Tutorial 2 Solutions – 2015

1.

a. $\int_0^\infty e^{-x}\,dx = \int_0^\infty (-1)e^{-x}d(-x) = -\int_0^\infty e^{-x}d(-x) = -e^{-x}\Big|_0^\infty = -e^{-\infty} - (-e^0) = 1$

b. $\int_0^\infty e^{-2x}\,dx = \int_0^\infty \left(-\frac{1}{2}\right)e^{-2x}d(-2x) = -\frac{1}{2}\int_0^\infty e^{-2x}d(-2x) = -\frac{1}{2}e^{-2x}\Big|_0^\infty = \frac{1}{2}$

c. $\int_0^\infty\int_0^\infty e^{-x-y}\,dx\,dy = \int_0^\infty\int_0^\infty e^{-x}e^{-y}\,dx\,dy = \int_0^\infty e^{-y}\left(\int_0^\infty e^{-x}\,dx\right)dy = \int_0^\infty e^{-y}\,dy = 1$

d. $\int_0^\infty\int_0^\infty xye^{-x-y}\,dx\,dy = \int_0^\infty\int_0^\infty xye^{-x}e^{-y}\,dx\,dy = \int_0^\infty ye^{-y}\left(\int_0^\infty xe^{-x}\,dx\right)dy$

(Integrate by parts)

$\int_0^\infty xe^{-x}\,dx = \int_0^\infty (-x)e^{-x}d(-x)$ is in $\int u\,dv$ form, where $dv = e^{-x}d(-x)$ so $v = e^{-x}$; $u = -x$

Therefore, $\int_0^\infty xe^{-x}\,dx = \int_0^\infty (-x)e^{-x}d(-x) = (-x)e^{-x}\Big|_0^\infty - \int_0^\infty e^{-x}d(-x) = (-x-1)e^{-x}\Big|_0^\infty = 1$

$\therefore \int_0^\infty\int_0^\infty xye^{-x-y}\,dx\,dy = \int_0^\infty ye^{-y}\left(\int_0^\infty xe^{-x}\,dx\right)dy = 1$

2. Based on the given relationships and equations, ψ_1 and ψ_2 are linearly independent and orthogonal to each other.

a.

$$\left|\langle\psi_1|\psi\rangle\right|^2 = \left|\left\langle\psi_1\middle|\left(\frac{1}{\sqrt{2}}\psi_1 + \frac{i}{\sqrt{2}}\psi_2\right)\right\rangle\right|^2 = \left(\frac{1}{\sqrt{2}}+0\right)^*\left(\frac{1}{\sqrt{2}}+0\right) = \frac{1}{2}$$

b.

$$\left|\langle\psi_2|\psi\rangle\right|^2 = \left|\left\langle\psi_2\middle|\left(\frac{1}{\sqrt{2}}\psi_1 + \frac{i}{\sqrt{2}}\psi_2\right)\right\rangle\right|^2 = \left(0+\frac{i}{\sqrt{2}}\right)^*\left(0+\frac{i}{\sqrt{2}}\right) = -\left(\frac{i}{\sqrt{2}}\right)^2 = \frac{1}{2}$$

c.

$$\left|\langle\psi|\psi\rangle\right|^2 = \left|\left\langle\left(\frac{1}{\sqrt{2}}\psi_1 + \frac{i}{\sqrt{2}}\psi_2\right)\middle|\left(\frac{1}{\sqrt{2}}\psi_1 + \frac{i}{\sqrt{2}}\psi_2\right)\right\rangle\right|^2 = \left|\frac{1}{2}\langle\psi_1|\psi_1\rangle + \frac{1}{2}\langle\psi_2|\psi_2\rangle\right|^2 = 1^2 = 1$$

d. $\hat{A}$ is Hermitian operator. $\hat{A}|\psi_1\rangle = a_1|\psi_1\rangle$; $\hat{A}|\psi_2\rangle = a_2|\psi_2\rangle$

$$\rho(a_2) = \left|\langle\psi_2|\psi\rangle\right|^2 = \frac{1}{2}$$