Electromagnetism

Paul

April 11, 2022

1 Field equations

For a given distribution of charges the fields are determined by

$$\begin{aligned}
\nabla \times \vec{E} &= -\frac{1}{c}\frac{\partial}{\partial t}\vec{B} \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{B} &= \frac{1}{c}\frac{\partial}{\partial t}\vec{E} + \frac{4\pi}{c}\vec{j} \\
\nabla \cdot \vec{E} &= 4\pi\rho
\end{aligned}$$

Conservation of charge

$$\frac{\partial}{\partial t}\rho + \nabla \cdot \vec{j} = 0$$

The motion of the charges is given by

$$\vec{k} = \rho\vec{E} + \frac{1}{c}\vec{j} \times \vec{B}$$

2 Potentials

It follows that $\vec{B}$ can be represented as $\vec{B} = \nabla \times \vec{A}$, where $\vec{A}$ is called the *vector potential.* Then

$$\nabla \times \left(\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\right) = 0$$

or

$$\vec{E} + \frac{1}{c}\frac{\partial}{\partial t}\vec{A} = -\nabla\phi$$

where ϕ represents a scalar function — the *scalar potential.* Then values of fields can be expressed as

$$\begin{aligned}\vec{B} &= \nabla\times\vec{A}\\ \vec{E} &= -\nabla\phi-\frac{1}{c}\frac{\partial}{\partial t}\vec{A}\end{aligned}$$

Making use of the general vector relation $\nabla\times\nabla\vec{A}=\nabla\nabla\cdot\vec{A}-\nabla^2\vec{A}$ differential equations for potentials can be written in the form

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}-\nabla^2\vec{A}+\nabla\left(\nabla\cdot\vec{A}+\frac{1}{c}\frac{\partial}{\partial t}\phi\right) &= \frac{4\pi}{c}\vec{j}\\ -\nabla^2\phi-\frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\vec{A} &= 4\pi\rho\end{aligned}$$

2.1 Gauge transformation

The vector $\vec{A}$ is not completely defined by the value of $\vec{B}$. Since for any scalar function χ it is true that $\nabla\times\nabla\chi=0$, transformation

$$\begin{aligned}\vec{A} &\to \vec{A}+\nabla\chi\\ \phi &\to \phi-\frac{1}{c}\frac{\partial}{\partial t}\chi\end{aligned}$$

does not change the values of fields $\left(\vec{E},\vec{B}\right)$.

2.2 Lorentz gauge

The condition

$$\nabla\cdot\vec{A}+\frac{1}{c}\frac{\partial}{\partial t}\phi=0$$

is called the *Lorent gauge*. Equations for potentials then become

$$\begin{aligned}\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A}-\nabla^2\vec{A} &= \frac{4\pi}{c}\vec{j}\\ \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\phi-\nabla^2\phi &= 4\pi\rho\end{aligned}$$

There is still freedom in selecting χ which satisfies the homogeneous wave equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\chi-\nabla^2\chi=0$$

2.3 Coulomb gauge

Another important gauge which is particularly important in quantum theory is

$$\nabla \cdot \vec{A} = 0$$

which is called the *Coulomb gauge*. Equations for potentials then become

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} + \frac{1}{c}\frac{\partial}{\partial t}\nabla\phi &= \frac{4\pi}{c}\vec{j} \\
-\nabla^2\phi &= 4\pi\rho
\end{aligned}$$

There is still freedom in selecting χ which can be any harmonic function

$$\nabla^2\chi = 0$$

3 Retarded potential

Special solutions for inhomogeneous wave equations are

$$\begin{aligned}
\vec{A}(t,x) &= \frac{1}{c}\int \frac{j\left(t - \frac{r_{xx'}}{c}, x'\right)}{r_{xx'}}\,\mathrm{d}\tau' \\
\phi(t,x) &= \int \frac{\rho\left(t - \frac{r_{xx'}}{c}, x'\right)}{r_{xx'}}\,\mathrm{d}\tau'
\end{aligned}$$

The Lorentz gauge is satisfied by the charge conservation condition.

For the part of the field, which satisfies the homogeneous wave equation, on can choose χ within the Lorentz gauge so that ϕ vanishes. The field which is independent of charges is given by

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\vec{A} - \nabla^2\vec{A} &= 0 \\
\nabla \cdot \vec{A} &= 0 \\
\vec{E} &= -\frac{1}{c}\frac{\partial}{\partial t}\vec{A} \\
\vec{B} &= \nabla \times \vec{A}
\end{aligned}$$