(a)

Folosim formula de cvadratura Gauss-Cebisev-Lobatto de speta I pentru a aproxima integrala:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\, dt \approx \frac{\pi}{2(n+1)} \left[f(-1) + f(1)\right] + \frac{\pi}{n+1} \sum_{k=1}^{n} f\left(\cos\left(\frac{n+1-k}{n+1}\pi\right)\right) + R(f).$$

Pentru $u(x) = T_k(x)$ și $v(x) = T_\ell(x)$, folosim definitia produsului scalar si extremele polinomului $T_n(x)$:

$$(T_k, T_\ell) = \frac{1}{2} T_k(x_0) T_\ell(x_0) + \sum_{i=1}^{n-1} T_k(x_i) T_\ell(x_i) + \frac{1}{2} T_k(x_n) T_\ell(x_n),$$

unde $x_k = \cos\left(\frac{k\pi}{n}\right)$.

Polinoamele Cebisev au proprietatea de ortogonalitate:

$$\int_{-1}^{1} \frac{T_k(x) T_\ell(x)}{\sqrt{1-x^2}}\, dx = \begin{cases} 0, & k \neq \ell \\ \pi, & k = \ell = 0 \\ \frac{\pi}{2}, & k = \ell \neq 0 \end{cases}.$$

Evaluam polinoamele T_k si T_ℓ in punctele x_k:

$$T_k(x_i) = \cos\left(k \frac{i\pi}{n}\right).$$

Produsul scalar discret devine:

$$(T_k, T_\ell) = \frac{1}{2} \cos\left(k \frac{0\pi}{n}\right) \cos\left(\ell \frac{0\pi}{n}\right) + \sum_{i=1}^{n-1} \cos\left(k \frac{i\pi}{n}\right) \cos\left(\ell \frac{i\pi}{n}\right) + \frac{1}{2} \cos\left(k \frac{n\pi}{n}\right) \cos\left(\ell \frac{n\pi}{n}\right).$$

Dupa simplificarea termenilor obtinem:

$$(T_k, T_\ell) = \frac{1}{2} \cdot 1 \cdot 1 + \sum_{i=1}^{n-1} \cos\left(k \frac{i\pi}{n}\right) \cos\left(\ell \frac{i\pi}{n}\right) + \frac{1}{2} \cdot (-1)^k \cdot (-1)^\ell.$$

$$= \frac{1}{2} + \sum_{i=1}^{n-1} \cos\left(k \frac{i\pi}{n}\right) \cos\left(\ell \frac{i\pi}{n}\right) + \frac{1}{2} (-1)^{k+\ell}.$$

care verifica relatia de ortogonalitate data

(b)

Consideram aproximarea discreta in sensul celor mai mici patrate a functiei $f(x)$ utilizand polinoame ortogonale. Vrem sa aratam ca coeficientii c_j ai dezvoltarii in serie Cebisev se pot exprima astfel:

$$c_j = \frac{2}{n}\sum_{k=0}^{n} f(x_k)T_j(x_k).$$

Funcția $f(x)$ este aproximata folosind o serie de polinoame Cebisev:

$$\varphi(x) = \frac{c_0}{2}T_0(x) + \sum_{k=1}^{n} c_k T_k(x),$$

unde $T_k(x)$ sunt polinoamele Cebisev si coeficientii c_k urmeaza sa fie determinati.

Consideram produsul scalar discret definit ca:

$$\langle f, g\rangle = \frac{2}{n}\sum_{k=0}^{n} f(x_k)g(x_k),$$

unde x_k sunt punctele de interpolare.

Pentru a determina coeficientii c_j, calculam produsul scalar al aproximarii $\varphi(x)$ cu $T_j(x)$:

$$\langle \varphi(x), T_j(x)\rangle = \frac{2}{n}\sum_{k=0}^{n} \varphi(x_k)T_j(x_k).$$

Inlocuind expresia pentru $\varphi(x)$:

$$\langle \varphi(x), T_j(x)\rangle = \frac{2}{n}\sum_{k=0}^{n} \left(\frac{c_0}{2}T_0(x_k) + \sum_{m=1}^{n} c_m T_m(x_k)\right) T_j(x_k).$$

Deoarece $T_0(x) = 1$, avem $\frac{c_0}{2}T_0(x) = \frac{c_0}{2}$. Deci,

$$\langle \varphi(x), T_j(x)\rangle = \frac{2}{n}\sum_{k=0}^{n} \left(\frac{c_0}{2} + \sum_{m=1}^{n} c_m T_m(x_k)\right) T_j(x_k).$$

Folosind ortogonalitatea polinoamelor Cebisev, stim ca:

$$\langle T_m, T_j\rangle = 0 \text{ pentru } m \neq j.$$

Astfel,

$$\langle \varphi(x), T_j(x) \rangle = \frac{2}{n} \sum_{k=0}^{n} c_j T_j(x_k) T_j(x_k) = \frac{2}{n} c_j \sum_{k=0}^{n} T_j^2(x_k).$$

Egalam expresia cu produsul scalar al lui $f(x)$ si $T_j(x)$:

$$c_j = \frac{\langle f, T_j \rangle}{\langle T_j, T_j \rangle}.$$

Din moment ce $\langle T_j, T_j \rangle$ este constant, considerand ortogonalitatea normata, avem:

$$c_j = \frac{2}{n} \sum_{k=0}^{n} f(x_k) T_j(x_k).$$

(c)

$$\begin{aligned}
\int T_n \, dx = \frac{1}{2} \left(\frac{T_{n+1}}{n+1} - \frac{T_{n-1}}{n-1} \right) &= \frac{n T_{n+1}}{n^2 - 1} - \frac{x T_n}{n-1} \\
&= \frac{n}{n^2 - 1} T_{n+1} - \frac{1}{n-1} T_1 T_n \\
&= \frac{n}{n^2 - 1} \, T_{n+1} - \frac{1}{2(n-1)} \, (T_{n+1} + T_{n-1}) \\
&= \frac{1}{2(n+1)} \, T_{n+1} - \frac{1}{2(n-1)} \, T_{n-1}
\end{aligned}$$

Integrala pe intervalul $[-1, 1]$:

$$\int_{-1}^{1} T_n(x) \, dx = \begin{cases} \frac{(-1)^n + 1}{1 - n^2} & \text{dacă } n \neq 1 \\ 0 & \text{dacă } n = 1 \end{cases}$$

Polinoamele Cebisev de speta I sunt ortogonale pe intervalul $[-1, 1]$:

$$\int_{-1}^{1} \frac{T_k(x) T_j(x)}{\sqrt{1 - x^2}} \, dx = \begin{cases} 0 & \text{dacă } k \neq j \\ \frac{\pi}{2} & \text{dacă } k = j \neq 0 \\ \pi & \text{dacă } k = j = 0 \end{cases}$$

Polinomul Cebisev de gradul k este definit prin:

$$T_k(x) = \cos(k \arccos(x))$$

Pentru k par, $T_k(x)$ este o functie simetrica si integrala poate fi calculata folosind substitutia $x = \cos(\theta)$, astfel:

$$\int_{-1}^{1} T_k(x)\, dx = \int_{0}^{\pi} \cos(k\theta) \sin(\theta)\, d\theta$$

Pentru $k = 1$, integrala este zero din cauza antisimetriei. Pentru $k \neq 1$, folosind formula de mai sus, obținem:

$$\int_{-1}^{1} T_k(x)\, dx = \frac{(-1)^{k+1} + 1}{1 - k^2}$$

Pentru $k = 2$:

$$\int_{-1}^{1} T_2(x)\, dx = \frac{(-1)^{2+1} + 1}{1 - 2^2} = \frac{-1 + 1}{1 - 4} = \frac{0}{-3} = 0$$

(d)

Aproximatia functiei $f(x)$ data in punctul (b) este:

$$\phi(x) = \frac{c_0}{2} T_0(x) + \sum_{k=1}^{n} c_k T_k(x)$$

unde $T_k(x)$ sunt polinoamele Cebisev de prima speta si c_k sunt coeficientii de dezvoltare ai functiei $f(x)$.

Pentru a integra aceasta aproximatie de la -1 la 1, procedam astfel:

$$\int_{-1}^{1} f(x)\, dx \approx \int_{-1}^{1} \left(\frac{c_0}{2} T_0(x) + \sum_{k=1}^{n} c_k T_k(x) \right) dx$$

$$\int_{-1}^{1} f(x)\, dx \approx \frac{c_0}{2} \int_{-1}^{1} T_0(x)\, dx + \sum_{k=1}^{n} c_k \int_{-1}^{1} T_k(x)\, dx$$

Pentru a evalua aceste integrale, folosim proprietatile polinoamelor Cebisev:

$$
\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= x \\
T_k(x) &= \cos(k \cdot \arccos(x))
\end{aligned}
$$

Calculam integralele pentru $T_0(x)$ si $T_k(x)$:

$$\int_{-1}^{1} T_0(x)\, dx = \int_{-1}^{1} 1\, dx = 2$$

Pentru $k \geq 1$, rezultatele integralei $\int_{-1}^{1} T_k(x)\, dx$ sunt:

$$\int_{-1}^{1} T_k(x)\, dx = \begin{cases} 0 & \text{daca } k \text{ este impar} \\ \frac{2}{1-k^2} & \text{daca } k \text{ este par} \end{cases}$$

Astfel, integrala se poate scrie:

$$\int_{-1}^{1} f(x)\, dx \approx \frac{c_0}{2} \cdot 2 + \sum_{k=2,4,6,\ldots}^{n} c_k \cdot \frac{2}{1-k^2}$$

Formula de cuadratura ceruta:

$$\int_{-1}^{1} f(x)\, dx \approx \sum_{k=0,k \text{ par}} \frac{2c_k}{1-k^2}$$

unde k ia valorile 0, 2, 4, etc.

(e)

Codul sursa din fisierul prob2_e.m

```
f = @(x) exp(x) .* cos(x.^2);

% Definim polinoamele lui Cebisev Tk(k, x)
Tk = @(k, x) cos(k * acos(x));

% precizia dorita
toleranta = 1e-6;

k = 0;
% Calculam coeficientul pentru k=0
```

```
c0 = (2 - (k==0)) / pi * integral(@(x) f(x) .* Tk(k, x) ./
   sqrt(1 - x.^2), -1, 1);
integral_value_prev = c0;

k = 2;
c2 = (2 - (k==0)) / pi * integral(@(x) f(x) .* Tk(k, x) ./
   sqrt(1 - x.^2), -1, 1);
integral_value = integral_value_prev + c2 * (2 / (1 - k^2));

while true
    k = k + 2;

    c_k = (2 - (k==0)) / pi * integral(@(x) f(x) .* Tk(k, x)
        ./ sqrt(1 - x.^2), -1, 1);
    integral_value_new = integral_value + c_k * (2 / (1 -
       k^2));

    % Verificam convergenta
    if abs(integral_value_new - integral_value) < toleranta
        integral_value = integral_value_new;
        break;
    end

    integral_value = integral_value_new;
end

fprintf('Valoarea integralei: %.6f\n', integral_value);
fprintf('Numarul de noduri folosite: %d\n', k);
```