# Simply Ramified Curves

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# 1 Introduction

# 2 Results

## 2.1 A Bertini theorem for simply ramified curves

**Theorem 2.1.** *Let* $V = (H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(d) \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(n)))$. $\mathbb{P}V$ *parametrizes curves of bidegree* $(d, n)$. *Let* $P^0$ *be the open subscheme parametrizing smooth curves.* $D \subseteq P^0$ *be the subscheme parametrizing smooth curves* $C$*, such that the projection* $\mathrm{pr}_1 : C \to \mathbb{P}^1$ *is not simply ramified. Try to show that* $D \subset P^0$ *is closed and* $D \neq P^0$.

*Proof.* The plan is to imitate the proof of Bertini theorem. We want to sieve out two subschemes, $D_1, D_2$. $D_1$ is the subscheme of $P^0$ parametrizing those curves who intersect some fiber $\mathrm{pr}_1^{-1}(\mathrm{pr}_1(p_0))$ at some point $p_0 \in \mathbb{P}^1 \times \mathbb{P}^1$ with multiplicity $\geq 3$. $D_2$ parametrizes those who intersect some fiber with multiplicity $\geq 2$ at least twice. Clearly $D$ is the union of the two. We show that both $D_1$ and $D_2$ are closed subschemes of dimension $\leq \dim \mathbb{P}V - 2$ and hence $D$ is closed and $D \neq P^0$.

To sieve out $D_1$ we consider the product space $(\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}V$. We want to define the locus $B$ of pairs $(p, C)$, where $C$ is "bad" at $p$. We describe $B$ locally. Using appropriate dehomogenization at $\mathbb{A}^1 \times \mathbb{A}^1$, we represent an element of $\mathbb{P}V$ as

$$f = \sum_{0 \leq i \leq n, 0 \leq j \leq d} a_{ij} x^i y^j$$

The coefficients $(a_{ij})$ are nothing but coordinates of $\mathbb{P}V$. The condition

$$f = \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial y^2} = 0$$

describes $B \cap (\mathbb{A}^1 \times \mathbb{A}^1) \times \mathbb{P}V$. $B$ is clearly closed. Now consider the fiber $B_{p_0}$ over a point $p_0 \in \mathbb{A}^1 \times \mathbb{A}^1$. $B_{p_0}$ imposes 3 linear equations on $\mathbb{P}V$, and therefore is of codimension 3, or dimension $\dim \mathbb{P}V - 4$. Hence $B$ has dimension $\leq \dim \mathbb{P}V - 2$. The projection of $B$ on $\mathbb{P}V$, i.e. $D$ has dimension bounded by $\dim \mathbb{P}V - 2$ while $\mathbb{P}V$ has dimension $\dim \mathbb{P}V - 1$.

To treat $D_2$ we sieve on pairs. Consider the space $(\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1)$. Label the coordinates as $((x_0 : x_1), (y_0 : y_1), (t_0 : t_1), (s_0 : s_1))$. Define a subscheme $T$ described by the polynomial $x_0t_1 - x_1t_0$ and additionally we take of the diagonal. Roughly speaking, $T$ parametrizes pairs of points on $\mathbb{P}^1 \times \mathbb{P}^1$ that lie on the same fiber of $\mathrm{pr}_1$. Now consider the product space $T \times \mathbb{P}V$. We want to describe a subscheme $B'$ that parametrizes the pairs $((p_0, p_1), C)$, where $(p_0, p_1) \times T$ and $C$ ramifies at both $p_0$ and $p_1$. Again we describe $B$ locally and assume $\{y_0, s_0 \neq 0\}$. Represent $f \in \mathbb{P}V$ as before, we define $B$ locally as:

$$f = \frac{\partial f}{\partial y} = 0 \ \ at \ y_0, s_0$$

When $p_0 \neq p_1$, the fiber $B_{(p_0,p_1)}$ imposes 4 linear conditions on $\mathbb{P}V$. Therefore $\mathrm{codim}_{\mathbb{P}V} B_{(p_0,p_1)} = 4$. Since $\dim T = 3$, $\dim B' \leq \dim \mathbb{P}V - 2$, just like $B$. Now the hard part is to show the image is closed. Or, is it really closed? $\square$

Some ideas and thoughts:

1. The number of distinct roots of a polynomial of degree $n$ is the same as the rank of the Vandermonde matrix $[r_i^j]_{1 \leq i,j \leq n}$, but unfortunately no minors of the matrix can be written as a polynomial in coefficients of $f$ except the determinant.

2. Since the projection is closed, the image is closed if and only if the preimage is closed. The preimage in $(\mathbb{P}^1 \times \mathbb{P}^1) \times (\mathbb{P}^1 \times \mathbb{P}^1) \times \mathbb{P}V$ of contains more points than $T \times \mathbb{P}V$. For example, $(p, C)$ where $C$ is ramified at $p$ on the diagonal once and is not simply ramified elsewhere.

3. How about consider the Cartier divisor $f/(f_y)$? Its positive degree is exactly the number of roots.

### 2.1.1 simply ramified without smoothness imposed

We first deal with the single-fiber case. For a fixed fiber $\mathrm{pr}_1^{-1}(X)$, where $X \in \mathbb{P}^1$ is a point corresponding to a homogeneous maximal ideal $\mathfrak{m}_X$, what is the probability that a curve will intersect it with multiplicity $\geq 3$? We start from a bi-homogeneous polynomial $f \neq 0 \in V$:

$$f = \sum_{0 \leq j \leq d} (\sum_{0 \leq i \leq n} a_{ij} x_0^i x_1^{n-i}) y_0^j y_1^{d-j}$$

Evaluate the polynomial in the parenthesis at $X$ we obtain a polynomial

$$f_X = \sum_{0 \le j \le d} f_j y_0^j y_1^{d-j}, f_j \in \kappa(X) = \mathbb{F}_{q^e}$$

where $e = \deg X$. The fiber $\mathrm{pr}_1^{-1}(X)$ is just $\mathbb{P}^1$ over $\kappa(X)$. We therefore turn our attention to homogeneous degree $d$ polynomials in $\mathbb{F}_{q^e}[y_0, y_1]$. At each point in $\mathbb{P}^1 = \mathbb{A}^1 \bigcup \{(0:1)\}$. (In scheme-theoretic language, $(0:1)$ is the closed point corresponding to the prime ideal $(y_0)$.) When $d = 3$ at each point in $\mathbb{P}^1$ there is at most one polynomial we want to sieve out, that is, the polynomial that has that point as a triple root. On $\mathbb{A}^1$ we first dehomogenize at $y_0$, and $f_X$ must have the form $\beta(y-\alpha)^3$ for some $\alpha \in \mathbb{F}_{q^e}, \beta \in \mathbb{F}_{q^e}^\times$. Therefore those $f_X$ that are "bad" at some point on $\mathbb{A}^1$ is parametrized by $\alpha, \beta$. On $(y_0)$ the "bad" polynomials are simply parametrized by $\beta y_0^3$ for some $\beta \in \mathbb{F}_{q^e}^\times$ of course. At the end we want to sieve out 0. The "evaluation at $X$" map is linear, so all polynomials in $\mathbb{F}_{q^e}[y_0, y_1]_3$ (in case there is confusion, this includes 0) are hit with the same probability. Among $\mathbb{F}_{q^e}[y_0, y_1]_3$ there are exactly $q^{2e}$ polynomials that we sieved out.

I want to state without proof something I believe is true. It is an analogue of Poonen's Lemma 2.1 and I am sure it might have been shown somewhere in Woods' paper. It is the intuitive reason for us to assume independence across fibers.

### 2.1.2 Simply ramified curves

Now let us also take smoothness into account, since it is part of the definition of being "simply ramified". Therefore instead of considering the fiber over $X$, which is a copy of $\mathbb{P}^1_{\kappa(X)}$, we consider the fiber over its first infinitestimal neighborhood, i.e. $\mathbb{P}^1_{\kappa(X)} \times_{\kappa(X)} \kappa(X)[\varepsilon] = \mathbb{P}^1_{\kappa(X)[\varepsilon]}$ where $\varepsilon^2 = 0$. Again, pick an affine neighborhood $\mathbb{A}^1$ of $X$, dehomogenize $f$ accordingly, and hence $f$ has the form

$$f = \sum_{0 \le j \le d} \Big( \sum_{0 \le i \le n} a_{ij} x^i \Big) y_0^j y_1^{d-j} = \sum_{0 \le j \le d} f_j(x) y_0^j y_1^{d-j}$$

"Evaluation at $X$" $\Gamma(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(d,n)) \to \Gamma(\mathbb{P}^1_{\kappa(X)[\varepsilon]}, \mathcal{O}(d))$ becomes

$$f \mapsto f_X = f_{X,1} + f_{X,2}\varepsilon$$

where

$$f_{X,1} = \sum_{0 \le j \le d} f_j(X) y_0^j y_1^{d-j} \text{ and } f_{X,2} = \sum_{0 \le j \le d} \frac{\partial f_j}{\partial x}(X) y_0^j y_1^{d-j}$$

$f_{X,1}$ and $f_{X,2}$ both lie in $\Gamma(\mathbb{P}^1_{\kappa(X)}, \mathcal{O}_{\mathbb{P}^1_{\kappa(X)}}(d))$. Let $d = 3$. We first compute the probability that that $H_f$ is smooth on $\mathrm{pr}_1^{-1}(X)$ and make sure that our computation agrees with that of Erman and Wood's. We want to sieve out those $f$ whose $f_{X,1} = 0$, or there is a point $Y \in \mathbb{P}^1_{\kappa(X)}$ that is a multiple root to $f_{X,1}$ and a root to $f_{X,2}$. We turn our attention to $\Gamma(\mathbb{P}^1_{\kappa(X)}, \mathcal{O}_{\mathbb{P}^1_{\kappa(X)}}(d)) + \Gamma(\mathbb{P}^1_{\kappa(X)}, \mathcal{O}_{\mathbb{P}^1_{\kappa(X)}}(d))\varepsilon$. We say a pair $(g, h)$ is bad at $Y \in \mathbb{P}^1_{\kappa(X)}$ if $Y$ is a multiple root to $g$ and a root to $h$. Conversely, a pair can be bad at at most one point and $\deg Y = 1$. Let $e = \deg X = [\kappa(X) : \mathbb{F}_q]$. For each $Y$, there are $(q^{2e} - 1)$ choices for $f_{X,1}$, and $q^{3e}$ choices for $f_{X,2}$. Therefore $(q^e + 1)$ such points. Finally we add in those with $f_{X,1} = 0$, for which we can choose $f_{X,2}$ freely. In total in $\Gamma(\mathbb{P}^1_{\kappa(X)}, \mathcal{O}_{\mathbb{P}^1_{\kappa(X)}}(d)) + \Gamma(\mathbb{P}^1_{\kappa(X)}, \mathcal{O}_{\mathbb{P}^1_{\kappa(X)}}(d))\varepsilon$ we want to sieve out

$$(q^e + 1)(q^{2e} - 1)q^{3e} + q^{4e} = q^{6e} + q^{5e} - q^{3e}$$

The probability of $f$ being bad at the fiber $\mathrm{pr}_1^{-1}(X)$ is hence

$$1 - q^{-2e} - q^{-3e} + q^{-5e} = (1 - q^{-2e})(1 - q^{-3e})$$

Once we take product over all $X \in \mathbb{P}^1$ we get $\zeta_{\mathbb{P}^1_{\mathbb{F}_q}}(2)^{-1}\zeta_{\mathbb{P}^1_{\mathbb{F}_q}}(3)^{-1}$ Given this, we additionally want to sieve out those $f_{X,1}$ is a cube. Again $f_X$ cannot be "bad" at two points. For each $Y$ we have then $q^e - 1$ choices for $f_{X,1}$ and we may choose those $f_{X,2}$ that does not vanish at $Y$ since otherwise we have already sieved them out. This gives us $q^{4e} - q^{3e}$ choices for $f_{X,2}$. In total we sieve out

$$(q^e + 1)((q^{2e} - 1)q^{3e} + (q^e - 1)(q^{4e} - q^{3e})) + q^{4e} = 2q^{6e} - q^{4e}$$

The probability of being good at $\mathrm{pr}_1^{-1}(X)$ is hence

$$1 - 2q^{-2e} - q^{-4e} = (1 - q^{-2e})^2$$

Take product over $\mathbb{P}^1_{\mathbb{F}_q}$ we yield $\zeta_{\mathbb{P}^1_{\mathbb{F}_q}}(2)^{-2}$.

#### 2.1.3 A concrete example