Fourier series

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1 Orthogonality

Let $n, m \in \mathbb{N}$. The computation of the Fourier series is based on the integral identities

$$\int\limits_{x=-\pi}^{\pi} \cos nx \cos mx \,\mathrm{d}x = \pi\delta_{nm}$$
$$\int\limits_{x=-\pi}^{\pi} \sin nx \sin mx \,\mathrm{d}x = \pi\delta_{nm}$$
$$\int\limits_{x=-\pi}^{\pi} \sin nx \cos mx \,\mathrm{d}x = 0$$

2 Series expansion

The Fourier series of a real function $f(x)$ for $x \in [-\pi, \pi]$ is given by

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} a_k \cos kx + \sum_{k=1}^{\infty} b_k \sin kx$$

where

$$a_k = \frac{1}{\pi} \int\limits_{x=-\pi}^{\pi} f(x) \cos kx \,\mathrm{d}x$$
$$b_k = \frac{1}{\pi} \int\limits_{x=-\pi}^{\pi} f(x) \sin kx \,\mathrm{d}x$$

3 Complex coefficients

The Fourier series of a real function $f(x)$ for $x \in [-\pi, \pi]$ is given by

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \, \mathrm{e}^{ikx}$$

where c_k are complex values for $k \in \mathbb{Z}$

$$c_k = \frac{1}{2\pi} \int\limits_{x=-\pi}^{\pi} f(x) \, \mathrm{e}^{-ikx} \, \mathrm{d}x$$

This expansion is based on orthogonality relations

$$\langle \phi_n, \phi_m \rangle = \delta_{nm}$$
$$\langle \phi_n, \phi_m \rangle = \frac{1}{2\pi} \int\limits_{x=-\pi}^{\pi} \overline{\phi_n(x)} \, \phi_m(x) \, \mathrm{d}x$$

for the set of complex functions $\phi_k(z) = \mathrm{e}^{ikz}, k \in \mathbb{Z}$

4 Parseval's theorem

The Fourier coefficients of function $f(x)$ will satisfy the relation, which is a generalized Pythagorean theorem for inner-product spaces

$$\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi} \int\limits_{x=-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x$$

For a complex Fourier series,

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{1}{2\pi} \int\limits_{x=-\pi}^{\pi} |f(x)|^2 \, \mathrm{d}x$$