Notes on Riemann Surfaces

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Suppose $U \subseteq \mathbb{C}^n$ be an open subset. We want to decide when f is holomorphic. We start by looking at the tangent bundles. Let $z_j = x_j + iy_j : U \to \mathbb{C}$ and $x_j, y_j : U \to \mathbb{R}$ are sections of T^*U. Here by T^*U we mean the real tangent space of $U \subseteq \mathbb{C}^n \cong \mathbb{R}^{2n}$. Similarly we want to say $dz_j = dx_j + idy_j$ is a section of $T^*U \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Hom}_{\mathbb{R}}(TU, \mathbb{C})$. We also have $d\overline{z} = dx_j - idy_j$ as sections of

$T^*U \otimes_{\mathbb{R}} \mathbb{C}$. We may often omit the subscript $\mathbb{R}$, but it is important to keep in mind that when we are dealing with these situations $\otimes_{\mathbb{R}}$ is assumed. Clearly dx, dy and $dz, d\overline{z}$ span the same space, since

$$[dz, d\overline{z}] = [dx, dy] \begin{bmatrix} 1 & 1 \\ i & -1 \end{bmatrix}$$

and matrix has determinant $-2i$. We may either use basis dx_j, dy_j's or $dz_j, d\overline{z}_j$'s. For the latter we set the dual basis

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}),\ \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$$

Now we may write

$$df = \sum_{j=1}^{n}(\frac{\partial f}{\partial z_j}dz_j + \frac{\partial f}{\partial \overline{z}_j}d\overline{z}_j)$$

We would like to endow the bundle TU with the structure of a complex vector bundle, so we need to give an action of $\mathbb{C}$ on fibers. We would like to say

$$i\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j}$$

We define $J : TU \to TU$ by

$$J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j},\ J(\frac{\partial}{\partial y_j}) = -i\frac{\partial}{\partial x_j}$$

Note that $J^2 = -1$. Hence we obtain an action of $\mathbb{C} \cong \mathbb{R}[J]/(J^2+1)$ on $V_{\mathbb{R}}$, where $V_{\mathbb{R}}$ is a fiber of TU.

Lemma 1.1. *Let V be a finite dimensional real vector space and $J : V \to V$ be a $\mathbb{R}$-linear map satisfying $J^2 = -1$. Then $V \otimes \mathbb{C} = V^1 \oplus V^2 \cong V \oplus iV$, where*

$$V' = \{v - iJv : v \in V\}$$
$$V'' = \{v + iJv : v \in V\}$$

Moreover $J|_{V'} = i, J|_{V''} = -i$.

Proof. Straightforward verification. □

Remark 1.2. Let $J : V \to V$ be as above. We want to define a conjugate operation $V \otimes \mathbb{C} \to V \otimes \mathbb{C}$ given by $a + ib \mapsto a - ib$, where $a, b \in V$. Note that $\overline{V'} = V''$.

Example Consider a fiber T_pU. Suppose we have already defined J. Then we can write $T_pU\otimes\mathbb{C} = T'_pU \oplus T''_pU$, where J acts on the first component by i and the second by $-i$. T'_pU is spanned by

$$\frac{\partial}{\partial x_j} - iJ(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j} = 2\frac{\partial}{\partial z_j}$$

and T''_pU is spanned by

$$\frac{\partial}{\partial x_j} + iJ(\frac{\partial}{\partial y_j}) = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j} = 2\frac{\partial}{\partial \overline{z}_j}$$

Lemma 1.3. *The composite map $\varphi' : V \to V\otimes\mathbb{C} \cong V' \oplus V'' \to V'$ is complex-linear. Similarly, the composite map $\varphi'' : V \to V''$ is conjugate-linear.*

Proof. Recall that i acts on V as J, so being complex linear just means

$$\varphi'(Jv) = \frac{1}{2}(Jv - iJ^2v) = \frac{1}{2}(Jv + iv) = \frac{1}{2}i(v - iJv) = i\varphi'(v)$$

□

Example Again we look at how this applies to a fiber T_pU. Consider $\varphi' : T_pU \to T'_pU$. We have

$$\frac{\partial}{\partial x_j} \mapsto \frac{\partial}{\partial z_j}$$
$$\frac{\partial}{\partial y_j} \mapsto \frac{\partial}{\partial \overline{z}_j}$$

J serves to permute the vectors in both columns.

1.2.2 Holomorphic functions

Definition 1.4. A smooth function $f : U \to \mathbb{C}$ is holomorphic at $p \in U$ if

$$Df : T_pU \to T_p\mathbb{C} = \mathbb{C}$$

is $\mathbb{C}$-linear with respect to J.

Equivalently, we could also require that the induced map $T_pU\otimes_{\mathbb{R}}\mathbb{C} \to T_p\mathbb{C}\otimes_{\mathbb{R}}\mathbb{C}$ commutes with J.

Note that $J : V \to V$ induces a natural map $J : V^* \to V^*$. J is self-ajoint in that

$$\langle J\varphi, v\rangle = \langle \varphi, Jv\rangle, \forall\varphi \in V^*, v \in V$$

which can be verified by straigtforward computation. We let J to act on the cotangent spaces as well, so by Lemma 1.1 we have a decomposition

$$T_p^*U \otimes \mathbb{C} = T_p^{1,0}U \oplus T_p^{0,1}U$$

where J acts on $T_p^{1,0}U$ as i and on $T_p^{0,1}U$ as $-i$. $T_p^{1,0}U$ is spanned by $dz_1, \cdots, dz_n$ and $T_p^{0,1}U$ is spanned by $d\overline{z}_1, \cdots, d\overline{z}_n$. Looking back to f, we see that Df_p pulls $dw \in T^*_{f(p)}\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ to $T_p^{1,0}U$ and $d\overline{w}$ to $T_p^{0,1}U$.

Write $w = f(z_1, \cdots, z_n)$. We see that

$$dw \mapsto df = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$$

and similarly $d\overline{w} = d\overline{f}$. Recall that d is a real operator, so $d\overline{f} = \overline{df}$. Now we make an important observation:

$$\begin{aligned} (Df)^* \textit{ commutes with } J &\iff Df^*(dw) \textit{ has type } (1,0) \\ &\iff df \textit{ has type } (1,0) \\ &\iff \frac{\partial f}{\partial \overline{z}_j} = 0,\ j = 1, \cdots, n \end{aligned}$$

Example Let us look at the case $n = 1$. Write $f = u + iv$, where $u, v : U \to \mathbb{C}$. Recall that

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

and hence

$$2\frac{\partial f}{\partial \overline{z}} = (u_x - v_y) + i(u_y + v_x)$$

Therefore f is holomorphic in our new definition if and only if it satisfies the Cauchy-Riemann equations. We can also readily make the following observation:

Lemma 1.5. *$f : U \to \mathbb{C}$ is holomorphic if and only if it is holomophic in each coordinate, i.e. for all $(a_1, \cdots, a_n) \in U$ and $j = 1, \cdots, n$ the function*

$$z \mapsto f(a_1, \cdots, a_{j-1}, z, a_{j+1}, \cdots, a_n)$$

is holomorphic.

We say a function is *analytic* if at each point it is given by a convergent power series. If $n = 1$, then it is well know that $f : U \to \mathbb{C}$ is holomorphic if and only if it is analytic. We would like to extend this property to higher dimensions. Let $\Delta(z_j)$ denote copy of open disk of radius r_j whose coordinate is labelled z_j. We have the following theorem.

Theorem 1.6. *A smooth function* $f : \Delta(z_1) \times \cdots \times \Delta(z_n) \to \mathbb{C}$ *is holomorphic if and only if there is a power series*

$$\sum_K c_K z_1^{k_1} \cdots z_n^{k_n},\ K = (k_1, \cdots, k_n) \in \mathbb{N}^n$$

that converges to $f(z_1, \cdots, z_n)$.

Proof. Suppose f is holomophic on U. Choose $(s_1, \cdots, s_n) \in \mathbb{R}^n_{>0}$ such that $|z_j| < s_j < r_j$. Then

$$\begin{aligned} f(z_1, \cdots, z_n) &= \frac{1}{2\pi i} \int_{|w_n|=s_n} \frac{f(z_1, \cdots, z_{n-1}, w_n)}{w_n - z_n} dw_n \\ &= (\frac{1}{2\pi i})^2 \int_{|w_n|=s_n} \int_{|w_{n-1}|=s_{n-1}} \frac{f(z_1, \cdots, w_{n-1}, w_n)}{(w_n - z_n)(w_{n-1} - z_{n-1})} dw_{n-1} dw_n \\ &= \cdots \\ &= (\frac{1}{2\pi i})^n \int \int \cdots \int \frac{f(w_1, \cdots, w_n)}{\prod_{j=1}^n (w_j - z_j)} dw_1 \cdots dw_n \end{aligned}$$

Therefore we only need to expand the denominators:

$$\frac{1}{w_j - z_j} = \frac{1}{w_j} \sum_{n=0}^{\infty} \frac{1}{w_j^n} z_j^n$$

The converse is straightforward by Lemma 1.5. $\square$

1.2.3 Differential forms

Let M be a manifold. We can form the kth exterior product $\bigwedge^k T^*M$ and let $E^k(M)$ be the set of C^∞ sections. Linear algebra says

$$\bigwedge^k (T^*U \otimes \mathbb{C}) = \bigoplus_{p+q=k} (\bigwedge^p T^{1,0}U) \otimes (\bigwedge^q T^{0,1}U)$$

Therefore $E^k_{\mathbb{C}}(U) = \{$*sections of* $T^*U \otimes_{\mathbb{R}} \mathbb{C}\}$,

$$E^k_{\mathbb{C}}(U) = \bigoplus_{p+q=k} E^{p,q}(U)$$

$E^{**}(U)$ is a bigraded algebra with operators $\partial, \overline{\partial}$. We define a map

$$d : E^0(X) \to E^{1,0} \oplus E^{0,1}$$

by $df = \partial f + \overline{\partial} f$, where

$$\partial f = \sum_{j=1}^n \frac{\partial f}{\partial z_j} dz_j,\ \overline{\partial} f = \sum_{j=1}^n \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j$$

Note that $f : X \to \mathbb{C}$ is a priori, only a C^∞ function. It is holomorphic if and only if $\overline{\partial} f = 0$. $\partial, \overline{\partial}$ extend to maps $E^{p,q}(X) \to E^{p+1,q}(X), E^{p,q}(X) \to E^{p,q+1}(X)$ and they still satisfy $\partial^2, \overline{\partial}^2 = 0$.

Theorem 1.7. *Every complex manifold has a natural orientation.*

Definition 1.8. A holomorphic 1-form on a Riemann surface X is a closed 1-form of type $(1, 0)$.

At the first spot of the definition, I find the requirement that the form must be closed a bit weird, but the following lemma tells us that closedness is exactly describing the condition of being "locally holomorphic".

Lemma 1.9. *Let $f(z)dz$ be a $(1, 0)$-form on the disk Δ. TFAE:*

a. $f(z)dz$ is holomorphic.

b. $f(z)$ is holomorphic.

c. $\overline{\partial}(f(z)dz) = 0$.

Proof.

$$d(fdz) = df \wedge dz = (\frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}) \wedge dz = \frac{\partial f}{\partial \overline{z}}d\overline{z} \wedge dz = 2i\frac{\partial f}{\partial \overline{z}}dx \wedge dy$$

Therefore

$$d(f(z)dz) = 0 \iff \frac{\partial f}{\partial \overline{z}} = 0 \iff f \text{ is holomorphic}$$

To show $a \iff c$, we first show that $\overline{\partial}(dz) = 0$. Since $d^2 z = 0$, we know that $(\partial + \overline{\partial})dz = 0$. $\partial(dz)$ is a $(2, 0)$ form, but there is none. Therefore $\overline{\partial}(dz) = 0$. Now

$$\overline{\partial}(fdz) = \overline{\partial} f \wedge dz = \frac{\partial f}{\partial \overline{z}}d\overline{z} \wedge dz = d(fdz)$$

□

Remark 1.10. More generally, on a Riemann surface X, if $f : X \to \mathbb{C}$ is a holomorphic function, then df is a holomorphic 1-form.

Example Label the coordinates of $\mathbb{C}^2$ by (x, y). Let $X \subseteq \mathbb{C}^2$ be the curve defined by $y^2 = p(x)$, where $p(x)$ is some square-free polynomial. Suppose the degree of $p(x)$ is $2g + 1$. We claim that

$$\frac{dx}{y}, \frac{xdx}{y}, \cdots, \frac{x^{g-1}dx}{y}$$

Note that $x : X \to \mathbb{C}$ is a holomorphic function, so dx is a holomorphic 1-form. Let $X' = X - y^{-1}(0)$. If

$$p(x) = \prod_{j=0}^{2g} (x - a_j)$$

$y^{-1}(0) = \{(a_j, 0), j = 0, 1, \cdots, 2g\}$. Clearly $x^n dx/y$ is holomorphic on X' for all $n \geq 0$, so we are reaally trying to show that these forms are holomorphic at each $(a_j, 0)$. Without loss of generality, we assume $a_0 = 0$.

2 Theory of Divisors

Let M be a C^∞ manifold, $f : M \to \mathbb{C}$ be a function. We write $f = u + iv$. $df = du + idv$ is a $\mathbb{C}$-valued 1-form. It is a section of

$$\mathrm{Hom}_{\mathbb{R}}(TM, \mathbb{C}) = \mathrm{Hom}_{\mathbb{C}}(TM \otimes \mathbb{C}, \mathbb{C}) \cong T^*M \otimes \mathbb{C} \cong T^*M \oplus iT^*M$$

We want to define $L(D) = \{f \in \mathfrak{M}(X) : (f) + D \geq 0\} \cup \{0\}$. Notation: $\mathrm{Div}(0) = \sum_p \infty[p]$ and hence $\mathrm{Div}(0) \geq D$ for all D.

Proposition 2.1. *Suppose $D \in \mathrm{Div}(X)$. Then $f \cdot s(D) \in H^0(X, \mathcal{L}_D)$ if and only if $f \in L(D)$.*

Proposition 2.2. *If X is compact and $\deg D = 0$, then $L(D) \neq 0$ if and only if D is principal.*