COMMENTS ON ANGULAR MOMENTUM

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Angular momentum is a vector, and so it is represented by a quantum mechanical operator with three components I_x, I_y and I_z. There is also the operator corresponding to the total angular momentum, $I^2 = I_x^2 + I_y^2 + I_z^2$. The fundamental relation is that the commutator has the following property: $[I_x, I_y] = i\, I_z$. We also define the raising and lowering operators as

$$\begin{aligned} I_+ &= I_x + i\, I_y \\ I_- &= I_x - i\, I_y \end{aligned} \tag{1}$$

Only the total angular momentum and its z component are measurable, so we can define a state by its total angular momentum, J, and its z component, m. We write this as a ket $|J\ m>$. This is an eigenfunction of I_z, as follows:

$$I_z \left| J\ m \right\rangle = m \left| J\ m \right\rangle \tag{2}$$

It is also an eigenfunction of I^2, but its eigenvalue is ***not*** J^2, as we will see.

The raising operator raises m by one unit, as shown by this trick (show for yourself that $[I_z\ , I_+] = I_+$).

$$\begin{aligned} I_z I_+ \left| J\ m \right\rangle &= I_z I_+ \left| J\ m \right\rangle - I_+ I_z \left| J\ m \right\rangle + I_+ I_z \left| J\ m \right\rangle \\ &= [I_z , I_+] \left| J\ m \right\rangle + I_+ I_z \left| J\ m \right\rangle \\ &= I_+ \left| J\ m \right\rangle + I_+ I_z \left| J\ m \right\rangle \\ &= I_+ \left| J\ m \right\rangle + I_+ m \left| J\ m \right\rangle \\ &= (m+1)\, I_+ \left| J\ m \right\rangle \end{aligned} \tag{3}$$

This means that the result of applying I_+ to $|J\ m>$ is an eigenfunction of I_z with eigenvalue (m+1), so it must be some multiple of $|J\ m+1>$.

From these rules we can derive most of what we need to know. First, we derive the eigenvalue of I^2. Let this be some number c, so that $I^2 \left| J\ m \right\rangle = c \left| J\ m \right\rangle$. The key to the argument is that there should be some maximum value of m, which we call $m_{\max}$. For this function, we can not raise the z component any more. Therefore,

$$I_+ \left| J\ m_{\max} \right\rangle = 0 \tag{4}$$

If that is true, then

$$I_- I_+ \left| J\ m_{\max} \right\rangle = 0 \tag{5}$$

And you can show that

$$I_- I_+ = I^2 - I_z^2 - I_z \tag{6}$$

Therefore

$$\begin{aligned} I_- I_+ \left| J\ m_{\max} \right\rangle &= \left(I^2 - I_z^2 - I_z \right) \left| J\ m_{\max} \right\rangle \\ &= \left(c - m_{\max}^2 - m_{\max} \right) \left| J\ m_{\max} \right\rangle \\ &= 0 \end{aligned} \tag{7}$$

This is true if

$$c = m_{\max} \left(m_{\max} + 1 \right) \tag{8}$$

Similarly, applying I_- to some $m_{\min}$ gives

$$c = m_{\min} \left(m_{\min} - 1 \right) \tag{9}$$

The only acceptable solution is that $m_{\min} = -m_{\max}$.

All this math leads us to the following conclusions:

1. The z component runs in integer steps from -J to +J, where J is the total angular momentum quantum number.

2. The length of the angular momentum vector (the square root of the eigenvalue of I^2) is not J, but rather $\sqrt{J(J+1)}$, so that even when the z component is at a maximum, the vector does not lie along the z axis.

3. Since the steps are integral, then $2m_{\max}$ must be an integer, so J must be an integer or a half-integer.

4. Further calculations show that

$$\begin{aligned} I_+\left|J\ m\right\rangle &= \sqrt{(J-m)(J+m+1)}\quad\left|J\ (m+1)\right\rangle \\ I_-\left|J\ m\right\rangle &= \sqrt{(J+m)(J-m+1)}\quad\left|J\ (m-1)\right\rangle \end{aligned} \tag{10}$$