Notes on Characteristic Classes

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Contents

1 Introduction

2 Algebraic Topology

2.1 A brief review of singular cohomology

Cross product Cross product is also called external cup product. The absolute version is

$$\times : H^k(X;R) \times H^l(Y;R) \to H^{k+l}(X \times Y;R)$$

and the relative version is

$$\times : H^k(X,A;R) \times H^l(Y,B;R) \to H^{k+l}(X \times Y, A \times Y \cup X \times B;R)$$

given by $a \times b = p_1^*(a) \cup p_2^*(b)$, where p_1, p_2 are projections from $X \times Y$ to X and Y.

Cohomology sequence of a triple
Given a triple $A \subseteq B \subseteq X$, It is not hard to convince ourselves that the following sequence is exact:

$$0 \to C^\bullet(B, A; G) \to C^\bullet(X, A; G) \to C^\bullet(X, B; G) \to 0$$

so by the snake lemma we obtain an exact sequence for a triple:

$$\cdots \to H^k(B, A; G) \to H^k(X, A; G) \to H^k(X, B; G) \to H^{k+1}(B, A; G) \to \cdots$$

In particular we obtain the exact sequence of pair as a special case by setting $A = \emptyset$.

2.2 Grassmann Manifolds

2.2.1 Universal bundle

We first explain the ideas in this section. Many proofs are pretty technical topological arguments and are delayed.

Given a curve $M^1 \subseteq \mathbb{R}^{k+1}$, we may define a map

$$t : M^1 \to S^k$$

that sends every point in M to a unit tangent vector. Of course, to do it consistently we need an orientation of M^1. However, when an orientation is not available or we may disregard it, it is always possible to define a map by passing to projective space, i.e. $t : M^1 \to \mathbb{P}^k$ that sends a point to its tangent space.

Similarly, given a hypersurface $M^k \subseteq \mathbb{R}^{k+1}$, we may define a map

$$n : M^k \to \mathbb{P}^k$$

that sends a point to the orthogonal complement of its tangent space.

More generally we would like a map for $M^n \subseteq \mathbb{R}^{n+k}$, so we need to space to parametrize the n-dimensional subspaces of $\mathbb{R}^{n+k}$. Therefore we introduce Grassmann manifolds. The underlying set $G_n(\mathbb{R}^{n+k})$ is the n-dimensional subspaces of $\mathbb{R}^{n+k}$ and we topologize it as follows: Let

$$V_n(\mathbb{R}^{n+k}) \subseteq \mathbb{R}^{n+k} \times \cdots \times \mathbb{R}^{n+k}$$

be the subset parametrizing the n-tuples of vectors $(v_1, \cdots, v_n)$ that are linearly independent (such tuple is called a n-frame). Clearly V_n is a Zariski open subset. Each n-tuple in V_n determines a n-dimensional subspace in $\mathbb{R}^{n+k}$. Therefore there is a canonical function

$$q : V_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{n+k})$$

This map gives $G_n(\mathbb{R}^{n+k})$ a quotient topology. Alternatively, we could have used the set of orthonomal n-frames V_n^0. V_n^0 is clearly compact, and hence $G_n(\mathbb{R}^{n+k})$ is compact. We will verify later that $G_n(\mathbb{R}^{n+k})$, which now is a priori only a topological space, is indeed a topological manifold.

In particular $\mathbb{P}^n$ can now be viewed as a special case of Grassmann manifolds, and we may define a canonical vector bundle on $G_n(\mathbb{R}^{n+k})$ in a completely analogous fashion:

$$\gamma^n(\mathbb{R}^{n+k}) = \{(n\text{-plane in } \mathbb{R}^{n+k}, \text{vectors in the plane})\}$$

With Grassmann mainfolds we can define generalized Gauss maps $\overline{g} : M \to G_n(\mathbb{R}^{n+k})$ by sending a point to its tangent space. However, Grassmann manifolds can absorb much more bundles. In some sense it behaves like a terminal object in a category.

Lemma 2.1. *For any n-plane bundle ξ over a compact base space B there exists a bundle map $\xi \to \gamma^n(\mathbb{R}^{n+k})$ provided that k is sufficiently large.*

In order to generalize to common manifolds that do not happen to be compact, we do two things: replace "compact" be "paracompact" and introduce infinite Grassmann manifolds. A topological space is paracompact if it is Hausdorff and for every open cover, there is a locally finite refinement. Every metric space is paracompact, a direct limit of compact spaces is compact, and nearly all familiar topological spaces are paracompact. We realize the infinite Grassmann manifolds as a direct limit of the sequence

$$G_n(\mathbb{R}^n) \subset G_n(\mathbb{R}^{n+1}) \subset G_n(\mathbb{R}^{n+1}) \subset \cdots$$

To make sense of "$\subset$": Define $\mathbb{R}^\infty = \oplus_{i=1}^\infty \mathbb{R}_i$ and view each $\mathbb{R}^n$ as a subset $\oplus_{i=1}^n \mathbb{R}_i$. We denote the above infinite Grassmann manifold as $G_n(\mathbb{R}^\infty)$. As a special case, we have infinite projective space $\mathbb{P}^\infty = G_1(\mathbb{R}^\infty)$ is the direct limit of $\mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots$.

A canonical bundle γ^n over G_n can be constructed just as in the finite case, i.e. as a subset of $G_n \times \mathbb{R}^\infty$ that consists of pairs of planes and vectors in the planes. It is topologized as a subset of the Catesian product. We will verify that it does satisfy the local triviality condition and is a n-plane bundle over G_n. Now Lemma 2.1 can be extended to the theorem:

Theorem 2.2. *Any $\mathbb{R}^n$-bundle ξ over a paracompact base admits a bundle map $\xi \to \gamma^n$. Moreover, any two such bundle maps are homotopic.*

In particular, the homotopic bundle maps induce homotopic maps on base spaces. Therefore any $\mathbb{R}^n$-bundle ξ over a paracompact space B determines a unique homotopy class of maps $\overline{f_\xi} : B \to G_n$.

2.2.2 Cell Structure

In this section we use D^p to denote the closed unit disk in $\mathbb{R}^p$. Let us recall the definition of a CW-complex:

Definition 2.3. A CW-complex is a Hausdorff space K, together with a partitiion of K into disjoint subsets $\{e_\alpha\}$, such that

1. Each e_α is topologically an open cell of dimension $n(e)$. For each e_α, there is a characteristic map $\varphi_\alpha : D^{n(\alpha)} \to K$ that carries $\operatorname{int} D^{n(\alpha)}$ homeomorphically onto e_α.

2. Each point $x \in \operatorname{cl}(e_\alpha) - e_\alpha$ must lie in a cell e_β of lower dimension.

3. Each point of K is contained in a finite subcomplex.

4. K is topologized as a direct limit of its finite subcomplexes.

The last two conditions always hold if the complex is finite (i.e. has finitely many cells), so we only use them to deal with infinite complexes. Note that the closure of a cell need not be a cell. For example, S^n can be considered a cell conplex with with a n-cell and a 0-cell, but the closure of the n-cell is the entire sphere. Every CW-complex is paracompact. We first study the cell structure for $G_n(\mathbb{R}^m)$. Any n-plane $X \subseteq \mathbb{R}^m$ is filtrated by

$$0 \subseteq X \cap \mathbb{R}^1 \subseteq X \cap \mathbb{R}^2 \subseteq \cdots \subseteq X \cap \mathbb{R}^m$$

The dimension of spaces in the above filtration increases from 0 and stabilizes at n. There are precisely n jumps. We use a Schubert symbol $\sigma = (\sigma_1, \cdots, \sigma_n)$ to record a jump type, i.e.

$$\dim X \cap \mathbb{R}^{\sigma_i} = i, \, \dim X \cap \mathbb{R}^{\sigma_i - 1} = i - 1$$

For each Schubert symbol σ, we use $e(\sigma)$ to denote the set of all n-planes with the same jump type. We will show that $e(\sigma)$ is an open cell of dimension $d(\sigma) = (\sigma_1 - 1) + (\sigma_2 - 2) + \cdots + (\sigma_n - n)$.

Let us use the standard basis for $\mathbb{R}^n$ and $\mathbb{R}^\infty$. For each n-plane $X \subseteq \mathbb{R}^m$ we can write down a $n \times m$ matrix whose row space is X, i.e. a generator matrix for X. Then we reduce the matrix to reduced echelon form, then the columns with pivotal entries are in the support of σ. With some linear algebra, it is not hard to show that each n-plane $X \in e(\sigma)$ possesses a unique othornormal basis $(x_1, \cdots, x_n) \in H^{\sigma_1} \times \cdots \times H^{\sigma_n}$.

Define

$$\widetilde{e}(\sigma) = V_n^0 \cap (H^{\sigma_1} \times \cdots \times H^{\sigma_n})$$

and

$$\overline{e}(\sigma) = V_n^0 \cap (\overline{H}^{\sigma_1} \times \cdots \times \overline{H}^{\sigma_n})$$

Lemma 2.4.

2.3 Orientation and Euler class

2.4 Thom Isomorphism

For the first half of the section, $\mathbb{Z}/2\mathbb{Z}$ coefficients are assumed. For $n = 1$, the cohomology group $H^1(\mathbb{R}, \mathbb{R}_0) \cong \mathbb{Z}/2\mathbb{Z}$. Of course, this can be shown by abstract means, but in order to see what the generator e^1 looks like I decide to do it by hand. Recall that

$$C^1(\mathbb{R}, \mathbb{R}_0) = \{\varphi : \{[a, b] : a, b \in \mathbb{R}\} \to \mathbb{Z}/2\mathbb{Z} : \forall [a, b] \subseteq \mathbb{R}_0, \varphi([a, b]) = 0\}$$

If in addition φ is a cocycle, then $\varphi([0, 0]) = 0$. In fact φ is fixed once $\varphi([0, 1])$ is fixed. If $\varphi([0, 1]) = 0$, then φ is trivial. If $\varphi([0, 1]) = 1$, then

$$\varphi([a, b]) = \begin{cases} 1, & \text{if } ab = 0, a \neq b \\ 0, & \text{otherwise} \end{cases}$$

and in this case $e^1 = [\varphi]$.

When $n > 1$, then consider the long exact sequence of the pair $(\mathbb{R}^n, \mathbb{R}^n_0)$. Since $H^{n-1}(\mathbb{R}^n) = H^n(\mathbb{R}^n) = 0$, we have that

$$H^n(\mathbb{R}^n, \mathbb{R}^n_0) = H^{n-1}(\mathbb{R}^n_0) = \mathbb{Z}/2\mathbb{Z}$$

3 Differential Geometry

4 Algebraic Geometry