**Researches about theory (R)**

**21\_R.What is a Brownian diffusion process. History, importance, definition and applications.**

In probability theory and statistics, a diffusion process is a solution to a stochastic differential equation. It is a continuous-time Markov process with almost surely continuous sample paths. Brownian motion, reflected Brownian motion and Ornstein-Uhlenbeck processes are examples of diffusion processes.

A sample path of a diffusion process models the trajectory of a particle embedded in a flowing fluid and subjected to random displacements due to collisions with other particles, which is called Brownian motion. The position of the particle is then random; its probability density function as a function of space and time is governed by an advection–diffusion equation.

Brownian diffusion is the characteristic random wiggling motion of small [airborne particles](https://www.sciencedirect.com/topics/engineering/airborne-particle) in still air, resulting from constant [bombardment](https://www.sciencedirect.com/topics/engineering/bombardment) by surrounding gas molecules. Such irregular motions of pollen grains in water were first observed by the botanist Robert Brown in 1827, and later similar phenomena were found for small smoke particles in air. In the early twentieth century, the relationships characterizing Brownian diffusion based on [kinetic theory of gases](https://www.sciencedirect.com/topics/engineering/kinetic-theory-of-gas) were first derived by Einstein, and later verified through experiments.

VIDEO

This is a simulation of the Brownian motion of 5 particles (yellow) that collide with a large set of 800 particles. The yellow particles leave 5 blue trails of (pseudo) random motion and one of them has a red velocity vector.

VIDEO

This is a simulation of the Brownian motion of a big particle (dust particle) that collides with a large set of smaller particles (molecules of a gas) which move with different velocities in different random directions.

The first person to describe the mathematics behind Brownian motion was Thorvald N. Thiele in a paper on the method of least squares published in 1880. This was followed independently by Louis Bachelier in 1900 in his PhD thesis "The theory of speculation", in which he presented a stochastic analysis of the stock and option markets. The Brownian motion model of the stock market is often cited, but Benoit Mandelbrot rejected its applicability to stock price movements in part because these are discontinuous

In mathematics, Brownian motion is described by the Wiener process, a continuous-time stochastic process named in honor of Norbert Wiener. It is one of the best known Lévy processes (càdlàg stochastic processes with stationary independent increments) and occurs frequently in pure and applied mathematics, economics and physics.

The Wiener process Wt is characterized by four facts:

1. W0 = 0
2. Wt is almost surely continuous
3. Wt has independent increments
4. Wt – Ws **~ N(0, t - s) (for 0 <= s <= t)**

The Wiener process can be constructed as the scaling limit of a random walk, or other discrete-time stochastic processes with stationary independent increments. This is known as Donsker's theorem. Like the random walk, the Wiener process is recurrent in one or two dimensions (meaning that it returns almost surely to any fixed neighborhood of the origin infinitely often) whereas it is not recurrent in dimensions three and higher. Unlike the random walk, it is scale invariant.

The Brownian motion can be modeled by a random walk. Random walks in porous media or fractals are anomalous.

In the general case, Brownian motion is a non-Markov random process and described by stochastic integral equations.

**22\_R.An "analog" of the CLT for stochastic process: the Brownian motion as limit of random walk and the functional CLT (Donsker theorem). Explain the intuitive meaning of this result.**

In probability theory, Donsker’s theorem (also known as Donsker’s invariance principle, or the functional central limit theorem), named after Monroe D. Donsker, is a functional extension of the central limit theorem.

VIDEO

Let be X1, X2, X3, … be a sequence of independent and identically distributed (i.i.d) random variables with mean 0 and variance 1.

Let IMAGE. The stochastic process IMAGE is known as a random walk. Define the diffusively rescaled random walk (partial-sum process) by

IMAGE

The Central Limit Theorem asserts that W(n)(1) converges in distribution to a Standard Gaussian random variable W(1) as n grows to infinite. Donsker’s invariance principle extends this convergence to the whole function W(n) := (W(n)(t)) w. t in [0,1]. More precisely, in its modern form, Donsker’s invariance principle states that: As random variables taking values in the Shorokhod space D[0,1], the random function W(n) converges in distribution to a standard Brownian Motion W := (W(t)) w. t in [0,1] as n grows to infinite.

**Applications / Practice (A)     [work on this at least 30' a day, all days]**  
**12\_A. Discover one of the most important stochastic process by yourself !  
  
Consider the general scheme we have used so far to simulate stochastic processes (such as the relative frequency of success in a sequence of trials, the sample mean, the random walk, the Poisson point process, etc.) and now add this new process to our simulator.  
  
Starting from value 0 at time 0, for each of m paths, at each new time compute P(t) = P(t-1) + Random step(t), for t = 1, ..., n,  
where the Random step(t) is now:  
  
σ \* sqrt(1/n) \* Z(t),  
  
where  Z(t) is a N(0,1) random variable (the "diffusion" σ is a user parameter, to scale the process dispersion).  
At time n (last time) and one (or more) other chosen inner time 1<j<n (j is a program parameter) create and represent with histogram the distribution of P(t). Observe the behavior of the process for large n.  
  
  
13\_A. Create the a distribution representation (histogram, or CDF ...) to represent the following:  
  
- Realizations taken from a Normal(0,1)  
- Realizations of the mean, obtained by averaging several times (say m times, m large) n of the above realizations  
- Realizations of the variance, obtained by averaging several times (say m times, m large) n of the above realizations  
- Realizations taken from exp(N(0,1)))  
- Realizations taken from N(0,1) squared  
- Realizations taken from a (squared N(0,1)) divided by another (squared N(0,1))**

**Researches about applications (RA))  
  
9\_RA  
  
Try to find on the web what are the names of the random variables that you just simulated in the applications, and see if the means and variances that you obtain in the simulation are compatible with the "theory". If not fix the possible bugs.**

**References**

<https://www.sciencedirect.com/topics/engineering/brownian-diffusion>

<https://en.wikipedia.org/wiki/Diffusion_process>

<https://en.wikipedia.org/wiki/Brownian_motion>

<https://en.wikipedia.org/wiki/Donsker%27s_theorem>