Meeting Record

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1 Meeting on Sept. 2nd

1. Prof: What do $\mathbb{P}^n$'s look like, for small n's?
   Me: (I drew the standard planar representation of $\mathbb{R}IP^2$ that identifies the boundary of a square using antipodal maps.) $\mathbb{P}^1$ is topologically the same as $\mathbb{S}^1$.

2. Prof: Is $\mathbb{P}^2$ orientable?
   Me: No. Ah, here is a quick proof. If $\mathbb{P}^2$ is orientable, then all open submanifolds of it are, but the Mobius strip is an open submanifold of $\mathbb{P}^2$.

3. Prof: How about $\mathbb{P}^3$? Is it orientable? Me: My impression is that it is not. (Then I spent a while trying to come up with a quick proof)

4. Me: So I am going to cheat and use the fact that a manifold M is orientable if and only if $w_1(M) = 0$.

5. Prof: All line bundle over paracompact spaces are isomorphic to their duals.
   Me: Emm, I can check that myself later.
   Prof: The space of inner products on $\mathbb{R}$ has a canonical orientation. Well, what is the dimension of the space of inner products of $\mathbb{R}^n$?
   Me: They correspond to $n \times n$ matrices, so I guess it is n^2?
   Prof: But they have to be symmetric.
   Me: Right. So the answer is $n(n+1)/2$.
   Prof: Then what is the dimension (of the inner product space) when $n = 1$? Me: 1.

6. Prof: We can give a natural orientation on the inner product space by choosing the positive ones, i.e. those that send (v, v) to a positive number, to be a basis. You can then use partition of unity to construct a nowhere vanishing section.

7. Prof: What are other common manifolds of dimension 3 do you know?
   Me: Not really.
   Prof: How about the space $O(n) = \{M \in \mathbb{R}^{n \times n} : M^T M = 1\}$? How many equations are used to describe the condition?
   Me: 9, but (trying to count how many equations are repeated)...
   Prof: $M^T M$ is symmetric, so...
   Me: Ah, so there are actually only 6 equations. (thinking for some seconds) The other equations are algebraically independent, since they each introduce a new variable. Therefore we cannot further reduce the number of equations.

8. Prof: Yes, but it may happen that the intersection is nasty somewhere. How do you want to show that it is smooth?
   Me: Write down the equations using coordinates and then compute the Jacobian. Show that the Jacobian has appropriate rank.
   Prof: Right, what is the theorem that you are using?
   Me: If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a smooth map and $p \in \mathbb{R}^n$ is a regular value, then $f^{-1}(p)$ is a smooth submanifold of $\mathbb{R}^m$.
   Prof: Right, what is the name of the theorem?
   Me: Regular value theorem? Or something like that?
   Prof: It's the implicit function theorem.

9. Me: My intuition is that $O(3)$ is homogeneous, so it suffices to check one point, for which we can simply use I.

10. Prof: That seems to be a lot of computations. Could you show that the derivative is surjective directly? So how would you compute the derivative?
    Me: The derivative of $M \to M^T T$ is
    $$\lim_{\|M\| \to 0} \frac{(I + M)^T (I + M) - I}{\|M\|}$$
    Prof: For a function $f : \mathbb{R}^n \to \mathbb{R}^m$, you should be thinking of this formula:
    $$f'(x)(v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}$$
    Me: But that computes the image of one v at a time. I guess if you let v run over a basis, you can reconstruct the derivative $f'(x)$.

11. Me: So using this formula, the answer is

$$\lim_{t\to 0}\frac{(I+tM)^T(I+tM)-I}{t}=\lim_{t\to\infty}\frac{I+tM+tM^T+t^2M^TM-I}{t}=M+M^T$$

References

[1] J. Milnor, J. Stasheff, *Characteristic classes*, Annals of mathematics studies, Princeton university press, Number 76, 1978