ECON 330 Computational Assignment 1

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Abstract

Run `MAIN.m` to get results and figures for all questions. There are following functions in the folder: `VFI.m` is for value function iteration. `uc.m` gives us utility and consumption plan given (k, k'). `kss_theory.m` and `kss_approx.m` produce theoretical and approximated steady state capital level, respectively.

(1) (a) The maximum sustainable amount of capital satisfies

$$z\overline{k}^{\alpha} + (1-\delta)\overline{k} = \overline{k} \tag{1}$$

which gives us $\overline{k} = (z/\delta)^{\frac{1}{1-\alpha}}$. Hence the relevant space is $[0, \overline{k}]$.

(b) The state variable is k, while the control variable is k'. The Bellman equation is

$$V(k) = \max_{k' \in \Gamma(k)} \log(zk^{\alpha} + (1-\delta)k - k') + \beta V(k') \tag{2}$$

where $\Gamma(k) = [0, zk^{\alpha} + (1-\delta)k]$.

(c) FOC is

$$\frac{1}{zk^{\alpha} + (1-\delta)k - k'} = \beta V'(k') \tag{3}$$

and envelop condition is

$$V'(k) = \frac{z\alpha k^{\alpha-1} + (1-\delta)}{zk^{\alpha} + (1-\delta)k - k'} \tag{4}$$

which give us the Euler equation

$$\frac{1}{zk^{\alpha} + (1-\delta)k - g(k)} = \beta \frac{z\alpha g(k)^{\alpha-1} + (1-\delta)}{zg(k)^{\alpha} + (1-\delta)g(k) - g(g(k))} \tag{5}$$

(d) The steady state captial solves

$$1 = \beta[z\alpha {k_s}^{\alpha-1} + 1 - \delta] \tag{6}$$

which gives us

$$k_s = \left[\frac{z\alpha}{\frac{1}{\beta}-(1-\delta)}\right]^{\frac{1}{1-\alpha}}, \quad c_s = z\left[\frac{z\alpha}{\frac{1}{\beta}-(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}} - \delta\left[\frac{z\alpha}{\frac{1}{\beta}-(1-\delta)}\right]^{\frac{1}{1-\alpha}} \tag{7}$$

(e) Let $C(X)$ be the space of bounded, continuous functions $V: X \to R$. Define the Bellman operator T : $C(X) \to C(X)$,

$$TV(x) = \max_{y\in\Gamma(x)} u(x,y) + \beta V(y) \tag{8}$$

(f) Suppose $f(x) < g(x)$ for all x.

i. Monotonicity

$$\begin{aligned} Tf(x) &= \max_{y\in\Gamma(x)} u(x,y) + \beta f(y) &(9)\\ &= u(x,y_f^*) + \beta f(y_f^*) &(10)\\ &\leq u(x,y_f^*) + \beta g(y_f^*) &(11)\\ &\leq \max_{y\in\Gamma(x)} u(x,y) + \beta g(y) = Tg(x) &(12)\end{aligned}$$

The first $=$ in the first and second line come by definition, the $\leq$ on the third line is by assumption and the $\leq$ on the last line is by definition.

ii. Discounting

$$[T(f+a)](x) = \max_{y\in\Gamma(x)} u(x,y) + \beta[f(y)+a] = \max_{y\in\Gamma(x)} u(x,y) + \beta f(y) + \beta a \leq Tf(x) + \beta a \tag{13}$$

Since $\beta \in (0,1)$, condition for discounting holds.

In sum, T is a contraction with modulus β.

(2) The computation time for value function iteration is in Table 1: a better guess or a lower β significantly shortens the time of iteration.

Table 1: Computation Time for Value Function Iteration

$V_0(k) = 0$ and $\beta = 0.99$	$V_0(k) = 0$ and $\beta = 0.1$	$V_0(k) = \frac{\log(zk^\alpha - \delta k)}{1-\beta}$ and $\beta = 0.99$
0.395498 second	0.003522 second	0.036066 second

(a) (i) See Figure 1 for value function iteration.

(ii) See Figure 2 for policy rule $g(k)$

(iii) For $z = 1$, the steady state capital in theory is 4.187 and it is 4.1828 in numerical approximation. See table 2

(b) See Figure 3. The value function converges quickly even after the first round of iteration. For each contraction, lower weight β is attached to the guessed function.

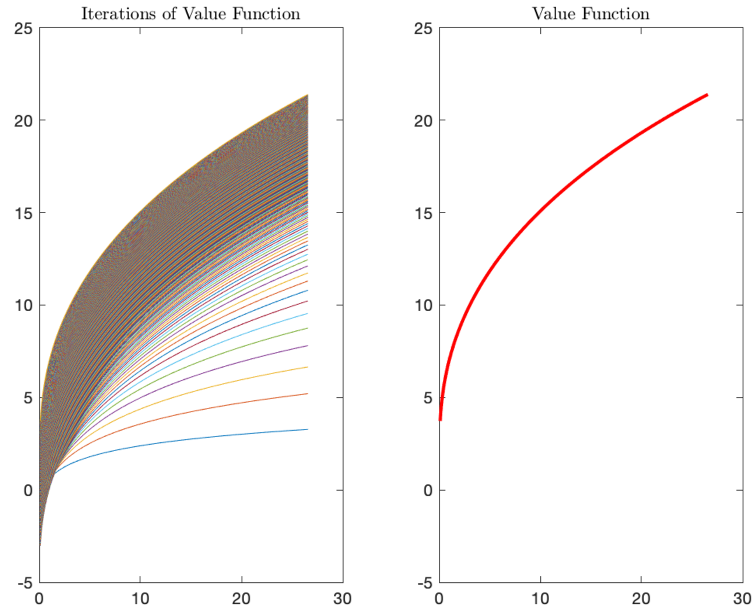

Figure 1: Value Function Iteration Using $V(k) = 0$

(c) A better guess converges faster, which is displayed in Figure 4.

(3) (a) Policy rules for $z = 1$ and $z = 2$ are in Figure 5. The intersection point of 45 degree line and two policy functions give us steady state values, which are listed in Table 2

Table 2: Steady State Capital for $z = 1$ and $z = 2$

	theory	approximation
$z = 1$	4.187	4.1828
$z = 2$	11.2705	11.2349

(b) The impulse response function in Figure 6 shows the trajectory from $z = 1$ steady state to $z = 2$ steady state. The change of capital slows down as it approaches new steady state.

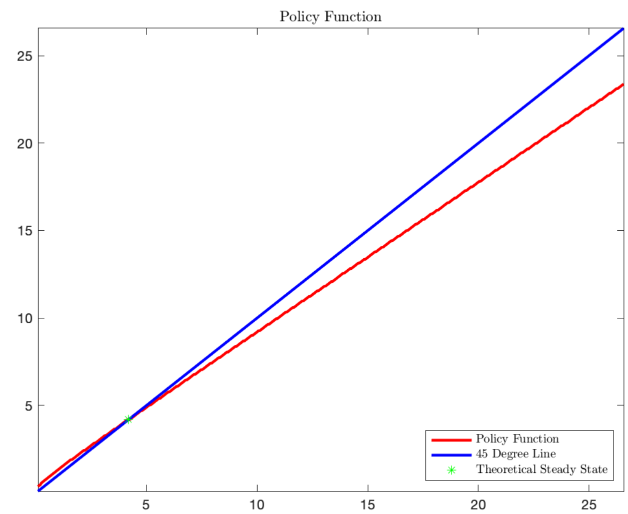

Figure 2: Policy Rule $g(k)$

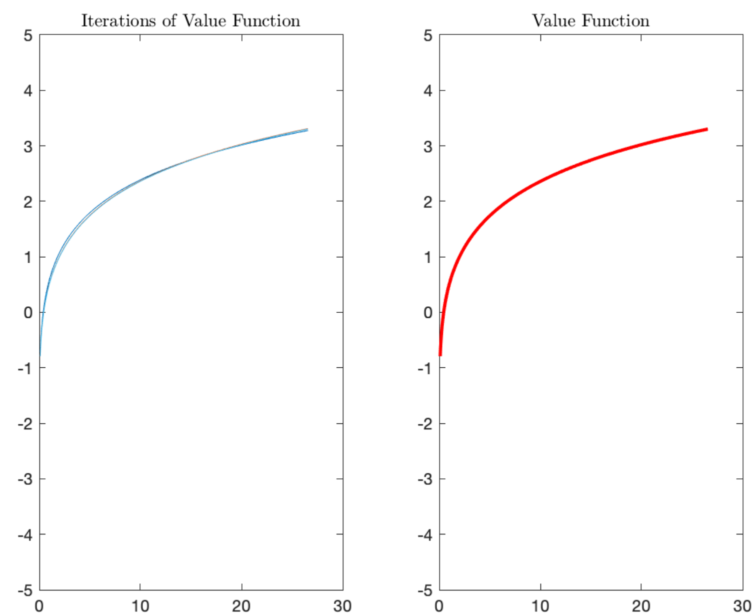

Figure 3: Value Function Iteration Using $V(k) = 0$ and $\beta = 0.1$

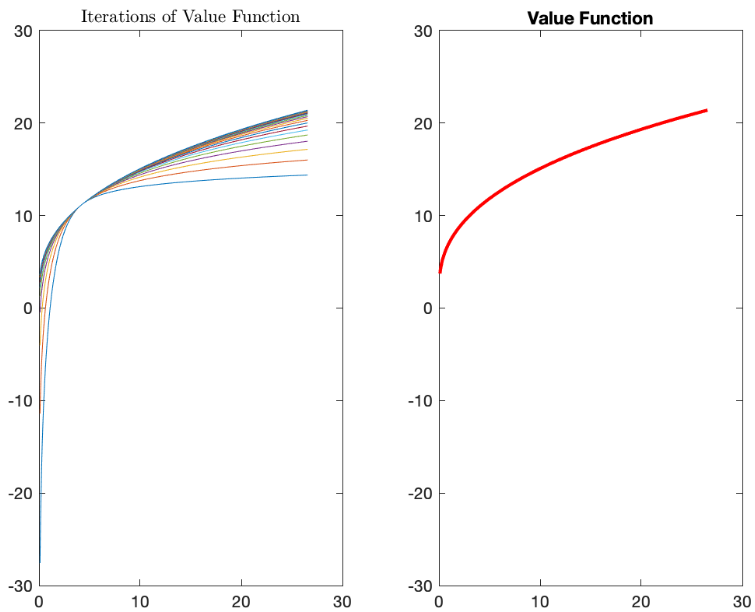

Figure 4: Value Function Iteration Using $V(k) = \frac{\log(zk^{\alpha} - \delta k)}{1-\beta}$

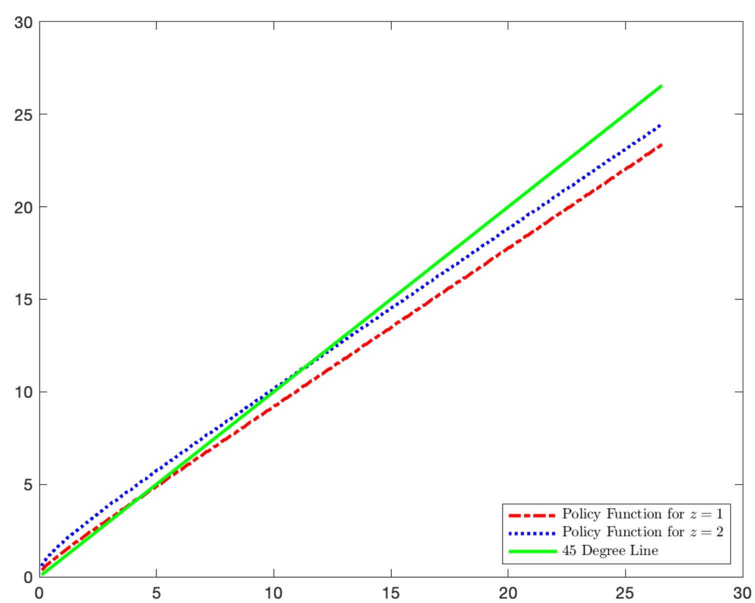

Figure 5: Policy Rule for $z = 1$ and $z = 2$

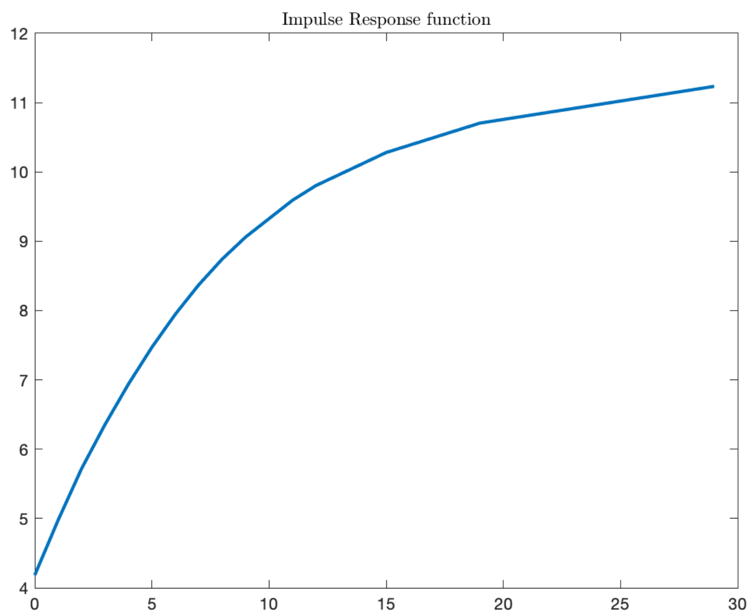

Figure 6: Impulse Response Function