① $E[\Psi] = \frac{\langle \Psi|\hat{A}|\Psi\rangle}{\langle\Psi|\Psi\rangle} \geq E_0$

$$\frac{\langle \sum_{k=0}^{\infty} c_k \Psi_k | \hat{A} | \sum_{l=0}^{\infty} c_l \Psi_l \rangle}{\langle \sum_{k=0}^{\infty} c_k \Psi_k | \sum_{l=0}^{\infty} c_l \Psi_l \rangle} - E_0 \geq 0$$

$$\frac{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_k^* c_l \langle \Psi_k|\hat{A}|\Psi_l\rangle}{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_k^* c_l \langle \Psi_k|\Psi_l\rangle} - E_0 = \frac{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_k^* c_l E_l \langle \Psi_k|\Psi_l\rangle}{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_k^* c_l \delta_{kl}} - E_0 \geq 0$$

$$\frac{\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} c_k^* c_l E_l \delta_{kl}}{\sum_{k=0}^{\infty} |c_k|^2} - E_0 = \frac{\sum_{k=0}^{\infty} |c_k|^2 E_k}{\sum_{k=0}^{\infty} |c_k|^2} - \frac{E_0 \sum_{k=0}^{\infty} |c_k|^2}{\sum_{k=0}^{\infty} |c_k|^2} \geq 0$$

$$\frac{\sum_{k=0}^{\infty} |c_k|^2 (E_k - E_0)}{\sum_{k=0}^{\infty} |c_k|^2} \geq 0 \quad \begin{cases} |c_k|^2 \geq 0 \\ (E_k - E_0) \geq 0 \end{cases} \longrightarrow E[\Psi] \geq E_0$$

② $\hat{H} = \hat{H}_0 + FV(x)$

$E(F) = E(0) + FE'(0) + \frac{F^2 E''(0)}{2} + \dots$

$\Psi(F) = \Psi(0) + F\Psi'(0) + \frac{F^2 \Psi''(0)}{2} + \dots$

$\hat{H}(F)|\Psi(F,x)\rangle = E(F)|\Psi(F,x)\rangle$

$(\hat{H}(F) - E(F))|\Psi(F,x)\rangle = 0$

$(\hat{H}_0(F) + FV(x) - [E(0) + FE'(0) + \frac{F^2 E''(0)}{2} + \dots])|\Psi(0) + F\Psi'(0) + \frac{F^2\Psi''(0)}{2} + \dots\rangle = 0$

* truncate $E(F)$ and $\Psi(F)$ up to second order and expand the expression:

$\hat{H}_0(F)|\Psi(0)\rangle + F\hat{H}_0|\Psi'(0)\rangle + \frac{1}{2}F^2\hat{H}_0|\Psi''(0)\rangle + FV(x)|\Psi(0)\rangle + F^2 V(x)|\Psi'(0)\rangle + \frac{F^3}{2}V(x)|\Psi''(0)\rangle$

$- E(0)|\Psi(0)\rangle - FE(0)|\Psi'(0)\rangle - \frac{1}{2}F^2 E(0)|\Psi''(0)\rangle - FE'(0)|\Psi(0)\rangle - F^2 E'(0)|\Psi'(0)\rangle - \frac{1}{2}F^3 E'(0)|\Psi''(0)\rangle$

$- \frac{F^2}{2}E''(0)|\Psi(0)\rangle - \frac{F^3}{2}E''(0)|\Psi'(0)\rangle - \frac{F^4}{2}E''(0)|\Psi''(0)\rangle = 0$

* Group terms with the same F^n:

$(\hat{H}_0(F) - E(0))|\Psi(0)\rangle + F[(V(x) - E'(0))|\Psi(0)\rangle + (\hat{H}_0 - E(0))|\Psi'(0)\rangle]$

$+ \frac{1}{2}F^2[(\hat{H}_0 - E(0))|\Psi''(0)\rangle + (2V(x) - 2E'(0))|\Psi'(0)\rangle - E''(0)|\Psi(0)\rangle] + \dots = 0$

* $(\hat{H}_0(F) - E(0))|\Psi(0)\rangle = 0$

$\hat{H}_0(F)|\Psi(0,x)\rangle = E(0)|\Psi(0,x)\rangle$

* $(V(x) - E'(0))|\Psi(0)\rangle + (\hat{H}_0 - E(0))|\Psi'(0)\rangle = 0$

$E'(0)|\Psi(0)\rangle - V(x)|\Psi(0)\rangle = \hat{H}_0|\Psi'(0)\rangle - E(0)|\Psi'(0)\rangle = (\hat{H}_0 - E(0))|\Psi'(0)\rangle$

Hermitian

$\langle\Psi(0)|E'(0)|\Psi(0)\rangle - \langle\Psi(0)|V(x)|\Psi(0)\rangle = \langle\Psi(0)|(\hat{H}_0 - E(0))|\Psi'(0)\rangle = \langle(\hat{H}_0 - E(0))\Psi(0)|\Psi'(0)\rangle \to 0$

$E'(0) = \frac{\langle\Psi(0)|V(x)|\Psi(0)\rangle}{\langle\Psi(0)|\Psi(0)\rangle} = \langle\Psi(0)|V(x)|\Psi(0)\rangle$ normalized Ψ

* $(\hat{H}_0 - E(0))|\Psi''(0)\rangle + 2(V(x) - E'(0))|\Psi'(0)\rangle - E''(0)|\Psi(0)\rangle = 0$

$\hat{H}_0|\Psi''(0)\rangle - E(0)|\Psi''(0)\rangle + 2V(x)|\Psi'(0)\rangle - 2E'(0)|\Psi'(0)\rangle - E''(0)|\Psi(0)\rangle = 0$

0, Hermitian

$\langle\Psi(0)|E''(0)|\Psi(0)\rangle = \langle\Psi(0)|(\hat{H}_0 - E(0))|\Psi''(0)\rangle + 2\langle\Psi(0)|V(x)|\Psi'(0)\rangle - 2E'(0)\langle\Psi(0)|\Psi'(0)\rangle \to 0$

$E''(0) = \frac{2\langle\Psi(0)|V(x)|\Psi'(0)\rangle}{\langle\Psi(0)|\Psi(0)\rangle} = 2\langle\Psi(0)|V(x)|\Psi'(0)\rangle$ normalized

$|\Psi'(0)\rangle = \sum_{m=1}^{\infty} \frac{\langle\Psi_m(0,x)|V(x)|\Psi_0(0,x)\rangle}{E_0(0) - E_m(0)} |\Psi_m(0,x)\rangle$

$E''(0) = 2\sum_{m=1}^{\infty} \frac{\langle\Psi(0)|V(x)|\Psi_m(0,x)\rangle\langle\Psi_m(0,x)|V(x)|\Psi_0(0,x)\rangle}{E_0(0) - E_m(0)}$

③ $\hat{A}|\Psi_1\rangle = \alpha_1|\Psi_1\rangle \to \langle\Psi_2|\hat{A}|\Psi_1\rangle = \alpha_1\langle\Psi_2|\Psi_1\rangle$

$A|\Psi_2\rangle = \alpha_2|\Psi_2\rangle \to [\langle\Psi_1|\hat{A}|\Psi_2\rangle = \alpha_2\langle\Psi_1|\Psi_2\rangle]^*$

$\hat{A}$ is Hermitian

$\langle\Psi_2|\hat{A}|\Psi_1\rangle - \langle\Psi_1|\hat{A}|\Psi_2\rangle^* = \alpha_1\langle\Psi_2|\Psi_1\rangle - \alpha_2\langle\Psi_1|\Psi_2\rangle^*$

$\langle\Psi_2|\hat{A}|\Psi_1\rangle - \langle\Psi_2|\hat{A}|\Psi_1\rangle = (\alpha_1 - \alpha_2)\langle\Psi_2|\Psi_1\rangle = 0$

$\alpha_1 \neq \alpha_2$, so $\langle\Psi_2|\Psi_1\rangle = 0$.

④ $\langle\hat{A}\Psi|\hat{B}\Psi\rangle = \langle\Psi|\hat{A}\hat{B}|\Psi\rangle$

$\langle\Psi_2|\hat{A}|\Psi_1\rangle = \langle\Psi_1|\hat{A}|\Psi_2\rangle^*$

$|\Psi_1\rangle = |\hat{B}\Psi\rangle$; $\langle\Psi_2| = \langle\Psi|$

$\langle\Psi|\hat{A}|\hat{B}\Psi\rangle = \langle\hat{B}\Psi|\hat{A}\Psi\rangle^* = \langle\hat{A}\Psi|\hat{B}\Psi\rangle$

$\langle\Psi|\hat{A}\hat{B}|\Psi\rangle = \langle\hat{A}\Psi|\hat{B}\Psi\rangle$

⑤ a) $|\phi\rangle = \sum_m c_m|m\rangle$

$\langle n|\phi\rangle = \langle n|\sum_m c_m m\rangle = \sum_m c_m\langle n|m\rangle = \sum_m c_m\delta_{nm} = c_n$

$c_n = \langle n|\phi\rangle$

b) $|\phi\rangle = \sum_n c_n|n\rangle = \sum_n \langle n|\phi\rangle|n\rangle = \sum_n |n\rangle\langle n|\phi\rangle$

c) $|\phi\rangle^* = \langle\phi| = [\sum_n c_n|n\rangle]^* = \sum_n c_n^*\langle n|$

$c_n^* = \langle n|\phi\rangle^* = \langle\phi|n\rangle$

$\langle\phi| = \sum_n \langle\phi|n\rangle\langle n|$

d) $\langle\phi|\phi\rangle = 1$

$\langle\phi|\phi\rangle = \sum_m \langle\phi|m\rangle\langle m| \sum_n |n\rangle\langle n|\phi\rangle = \sum_m\sum_n \langle\phi|m\rangle\langle m|n\rangle\langle n|\phi\rangle$.

$= \sum_m\sum_n \langle\phi|m\rangle\delta_{mn}\langle n|\phi\rangle = \sum_n \langle\phi|n\rangle\langle n|\phi\rangle = \langle\phi|(\sum_n |n\rangle\langle n|)|\phi\rangle = \langle\phi|\phi\rangle$

$\sum_n |n\rangle\langle n| = 1$