Solutions 1

Consider a single particle in a ring. The position of the particle corresponds to an angle, θ, which varies from 0 to 2π. The states of this particle are functions of θ over the interval, $(0, 2\pi)$. In addition because the ring extends back on itself - i.e., going beyond $\theta = 2\pi$ corresponds to θ returning back to 0 - and the wavefunctions (states of the system) must be continuous, we must have

$$\psi(0) = \psi(2\pi).$$

This is called periodic boundary conditions. The Hamiltonian for this system takes the form,

$$\hat{H} = -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}.$$

There is one angular degree of freedom, θ. The Hamiltonian (there is only kinetic energy) in classical mechanics is

$$H = \frac{L^2}{2mR^2},$$

where L is the angular momentum and mR^2 is the moment of inertia of the particle about the center of the ring. The classical Hamiltonian becomes the quantum Hamiltonian operator when we insert the angular momentum operator,

$$\hat{L} = -i\hbar\frac{d}{d\theta}.$$

The states,

$$\psi_{c1}(\theta) = \frac{1}{\sqrt{\pi}}\cos(\theta)$$

and

$$\psi_{s2}(\theta) = \frac{1}{\sqrt{\pi}}\sin(2\theta)$$

are energy eigenstates - i.e., eigenfunctions of the Hamiltonian operator.

1. What are the energy eigenvalues associated with ψ_{c1} and ψ_{s2}?

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We could evaluate the expectation value of the Hamiltonian for these states. It will give the energy eigenvalue for each state. (Note that this means doing an integral. However, it is no trouble if we recognize it is just a constant - the eigenvalue - $\times$ the normalization integral which equals 1.) Alternatively, we can simply apply the Hamiltonian operator to both states and identify the result as a constant - the eigenvalue - $\times$ the orginal function. This gives the energy eigenvalues, and verifies that the states are indeed eigenfunctions of $\hat{H}$. Specifically, since

$$
\begin{aligned}
\left(\hat{H}\psi_{c1}\right)(\theta) &= -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}\psi_{c1}(\theta) \\
&= -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}\frac{1}{\sqrt{\pi}}\cos(\theta) \\
&= -\frac{\hbar^2}{2mR^2}\frac{1}{\sqrt{\pi}}(-\cos(\theta)) \\
&= \frac{\hbar^2}{2mR^2}\psi_{c1}(\theta) \\
&= E_1\psi_{c1}(\theta),
\end{aligned}
$$

the energy eigenvalue associated with ψ_{c1} is

$$
E_1 = \frac{\hbar^2}{2mR^2}.
$$

Similarly, since

$$
\begin{aligned}
\left(\hat{H}\psi_{s2}\right)(\theta) &= -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}\psi_{s2}(\theta) \\
&= -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}\frac{1}{\sqrt{\pi}}\sin(2\theta) \\
&= -\frac{\hbar^2}{2mR^2}\frac{1}{\sqrt{\pi}}(-4\sin(\theta)) \\
&= \frac{2\hbar^2}{mR^2}\psi_{s2}(\theta) \\
&= E_2\psi_{s2}(\theta),
\end{aligned}
$$

the energy eigenvalue associated with ψ_{s2} is

$$
E_2 = \frac{2\hbar^2}{mR^2}.
$$

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2. Show that ψ_{c1} and ψ_{s2} are orthogonal - i.e.,

$$
\begin{aligned}
\langle\psi_{c1}|\psi_{s2}\rangle &= \int_0^{2\pi}\psi_{c1}^*(\theta)\psi_{s2}(\theta)\,d\theta \\
&= 0.
\end{aligned}
$$

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$$
\begin{aligned}
\langle \psi_{c1} | \psi_{s2} \rangle &= \int_0^{2\pi} \psi_{c1}^*(\theta) \psi_{s2}(\theta)\, d\theta \\
&= \int_0^{2\pi} \frac{1}{\sqrt{\pi}} \cos(\theta) \frac{1}{\sqrt{\pi}} \sin(2\theta)\, d\theta \\
&= \frac{1}{\pi} \int_0^{2\pi} \sin(2\theta)\cos(\theta)\, d\theta \\
&= \frac{1}{\pi} \int_0^{2\pi} [\sin(2\theta + \theta) - \sin(2\theta - \theta)]\, d\theta \qquad \text{sum and difference formulas for sine} \\
&= \frac{1}{\pi} \int_0^{2\pi} [\sin(3\theta) - \sin(\theta)]\, d\theta \\
&= \frac{1}{\pi} \left\{ \left[-\frac{1}{3}\cos(3\theta) \right]_0^{2\pi} - [-\cos(\theta)]_0^{2\pi} \right\} \\
&= \frac{1}{\pi} \left\{ -\frac{1}{3}[\cos(6\pi) - \cos(0)] + [\cos(2\pi) - \cos(0)] \right\} \\
&= \frac{1}{\pi} \left\{ -\frac{1}{3}[1-1] + [1-1] \right\} = 0
\end{aligned}
$$

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3. What is the expectation value of angular momentum for the system in state, ψ_{s2}?

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\begin{aligned}
\langle \psi_{s2} | \hat{L} \psi_{s2} \rangle &= \int_0^{2\pi} \psi_{s2}^*(\theta) \hat{L} \psi_{s2}(\theta)\, d\theta \\
&= \int_0^{2\pi} \frac{1}{\sqrt{\pi}} \sin(2\theta) \left(-i\hbar \frac{d}{d\theta} \right) \frac{1}{\sqrt{\pi}} \sin(2\theta)\, d\theta \\
&= \frac{-i\hbar}{\pi} \int_0^{2\pi} 2\sin(2\theta)\cos(2\theta)\, d\theta \\
&= \frac{-i\hbar}{\pi} \int_0^{2\pi} \sin(4\theta)\, d\theta \\
&= \frac{-i\hbar}{\pi} \left[-\frac{1}{4}\cos(4\theta) \right]_0^{2\pi} \\
&= \frac{i\hbar}{4\pi} [\cos(8\pi) - \cos(0)] \\
&= \frac{i\hbar}{4\pi} [1-1] = 0.
\end{aligned}
$$

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4. Show that ψ_{c1} does not have a well-defined value of angular momentum - i.e., show that $\psi_{c1}(\theta)$ is not an eigenfunction of the angular momentum operator, $\hat{L}$.

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Apply $\hat{L}$ to $\psi_{c1}(\theta)$;

$$\begin{aligned}\hat{L}\psi_{c1}(\theta) &= -i\hbar\frac{d}{d\theta}\frac{1}{\sqrt{\pi}}\cos(\theta)\\ &= i\hbar\frac{1}{\sqrt{\pi}}\sin(\theta).\end{aligned}$$

$\sin(\theta)$ is not a multiple of $\cos(\theta)$. Therefore, ψ_{c1} is not an eigenfunction of $\hat{L}$.

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5. What is the expectation value of angular momentum for a system in state, ψ_{c1}?

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$$\begin{aligned}\langle\psi_{c1}|\hat{L}\psi_{c1}\rangle &= \int_0^{2\pi}\psi_{c1}^*(\theta)\hat{L}\psi_{c1}(\theta)\,d\theta\\ &= \int_0^{2\pi}\frac{1}{\sqrt{\pi}}\cos(\theta)\left(-i\hbar\frac{d}{d\theta}\right)\frac{1}{\sqrt{\pi}}\cos(\theta)\,d\theta\\ &= \frac{i\hbar}{\pi}\int_0^{2\pi}\cos(\theta)\sin(\theta)\,d\theta\\ &= \frac{i\hbar}{\pi}\int_0^{2\pi}\sin(2\theta)\,d\theta\\ &= \frac{i\hbar}{\pi}\left[-\frac{1}{2}\cos(2\theta)\right]_0^{2\pi}\\ &= \frac{-i\hbar}{2\pi}[\cos(4\pi)-\cos(0)]\\ &= \frac{-i\hbar}{2\pi}[1-1] = 0.\end{aligned}$$

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6. Show that

$$\psi_{+1}(\theta) = \frac{1}{\sqrt{2\pi}}\exp(i\theta)$$

has a well-defined value of angular momentum. What is this value?

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To have a well-defined value of angular momentum, $\psi_{+1}(\theta)$ must be an eigenfunction of $\hat{L}$. Check:

$$\begin{aligned}\hat{L}\psi_{+1}(\theta) &= -i\hbar\frac{d}{d\theta}\frac{1}{\sqrt{2\pi}}\exp(i\theta)\\ &= \frac{-i\hbar}{\sqrt{2\pi}}i\exp(i\theta)\\ &= \hbar\frac{1}{\sqrt{2\pi}}\exp(i\theta)\\ &= \hbar\psi_{+1}(\theta).\end{aligned}$$

We see that $\psi_{+1}(\theta)$ is an eigenfunction of $\hat{L}$. The associated eigenvalue is $\hbar$.

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7. What is the probability that a system in state, ψ_{c1}, has angular momentum, $\hbar$?

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The eigenfunction of $\hat{L}$ associated with eigenvalue, $\hbar$, is $\psi_{+1}(\theta)$. The probability that angular momentum has this value is given by

$$\rho_{+1} = |\langle\psi_{+1}|\psi_{c1}\rangle|^2 = \left|\int_0^{2\pi} \psi_{+1}^*(\theta)\psi_{c1}(\theta)\,d\theta\right|^2,$$

where

$$\begin{aligned}
\int_0^{2\pi} \psi_{+1}^*(\theta)\psi_{c1}(\theta)\,d\theta &= \int_0^{2\pi} \frac{1}{\sqrt{2\pi}}\exp(-i\theta)\frac{1}{\sqrt{\pi}}\cos(\theta)\,d\theta \\
&= \frac{1}{\sqrt{2}\,\pi}\int_0^{2\pi}\exp(-i\theta)\frac{\exp(i\theta)+\exp(-i\theta)}{2}\,d\theta \\
&= \frac{1}{2\sqrt{2}\,\pi}\int_0^{2\pi}[1+\exp(-2i\theta)]\,d\theta \qquad \text{sum and difference formulas for sine} \\
&= \frac{1}{2\sqrt{2}\,\pi}\left\{[\theta]_0^{2\pi} + \left[\frac{1}{-2i}\exp(-2i\theta)\right]_0^{2\pi}\right\} \\
&= \frac{1}{2\sqrt{2}\,\pi}\left\{2\pi + \frac{i}{2}[\exp(-4i\pi)-\exp(0)]\right\} \\
&= \frac{1}{2\sqrt{2}\,\pi}\left\{2\pi + \frac{i}{2}[1-1]\right\} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

Therefore,

$$\rho_{+1} = |\langle\psi_{+1}|\psi_{c1}\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$

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8. Show that ψ_{+1} is also an energy eigenstate. What is the associated energy eigenvalue?

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\begin{aligned}
\left(\hat{H}\psi_{+1}\right)(\theta) &= -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}\psi_{+1}(\theta) \\
&= -\frac{\hbar^2}{2mR^2}\frac{d^2}{d\theta^2}\frac{1}{\sqrt{2\pi}}\exp(i\theta) \\
&= -\frac{\hbar^2}{2mR^2}\frac{1}{\sqrt{2\pi}}\left(i^2\exp(i\theta)\right) \\
&= \frac{\hbar^2}{2mR^2}\psi_{+1}(\theta) \\
&= E_1\psi_{+1}(\theta)
\end{aligned}
$$

So, $\psi_{+1}(\theta)$ is an energy eigenstate associated with energy eigenvalue,

$$
E_1 = \frac{\hbar^2}{2mR^2}.
$$

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