(☆) Worked with Zhenghao Li

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12/05/22
ECO384G
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Note: I apologize for only being able to complete problem 1 & its parts. I unfortunately am a slow writer for papers, & have to put words for the 2YP now. ☹

ECO384 Problem Set 5 (Problem Set 2, Spring 2022; Problem Set 2, Nitya)

1.1). Lagrangian $\mathcal{L} = \int_0^N (1-e^{-\alpha q_i})di + \lambda\left[E - \int_0^N p_i q_i di\right]$

FOCs:

$[q_i]: \alpha e^{-\alpha q_i} - \lambda p_i = 0$

$[\lambda]: E - \int_0^N p_i q_i d_i = 0$

Observe that for some i & $\forall j$, we have $e^{-\alpha(q_i - q_j)} = \frac{p_i}{p_j}$

$\Rightarrow -\alpha(q_i - q_j) = \ln(p_i) - \ln(p_j)$

$\Rightarrow q_j = q_i + \frac{1}{\alpha}(\ln(p_i) - \ln(p_j))$

$\therefore p_i q_j = p_i q_i + \frac{1}{\alpha}(p_i \ln(p_i) - p_i \ln(p_j))$

$\Rightarrow \int_0^N p_i q_j di = \int_0^N p_i q_i d_i + \frac{1}{\alpha}\int_0^N p_i \ln(p_i) d_i - \frac{1}{\alpha}\ln(p_j)\int_0^N p_i d_i$

$$\boxed{\therefore q_j = \frac{E + \frac{1}{\alpha}\int_0^N p_i \ln(p_i) d_i}{\int_0^N p_i d_i} - \frac{1}{\alpha}\ln(p_j)}$$

1.2). Firm maximises profits: $\max\{L q_i (p_i - mw)\}$

$\Rightarrow \max\left\{L\left[\frac{E + 1/\alpha \int_0^N p_j \ln(p_j) d_j}{\int_0^N p_j d_j} - \frac{1}{\alpha}\ln(p_i)\right](p_i - mw)\right\}$

$\therefore [FOC]: L\left[\frac{E + \frac{1}{\alpha}\tilde{p}}{\bar{p}} - \frac{1}{\alpha}\ln(p_i)\right] + L(p_i - mw)\left(\frac{-1}{\alpha p_i}\right) = 0$, where

$\tilde{p} = \int_0^N p_j \ln(p_j) d_j$; $\bar{p} = \int_0^N p_j d_j$.

1.3). Assume equilibrium is symmetric & all firms charge the same price p.

$\therefore p_i = p_j = p, \forall i, j$. $\therefore \tilde{p} = N p \ln(p)$, $\bar{p} = Np$.

$\therefore [FOC]: L\left[\frac{E + 1/\alpha N p \ln(p)}{Np} - \frac{1}{\alpha}\ln(p)\right] + L(p - mw)\left(\frac{1}{-\alpha p}\right) = 0$

$\Rightarrow \frac{E\alpha}{N} + p\ln(p) - p\ln(p) = p - mw$

$$\boxed{\therefore p = mw + \frac{E\alpha}{N}}$$

1.3 (cont). $\therefore$ we see that the price does feature a markup over the MC. Specifically, $p - mw = \frac{E\alpha}{N} > 0$. We also see that this markup decreases when the number of firms, N, increases.

Regarding CES utility, if we replace the utility function with $\left[\int_0^N (\alpha^{1/\sigma} q^{\sigma-1/\sigma}) di\right]^{\sigma/\sigma-1}$, we would obtain $p = \frac{m\omega}{\sigma-1}$ & $\frac{p - m\omega}{p} = \frac{1}{\sigma}$. This means in the CES case, price doesn't depend on E & N, & the the markup over price only depends on the elasticity of substitution.

In our original case, $\frac{p - mw}{p} = \frac{\frac{E\alpha}{N}}{mw + \frac{E\alpha}{N}}$.

1.4). Recall $p = mw + \frac{E\alpha}{N}$.

$$q_i = \frac{E + \frac{1}{\alpha}\tilde{p}}{\tilde{P}} - \frac{1}{\alpha}\ln(p) = \frac{E + \frac{1}{\alpha}N\ln(p)}{Np} - \frac{1}{\alpha}\ln(p) = \frac{E}{Np}$$

$$= \frac{E}{N\left(mw + \frac{E\alpha}{N}\right)} = \boxed{\frac{E}{Nmw + E\alpha}}$$

Profits:

$$\Pi_i = Lq_i(p - mw) = L\left(\frac{E}{Nmw + E\alpha}\right)\left(\frac{E\alpha}{N}\right) = \boxed{\frac{\alpha L E^2}{mwN^2 + \alpha E N}}$$

1.5). Each producer pays fixed cost equal to F units of labour. In free entry equilibrium, profits $- F = 0$ & labour markets clear:

$\Pi_i = Lq_i(p_i - mw) - Fw = 0$; $\int_0^N (mLq_i + F) di = L$

Solve for eq^m number of firms N.

By the free entry condition, we know $\Pi_i = Fw \Rightarrow \frac{\alpha L E^2}{mwN^2 + \alpha E N} = Fw$

$\Rightarrow \alpha L E^2 = Fmw^2N^2 + \alpha EFwN \Rightarrow Fmw^2N^2 + \alpha EFwN - \alpha L E^2 = 0$

With N as our variable of interest, we get:

$$N = \frac{\sqrt{(\alpha EFw)^2 + 4\alpha L E^2 Fmw^2} - \alpha EFw}{2Fmw^2}$$

If we apply $E = w$, we get:

$$N = \frac{\sqrt{\alpha^2F^2w^4 + 4\alpha L w^4} - \alpha F w^2}{2Fmw^2} = \boxed{\frac{\sqrt{\alpha^2F^2 + 4\alpha L} - \alpha F}{2Fm}}$$

1.6). Solve for welfare $\frac{w}{p}$. How does it depend on the number of firms?

$$\frac{w}{p} = \frac{w}{mw + \frac{E\alpha}{N}} = \frac{Nw}{Nmw + E\alpha}$$

Recall $E = w$. Then:

$$\frac{w}{p} = \frac{Nw}{Nmw + w\alpha} = \boxed{\frac{N}{Nm + \alpha}}$$

$$\frac{\partial\left(\frac{w}{p}\right)}{\partial N} = \boxed{\frac{Nm + \alpha - Nm}{(Nm+\alpha)^2} > 0}$$

$\boxed{\therefore \text{ As } N\uparrow, \text{ welfare } \frac{w}{p} \text{ will also } \uparrow.}$