Assignment (4)

(1) $k = 319\ \text{N}\cdot\text{m}^{-1}$

$$\mu = \frac{[35(1.661\times10^{-27}\text{ kg})][35(1.661\times10^{-27}\text{ kg})]}{35(1.661\times10^{-27}\text{ kg}) + 35(1.661\times10^{-27}\text{ kg})} = \frac{1}{2}(35)(1.661\times10^{-27}\text{ kg}) = 2.907\times10^{-26}\text{ kg}$$

$$\tilde{\nu}_{obs} = \frac{1}{2\pi c}\left(\frac{k}{\mu}\right)^{1/2} = \frac{1}{2\pi(2.998\times10^{8}\text{ m/s})}\sqrt{\frac{319\ \text{N}\cdot\text{m}^{-1}}{2.907\times10^{-26}\text{ kg}}} = 55611\ \text{m}^{-1} = 556\ \text{cm}^{-1}$$

$$E_0 = \frac{1}{2}h\nu = \frac{1}{2}hc\tilde{\nu} = \frac{1}{2}(6.626\times10^{-34}\text{ J}\cdot\text{s})(2.998\times10^{8}\text{ m/s})(55611\text{ m}^{-1}) = 5.52\times10^{-21}\text{ J}$$

(2) $\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}kx^2$; $\Psi_0\{x\} = A\exp\{-\frac{\sqrt{km}}{2\hbar}x^2\}$

a) $\int_{-\infty}^{\infty}\Psi_0^*\{x\}\Psi_0\{x\}dx = \int_{-\infty}^{\infty}A^2\exp\{-\frac{\sqrt{km}}{\hbar}x^2\}dx = 1$

$A^2\int_{-\infty}^{\infty}\exp\{-\frac{\sqrt{km}}{\hbar}x^2\}dx = 1$

$A^2\left(\frac{\pi}{\sqrt{km}/\hbar}\right)^{1/2} = 1$

$A^2 = \left(\frac{\sqrt{km}}{\hbar\pi}\right)^{1/2}$

$A = \pm\left(\frac{\sqrt{km}}{\hbar\pi}\right)^{1/4}$

b) $E = \int\Psi_0^*\{x\}\hat{H}\Psi_0\{x\}dx = \hbar\sqrt{\frac{k}{m}}\left(\frac{1}{2}\right) = \frac{\hbar}{2}\sqrt{\frac{k}{m}}$

c) $\frac{dE}{d\hbar} = \int\Psi_0^*\{x\}\frac{\partial\hat{H}}{\partial\hbar}\Psi_0\{x\}dx$

$\frac{1}{2}\sqrt{\frac{k}{m}} = \int\Psi_0^*\{x\}\left(-\frac{2\hbar}{2m}\frac{d^2}{dx^2}\right)\Psi_0\{x\}dx$

$\int\Psi_0^*\{x\}\frac{d^2}{dx^2}\Psi_0\{x\}dx = \left(-\frac{m}{\hbar}\right)\left(\frac{1}{2}\sqrt{\frac{k}{m}}\right) = -\frac{\sqrt{km}}{2\hbar}$

$\langle T\rangle = \int\Psi_0^*\{x\}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)\Psi_0\{x\}dx = -\frac{\hbar^2}{2m}\int\Psi_0^*\{x\}\frac{d^2}{dx^2}\Psi_0\{x\}dx = -\frac{\hbar^2}{2m}\left(-\frac{\sqrt{km}}{2\hbar}\right) = \frac{\hbar\sqrt{k}}{4\sqrt{m}} = \frac{\hbar}{4}\sqrt{\frac{k}{m}}$

d) $\langle\hat{p}^2\rangle = \int\Psi_0^*\{x\}\left(-i\hbar\frac{d}{dx}\right)^2\Psi_0^*\{x\}dx = \int\Psi_0^*\{x\}\left(-\hbar^2\frac{d^2}{dx^2}\right)\Psi_0\{x\}dx = -\hbar^2\left(-\frac{\sqrt{km}}{2\hbar}\right)$

$\langle\hat{p}^2\rangle = \hbar\sqrt{km}$

e) $\sigma_x = \sqrt{\frac{\hbar(1/2)}{\sqrt{km}}}$

$\sigma_x\sigma_{p_x} = \frac{[\hbar(\frac{1}{2})]^{1/2}}{(km)^{1/4}}\left(\hbar^{1/2}(km)^{1/4}\right) = \frac{\hbar}{\sqrt{2}} \geq \frac{\hbar}{2}$

$\langle\hat{p}\rangle = 0 \rightarrow \sigma_{p_x} = \sqrt{\langle\hat{p}^2\rangle} = \sqrt{\hbar\sqrt{km}}$

Recall that $\hat{p}_x$ and $\hat{x}$ don't commute, so eigenvalues of both operators cannot be specified simultaneously {Heisenberg's uncertainty principle}.

$\sigma_A\sigma_B \geq \frac{1}{2}\hbar$ if $[\hat{A},\hat{B}] \neq 0$.

(2) f) $$\langle x\hat{p}+\hat{p}x\rangle = \int_{-\infty}^{\infty}\left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}\exp\{-\frac{\sqrt{km}}{2\hbar}x^2\}\left[-(i\hbar x)\left(\frac{d}{dx}\right)-\left(i\hbar\frac{d}{dx}\right)(x)\right]\exp\{-\frac{\sqrt{km}}{2\hbar}x^2\}dx$$

$$= \int_{-\infty}^{\infty}\left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}(-i\hbar)\left[\exp\{-\frac{2\sqrt{km}}{2\hbar}x^2\}(x)\left(-\frac{\sqrt{km}}{\hbar}x\right)+\exp\{-\frac{2\sqrt{km}}{2\hbar}x^2\}\left(1+x\left(-\frac{\sqrt{km}}{\hbar}x\right)\right)\right]dx$$

$$= \left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}(-i\hbar)\left[\int_{-\infty}^{\infty}\left(-\frac{\sqrt{km}}{\hbar}\right)(x^2)(2)\exp\{-\frac{\sqrt{km}}{\hbar}x^2\}dx + \int_{-\infty}^{\infty}\exp\{-\frac{\sqrt{km}}{\hbar}x^2\}dx\right]$$

$$= \left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}(-i\hbar)\left[\left(-\frac{\sqrt{km}}{\hbar}\right)(2)\left(\frac{\hbar}{2\sqrt{km}}\right)\left(\sqrt{\frac{\pi\hbar}{\sqrt{km}}}\right)+\left(\sqrt{\frac{\pi\hbar}{\sqrt{km}}}\right)\right]$$

$$= (-i\hbar)\left[\left(-\frac{\sqrt{km}}{\hbar}\right)(2)\left(\frac{\hbar}{2\sqrt{km}}\right)+1\right] = (-i\hbar)[-1+1] = 0$$

$\langle\hat{p}\rangle = 0$

$\langle\hat{x}\rangle = 0$

$$\langle x\hat{p}-\hat{p}x\rangle = \int_{-\infty}^{\infty}\left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}\exp\{-\frac{\sqrt{km}}{2\hbar}x^2\}\left[-(i\hbar x)\left(\frac{d}{dx}\right)+\left(i\hbar\frac{d}{dx}\right)(x)\right]\exp\{-\frac{\sqrt{km}}{2\hbar}x^2\}dx$$

$$= \left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}(-i\hbar)\left[\int_{-\infty}^{\infty}\left(-\frac{\sqrt{km}}{\hbar}\right)(x^2)\underbrace{(1-1)}_{0}\exp\{-\frac{\sqrt{km}}{\hbar}x^2\}dx - \int_{-\infty}^{\infty}\exp\{-\frac{\sqrt{km}}{\hbar}x^2\}dx\right]$$

$$= \left(\frac{\sqrt{km}}{\pi\hbar}\right)^{1/2}(-i\hbar)\left(\frac{\pi\hbar}{\sqrt{km}}\right)^{1/2}(-1) = +i\hbar$$

$$\sigma_x^2\sigma_{p_x}^2 = \frac{\hbar^2}{2} \geq \left|\frac{1}{2}\langle x\hat{p}+\hat{p}x\rangle - \langle\hat{p}\rangle\langle\hat{x}\rangle\right|^2 + \left|\frac{1}{2i}\langle x\hat{p}-\hat{p}x\rangle\right|^2$$

$$\frac{\hbar^2}{2} \geq \left|\frac{1}{2i}(+i\hbar)\right|^2$$

$$\frac{\hbar^2}{2} \geq \frac{\hbar^2}{4}$$