Notes on Homotopy Groups

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Homotopy extension property

A non-example Here is an example of a pair (X, A) where A is closed in X that does not have the homotopy extension property. Let $X = I$ and $A = \{0, 1, 1/2, 1/3, \cdots\}$. We show that there does not exist a retraction $r : I \times I \to A \times I$. Suppose there is such a r. Consider the horizontal path

$$\varepsilon_n^1 = \{(t, 1) : \frac{1}{n+1} \leq t \leq \frac{1}{n}\}$$

The image $r(\varepsilon_n)$ is a path in $A \times I$ and it must surject to

$$\varepsilon_n^0 = \{(t, 0) : \frac{1}{n+1} \leq t \leq \frac{1}{n}\}$$

Now, select one point $y_n \in \varepsilon_n^0$ and a point $x_n \in \varepsilon_n^1$ such that $r(x_0) = y_0$. Clearly $x_n \to (0, 1)$ and $y_n \to (0, 0)$. However, $r : (0, 1) \mapsto (0, 1)$, contradicting with the continuity of r.

Relative homotopy group They fit into a longe exact sequence

$$\cdots \to \pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \to \cdots$$

Compression lemma The homotopy groups naturally encode information on the obstruciton we encounter when trying to homotope a map. The following conditions are equivalent:

1. Every map $(D^i, \partial D^i) \to (X, A)$ is homotopic rel ∂D^i to a map $D^i \to A$.
2. Every map $(D^i, \partial D^i) \to (X, A)$ is homotopic through such maps to a map $D^i \to A$.

3. Every map $(D^i, \partial D^i) \to (X, A)$ is homotopic through such maps to a constant map $D^i \to A$.

4. $\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$.

Clearly $3, 4$ are saying the same thing. 2 implies 3 since D^i is contractible. We can homotope the map to constant map by pre-composing with the deformation retraction.

The compression lemma says: Let (X, A) be a CW pair and $(Y, B \neq \emptyset)$ be any pair. If for each n such that $X - A$ has a n-cell, $\pi_n(Y, B, y_0) = 0$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic rel A to a map $X \to B$.

Assume inductively that f has already been homotoped to map that sends X^{k-1} to B. We want to further deform it so that it sends the k-skeleton into B as well.

Let Φ be the characteristic map of a cell e^k of $X - A$, i.e $\Phi : (D^k, \partial D^k) \to X$ tha sends $\int D^k$ homeomorphically onto e^k. ∂D^k lies in the $(k-1)$-skeleton of X, so $f \circ \Phi$ sends $(D^k, \partial D^k)$ to (Y, B). Since we assumed that $\pi_k(Y, B, y_0) = 0$ for all $y_0 \in B$, the map $f \circ \Phi$ can be homotoped to map whose image lies in B. This homotopy induces a homotopy of f on the quotient $X^{k-1} \cup e^k$ of $X^{k-1} \coprod D^k$. By the homotopy extension property, the homotopy can be extended to all of X. Therefore the induction step is completed.

If X is finite dimensional, then we are done. If not, we can use the geometric sequence trick to concatenate the steps.

As we can see, the core of the proof uses nothing but the simple observations we made eariler, which are not very far from the very definition of relative homotopy groups.

Whitehead's theorem The Whitehead's theorem says that if a map $f : X \to Y$ induces isomorphisms on all homotopy groups, then it must be a homotopy equivalence. Furthermore, if f is an inclusion, then Y deformation retracts to X.

Fiber bundles A map $p : E \to B$ is said to have homotopy lifting property with respect to X if given a homotopy $g_t : X \to B$ and a life $\widetilde{g}_0$ of g_0, then we can life g_t to $\widetilde{g}_t$ for each t. A fibration is a map $p : X \to B$ has the homotopy lifting property with respect to all spaces X.