Name:__________________________ Student #:__________________________

Quiz 3

1-3. Consider the following matrix,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 5 & 4 \\ 0 & 3+i & 1 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & i \end{bmatrix}$$

1. ______ What is the value of the trace of the matrix, Tr[A]?

2. ______ What is the value of the determinant of the matrix |A|?

3. What are the eigenvalues of the matrix A?

4. What is the determinant of the following matrix, |B|?

$$\mathbf{B} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 2 & 2 & 1 & 2 \\ -1 & 2 & -2 & 0 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

5. **What are the eigenvalues of the following matrix?**

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

6-7. One of the eigenvalues of the following matrix is equal to –2.

$$\mathbf{F} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

6. What is the left-eigenvector of F with eigenvalue –2?

7. What is the right-eigenvector of F with eigenvalue –2?

8-10. You are given the following matrix decomposition:

$$\mathbf{G} = \begin{bmatrix} 3 & 3 & -3 & 6 \\ 0 & 6 & 0 & 0 \\ 1 & -5 & 1 & -2 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & 1 & -1 \\ 3 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

8-9. What are the right-eigenvalues and eigenvectors for the matrix G.

10. What is the exponential of the matrix, e^G?

BONUS (10 pts): Find the other eigenvalues/eigenvectors of **F** (from problems 6-7).

BONUS (5 pts): It has come to my attention that most of you have class during my office hours. Check-off the times you are available below:

	11:30-12:30	**12:30-1:30**	**2:30-3:30**	**3:30-4:30**	**4:30-5:30**	**5:30-6:30**
Monday						
Wednesday						
Thursday						

Quiz 3

1-3. Consider the following matrix,

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 5 & 4 \\ 0 & 3+i & 1 & 0 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 0 & i \end{bmatrix}$$

1. __2+2*i*____ **What is the value of the trace of the matrix, Tr[A]?**

$$\mathrm{Tr}\left[\mathbf{A}\right] = 1 + \left(3+i\right) + \left(-2\right) + \left(i\right) = 2 + 2i$$

2. __2-6*i*____ **What is the value of the determinant of the matrix |A|?**

$$\left|\mathbf{A}\right| = 1\left(3+i\right)\left(-2\right)\left(i\right) = 1\left(3+i\right)\left(-2i\right) = -6i - 2i^2 = 2 - 6i$$

3. What are the eigenvalues of the matrix A?

The eigenvalues are $1, 3+i, -2, i$ because these are the solutions to

$$0 = \left|\mathbf{A} - \lambda\mathbf{I}\right| = \left(1-\lambda\right)\left(3+i-\lambda\right)\left(-2-\lambda\right)\left(i-\lambda\right)$$

4. What is the determinant of the following matrix, |B|?

$$\mathbf{B} = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 2 & 2 & 1 & 2 \\ -1 & 2 & -2 & 0 \\ 0 & 0 & -2 & 4 \end{bmatrix}$$

It is slightly easier to make this lower-triangular than upper triangular. So we (a) factor out 2 from the last row, add -1 times the last row to the second row, ad the second and third rows, add -1 times the second row to the first row. The determinant we find is zero.

$$\begin{vmatrix} 1 & 4 & 0 & 0 \\ 2 & 2 & 1 & 2 \\ -1 & 2 & -2 & 0 \\ 0 & 0 & -2 & 4 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 & 0 & 0 \\ 2 & 2 & 1 & 2 \\ -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 4 & 0 & 0 \\ 2 & 2 & 2 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 4 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 \end{vmatrix}$$

$$= 2\begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & 0 & -1 & 2 \end{vmatrix} = 0$$

$$|\mathbf{B}| = 0$$

5. What are the eigenvalues of the following matrix?

$$\mathbf{C} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues are found by finding the roots of the characteristic polynomial. In the first step we interchange the first and second rows, then we add λ times the first row to the second row, then we interchange the second and third rows, then we add λ^2-1 times the second row to the third row.

$$0 = |\mathbf{C} - \lambda\mathbf{I}| = \begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\begin{vmatrix} 1 & -\lambda & 1 \\ -\lambda & 1 & 0 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & -\lambda & 1 \\ 0 & 1-\lambda^2 & \lambda \\ 0 & 1 & -\lambda \end{vmatrix} = -(-1)\begin{vmatrix} 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \\ 0 & 1-\lambda^2 & \lambda \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \\ 0 & 0 & -\lambda(\lambda^2-1)+\lambda \end{vmatrix}$$

$$0 = \lambda(1-\lambda^2+1)$$

$$0 = \lambda(2-\lambda^2)$$

$$\lambda = 0, \pm\sqrt{2}$$

6-7. One of the eigenvalues of the following matrix is equal to –2.

$$\mathbf{F} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

6. What is the left-eigenvector of F with eigenvalue –2?

We need to solve,

$$\mathbf{u}^\dagger\left(\mathbf{F} - (-2)\mathbf{I}\right) = \mathbf{0}^\dagger$$

$$\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

It is a bit easier (or at least more "conventional") to solve this by taking the transpose of both sides. Then,

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we add -1 times row 2 to row 3. Then we add ½ row 1 to row 2. This gives

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Add 2 times row 2 to row 3,

$$\begin{bmatrix} 2 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Choose $u_3 = 1$. Then

$$\begin{aligned} \tfrac{3}{2}u_2 - \tfrac{3}{2}u_3 &= 0 \\ u_2 = u_3 &= 1 \\ 2u_1 - u_2 - u_3 &= 0 \\ 2u_1 - 1 - 1 &= 0 \\ u_1 &= 1 \end{aligned}$$

and so,

$$\mathbf{u}^\dagger_{\lambda=2} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

7. What is the right-eigenvector of F with eigenvalue –2?

The matrix is symmetric so the left-eigenvectors and the right-eigenvectors are the same. This is obvious if you consider that the equation for the right-eigenvector can be rewritten as,

$$\left(\mathbf{F} - \lambda\mathbf{I}\right)\mathbf{v} = \mathbf{0}$$

$$\mathbf{v}^{\dagger}\left(\mathbf{F} - \lambda\mathbf{I}\right)^{\dagger} = \mathbf{0}^{\dagger}$$

$$\mathbf{v}^{\dagger}\left(\mathbf{F}^{\dagger} - \lambda\mathbf{I}^{\dagger}\right) = \mathbf{0}^{\dagger}$$

$$\mathbf{v}^{\dagger}\left(\mathbf{F} - \lambda\mathbf{I}\right) = \mathbf{0}^{\dagger}$$

which is the same that considered in problem #6.

8-10. You are given the following matrix decomposition:

$$\mathbf{G} = \begin{bmatrix} 3 & 3 & -3 & 6 \\ 0 & 6 & 0 & 0 \\ 1 & -5 & 1 & -2 \\ 1 & 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & 1 & -1 \\ 3 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

8-9. What are the right-eigenvalues and eigenvectors for the matrix G.

I screwed up this problem because I forgot to take the transpose of the first matrix. So there are two ways to solve this.

#1. Multiply out the matrix. Then find the eigenvectors/eigenvalues. If you do this, you get, something rather hideous. You can confirm that the eigenvalues/eigenvectors are:

$$\lambda_1 = 16.89; \lambda_2 = -3.44 + 5.15i; \lambda_3 = -3.44 - 5.15i, \lambda_4 = 0$$

$$\begin{bmatrix} 2.57 \\ 1.67 \\ -1.10 \\ 1 \end{bmatrix}, \begin{bmatrix} 1.11 - .37i \\ -1.08 + .33i \\ .20 - .70i \\ 1 \end{bmatrix}, \begin{bmatrix} 1.11 + .37i \\ -1.08 - .33i \\ .20 + .70i \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -1 \\ 2 \end{bmatrix}$$

#2. If I had set this up correctly, it would have said,

$$\mathbf{G} = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 3 & 6 & -5 & 1 \\ -3 & 0 & 1 & 1 \\ 6 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & 1 & -1 \\ 3 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

and what you would have needed to remember was that the matrix can be decomposed as $\mathbf{G} = \mathbf{PDQ}$, where **P** contains the right-eigenvectors of **G** as columns, **Q** contains the left-eigenvectors of **G** as rows, and **D** is diagonal. I.e.,

$$\mathbf{G} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 & \cdots \\ \mathbf{u}_2 & \cdots \\ \mathbf{u}_3 & \cdots \\ \mathbf{u}_4 & \cdots \end{bmatrix}$$

The normalization of the left- and right-eigenvectors is arbitrary, but we can choose them so that $\mathbf{Q} = \mathbf{P}^{-1}$ and if we do this, then the entries of the diagonal matrix are the eigenvalues. That is, **PQ** = **QP** = c**I**. I.e., the product of **P** and **Q** is proportional to the identity matrix. So we take **P** times **Q** and find,

$$\mathbf{PQ} = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 3 & 6 & -5 & 1 \\ -3 & 0 & 1 & 1 \\ 6 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & 1 & -1 \\ 3 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} = 6\mathbf{I}$$

This lets us rewrite the matrix as

$$\mathbf{G} = \begin{bmatrix} 3 & 0 & 1 & 1 \\ 3 & 6 & -5 & 1 \\ -3 & 0 & 1 & 1 \\ 6 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 & \frac{-1}{6} & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2} & 0 & \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$= \mathbf{PDP}^{-1}$$

The eigenvectors are now obviously 12, 6, 0, –6. The right-eigenvectors are the columns,

$$\begin{bmatrix} 3 \\ 3 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 4 \end{bmatrix}$$

10. What is the exponential of the matrix, $e^{\mathbf{G}}$?

The exponential can be easily evaluated as,

$$e^{\mathbf{G}} = \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1}$$

$$= \begin{bmatrix} 3 & 0 & 1 & 1 \\ 3 & 6 & -5 & 1 \\ -3 & 0 & 1 & 1 \\ 6 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} e^{12} & 0 & 0 & 0 \\ 0 & e^{6} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-6} \end{bmatrix} \begin{bmatrix} \frac{1}{6} & 0 & \frac{-1}{6} & 0 \\ \frac{1}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \\ \frac{1}{2} & 0 & \frac{1}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3 & 0 & 1 & 1 \\ 3 & 6 & -5 & 1 \\ -3 & 0 & 1 & 1 \\ 6 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} e^{12} & 0 & 0 & 0 \\ 0 & e^{6} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-6} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 2 & 1 & 1 & -1 \\ 3 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3 & 0 & 1 & 1 \\ 3 & 6 & -5 & 1 \\ -3 & 0 & 1 & 1 \\ 6 & 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} e^{12} & 0 & -e^{12} & 0 \\ 2e^{6} & e^{6} & e^{6} & -e^{6} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2e^{-6} & e^{-6} \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3e^{12} & 0 & -3e^{12}+2e^{-6} & e^{-6} \\ 3e^{12}+12e^{6} & 6e^{6} & -3e^{12}+6e^{6}+2e^{-6} & -6e^{6}+e^{-6} \\ -3e^{12} & 0 & 3e^{12}+2e^{-6} & e^{-6} \\ 6e^{12} & 0 & -6e^{12}+8e^{-6} & 4e^{-6} \end{bmatrix}$$

BONUS (10 pts): Find the other eigenvalues/eigenvectors of **F** (from problems 6-7).

The left/right eigenvectors are equal. So the only thing we need to solve for is the right-eigenvectors (or the left ones).

We know that one eigenvalue is -2. We could find the eigenvalues in the standard way, but we also know that,

$$\mathrm{Tr}\left[\mathbf{F}\right] = \lambda_1 + \lambda_2 + \lambda_3 = 0$$

$$\left|\mathbf{F}\right| = \lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{vmatrix}_{\text{swap rows 3 and 1}}$$

$$= (-1)^2\begin{vmatrix} 1 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & -1 \end{vmatrix}_{\text{factor out -1 from row 1}} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & -1 \end{vmatrix}_{\text{add row 1 to row 2}}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -2 \end{vmatrix}_{\text{add row 2 to row 3}}$$

$$= 1 \cdot 1 \cdot -2$$

$$= -2$$

So I have,

$$\lambda_1 + \lambda_2 + (-2) = 0$$

$$\lambda_1\lambda_2(-2) = -2$$

$$\lambda_2 = 2 - \lambda_1$$

$$\lambda_1(2 - \lambda_1) = 1$$

$$\lambda_1^2 - 2\lambda_1 + 1 = 0$$

$$(\lambda_1 - 1)^2 = 0$$

$$\lambda_1 = 1$$

$$\lambda_2 = 2 - \lambda_1 = 2 - 1 = 1$$

$$\lambda_2 = 1$$

So the other eigenvalues are both equal to 1.

The eigenvectors for these eigenvalues are obtained by solving,

$$(\mathbf{F}-\lambda\mathbf{I})\mathbf{v}=\mathbf{0}$$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}\mathbf{v}=0$$

There is only one equation here. If we choose $v_3 = t$, and $v_2 = s$ $v_1 = -s-t$. So the eigenvectors are any vectors of the form,

$$\begin{bmatrix} -s-t \\ s \\ t \end{bmatrix}$$

For example, we could choose,

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$