Tutorial #4.

#1 $\quad [x, \frac{1}{2m}\hat{p}^2] \qquad \text{where } \hat{p} = -i\hbar \frac{d}{dx}$

$$\Rightarrow \quad = x \cdot \frac{\hat{p}^2}{2m}\psi(x) - \frac{\hat{p}^2}{2m} x \cdot \psi(x)$$

$$= x \cdot \frac{(-i\hbar)^2}{2m} \frac{d^2}{dx^2}\psi(x) - \frac{(-i\hbar)^2}{2m} \cdot \frac{d^2}{dx^2} x \cdot \psi(x)$$

$$= \frac{-\hbar^2}{2m} x \cdot \frac{d^2}{dx^2}\psi(x) - \frac{-\hbar^2}{2m} \frac{d}{dx}\left[\frac{d}{dx} x \cdot \psi(x)\right]$$

$$= \frac{-\hbar^2}{2m} x \cdot \frac{d^2}{dx^2}\psi(x) - \frac{-\hbar^2}{2m} \frac{d}{dx}\left[\psi(x) + x \frac{d\psi(x)}{dx}\right]$$

$$= \frac{-\hbar^2}{2m} x \cdot \frac{d^2\psi}{dx^2}(x) - \frac{-\hbar^2}{2m}\left[\frac{d}{dx}\psi(x) + \frac{d}{dx}\psi(x) + x \cdot \frac{d^2}{dx^2}\psi(x)\right]$$

$$= \cancel{\frac{-\hbar^2}{2m} \cdot x \cdot \frac{d^2}{dx^2}\psi(x)} - \frac{-\hbar^2}{\cancel{2m}} \cdot \cancel{2} \cdot \frac{d}{dx}\psi(x) - \cancel{\frac{-\hbar^2}{2m} \cdot x \cdot \frac{d^2}{dx^2}\psi(x)}$$

$$= \frac{\hbar^2}{m}\frac{d}{dx}\psi(x)$$

#2 $\psi_1 = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$

a. $\langle x^2 \rangle = \langle \psi_1 | x^2 \psi_1 \rangle$

$$= \int_0^L \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \cdot x^2 \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \cdot dx$$

$$= \int_0^L \cdot \frac{2}{L} \cdot x^2 \cdot \sin^2\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^L x^2 \sin^2\left(\frac{\pi x}{L}\right) dx \quad (*)$$

let $\frac{\pi x}{L} = u \quad x = \frac{L}{\pi} u \quad du = \frac{\pi}{L} dx \quad dx = \frac{L}{\pi} du$

$$(*) = \frac{2}{L} \int_0^\pi \left(\frac{L}{\pi} u\right)^2 \cdot \sin^2 u \cdot \left(\frac{L}{\pi} du\right)$$

$$= \frac{2}{\cancel{L}} \cdot \frac{L^2}{\pi^2} \cdot \frac{\cancel{L}}{\pi} \int_0^\pi u^2 \sin^2 u \, du \qquad \left(\int_0^\pi u^2 \sin^2(u) \, du = \frac{\pi}{2}\left(\pi^2 - \frac{1}{2}\right) \right)$$

$$= \frac{\cancel{2} L^2}{\pi^3} \cdot \left[\frac{\pi}{\cancel{2}} \cdot \left(\pi^2 - \frac{1}{2}\right) \right]$$

$$= \frac{L^2}{\pi^2}\left(\pi^2 - \frac{1}{2}\right)$$

b. $\langle \hat{p}^2 \rangle = \langle \psi_1 | \hat{p}^2 \psi_1 \rangle \qquad \hat{H} = \frac{\hat{p}^2}{2m} \Rightarrow \hat{p}^2 = 2m \cdot \hat{H}$

$$= \langle \psi_1 | 2m \hat{H} \psi_1 \rangle$$

$$= \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \cdot \cancel{2m} \cdot \left(-\frac{\hbar^2}{\cancel{2m}} \frac{d^2}{dx^2}\right) \cdot \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{-2\hbar^2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \cdot \frac{d^2}{dx^2} \sin\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{-2\hbar^2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \cdot \left(\frac{\pi}{L}\right) \cdot \frac{d}{dx} \cdot \cos\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{-2\hbar^2}{L} \int_0^L \sin\left(\frac{\pi x}{L}\right) \cdot \left(\frac{\pi}{L}\right) \cdot \left(-\frac{\pi}{L}\right) \cdot \sin\left(\frac{\pi x}{L}\right) dx$$

$$= \left(\frac{-2\hbar^2}{L}\right) \cdot \left(-\frac{\pi^2}{L^2}\right) \cdot \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{2\pi^2 \hbar^2}{L^3} \int_0^L \sin^2\left(\frac{\pi x}{L}\right) dx \quad (*) \qquad \text{let } \frac{\pi x}{L} = u.$$

$$(*) = \frac{2\pi^2\hbar^2}{L^3} \cdot \frac{L}{\pi} \int_0^\pi \sin^2 u \, du .$$

$$= \frac{2\pi\hbar^2}{L^2} \cdot \int_0^\pi \frac{(1-\cos 2u)}{2} \, du .$$

$$= \frac{\pi\hbar^2}{L^2} \cdot \int_0^\pi (1-\cos 2u) \, du$$

$$= \frac{\pi\hbar^2}{L^2} \cdot \left[u - \frac{1}{2}\sin 2u \right]_0^\pi$$

$$= \frac{\pi\hbar^2}{L^2} \cdot (\pi - 0) = \frac{\pi^2\hbar^2}{L^2} . //$$

Shortcut $\hat{p}^2 = 2m\hat{H}$

$\hat{H}\varphi_1 = E\varphi_1$ where $E = \dfrac{\hbar^2\pi^2}{2mL^2}$.

$\underline{\hat{p}^2\varphi_1 = 2m\hat{H}\varphi_1}$

$$\langle \varphi_1 | \hat{p}^2 \varphi_1 \rangle$$

$$= \cancel{2m} \cdot \frac{\hbar^2\pi^2}{\cancel{2m}L^2} \cancel{\langle \varphi_1 | \varphi_1 \rangle}^{1}$$

$$= \frac{\pi^2\hbar^2}{L^2} . //$$

C. Need to show if φ_1 satisfies $\sigma_x^2 \sigma_p^2 \geq \dfrac{\hbar^2}{4}$.

$\sigma_x^2 = \langle \varphi | (\Delta x)^2 \varphi \rangle$ where $\Delta x = x - \langle x \rangle$

$$\Rightarrow \sigma_x^2 = \langle \varphi | (x - \langle x \rangle)^2 \varphi \rangle$$

$$= \langle \varphi | x^2 \varphi \rangle - \langle \varphi | x \varphi \rangle^2$$

} Details see Pg. 39

$$= \frac{L^2}{\pi^2}\left(\pi^2 - \frac{1}{2}\right) - \left(\frac{L}{2}\right)^2$$

$$= \frac{(3\pi^2 - 2)L^2}{4\pi^2}$$

Similarly,

$$\begin{aligned}
\sigma_p^2 &= \langle \psi | (\Delta p)^2 \psi \rangle \\
&= \langle \psi | (p - \langle p \rangle)^2 \psi \rangle \\
&= \langle \psi | p^2 \psi \rangle - \langle \psi | p \psi \rangle^2 \\
&= \frac{\pi^2 \hbar^2}{L^2} - 0^2 = \frac{\pi^2 \hbar^2}{L^2}
\end{aligned}$$

$$\Rightarrow \sigma_x^2 \cdot \sigma_p^2 = \frac{(3\pi^2 - 2)\cancel{L^2}}{4\cancel{\pi^2}} \cdot \frac{\cancel{\pi^2}\hbar^2}{\cancel{L^2}}$$

$$= \frac{3\pi^2 - 2}{4} \hbar^2 > \frac{\hbar^2}{4}$$

$\therefore$ ψ_1 (ground state) satisfies the uncertainty principle.