + \beta R^e_{mt} + \varepsilon_t$, gives

$$R^e_t = a + bR^e_{mt} + c(R^e_{mt})^2 + \varepsilon_t, \tag{11.17}$$

where $c$ captures the ability to "time" the market. That is, if the investor systematically exits the market prior to periods of low returns and vice versa, then the slope coefficient $c$ is positive. The interpretation is not clear cut, however. If we still regard the market portfolio as the benchmark, then $a + c(R^e_{mt})^2$ could be counted as performance. In contrast, if we think that this sort of market timing is straightforward to implement, that is, if the benchmark is the market plus market timing, then only $a$ should be counted as performance.

A recent way to merge the ideas of market timing and multi-factor models is to allow the coefficients to be time-varying according to some predetermined information variable, $z_{t-1}$. To illustrate this, suppose $z_{t-1}$ is a single variable, so the time-varying (or "conditional") CAPM regression is

$$\begin{aligned} R^e_t &= (a + \gamma z_{t-1}) + (b + \delta z_{t-1})\, R^e_{mt} + \varepsilon_t \\ &= \theta_1 + \theta_2 z_{t-1} + \theta_3 R^e_{mt} + \theta_4 z_{t-1} R^e_{mt} + \varepsilon_t. \end{aligned} \tag{11.18}$$

Similar to the market timing regression, there are two possible interpretations of the results: if we still regard the market portfolio as the benchmark, then the other three terms should be counted as performance. In contrast, if the benchmark is a dynamic strategy in the market portfolio (where $z_{t-1}$ is allowed to affect the choice market portfolio/risk-free asset), then only the first two terms are performance. In either case, the performance is time-varying.

## 11.2 Holdings-Based Performance Measurement

As a complement to the purely return-based performance measurements discussed, it may be of interest to study how the portfolio weights change (if that information is available). This highlights *how* the performance has been achieved.

Grinblatt and Titman's measure (in period $t$) is

$$GT_t = \sum\nolimits_{i=1}^{n} (w_{i,t-1} - w_{i,t-2}) R_{it}, \tag{11.19}$$

where $w_{i,t-1}$ is the weight on asset $i$ in the portfolio chosen (at the end of) in period $t-1$ and $R_{it}$ is the return of that asset between (the end of) period $t-1$ and (end of) $t$. A positive value of $GT_t$ indicates that the fund manager has moved into assets that turned out to give positive returns.

Researchers commonly report the time-series average of $GT_t$.

## 11.3 Performance Attribution

The performance of an investment fund often depends on decisions taken on several levels of the organisation. To get a better understanding of where the performance was generated, a performance attribution calculation can be helpful. It uses information on portfolio weights (often in-house information) to decompose overall performance according to a number of criteria (typically related to different levels of decision making).

For instance, we could decompose the return into the effects of *(a)* allocation to asset classes (equities, bonds, bills); and *(b)* security choice within each asset class. Alternatively, for a pure equity portfolio, it could be the effects of *(a)* allocation to industries; and *(b)* security choice within each industry.

Consider portfolios $p$ and $b$ (for benchmark) from the same set of assets. Let $n$ be the number of asset classes (or industries). Returns are

$$R_p = \sum_{i=1}^{n} w_i R_{pi} \text{ and } R_b = \sum_{i=1}^{n} v_i R_{bi}, \tag{11.20}$$

where $w_i$ is the weight on asset class $i$ (for instance, T-bonds) in portfolio $p$, and $v_i$ is the corresponding weight in the benchmark $b$. Analogously, $R_{pi}$ is the return that the portfolio earns on asset class $i$, and $R_{bi}$ is the return the benchmark earns. In practice, the benchmark returns are typically taken from well established indices.

Form the difference and rearrange ($\pm w_i R_{bi}$) to get

$$R_p - R_b = \underbrace{\sum_{i=1}^{n} (w_i - v_i) R_{bi}}_{\text{allocation effect}} + \underbrace{\sum_{i=1}^{n} w_i \left(R_{pi} - R_{bi}\right)}_{\text{selection effect}}. \tag{11.21}$$

The first term is the *allocation effect*, that is, the importance of allocation across asset classes, measured using the benchmark return of that asset class. If decisions on allocation to different asset classes are taken by senior management (or a board), then this is the contribution of that level. Instead, the second term is the *selection effect*, that is, the importance of selecting the individual securities within an asset class, as it depends on difference in returns that the fund and the benchmark earns in a given asset class. This contribution is more likely to come from the the trading desk.

**Remark 11.6** *(Alternative expression for the allocation effect*) The allocation effect is sometimes defined as* $\sum_{i=1}^{n} (w_i - v_i)(R_{bi} - R_b)$*, where* $R_b$ *is the benchmark return. This is clearly the same as in (11.21) since* $\sum_{i=1}^{n} (w_i - v_i) R_b = R_b \sum_{i=1}^{n} (w_i - v_i) = 0$.

### 11.3.1 What Drives Differences in Performance across Funds?

Reference: Ibbotson and Kaplan (2000)

Plenty of research shows that the asset allocation (choice between markets or large market segments) is more important for mutual fund returns than the asset selection (choice of individual assets within a market segment). For other investors, including hedge funds, leverage also plays a main role.

## 11.4 Style Analysis

Reference: Sharpe (1992)

Style analysis is a way to use econometric tools to find out the portfolio composition from a series of the returns, at least in broad terms.

The key idea is to identify several return indices (typically 5 to 10) expected to account for the majority of the portfolio's returns, followed by running a regression to find the portfolio "weights." It is essentially a multi-factor regression without a intercept and where the coefficients are constrained to sum to unity (and, optionally, to be positive)

$$R^e_{pt} = \sum_{j=1}^{K} b_j R^e_{jt} + \varepsilon_{pt}, \text{ with} \tag{11.22}$$
$$\sum_{j=1}^{K} b_j = 1 \text{ and } b_j \geq 0 \text{ for all } j.$$

Clearly, the restrictions could be changed to $U_j \leq b_j \leq L_j$, which could allow for some short positions.

The coefficients are typically estimated by minimizing the sum of squared residuals. In case the only restriction is that the coefficient should sum to one, then this can be solved with basic linear algebra (see the Remark below). With restrictions on the individual coefficient (for instance, no short sales), this is a non-linear least squares problem, but there are very efficient numerical methods for it.

**Remark 11.7** *(Restricted OLS*) If we want to impose the restrictions $Rb = q$ on OLS where $R$ is an $L \times K$ matrix and $q$ is an $L \times 1$ vector, then the closed form solution is*

$$\hat{b} = b_{OLS} - S_{xx}^{-1} R' (R S_{xx}^{-1} R')^{-1} (R b_{OLS} - q),$$

*where $S_{xx} = \Sigma_{t=1}^{T} x_t x_t'$ (that is, $X'X$ if row $t$ of $X$ contains $x_t'$) and $b_{OLS}$ is the unrestricted OLS estimate. For instance, $R = \mathbf{1}_K'$ and $q = 1$ gives the style analysis solution (11.22) except that short sales are allowed.*

A pseudo-$R^2$ (the squared correlation of the fitted and actual values) is sometimes used to gauge how well the regression captures the returns of the portfolio. The residuals can be thought of as the effect of stock selection, or possibly changing portfolio weights more generally. One way to get a handle of the latter is to run the regression on a moving data sample. The time-varying weights are often compared with the returns on the indices to see if the weights were moved in the right direction.

**Empirical Example 11.8** *See Figure 11.3 for an example of style analysis for the two mutual funds studied earlier. The results indicate that the coefficients move considerably over time (based on estimations from rolling data windows) and that the $R^2$ is above 95% except for the first few years.*

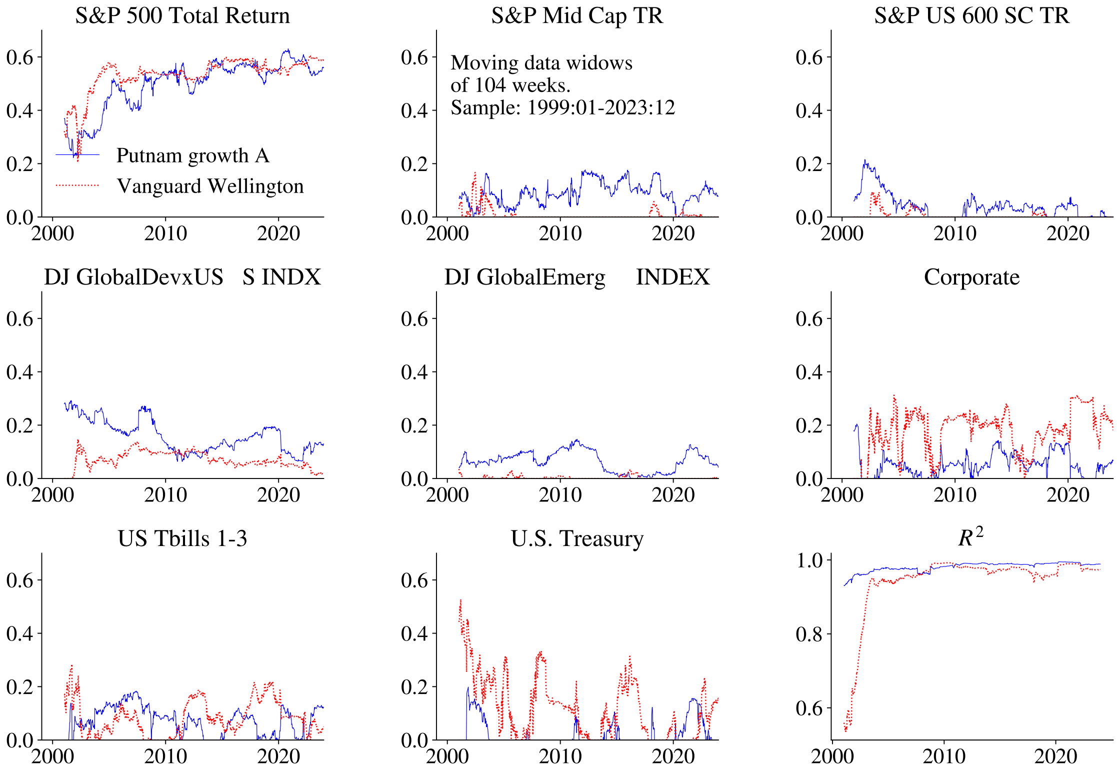



Figure 11.3: Example of style analysis, rolling data window

# Chapter 12

# Investment for the Long Run

## 12.1 Time Diversification

This section discusses the notion of "time diversification," which essentially amounts to claiming that equity is safer for long run investors than for short run investors. The argument comes in two forms: that Sharpe ratios increase with the investment horizon and that the probability that equity returns outperforming bond returns increases with the horizon. The discussion will compare these findings with results from mean-variance (MV) analysis. For simplicity, this section assumes that the investor picks a portfolio to hold for $q \geq 1$ periods without rebalancing. The next chapter will relax this assumption.

**Empirical Example 12.1** *Figure 12.1 shows how, for the U.S. equity market index, the Sharpe ratio and the probability of outperforming a safe asset differ across investment horizons.*

### 12.1.1 Long-Run Return as a Sum of Short-Run Returns

This section explains how a long-run return can be expressed in terms of multiple short-run returns. In particular, we use logarithmic returns, since they can be easily cumulated over time.

The gross (buy-and-hold) return on a $q$-period investment made in period 0 can be written

$$1 + Z(q) = \prod_{t=1}^{q}(1 + R_t), \tag{12.1}$$

where $R_t$ is the net portfolio return in period $t$. Taking logs (and using lower case letters to denote them), we have the log $q$-period return

$$z(q) = \sum_{t=1}^{q} r_t, \tag{12.2}$$

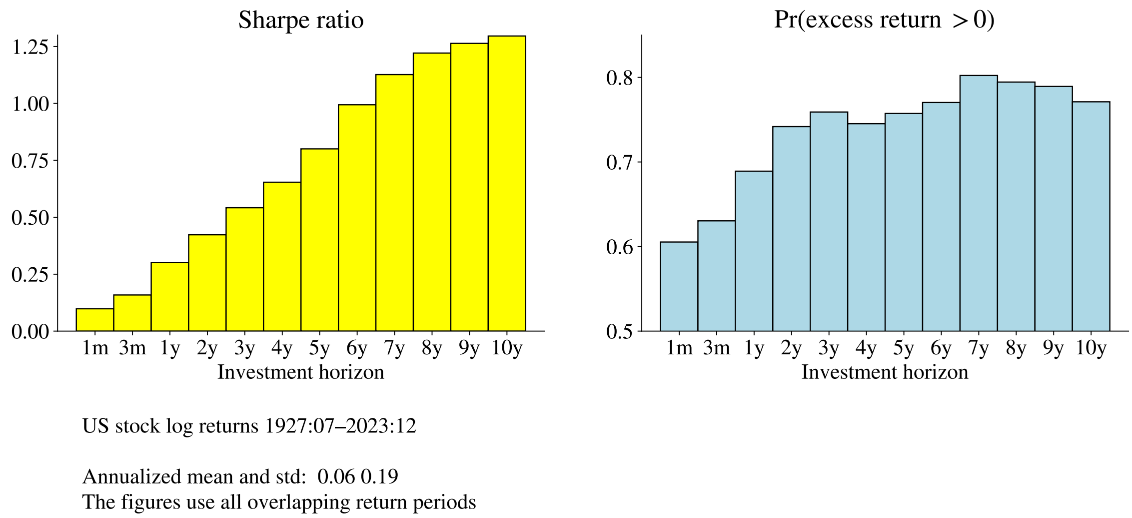



Figure 12.1: Empirical evidence on SR and probability of excess return > 0

where $z(q) = \ln(1 + Z(q))$ and where the log one-period return is $r_t = \ln(1 + R_t)$. Notice that if $R$ is small, then $\ln(1 + R) \approx R$. We use $r_t^e$ to denote the excess long return, $r_t^e = r_t - r_f$, where $r_f = \ln(1 + R_f)$, and similarly for $z^e(q)$.

**Remark 12.2** *(Approximating $q$-period returns*) It is sometimes convenient to approximate the $q$-period net return $Z(q)$ as*

$$Z(q) \approx \textstyle\sum_{t=1}^{q} R_t.$$

*This approximation often works well unless there are numerous periods or extreme one-period returns. For instance, if $R_1 = 0.9$ and $R_2 = -0.9$ (indeed very extreme returns), then the two-period net return is $Z(2) = (1 + 0.9)(1 - 0.9) - 1 = -0.81$, while the approximation gives $Z(2) \approx R_1 + R_2 = 0$. The difference is dramatic. If the two net returns instead are $R_1 = 0.09$ and $R_2 = -0.09$, then $Z(2) = (1+0.09)(1-0.09)-1 = -0.01$ and the approximation is still zero: the difference is much smaller.*

**Remark 12.3** *(Geometric and arithmetic average returns*) The average log return, $\Sigma_{t=1}^{q} r_t/q$, is closely related to the geometric mean return. To see that, notice that a geometric mean return $\tilde{r} = [\Pi_{t=1}^{q}(1 + R_t)]^{1/q} - 1$, or $1 + \tilde{r} = [1 + Z(q)]^{1/q}$. Take logs and divide by q to get $\ln(1 + \tilde{r}) = \Sigma_{t=1}^{q} r_t/q$. This is approximately the same as $\tilde{r}$ for means close to 0.*

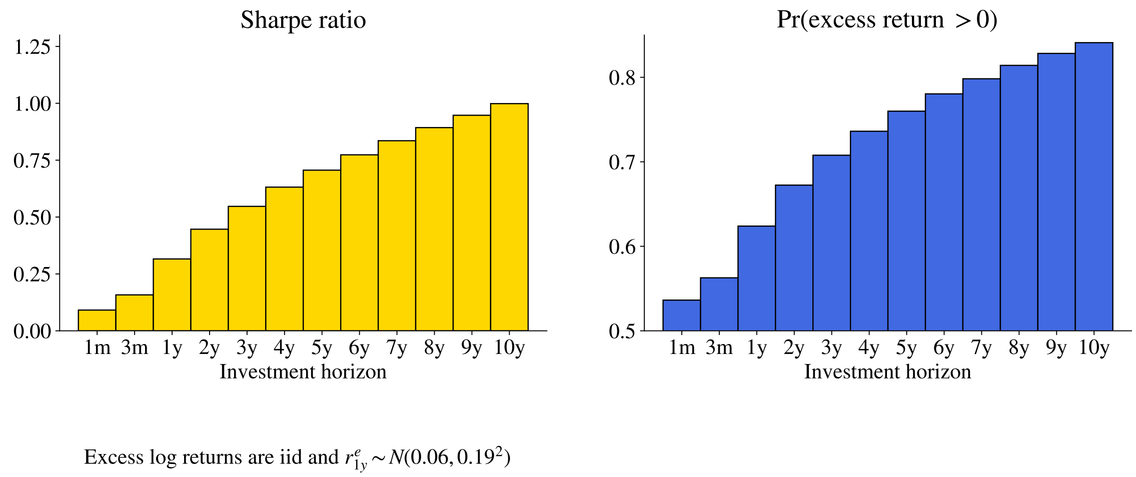



Figure 12.2: SR and probability of excess return > 0, iid returns

### 12.1.2 Increasing Sharpe Ratios

This section demonstrates that with iid (independently and identically distributed) returns, the expected return and variance of a portfolio both grow linearly with the investment horizon. Consequently, the Sharpe ratio, defined as the expected excess return divided by the standard deviation, increases with the square root of horizon. This needs to be considered when comparing Sharpe ratios across investment horizons.

As before, let $z(q)$ be the log return on a $q$-period investment. *If log returns are iid*, the Sharpe ratio of $z(q)$ is

$$SR(z(q)) = \sqrt{q} SR(r), \tag{12.3}$$

where $SR(r) = SR(z(1))$ is the Sharpe ratio of the *one-period* log return. The Sharpe ratio of the $q$-period return is clearly increasing with the horizon, $q$, provided $SR(r) > 0$. The significance of this result will be discussed later.

*Proof* of (12.3). The $q$-period log return is as in (12.2). If one-period excess log returns are iid with mean $\mathrm{E}\, r^e$ and variance $\mathrm{Var}(r)$, then the mean and variance of the $q$-period excess log returns are $\mathrm{E}\, z^e(q) = q\, \mathrm{E}\, r^e$ and $\mathrm{Var}(z(q)) = q\, \mathrm{Var}(r)$. $\square$

### 12.1.3 Probability of Outperforming a Risk-free Asset

Under the assumption of normally distributed returns, the increasing Sharpe ratios imply higher probabilities of out-performing a risk-free asset.

In particular, assume that the log one-period returns are jointly normally distributed, which carries over to the $q$-period excess log return, $z^e(q)$. Then, we have

$$\Pr(z^e(q) > 0) = \Phi[SR(z(q))], \tag{12.4}$$

where $\Phi()$ is the cumulative distribution function of a standard normal variable, $N(0,1)$. See Figure 12.2 for an illustration.

Together with the results in (12.3) this suggests that the empirical evidence in Figure 12.1 could potentially be explained by iid returns.

*Proof* of (12.4). By standard manipulations we have

$$\Pr(x \leq 0) = \Pr\left(\frac{x - \mathrm{E}\,x}{\mathrm{Std}(x)} \leq \frac{-\mathrm{E}\,x}{\mathrm{Std}(x)}\right) = \Phi\left(\frac{-\mathrm{E}\,x}{\mathrm{Std}(x)}\right),$$

since $(x - \mathrm{E}\,x)/\,\mathrm{Std}(x)$ is an $N(0,1)$ variable. Clearly, $\Pr(x > 0) = 1 - \Pr(x \leq 0)$. Use the fact that $\Phi(z) + \Phi(-z) = 1$ (since the standard normal distribution is symmetric around zero) and substitute $z^e(q)$ for $x$ to get (12.4). $\square$

### 12.1.4 Why These Arguments Are Not Enough

Although increasing Sharpe ratios (at longer investment horizons) suggest a higher probability of out-performing a risk-free asset, that does not necessarily imply that the risky asset is safer for a long-run investor. In fact, we also have to take into account the size of the loss—in case the portfolio underperforms. With a longer horizon (and therefore higher dispersion), *really* bad outcomes are more likely: the expected loss, conditional of having one, is increasing with the investment horizon. See Figure 12.3 for an illustration.

**Remark 12.4** *(Expected excess return conditional on a negative one*) If* $x \sim N(\mu, \sigma^2)$, *then* $\mathrm{E}(x|x \leq b) = \mu - \sigma\phi(b_0)/\Phi(b_0)$ *where* $b_0 = (b-\mu)/\sigma$ *and where* $\phi()$ *and* $\Phi()$ *are the pdf and cdf of a* $N(0,1)$ *variable respectively. To apply this, use* $b = 0$ *so* $b_0 = -\mu/\sigma$. *This gives* $\mathrm{E}(x|x \leq 0) = \mu - \sigma\phi(-\mu/\sigma)/\Phi(-\mu/\sigma)$.

To further explore how the investment horizon affects the portfolio weights, it is necessary to clarify the investor's preferences, specifically how risks and opportunities are compared.

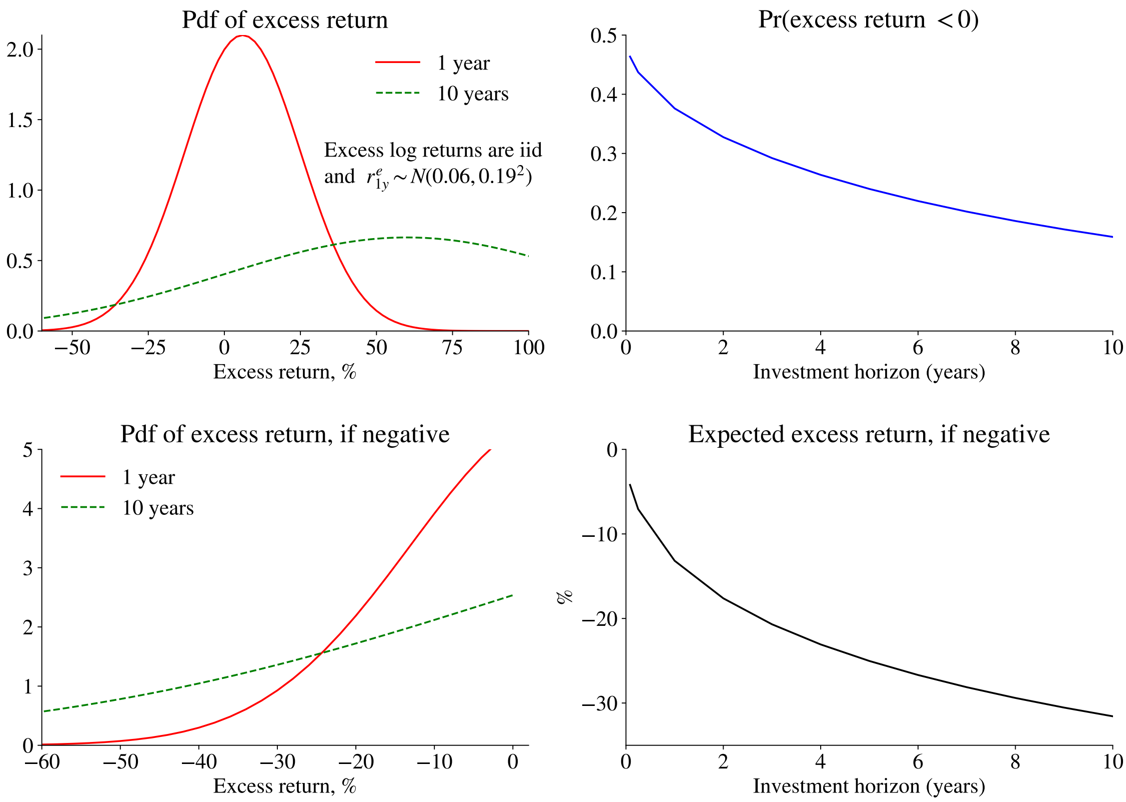



Figure 12.3: Time diversification, normally distributed returns

## 12.2 Mean-Variance Portfolio Choice

### 12.2.1 Approximating the Log Portfolio Return

Logarithmic portfolio returns are convenient in a dynamic setting since they are additive across time. However, they have a drawback on the portfolio formation stage: the logarithmic portfolio return is a *non-linear* function of the logarithmic returns of the assets. Therefore, we will use an approximation.

If there is only one risky asset with return $Z$ and risk-free asset with return $Z_f$, then the portfolio return is $Z_p = vZ + (1-v)Z_f$. For notational simplicity, we drop the indicator $q$ for the investment horizon.

Campbell and Viceira (2002) approximate the log portfolio return by

$$z_p = \ln[ve^z + (1-v)e^{z_f}] \tag{12.5}$$

$$\approx z_f + v(z - z_f) + v\sigma_z^2/2 - v^2\sigma_z^2/2, \tag{12.6}$$

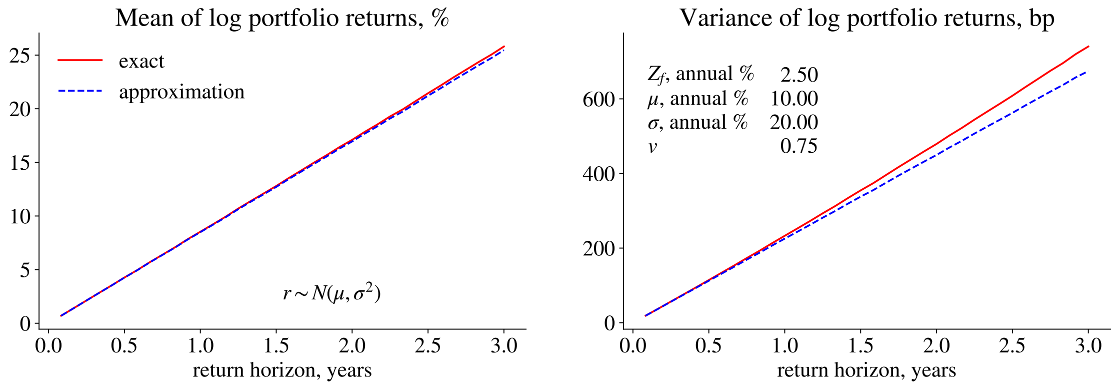



Figure 12.4: Mean and variance of log portfolio return for different return horizons, exact and according the approximation (12.6).

where $\sigma_z^2$ is the variance of $z$ over the relevant investment horizon. As usual, all moments represent the beliefs of the investor, conditional on the information available at the time of investment. As mentioned before, we suppress the indicator $q$ for the investment horizon.

The approximation error from (12.5) is straightforward to calculate, and it will differ across return levels. What is more important, however, is how the mean and variance differ between the exact calculation and the approximation. To assess that we need to assume a distribution of the returns. Figure 12.4 provides an illustration, where the basic assumption is that the log return on the risky asset is normally distributed with a mean and standard deviation that are broadly in line with the U.S. equity market index. The figure suggests that both the mean and variance scale fairly linearly with the return horizon, and that the approximation works well up to 2–3 years. After that, we see a deviation, where the approximation underestimates the variance somewhat so a manual adjustment of the risk aversion parameter $k$ might be sensible.

*Proof* of (12.6). The portfolio return $Z_p = vZ + (1-v)Z_f$ can be used to write

$$\frac{1+Z_p}{1+Z_f} = 1 + v\left(\frac{1+Z}{1+Z_f} - 1\right).$$

The logarithm is

$$z_p - z_f = \ln\{1 + v[\exp(z - z_f) - 1]\}.$$

The function $f(x) = \ln\{1 + v[\exp(x) - 1]\}$, where $x = z - z_f$, has the following derivatives (evaluated at $x = 0$): $df(x)/dx = v$ and $d^2 f(x)/dx^2 = v(1-v)$, and notice that $f(0) = 0$. A second order Taylor approximation of the log portfolio return around $z - z_f = 0$ is then

$$z_p - z_f \approx v(z - z_f) + \frac{1}{2}v(1 - v)(z - z_f)^2.$$

In a continuous time model, the square would equal its expectation, $\sigma_z^2$, so this further approximation is used to give (12.6). $\square$

### 12.2.2 Mean-Variance Optimization

The investor solves a traditional mean-variance problem, but expressed in terms of log returns

$$\max_v \mathrm{E}\, z_p(q) - \frac{k}{2}\operatorname{Var}(z_p(q)), \tag{12.7}$$

where $z_p(q)$ is the $q$-period log return of a portfolio of a risky and a risk-free asset. Using the approximation (12.6), the optimal weight on the risky asset is

$$v = \frac{\mu_z^e(q)}{(1 + k)\sigma_z^2(q)} + \frac{1}{2(1 + k)}. \tag{12.8}$$

This is, of course, very similar to the traditional MV results based on net returns. In particular, the key driver is the mean excess return divided by the variance (and the risk aversion).

**Example 12.5** *(of (12.8)) With* $(\mu_z^e, \sigma_z^2(q), k) = (0.008, 0.05^2, 5)$ *we get* $v \approx 0.62$*, but with* $(\mu_z^e, \sigma_z^2(q)) = (0.016, 0.0594^2)$ *we get* $v \approx 0.84$.

*Proof* of (12.8). Using (12.6), (12.7) is approximately the same as

$$\max_v z_f + v\mu_z^e + v\sigma_z^2/2 - v^2\sigma_z^2/2 - \frac{k}{2}v^2\sigma_z^2,$$

where $z_f$ is the risk-free rate over the investment horizon and $(\mu_z^e, \sigma_z^2)$ are mean and variance of the excess log return of the risky asset. For notational simplicity, we drop the indicator $q$ for the investment horizon. The first order condition is $\mu_z^e + \sigma_z^2/2 - v(1 + k)\sigma_z^2 = 0$, which gives (12.8). $\square$

### 12.2.3 Mean-Variance Optimization with iid Logarithmic Returns

When log returns are iid, then both the mean and the variance scaled with the investment horizon

$$\mu_z^e(q) = q\,\mathrm{E}\, r^e \text{ and } \sigma_z^2(q) = q\operatorname{Var}(r), \tag{12.9}$$

where $\mathrm{E}\, r^e$ is the expected excess log 1-period return and $\operatorname{Var}(r)$ its variance.

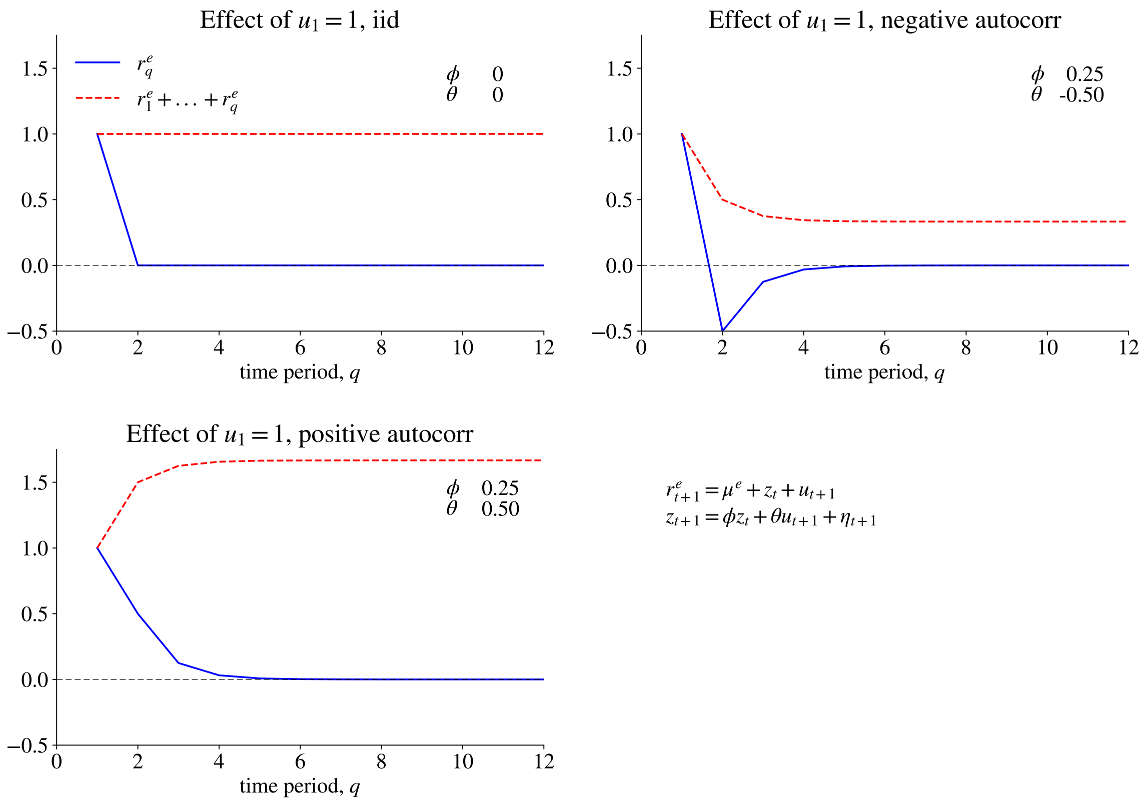



Figure 12.5: Impulse responses to an $u_1$ shock in the time series model (12.10)

This result implies that, with iid log returns, the portfolio weight on the risky asset (12.8) is the same for all investment horizons, $q$. This is in stark contrast to what the increasing Sharpe ratios and probability of beating a risk-free asset would suggest, as discussed in earlier sections. The key reason is that the mean-variance preferences consider also the magnitude of a loss, not just the probability of one.

### 12.2.4 A Time Series Model for Autocorrelated Logarithmic Returns

This section builds a simple time series model for log returns which allows for predictability

$$\begin{bmatrix} r^e_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} r^e_t \\ z_t \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ \theta & 1 \end{bmatrix} \begin{bmatrix} u_{t+1} \\ \eta_{t+1} \end{bmatrix} \tag{12.10}$$

where $u_{t+1}$ and $\eta_{t+1}$ are iid and uncorrelated shocks, with standard deviations $\sigma_u$ and $\sigma_\eta$, respectively. The $z_t$ variable can be thought of as a state variable which affects future average returns. The $\theta$ parameter controls how return shocks spill over to future average returns, which will turn out to be a key feature of the model. In short, with $\theta < 0$ a positive return in $t + 1$ will tend to be followed by a sequence of negative returns, that is, a long-run reversal. See Figure 12.5 for an illustration. In particular, notice how the response of long-run return (the sum of 1-period returns $r_1^e + \ldots r_q^e$), is muted by the long-run reversal. This will imply that the volatility of long-run returns is low. In contrast, $\theta > 0$ will cause movements in the same direction (momentum). how the current and future returns react to a return shock in period 1.

Formally, (12.10) is a state space model with correlated shocks, but it is written on VAR(1) form. For our purposes, the model has the advantage that is can generate both long-run momentum and reversals. (In contrast, an AR(1) with a negative coefficient has an oscillating forecast for future values.)

Since the shocks are iid and uncorrelated, the expectation and variance of $r_{t+1}^e$, *conditional on the information available* at the time of investment in $t$, are straightforward to calculate. Figure 12.6, which is roughly calibrated to monthly data although the autocorrelations are exaggerated to illustrate a point, provides an illustration. In particular, it shows how the variance of $q$-period returns, conditional on the information available at the time of investment ($t = 0$ in the figure), scale with the investment horizon $q$, but slower when the model exhibits mean reversion ($\theta < 0$). Details on the calculations are in an Appendix to this chapter.

### 12.2.5 Mean-Variance Optimization with Autocorrelated Logarithmic Returns

Autocorrelation can affect both the expectations and the uncertainty of future returns. However, the analysis here will focus on how the uncertainty depends on the investment horizon, disregarding the "market timing" issue. That is, we assume a neutral *initial* state, $z_t = 0$, but allow for future shocks to the state. A later chapter will relax this assumption.

In general, positive autocorrelation will make the sum of returns, $z(q)$, have a variance that scales quicker than the return horizon $q$ as shocks "build up" over time. The opposite holds for a negative autocorrelation.

Figure 12.6 suggests that (12.10) can replicate the iid case (12.9). But it also shows that with long-run reversal, uncertainty increases slower than the investment horizon, so the equity is safer for a long-run investor. See Figure 12.7 for an illustration of the optimal portfolio weight on the risky asset, based on the same rough calibration to monthly data as before as well as the same exaggeration of the autocorrelation.

In summary, this analysis suggests that iid log returns are *not* sufficient to make equity

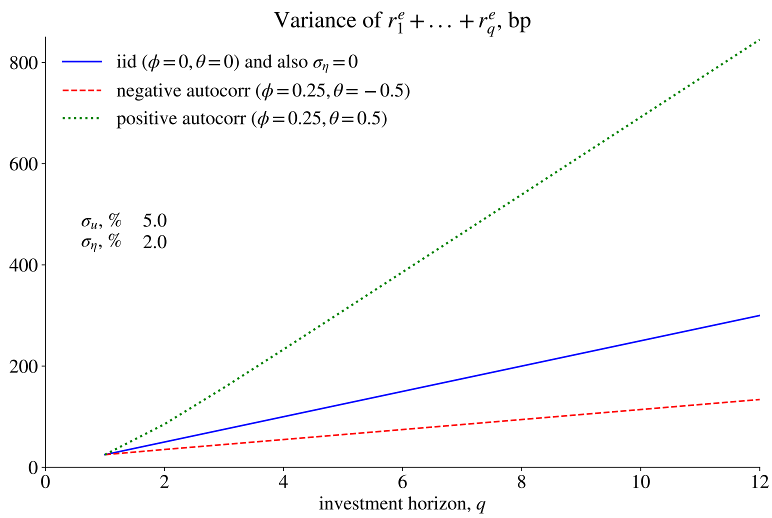



Figure 12.6: Variances of $q$-period return in the time series model (12.10)

relatively safer for a long-run investor—if we use a mean-variance approach to portfolio choice. Rather, it requires long-run reversals (negative autocorrelation). Empirical evidence suggests that there might be some reversals, but not very much, questioning the notion of equity being safe in the long run.

Note, however, that the analysis in this chapter relies on the assumption that the investor makes *one* portfolio choice, irrespective of investment horizon. That is, no rebalancing. A later chapter will look at that issue in more detail as well as discuss the optimal response to differences in the initial state.

### 12.2.6 Utility Based Portfolio Choice

To study whether the conclusions from the MV approach are robust, this section considers utility based portfolio choice.

A *logarithmic utility function* means setting $k = 0$ in (12.7 )–(12.8).

Also, for a CRRA utility function and normally distributed log portfolio returns, we know that maximizing $\mathrm{E}(1 + Z_p)^{1-\gamma}/(1-\gamma)$ is equivalent to maximizing

$$\mathrm{E}\, z_p - (\gamma - 1)\,\mathrm{Var}(z_p)/2, \tag{12.11}$$

which is once again of the same form as (12.7)–(12.8), but with $k = \gamma - 1$. (See the chapter on utility theory for a proof.)

Both these examples lend support to the conclusion from the MV approach: to make

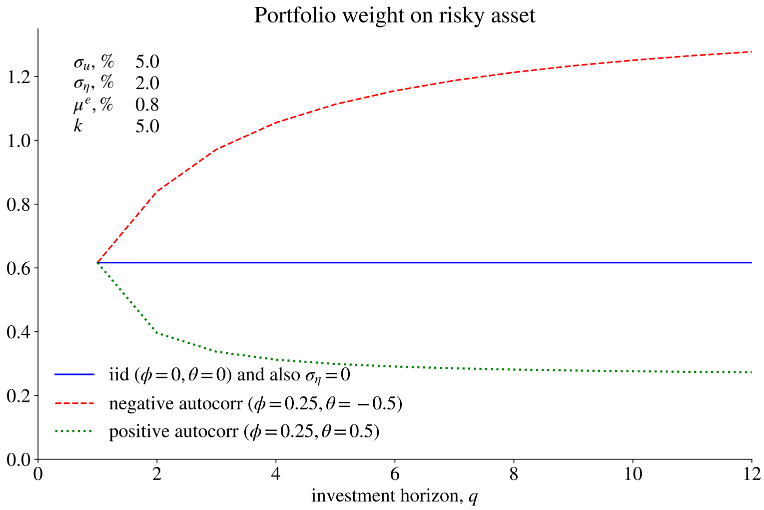



Figure 12.7: Portfolio weight on risky asset based on the time series model (12.10)

equity safer for a long-run investor, returns must be mean-reverting.

## 12.3 Appendix – Calculating the Conditional Variances*

*First*, simulate the system by setting $u_t = 1$ and all other shock to zero. Trace out the dynamic effects on $r_t, r_{t+1}, \ldots$. Let the first element in $\psi_0$ be the effect on $r_t$, the first element in $\psi_1$ be the effect on $r_{t+1}$. Now, instead set $\eta_t = 1$ and trace of the dynamic effects to fill the second elements in the $\psi$ vectors. We thus have the MA representation of a return in period $t$

$$r_t = \psi_0' \varepsilon_t + \psi_1' \varepsilon_{t-1} + \psi_2' \varepsilon_{t-2} + \ldots, \text{ where } \varepsilon_t = \begin{bmatrix} u_t \\ \eta_t \end{bmatrix}.$$

*Second*, notice that the innovations of future returns, compared to the information available in $t$, can be written

$$r_{t+1} = \begin{bmatrix} \mathbf{0}' \varepsilon_{t+3} \\ \mathbf{0}' \varepsilon_{t+2} \\ \psi_0' \varepsilon_{t+1} \end{bmatrix}, r_{t+2} = \begin{bmatrix} \mathbf{0}' \varepsilon_{t+3} \\ \psi_0' \varepsilon_{t+2} \\ \psi_1' \varepsilon_{t+1} \end{bmatrix}, r_{t+2} = \begin{bmatrix} \psi_0' \varepsilon_{t+3} \\ \psi_1' \varepsilon_{t+2} \\ \psi_2' \varepsilon_{t+1} \end{bmatrix}.$$

Note that shocks in $t$ or earlier are known in $t$, so they do not enter the expressions. Also, future shocks cannot affect current returns, which explains the zeros.

*Third*, since $\varepsilon_{t+1}$ is uncorrelated across time, variances and covariances are straightforward to calculate, for instance,

$$\mathrm{Cov}_t(r_{t+2}, r_{t+3}) = \psi_0' \Omega \psi_1 + \psi_1' \Omega \psi_2, \text{ where } \Omega = \mathrm{Var}(\varepsilon_t).$$

*Fourth*, once we have the variance-covariance matrix of $(r_{t+1}, r_{t+2}, r_{t+3})$, the variance of $r_{t+1} + r_{t+2} + r_{t+3}$ can be calculated as the sum of all the elements.

# Chapter 13

# Dynamic Portfolio Choice

This chapter discusses portfolio choice of a long-run investor who can rebalance in each period.

## 13.1 Logarithmic Utility

Let the objective in period $t$ be to maximize the expected log wealth in some future period

$$\max \mathrm{E}_t \ln W_{t+q} = \max(\ln W_t + \mathrm{E}_t\, r_{p,t+1} + \mathrm{E}_t\, r_{p,t+2} + \ldots + \mathrm{E}_t\, r_{p,t+q}), \tag{13.1}$$

where $r_{pt}$ is the log portfolio return, $r_{pt} = \ln(1 + R_{pt})$ with $R_{pt}$ being the net portfolio return. The investor can rebalance the portfolio weights every period.

Since the returns in the different periods enter separably in the utility function, the best an investor can do in period $t$ is to choose a portfolio that solves $\max_v \mathrm{E}_t\, r_{p,t+1}$.That is, to choose the one-period growth-optimal portfolio. This *myopic* approach is thus the optimal *dynamic* portfolio choice. This means that the investment horizon $q$ does not matter: short-run and long-run investors choose the same portfolio. This is specific to the logarithmic utility function.

Note, however, that the portfolio choice may change over time ($t$), if the distribution of the returns changes; that is, when returns are *not iid*.

The net portfolio return is $R_p = v'R + (1 - v'\mathbf{1})R_f$, where $v$ is an $n$-vector of portfolio weights and $R$ a corresponding vector of returns of risky assets. To approximate the *log* portfolio return, $r_p = \ln(1 + R_p)$, we use the approach of Campbell and Viceira (2002). (An earlier chapter includes a proof and an application.)

**Remark 13.1** *(Approximate log portfolio return) The log portfolio return is approximately*

$$r_{pt} \approx r_{ft} + v'(r_t - r_{ft}) + v' \operatorname{diag}(\Sigma)/2 - v'\Sigma v/2, \tag{13.2}$$

*where $\Sigma$ is the $n \times n$ variance-covariance matrix of $r_t$ and* $\operatorname{diag}(\Sigma)$ *is the $n$-vector of the variances (that is, the diagonal elements of $\Sigma$). With a single risky asset, this can be simplified as*

$$r_{pt} \approx r_{ft} + v(r_t - r_{ft}) + v\sigma^2/2 - v^2\sigma^2/2, \tag{13.3}$$

*where $\sigma^2$ is variance of $r_t$.*

Maximizing $\mathrm{E}_t\, r_{p,t+1}$ gives the optimal $n$-vector of portfolio weights as

$$v = \Sigma^{-1}(\mu^e + \operatorname{diag}(\Sigma)/2), \tag{13.4}$$

where $\mu^e$ is the vector of excess log returns of the risky assets, $\Sigma$ is their variance-covariance matrix and $\operatorname{diag}(\Sigma)$ picks out the diagonal of $\Sigma$, that is, the vector of variances. (The proof is at the end of the section.) The weight on the risk-free asset is the remainder, $1 - v'\mathbf{1}$. The case of a single risky asset was solved in an earlier chapter, yielding $v = \mu^e/\sigma^2 + 1/2$.

Clearly, the portfolio weight $v$ changes over time if the expected excess returns and/or the variance-covariance matrix change; that is, when returns are not iid. We could think of this as a *managed portfolio*.

**Example 13.2** *(One risky asset) Suppose there is one risky asset with $\sigma = 5\%$, and the expected excess returns are different the two"scenarios" A and B: $\mu^e_A = 0.8\%$ or $\mu^e_B = 0.2\%$. Then (13.4) gives $v_A = 3.7$ and $v_B = 1.3$ in the two scenarios.*

**Example 13.3** *(Three risky assets) Suppose we have three assets with the variance-covariance matrix (which is the same in both states)*

$$\Sigma = \begin{bmatrix} 83 & 17 & 29 \\ 17 & 32 & 2 \\ 29 & 2 & 50 \end{bmatrix} bp,$$

*and the means (in scenario A and B, respectively)*

$$\mu^e_A = \begin{bmatrix} 0.8 \\ 0.9 \\ 0.3 \end{bmatrix} \% \text{ and } \mu^e_B = \begin{bmatrix} 0.4 \\ 0.45 \\ 0.15 \end{bmatrix} \%,$$

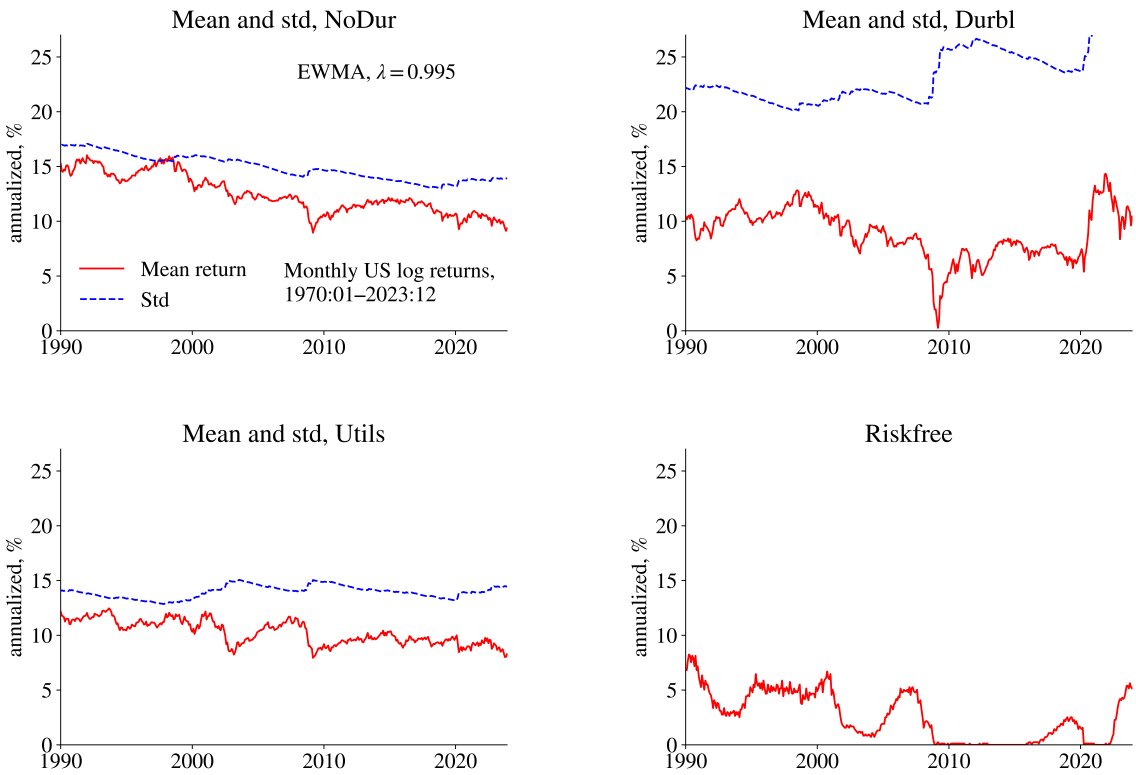



Figure 13.1: Dynamically updated estimates, 5 U.S. industries

*In this case, the portfolio weights in the two states are*

$$v_A \approx \begin{bmatrix} 0.65 \\ 2.93 \\ 0.60 \end{bmatrix} \text{ and } v_B \approx \begin{bmatrix} 0.49 \\ 1.62 \\ 0.45 \end{bmatrix}.$$

**Empirical Example 13.4** *Figure 13.1 illustrates mean returns and standard deviations, estimated by exponentially weighted moving averages. Figure 13.2 shows how the optimal portfolio weights change. It is clear that the portfolio weights can be fairly extreme and also change a lot—perhaps too much to be realistic.*

*Proof* of (13.4). From (13.2) we have that the objective function can be written $r_f + v'\mu^e + v' \operatorname{diag}(\Sigma)/2 - v'\Sigma v/2$, so the first order conditions are $\mu^e + \operatorname{diag}(\Sigma)/2 - \Sigma v = \mathbf{0}_{n\times 1}$, which gives (13.4). $\square$

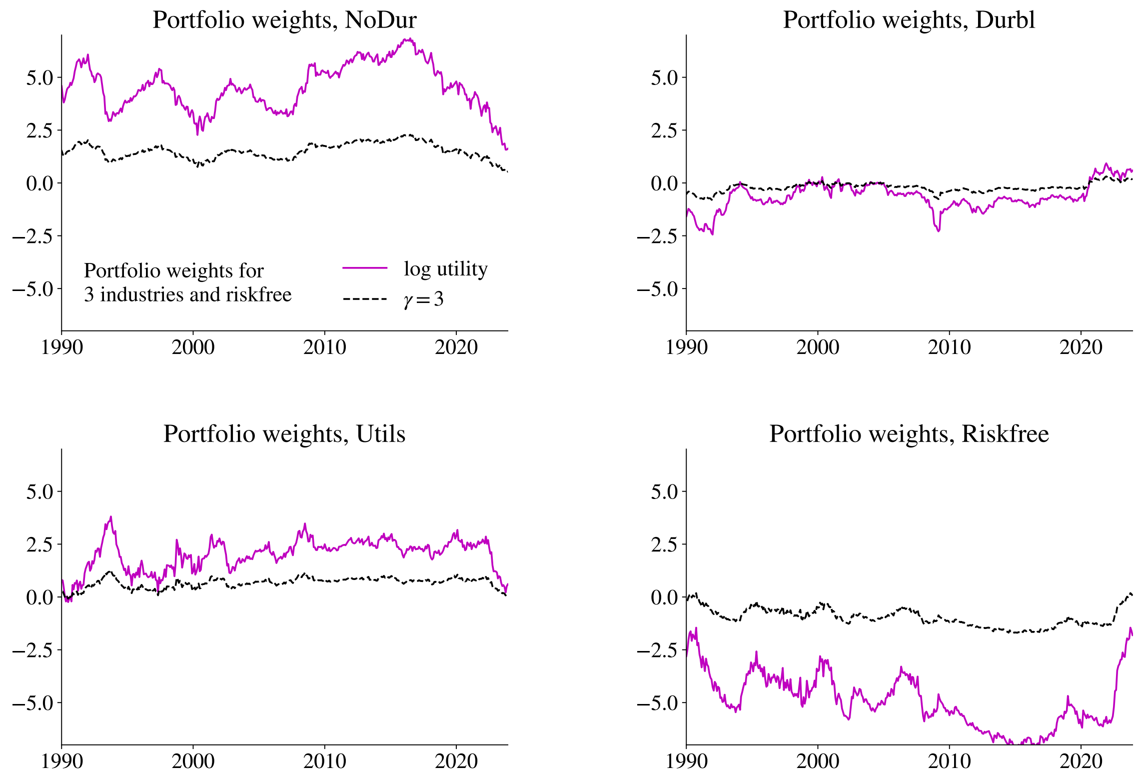



Figure 13.2: Dynamically updated portfolio weights, T-bill and 5 U.S. industries

## 13.2 CRRA Utility

The previous section has shown that logarithmic utility leads to myopic behaviour where the optimal portfolio depends only on beliefs about the next-period return. This clearly simplifies the choice, but it is unclear if logarithmic utility is a good representation of preferences. We therefore extend the analysis to the general constant relative risk aversion (CRRA) case.

An earlier chapter has already established that if the log portfolio return, $r_p = \ln(1 + R_p)$, is normally distributed, then maximizing $\mathrm{E}(1 + R_p)^{1-\gamma}/(1-\gamma)$ is equivalent to maximizing

$$\mathrm{E}\, r_p + (1-\gamma)\,\mathrm{Var}(r_p)/2. \tag{13.5}$$

Note that this is a one-period (myopic) optimum. A dynamic optimum is discussed later on.

Using the approximation (13.2) gives optimal portfolio weights for the one-period (myopic) case as

$$v = \Sigma^{-1}(\mu^e + \mathrm{diag}(\Sigma)/2)/\gamma. \tag{13.6}$$

(The proof is at the end of the section.) These are the weights from the log utility case, but now divided by the risk aversion $\gamma$.

**Example 13.5** *(One risky asset) Using the same figures as in Example 13.2 and $\gamma = 6$ gives $v_A = 0.62$ and $v_B = 0.22$.*

**Example 13.6** *(Three risky assets) Using the same figures as in Example 13.3 and $\gamma = 6$ gives*

$$v_A \approx \begin{bmatrix} 0.11 \\ 0.49 \\ 0.10 \end{bmatrix} \text{ and } v_B \approx \begin{bmatrix} 0.08 \\ 0.27 \\ 0.07 \end{bmatrix}.$$

**Empirical Example 13.7** *See Figure 13.2 for a comparison of the solutions from log utility and from CRRA with $\gamma = 3$. The changes are more muted with the higher risk aversion.*

*Proof* of (13.6). From (13.2) we have that the objective function can be written $r_f + v'\mu^e + v'\,\mathrm{diag}(\Sigma)/2 - v'\Sigma v/2 + (1-\gamma)v'\Sigma v/2$, so the first order conditions are $\mu^e + \mathrm{diag}(\Sigma)/2 - \gamma\Sigma v = \mathbf{0}_{n\times 1}$, which gives (13.6). □

## 13.3 Intertemporal Hedging (CRRA Utility and non-iid Returns)

Reference: Campbell and Viceira (2002) and Merton (1973)

### 13.3.1 An Autocorrelated Return Process

The *combination* of a CRRA utility function (with $\gamma \neq 1$) and non-iid returns makes portfolio choice more challenging. If there is a link between returns in different periods, then a long-run investor might want to take this into account as it provides "diversification" across periods. This is *intertemporal hedging*. We illustrate this below by using a simple model. (See Campbell and Viceira (1999) for a more elaborate approach.)

An earlier chapter used the following simple time series model for a single excess log return, which encompasses both the iid case, as well as long-run reversal or momentum.

When $r^e$ and $z$ are $n$-vectors, then the corresponding multivariate system is

$$\begin{bmatrix} r^e_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} a \\ \mathbf{0}_n \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n\times n} & I_n \\ \mathbf{0}_{n\times n} & \phi \end{bmatrix} \begin{bmatrix} r^e_t \\ z_t \end{bmatrix} + \begin{bmatrix} I_n & \mathbf{0}_{n\times n} \\ \theta & I_n \end{bmatrix} \begin{bmatrix} u_{t+1} \\ \eta_{t+1} \end{bmatrix}. \tag{13.7}$$

where $r^e_{t+1}$ and $z_{t+1}$ are $n$-vectors. We assume that $u_{t+1}$ is uncorrelated with $\eta_{t+1}$, but there may be correlations within each vector. The respective variance-covariance matrices are $\Sigma_{uu}$ and $\Sigma_{\eta\eta}$. Note that $a$ is an $n$-vector and that $\phi$ and $\theta$ are both $n \times n$ matrices.

### 13.3.2 Myopic Portfolio Choice

The expectation and variance of $r^e_{t+1}$, conditional on the information available at the time of investment in $t$, are

$$\mathrm{E}_t\, r^e_{t+1} = a + z_t \tag{13.8}$$

$$\mathrm{Var}_t(r^e_{t+1}) = \Sigma_{uu}. \tag{13.9}$$

The optimal portfolio of a *myopic investor* is the same as in (13.6), but where the expected return is $\mu^e = a + z_t$ and the variance-covariance matrix is $\Sigma = \Sigma_{uu}$. Figures 13.3–13.5 indicate the myopic portfolio weights, mostly to make a comparison with the other solutions discussed below.

### 13.3.3 A Two-Period Investor (No Rebalancing)

We now consider a two-period investor who does not rebalance. This investor also maximizes (13.5), but the expectation and variance are for a 2-period return

$$\max_v \mathrm{E}_t(r_{p,t+1} + r_{p,t+2}) + (1-\gamma)\,\mathrm{Var}_t(r_{p,t+1} + r_{p,t+2})/2. \tag{13.10}$$

By using (13.8)–(13.9) and the results in the proof below, the two terms can be written

$$\mathrm{E}_t(r_{p,t+1} + r_{p,t+2}) = 2a + (I + \phi)z_t \tag{13.11}$$

$$\mathrm{Var}_t(r_{p,t+1} + r_{p,t+2}) = 2\Sigma_{uu}(I + \theta') + \theta\Sigma_{uu}\theta' + \Sigma_{\eta\eta}. \tag{13.12}$$

(The proof is at the end of the section.) For a single asset the two equations can be simplified as $2a + (1+\phi)z_t$ and $\sigma_u^2(2 + 2\theta + \theta^2) + \sigma_\eta^2$, respectively.

The optimal $v$ is obtained by substituting (13.11) for $\mu^e$ and (13.12) for $\Sigma$ in (13.6). This would equal the myopic choice when returns are iid ($\phi = \mathbf{0}, \theta = \mathbf{0}, \Sigma_{\eta\eta} = \mathbf{0}$).

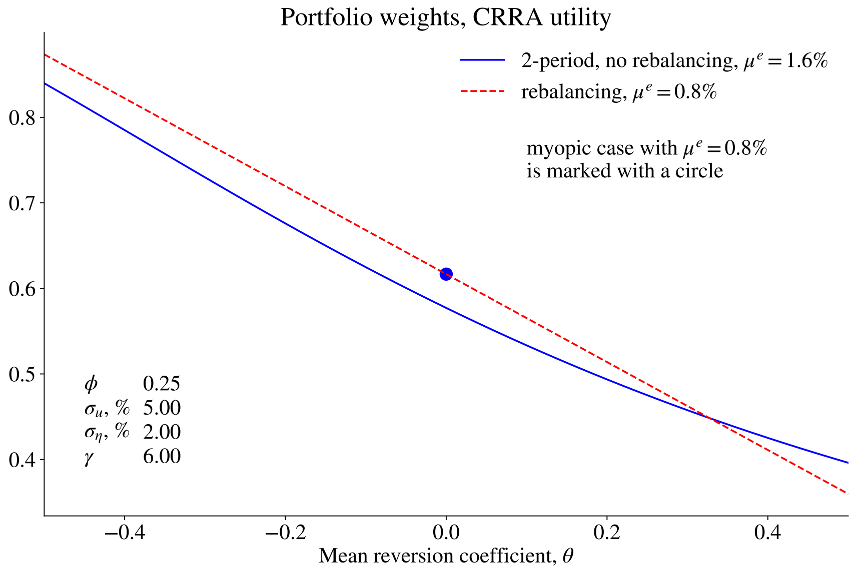



Figure 13.3: Weight on a single risky asset, two-period investor with CRRA utility for three cases: myopic, 2-period optimization without rebalancing and also with rebalancing. The return process is a scalar version of (13.7).

See Figure 13.3 for an illustration of how the portfolio weight on a single risky asset is higher with reversal ($\theta < 0$) than according to the myopic optimum. Also, see Figure 13.4 for an illustration of a case with several risky assets. The latter figure shows that the portfolio weight on the asset with non-iid returns (asset 2 in the figure) reacts strongly to variation in the reversal/momentum ($\theta$), but that there are some spillover effects on the other assets because of the correlations.

*Proof* of (13.11)–(13.12). By combining the equations in (13.7) we can also write

$$r^e_{t+2} = a + \phi z_t + \theta u_{t+1} + \eta_{t+1} + u_{t+2}.$$

The conditional moments are therefore

$$\begin{aligned}
\mathrm{E}_t\, r^e_{t+2} &= a + \phi z_t \\
\mathrm{Var}_t(r^e_{t+2}) &= \theta \Sigma_{uu} \theta' + \Sigma_{uu} + \Sigma_{\eta\eta} \\
\mathrm{Cov}_t(r^e_{t+1}, r^e_{t+2}) &= \Sigma_{uu} \theta'.
\end{aligned}$$

For a scalar return the latter two equations can also be written $\mathrm{Var}_t(r^e_{t+2}) = (1+\theta^2)\sigma_u^2 + \sigma_\eta^2$ and $\mathrm{Cov}_t(r^e_{t+1}, r^e_{t+2}) = \theta\sigma_u^2$. $\square$

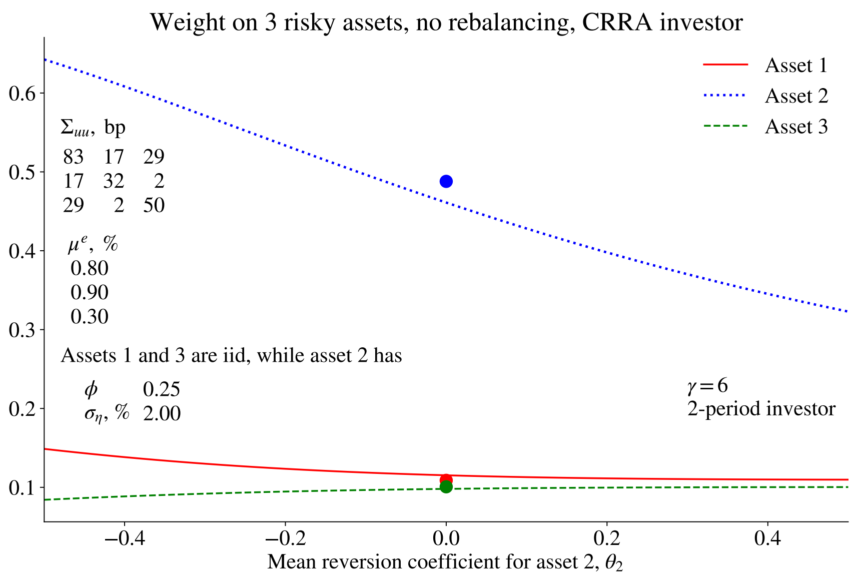



Figure 13.4: Weight on a three risky assets, for two cases: myopic, 2-period optimization without rebalancing. The return process is (13.7).

### 13.3.4 Two-Period Investor (with Rebalancing)

It is perhaps more reasonable to assume that the two-period investor can rebalance in each period. Rewrite the 2-period problem (13.10 ), but drop the terms that are not affected by the portfolio choice in period $t$, that is, $\mathrm{E}_t\, r_{p,t+2}$ and $\mathrm{Var}_t(r_{p,t+2})$, to get

$$\mathrm{E}_t\, r_{p,t+1} + (1-\gamma)[\mathrm{Var}_t(r_{p,t+1}) + 2\,\mathrm{Cov}_t(r_{p,t+1}, r_{p,t+2})]/2. \tag{13.13}$$

The new aspect is clearly the covariance term. Intuitively, if some assets returns in $t+1$ provide a hedge against returns in $t+2$, then this might affect the portfolio choice in $t$.

In principle, the covariance term is

$$\mathrm{Cov}_t(r_{p,t+1}, r_{p,t+2}) = v'\,\mathrm{Cov}_t(r^e_{t+1}, r^e_{t+2} v_{t+1}), \tag{13.14}$$

where $v$ is the portfolio choice in $t$ (and can therefore be moved outside the covariance operator) and $v_{t+1}$ is the portfolio choice in $t+1$. The latter must be a 1-period (myopic) choice, but applied to $t+1$. Clearly, those weights are not known in $t$, so we apply an *approximation* by replacing $v_{t+1}$ by its expected value, $\mathrm{E}_t\, v_{t+1}$, to get

$$\mathrm{Cov}_t(r_{p,t+1}, r_{p,t+2}) \approx v'\Sigma_{uu}\theta'\,\mathrm{E}_t\, v_{t+1}, \tag{13.15}$$

where the covariance follows directly from the earlier proof of (13.11)–(13.12).

The optimal portfolio choice is now

$$v = \Sigma_{uu}^{-1}(\mu^e + \text{diag}(\Sigma_{uu})/2 + (1-\gamma)\Sigma_{uu}\theta' \, \text{E}_t \, v_{t+1})/\gamma, \tag{13.16}$$

where $\mu^e = a + z_t$. (The proof is at the end of the section.) This equals the myopic portfolio in two cases: *(1)* when $\gamma = 1$ (log utility); and/or when *(2)* $\theta = \mathbf{0}$ (no reversal or momentum).

The value of $\text{E}_t \, v_{t+1}$ is found from using applying (13.8)–(13.9) but for $t+1$ in (13.6) and then taking expectations

$$\text{E}_t \, v_{t+1} = \Sigma_{uu}^{-1}(a + \phi z_t + \text{diag}(\Sigma_{uu})/2)/\gamma \tag{13.17}$$

See Figure 13.3 for an illustration of the case of a scalar risky return. The general pattern is similar to that of a 2-periods investor who cannot rebalance.

See also Figure 13.5 for the case of three risky assets. Again, the results are broadly in line with those for the 2-period investor who cannot rebalance, as in Figure 13.4.

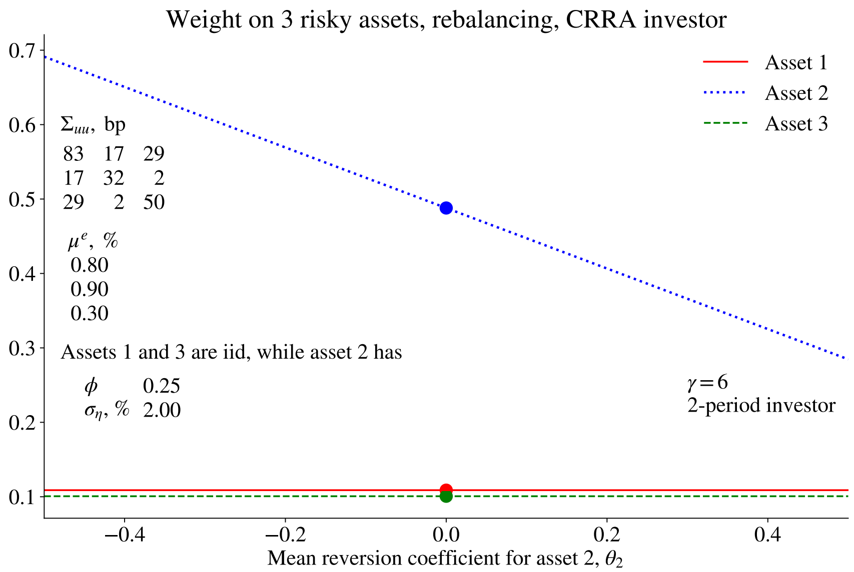



Figure 13.5: Weight on a three risky assets, for two cases: myopic and 2-period optimization with rebalancing. The return process is (13.7 ).

*Proof* of (13.16). Add the $(1-\gamma)\,\text{Cov}_t(r_{p,t+1}, r_{p,t+2})$ term from (13.15) to (13.5) and (for simplicity) use $\Sigma$ to denote $\Sigma_{uu}$. Then, following the same approach as in the proof of (13.6)

$$r_f + v'\mu^e + v' \operatorname{diag}(\Sigma)/2 - v'\Sigma v/2 + (1-\gamma)v'\Sigma v/2 + (1-\gamma)v'\Sigma\theta' \operatorname{E}_t v_{t+1}.$$

The first order conditions are

$$\mu^e + \operatorname{diag}(\Sigma)/2 - \gamma\Sigma v + (1-\gamma)\Sigma\theta' \operatorname{E}_t v_{t+1} = \mathbf{0}_{n\times 1}.$$

Substitute $\Sigma_{uu}$ for $\Sigma$ and solve for $v$. $\square$

### 13.3.5 Summary

The analysis in this section has shown that the optimal portfolio choice for a long-run (here, two-period) investor may differ substantially from that of a one-period investor *if* returns are non-iid. In particular, assets with long-run reversals are "safe" for a long run investor. This holds irrespective of whether the investor rebalances or not, although the mechanisms differ somewhat.

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