Notes on Characteristic Classes

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August 15, 2015

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2.1 A brief review of singular cohomology

The algebraic topological view of characteristic classes is built upon singular cohomology. Therefore we brief recap some important facts about singular cohomology. Singular homology and singular cohomology are dual to each other. Indeed, cohomology groups can be computed once we know the homology groups, using *universal coefficient theorem*, which says the following sequence

$$0 \to \mathrm{Ext}(H_{n-1}(C), G) \to H^n(C; G) \to \mathrm{Hom}(H_n(C), G) \to 0$$

for a chain complex C of free abelian groups. UCT is a basic fact in homological algebra and there is not much topology in it. However, singular cohomology has some advantage over homology in that there is a ring structure on $H^*(X;G)$.

Long exact sequence of a pair

Then short exact sequence

$$0 \to C_n(A) \xrightarrow{i} C_n(X) \to C_n(X,A) \to 0$$

dualizes to

$$0 \to C^n(X,A;G) \to C^n(X;G) \xrightarrow{i^*} C^n(A;G) \to 0$$

Recall that the function $\mathrm{Hom}(\cdot, G)$ is always left exact, but since the above groups are all free abelian, we do have surjectivity onto $C^n(A;G)$. Indeed, given $\varphi \in C^n(A;G)$, we can extend it by 0 on simplicies whose images are not contained in A.

i^* is naturally interpreted as the restriction of a map on $C_n(X)$ to a map on $C_n(A)$, and the kernel consists of those maps that are zero on $C_n(A)$, which are naturally identified with maps in $\mathrm{Hom}(C_n(X)/C_n(A), G)$, which is precisely $C^n(X,A;G)$. In particular, unlike $C_n(X,A;G)$, $C^n(X,A;G)$ can be naturally viewed as a subgroup of $C^n(X;G)$, i.e. those that are zero on $C(A)$, and the boundary map $\delta : C^n(X,A;G) \to C^{n+1}(X,A;G)$ is simply the restriction of $\delta : C^n(X;G) \to C^{n+1}(X;G)$.

2.2 Grassmann Manifolds

2.2.1 Universal bundle

We first explain the ideas in this section. Many proofs are pretty technical topological arguments and are delayed.

Given a curve $M^1 \subseteq \mathbb{R}^{k+1}$, we may define a map

$$t : M^1 \to S^k$$

that sends every point in M to a unit tangent vector. Of course, to do it consistently we need an orientation of M^1. However, when an orientation is not available or we may disregard it, it is always possible to define a map by passing to projective space, i.e. $t : M^1 \to \mathbb{P}^k$ that sends a point to its tangent space.

Similarly, given a hypersurface $M^k \subseteq \mathbb{R}^{k+1}$, we may define a map

$$n : M^k \to \mathbb{P}^k$$

that sends a point to the orthogonal complement of its tangent space.

More generally we would like a map for $M^n \subseteq \mathbb{R}^{n+k}$, so we need to space to parametrize the n-dimensional subspaces of $\mathbb{R}^{n+k}$. Therefore we introduce Grassmann manifolds. The underlying set $G_n(\mathbb{R}^{n+k})$ is the n-dimensional subspaces of $\mathbb{R}^{n+k}$ and we topologize it as follows: Let

$$V_n(\mathbb{R}^{n+k}) \subseteq \mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$$

be the subset parametrizing the n-tuples of vectors $(v_1, \cdots, v_n)$ that are linearly independent (such tuple is called a n-frame). Clearly V_n is a Zariski open subset. Each n-tuple in V_n determines a n-dimensional subspace in $\mathbb{R}^{n+k}$. Therefore there is a canonical function

$$q : V_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})$$

This map gives $G_n(\mathbb{R}^{n+k})$ a quotient topology. Alternatively, we could have used the set of orthonomal n-frames V_n^0. V_n^0 is clearly compact, and hence $G_n(\mathbb{R}^{n+k})$ is compact. We will verify later that $G_n(\mathbb{R}^{n+k})$, which now is a priori only a topological space, is indeed a topological manifold.

In particular $\mathbb{P}^n$ can now be viewed as a special case of Grassmann manifolds, and we may define a canonical vector bundle on $G_n(\mathbb{R}^{n+k})$ in a completely analogous fashion:

$$\gamma^n(\mathbb{R}^{n+k}) = \{(n\text{-plane in } \mathbb{R}^{n+k}, \text{vectors in the plane})\}$$

With Grassmann mainfolds we can define generalized Gauss maps $\overline{g} : M \to G_n(\mathbb{R}^{n+k})$ by sending a point to its tangent space. However, Grassmann manifolds can absorb much more bundles. In some sense it behaves like a terminal object in a category.

Lemma 2.1. *For any n-plane bundle ξ over a compact base space B there exists a bundle map $\xi \to \gamma^n(\mathbb{R}^{n+k})$ provided that k is sufficiently large.*

Proof. It suffices to define a map $\widetilde{f} : E(\xi) \to \mathbb{R}^m$ that is linear and injective on each fiber. Since we may then define a bundle map $f : E(\xi) \to \gamma^n(\mathbb{R}^{n+k})$ by

$$f(e) = (\widetilde{f}(\text{fiber through } e), \widetilde{f}(e))$$

□

In order to generalize to common manifolds that do not happen to be compact, we do two things: replace "compact" be "paracompact" and introduce infinite Grassmann manifolds. A topological space is paracompact if it is Hausdorff and for every open cover, there is a locally finite refinement. Every metric space is paracompact, a direct limit of compact spaces is compact, and nearly all familiar topological spaces are paracompact. We realize the infinite Grassmann manifolds as a direct limit of the sequence

$$G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+1}) \subset \cdots$$

To make sense of "$\subset$": Define $\mathbb{R}^\infty = \oplus_{i=1}^\infty \mathbb{R}_i$ and view each $\mathbb{R}^n$ as a subset $\oplus_{i=1}^n \mathbb{R}_i$. We denote the above infinite Grassmann manifold as $G_n(\mathbb{R}^\infty)$. As a special case, we have infinite projective space $\mathbb{P}^\infty = G_1(\mathbb{R}^\infty)$ is the direct limit of $\mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots$.

A canonical bundle γ^n over G_n can be constructed just as in the finite case, i.e. as a subset of $G_n \times \mathbb{R}^\infty$ that consists of pairs of planes and vectors in the planes. It is topologized as a subset of the Catesian product. We will verify that it does satisfy the local triviality condition and is a n-plane bundle over G_n. Now Lemma 2.1 can be extended to the theorem:

Theorem 2.2. *Any $\mathbb{R}^n$-bundle ξ over a paracompact base admits a bundle map $\xi \to \gamma^n$. Moreover, any two such bundle maps are homotopic.*

In particular, the homotopic bundle maps induce homotopic maps on base spaces. Therefore any $\mathbb{R}^n$-bundle ξ over a paracompact space B determines a unique homotopy class of maps $\overline{f_\xi} : B \to G_n$.

2.2.2 Grassmann manifolds

Now we take a closer loot at the structure of Grassmann manifolds. We verify that is indeed a manifold and give a decomposition into CW complex. Finally we talk about the cohomology of Grassmann manifolds.

A convenient open chart

Recall that when working with projective space we would like to choose an affine chart and work locally. At each point $p = (x_0 : \cdots : x_n) \in \mathbb{P}^n$ there is some coordinate $x_j \neq 0$, say $j = 0$. Then we can choose $A_0 = \{x_0 \neq 0\}$ to be an open neighborhood containing p, which is invariably easier to work with. Therefore with Grassmann manifolds $G_n(\mathbb{R}^{n+k})$ the first thing we do is to describe how to conveniently choose an open neighborhood of a point, which in our case is some n plane $X \in \mathbb{R}^{n+k}$. Thanks to the natural inner product in $\mathbb{R}^{n+k}$ we may consider the orthogonal projection map proj_X onto X and let U be the set of planes that project injectively onto X, i.e.

$$U = \{X' \in G_n(\mathbb{R}^{n+k}) : X' \cap \ker \mathrm{proj}_X = 0\}$$

We verify that U is an open subset of $G_n(\mathbb{R}^{n+k})$. Since the topology on $G_n(\mathbb{R}^{n+k})$ is the quotient topology from $V_n(\mathbb{R}^{n+k})$, it is equivalent to verifying that the preimage of U in $V_n(\mathbb{R}^{n+k})$ is open. Choose a basis $w_1, \cdots, w_k$

of $X^\perp = \ker \mathrm{proj}_X = 0$, a n-frame $(v_1, \cdots, v_n) \in V_n(\mathbb{R}^{n+k})$ has a span disjoint from $X^\perp$ if and only if $\{v_1, \cdots, v_n, w_1, \cdots, w_k\}$ is a basis of $\mathbb{R}^{n+k}$. Therefore we define a map $\det : V_n(\mathbb{R}^{n+k}) \to \mathbb{R}$ by sending $(v_1, \cdots, v_n)$ to $\det([v_1; \cdots; v_n; w_1; \cdots; w_k])$. The map is clearly continuous and U is evidently $\det^{-1}(\mathbb{R} - \{0\})$ and hence is open. Just as those standard affine charts on $\mathbb{P}^n$, U is also Zariski dense in $G_n(\mathbb{R}^{n+k})$.

We can view planes in U as the graph of some linear map $X \to X^\perp$, so there is a natural bijective correspondence between U and $\mathrm{Hom}(X, X^\perp) \cong M^{n\times k} = \mathbb{R}^{nk}$. Indeed, we verify that it is a homeomorphism. Actually this is clear once we write down such a correspondence $T : U \to M^{n\times k}$ explicitly using coordinates. Let $v_1, \cdots, v_n$ be an orthonormal basis for X. For each $X' \in U$, there is a unique basis $\{w_1, \cdots, w_n\}$ such that $\mathrm{proj}_X w_j = v_j$ for all j. Then we set

$$T(X') = [\mathrm{proj}_{X^\perp} w_1, \cdots, \mathrm{proj}_{X^\perp} w_n]$$

i.e. the matrix whose columns are $\mathrm{proj}_{X^\perp} w_j$. w_j's evidently depends continuously on X' and vice versa, so T is indeed a homeomorphism. In particular, it implies that $G_n(\mathbb{R}^{n+k})$ is Hausdorff, which could easily shown independently as well.

Local trivialization of $\gamma^n(\mathbb{R}^{n+k})$

With U it is also not hard to show that $\gamma^n(\mathbb{R}^{n+k})$ is a bundle. Of course, the only interesting thing to verify is local triviality. On U, $\gamma^n(\mathbb{R}^{n+k})$ restricts to pairs

$$\{(X', v) : X' \cap \ker \mathrm{proj}_X = 0, v \in X'\}$$

Therefore we may use $h : \pi^{-1}(U) \to U \times X$ defined by $h(X', v) = (X', \mathrm{proj}_X v)$ as a local trivialization. That $X' \in U$ guarantees that $v \to \mathrm{proj}_X v$ is injective. It is not hard to verify that it is indeed a homeomorphism.

In this section we use D^p to denote the closed unit disk in $\mathbb{R}^p$. Let us recall the definition of a CW-complex:

Definition 2.3. A CW-complex is a Hausdorff space K, together with a partitiion of K into disjoint subsets $\{e_\alpha\}$, such that

1. Each e_α is topologically an open cell of dimension $n(e)$. For each e_α, there is a characteristic map $\varphi_\alpha : D^{n(\alpha)} \to K$ that carries $\mathrm{int}\, D^{n(\alpha)}$ homeomorphically onto e_α.

2. Each point $x \in \mathrm{cl}(e_\alpha) - e_\alpha$ must lie in a cell e_β of lower dimension.

3. Each point of K is contained in a finite subcomplex.

4. K is topologized as a direct limit of its finite subcomplexes.

The last two conditions always hold if the complex is finite (i.e. has finitely many cells), so we only use them to deal with infinite complexes. Note that the closure of a cell need not be a cell. For example, S^n can be considered a cell conplex with with a n-cell and a 0-cell, but the closure of the n-cell is the entire sphere. Every CW-complex is paracompact. We first study the cell structure for $G_n(\mathbb{R}^m)$. Any n-plane $X \subseteq \mathbb{R}^m$ is filtrated by

$$0 \subseteq X \cap \mathbb{R}^1 \subseteq X \cap \mathbb{R}^2 \subseteq \cdots \subseteq X \cap \mathbb{R}^m$$

The dimension of spaces in the above filtration increases from 0 and stabilizes at n. There are precisely n jumps. We use a Schubert symbol $\sigma = (\sigma_1, \cdots, \sigma_n)$ to record a jump type, i.e.

$$\dim X \cap \mathbb{R}^{\sigma_i} = i, \, \dim X \cap \mathbb{R}^{\sigma_{i-1}} = i - 1$$

For each Schubert symbol σ, we use $e(\sigma)$ to denote the set of all n-planes with the same jump type.

Let us use the standard basis for $\mathbb{R}^n$ and $\mathbb{R}^\infty$. For each n-plane $X \subseteq \mathbb{R}^m$ we can write down a $n \times m$ matrix whose row space is X, i.e. a generator matrix for X. Then we reduce the matrix to echelon form, then the columns with pivotal entries are in the support of σ.

2.3 Orientation and Euler class

3 Differential Geometry

4 Algebraic Geometry