Solutions 2

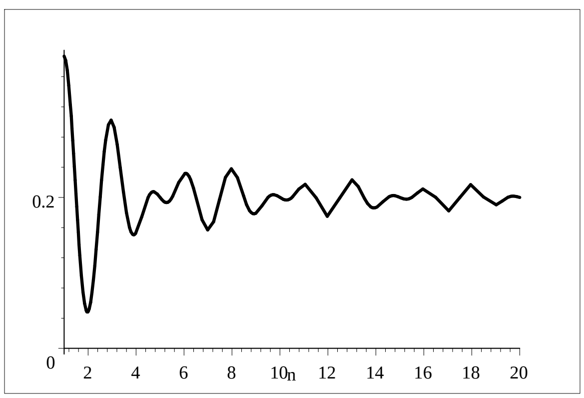
1. Consider an electron in a 1D box, in energy eigenstate $\psi_n(x)$.

a. Determine an expression for the probability that the electron is found to be within the interval, $(\frac{2}{5}L, \frac{3}{5}L)$?

– – – – – – – –

$$\begin{aligned}
\text{probability} &= \int_{2/5}^{3/5} |\psi_n(x)|^2 dx \\
&= \frac{2}{L}\int_{2L/5}^{3L/5} \sin^2\left(\frac{n\pi x}{L}\right)dx \\
&= \frac{2}{n\pi}\int_{2n\pi/5}^{3n\pi/5} \sin^2(u)du \\
&= \frac{1}{n\pi}\int_{2n\pi/5}^{3n\pi/5} (1-\cos(2u))du \\
&= \frac{1}{n\pi}\left[u - \frac{1}{2}\sin(2u)\right]_{2n\pi/5}^{3n\pi/5} \\
&= \frac{1}{5} - \frac{1}{2n\pi}\left[\sin\left(\frac{6n\pi}{5}\right) - \sin\left(\frac{4n\pi}{5}\right)\right]
\end{aligned}$$

Here is a graph of the above probability - as a function of n.



The probability is largest for $n = 1$, smallest for $n = 2$, and next largest for $n = 3$. Beyond that value, the probability oscillates about 0.2, the value one would expect if the particle were evenly likely to be anywhere within the box.

– – – – – – – –

b. Evaluate your expression for $n = 1$ and 2.

– – – – – – – –

$$\text{probability}_{n=1} = \frac{1}{5} - \frac{1}{2\pi}\left[\sin\left(\frac{6\pi}{5}\right) - \sin\left(\frac{4\pi}{5}\right)\right] = 0.3871$$

$$\text{probability}_{n=2} = \frac{1}{5} - \frac{1}{4\pi}\left[\sin\left(\frac{12\pi}{5}\right) - \sin\left(\frac{8\pi}{5}\right)\right] = 0.04863$$

– – – – – – – –

2. Suppose that the particle in a 1D box is in the state,

$$\psi(x) = Ax(L-x).$$

a. Determine real positive A such that $\psi(x)$ is normalized.

– – – – – – – –

$$\begin{aligned} 1 &= \int_0^L |\psi(x)|^2 dx \\ &= A^2 \int_0^L x^2(L-x)^2 dx \\ &= A^2 \int_0^L (L^2x^2 - 2Lx^3 + x^4) dx \\ &= A^2 \left[\frac{1}{3}L^2x^3 - \frac{1}{2}Lx^4 + \frac{1}{5}x^5 \right]_0^L \\ &= A^2L^5 \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{5} \right] = \frac{A^2L^5}{30} \end{aligned}$$

Therefore,

$$A = \sqrt{30}\, L^{-5/2}.$$

– – – – – – – –

b. Expand $\psi(x)$ as a sum over energy eigenstates - i.e., find the coefficients in the expansion.

– – – – – – – –

$$\psi(x) = \sum_{n=1}^{\infty} c_n \psi_n(x),$$

where

$$\begin{aligned} c_n &= \langle \psi_n | \psi \rangle \\ &= \int_0^L \psi_n^*(x)\psi(x) dx \\ &= \int_0^L \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L} \right) \sqrt{30}\, L^{-5/2} x(L-x) dx \\ &= \sqrt{60}\, L^{-3} \int_0^L \sin\left(\frac{n\pi x}{L} \right) x(L-x) dx \\ &= \sqrt{60}\, L^{-3} \left[L \int_0^L \sin\left(\frac{n\pi x}{L} \right) x dx - \int_0^L \sin\left(\frac{n\pi x}{L} \right) x^2 dx \right] \end{aligned}$$

Since

$$
\begin{aligned}
\int_0^L \sin\left(\frac{n\pi x}{L}\right) x\,dx &= \left(\frac{L}{n\pi}\right)^2 \int_0^{n\pi} \sin(u) u\,du \\
&= -\left(\frac{L}{n\pi}\right)^2 \int_{u=0}^{u=n\pi} u\,d\cos(u) \\
&= -\left(\frac{L}{n\pi}\right)^2 \left\{ [u\cos(u)]_0^{n\pi} - \int_0^{n\pi} \cos(u)\,du \right\} \\
&= -\left(\frac{L}{n\pi}\right)^2 \{ n\pi \cos(n\pi) - [\sin(u)]_0^{n\pi} \} \\
&= -\left(\frac{L}{n\pi}\right)^2 \{ n\pi(-1)^n - 0 \} \\
&= \frac{L^2(-1)^{n+1}}{n\pi}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^L \sin\left(\frac{n\pi x}{L}\right) x^2\,dx &= \left(\frac{L}{n\pi}\right)^3 \int_0^{n\pi} \sin(u) u^2\,du \\
&= -\left(\frac{L}{n\pi}\right)^3 \int_{u=0}^{u=n\pi} u^2\,d\cos(u) \\
&= -\left(\frac{L}{n\pi}\right)^3 \left\{ [u^2\cos(u)]_0^{n\pi} - \int_{u=0}^{u=n\pi} \cos(u)\,du^2 \right\} \\
&= -\left(\frac{L}{n\pi}\right)^3 \left\{ n^2\pi^2 \cos(n\pi) - 2\int_0^{n\pi} \cos(u) u\,du \right\} \\
&= \frac{L^3(-1)^{n+1}}{n\pi} + 2\left(\frac{L}{n\pi}\right)^3 \int_{u=0}^{u=n\pi} u\,d\sin(u) \\
&= \frac{L^3(-1)^{n+1}}{n\pi} + 2\left(\frac{L}{n\pi}\right)^3 \left\{ [u\sin(u)]_0^{n\pi} - \int_0^{n\pi} \sin(u)\,du \right\} \\
&= \frac{L^3(-1)^{n+1}}{n\pi} + 2\left(\frac{L}{n\pi}\right)^3 \{ 0 + [\cos(u)]_0^{n\pi} \} \\
&= \frac{L^3(-1)^{n+1}}{n\pi} + 2\left(\frac{L}{n\pi}\right)^3 (-1)^n,
\end{aligned}
$$

$$
\begin{aligned}
c_n &= \sqrt{60}\,L^{-3} \left[L\frac{L^2(-1)^{n+1}}{n\pi} - \frac{L^3(-1)^{n+1}}{n\pi} + 2\left(\frac{L}{n\pi}\right)^3 (-1)^{n+1} \right] \\
&= 2\sqrt{60}\,\frac{(-1)^{n+1}}{(n\pi)^4}
\end{aligned}
$$

– – – – – – – –

c. What is the probability that the energy of the particle in state $\psi(x)$ is measured to be $E_3 = 9\hbar^2\pi^2/(2m)$?

– – – – – – – –

The probability of observing energy, E_3, is given in terms of the inner product of the associated energy eigenstate, ψ_3, with ψ,

$$
\begin{aligned}
\text{prob}_{n=3} &= |\langle\psi_3|\psi\rangle|^2 \\
&= |c_3|^2 \\
&= \left| 2\sqrt{60}\,\frac{(-1)^{3+1}}{(3\pi)^4} \right|^2 \\
&= \frac{4\times 60}{3^8\pi^8} = 3.9\times 10^{-6}
\end{aligned}
$$

– – – – – – – –