Direction-of-Arrival Estimation by Subspace Rotation Methods – *ESPRIT*[1]

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Abstract—Results of simulations comparing the performance of *ESPRIT* (Estimation of Signal Parameters via Rotational Invariance Techniques) and the MUSIC (MUltiple SIgnal Classification) algorithm are presented. *ESPRIT* exploits an underlying rotational invariance among signal subspaces induced by an array of sensors with a translational invariance structure. In contrast, the MUSIC algorithm uses intersections between the array manifold and the signal subspace to estimate the directions. *ESPRIT* is shown to have performance advantages over MUSIC in certain scenarios apart from its previously reported implementational advantages.

I. INTRODUCTION

High resolution direction-of-arrival (DOA) estimation is important in many applications and over the years, many techniques have been proposed. Of the currently favored techniques, the signal subspace method (MUSIC) due to Schmidt [1] has been the most popular and is known to yield asymptotically unbiased and efficient estimates. The MUSIC algorithm first estimates the so-called *signal subspace* from the array measurements, then estimates the parameters of interest from the intersections between the *array manifold* and the estimated signal subspace.

A new approach (*ESPRIT*) to the signal parameter estimation problem was recently proposed in [2,3]. *ESPRIT* is similar to MUSIC in that it exploits the underlying data model and generates estimates that are asymptotically unbiased and efficient. In addition, it has several important advantages over MUSIC.

- The algorithm has improved performance over that of MUSIC in certain scenarios.
- The algorithm does not require knowledge of the array geometry and element characteristics; thus array calibration is not required, eliminating the need for the associated storage of the array manifold.
- It is computationally less complex since it does not need the search procedure inherent in MUSIC (and other algorithms).

However, restrictions to planar wavefronts and pairwise matched co-directional doublets are present, and in this sense, *ESPRIT* is not completely general. In the interest of completeness, a brief discussion of the details of *ESPRIT* is presented after which a description of the simulations performed and the results obtained is given.

II. PROBLEM FORMULATION

The basic problem under consideration is that of estimation of parameters of finite dimensional signal processes given measurements from an array of sensors. Though the technique is quite general, herein specific attention is given to the problem of direction-of-arrival (DOA) estimation. The application of *ESPRIT* to estimation of frequencies of cisoids is discussed in [4].

Consider a planar array of arbitrary geometry composed of m matched sensor doublets whose elements are translationally separated by a known constant displacement vector as shown in Figure 1. The element characteristics may be arbitrary for each doublet as long as the elements are *pairwise identical.* Assume there are $d < m$ narrowband stationary zero-mean sources centered at frequency ω_0 located sufficiently far from the array such that the wavefronts are planar. Additive noise is present at all $2m$ sensors and is assumed to be a stationary zero-mean random process uncorrelated from sensor to sensor.

It is convenient to describe the array as being comprised of two identical subarrays, X and Y, displaced by a known vector. The signals received at the i^{th} doublet can then be expressed as:

$$\begin{aligned} x_i(t) &= \sum_{k=1}^{d} s_k(t) a_i(\theta_k) + n_{x_i}(t) \,, \\ y_i(t) &= \sum_{k=1}^{d} s_k(t) e^{j\omega_0 \Delta \sin\theta_k / c} a_i(\theta_k) + n_{y_i}(t) \,; \end{aligned} \tag{1}$$

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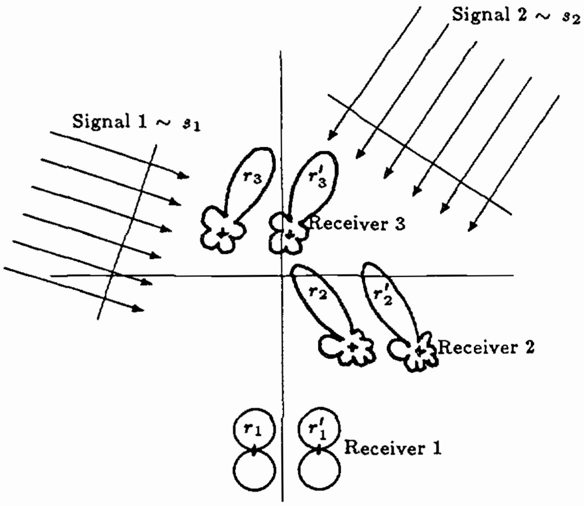

Figure 1: Multiple Source DOA Estimation using *ESPRIT*

where $s_k(\cdot)$ is the k^{th} signal (wavefront) as received at sensor 1 (the reference sensor) of the X subarray, and θ_k is the direction of arrival of the k^{th} source relative to the direction of the translational displacement vector whose magnitude is Δ. The response of the i^{th} sensor of either subarray (to a single wavefront impinging at an angle θ_k) relative to its response at sensor 1 of the same subarray is $a_i(\theta_k)$. The speed of wave propagation is c, and $n_{x_i}(\cdot)$ and $n_{y_i}(\cdot)$ are the additive noises at the elements of the X and Y subarrays.

Combining the outputs of each of the sensors in the two subarrays, the received data vectors can be written as follows:

$$\begin{aligned} \mathbf{x}(t) &= \mathbf{A}\mathbf{s}(t) + \mathbf{n}_x(t)\,, \\ \mathbf{y}(t) &= \mathbf{A}\Phi\mathbf{s}(t) + \mathbf{n}_y(t)\,; \end{aligned} \tag{2}$$

where:

$$\begin{aligned} \mathbf{x}^T(t) &= [x_1(t), \ldots, x_m(t)], \\ \mathbf{n}_x^T(t) &= [n_{x_1}(t), \ldots, n_{x_m}(t)], \end{aligned} \tag{3}$$

and $\mathbf{y}(t)$ and $\mathbf{n}_y(t)$ are similarly defined. The vector $\mathbf{s}(t)$ is a $d \times 1$ vector of impinging signals (wavefronts) as observed at the reference sensor of subarray X. The matrix Φ is a diagonal $d \times d$ matrix of the phase delays between the doublet sensors for the d wavefronts, and can be written as:

$$\Phi = diag[e^{j\phi_1}, \ldots, e^{j\phi_d}]; \quad \phi_k = \omega_0 \Delta \sin\theta_k / c\,. \tag{4}$$

Note that Φ is a unitary matrix (operator) that relates the measurements from subarray X to those from subarray Y. The $m \times d$ matrix $\mathbf{A}$ is the *direction matrix* whose columns $\{\mathbf{a}(\theta_k), \quad k = 1, \ldots, d\}$ are the signal *direction vectors* for the d wavefronts.

$$\mathbf{a}^T(\theta_k) = [a_1(\theta_k), \ldots, a_m(\theta_k)]. \tag{5}$$

The auto-covariance of the data received by subarray X is given by:

$$\mathbf{R}_{xx} = E[\mathbf{x}(t)\mathbf{x}^*(t)] = \mathbf{ASA}^* + \sigma^2\mathbf{I}\,, \tag{6}$$

where $\mathbf{S} = E[\mathbf{s}(t)\mathbf{s}^*(t)]$ is the $d \times d$ covariance matrix of *the signals* $\mathbf{s}(t)$*, and* σ^2 *is the covariance of the additive* uncorrelated white noise. Herein, $(\cdot)^*$ denotes Hermitean conjugate. Similarly, the cross-covariance between measurements from subarrays X and Y is given by:

$$\mathbf{R}_{xy} = E[\mathbf{x}(t)\mathbf{y}^*(t)] = \mathbf{AS}\Phi^*\mathbf{A}^*\,. \tag{7}$$

Having defined the signal and noise model, the problem can now be stated as follows: *Given measurements* $\mathbf{x}(t)$ *and* $\mathbf{y}(t)$*, and making no assumptions about array geometry, element characteristics, DOA's, noise powers, or signal correlations, estimate the signal DOA's.*

III. Invariant Subspace Approach

The following theorem provides the foundation for the results presented herein.

Theorem: *Define* Γ *as the generalized eigenvalue matrix associated with the matrix pencil* $\{(\mathbf{R}_{xx} - \lambda_{\min}\mathbf{I}), \mathbf{R}_{xy}\}$ *where* $\lambda_{\min}$ *is the minimum (repeated) eigenvalue of* $\mathbf{R}_{xx}$*. Then, if* $\mathbf{S}$ *is nonsingular, the matrices* Φ *and* Γ *are related by*

$$\Gamma = \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

to within a permutation of the elements of Φ.

Proof: From linear algebra we can see that $\mathbf{ASA}^*$ is rank d and $\mathbf{R}_{xx}$ has a multiplicity $(m-d)$ of eigenvalues all equal to σ^2. If $\{\lambda_1 > \lambda_2 > \ldots > \lambda_m\}$ are the ordered eigenvalues of $\mathbf{R}_{xx}$, then $\lambda_{d+1} = \ldots = \lambda_m = \sigma^2$. Hence,

$$\mathbf{R}_{xx} - \lambda_{\min}\mathbf{I} \;=\; \mathbf{R}_{xx} - \sigma^2\mathbf{I} \;=\; \mathbf{ASA}^*\,. \tag{9}$$

Now consider the matrix pencil

$$\mathbf{C}_{xx} - \gamma\mathbf{R}_{xy} = \mathbf{ASA}^* - \gamma\mathbf{AS}\Phi^*\mathbf{A}^* = \mathbf{AS}(\mathbf{I} - \gamma\Phi^*)\mathbf{A}^*; \tag{10}$$

where $\mathbf{C}_{xx} \doteq \mathbf{R}_{xx} - \lambda_{\min}\mathbf{I}$. By inspection, the column space of both $\mathbf{ASA}^*$ and $\mathbf{AS}\Phi^*\mathbf{A}^*$ are identical. Therefore, $\rho(\mathbf{ASA}^* - \gamma\mathbf{AS}\Phi^*\mathbf{A}^*)$ will in general be equal to d. However, if

$$\gamma = e^{j\omega_0\Delta\sin\theta_i/c}, \tag{11}$$

the i^{th} row of $(\mathbf{I} - e^{j\omega_0\Delta\sin\theta_i/c}\Phi^*)$ will become zero. Thus,

$$\rho(\mathbf{I} - e^{j\omega_0\Delta\sin\theta_i/c}\Phi^*) = d - 1\,. \tag{12}$$

Consequently, the pencil $(\mathbf{C}_{xx} - \gamma\mathbf{R}_{xy})$ will also decrease in rank to $d-1$ whenever γ assumes values given by (11). However, by definition these are exactly the *generalized eigenvalues* (GE's) of the matrix pair $\{\mathbf{C}_{xx}, \mathbf{R}_{xy}\}$. Also, since both matrices in the pair span the same subspace, the GE's corresponding to the common null space of the two matrices will be zero, *i.e.*, d GE's lie on the unit circle and are equal to the diagonal elements of the rotation matrix Φ, and the remaining $m-d$ GE's are at the origin. This completes the proof of the theorem.

ESPRIT Algorithm – Covariance Formulation

Due to errors in estimating $\mathbf{R}_{xx}$ and $\mathbf{R}_{xy}$ from finite data as well as errors introduced during the subsequent finite precision computations, relations such as

(8) will not be exactly satisfied. Therefore, a procedure is proposed which is not globally optimal, but utilizes some well established, stepwise-optimal techniques to deal with such issues.

1. Find the $2m \times 2m$ covariance matrix of the complete $2m$ sensor array and then estimate the number of sources $\hat{d}$ and the noise variance $\hat{\sigma}^2$ (this can be estimated as the average of the noise eigenvalues).

2. Compute rank $\hat{d}$ approximations to this full array covariance matrix and subtract out $\hat{\sigma}^2$. $\mathbf{ASA}^*$ and $\mathbf{AS\Phi^*A}^*$ are then the top-left and top-right blocks.

3. The $\hat{d}$ GE's of the estimates of $(\mathbf{ASA}^*, \mathbf{AS\Phi^*A}^*)$ that lie close to the unit circle determine the subspace rotation operator $\mathbf{\Phi}$.[2]

REMARKS AND EXTENSIONS

Estimation of the Number of Signals

In the ***ESPRIT*** algorithm described above, an estimate of the number of sources present is used to obtain estimates of σ^2 and perform rank reduction on the estimated covariance matrix. Simulation results have shown that the estimates of Φ are not seriously degraded if rank $\hat{d}$ estimates of $\mathbf{ASA}^*$ and $\mathbf{AS\Phi^*A}^*$ are dispensed with and the GEV's computed directly from the matrix pair $\{\hat{\mathbf{R}}_{xx} - \lambda_{min}, \hat{\mathbf{R}}_{xy}\}$. This ability to simultaneously estimate d and the parameters of interest is a distinct advantage of ***ESPRIT*** over MUSIC.

Extensions to Multiple Dimensions

To extend ***ESPRIT*** to multidimensional parameter vectors, measurements must be made by arrays manifesting the the shift invariant structure in the appropriate dimension. For example, co-directional sensor doublets are used to estimate DOA's in a plane (*e.g.,* azimuth) containing the doublet axes. To obtain elevation estimates, another pair of subarrays sensitive to elevation angle is necessary. Similarly, ***ESPRIT*** can be extended to estimate temporal frequencies by including temporal in addition to spatial sampling. It is important to note that estimation in each dimension is effectively decoupled, resulting in the computational load in ***ESPRIT*** growing *linearly* with the dimension of the signal parameter vector, whereas in MUSIC it increases *exponentially.*

Array Response Vector Estimation

Let $\mathbf{e}_i$ be the generalized eigenvector (GEV) corresponding to the generalized eigenvalue (GE) γ_i. By definition, $\mathbf{e}_i$ satisfies the relation

$$\mathbf{AS}(\mathbf{I} - \gamma_i \mathbf{\Phi}^*)\mathbf{A}^*\mathbf{e}_i = 0\,. \tag{13}$$

[2]Since $\mathbf{ASA}^*$ and $\mathbf{AS\Phi^*A}^*$ are singular, care is necessary in the computation of the GE's.

Since the column space of the pencil $\mathbf{AS}(\mathbf{I} - \gamma_i \mathbf{\Phi}^*)\mathbf{A}^*$ is same as the subspace spanned by the vectors $\{\mathbf{a}_j, j \neq i\}$, it follows that $\mathbf{e}_i$ is orthogonal to all direction vectors, except $\mathbf{a}_i$. Assuming for now that the sources are uncorrelated, *i.e.,* , $\mathbf{S} = diag[\sigma_1^2, \ldots, \sigma_d^2]$, multiplying $\mathbf{C}_{xx}$ by $\mathbf{e}_i$ yields the desired result:

$$\begin{aligned}\mathbf{C}_{xx}\mathbf{e}_i &= \mathbf{AS}[0, \ldots, 0, \mathbf{a}_i^*\mathbf{e}_i, 0, \ldots, 0]^T \\ &= (\sigma_i^2 \mathbf{a}_i^* \mathbf{e}_i)\mathbf{a}_i\,.\end{aligned} \tag{14}$$

Source Power Estimation

The source powers follow from (14):

$$\sigma_i^2 = \frac{|\mathbf{u}^T\mathbf{C}_{xx}\mathbf{e}_i|^2}{\mathbf{e}_i^*\mathbf{C}_{xx}\mathbf{e}_i}\,. \tag{15}$$

Signal Copy (SC)

Signal copy refers to the weighted combination of the sensor measurements such that the output contains the desired signal while completely rejecting the other $d-1$ signals. From (13), $\mathbf{e}_i$ is orthogonal to all wavefront direction vectors except the i^{th} wavefront, and is therefore the desired weight vector for signal copy of the i^{th} signal. Note that this is true even for correlated signals.

$$\mathbf{w}_i^{SC} = \mathbf{e}_i \left\{ \frac{|\mathbf{u}^T\mathbf{C}_{xx}\mathbf{e}_i|}{\mathbf{e}_i^*\mathbf{C}_{xx}\mathbf{e}_i} \right\}. \tag{16}$$

In the presence of correlated signals (*cf.* multipath), the maximum likelihood (ML) beamformer [5] given by:

$$\mathbf{w}_i^{ML} = \mathbf{R}_{xx}^{-1}\mathbf{C}_{xx}\mathbf{e}_i\,, \tag{17}$$

improves performance by optimally combining the information from the multiple DOAs. In the absence of noise, $\mathbf{R}_{xx} = \mathbf{C}_{xx}$ and $\mathbf{w}_i^{ML} = \mathbf{w}_i^{SC}$.

ESPRIT Algorithm – GSVD Formulation

The covariance formulation of ***ESPRIT*** used auto- and cross-covariance estimates of the subarray sensor data. Since the algorithm involves determining the GE's of a singular matrix pair, it is preferable to avoid using covariance matrices, choosing instead to operate directly on the *data.* This approach leads to a generalized singular value decomposition (GSVD) of data matrices.

Let $\mathbf{X}$ and $\mathbf{Y}$ be $m \times N$ data matrices containing N simultaneous snapshots $\mathbf{x}(t)$ and $\mathbf{y}(t)$ respectively;

$$\begin{aligned}\mathbf{X} &= [\mathbf{x}(t_1), \mathbf{x}(t_2), \ldots, \mathbf{x}(t_N)]\,, \\ \mathbf{Y} &= [\mathbf{y}(t_1), \mathbf{y}(t_2), \ldots, \mathbf{y}(t_N)]\,.\end{aligned} \tag{18}$$

The GSVD of the matrix pair $(\mathbf{X}, \mathbf{Y})$ is given by:

$$\mathbf{X} = \mathbf{U}_X \Sigma_X \mathbf{V}^*\,; \quad \mathbf{Y} = \mathbf{U}_Y \Sigma_Y \mathbf{V}^*\,; \tag{19}$$

where $\mathbf{U}_X$ and $\mathbf{U}_Y$ are unitary matrices containing the left generalized singular vectors, Σ_X and Σ_Y are $m \times N$ real rectangular matrices with zero entries everywhere except on the main diagonal, and $\mathbf{V}$ is a nonsingular matrix. Using similar arguments as in the proof of the main result, the desired Φ_{ii} are the generalized eigenvalues of the pair $(\mathbf{U}_X\Sigma_X, \mathbf{U}_Y\Sigma_Y)$.

IV. Simulation Results

Computer simulations were carried out to investigate the performance of ***ESPRIT*** compared to that of MUSIC under similar conditions. The simulation consisted of two colinear arrays of omni-directional elements and two sources. The 6 elements in each array were uniformly spaced one-half wavelength apart, and the arrays were spaced 6 wavelengths apart (*i.e.,* $\Delta = 6\lambda$). Two planar uncorrelated signal wavefronts impinged on the array at angles of 26° and 27°, with SNRs of 20 dB and 15 dB relative to the noise. Covariance estimates were computed from 100 snapshots of data and parameter estimates using the covariance version of ***ESPRIT*** were obtained using independent data sets. Figure 2 shows a plot of the GE estimates obtained from 100 trials. The two sources 1° in DOA space apart are easily resolved. Note that the angular separation of the true GE estimates in Figure 2 is given by $2\pi\Delta[\sin(27°) - \sin(26°)]/\lambda$ in *radians*, which for $\Delta = 6\lambda$ is approximately 34°. The sample means and sigmas of the ***ESPRIT*** estimates of $\sin(\theta)$ are $(0.4381 \pm 0.0011, 0.4540 \pm 0.0021)$ which compare favorably with the actual values (0.4384, 0.4540).

Figure 3 contains MUSIC spectral estimates obtained using the sample covariances from the first 20 trials. In all cases, the number of sources was assumed to be known ($d = 2$), and the subspace estimates generated appropriately. Letting $\mathbf{E}_n$ denote the estimated noise subspace, the MUSIC *spectrum* is given by $[\mathbf{a}^*(\theta)\mathbf{E}_n\mathbf{E}_n^*\mathbf{a}(\theta)]^{-1}$. In a majority of the trials, two *spectral peaks* were not resolvable in the search region [25°, 28°].

ESPRIT **Simulation Results**

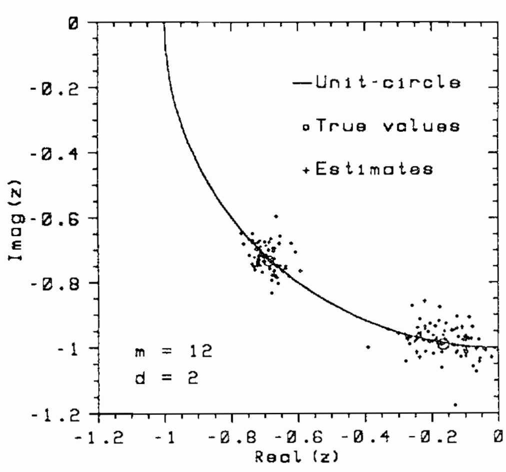

Figure 2: GE Estimates using *ESPRIT*

V. Concluding Remarks

In this paper, the ***ESPRIT*** algorithm applied to DOA estimation using an array with a translational invariance structure has been described and its comparative performance to the MUSIC technique studied. For the scenario chosen, ***ESPRIT*** substantially outperformed MUSIC. MUSIC was unable to resolve two sources in many instances in contrast to ***ESPRIT***. It should be pointed out that the *loss of resolution* MUSIC suffers in such scenarios is the consequence of a one-dimensional search being conducted for d parameters. An appropriate multi-dimensional performance measure mitigates this problem entirely. In other scenarios investigated, specifically with $\Delta < \lambda$, ***ESPRIT*** yielded unbiased estimates, whereas MUSIC estimates had a bias on the order of the sample sigma. In these cases, sample sigmas from ***ESPRIT*** were approximately twice those from MUSIC. In general, as the number of elements in the array (*i.e.,* array aperature) increases, MUSIC manifests a substantial advantage in sample variance over ***ESPRIT*** for small $\Delta's$. However, ***ESPRIT*** is known to have many implementational advantages including speed, storage, and indifference to array calibration, *etc.*, making it attractive in many practical applications.

MUSIC Simulation Results

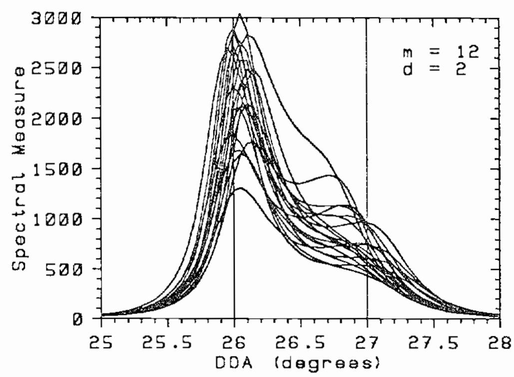

Figure 3: MUSIC *Spectral* Estimates

References

[1] R. O. Schmidt, Multiple emitter location and signal parameter estimation, In *Proc. RADC Spectrum Estimation Workshop*, pages 243–258, Griffiths AFB, N.Y., 1979.

[2] A. Paulraj, R. Roy, and T. Kailath, Estimation of signal parameters via rotational invariance techniques – *ESPRIT*, In *Proc. Nineteenth Asilomar Conf. on Circuits, Systems and Comp.*, Asilomar, CA., November 1985.

[3] A. Paulraj, R. Roy, and T. Kailath, Patent Application: *Methods and Means for Signal Reception and Parameter Estimation*, Stanford University, Stanford, Ca., 1985.

[4] R. Roy, A. Paulraj, and T. Kailath, *ESPRIT* – a subspace rotation approach to estimation of parameters of cisoids in noise, *Submitted to IEEE Trans. on ASSP*, 1986.

[5] A. Paulraj and T. Kailath, On beamforming in the presence of multipath, In *Proc. IEEE ICASSP*, pages 15.4.1–15.4.4, Tampa, Fla., March 1985.