Tutorial

#1 a. $\langle \psi_0 | x \psi_0 \rangle$

$$= \int_{-\infty}^{\infty} \pi^{-1/2} \cdot \alpha^{-1} \cdot x \cdot e^{-\frac{x^2}{\alpha^2}} dx$$

$$= \pi^{-1/2} \cdot \alpha^{-1} \int_{-\infty}^{\infty} x \cdot e^{-\frac{x^2}{\alpha^2}} dx \quad (*)$$

$\because$ $f(x) = x \cdot e^{-\frac{x^2}{\alpha^2}}$ is odd function.

$\therefore$ $\int_{-\infty}^{\infty} f(x) dx = 0$

$\Rightarrow$ $(*) = \pi^{-1/2} \cdot \alpha^{-1} \cdot 0 = 0$ //

b. $\langle \psi_0 | x^2 \psi_0 \rangle$

$$= \int_{-\infty}^{\infty} \pi^{-1/2} \cdot \alpha^{-1} \cdot x^2 \cdot e^{-\frac{x^2}{\alpha^2}} dx$$

$$= \pi^{-1/2} \cdot \alpha^{-1} \cdot \int_{-\infty}^{\infty} x^2 \cdot e^{-\frac{x^2}{\alpha^2}} dx$$

(since $\int_{-\infty}^{\infty} x^2 \cdot e^{-\frac{x^2}{\alpha^2}} dx = \frac{\pi^{1/2}}{2} \alpha^3$)

$$= \cancel{\pi^{-1/2}} \cdot \alpha^{-1} \cdot \frac{1}{2} \cancel{\pi^{1/2}} \cdot \alpha^3$$

$$= \frac{\alpha^2}{2}$$ //

c. $\sigma_x = \sqrt{\frac{\alpha^2}{2} - 0^2} = \frac{\alpha}{\sqrt{2}}$ //

#2 a. $\langle \varphi_0 | \hat{p}_x \varphi_0 \rangle$

$$= \int_{-\infty}^{\infty} \pi^{-1/2} \cdot \alpha^{-1} \cdot e^{-\frac{x^2}{2\alpha^2}} \cdot \left(-i\hbar \frac{d}{dx} \cdot e^{-\frac{x^2}{2\alpha^2}}\right) dx$$

$$= \pi^{-1/2} \cdot \alpha^{-1} \cdot (-i\hbar) \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}} \cdot \left(-\frac{2x}{2\alpha^2}\right) \cdot e^{-\frac{x^2}{2\alpha^2}} dx$$

$$= \pi^{-1/2} \cdot \alpha^{-1} \cdot i\hbar \cdot \frac{1}{\alpha^2} \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}} x \cdot e^{-\frac{x^2}{2\alpha^2}} dx$$

$$= \pi^{-1/2} \cdot \alpha^{-3} \cdot i\hbar \int_{-\infty}^{\infty} x \cdot e^{-\frac{x^2}{\alpha^2}} dx$$

$\because$ $f(x) = x \cdot e^{-\frac{x^2}{\alpha^2}}$ is odd function

$\therefore$ $\int_{-\infty}^{\infty} f(x) dx = 0$

$\Rightarrow \langle \varphi_0 | \hat{p}_x \varphi_0 \rangle = 0$ //

b. $\langle \varphi_0 | \hat{p}_x^2 \varphi_0 \rangle$

$$= \int_{-\infty}^{\infty} \pi^{-1/2} \alpha^{-1} \cdot e^{-\frac{x^2}{2\alpha^2}} \left(-i\hbar \frac{d}{dx}\right)^2 e^{-\frac{x^2}{2\alpha^2}} dx$$

$$= \pi^{-1/2} \cdot \alpha^{-1} (-i\hbar)^2 \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}} \cdot \frac{d^2}{dx^2} e^{-\frac{x^2}{2\alpha^2}} dx \qquad (*)$$

where $\frac{d^2}{dx^2} e^{-\frac{x^2}{2\alpha^2}}$

$$= \frac{d}{dx}\left[e^{-\frac{x^2}{2\alpha^2}} \cdot \left(-\frac{\cancel{2}x}{\cancel{2}\alpha^2}\right)\right] \qquad \text{(product rule)}$$

$$= -\frac{1}{\alpha^2}\left[e^{-\frac{x^2}{2\alpha^2}} \left(-\frac{\cancel{2}x}{\cancel{2}\alpha^2}\right) \cdot x + e^{-\frac{x^2}{2\alpha^2}}\right]$$

$$= -\frac{1}{\alpha^2}\left[\left(-\frac{1}{\alpha^2}\right) x^2 \cdot e^{-\frac{x^2}{2\alpha^2}} + e^{-\frac{x^2}{2\alpha^2}}\right]$$

$$(*) = \pi^{-1/2} \cdot \alpha^{-1} \cdot (-\hbar^2) \cdot \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\alpha^2}} \cdot \left(-\frac{1}{\alpha^2}\right)\left[\left(-\frac{1}{\alpha^2}\right) x^2 \cdot e^{-\frac{x^2}{2\alpha^2}} + e^{-\frac{x^2}{2\alpha^2}}\right] dx$$

⋮ (see next page)

$\vdots$

③

$$= \pi^{-1/2} \cdot \alpha^{-1} \cdot (-\hbar^2) \left[\left(\frac{1}{\alpha^4}\right) \cdot \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{\alpha^2}} dx - \left(\frac{1}{\alpha^2}\right) \int_{-\infty}^{\infty} e^{-\frac{x^2}{\alpha^2}} dx \right]$$

$$= \cancel{\pi^{-1/2}} \cdot \alpha^{-1} (-\hbar^2) \cdot \left[\frac{1}{\alpha^4} \cdot \frac{\cancel{\pi^{1/2}}}{2} \alpha^3 - \frac{1}{\alpha^2} \cdot \cancel{\pi^{1/2}} \cdot \alpha \right].$$

$$= \frac{-\hbar^2}{\alpha} \cdot \left(\frac{1}{2\alpha} - \frac{1}{\alpha} \right)$$

$$= \frac{\hbar^2}{2\alpha^2} \quad //$$

c. $$\sigma_p = \sqrt{\left(\frac{\hbar^2}{2\alpha^2}\right) - 0^2}$$

$$= \frac{\hbar}{\sqrt{2}\alpha}.$$

#3 $$\sigma_x \sigma_p = \frac{\cancel{\alpha}}{\sqrt{2}} \cdot \frac{\hbar}{\sqrt{2}\cancel{\alpha}}$$

$$= \frac{\hbar}{2}$$

$\hookrightarrow$ satisfies the uncertainty principle.