Mathematical pendulum

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1 Hamiltonian

$$\mathcal{H} = \frac{1}{2}p^2 + 2\sin^2\frac{1}{2}\theta \tag{1}$$

2 Equations of motion

Select initial conditions so that $\theta\Big|_{t=0} = 0$

2.1 Librations

Case $\mathcal{E} < 2$. Assign $k^2 = \dfrac{1}{2}\mathcal{E}$

$$\theta(t) = 2\arcsin(k\,\mathrm{sn}(t;k)) \tag{2}$$
$$p(t) = 2k\,\mathrm{cn}(t;k) \tag{3}$$

Period of librations

$$T = 4\mathrm{K}(k) \tag{4}$$

2.2 Rotations

Case $\mathcal{E} > 2$. Assign $k^2 = 2\,\mathcal{E}^{-1}$

$$\theta(t) = 2\,\mathrm{am}(k^{-1}t;k) \tag{5}$$
$$p(t) = 2k^{-1}\,\mathrm{dn}(k^{-1}t;k) \tag{6}$$

Period of rotations

$$T = 2k\,\mathrm{K}(k) \tag{7}$$

3 Elliptic integral via AGM

The *elliptic integral of the first kind* $\mathrm{F}(\phi; k)$ and the *complete elliptic integral of the first kind* $\mathrm{K}(k)$ are defined as

$$\mathrm{F}(\phi; k) = \int_0^{\phi} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \tag{8}$$

$$\mathrm{K}(k) = \mathrm{F}\left(\frac{\pi}{2}; k\right) \tag{9}$$

The complete elliptic integral of the first kind can be evaluated via the *arithmetic-geometric mean* (AGM)

$$\mathrm{K}(k) = \frac{\pi}{2} \frac{1}{\mathcal{M}\left(1, \sqrt{1 - k^2}\right)} \tag{10}$$

4 Jacobi elliptic functions

Elliptic functions can be represented via Jacobi amplitude function

$$\mathrm{sn}(u; k) = \sin \mathrm{am}(u; k) \tag{11}$$

$$\mathrm{cn}(u; k) = \cos \mathrm{am}(u; k) \tag{12}$$

$$\mathrm{dn}(u; k) = \frac{\partial}{\partial u} \mathrm{am}(u; k) \tag{13}$$

where the amplitude function is defined via inverse of the elliptic integral

$$\mathrm{F}(\mathrm{am}(u; k); k) = u \tag{14}$$