Problems 1

1. Consider a single particle in a ring. The position of the particle corresponds to an angle, θ, which varies from 0 to 2π. The states of this particle are functions of θ over the interval, $(0, 2\pi)$. In addition because the ring extends back on itself - i.e., going beyond $\theta = 2\pi$ corresponds to θ returning back to 0 - and the wavefunctions (states of the system) must be continuous, we must have

$$\psi(0) = \psi(2\pi).$$

This is called periodic boundary conditions. The Hamiltonian for this system takes the form,

$$H = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}.$$

There is one angular degree of freedom, θ. The Hamiltonian (there is only kinetic energy) in classical mechanics is

$$H = \frac{L^2}{2mR^2},$$

where L is the angular momentum and mR^2 is the moment of inertia of the particle about the center of the ring. The classical Hamiltonian becomes the quantum Hamiltonian operator when we insert the angular momentum operator,

$$\hat{L} = -i\hbar\frac{d}{d\theta}.$$

The states,

$$\psi_{c1}(\theta) = \frac{1}{\sqrt{\pi}}\cos(\theta)$$

and

$$\psi_{s2}(\theta) = \frac{1}{\sqrt{\pi}}\sin(2\theta)$$

are energy eigenstates - i.e., eigenfunctions of the Hamiltonian operator.

2. What are the energy eigenvalues associated with ψ_{c1} and ψ_{s2}?

3. Show that ψ_{c1} and ψ_{s2} are orthogonal - i.e.,

$$\begin{aligned}\langle\psi_{c1}|\psi_{s2}\rangle &= \int_0^{2\pi}\psi_{c1}^*(\theta)\psi_{s2}(\theta)\,d\theta \\ &= 0.\end{aligned}$$

4. What is the expectation value of angular momentum for the system in state, ψ_{s2}?

5. Show that ψ_{c1} does not have a well-defined value of angular momentum - i.e., show that $\psi_{c1}(\theta)$ is not an eigenfunction of the angular momentum operator, $\hat{L}$.

6. What is the expectation value of angular momentum for a system in state, ψ_{c1}?

7. Show that

$$\psi_{+1}(\theta) = \frac{1}{\sqrt{2\pi}} \exp(i\theta)$$

has a well-defined value of angular momentum. What is this value?

8. What is the probability that a system in state, ψ_{c1}, has angular momentum, $\hbar$?

9. Show that ψ_{+1} is also an energy eigenstate. What is the associated energy eigenvalue?