Worksheet 4.

1. The fourth derivative, $\frac{d^4}{dx^4}$, is an operator that appears in some (approximate) relativistic quantum mechanical treatments. **Is this a Hermitian operator?**

2. Suppose that you are given two linear Hermitian operators, each of which has discrete, nondegenerate, spectrum:
$$\hat{A}\Psi_k(\boldsymbol{\tau}) = a_k \Psi_k(\boldsymbol{\tau}) \qquad a_0 < a_1 < a_2 < \cdots$$
$$\hat{B}\Phi_l(\boldsymbol{\tau}) = b_l \Phi_l(\boldsymbol{\tau}) \qquad b_0 < b_1 < b_2 < \cdots$$
Show that these operators have the same eigenfunctions (i.e., $\Psi_k = \Phi_k$) if and only if they commute (i.e., $\left[\hat{A}, \hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A} = \hat{0}$, where $\hat{0}$ is the "zero operator").

3. The *variance* in the expectation value of an operator is defined as
$$\sigma_{\hat{C}}^2 \equiv \left\langle \Psi \middle| \left(\hat{C} - \left\langle \hat{C} \right\rangle\right)^2 \middle| \Psi \right\rangle$$
$$\left\langle \hat{C} \right\rangle = \left\langle \Psi \middle| \hat{C} \middle| \Psi \right\rangle$$
Show that the following equation for the variance is equivalent to the preceding definition
$$\sigma_{\hat{C}}^2 \equiv \left\langle \Psi \middle| \hat{C}^2 \middle| \Psi \right\rangle - \left(\left\langle \Psi \middle| \hat{C} \middle| \Psi \right\rangle\right)^2.$$
In quantum mechanics, $\sigma_{\hat{C}}^2$ is sometimes called the *dispersion* of an operator and $\sigma_{\hat{C}} = \sqrt{\sigma_{\hat{C}}^2}$ is considered to represent the inherent uncertainty in measurements of the property associated with $\hat{C}$.

4. One of the Heisenberg Uncertainty Principles states that the uncertainty in the position and the momentum of a particle are coupled, and have the lower bound,
$$\sigma_p \sigma_x \geq \tfrac{1}{2}\hbar.$$
Consider a particle with unit mass in an infinite box of unit length. **Show, by explicit computation of the integrals involved, that the uncertainty principle is satisfied.** How tight is the lower bound?

5. The *dispersion condition* states: $\sigma_{\hat{C}}^2 = 0$ for a wavefunction, $\Psi(\boldsymbol{\tau})$, if and only if $\Psi(\boldsymbol{\tau})$ is an eigenfunction of $\hat{C}$. **Derive this theorem.**