Solutions to Ex. 1

Have you tried the questions yet? If not, I recommend you make a serious attempt at the questions before looking at these solutions. You can learn a lot by trying and failing in maths. If you read the solution first, you lose that experience.

1.

$$Y_{i1} - Y_{i2} = \mu + b_i + \epsilon_{i1} - \mu - \beta - b_i - \epsilon_{i2}.$$

Thus

$$\begin{aligned} E[Y_{i1} - Y_{i2}] &= -\beta, \\ \text{var}[Y_{i1} - Y_{i2}] &= \text{var}(\epsilon_{i1} - \epsilon_{i2}) \\ &= \text{var}(\epsilon_{i1}) + \text{var}(\epsilon_{i2}) \\ &= 2\sigma^2. \end{aligned}$$

Normality of $Y_{i1} - Y_{i2}$ follows since they are differences of Normal variates, and we have shown that the variance of the difference is constant, independent of i. Thus we have established the required conditions.

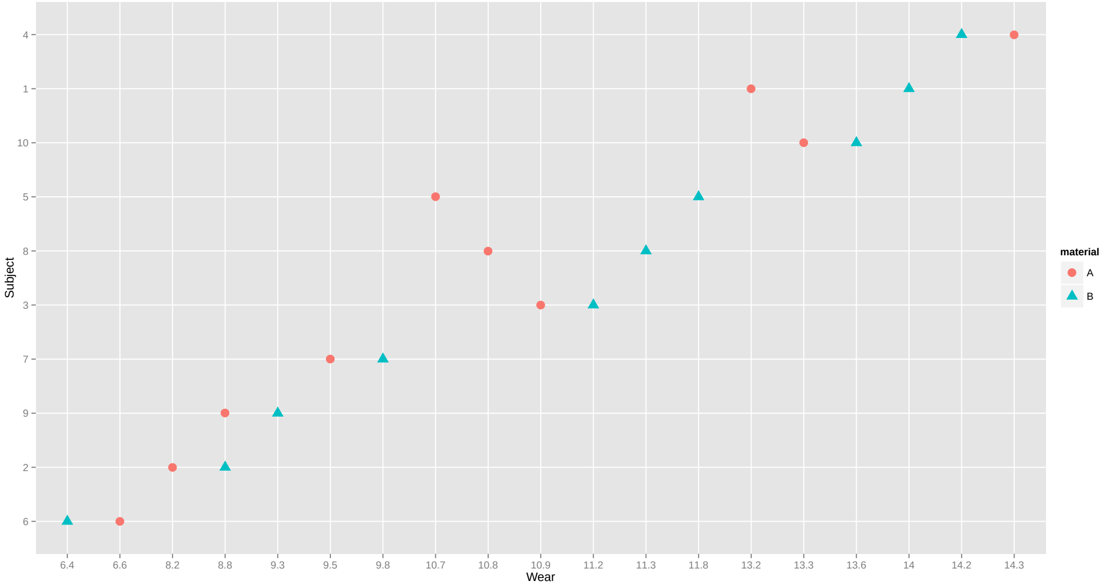
2.
```
 ++++++ Paired t-test ++++++++++
  > attach(BHHshoes)
> t.test(wear[material=="A"],wear[material=="B"],paired=T)

Paired t-test

data:  wear[material == "A"] and wear[material == "B"]
t = -3.3489, df = 9, p-value = 0.008539
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -0.6869539 -0.1330461
sample estimates:
mean of the differences
                  -0.41
```

The paired t-test has given reasonably strong evidence against the null that the mean difference is 0. The sample mean difference is -0.41, suggesting mean wear is higher in material B.

3. A single plot that will suffice here is plot of wear against subject, using different symbols and colours for the two material types. See the supporting R script for the plotting commands.



We can see that there is large variation between subjects relative to variation within subjects, and that material B has higher wear for most subjects.

4. The model fitting command and output are below. `material` is modelled as a fixed effect, as the interest is in the two particular material types chosen for the study, and `Subject` is modelled as a random effect, as the individuals in the study are not in themselves of interest, and are a sample from a larger population of individuals.

```
> fm1<-lmer(wear~material+(1|Subject), data = BHHshoes)
> summary(fm1)
Linear mixed model fit by REML ['lmerMod']
Formula: wear ~ material + (1 | Subject)
   Data: BHHshoes

REML criterion at convergence: 54.9

Scaled residuals:
     Min       1Q   Median       3Q      Max
-1.25097 -0.27027 -0.00901  0.26815  1.26948

Random effects:
 Groups   Name        Variance Std.Dev.
 Subject  (Intercept) 6.10089  2.4700
 Residual             0.07494  0.2738
Number of obs: 20, groups:  Subject, 10

Fixed effects:
            Estimate Std. Error t value
(Intercept)  10.6300     0.7859  13.527
materialB     0.4100     0.1224   3.349

Correlation of Fixed Effects:
          (Intr)
materialB -0.078
```

Parameter estimate are $\hat{\mu} = 10.63$, $\hat{\beta}_2 = 0.41$, $\hat{\sigma}_b^2 = 6.1$ and $\hat{\sigma}^2 = 0.075$.

5. The estimate of β_2 is 0.41, which is the difference between the sample means for the two materials (given in the t-test output). Calculating the sample mean for material A

```
> mean(wear[material=="A"])
[1] 10.63
```

we see that the estimate of μ is the sample mean for material A. Now, noting that $Var(Y_{ij}) = \sigma_b^2 + \sigma^2$,

$$\begin{aligned} \hat{\mu} &= \frac{1}{10}\sum_{i=1}^{10} Y_{ij}, \\ \Rightarrow Var(\hat{\mu}) &= \frac{\sigma_b^2 + \sigma^2}{10}, \end{aligned}$$

and so the estimated standard error is

$$\sqrt{\frac{6.10089 + 0.07494}{10}} = 0.7859,$$

to 4 d.p. Also,

$$\begin{aligned} \hat{\beta}_2 &= \frac{1}{10}\left(\sum_{i=1}^{10} Y_{i2} - \sum_{i=1}^{10} Y_{i1}\right), \\ &= \frac{1}{10}\left(\sum_{i=1}^{10}(\mu + \beta_2 + b_i + \epsilon_{i2}) - \sum_{i=1}^{10}(\mu + b_i + \epsilon_{i1})\right), \\ \Rightarrow Var(\hat{\beta}_2) &= \frac{2\sigma^2}{10}, \end{aligned}$$

and so the estimated standard error is

$$\sqrt{\frac{2 \times 0.07494}{10}} = 0.1224,$$

to 4 d.p.

6. The `t value` is the same as the (absolute value of the) t-statistic in the paired t-test, as in both cases they are difference in sample means divided by the estimated standard error for the difference. The analysis with the mixed effects model (regarding the effect of `material`) is equivalent to the paired t-test. The mixed effects model does, however, give a little more information, as it also give estimates of between and within subject variability.