Simply Ramified Curves in $\mathbb{P}^1 \times \mathbb{P}^1$

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1 Introduction

How should I do the introduction?

Notation Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ over a ground field κ. We label its coordinates as $((s_0, s_1), (t_0, t_1))$. When we need to differentiate, the first copy of $\mathbb{P}^1$ is labelled $\mathbb{P}^1_s$ and the second one $\mathbb{P}^1_t$. Let $\pi : X \to \mathbb{P}^1_t$ be the natural projection map. Given $P \in \mathbb{P}^1_t$, let X_Q denotes the fiber $\pi^{-1}P$. $R_{n,d} = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(n, d))$. For a section $f \in R_{n,d}$ or $\mathbb{P}R_{n,d}$, the hypersurface in X cut out by f is denoted by H_f. If $C \subseteq X$ is a curve and $Q \in X$ is a closed point, then $e_Q(C)$ denotes the ramification degree of C with respect to π.

2 An analogue of Bertini's theorem

Theorem 2.1. *When κ is algebraically closed, then for almost all $f \in R_{n,d}$, H_f is simply ramified. More precisely, let P^0 be the subscheme of $\mathbb{P}R_{n,d}$ parametrizing non-singular curves, then the subset*

$$D = \{f \in \mathbb{P}R_{n,d} : H_f \textit{ is not simply ramified w.r.t. } \pi\}$$

is contained in a proper closed subscheme of P^0.

Proof. To simplify notation in this proof we write $V = R_{n,d}$. There are two types of curves in D. Type I curves are those that have a ramfication degree ≥ 3 at some point. Type II curves are those that ramify at more than 2 points along some fiber. We want to define subschemes of $\mathbb{P}V$ that can be informally described as

$$D_1 = \{f \in \mathbb{P}V : \exists Q \in X, \text{ s.t. } e_Q(H_f) \geq 3\}$$
$$D_2 = \{f \in \mathbb{P}V : \exists P \in \mathbb{P}^1_t, \text{ s.t. } e_Q(H_f), e_{Q'}(H_f) \geq 2 \textit{ for some } Q \neq Q' \in X_P\}$$

Clearly D corresponds the set of closed points in the scheme $P^0 \cap (D_1 \cup D_2)$ and hence it suffices to prove the following two claims.

Claim 1 : D_1 is a proper closed subscheme of $\mathbb{P}V$.
This is more or less a standard application of Bertini's arguments. We want to define a subscheme $\widetilde{D}_1 \subseteq X \times \mathbb{P}V$ to parametrize pairs (P, f) such that $e_Q(H_f) \geq 3$. Suppose $Q \in \mathbb{A}^1 \times \mathbb{A}^2 = \operatorname{Spec} \kappa[\mathrm{s}, \mathrm{t}] \subset \mathrm{X}$ where $s = s_0/s_1, t = t_0/t_1$. Then $f \in \mathbb{P}V$ can be written in the form

$$f = \sum_{0 \leq j \leq n} \sum_{0 \leq i \leq d} a_{ij} s^i t^j$$

where (a_{ij}) is homogeneous coordinates for $\mathbb{P}V$. $e_Q(H_f) \geq 3$ if and only if

$$f(Q) = \frac{\partial}{\partial s} f(Q) = \frac{\partial^2}{\partial s^2} f(Q) = 0$$

Hence on $\mathbb{A}^1 \times \mathbb{A}^1 \times X$, $\widetilde{D}_1$ can be described as $\{f = f_s = f_{ss} = 0\}$. Clearly the $e_Q(H_f)$ is independent of the affine chart of X that we choose to cover Q, so if we define $\widetilde{D}_1$ this way on other charts, they patch up. For each $Q \in X$, $f \in (\widetilde{D}_1)_Q$ amounts to 3 linear conditions on $\mathbb{P}V$. Therefore we can easily compute that $\dim \widetilde{D}_1 \leq \mathbb{P}V - 1$. Let D_1 be the image of $\widetilde{D}_1$ under the projection to $\mathbb{P}V$. $\widetilde{D}_1$ is closed in $X \times \mathbb{P}V$ and hence by elimination theory D_1 is closed in $\mathbb{P}V$. It is proper since $\dim D_1 \leq \mathbb{P}V - 1$.

Claim 2 : D_2 is a contained in a proper closed subscheme of $\mathbb{P}V$.
The proof is not as straightforward as the previous one, since the question is no longer local at a point of X. Nonetheless we may sieve on pairs of points along the same fiber. Consider the space $X \times X$. We label its coordinates as $((x_0 : x_1), (y_0 : y_1), (s_0 : s_1), (t_0 : t_1))$. Let T' be the subscheme defined by $y_0 t_1 - y_1 t_0 = 0$ and let Δ be the diagonal of $X \times X$, i.e. $\{y_0 t_1 - y_1 t_0 = 0, x_0 s_1 - x_1 s_0 = 0\}$. Define $T = T' - \Delta$. Clearly T is simply parametrizing pairs on X that project to the same point under π. This time we let B be the subscheme of $T \times \mathbb{P}V$ parametrizing pairs $((Q, Q'), f)$ such that $e_Q(H_f), e_{Q'}(H_f) \geq 2$. More precisely, on $(\mathbb{A}^1 \times \mathbb{A}^1) \times (\mathbb{A}^1 \times \mathbb{A}^1) \times \mathbb{P}V$, where $(\mathbb{A}^1 \times \mathbb{A}^1) \times (\mathbb{A}^1 \times \mathbb{A}^1) \subset X \times X$ is the affine chart with coordinates $x = x_0/x_1, y = y_0/y_1, s = s_0/s_1, t = t_0/t_1$, $\widetilde{D}_2$ is defined as

$$f(x, y) = \frac{\partial}{\partial x} f(x, y) = f(s, t) = \frac{\partial}{\partial s} f(s, t) = 0 \tag{*}$$

To make sense of $f(x, y)$, we simply associate the point $f \in \mathbb{P}V$ with homogeneous coordinates (a_{ij}) with

$$f = \sum_{0 \leq j \leq n} \sum_{0 \leq i \leq d} a_{ij} x^i y^j$$

Therefore (*) is nothing but a concise way to write down polynomial equations in x, y, s, t and a_{ij}'s. For each point $(Q, Q') \in X \times X$, $f \in (\widetilde{D}_2)_{(Q,Q')}$ amounts to 4 linear conditions on a_{ij}'s. Therefore we easily obtain $\dim \widetilde{D}_2 = \mathbb{P}V - 1$. D_2 is defined as the image of $\widetilde{D}_2$ under the projection to $\mathbb{P}V$, and there D_2 is a locally closed subscheme of $\mathbb{P}V$ with closure $\overline{D}_2 \neq \mathbb{P}V$. This proves Claim 2. □

3 A density result

If $\mathcal{P} \subseteq \bigcup_d R_{3,d}$ is subset, we define

$$\mathrm{Prob}(f \in \mathcal{P}) = \lim_{d\to\infty} \mathrm{Prob}(f_d \in \mathcal{P})$$

where f and f_d are randomly chosen from $\bigcup_d R_{3,d}$ and $R_{3,d}$ respectively.

Theorem 3.1. *Suppose $p = \mathrm{char}\mathbb{F}_q > 2$. Let $\mathcal{D} \subset \bigcup_d R_{3,d}$ be the subset of sections f such that H_f is simply ramified with respect to π. Then*

$$\mathrm{Prob}(f \in \mathcal{D}) = \zeta_{\mathbb{P}_{\mathbb{F}_q}}(2)^{-2}$$

In particular, the conditional probability for a randomly chosen non-singular curve to be simply ramified is $\zeta_{\mathbb{P}_{\mathbb{F}_q}}(2)/\zeta_{\mathbb{P}_{\mathbb{F}_q}}(3)$.

We say $f \in R_{n,d}$ is "good" at $Q \in X$, if H_f is both non-singular and simply ramified at Q, and "bad" otherwise. For a fixed $e_0 \in \mathbb{N}$, we define

$$\mathcal{P}_{e_0}^{\mathrm{low}} = \bigcup_{d\geq 0}\{f \in R_{3,d} : f \textit{ is good at all } Q \in X, \deg \pi(Q) < e_0\}$$

$$\mathcal{Q}_{e_0}^{\mathrm{med}} = \bigcup_{d\geq 0}\{f \in R_{3,d} : f \textit{ is bad at some } Q,\ \deg \pi(Q) \in [e_0, \lfloor d/p \rfloor]\}$$

$$\mathcal{Q}^{\mathrm{high}} = \bigcup_{d\geq 0}\{f \in R_{3,d} : f \textit{ is bad at some } Q,\ \deg \pi(Q) > d/p\}$$

Lemma 3.2.

$$\mathrm{Prob}(f \in \mathcal{P}_{e_0}^{\mathrm{low}}) = \prod_{\deg(P)<e_0} \mathrm{Prob}(H_f \textit{ is good at all points of } \pi^{-1}(P))$$

Now we want to do some local analysis on the fiber. Let $P \in \mathbb{P}^1_t$ be a closed point. Without loss of generality, we assume $t_1 \neq 0$ at P and and lies in $\mathbb{A}^1 = \mathrm{Spec}\,\mathbb{F}_\mathrm{q}[\mathrm{t}]$ where $t = t_0/t_1$. Let $\mathfrak{m} = (r(t)) \subseteq \mathbb{F}_q[t]$ be the maximal ideal corresponding to P, so that $P^{(2)} = \mathrm{Spec}\,\mathbb{F}_\mathrm{q}[\mathrm{t}]/\mathfrak{m}^2$ and $\kappa(P) = \mathbb{F}_q[t]/\mathfrak{m}$. Denote the reduction map $\mathbb{F}_q[t]/\mathfrak{m}^2 \to \mathbb{F}_q[t]/\mathfrak{m}$ by $g \mapsto \overline{g}$. We extend it to a map

$H^0(\mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}^2}, \mathcal{O}(3)) \to H^0(\mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}}, \mathcal{O}(3))$ by applying $\mathbb{F}_q[t]/\mathfrak{m}^2 \to \mathbb{F}_q[t]/\mathfrak{m}$ to each coefficients. Let $\varphi_P : R_{3,d} \to H^0(\mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}^2}, \mathcal{O}(3))$ be the restriction map. H_f is smooth at a point $Q \in \pi^{-1}(P)$ if and only if $\varphi_P(f)$ does not vanish at $Q^{(2)} \in \mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}^2}$, and $e_Q(H_f) \geq 3$ if and only if $\overline{\varphi_P(f)}$ does not have a zero of multiplicity ≥ 3 at Q on $\mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}}$. In other words, we can determine if H_f is good at all points on $\pi^{-1}(P)$ by looking at the image of f in $H^0(\mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}^2}, \mathcal{O}(3))$, so we observe that:

Lemma 3.3.

$$\mathrm{Prob}(f \in \mathcal{P}^{\mathrm{low}}_{e_0}) = \prod_{\deg(P) < e_0} \mathrm{Prob}(H_f \textit{ is good at all points } Q \in H_f \cap X_P)$$

Proof. More generally, if $\{P_1, P_2, \cdots, P_s\}$ is a set of finitely many closed points in $\mathbb{P}^1$, the restriction map

$$R_{3,d} = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(3,d)) \to \prod_{i=1}^{s} H^0(X_{P_i^{(2)}}, \mathcal{O}(3))$$

is surjective for $d \geq \sum_i 2\deg P_i + 1$. $\square$

We make the following convention: We call a pair $(F_1, F_2) \in H^0(\mathbb{P}^1_{\kappa(P)}, \mathcal{O}(3))^2$ "bad" if it is one of the following 3 types:

1. F_1 has a root of multiplicity ≥ 2 at a point where F_2 also vanishes.

2. F_1 has a root of multiplicity 3 at a point where F_2 does not vanish.

3. $F_1 \equiv 0$.

Let $\widetilde{\ }: \mathbb{F}_q[t]/\mathfrak{m} \to \mathbb{F}_q[t]/\mathfrak{m}^2$ be a $\mathbb{F}_q$-linear map that is a section to the reduction map $\overline{\ }: \mathbb{F}_q[t]/\mathfrak{m}^2 \to \mathbb{F}_q[t]/\mathfrak{m}$. We extend it to a map $H^0(\mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}^2}, \mathcal{O}(3)) \to H^0(\mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}}, \mathcal{O}(3))$ in the same way we extended the reduction map. Then we have the following:

Lemma 3.4.

$$H^0(\mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}}, \mathcal{O}(3))^2 \to H^0(\mathbb{P}^1_{\mathbb{F}_q[t]/\mathfrak{m}^2}, \mathcal{O}(3))$$

given by $(F_1, F_2) \mapsto \widetilde{F_1} + uF_2$ is a bijection. Moreover, $f \in R_{3,d}$ is "good" at all points $Q \in \pi^{-1}(P)$ if and only if $\varphi_P(f)$ does not correspond to a bad pair.

Proof. The map $f \mapsto (\overline{f}, (f - \widetilde{\overline{f}})/r(t))$ is an inverse to the map given in the lemma. Let (F_1, F_2) be the pair corresponding to $\varphi_P(f)$. $\varphi_P(f)$ vanishes on $Q^{(2)}$, i.e. H_f is singular at Q, if and only if Q is a double root of F_1 and a root to F_2. Similarly, $e_Q(H_f) \geq 3$ if and only if Q is root to F_1 of multiplicity ≥ 3. $\square$

Lemma 3.5. *The density of good pairs in* $H^0(\mathbb{P}^1_{\kappa(P)}, \mathcal{O}(3))^2$ *is*

$$(1 - q^{-2e})^2$$

Proof. We first count type 1 pairs. Let P be the point that is a double root to F_1 and a root to F_2. Note that $\deg P = 1$. F_1 is fixed up to rescaling by $\mathbb{F}^*_{q^e}$ after we choose a third root, which can be any point of degree 1. Therefore there are $(q^e + 1)(q^e - 1)$ choices for F_1. The probability that F_2 vanishes at P is q^{-e}, so we have q^3 choices for F_2. Since we have $(q^e + 1)$ choices for P, there are $q^{3e}(q^e + 1)^2(q^e - 1)$ type 1 pairs.

To count type 2 pairs, let P be the triple root to F_1. Again there are $q^e + 1$ choices. At each P, there are $(q^e - 1)$ homogeneous polynomials of degree 3 which have P as a triple root, since the leading coefficient will uniquely determine such a polynomial. We have $q^{4e} - q^{3e}$ choices for F_2. In total there are $(q^e + 1)(q^e - 1)(q^{4e} - q^{3e})$ pairs of type 2.

Finally when $F_1 = 0$, F_2 can be anything, so we have q^{4e} type 3 pairs. Therefore the density of good pairs is

$$1 - q^{-8e}(q^{3e}(q^e + 1)^2(q^e - 1) + (q^e + 1)(q^e - 1)(q^{4e} - q^{3e}) + q^{4e}) = (1 - q^{-2e})^2$$

□

Lemma 3.6.

$$\lim_{e_0 \to \infty} \operatorname{Prob}(f \in \mathcal{Q}^{\text{med}}_{e_0}) = 0$$

Proof. Let P be a point of degree $e < \lfloor d/p \rfloor$ on C and let B_P denote the event that a randomly chosen $f \in R_{3,d}$ is "bad" at some point in the fiber X_P. By the proof of Lemma 3.3, the restriction map $R_{3,d} \to H^0(P^{(2)}, \mathcal{O}(3))$ is surjective since $p > 2$ and $e < \lfloor d/p \rfloor < (d-1)/2$ when d is large. Lemma 3.5 hence implies that probability that f is "bad" at some point in the fiber X_P is $1 - (1 - q^{-2e})^2 < 2q^{-2e}$.

$$\begin{aligned}
\operatorname{Prob}(f \in \mathcal{Q}^{\text{med}}_{e_0}) &\leq \sum_{e=e_0}^{\lfloor d/p \rfloor} (\text{number of points of degree } e \text{ in } \mathbb{P}^1)(2q^{-2e}) \\
&\leq O(\sum_{e=e_0}^{\lfloor d/p \rfloor} q^e q^{-2e}) \\
&= O(\frac{cq^{-e_0}}{1 - q^{-1}})
\end{aligned}$$

Therefore as $e_0 \to \infty$, $\operatorname{Prob}(f \in \mathcal{Q}^{\text{med}}_{e_0}) \to 0$. □

Lemma 3.7. *Let $j > 2$ be an integer. For a randomly chosen $f \in R_{3,d}$, the probability that there exists a point $Q \in \mathbb{P}^1 \times \mathbb{P}^1$ with $\deg \pi(Q) \geq j$ such that $e_Q(H_f) \geq 3$ or H_f is singular at Q is at most*

$$O(d^2 q^{-\min(\lfloor d/p \rfloor + 1, j)})$$

Proof. Since $\mathbb{P}^1 \times \mathbb{P}^1$ can be covered by 4 affine charts $\mathbb{A}^1 \times \mathbb{A}^1$, it suffices for us to show that for each $\mathbb{A}^1 \times \mathbb{A}^1$, we have that for a randomly chosen $f \in R_{3,d}$, the probability that there exists a point $Q \in \mathbb{A}^1 \times \mathbb{A}^1$ with $\deg \pi(Q) \geq j$ such that $e_Q(H_f) \geq 3$ or H_f is singular at Q is at most $O(d^2 q^{-\min(\lfloor d/p \rfloor + 1, j)})$. Therefore we may work affine-locally.

Without loss of generality, we may assume $s_1 \neq 0, t_1 \neq 0$ on $\mathbb{A}^1 \times \mathbb{A}^1$ and work with coordinates $s = s_0/s_1, t = t_0/t_1$. Let $A_{n,d}$ be the polynomials that are of degree $\leq n$ in s and $\leq d$ in t. Clearly by dehomogenizing sections in $S_{3,d}$, we may naturally identify $S_{3,d} = A_{3,d}$. Accordingly, we replace H_f by $H_f \cap \mathbb{A}^1 \times \mathbb{A}^1$. We call a closed point $Q \in \mathbb{A}^1 \times \mathbb{A}^1$ *admissible* if $\deg \pi(Q) \geq j$ and a subscheme $W \subseteq \mathbb{A}^1 \times \mathbb{A}^1$ *admissible* if it contains an admissible point. By $(W)_{\mathrm{ad}}$ we denote the union of admissible irreducible component of $(W)_{\mathrm{red}}$.

We first deal with the probability that for a randomly chosen $f \in A_{3,d}$, $e_Q(H_f) \geq 3$ for some admissible $Q \in \mathbb{A}^1 \times \mathbb{A}^1$, which happens if and only if

$$f(Q) = f_s(Q) = f_{ss}(Q) = 0 \tag{3.8}$$

Define $W_2 = \{f_{ss} = 0\}, W_1 = W_2 \cap \{f_s = 0\}$ and $W_0 = W_1 \cap \{f = 0\}$. We want to bound the probability that for a randomly chosen $f \in A_{3,d}$, W_0 contains an admissible point.

Following Poonen's idea, we write f is such a way so that the first and second order partial derivatives are largely independent. If $f_0 \in A_{3,d}$ and $g_1, g_2, h \in A_{0,\lfloor d/p \rfloor}$ are selected uniformly and independently at random, then the distribution of

$$f = f_0 + g_1^p s^2 + g_2^p s + h^p$$

is uniform over $A_{3,d}$. Direct computation shows that

$$\begin{aligned} f_s &= f_{0,s} + 2g_1^p s + g_2^p \\ f_{ss} &= f_{0,ss} + 2g_1^p \end{aligned}$$

Note that W_2 depends only on the choice of f_0, g_1 and W_1 only on f_0, g_1, g_2. Let E denote the event that

a. The *admissible* irreducible components of W_1 are of dimension 0.

b. f does not vanish at any of these irreducible components.

Clearly if E holds for f, then W_0 does not contain any admissible point. Therefore it suffices to show that for a randomly chosen $f \in A_{3,d}$,

$$\mathrm{Prob}(E) = 1 - O(d^2 q^{-\min(\lfloor d/p \rfloor + 1, j)})$$

as $d \to \infty$. Now we bound $\mathrm{Prob}(E)$ in three steps:

Step 1: Conditioned on a choice of f_0, the probability that $\dim W_2 = 2$ is at most $q^{-(\lfloor d/p \rfloor + 1)}$, since $\dim W_2 = 2$ if and only if $g_1^p = -f_{0,ss}/2$, for which there is at most one choice of g_1.

Step 2: Conditioned on a choice of f_0 and g_1 such that $\dim W_2 = 1$, the probability that $\dim (W_1)_{\mathrm{ad}} = 1$ is at most $O(dq^{-\min(\lfloor d/p \rfloor + 1, j)})$. Let $V_1, \cdots, V_\ell$ be the $\mathbb{F}_q$-irreducible component of $\{W_2\}_{\mathrm{red}}$. View $\mathbb{P}^1 \times \mathbb{P}^1$ as a subscheme of $\mathbb{P}^3$ via Segre embedding, we may apply Bézout's theorem to obtain that $\ell = O(d)$. $\dim W_1 = 1$ and $\pi(W_1)$ contains a point P with $\deg P \geq j$ if and only if f_s vanishes identically on V_i for some i. We need to bound the set

$$G_i^{\mathrm{bad}} = \{g_2 \in A_{0,\lfloor d/p \rfloor} : f_{0,s} + 2g_1^p s + g_2^p \textit{ vanishes identically on } V_i\}$$

If $g, g' \in G_i^{\mathrm{bad}}$, then $g - g'$ vanihes identically on V_i. Therefore G_i^{bad} is a coset of $\ker \varphi_i$, where φ_i is the $\mathbb{F}_q$-linear map $\varphi_i : A_{0,\lfloor d/p \rfloor} \to H^0(V_i, \mathcal{O}_{V_i})$. Now we divide it into two cases.

Case 1: If $\dim \pi(V_i) > 0$, then the function t, and hence any nonzero polynomial in t, does not vanish identically on V_i. Therefore the codimension of $\ker \varphi_i$ in $A_{0,\lfloor d/p \rfloor}$ is $\lfloor d/p \rfloor + 1$, and the probability that f_s vanishes identically on V_i is at most $q^{-(\lfloor d/p \rfloor + 1)}$.

Case 2: If $\dim \pi(V_i) = 0$, then since V_i is assumed to be admissible, $\deg \pi(V_i) \geq j$. φ_i factors as

$$A_{0,\lfloor d/p \rfloor} = \mathbb{F}[t]_{\leq \lfloor d/p \rfloor} \to H^0(\pi(V_i), \mathcal{O}_{\pi(V_i)}) \xrightarrow{\pi^*} H^0(V_i, \mathcal{O}_{V_i})$$

The pullback map π^* is clearly injective. Let $B_{0,i}$ be the image of $A_{0,i}$ in $H^0(\pi(V_i), \mathcal{O}_{\pi(V_i)})$. Suppose $\pi(V_i) = \mathrm{Spec}\, \mathrm{F_q[t]/(r(t))}$, where $\deg r(t) = \deg \pi(V_i)$. Then it is clear that $\dim_{\mathbb{F}_q} B_{0,i}$ increases in dimension with each increase of i until it stabilizes at $\deg r$. Therefore $\dim \mathrm{Im}\, \varphi_i \geq \min(\lfloor d/p \rfloor + 1, j)$, and the probability that f_s vanishes identically on V_i is at most $q^{-\min(\lfloor d/p \rfloor + 1, j)}$. Since $q^{-(\lfloor d/p \rfloor + 1)} \leq q^{-\min(\lfloor d/p \rfloor + 1, j)}$, and there are at most $O(d)$ such components V_i, in either case we obtain the desired conclusion of this step.

Step 3: Conditioned on a choice of f_0, g_1 and g_2 such that $\dim (W_1)_{\mathrm{ad}} = 0$, the probability that $(W_0)_{\mathrm{ad}} \neq \emptyset$ is at most $O(d^2 q^{-\min(\lfloor d/p \rfloor + 1, j)})$. Let $Q_1, Q_2, \cdots, Q_r$ be all irreducible components of $(W_1)_{\mathrm{ad}}$, where $r = |(W_1)_{\mathrm{ad}}|$.

Since W_1 is cut out by f_s and $f_s s$, and $\deg f_s, \deg f_{ss} = O(d)$, by Bézout theorem $r = O(d^2)$, and the same argument as in the previous paragraph shows that at each point in W_1, the probability that $f = 0$ at the point is at most $q^{-\min(\lfloor d/p \rfloor + 1, j)}$.

Finally *Step 1* and *2* combined to give that

$$\text{Prob}(E_a) \geq (1 - q^{-(\lfloor d/p \rfloor + 1)})(1 - q^{-\min(\lfloor d/p \rfloor + 1, j)}) = 1 - O(dq^{-\min(\lfloor d/p \rfloor + 1, j)})$$

And *Step 3* gives

$$\text{Prob}(E) \geq \text{Prob}(E_a)(1 - O(d^2 q^{-\min(\lfloor d/p \rfloor + 1, j)})) = 1 - O(d^2 q^{-\min(\lfloor d/p \rfloor + 1, j)})$$

Now we deal with the probability that H_f is singular at Q for some $Q \in \mathbb{A}^1 \times \mathbb{A}^1$, which happens if and only if $f(Q) = f_s(Q) = f_t(Q) = 0$. This time we may write f in the form

$$f = f_0 + g_1^p s + g_2^p t + h^p$$

for some randomly chosen $f_0 \in A_{3,d}$ and $g_1, g_2, h \in A_{0, \lfloor d/p \rfloor}$. The rest of the proof is completely analogous to the above. $\square$

Lemma 3.9.

$$\text{Prob}(f \in \mathcal{Q}^{\text{high}}) = 0$$

Proof. Apply Lemma 3.7 with $j = \lfloor d/p \rfloor$. $\square$

Proof of Theorem 3.1. For each e_0, we have that

$$\mathcal{P}_{e_0}^{\text{low}} \subseteq \mathcal{D} \subseteq \mathcal{P}_{e_0}^{\text{low}} \cup \mathcal{Q}_{e_0}^{\text{med}} \cup \mathcal{Q}^{\text{high}}$$

Therefore

$$\text{Prob}(f \in \mathcal{P}_{e_0}^{\text{low}}) \leq \text{Prob}(f \in \mathcal{D}) \leq \text{Prob}(f \in \mathcal{P}_{e_0}^{\text{low}} \cup \mathcal{Q}_{e_0}^{\text{med}} \cup \mathcal{Q}^{\text{high}})$$

Now take $e_0 \to \infty$, Lemma 3.3, 3.5, 3.6, 3.9 combine the give the result. $\square$

References

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