Axioms of Quantum Mechanics

Paul

March 29, 2022

1 States

Axiom 1 *With every quantum system there is associated a complex separable Hilbert space* $(\mathfrak{H}, +, \cdot, \langle \quad \rangle)$*. The states of the system are all positive trace-class linear maps* $\varrho : \mathfrak{H} \to \mathfrak{H}$ *for which* $\mathrm{Tr}\,\varrho = 1$

Remark 1 *Almost everywhere it is stated: The normalized elements* $\psi \in \mathfrak{H}$ *are the states of the quantum system - it is false.*

Definition 1 *A state is called a pure state (not pure = mixed) if there exists* $\psi \in \mathfrak{H}$ *such that*

$$\varrho : \mathfrak{H} \to \mathfrak{H}, \qquad\qquad \alpha \mapsto \varrho(\alpha) = \frac{\langle \psi, \alpha \rangle}{\langle \psi, \psi \rangle} \psi$$

Remark 2 *Thus for pure states it is true that state* ϱ *is associated with the element of the Hilbert space* ψ*.*

Complex Hilbert space $(\mathfrak{H}, +, \cdot, \langle \quad \rangle)$

- $\mathfrak{H}$ is a set that satisfies the axioms of complex vector space
  - $+ : \mathfrak{H} \times \mathfrak{H} \to \mathfrak{H}$
  - $\cdot : \mathbb{C} \times \mathfrak{H} \to \mathfrak{H}$
- Sesquilinear map $\langle .,. \rangle : \mathfrak{H} \times \mathfrak{H} \to \mathbb{C}$ satisfying
  - $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$ (complex conjugate)
  - $\langle \phi, \psi_1 + \alpha\psi_2 \rangle = \langle \phi, \psi_1 \rangle + \alpha \langle \phi, \psi_2 \rangle$, $\forall \alpha \in \mathbb{C}$
  - $\langle \psi, \psi \rangle \geq 0$, $\forall \psi \in \mathfrak{H}$ and $\langle \psi, \psi \rangle = 0 \Leftrightarrow \psi = 0$
- $\mathfrak{H}$ is complete

If one has a sequence in $\mathfrak{H}$, $\phi : \mathbb{N} \to \mathfrak{H}$, which satisfies the Cauchy property $\forall \epsilon > 0\, \exists N \in \mathbb{N}$ such that $\forall n, m \geq N$: $\|\phi_n - \phi_m\| < \epsilon$, where $\|\phi\| = \sqrt{\langle \phi, \phi \rangle}$, one may already conclude that the sequence ϕ converges in $\mathfrak{H}$ i.e. $\exists \phi \in \mathfrak{H}$ such that $\forall \epsilon > 0\, \exists N \in \mathbb{N}$ such that $\forall n \geq N$: $\|\phi - \phi_n\| < \epsilon$.

Remark 3 *For linear map $A : \mathfrak{H} \supset \mathcal{D}_A \to \mathfrak{H}$ we will only look at densely defined linear maps*

$$
\begin{aligned}
&\forall \psi \in \mathfrak{H}, \forall \epsilon > 0 \, \exists \chi \in \mathcal{D}_A : && \|\chi - \psi\| < \epsilon \\
&A(\phi + \alpha\psi) = A\phi + \alpha A\psi, && \forall \alpha \in \mathbb{C}
\end{aligned}
$$

Definition 2 *Positive linear map is a map A such that $\forall \psi \in \mathcal{D}_A$: $\langle \psi, A\psi \rangle \geq 0$*

Definition 3 *Trace-class linear map is a map $A : \mathfrak{H} \to \mathfrak{H}$ (defined on the entire Hilbert space) such that $\forall$ orthonormal basis $\{e_n\}$ of $\mathfrak{H}$ the sum/series $\sum_n \langle e_n, Ae_n \rangle < \infty$, $\operatorname{Tr} A = \sum_n \langle e_n, Ae_n \rangle$*

Remark 4 *Hilbert space $\mathfrak{H}$ has to be separable.*

2 Observables

Axiom 2 *The observables of a quantum system are the self-adjoint linear maps $A : \mathcal{D}_A \longrightarrow \mathfrak{H}$*

Definition 4 *A linear map $A : \mathcal{D}_A \longrightarrow \mathfrak{H}$ densely defined on its domain is called self-adjoint if it coincides with its adjoint map $A^* : \mathcal{D}_{A^*} \longrightarrow \mathfrak{H}$*

$$
\mathcal{D}_{A^*} = \mathcal{D}_A \qquad\qquad A^*\psi = A\psi
$$

Definition 5 *The adjoint $A^* : \mathcal{D}_{A^*} \longrightarrow \mathfrak{H}$ of a linear map $A : \mathcal{D}_A \longrightarrow \mathfrak{H}$ is defined by*

$$
\begin{aligned}
&\mathcal{D}_{A^*} = \left\{ \psi \in \mathfrak{H} \,\middle|\, \forall \alpha \in \mathcal{D}_A \, \exists \eta \in \mathfrak{H} : \langle \psi, A\alpha \rangle = \langle \eta, \alpha \rangle \right\} \\
&A^*\psi = \eta
\end{aligned}
$$

A^* is well defined if there is unique η

3 Measurements

Axiom 3 *The probability that a measurement of an observable A on a system that is in the state ϱ yields a result in the Borel set $E \subseteq \mathbb{R}$ is given by*

$$
\mu_\varrho^A(E) = \operatorname{Tr}\left(P_A(E) \circ \varrho\right)
$$

$P_A(E)$ is a bounded operator. The composition of trace-class operator with the bounded operator is again trace-class operator.

$$P_A : \mathrm{Borel}(\mathbb{R}) \longrightarrow \mathcal{L}(\mathfrak{H}) : \text{Banach space of bounded linear maps on } \mathfrak{H}$$

is the unique projection-valued measure that is associated with a self-adjoint map A accordingly to the spectral theorem

$$A = \int_{-\infty}^{\infty} \lambda \, \mathrm{d}P_A(\lambda)$$

4 Unitary dynamics

Time intervals (t_1, t_2) during which no measurement occurs

Axiom 4 *State $\varrho(t_1)$ and state $\varrho(t_2)$ are related through*

$$\varrho(t_2) = \mathcal{U}(t_2 - t_1)\, \varrho(t_1)\, \mathcal{U}^{-1}(t_2 - t_1)$$

where

$$\mathcal{U}(t) = \exp\Big(-\frac{i}{\hbar}\mathcal{H}t\Big)$$

$\mathcal{H}$ is the energy observable

$$f(A) = \int_{-\infty}^{\infty} f(\lambda) \, \mathrm{d}P_A(\lambda)$$

5 Projective dynamics

Step in when a measurement is made at time t_m

Axiom 5 *The state ϱ_{after} immediately after the measurement of an observable A is*

$$\varrho_{after} = \frac{P_A(E) \circ \varrho_{before} \circ P_A(E)}{\mathrm{Tr}\,\big(P_A(E) \circ \varrho_{before} \circ P_A(E)\big)}$$

where E is the smallest Borel set in which the actual outcome of the measurement happened to lie.