A Bertini Density Theorem on Elliptic Surfaces

Ziquan Yang

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1 Introduction

Consider hypersurfaces in $\mathbb{P}^2 \times \mathbb{P}^1$ of bidegree $(3, d)$ over some base field k. These hypersurfaces are parametrized by $P = \mathbb{P}V$, where

$$V = H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3, d))$$

Let P^0 be the subscheme parametrizing those non-singular ones:

$$P^0 = \{f \in P : H_f \textit{ is non-singular }\}$$

Let $\pi : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$ be projection to the second component. If H_f is non-singular, then fibers of $\pi|_{H_f}$ are cubic curves in $\mathbb{P}^2$. This makes H_f an elliptic surface. The analogy of being simply ramified for H_f has to do with singular fibers of the map $\pi : H_f \to \mathbb{P}^1$. Smooth fibers are all isomorphic to the smooth cubic curve. It is customary for some literature to call an irredcucible cubic curve, together with a specified point as the base point, an elliptic curve. There are many types of singular fibers. If the fiber is irreducible, then it is either a nodal curve (e.g. $y^2 = x^2 - x^3$), or a cuspidal curve (e.g. $y^2 = x^3$). If the fiber is not irreducible, then it may be a union of a conic curve and a line, or a union of three lines and there are many different configurations of irreducible components. An analogue of a simply ramified curve would be a hypersurface whose singular fibers are all nodal curves, since nodal curves have the simplest type of singularity and lowest Euler characteristics among singular cubics. To simplify expression, we are going to refer to these hypersurfaces as having the "desired type". Hence we are primarily concerned with the following subset of P^0:

$$D = \{f \in P^0 : \textit{all singular fibers of } H_f \textit{ are nodal curves}\}$$

We will first show that the complement of D is a locally closed subscheme of codimension ≥ 1. Therefore when κ is algebraically closed, D is in fact a nonempty dense subset of $\mathbb{P}V$.

Then we deal with the case when the base field κ is a finite field $\mathbb{F}_q$. Instead of looking at subsets of $H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3,d))$ for a fixed d, we are going to study the asympotic density of hypersurfaces of the desired type as $d \to \infty$. Let $p = \operatorname{char} \mathbb{F}_q$. We will assume $p \neq 2, 3$. Let $R_{3,d} = H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3,d))$. In general, if $S \subseteq \bigcup_d R_{3,d}$, we say that the probability of $f \in S$ as $d \to \infty$ is:

$$\operatorname{Prob}(f \in S) = \lim_{d\to\infty} Prob(f_d \in S \cap R_{3,d}) = \lim_{d\to\infty} \frac{|S \cap R_{3,d}|}{|R_{3,d}|}$$

Let $\mathcal{D} \subseteq \bigcup_d R_{3,d}$ be the subset consisting of those f such that H_f is a smooth hypersurface in $\mathbb{P}^2 \times \mathbb{P}^1$ and has no singular fibers over $\mathbb{P}^1$ other than nodal curves. We will prove the following theorem:

Theorem 1.1. $\operatorname{Prob}(f \in \mathcal{D}) > 0$ *and hence when d is sufficiently large, there always exists some smooth hypersurface of bidegree $(3,d)$ of the desired type. Moreover, if we replace $\mathbb{F}_q$ by its extension $\mathbb{F}_{q^\alpha}$ and label $\mathcal{D}$ as $\mathcal{D}_\alpha$ to emphasize its dependence on the ground field, then*

$$\lim_{\alpha\to\infty} \operatorname{Prob}(f \in D_\alpha) = 1$$

For future use we introduce some notation. Let X be a scheme and $f : X \to Y$ be a morphism. For $P \in Y$, we denote the fiber $f^{-1}(P)$ by X_P when f is clear in the context. We denote the singular locus of X by X_{sing}.

2 Euler Characteristic and Singular Fibers

Let $f \in P^0$ and H_f be the corresponding hypersurface. Suppose H_f is non-singular. The Euler characteristic of H_f tells us that H_f will always contain some singular fibers with respect to the projection $\pi : H_f \to \mathbb{P}^1$. Let $C = \{P \in \mathbb{P}^1 : (H_f)_P \text{ } \textit{is singular} \}$, $Y^0 = H_f - \pi^{-1}(C)$ and $X^0 = \mathbb{P}^1 - C$. Let F denote a nonsingular cubic curve in $\mathbb{P}^2$, which is unique up to isomorphism. We see that Y^0 is a F-bundle of X^0, and their Euler characteristics are related by

$$e(Y^0) = e(X^0) \cdot e(F)$$

Since $e(F) = 0$ we have that $e(Y^0) = 0$. Now since $H_f = Y^0 \coprod \pi^{-1}(C)$, we see that $e(H_f) = e(Y^0) + e(\pi^{-1}(C)) = e(\pi^{-1}(C))$. Note that C is a finite set of points, and hence $\pi^{-1}(C)$ is a disjoint union of singular cubic curves.

From the following table we see that if $f \in D$, then $e(H_f)$ is exactly the number of points in $\mathbb{P}^1$ over which the fibers are singular.

Now we compute $e(H_f)$. In fact we do a more general computation, since it is not harder. Let $H_f \subseteq \mathbb{P}^2 \times \mathbb{P}^1$ be a non-singular hypersurface of bidegree

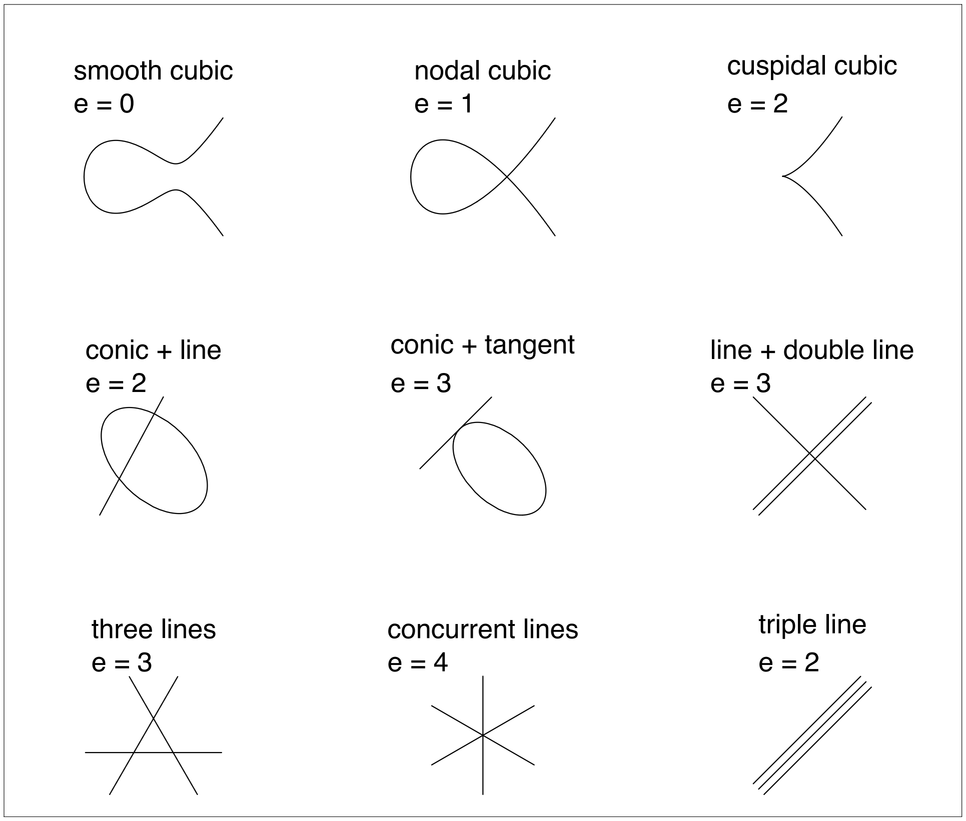

Figure 1: Cubic plane curves and their Euler characteristics [1]

(n, d). We denote the projections of $\mathbb{P}^2 \times \mathbb{P}^1$ onto the first and second components by π_1, π_2 respectively. Let H_1 be a hyperplane divisor of $\mathbb{P}^2$ and H_2 be a hyperplane divisor of $\mathbb{P}^1$. We define

$$A = \pi_1^*(H_1),\ E = \pi_2^*(H_2)$$

and

$$h_1 = A \cap H_f,\ h_2 = E \cap H_f$$

We compute the Chern classes of H_f using the normal exact sequence:

$$0 \to T_{H_f} \to T_{\mathbb{P}^2 \times \mathbb{P}^1|_{H_f}} \to N_{H_f/\mathbb{P}^2 \times \mathbb{P}^1} \to 0$$

By Whitney product formula, their total Chern classes are related by

$$c(T_{\mathbb{P}^2 \times \mathbb{P}^1|_{H_f}}) = c(T_{H_f})c(N_{H_f/\mathbb{P}^2 \times \mathbb{P}^1})$$

Now $T_{\mathbb{P}^2 \times \mathbb{P}^1|_{H_f}}$ is $T_{\mathbb{P}^2 \times \mathbb{P}^1} = \pi_1^* T_{\mathbb{P}^2} \oplus \pi_2^* T_{\mathbb{P}^1}$ restricted to H_f, so we obtain

$$c(T_{\mathbb{P}^2 \times \mathbb{P}^1|_{H_f}}) = (1 + 3h_1 + 3h_1^2)(1 + 2h_2)$$

If $\iota : H_f \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^1$ is the inclusion, then

$$N_{H_f/\mathbb{P}^2\times\mathbb{P}^1} = \iota^*\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^1}(nA + dE) = (1 + nh_1 + dh_2)$$

Therefore

$$\begin{aligned}
c(T_{H_f}) &= c(T_{\mathbb{P}^2\times\mathbb{P}^1|_{H_f}})c(N_{H_f/\mathbb{P}^2\times\mathbb{P}^1})^{-1} \\
&= (1 + 3h_1 + 3h_1^2)(1 + 2h_2)(1 + nh_1 + dh_2)^{-1} \\
&= (1 + 3h_1 + 3h_1^2)(1 + 2h_2)(1 - (nh_1 + dh_2) + (nh_1 + dh_2)^2 - \cdots)
\end{aligned}$$

In particular we obtain

$$c_2(T_{H_f}) = (n^2 - 3n + 3)h_1^2 + (6 + 2nd - 2n - 3d)h_1h_2$$

which is the top Chern class. Now we compute

$$\begin{aligned}
\deg h_1^2 &= A \cdot A \cdot (nA + dE) = d \\
\deg h_1h_2 &= A \cdot E \cdot (nA + dE) = n
\end{aligned}$$

Finally we obtain

$$e(H_f) = \deg c_2(T_{H_f}) = 6n + 3d + 3n^2d - 6dn - 2n^2$$

In particular, for future use we want to fix $n = 3$, so $e(H_f)$ depends only on d. In this case $e(H_f) = 12d$.

3 Dimension Counting

We use the standard affine covering $\mathbb{A}_i^2, i = 0, 1, 2$ on $\mathbb{P}^2$ and $\mathbb{A}_j^1, j = 0, 1$ on $\mathbb{P}^1$. Then $\mathbb{P}^2 \times \mathbb{P}^1$ can be covered by six affine charts $\mathbb{A}_i^2 \times \mathbb{A}_j^1$. Our plan is to construct subschemes $G_{ij} \subseteq \mathbb{P}^1 \times \mathbb{A}_i^2 \times \mathbb{A}_j^1 \times \mathbb{P}V$ for each i, j such that if H_f is non-singular and contains a singular fiber other than a nodal curve, then f is "hit" by some G_{ij} under the projection map $p : \mathbb{P}^1 \times \mathbb{A}_i^2 \times \mathbb{A}_j^1 \times \mathbb{P}V \to \mathbb{P}V$. In other words,

$$P^0 - D \subseteq \bigcup_{i,j} p(G_{ij})$$

Furthermore, we will show that each G_{ij} is of codimension 5 in $\mathbb{P}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{P}V$, so each $p(G_{ij})$, and hence $P^0 - D$, is of codimension at least 1 in $\mathbb{P}V$.

We will construct these G_{ij}'s in exactly the same manner for each i, j, except that we dehomogenize at different coordinates each time. Therefore from now on we concentrate on one of them and suppress the supscripts and G may denote any G_{ij}.

Consider the open subset $\mathbb{P}^2 \times \mathbb{A}^1 \subseteq \mathbb{P}^2 \times \mathbb{P}^1$. Let us denote its coordinates by $((x, y, z), t)$. After dehomogenization at $\mathbb{A}^1$, f can be written in the form

$$f = \sum_{0 \le i+j \le 3} f_{ij}(t) x^i y^j z^{3-i-j}$$

where

$$f_{ij} = \sum_{0 \le k \le d} a_{ijk} t^k$$

Fix $P \in \mathbb{A}$ and consider $f_P = \sum f_{ij}(P) x^i y^j z^{3-i-j} \in H^0(\mathbb{P}^2_{\kappa(P)}, \mathcal{O}(3))$. If $f_P \neq 0$, which is guranteed when H_f is non-singular, $\deg f_P = 3$. Note that $\{f_P = 0\}$ describes the fiber $(H_f)_P$ in $\mathbb{P}^2_{\kappa(P)}$.

If $(H_f)_P$ is singular at a point $Q \in \mathbb{P}^2_{\kappa(P)}$, then by degree considerations $\deg Q = 1$ over $\kappa(P)$. Now without loss of generality, we assume that $Q \in \mathbb{A}^2$, where $\mathbb{A}^2 \subseteq \mathbb{P}^2$ is the affine chart $\{z \neq 0\}$, and $Q = (0, 0) \in \mathbb{A}^2$ to illustrate some geometric observations.

We first dehomogenize f_P on $\mathbb{A}^2$ by setting $z = 1$ and donote $g := f(x, y, 1)$. Clearly $V_g := \{g = 0\} \subseteq \mathbb{A}^2$ is the restriction of $(H_f)_P$ on $\mathbb{A}^2$. By considering the partial derivatives of g at Q, we may partially classify the curve $\{f_P = 0\}$, i.e. the fiber $(H_f)_P$. In particular, we may tell the nodal curves from other types of singular curves.

We write g in the form:

$$g = g_0 + g_1 + g_2 + g_3$$

where g_i is homogeneous component of g of degree i. V_g is singular at Q if and only if $g_0 = g_1 = 0$. In the following discussion, by $\mathbb{P}^1$ we are thinking of $\operatorname{Proj} \kappa[x, y]$. When we say the tangent cone is something, we are talking about its picture in $V_g \times \overline{\kappa}$, so we can use Figure 1 as a reference.

1. If $g_2 \neq 0$ but as a homogeneous polynomial of degree 2 in x and y it vanishes twice at a point in $\mathbb{P}^1$, then the tangent cone of V_g at P is a double line. V_g might be a cusp, or a conic + a tangent.

2. If $g_2 \neq 0$ and has two distinct roots on $\mathbb{P}^1$ and $\{g_2 = 0\} \cap \{g_3 = 0\} \neq \emptyset$ as subschemes of $\mathbb{P}^1$, then the tangent cone of V_g at P is two lines, at least one of which is an irreducible component of V_g. In this case V_g might be a conic plus a line (not tangent to it).

3. If $g_2, g_3 \neq 0$, and g_2 has two distinct roots in $\mathbb{P}^1$, neither of which is root to g_3, then it must be a nodal curve.

4. If $g_2 = 0$ and $g_3 \neq 0$, the tangent cone of V_g at Q is a triple line, concurrent lines or a line plus a double line. V_g itself can be identified with its tangent cone at Q.

If Q is other point in $\mathbb{A}^2$, we can "shift it to the origin" by Taylor expansion and the effect is the same. Motivated by the above observations, now we begin to construct subschemes $G \subseteq \mathbb{P}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{P}V$. We write its coordinates as $((u:v),(x,y),t,f)$. Here we represent f in its dehomogenized form on the chart $\mathbb{A}^2 \times \mathbb{A}^1 \subseteq \mathbb{P}^2 \times \mathbb{P}^1$, i.e.

$$f = \sum_{0 \le i+j \le 3} f_{ij}(t) x^i y^j$$

to make sense of partial derivatives with respect to x and y.

Intuitively G is the subscheme parametrizing 5-tuples $((u_0:v_0),(x_0,y_0),t_0,f)$ where

a. $\{f_{t_0} = 0\}$ is singular at (x_0, y_0).

b. The tangent cone of $\{f_t = 0\}$ at (x_0, y_0) is a bounded line at $\{v_0(x - x_0) + u_0(y - y_0) = 0\}$.

c. $\{v(x - x_0) + u(y - y_0) = 0\}$ is contained in the tangent cone of $\{f_t = 0\}$ at (x, y) and is in fact an irreducible component of $\{f_{t_0} = 0\}$.

The image of G under the projection to $\mathbb{P}V$ contains those hypersufaces that contain singular fibers other than a nodal curve. Conversely, these hypersurfaces lie in the image of some G corresponding to the affine chart $\mathbb{A}^2 \times \mathbb{A}^1$. Since X is covered by finitely many $\mathbb{A}^2 \times \mathbb{A}^1$, we reduce to studying one such G.

We define $T^k_{(x,y)} f_t(u,v), 0 \le k \le 3$ to be terms in the Taylor expansion of f_t at (x,y):

$$T^k_{(x,y)} f_t(u,v) = \sum_{0 \le i \le k} \binom{k}{i} \frac{\partial^k f_t}{\partial x^i \partial y^j}(x,y) u^i v^{k-i}$$

Let $G_s \subseteq \mathbb{P}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{P}V$ be described by the following 3 conditions:

$$f_t(x,y) = 0; \ \frac{\partial f_t}{\partial x}(x,y) = 0; \ \frac{\partial f_t}{\partial y}(x,y) = 0$$

Note that G_s is independent of $(u:v)$. Now define $G_1 \subseteq G_s$ by appending two more conditions:

$$T^2_{(x,y)} f_t(u,v) = 0$$
$$T^3_{(x,y)} f_t(u,v) = 0$$

To define $G_2 \subseteq G_s$, we work affine-locally. On the affine chart $\mathbb{A}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{P}V$ where $u \neq 0$, we append two more linear conditions to G_s:

$$T^2_{(x,y)} f_t(1,v) = 0; \ \frac{\partial}{\partial v} T^2_{(x,y)} f_t(1,v) = 0$$

Finally we define $G = G_1 \cup G_2$. By dimension counting, we see that G_1 and G_2 are locally described by 5 equations, each of which decreases the dimension by 1. Therefore G is of codimension 5 in $\mathbb{P}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{P}V$.

Example Since we have written down quite complicated equations, an example may make the argument clearer. For simplicity, let $d = 1$. Suppose on an affine cover $\mathbb{A}^2 \times \mathbb{A}^1$, we may write down $f \in H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3,1))$ locally as

$$f(x, y, t) = x^3 - 3x^2 + 3x - ty^2 + 2ty - 2(2t - 1)$$

Below is the graph of the family of elliptic curves given by f near $t = 1$ over groud field $\kappa = \mathbb{R}$. As t increases, the curve shifts leftwards. We note that the

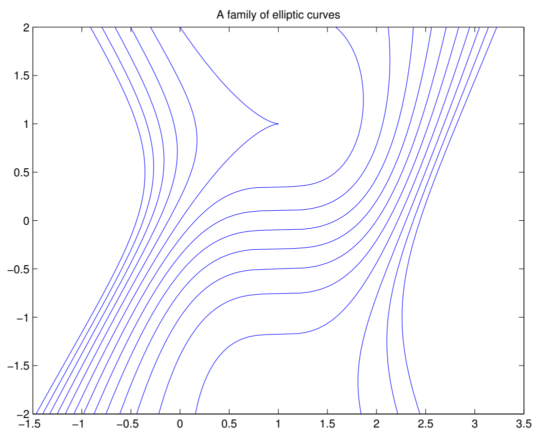

Figure 2: The family of elliptic curves near $t = 1$

fiber of H_f over $t = 1$ is described by

$$f(x, y, 1) = x^3 - 3x^2 + 3x - y^2 + 2y - 2$$

It is the cupsidal curve in the above graph. We check that the image of f in $\mathbb{P}V$, which we denote by $\overline{f}$, is contained in $p(G)$. We first compute

$$\begin{aligned} T^1_{(x,y)} f_t(u, v) &= (3x^2 - 6x + 3)u + (-2y + 2)v \\ T^2_{(x,y)} f_t(u, v) &= (6x - 6)u^2 - 2v^2 \\ T^3_{(x,y)} f_t(u, v) &= 6u^3 \end{aligned}$$

That $\{f(x, y, 1) = 0\}$ is singular at $(x - 1, y - 1)$ can be easily checked. Note that

$$T^2_{(x,y)} f_t(1, v) = (6x - 6) - 2v^2; \ \frac{\partial}{\partial v} T^2_{(x,y)} f_t(1, v) = (6x - 6) - 4v$$

both vanish at the point $(x-1, y-1, v)$. This implies that the tangent cone of $\{f(x, y, 1) = 0\}$ at $(x-1, y-1)$ is a doubled line. Together we see that $(v, x-1, y-1, t-1, \overline{f}) \in \mathbb{P}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{P}V$ lies in G. Therefore $\overline{f} \in p(G)$.

Remark 3.1. As we have seen, the copy of $\mathbb{P}^1$ in the product space $\mathbb{P}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{P}V$ is set up to parametrize the tangent lines through a point in $\mathbb{A}^2 \subseteq \mathbb{P}^2$. The reason that we insist on working affine-locally is really a topological one: We cannot parametrize pairs "(a point in $\mathbb{P}^2$, a tangent line through the point)" with $\mathbb{P}^2 \times \mathbb{P}^1$ as the projectivization of the tangent bundle of $\mathbb{P}^2$ is not a trivial $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.

4 Lemmas and Proof of Main Result

The above dimension counting arguments provide the basic strategy of our sieve method, so we will keeping using the notations defined in the previous section.

4.1 Lemmas

Let $P \in \mathbb{P}^1_{\mathbb{F}_q}$ be a point. We say that a hypersuface H_f, or sometimes the polynomial $f \in H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3, d))$ defining it, is *good* at the point P if H_f is smooth at all points $Q \in \pi^{-1}(P)$, and the fiber $(H_f)_P \subseteq \mathbb{P}^2_{\kappa(P)}$ is either smooth or a nodal curve. Otherwise, we say H_f, or f, is *bad* at P.

Consider the vector space $H^0(\mathbb{P}^2_{\mathbb{F}_{q^e}}, \mathcal{O}(3))^2$, we say that a pair (F_1, F_2) is good if, it satisfies one the following conditions:

1. F_1 describes a smooth curve in $\mathbb{P}_{\mathbb{F}_{q^e}}$.
2. F_1 describes a nodal curve but F_2 does not vanish at the node.

Lemma 4.1. *Let $P \in \mathbb{P}^1$ be a fixed point and let $e = \deg P$. Let $r(s) \in \mathbb{F}_q[s]$ be an irreducible polynomial such that the second infinitestimal neighborhood $P^{(2)} = \mathrm{Spec}\, \mathrm{F_q[s]/r(s)^2}$. Let $\tilde{\ }\colon \mathbb{F}_q[s]/r(s) \to \mathbb{F}_q[s]/r(s)^2$ be a section of the reduction map $\bar{\ }\colon \mathbb{F}_q[s]/r(s)^2 \to \mathbb{F}_q[s]/r(s)$. Then the map*

$$\varphi : H^0(\mathbb{P}^2_{\mathbb{F}_{q^e}}, \mathcal{O}(3))^2 \to H^0(\mathbb{P}^2 \times P^{(2)}, \mathcal{O}(3))$$

defined by

$$(F_1, F_2) \mapsto \widetilde{F_1} + r(s) F_2$$

is an isomorphism of vector spaces. Moveover, f is good at P if and only if the image of f in $H^0(\mathbb{P}^2_{\mathbb{F}_{q^e}}, \mathcal{O}(3))^2$ is a good pair.

Proof. This is essentially the same as Lemma 9.7 in [2]. For reader's convenience we give a proof. The above map has an inverse $f \mapsto (\overline{f}, (f - \widetilde{\overline{f}}))/r(s))$. For f to be good at P, we need that its image in $H^0(\mathbb{P}^2_{\kappa(P)}, \mathcal{O}(3))$, which ends up being F_1, to describe a smooth or nodal curve. Let $Q \in \mathbb{P}^2_{\mathbb{F}_{q^e}} \times P^{(2)}$ be a closed point. To check whether $f \in H^0(\mathbb{P}^2 \times P^{(2)}, \mathcal{O}(3))$ vanishes on $Q^{(2)}$, we check whether f and its partial derivatives all vanish at Q. We can replace the local parameter s by $r(s)$ and thus we are checking the simultaneous vanishing of $F_1, \operatorname{grad} F_1, F_2$. $F_1, \operatorname{grad} F_1$ only vanish simultaneously at the node if F_1 describes a nodal curve, and hence in order for f to not vanish on $Q^{(2)}$ it suffices to require that F_2 does not vanish at the node. $\square$

Lemma 4.2. *Let $P \in \mathbb{P}^1_{\mathbb{F}_q}$ be a point of degree e. Then map $\mathrm{ev}_P : H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3,d)) \to H^0(\mathbb{P}^2 \times P^{(2)}, \mathcal{O}(3))$ is surjective for $d \geq 2e+1$.*

Proof. Let $\mathbb{A}^1$ be an affine chart that contains P. We dehomogenize $f \in H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3,d))$ and write

$$f = \sum_{0 \leq i+j \leq 3} f_{ij}(t) x^i y^j$$

Suppose $r(t) \in \mathbb{F}_q[t]$ is a polynomial such that $P^{(2)} = \operatorname{Spec} \mathrm{\mathbb{F}_q[t]/r(t)}^2$. Then ev_P is nothing but reduce each coefficient modulo $r(t)^2$:

$$\mathrm{ev}_P(f) = \sum_{0 \leq i+j \leq 3} \overline{f_{ij}(t)} x^i y^j$$

where $\overline{f_{ij}(t)} \in \mathbb{F}_q[t]/r(t)^2$. Therefore by linear algebra, when $d \geq 2e+1$, we have that ev_P is surjective. $\square$

Lemma 4.3. *The fraction of good pairs in $H^0(\mathbb{P}^2_{\mathbb{F}_{q^e}}, \mathcal{O}(3))^2$ is lower bounded by $1 - cq^{-2e}$ for some c that is independent of e.*

Proof. In this proof schemes involved are assumed to be over $\mathbb{F}_{q^e}$. Let $D \subseteq \mathbb{P}^9 = \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))$ be the discriminant locus that parametrizes singular cubic curves. D is the image of

$$\widetilde{D} = \{(P, F) \in \mathbb{P}^2 \times \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3)) : F(P) = 0 \text{ and } \operatorname{grad} F|_P = 0\}$$

under the projection map to $\mathbb{P}^9$. It is not hard to observe that $\dim D = 8$. Let $D_{red}, D_{cusp}, D_{nod}$ be subschemes of D that parametrize reducible, cuspidal and nodal curves respectively. We will give an upper bound on the complement instead, i.e. pairs (F_1, F_2) that satisfies one of the following:

1. $F_1 \in D_{red} \cup D_{cusp}$.

2. $F_1 \in D_{nod}$ and F_2 vanishes at the node of F_1.

Since a homogeneous polynomial of degree 3 always factors into a linear term and a quadratic term if it is reducible, D_{red} is the image of a the natural map

$$\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(1)) \times \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(2)) \to \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3))$$

$\dim D_{red} \leq 7$ since $\dim \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(1)) = 2$ and $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(2)) = 5$. Now we consider D_{cusp}. D_{cusp} is evidently contained in the subscheme of D parametrizing those curves whose tangent cones are double lines somwhere. Just as in the proof of Theorem **??**, the tangent cone of $\{f = 0\}$ at a point (x, y) is a double line if $\{f = 0\}$ is singular at (x, y) and $T^2_{(x,y)}f(u, v)$ has a double root for some $(u : v)$. We may describe the last condition alternatively using the discriminant of a quadratic equation, without resorting to auxilliary variables $(u : v)$. Therefore, D_{cusp} is contained in the image under projection to D of the following subscheme of $\mathbb{P}^2 \times D$:

$$\widetilde{D}_{double} = \{(P, F) \in \mathbb{P}^2 \times D : (f_{xy}(P))^2 = (f_{xx}(P))(f_{yy}(P))\}$$

Note that in the equation $(f_{xy}(P))^2 = (f_{xx}(P))(f_{yy}(P))$, if $f_{xx}(P) \neq 0$, then the tangent cone is indeed a double line. If $f_{xx}(P) = 0$, then in fact f vanishes at the second infinitestimal neighborhood of P and the tangent cone is given by some homogeneous degree 3 polynomial in $\mathbb{F}_q[x, y]$. Therefore $\mathrm{proj}_D(\widetilde{D}_{double})$ in fact contains both cuspidal curves and some reducible curves, and hence

$$\mathrm{proj}_D \widetilde{D}_{double} \cup D_{red} = D_{red} \cup D_{cusp}$$

Note that $f \in H^0(\mathbb{P}^2, \mathcal{O}(3))$ corresponds to points of degree 1 in the projectivization $\mathbb{P}^9$ up to rescaling. $\dim D_{cusp}$ and $\dim D_{red}$ are each bounded by 7, and hence by Lang-Weil bound, there is a constant c', such that

$$|\{\text{points of degree } 1 \in D_{red} \cup D_{cusp} \subseteq \mathbb{P}^9\}| \leq c' q^{7e}$$

The number of degree 1 points in $\mathbb{P}^9$ is $\geq q^{9e}$. Therefore

$$\mathrm{Prob}(F_1 \in D_{red} \cup D_{cusp}) \leq c' q^{-2e}$$

Given the above discussion we see that $D_{nod} \subseteq D$ is open since $D_{nod} = D - D_{red} \cup D_{cusp}$. By Lang-Weil bound again, we see that $\mathrm{Prob}(F_1 \in D_{nod}) = c'' q^{-e}$ for some constant c''. Only points in $\mathbb{P}^2_{\mathbb{F}_{q^e}}$ of degree 1 could possibly be the node of some nodal curve. If some nodal curve $C \subseteq \mathbb{P}^2_{\mathbb{F}_{q^e}}$ has a node at some point P, $\deg P = r > 1$, then $C \times \mathrm{Spec}\, \mathbb{F}_{\mathrm{q}^{\mathrm{er}}}$ will be a nodal curve in $\mathbb{P}^2_{\mathbb{F}_q} \times \mathrm{Spec}\, \mathbb{F}_{\mathrm{q}^{\mathrm{er}}}$ with r nodes, counted with multiplicity, which is impossible. Conversely, every point of degree 1 is equally likely to be the node of some nodal curve. Given a point N of degree 1, the probability that a randomly chosen F_2 will vanish at N is q^{-e}, since the restriction map

$$H^0(\mathbb{P}^2_{\mathbb{F}_{q^e}}, \mathcal{O}(3)) \to H^0(Q, \mathcal{O}_Q)$$

is clearly surjective. Since F_1 and F_2 are chosen independently, we may conclude that

$$\mathrm{Prob}(F_1 \in D_{nod} \text{ and } F_2 \text{ vanishes at the node of } F_1) \leq c'' q^{-2e}$$

The desired conclusion follows by combining the two bounds. We may take $c = c' + c''$. It works for all e since c', c'' are both given by Lang-Weil bounds, which are independent of e. $\square$

Lemma 4.4. *Let $W \subseteq \mathbb{A}^N \times \mathbb{A}^M$ be a closed subscheme of dimension m. Let the coordinates of $\mathbb{A}^N \times \mathbb{A}^M$ be $(t_1, \cdots, t_N, s_1, \cdots, s_M)$ and let $S_{n,d}$ denote the polynomials that are of degree $\leq n$ in t_i's and degree $\leq d$ in s_j's. Let $\pi_1 : \mathbb{A}^N \times \mathbb{A}^M \to \mathbb{A}^N$, $\pi_1 : \mathbb{A}^N \times \mathbb{A}^M \to \mathbb{A}^M$ be the natural projections. Suppose $\pi_2(W)$ is a closed point P of degree e and let $r = \deg(\pi_2(W))$. Consider the restriction map*

$$\varphi : S_{n,d} \to H^0(W, \mathcal{O}_W)$$

Then for any $n, d \geq 0$, we have

$$|\mathrm{im}\, \varphi| \geq q^{\min(d+1, r)}$$

Proof. We have a commutative diagram of $\mathbb{F}_q$-algebra:

$$\begin{array}{ccc}
\mathbb{F}_q[t_1, \cdots, t_N, s_1, \cdots, s_M] & \longrightarrow & H^0(W, \mathcal{O}_W) \\
\uparrow & & \uparrow \\
\mathbb{F}_q[s_1, \cdots, s_M] & \longrightarrow & H^0(\pi_2(W), \mathcal{O}_{\pi_2(W)})
\end{array}$$

where both vertical arrows are injections and both horizontal arrows are surjections. Let $B_{n,i} = \mathrm{im}\, S_{n,i} \subseteq H^0(W, \mathcal{O}_W)$, the above diagram restricts to

$$\begin{array}{ccc}
S_{n,i} & \longrightarrow & B_{n,i} \\
\uparrow & & \uparrow \\
S_{0,i} & \longrightarrow & B_{0,i}
\end{array}$$

Since $B_{0,i+1} = \sum_j t_j B_{0,i} + B_{0,i}$, $B_{0,i}$ increases in dimension with each increase of i until it stabilizes to $H^0(\pi_2(W), \mathcal{O}_{\pi_2(W)})$. The statement follows since $\dim_{\mathbb{F}_q} B_{0,i} = i + 1$ and $\dim_{\mathbb{F}_q} H^0(\pi_2(W), \mathcal{O}_{\pi_2(W)}) = r$. $\square$

The above lemma is essentially half of lemma 5.3 in [2]. Note that we dropped the assumption that π_1 restricts to an isomorphism on W yet since the part of the lemma has not used this assumption and it is very important for us to drop it. To avoid confusion I copied the proof.

Lemma 4.5. *Let $j > 0$ be an integer. The probability that for a randomly chosen H_f there is some $Q \in \mathbb{A}^2$, such that $P = \pi(Q)$ has degree $e \geq j$ and*

$$((u : v), Q, P, f) \in G_1 \cup G_2$$

is at most

$$O((d^4)q^{-\min(\lfloor d/p \rfloor + 1, j)})$$

Proof. In this proof schemes involved are assumed to be over $\mathbb{F}_{q^e}$. We may work locally and consider $\mathbb{A}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \subseteq \mathbb{P}^1 \times \mathbb{A}^2 \times \mathbb{A}^1$. Let u be the coordiate of the first $\mathbb{A}^1$ component. Let $\pi : \mathbb{A}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \to \mathbb{A}^1$ be the projection to last component. We label the coordinates of $\mathbb{A}^1 \times \mathbb{A}^2 \times \mathbb{A}^1 \times \mathbb{A}^1$ by $(u, (x, y), t)$ and use $S_{l,n,d}$ to denote the set of polynomials in $\mathbb{F}_q[u, x, y]$ that are of degree $\leq l$ in u, $\leq n$ in x, y and $\leq d$ in t.

We first treat G_2. Following Poonen's decoupling idea, we write f in such a way that all the partial derivatives involved are largely independent. We choose $f_0 \in S_{0,3,d}$ and $g_{ij} \in S_{0,0,\lfloor d/p \rfloor}$ uniformly at random and put

$$f = f_0 + \sum_{0 \leq i+j \leq 2} g_{ij}^p x^i y^j$$

$$\begin{aligned}
W_1 &= \{2uf_{xx} + 2uf_{xy} = 0\} \\
W_2 &= W_1 \cap \{u^2 f_{xx} + 2uf_{xy} + f_{yy} = 0\} \\
W_3 &= W_2 \cap \{f_x = 0\} \\
W_4 &= W_3 \cap \{f_y = 0\} \\
W_5 &= W_4 \cap \{f = 0\}
\end{aligned}$$

□

Suppose f_0 is chosen. Then the dependence relations of W_i's on g_{ij}'s are shown below:

i	dependence set D_i
W_1	$g_{2,0}, g_{1,1}$
W_2	$g_{2,0}, g_{1,1}, g_{0,2}$
W_3	$g_{2,0}, g_{1,1}, g_{0,2}, g_{1,0}$
W_4	$g_{2,0}, g_{1,1}, g_{0,2}, g_{1,0}, g_{0,1}$
W_5	$g_{2,0}, g_{1,1}, g_{0,2}, g_{1,0}, g_{0,1}, g_{0,0}$

The above table reads: W_i is fixed by a choice of g_{ij}'s in D_i. We first show that for $1 \leq i \leq 4$ conditioned on a choice of D_i such that $\dim W_i \leq 4 - i$, the probability that a randomly chosen $g_{ij} \in D_{i+1} - D_i$ will make $\dim W_{i+1} > 4 - i - 1$ and contains a point Q for which $\deg \pi(Q) \geq j$ is at most

$$1 - O((d^i)q^{-\min(\lfloor d/p \rfloor + 1, j)})$$

We show the first two steps $i = 1, 2$, and the rest of steps are entirely similar.

i = 1: The equation defining W_1 breaks down to

$$u(f_{0,xx} + 2g_{2,0}) + (f_{0,xy} + g_{1,1}) = 0$$

It cuts off one dimension in $\mathbb{A}^1 \times \mathbb{A}^2 \times \mathbb{A}^1$ so long as it is not identically 0. We know that a polynomial in $\mathbb{F}_q[u, x, y]$ does not vanish identically on $\mathbb{A}^1 \times \mathbb{A}^2 \times \mathbb{A}^1$ unless it is zero. As the second term in the above equation contains no u, the above equation breaks down to two equations:

$$f_{0,xx} + 2g_{2,0} = 0; \; f_{0,xy} + g_{1,1} = 0$$

Therefore, since f_0 has been chosen, there is at most one choice of $g_{2,0}$ and $g_{1,1}$. (It is entirely possible that $f_{0,xx}, f_{0,xy} \notin S_{0,0,\lfloor n/p \rfloor}$) Therefore the probability that $\dim W_1 = 4$ is at most

$$\left(\frac{1}{q^{\lfloor d/p \rfloor}}\right)^2$$

i = 2: So now we suppose $g_{2,0}, g_{1,1}$ have been chosen such that $\dim W_1 \leq 3$. The equation defining W_2 in W_1 is

$$u^2 f_{xx} + 2u f_{xy} + f_{yy} = 0$$

which breaks down to

$$u^2(f_{0,xx} + 2g_{2,0}) + 2u(f_{0,xy} + g_{1,1}) + (f_{0,yy} + 2g_{0,2}) = 0$$

Let $V_1, \cdots, V_l$ be the irreducible components of $(W_1)_{\text{red}}$. By Bezout's theorem, $l \sim O(d)$. The above equation fails to cut off one dimension in W_1 if and only if for some V_k, LHS vanishes identically on V_k (denote the probability of this even by $\mathcal{P}_k$). Therefore the set of $g_{0,2}$ that will make LHS vanishes identically on V_k forms a coset of the restriction map

$$\varphi_k : S_{0,0,\lfloor d/p \rfloor} \to H^0(V_k, \mathcal{O})$$

We will give a lower bound on $|\operatorname{im} \varphi_k|$, and thus an upper bound on $\mathcal{P}_k = |\operatorname{im} \varphi_k|^{-1}$.

If $\dim \pi(V_k) \geq 1$, then t, and hence any nonzero polynomial in t, does not vanish on V_k, and so $|\mathrm{im}\,\varphi_k| \geq q^{\lfloor d/p \rfloor+1}$. If $\dim \pi(V_k) = 0$, then by lemma 4.5,

$$|\mathrm{im}\,\varphi_k| \geq q^{\min(\lfloor d/p \rfloor+1, \deg(\pi(V_k)))} \geq q^{\min(\lfloor d/p \rfloor+1, j)}$$

The conclusion of this step follows since

$$\prod_{k=1}^{l}(1-\mathcal{P}_k) \sim 1 - \mathcal{O}(d^2 q^{-\min(\lfloor d/p \rfloor+1, j)})$$

The steps $i = 3, 4$ are similar to $i = 2$.

Now suppose we have chosen g_{ij}'s in D_4 such that $\dim W_4 = 0$. There are $O(d^4)$ points by previous steps. We are to choose $g_{0,0}$ such that $\{f = 0\} \cap W_4$ contains no point Q, such that $\deg \pi(Q) \geq j$. For each such Q, let $\varphi : S_{0,0,\lfloor d/p \rfloor} \to H^0(Q, \mathcal{O})$ be the evaluation map, then by Lemma 4.5 again

$$|\mathrm{im}\,\varphi| \geq q^{\min(\lfloor d/p \rfloor+1, j)}$$

and the conclusion follows.

Finally we treat G_1. This time we decouple f as follows:

$$f = f_0 + \sum_{0 \leq i+j \leq 3} g_{ij}^p x^i y^j$$

and again $f_0 \in S_{0,3,d}$ and $g_{ij} \in S_{0,0,\lfloor d/p \rfloor}$ Like G_2, G_1 is also filtrated by $W_1, \cdots, W_5$ except W_1, W_2 are replaced by the following

$$\begin{aligned} W_1 &= \{f_{xxx}u^3 + 3f_{x^2y}u^2 + 3f_{xy^2}u + f_{yyy} = 0\} \\ W_2 &= \{u^2 f_{xx} + 2u f_{xy} + f_{yy} = 0\} \end{aligned}$$

Fixing f_0, the table of dependence relations of W_i's on g_{ij}'s for G_1 is:

i	dependence set D_i
W_1	$g_{3,0}, g_{2,1}, g_{1,2}, g_{0,3}$
W_2	$g_{3,0}, g_{2,1}, g_{1,2}, g_{0,3}, g_{2,0}, g_{1,1}, g_{0,2}$
W_3	$g_{3,0}, g_{2,1}, g_{1,2}, g_{0,3}, g_{2,0}, g_{1,1}, g_{0,2}, g_{1,0}$
W_4	$g_{3,0}, g_{2,1}, g_{1,2}, g_{0,3}, g_{2,0}, g_{1,1}, g_{0,2}, g_{1,0}, g_{0,1}$
W_5	$g_{3,0}, g_{2,1}, g_{1,2}, g_{0,3}, g_{2,0}, g_{1,1}, g_{0,2}, g_{1,0}, g_{0,1}, g_{0,0}$

The situation is entirely analogous to that of G_1 except we need take some special care of the second step, in which we appended three g_{ij}'s at once. We suppose D_1 has been chosen such that $\dim W_1 \leq 3$. Let $V_1, \cdots, V_l$ be irreducible components of $(W_1)_{\mathrm{red}}$ and again by Bezout' theorem $l \sim \mathcal{O}(d)$.

We have the following:

$$
\begin{aligned}
f_{xx} &= f_{0,xx} + 2g_{2,0}^p + 2g_{2,1}^p y + 6g_{3,0}^p x \\
f_{yy} &= f_{0,yy} + 2g_{0,2}^p + 2g_{1,2}^p x + 6g_{3,0}^p y \\
f_{xy} &= f_{0,xy} + 2g_{1,1}^p + 2g_{2,1}^p x + 2g_{1,2}^p y
\end{aligned}
$$

If for some V_k that contains a point Q with $\deg Q \geq j$,

$$u^2 f_{xx} + 2u f_{xy} + f_{yy}$$

vanishes identically on V_k, then the set of the tuples $(g_{0,2}, g_{1,1}, g_{2,0})$ forms a coset, after some rescaling, of the kernel of the map φ:

$$\varphi : \oplus_{r=0}^{i} S_{0,0,\lfloor n/p \rfloor} \xrightarrow{\psi} S_{2,0,\lfloor n/p \rfloor} \to H^0(V_k, \mathcal{O})$$

where the intermediate map ψ is given by

$$(h_1, h_2, h_3) \mapsto u^2 h_1 + u h_2 + h_3$$

It is not hard to observe that ψ is surjective, therefore we may reduce to studying the second map $S_{2,0,\lfloor n/p \rfloor} \to H^0(V_k, \mathcal{O})$. By a similar argument as before, the size of the image is bounded by $q^{\min(\lfloor d/p \rfloor + 1, j)}$ and the rest of the arguments follows.

4.2 Proof of Main Result

For a fixed $e_0 \in \mathbb{N}$, we define

$$
\begin{aligned}
\mathcal{P}_{e_0}^{\text{low}} &= \bigcup_{d \geq 0} \{f \in R_{n,d} : H_f \textit{ is good at all } P \in \mathbb{P}^1_{\mathbb{F}_q}, \deg P < e_0\} \\
\mathcal{Q}_{e_0}^{\text{med}} &= \bigcup_{d \geq 0} \{f \in R_{n,d} : H_f \textit{ is bad at some } P \in \mathbb{P}^1_{\mathbb{F}_q}, \deg P \in [e_0, \lfloor d/p \rfloor]\} \\
\mathcal{Q}^{\text{high}} &= \bigcup_{d \geq 0} \{f \in R_{n,d} : H_f \textit{ is bad at some } P \in \mathbb{P}^1_{\mathbb{F}_q}, \deg P \in > d/p\}
\end{aligned}
$$

We prove the following three lemmas:

Lemma 4.6.

$$\operatorname{Prob}(f \in \mathcal{P}_{e_0}^{\text{low}}) = \prod_{\deg P < e_0} \operatorname{Prob}(f \textit{ is good at } P)$$

Proof. Whether f is good at P is decided by its image in $H^0(\mathbb{P}^2 \times P^{(2)}, \mathcal{O}(3))$. Suppose $P_1, \cdots, P_s$ are the points with degree $< e_0$. By a similar linear-algebraic argument as in the proof of Lemma 4.2, we see that if

$$n \geq \sum_{i=1}^{s} 2\deg P_i + 1$$

then $H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3,n)) \to \oplus_{i=1}^{s} H^0(\mathbb{P}^2 \times P_i^{(2)}, \mathcal{O}(3))$ is surjective. Hence the conclusion follows. □

Lemma 4.7.

$$\lim_{e_0 \to \infty} \mathrm{Prob}(f \in \mathcal{Q}_{e_0}^{\mathrm{med}}) = 0$$

Proof. By Lemma 4.2, when $\deg P \in [e_0, \lfloor d/p \rfloor]$ we still have that

$$H^0(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(3,d)) \to H^0(\mathbb{P}^1_{q^e} \times P^{(2)}, \mathcal{O}(3)) \cong H^0(\mathbb{P}^1_{q^e}, \mathcal{O}(3))^2$$

is surjective since we have assumed $p > 2$ and $H^0(\mathbb{P}^1_{q^e} \times P^{(2)}, \mathcal{O}(3))$ is enough for us to decide whether H_f is good at P. In Lemma 4.3 we have already given an upper bound on bad pairs, which implies that H_f is bad at P. Therefore

$$\frac{|\mathcal{Q}_{e_0}^{\mathrm{med}} \cap S_{3,d}|}{|S_{3,d}|} \leq \sum_{e=e_0}^{\lfloor d/p \rfloor} |\mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^e})|(cq^{-2e}) \leq \sum_{e=e_0}^{\infty} cq^{-e}$$

Clearly it converges to zero as $e_0 \to \infty$. □

Lemma 4.8.

$$\mathrm{Prob}(f \in \mathcal{Q}^{\mathrm{high}}) = 0$$

Proof. Apply Lemma 4.5 with $j = \lfloor d/p \rfloor$. Then push $d \to \infty$. □

For each e_0, $D \subseteq \mathcal{P}_{e_0}^{\mathrm{low}} \cup \mathcal{Q}_{e_0}^{\mathrm{med}} \cup \mathcal{Q}^{\mathrm{high}}$. Therefore, the above three lemmas combine to give that

$$\mathrm{Prob}(f \in D) = \prod_{P \in \mathbb{P}^1_{\mathbb{F}_q}} \mathrm{Prob}(f \textit{ is good at } P)$$

Finally we treat the convergence of products. We invoke the following well known lemma:

Lemma 4.9. *Suppse that $p_e, e = 1, 2, \cdots$ satisfy $0 \leq p_e < 1$ and $\sum p_e < \infty$, then $\prod (1 - p_e) > 0$.*

We apply the lemma to

$$p_e = \text{Prob}(f \textit{ is bad at some } P \textit{ of degree } e) = 1 - \prod_{P \in \mathbb{P}^1_{\mathbb{F}_q}} \text{Prob}(f \textit{ is good at } P)$$

By lemma 4.3, we can bound p_n by

$$p_e \leq \sum_{\deg P = e} |\mathbb{P}^1_{\mathbb{F}_q}(\mathbb{F}_{q^e})|(cq^{-2e}) \leq \sum_{\deg P = e} (cq^{-e})$$

There are $q^e + 1$ points instead of q^e points of degree e in the projective line when $e = 1$, but we can extend c slightly larger if needed. Therefore $\sum_e p_e < \infty$. Clearly at each $P \in \mathbb{P}^1_{\mathbb{F}_q}$, there are some hypersurface that is good at P. Therefore, none of the terms $(1 - p_e)$ is zero. We may now conclude that $\text{Prob}(f \in D) > 0$. Finally we note that

$$\prod_{e=1}^{\infty}(1 - p_e) \geq 1 - \sum_{p=1}^{\infty} p_e \geq 1 - c\sum_{e=1}^{\infty} q^{-e} = 1 - \frac{cq^{-1}}{1 - q^{-1}}$$

If we replace $\mathbb{F}_q$ with $\mathbb{F}_{q^\alpha}$, then the last term is replaced with

$$1 - \frac{cq^{-\alpha}}{1 - q^{-\alpha}}$$

which clearly converges to 1 as $\alpha \to \infty$.

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