Tomographic imaging & reconstruction I

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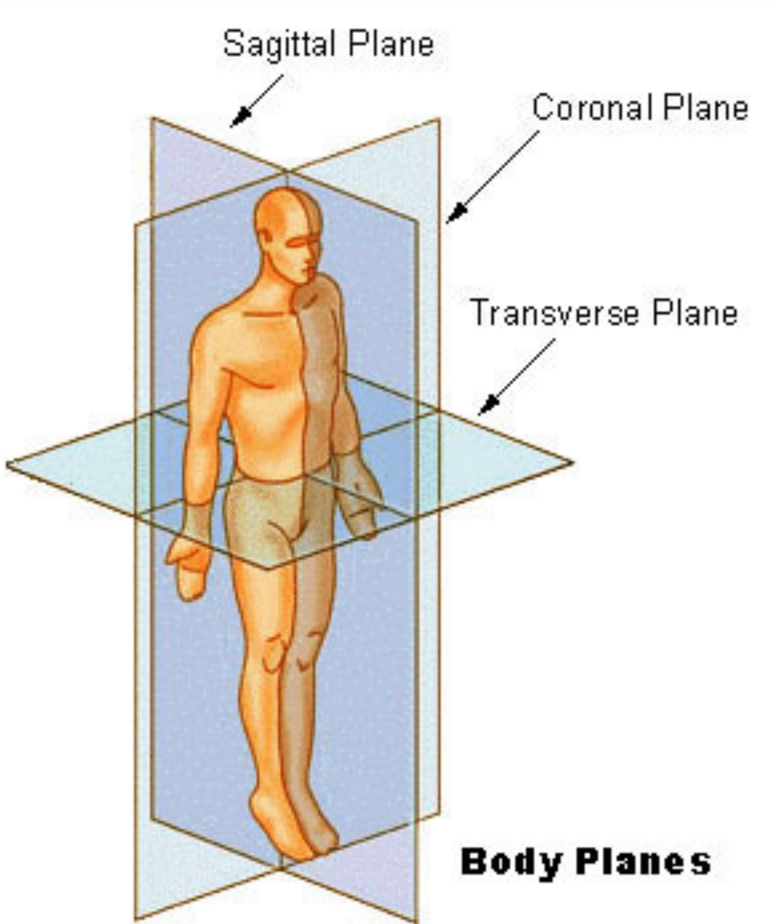
Overview

- Introduction to tomography
- Tomographic reconstruction (FBP today, ART next week)
- Artifacts
- Display



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Tomography

from ancient Greek tome “cut”,
tomos “section”
graphein “to write”,”to describe”

Literature

- A.C. Kak, M. Slaney,
  "Principles of Computerized Tomographic imaging",
  Prentice Hall International, 3rd edition (2008), IEEE Press,
  ISBN 0-87942-198-3, free electronic copy at www.slaney.org

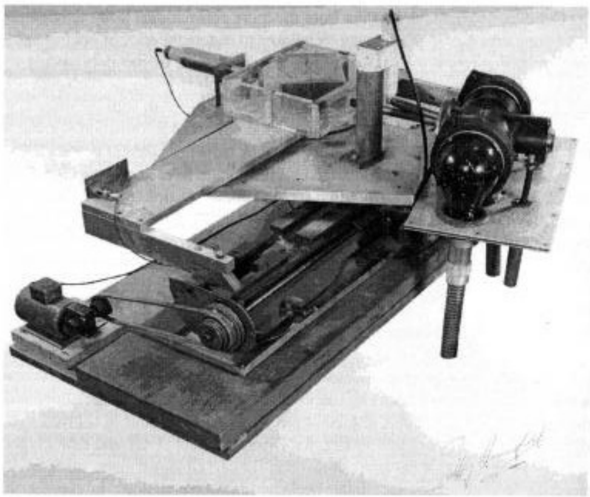
- T.M. Buzug,
  "Computed Tomography. From Photon Statistics to Modern Cone-Beam CT", Springer, 1st edition (2008), ISBN-13: 978-354039407,
  link.springer.com/book/10.1007/978-3-540-39408-2

- Philip Willmott, Chapter 7.2 „Computed Microtomography"
  An introduction to Synchrotron Radiation, Wiley, 2011
  https://ebookcentral.proquest.com/lib/Munchentech/detail.action?docID=819229#

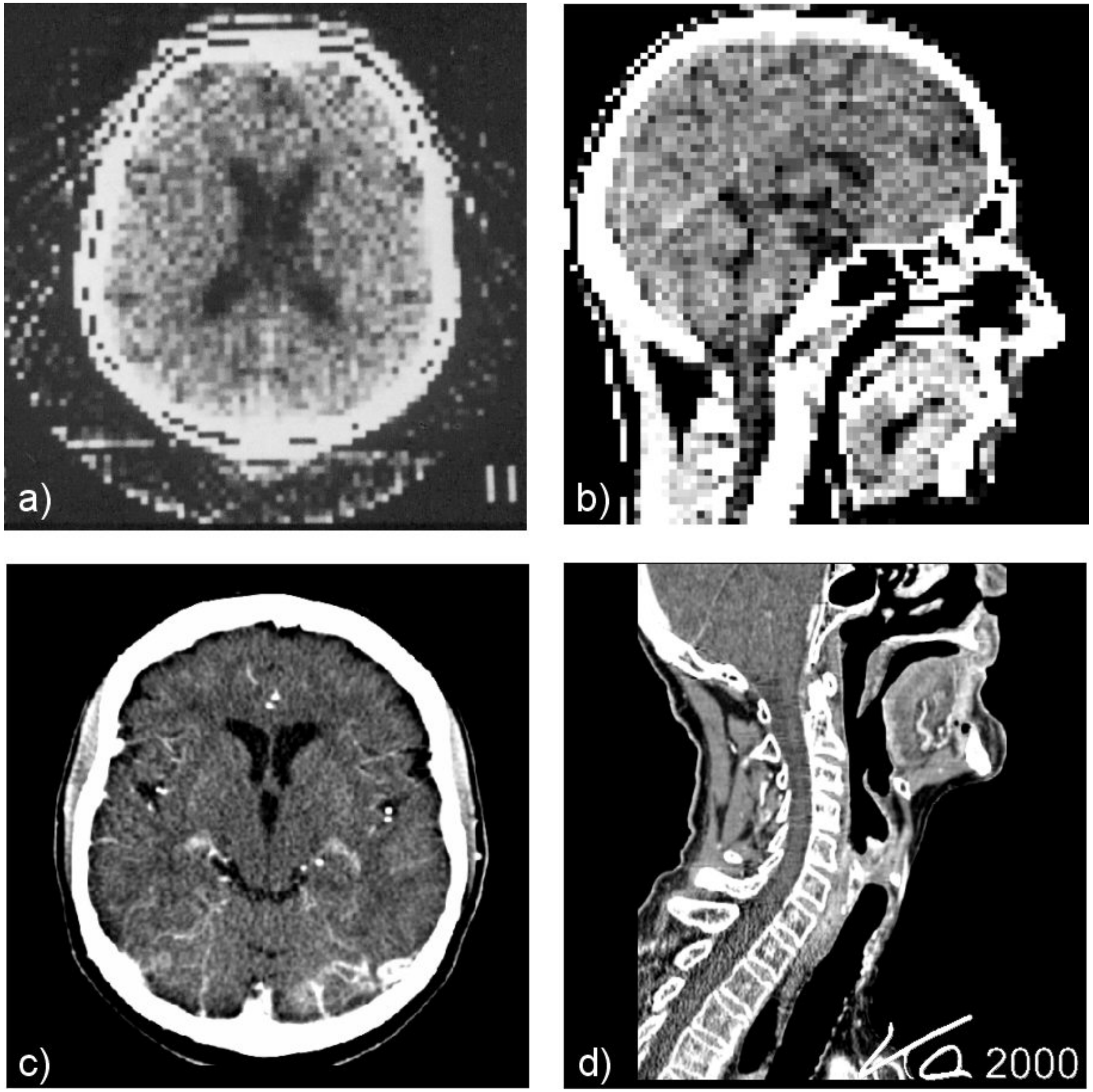
- W.A. Kalender, "Computed tomography",
  Publicis publishing, 3rd edition (2011),
  ISBN-13: 978-0471292883
  https://opac.ub.tum.de/TouchPoint/search.do?methodToCall=selectLanguage&Language=en

History of tomographic imaging

Computed (X-ray) Tomography (CT)



Hounsfield's original test lathe

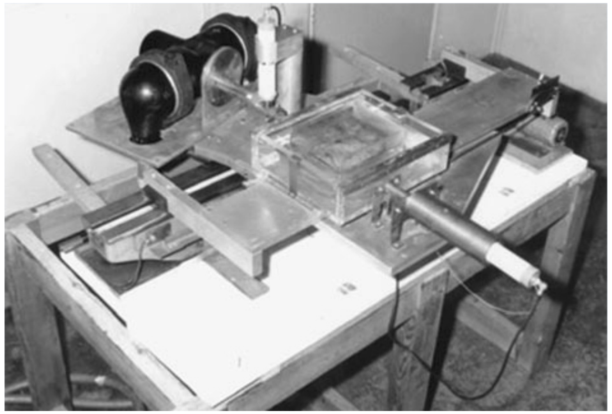


1974, 80x80 pixels

2000, 512x512 pixels, spiral CT

source: W. Kalender, Publicis, 3rd ed. 2011

History of tomographic imaging

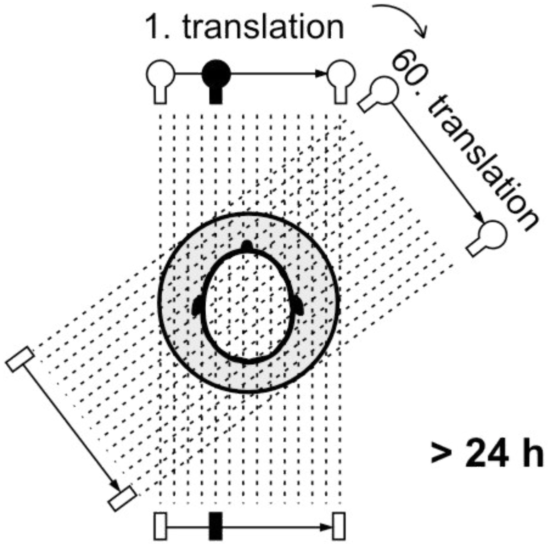


W. Bautz und W. Kalender, Radiologe 2005 · 45:350–355

pencil beam (1970)

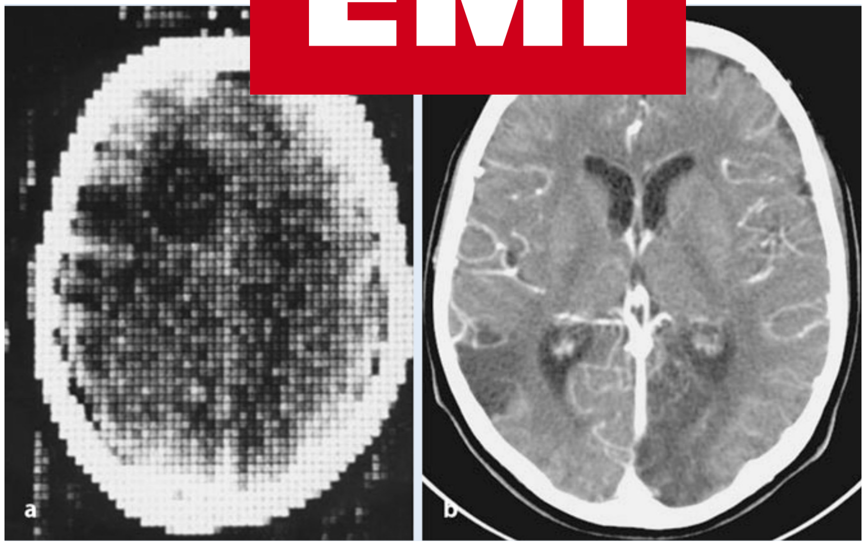
Godfrey Newbold Hounsfield
1919-2004
1979 Nobel price in medizine
W. Bautz and W. Kalender, Radiologe 2005 · 45:350–355

EMI



1st generation: translation / rotation

Willi Kalender, Computed Tomography, Publicis Publishing



W. Bautz und W. Kalender, Radiologe 2005 · 45:350–355

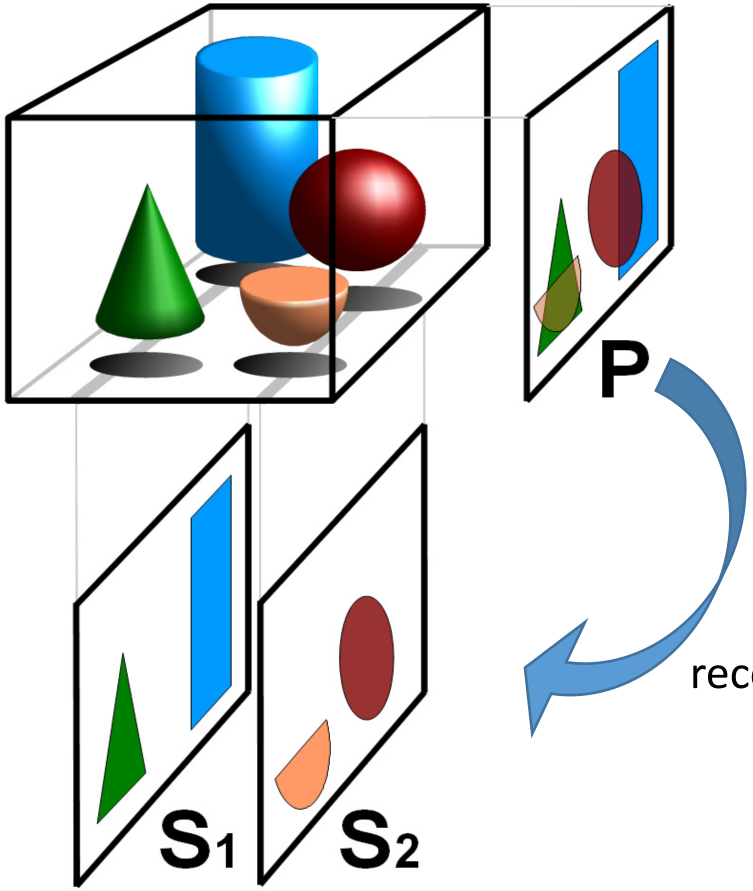
Reconstructions from projections

Reconstruction of volume from projections

Digitization into voxels

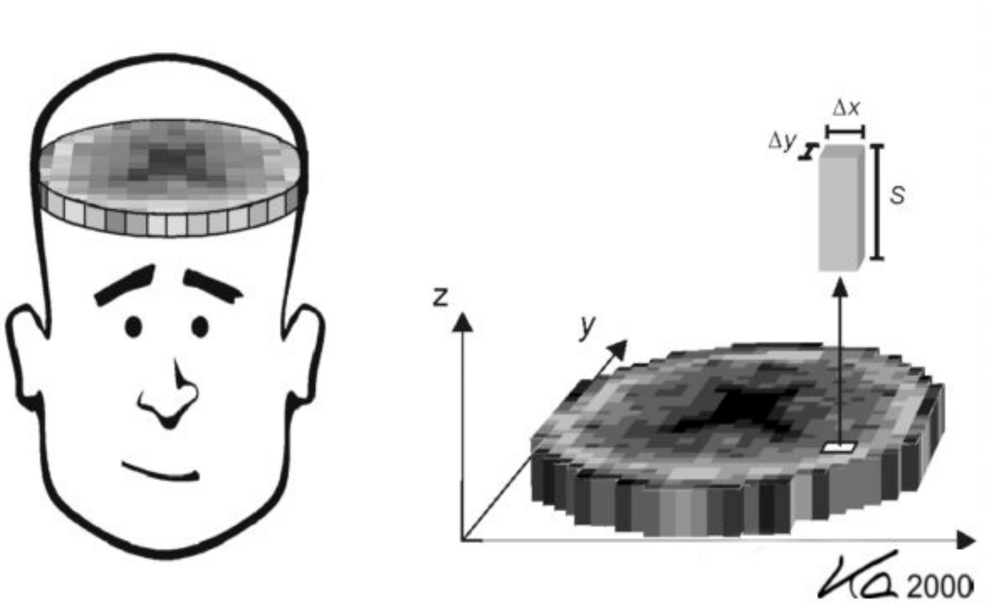
3D unknown volume

2D projection



reconstruction

2D slices after reconstruction

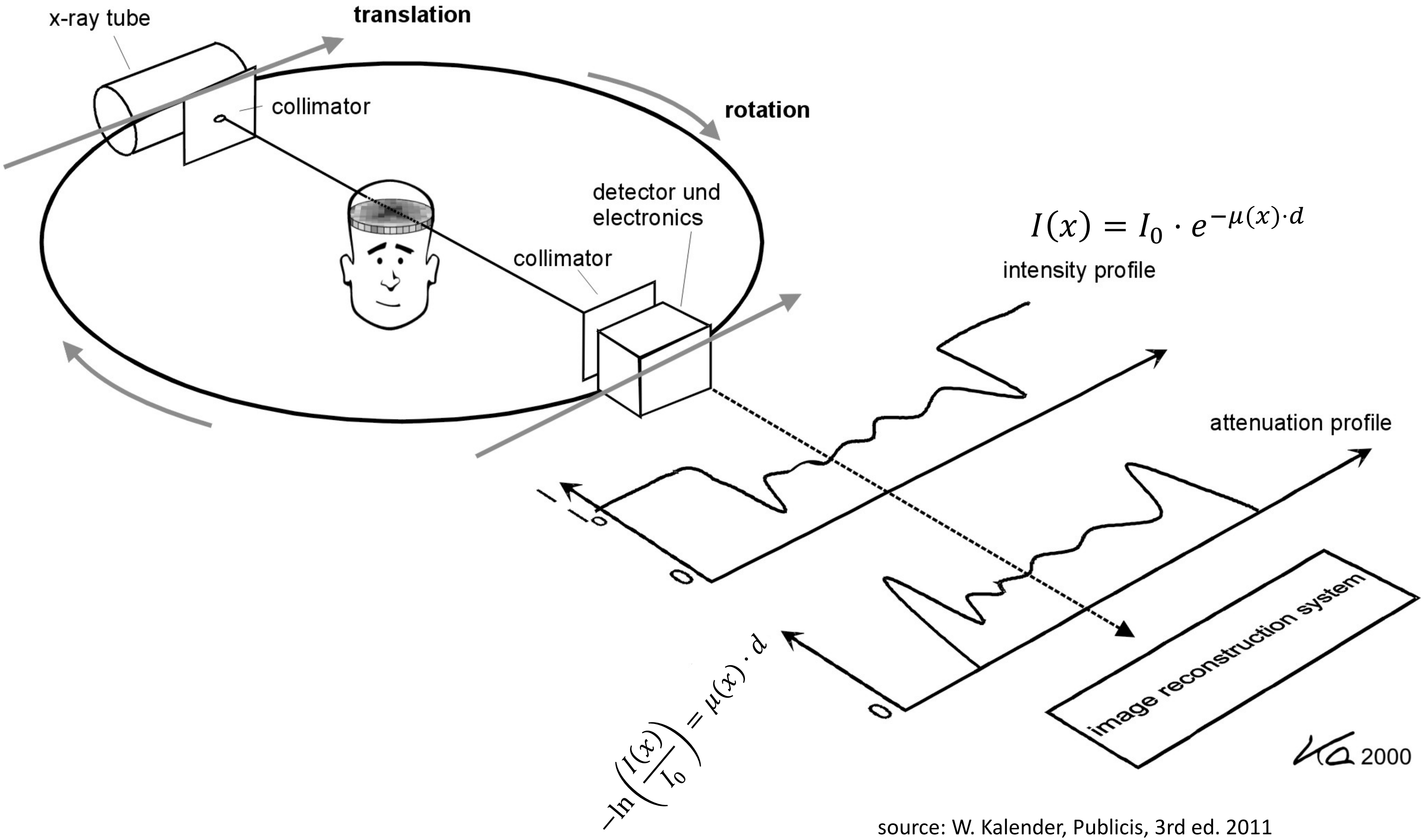


often: $\Delta x = \Delta y$

but $\Delta x \neq \Delta S$

source: W. Kalender, Publicis, 3rd ed. 2011

Principle of X-ray CT



source: W. Kalender, Publicis, 3rd ed. 2011

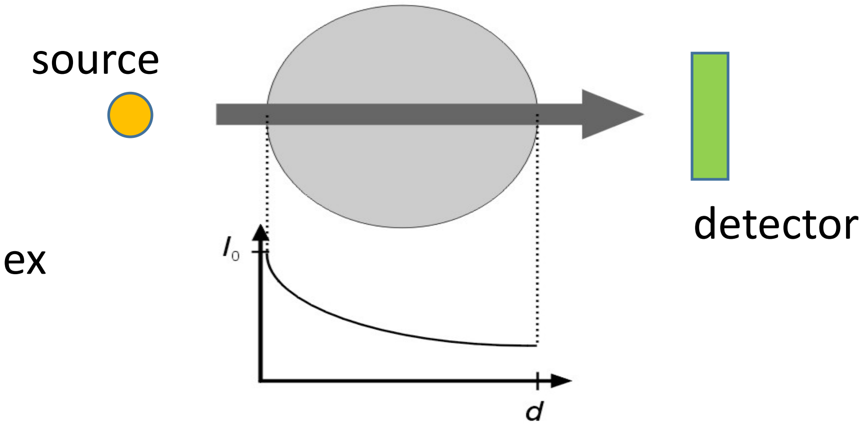
Projections & Line integrals

A single material, monochromatic

$$I = I_0 \cdot e^{-\mu d}$$

μ attenuation coefficient
$\mu = 2k\beta$ with β
imaginary part of refraction index

$$p = -ln\frac{I}{I_0} = \mu d \quad \rightarrow \quad \mu = \frac{p}{d}$$

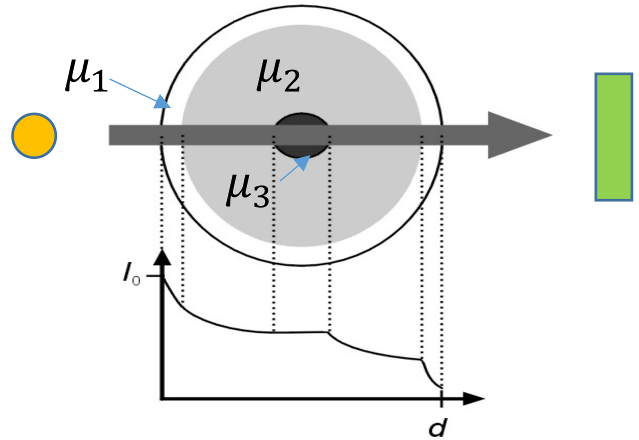


B multiple materials, monochromatic

$$I = I_0 \cdot e^{-\sum_i \mu_i d_i}$$

$$p = -ln\frac{I}{I_0} = \sum_i \mu_i d_i$$

$\mu_i = ?$ d_i unknown

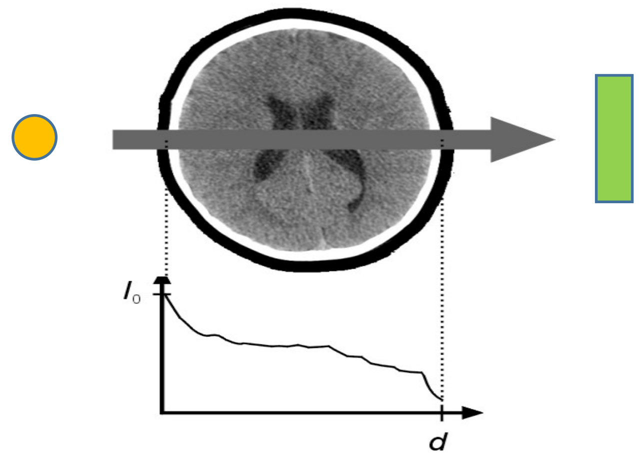


C multiple materials, polychromatic

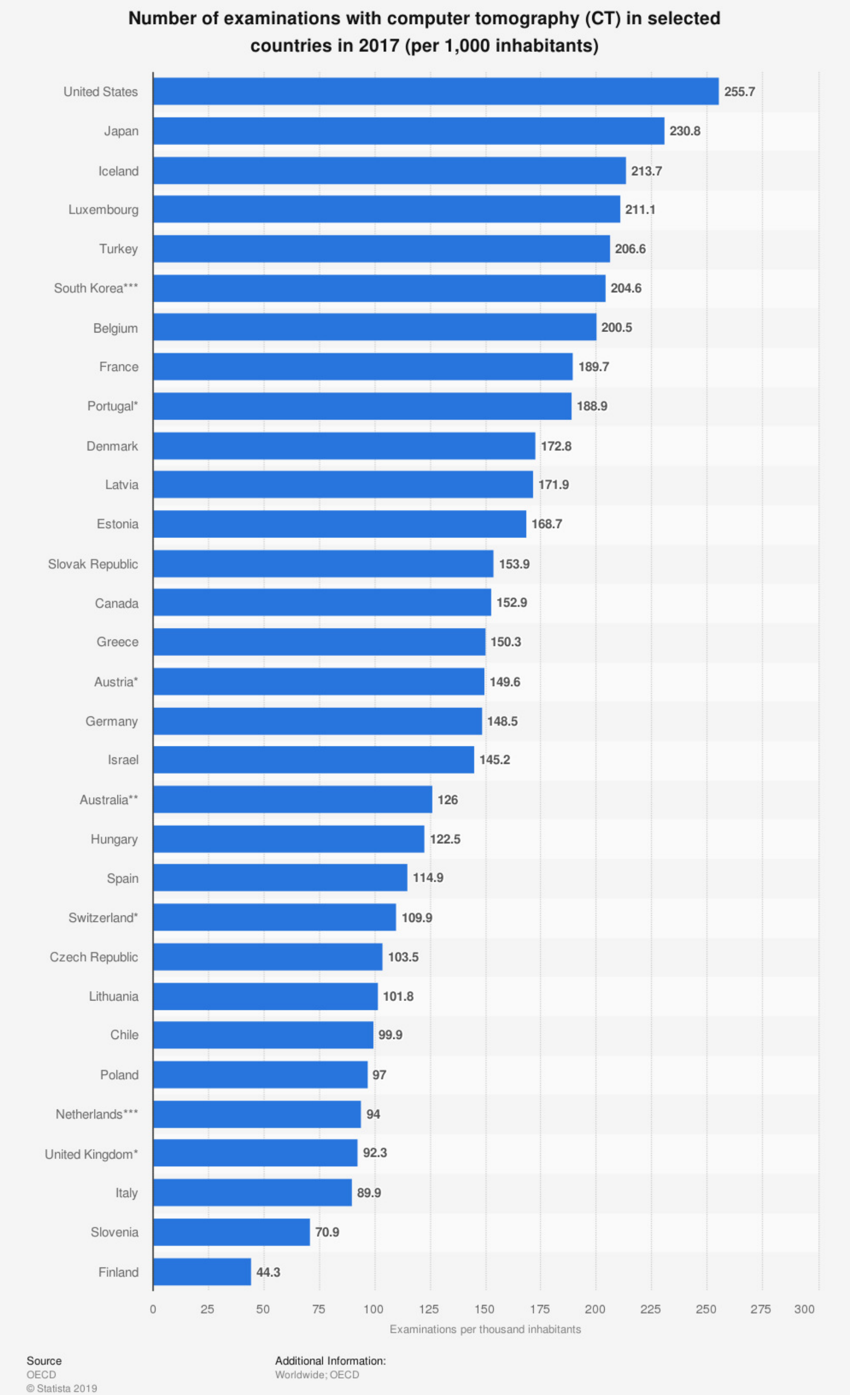
$$I = \int I_0(E) \cdot e^{-\int \mu(E)\, ds}\, dE$$

$\mu_i = ?$

spectrum of source
detection efficiency of detector
absorption of windows …



Computed Tomography, CT



PHILIPS iCT

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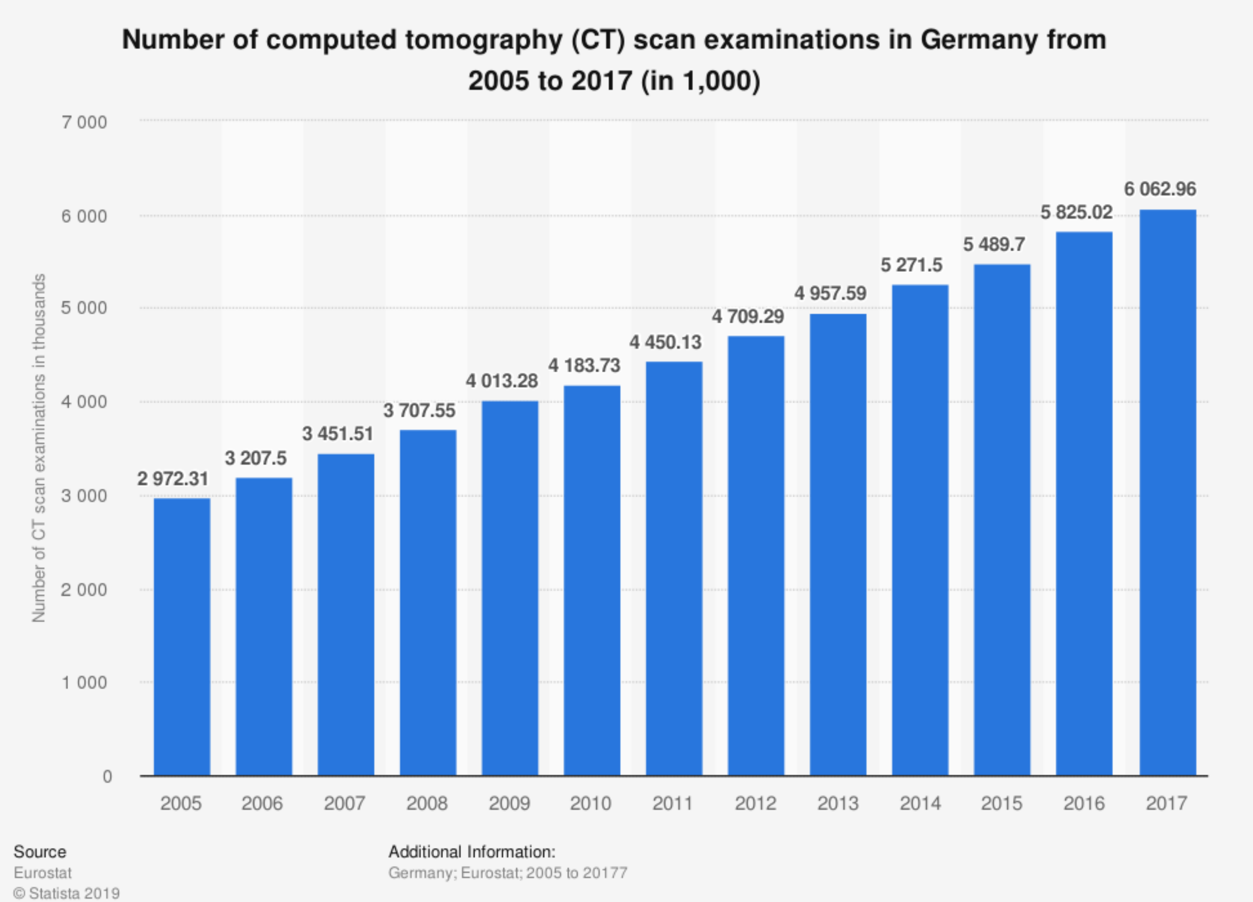
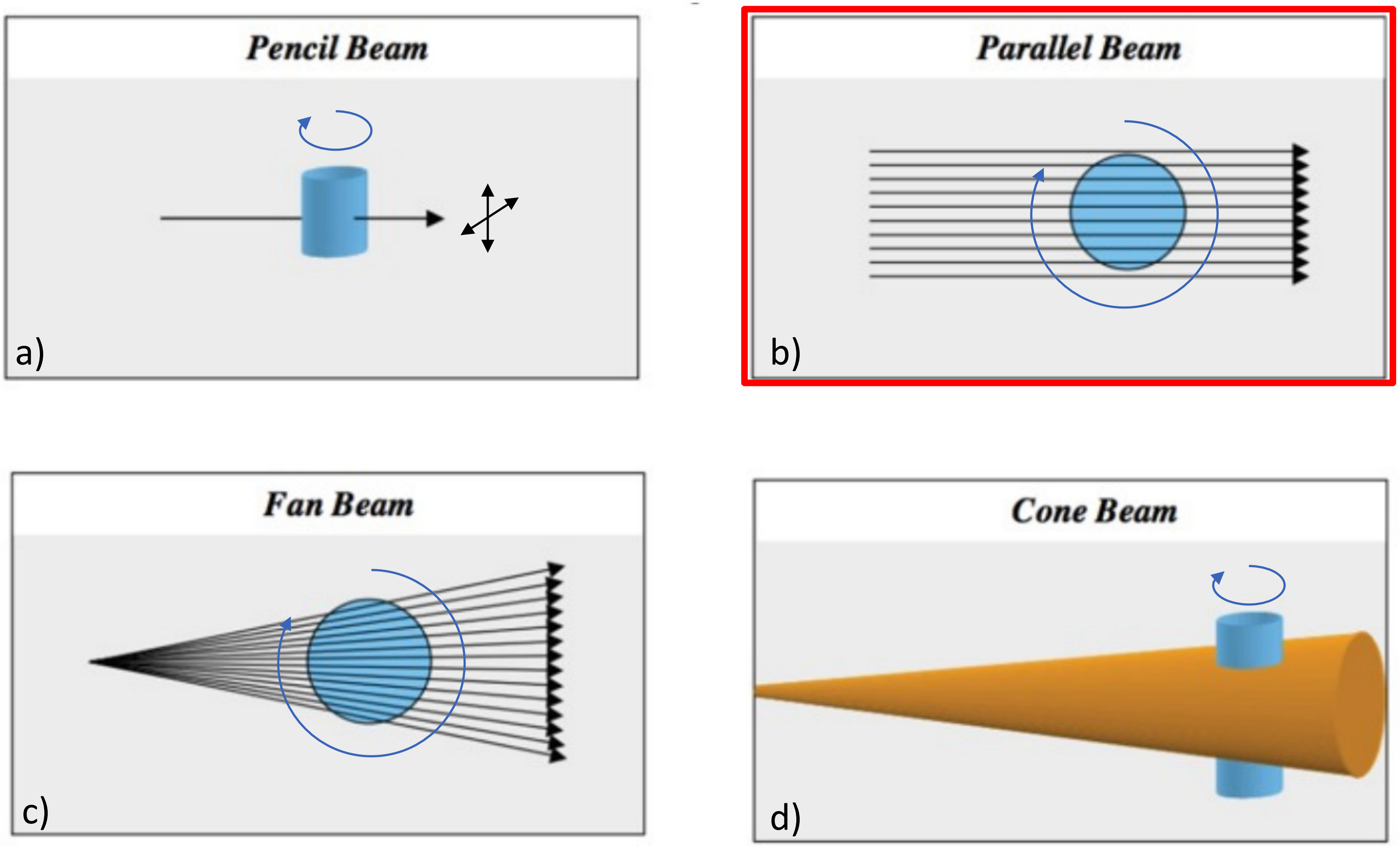


Basic ideas of CT

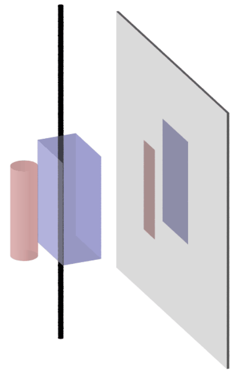
Imaging geometries

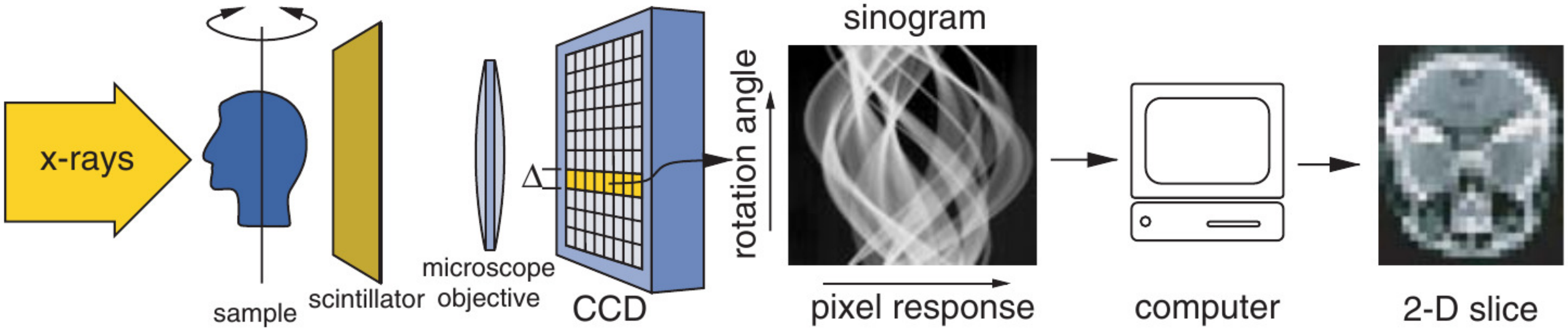


Basic ideas of CT

Parallel beam geometry

- Computed tomography (CT)
  - in parallel-beam geometry:
    slices (detector lines) are independent
    → look just at one slice





Willmott, Fig. 7.5

Basic ideas of CT

Parallel beam geometry

- Computed tomography (CT)
  - in parallel-beam geometry: slices (detector lines) are independent
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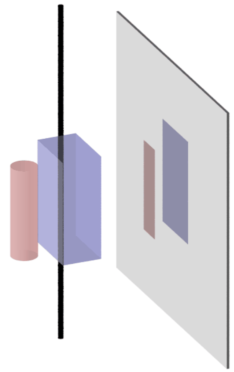
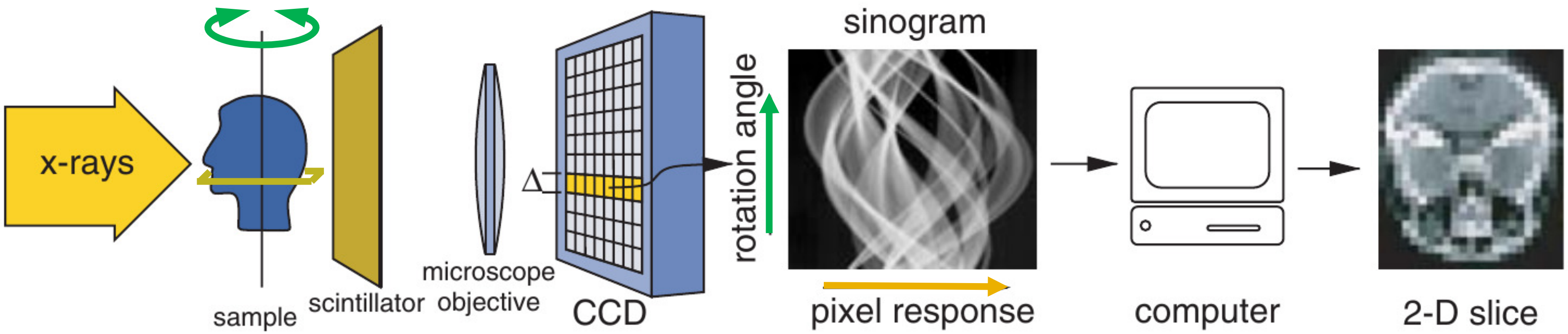
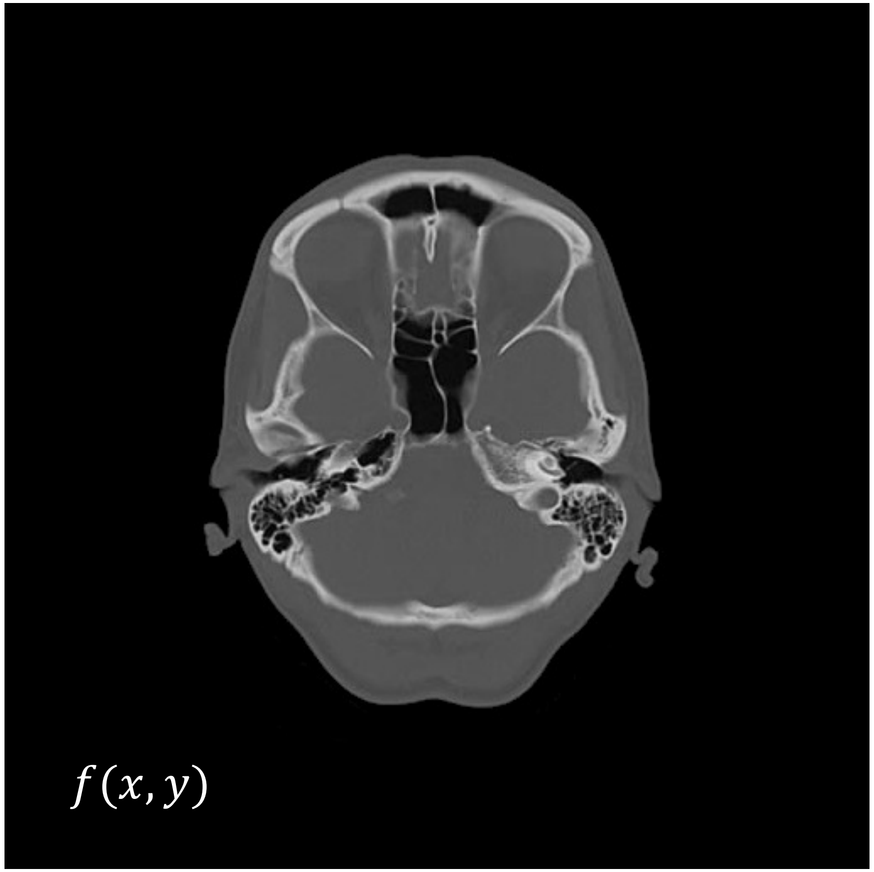


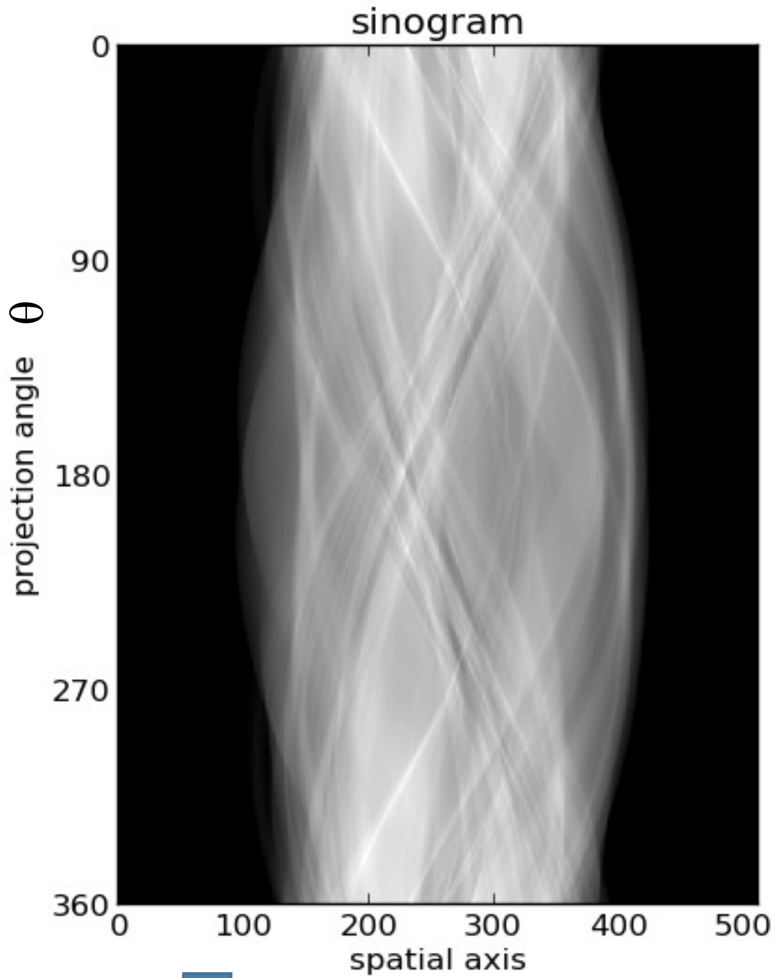


Willmott, Fig. 7.5

Sinograms & Forward projection

Sinogram: representation of projection-data measured by a single detector line



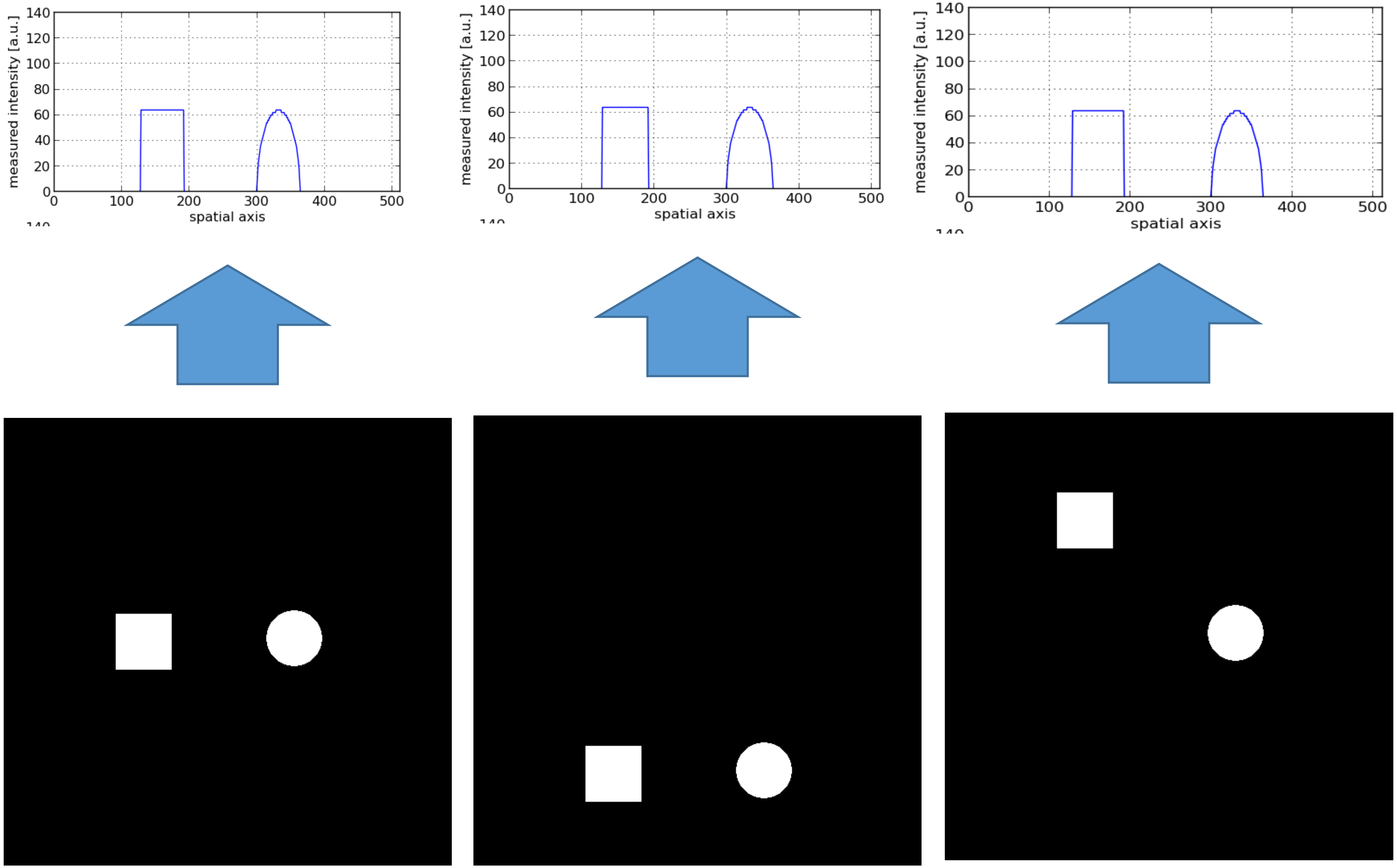


detector row

reconstruction ?

Sinograms & Forward projection

A single projection is not enough to reconstruct sample distribution

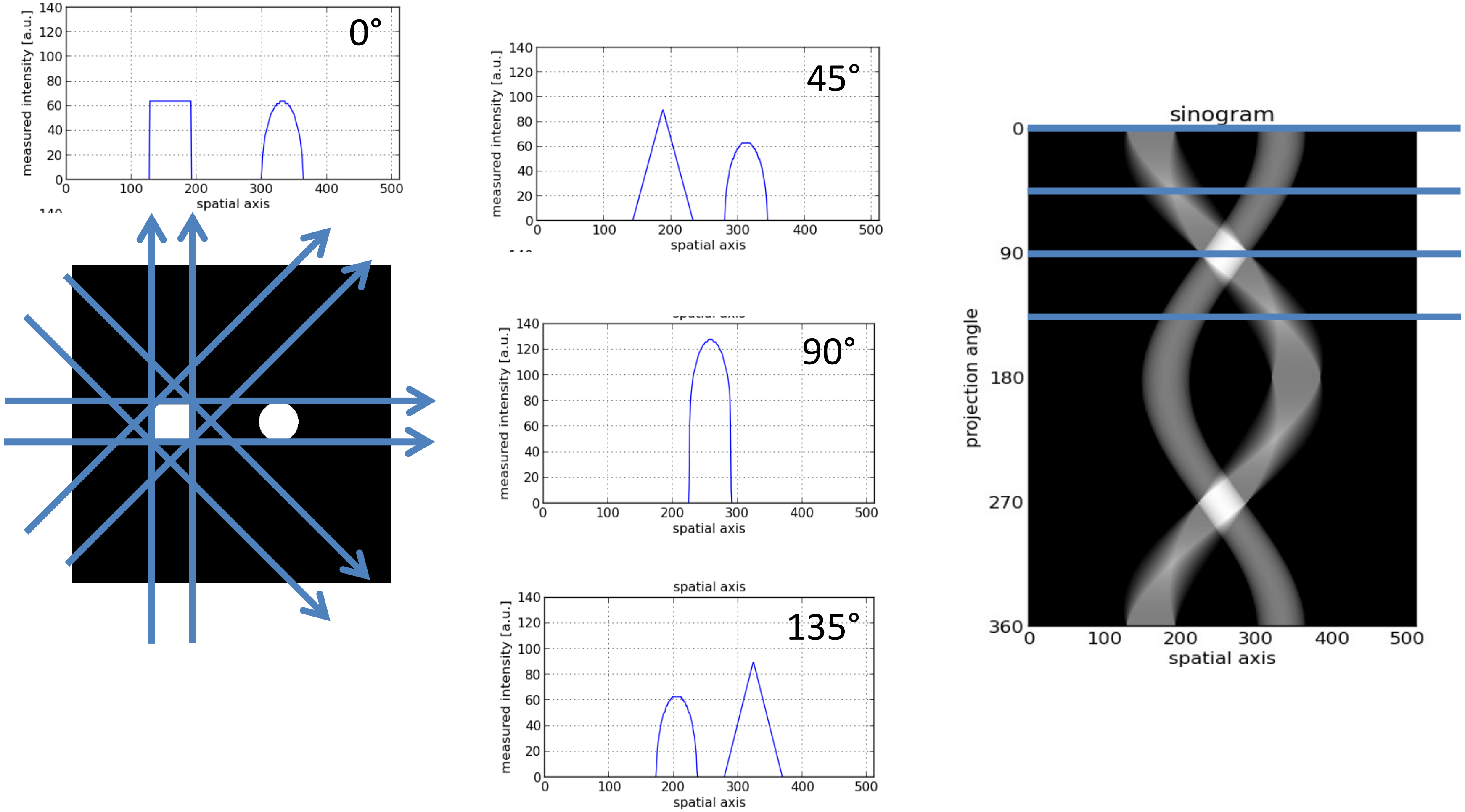


Projection within a detector row

Slice

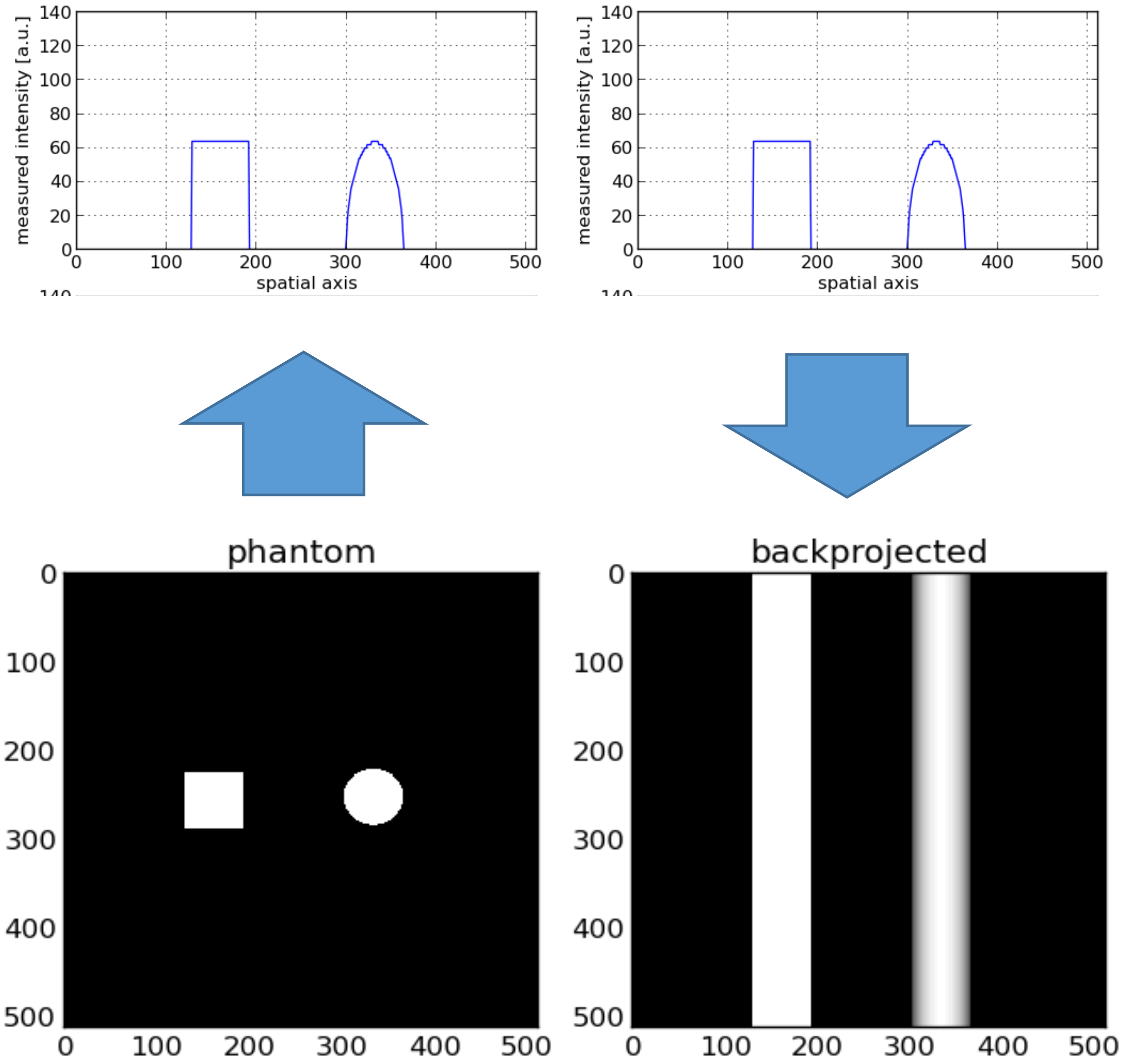
Sinograms & Forward projection

Sinogram: representation of projection-data measured by a single detector line



Back projection

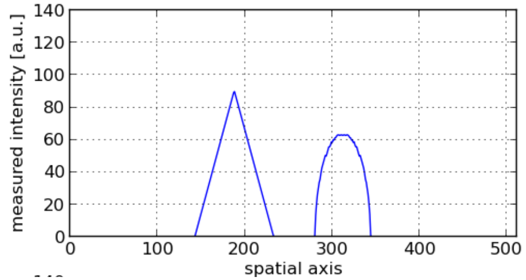
Back project intensity profiles evenly over full slice along the projection direction

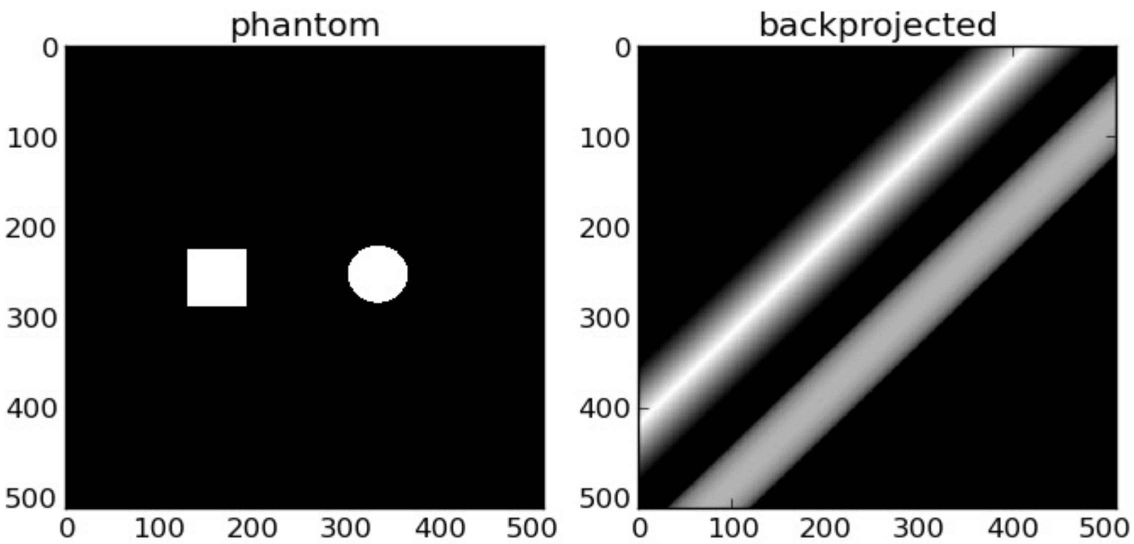


← gray values within a detector row

Back projection

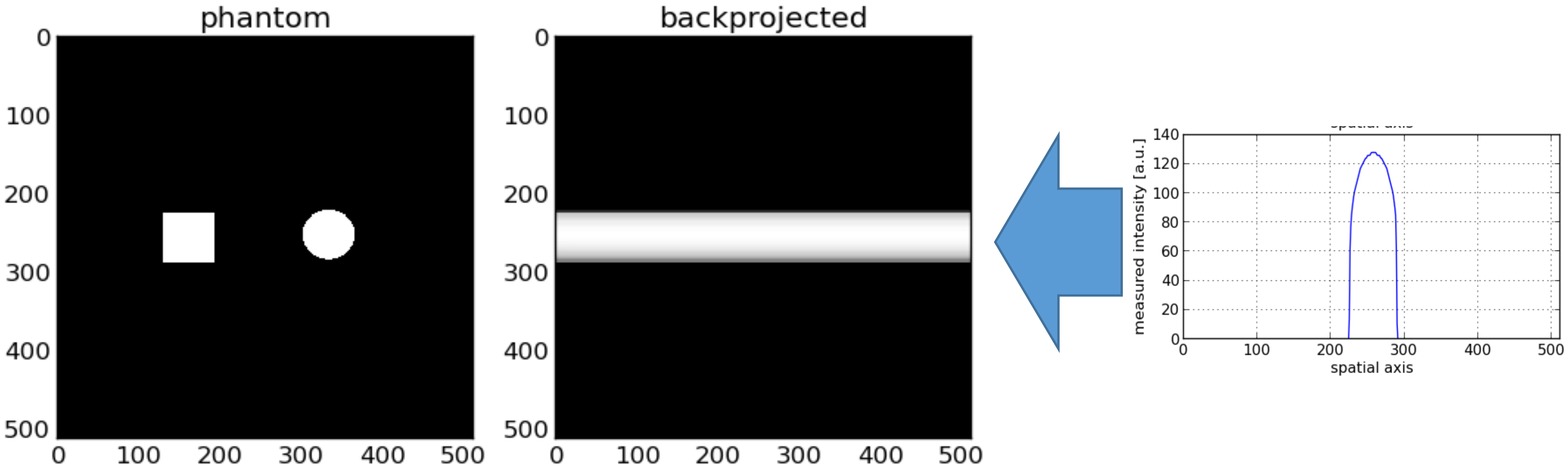
Back project intensity profiles evenly over full slice along the projection direction





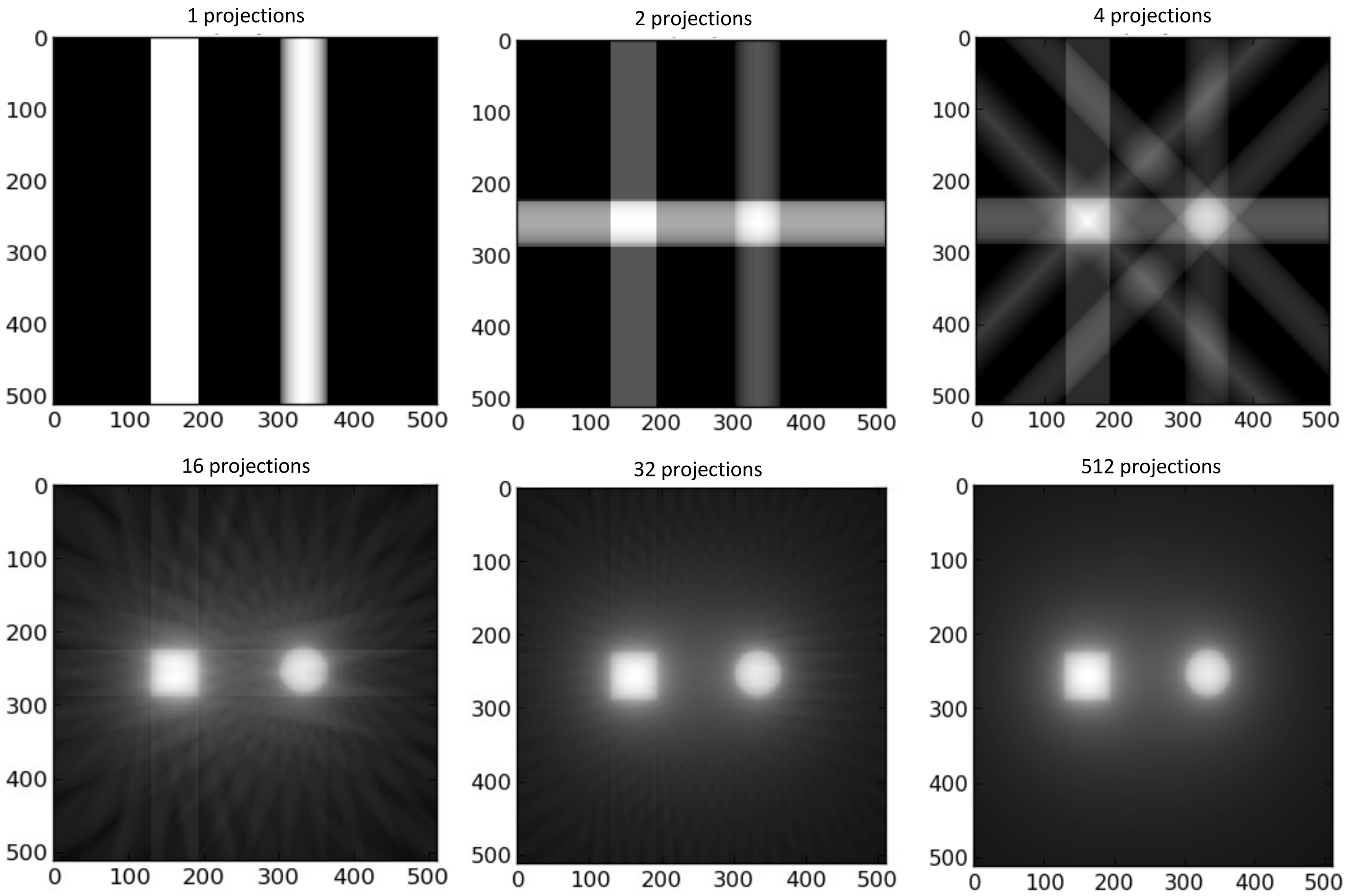
Back projection

Back project intensity profiles evenly over full slice along the projection direction



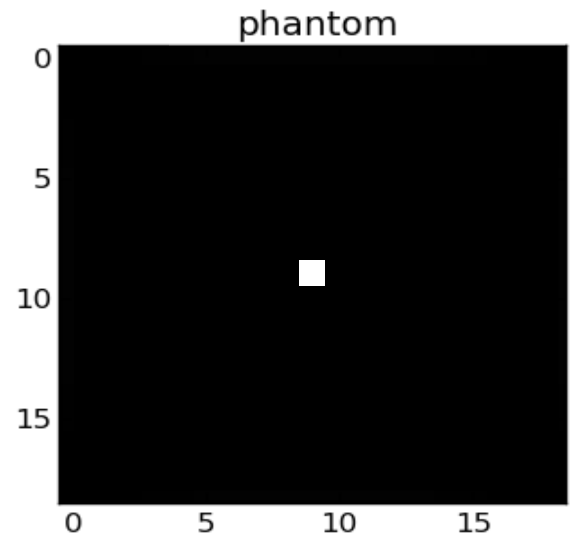
Back projection

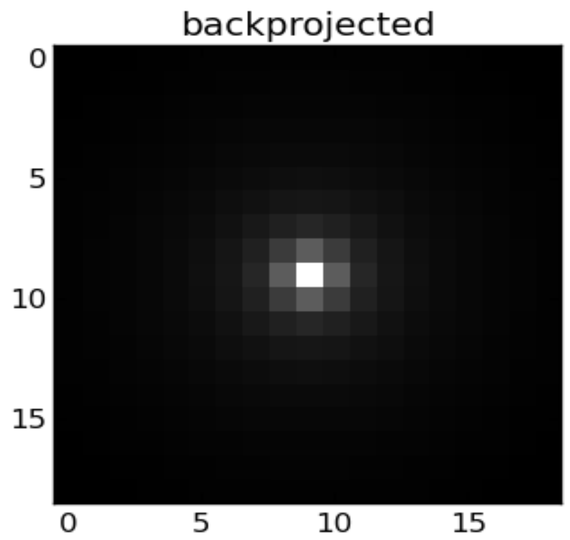
Sum up all backprojections

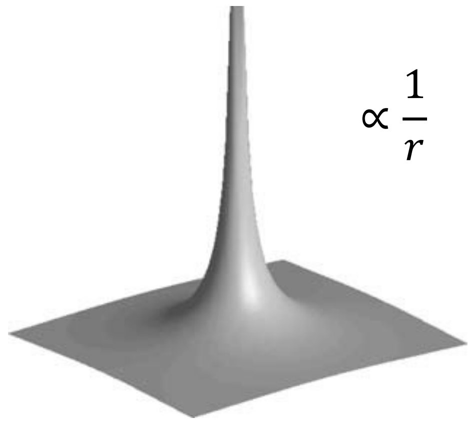


PSF of back projection

Backprojection algorithm causes smearing

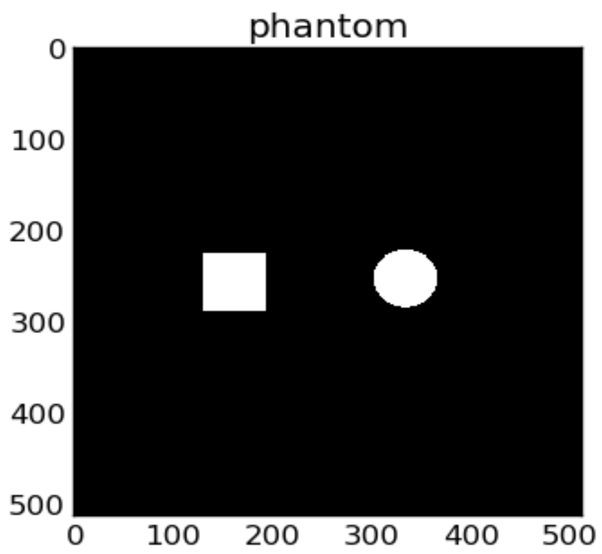


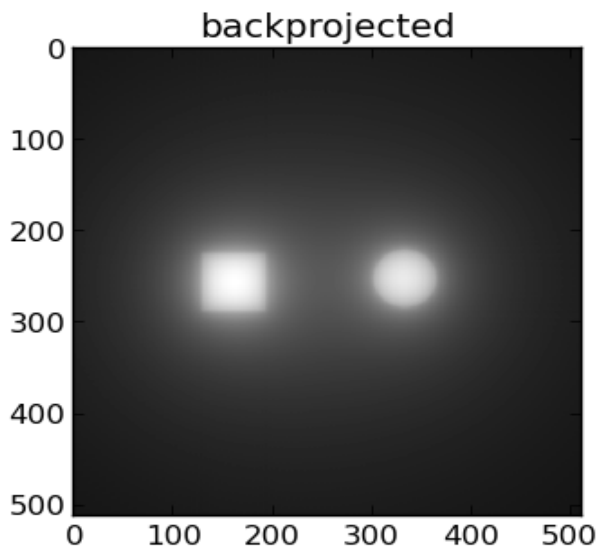




$$\propto \frac{1}{r}$$

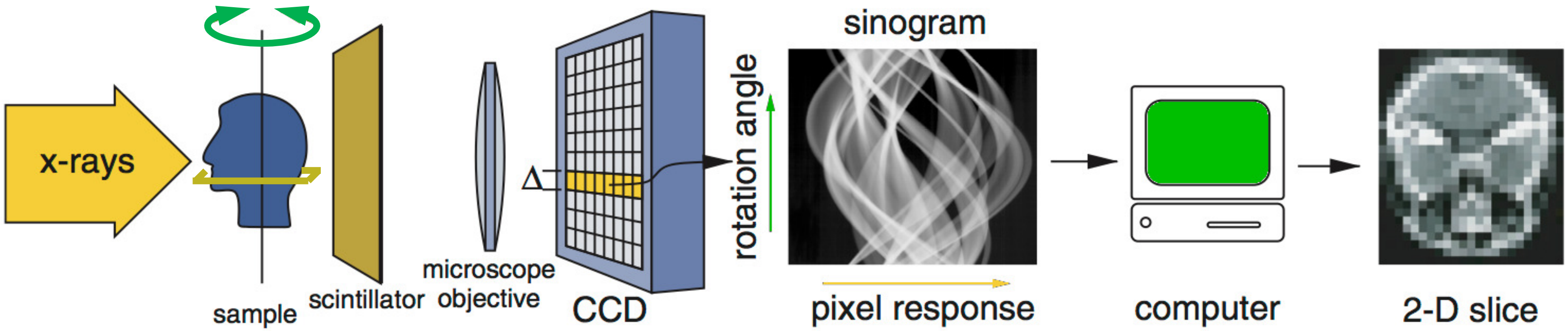
point spread function, PSF





$$f(x, y) \otimes \frac{1}{r}$$

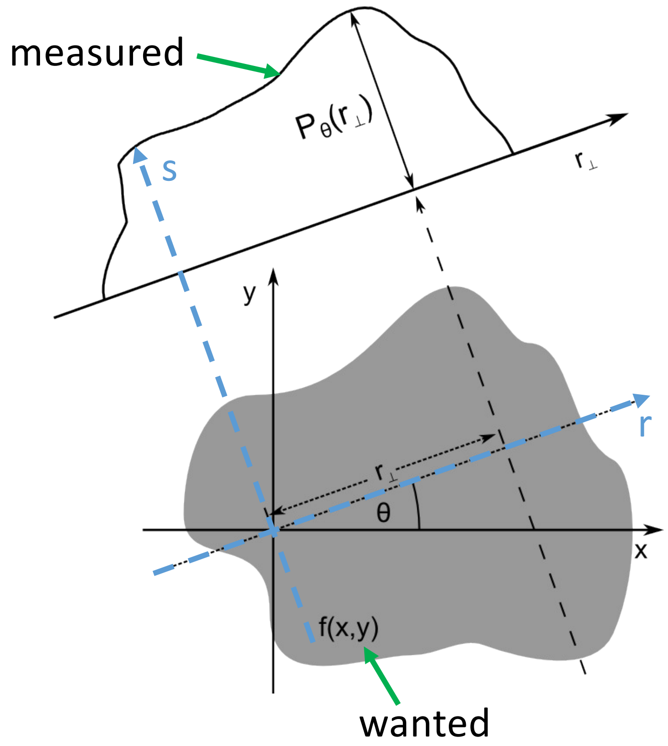
- Computed tomography (CT)
  in parallel-beam geometry: slices
  (detector lines) are independent
  → look just at one slice

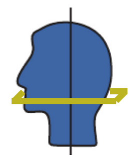


Willmott, Fig. 7.5

Basic ideas of CT

$$P_\theta(r_\perp) = -\ln\left(\frac{I(\theta, r_\perp)}{I_0}\right)$$





Computed tomography (CT)

in parallel-beam geometry: slices (detector lines) are independent

→ look just at one slice

→ object in slice described by

- 2D function $f(x,y)$

→ projection is 1D line $P_\theta(r_\perp)$,

- related to $f(x,y)$ by Radon transform

Radon transform

- rotated coordinate system

$$r = x \cdot cos\theta + y \cdot sin\theta$$

$$s = -\, x \cdot sin\theta + y \cdot cos\theta$$

- Radon transform

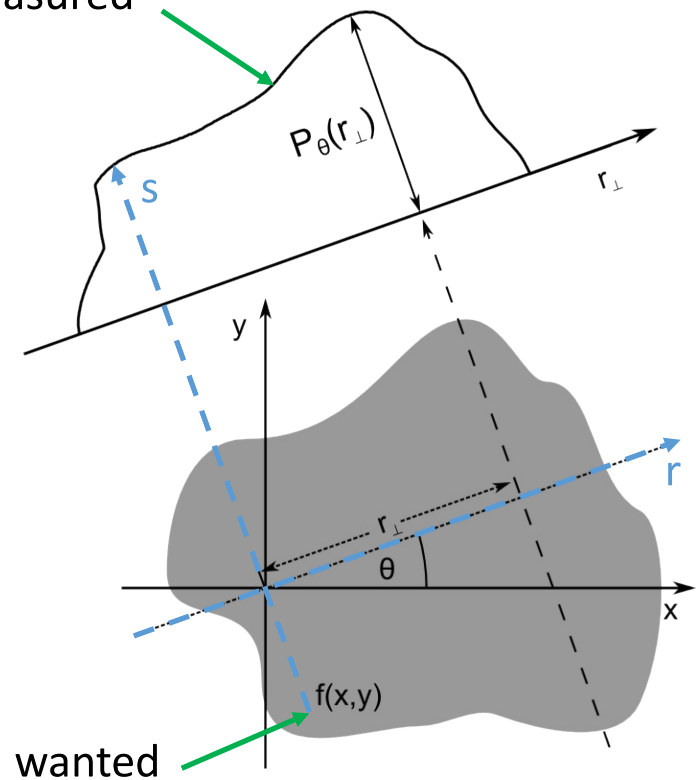
$$P_\theta(r_\perp) = R_\theta\{f(x,y)\}$$

$$= \int_{-\infty}^{\infty} f(r_\perp, s)\, ds$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, \underbrace{\delta(x \cdot cos\theta + y \cdot sin\theta \, - r_\perp)}_{\text{line perpendicular to } \vec{r} \text{ in a distance } r_\perp \text{ from the centre}}\, dxdy$$

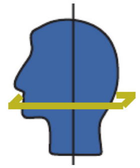
$$P_\theta(r_\perp) = -\ln\left(\frac{I(\theta, r_\perp)}{I_0}\right)$$

measured



wanted

$f(x,y) = \mu(x,y)$



Radon transform and Fourier transform

$$P_\theta(r_\perp) = \int_{-\infty}^{\infty} f(r_\perp, s)\, ds = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, \delta(x \cdot cos\theta + y \cdot sin\theta\ - r_\perp)\, dxdy$$

Fourier transform of Radon transform

$$FT(P_\theta(r_\perp)) = \tilde{P}_\theta(q) = \int_{-\infty}^{\infty} P_\theta(r_\perp)\ \cdot\ e^{-2\pi i \cdot q r_\perp}\, dr_\perp\ = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(r_\perp, s)\, ds\right) \cdot\ e^{-2\pi i \cdot q r_\perp}\, dr_\perp$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, \delta(x \cdot cos\theta + y \cdot sin\theta\ - r_\perp)\, dxdy\right) \cdot\ e^{-2\pi i \cdot q r_\perp}\, dr_\perp$$

we only have contributions for $x \cdot cos\theta + y \cdot sin\theta\ - r_\perp = 0$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \cdot\ e^{-2\pi i \cdot q(x \cdot cos\theta + y \cdot sin\theta)}\, dxdy$$

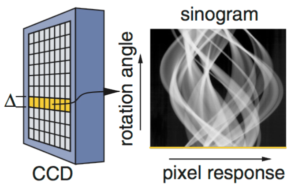
integration over $r_\perp$ reduces the integration over x,y to the lines with distances $r_\perp$

with $u = q \cdot cos\theta, v = q \cdot sin\theta$

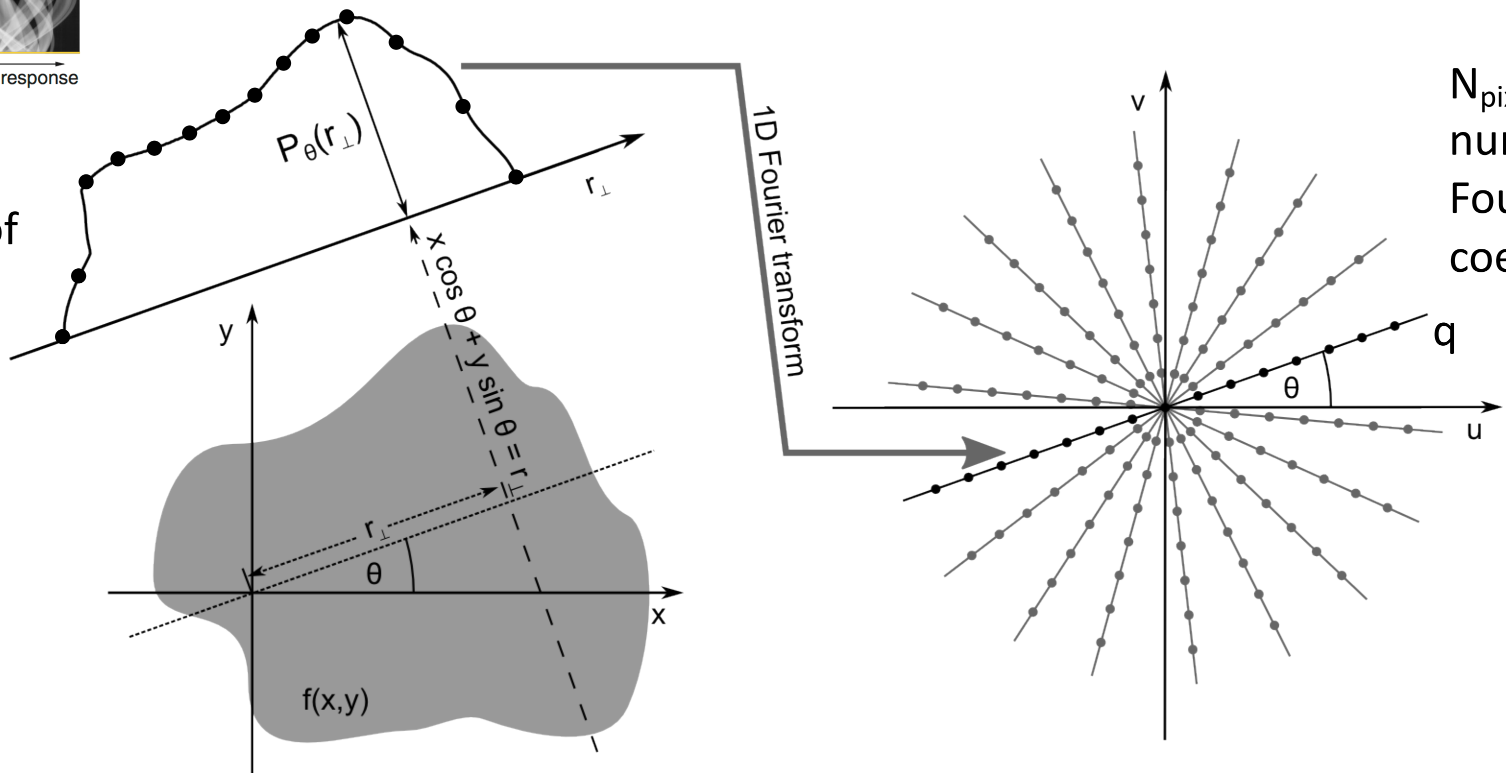
$$= \boxed{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \cdot\ e^{-2\pi i \cdot xu}\ \cdot\ e^{-2\pi i \cdot yv}\, dxdy}\ = \tilde{P}_\theta(u,v)\ = \tilde{P}_\theta(q \cdot cos\theta, q \cdot sin\theta)$$

This is a two dimensional Fourier transform in the coordinates u,v but only on a line through the center of the coordinate system. The coordinate on the line is q, the line has an angle θ to the u-axis and x-axis.

Radon transform related to Fourier transform



N_{pix}
number of
pixels

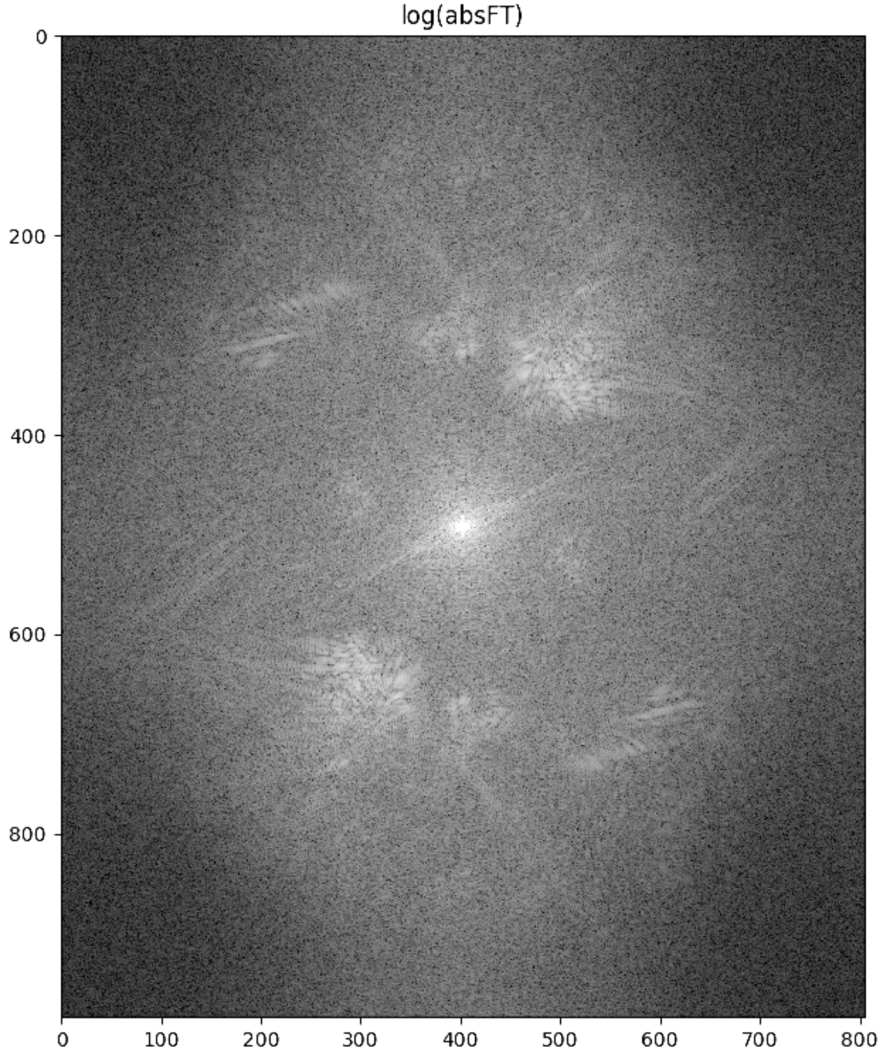


N_{pix}
number of
Fourier
coefficience

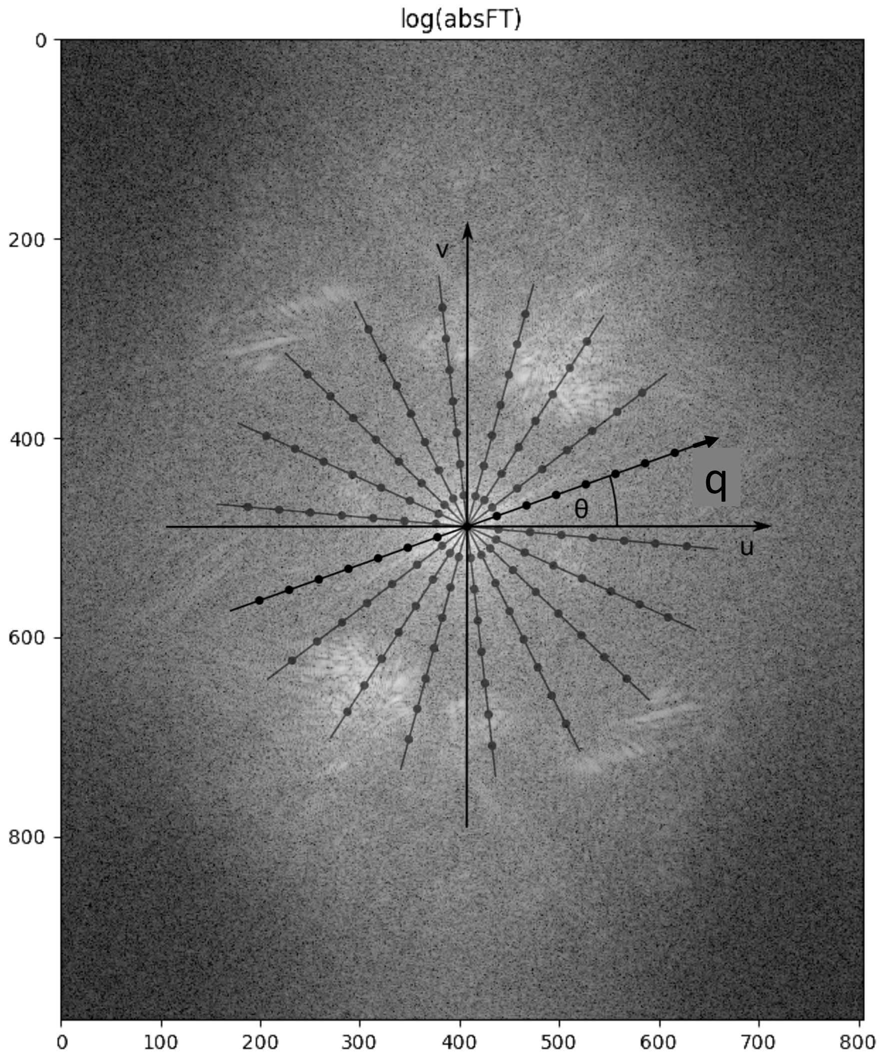
1D-FT of the projection $P_\theta(r_\perp)$= axial line through the 2D FT of the original function $f(x,y)$ at same angle.

→ get full 2D-FT of $f(x,y)$ by combining 1D-FTs of many angles.

„slice“ in real space



DFT of „slice“



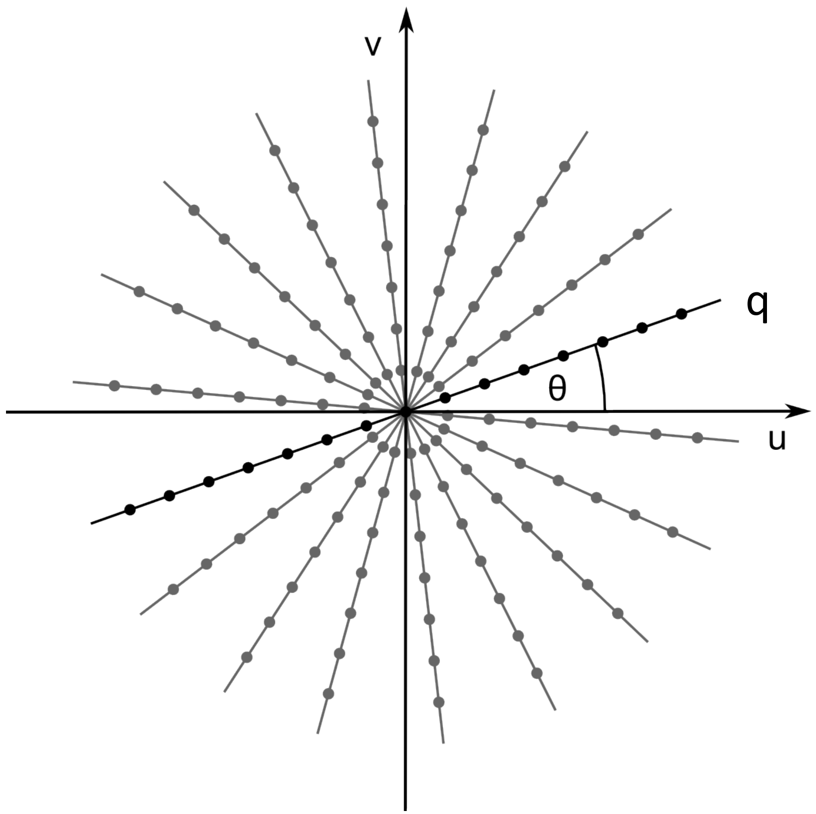
DFT of „slice“

only the gray values (= Fourier coefficients) at the black points are used for the inverse Fourier transform

Problems:

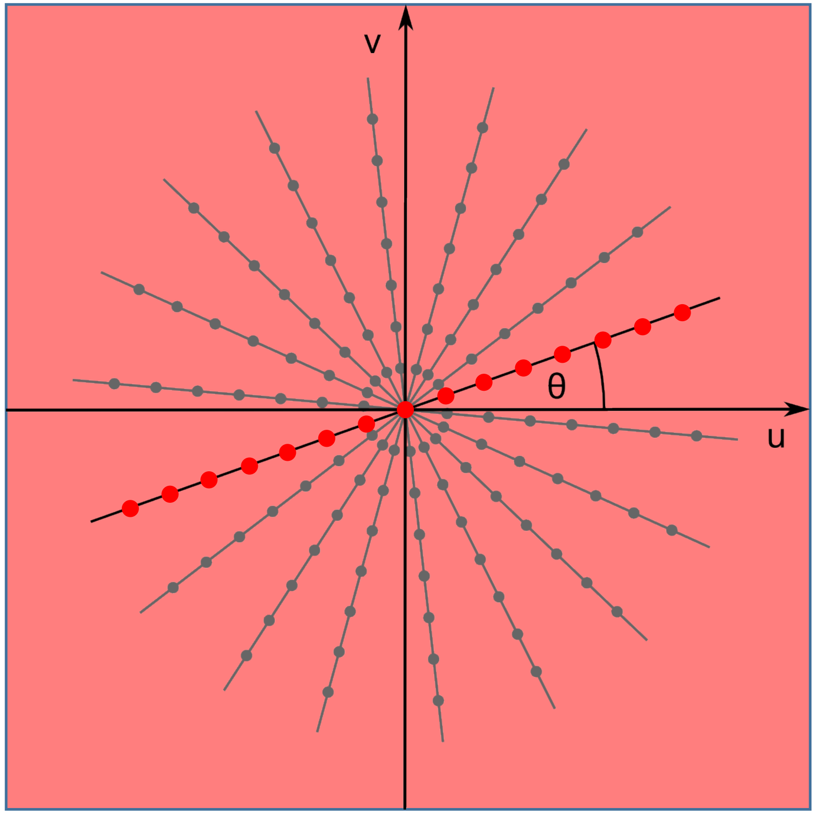
- information on polar coordinates
- information dense in the center (at low frequencies)
  information thinned out at high freq. (lose sharpness)
- in this excample the highest frequencies are lost

From Fourier space back to real space



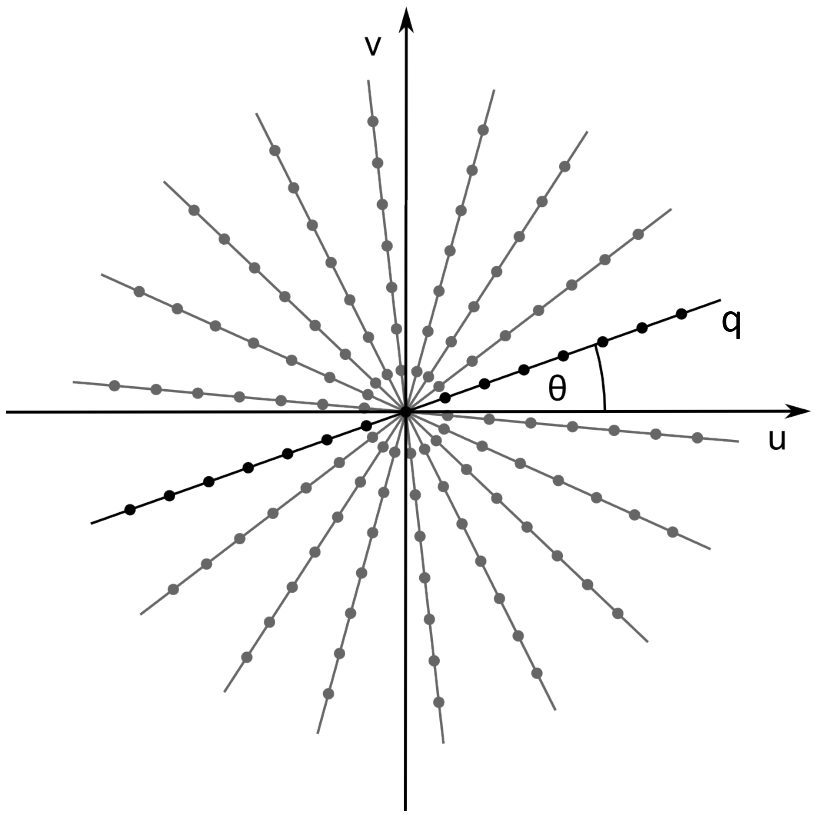
From Fourier space back to real space

$$f(x,y) = iFT\big(F(u,v)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v) \cdot e^{+2\pi i(ux+vy)}\, du\, dv$$



$F(u,v)$

From Fourier space back to real space



$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\cdot e^{+2\pi i(ux+vy)}du\,dv$$

<u>Problem:</u> points in Fourier space measured on polar grid

<u>Solution:</u> express inverse FT in polar coordinates (q,θ)

$$(u,v) = (q\,cos\theta, q\,sin\theta)$$

and the 2D volume $du\,dv = |q|\,dq\,d\theta$

with $q \in (-\infty,\infty)$

$\theta \in [0,\pi)$

From Fourier space back to real space

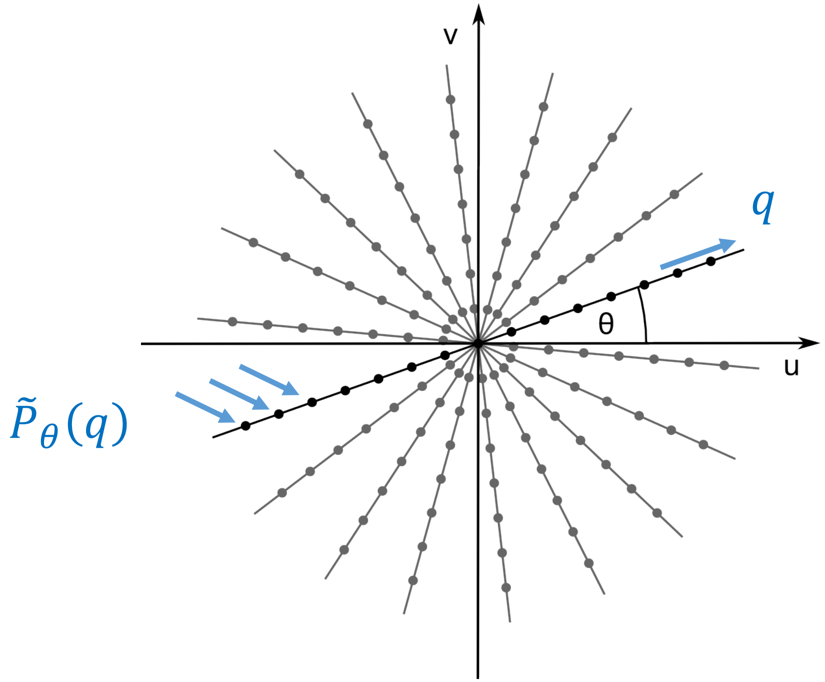
$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\cdot\ e^{2\pi i(ux+vy)}\ du\,dv$$

$$= \int_{0}^{\pi}\int_{-\infty}^{\infty} F(q\cdot cos\theta, q\cdot sin\theta)\cdot\ e^{2\pi iq(x\cdot cos\theta+y\cdot sin\theta)}\,|q|\ dq\ d\theta$$

$$r_\perp = x\cdot cos\theta + y\cdot sin\theta$$

the function F(q, θ) is the Fourier transform $\tilde{P}_\theta(q)$ of the projection $P_\theta(r_\perp)$ at an angle θ

$$= \int_{0}^{\pi}\left[\int_{-\infty}^{\infty} \tilde{P}_\theta(q)\cdot|q|\cdot e^{2\pi iqr_\perp}\ dq\right]d\theta$$



From Fourier space back to real space

$$f(x,y) = \int_0^{\pi} \underbrace{\left[\int_{-\infty}^{\infty} \tilde{P}_\theta(q) \cdot |q| \cdot e^{2\pi i q r_\perp} \, dq \right]}_{\text{FT}^{-1}\{(\text{FT of projection}) \cdot \text{„filter“}\} = \text{filtered projection}} d\theta$$

FT^{-1}{(FT of projection) · „filter“} = filtered projection
it is the FT from „q“ to „$r_\perp$“ space

$$= \int_0^{\pi} P_\theta^F(r_\perp) \, d\theta$$

F: filtered

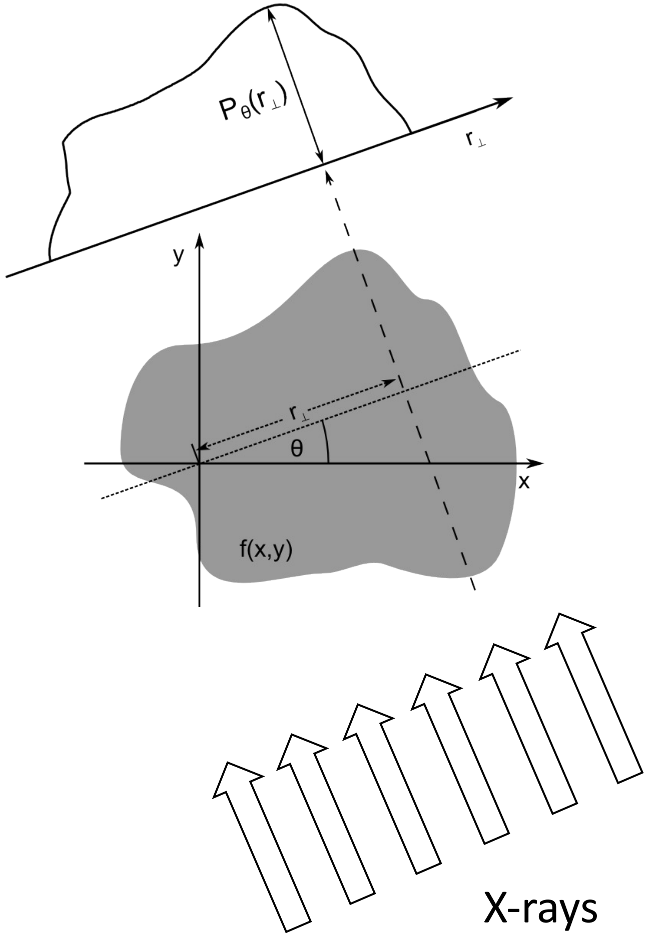
„added up“ for all projection angles θ

“filtered backprojection”

„FBP“

CT with filtered backprojection: steps

1.) Measure projections $P_\theta(r_\perp)$

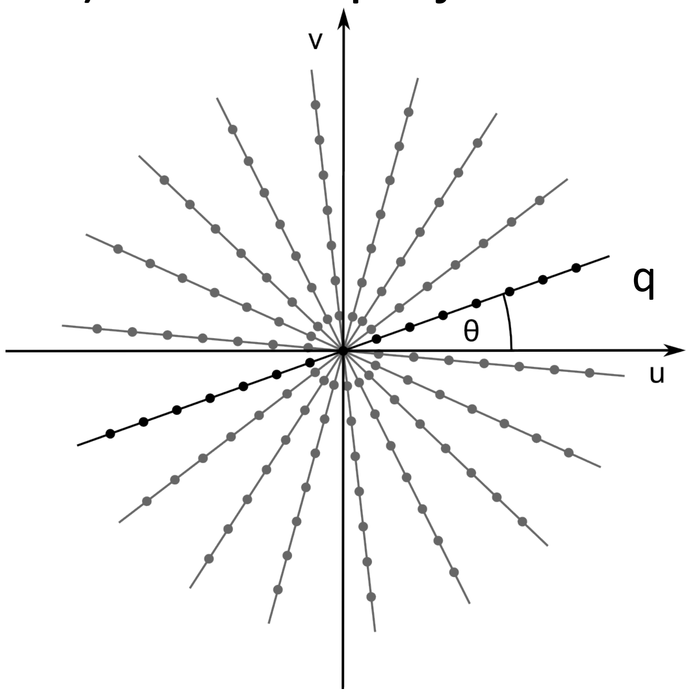


CT with filtered backprojection: steps

1.) Measure projections $P_\theta(r_\perp)$

2.) Filtering in Fourier space:

a) 1D-FT of projection: $\tilde{P}_\theta(q) = FT\{P_\theta(r_\perp)\}$



CT with filtered backprojection: steps

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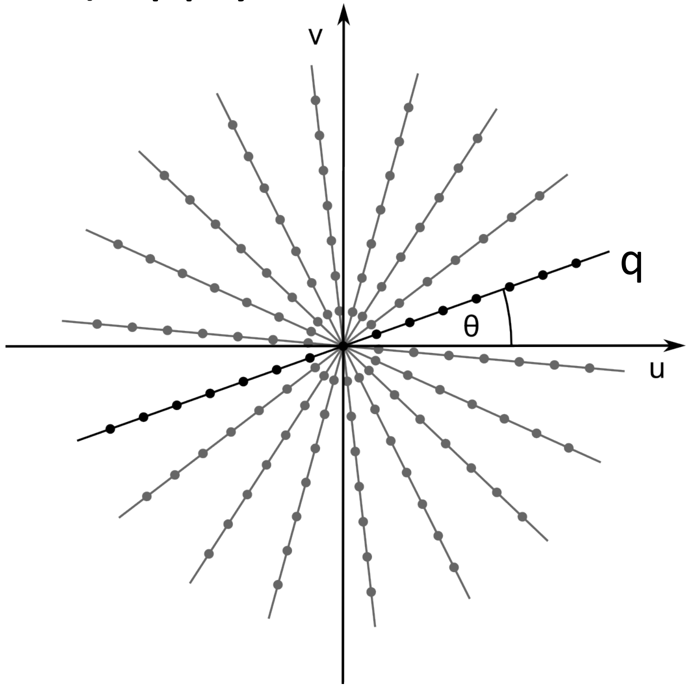
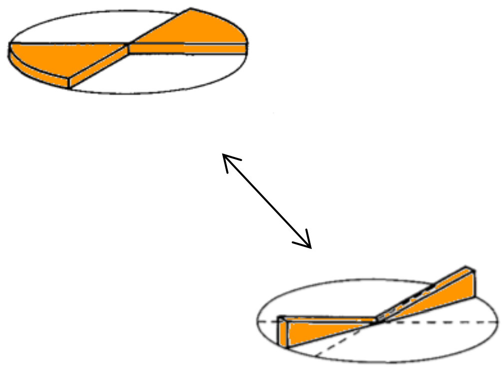
b) apply filter: $\tilde{P}_\theta(q) \cdot |q|$, or more general $\tilde{P}_\theta(q) \cdot \tilde{h}(q)$





A.C. Kak, M. Slaney,
IEEE press 1987

CT with filtered backprojection: steps

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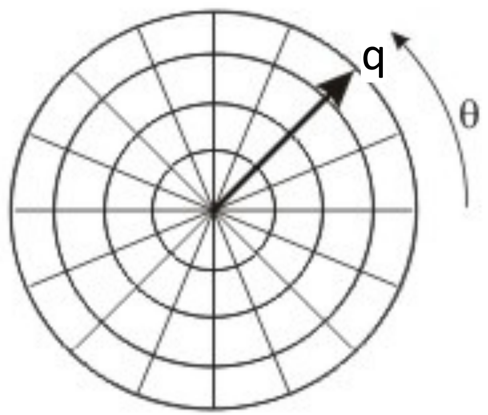
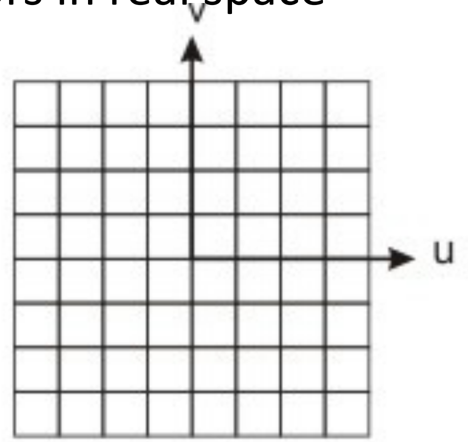
c) 1D-FT^{-1} of filtered projection: $P_\theta^F(r_\perp) = FT^{-1}\{\tilde{P}_\theta(q) \cdot |q|\}$

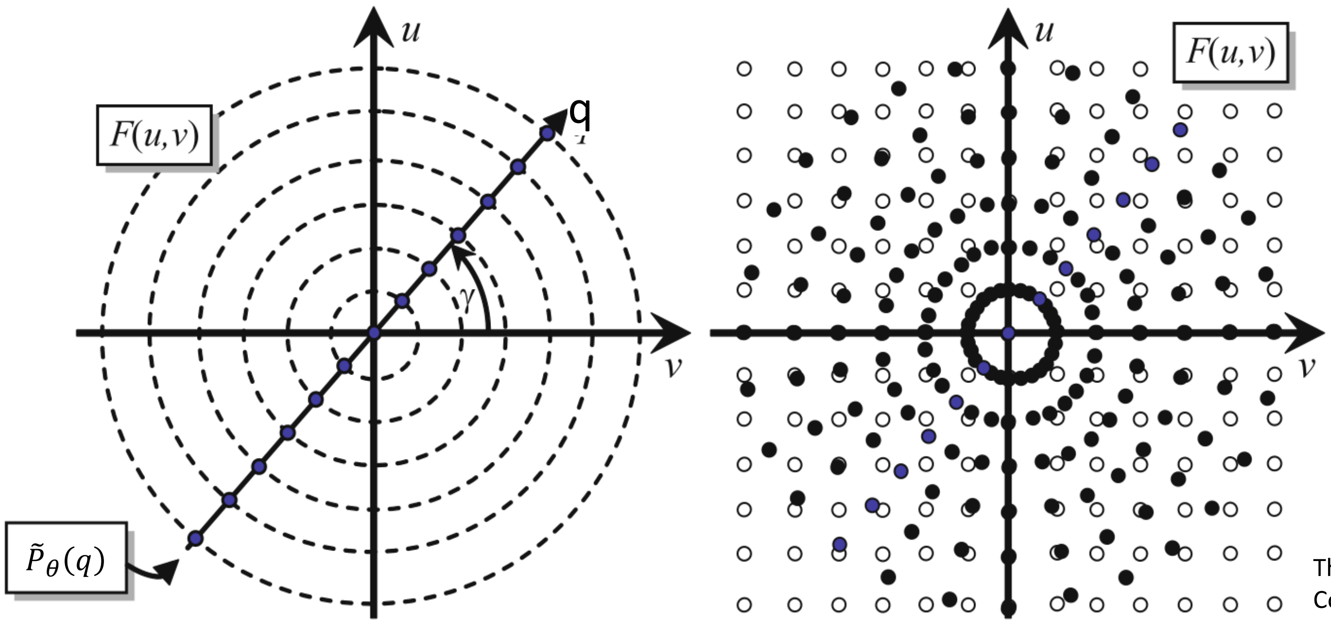
3.) Backprojection:
(requires interpolation!)

$$f(x,y) = \int_0^\pi P_\theta^F(r_\perp)\, d\theta$$

Change of sampling grid from polar to rectangular requires interpolation
+ in real space after FBP
- in Fourier space followed by a 2D FT^{-1} .
  Local interpolation error in Fourier space give global errors in real space





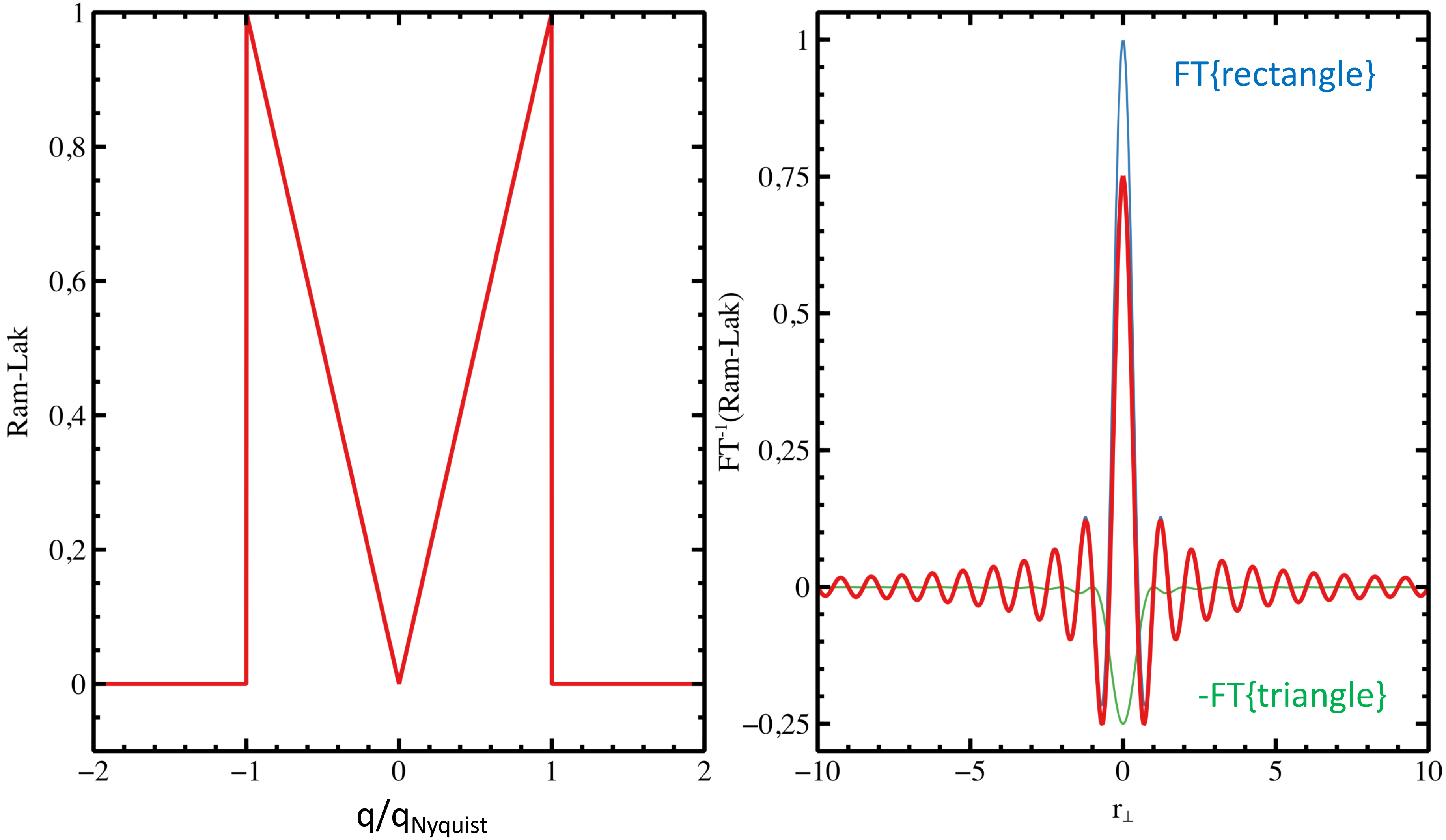


a b

Thorsten M. Buzug,
Computed Tomography,
Springer Publishers

for further reading:
F. Marone and M. Stampanoni
Regridding reconstruction
algorithm for real-time tomographic imaging
J. Synchrotron Rad. (2012). 19, 1029–1037 doi:10.1107/S0909049512032864

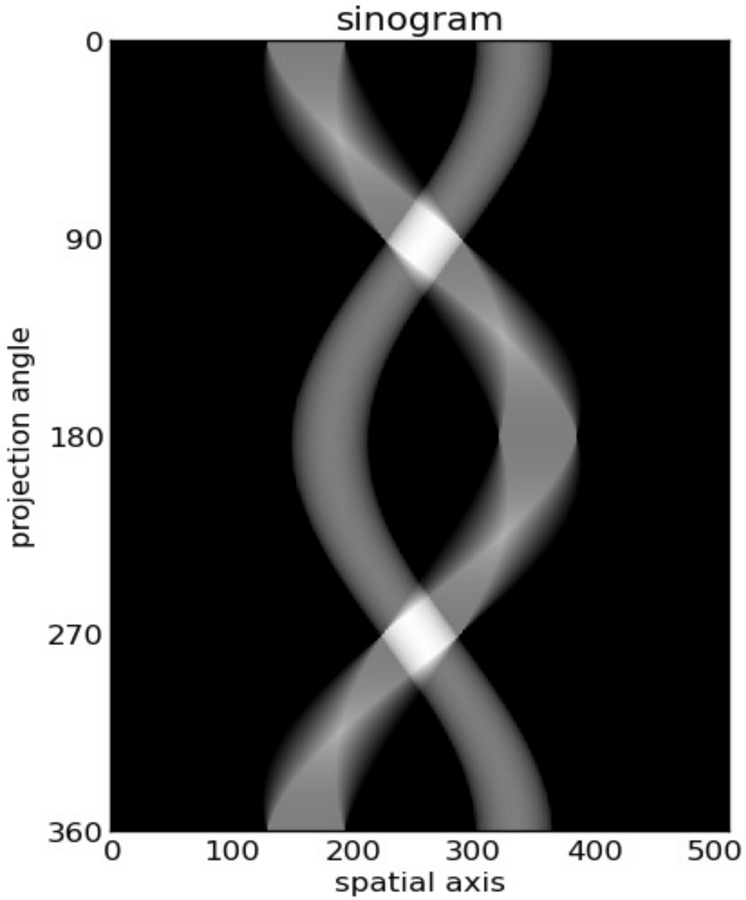
Ram-Lak Filter in Fourier space and in real space
(Ramachandran-Lakshminarayanan, PNAS 1971)



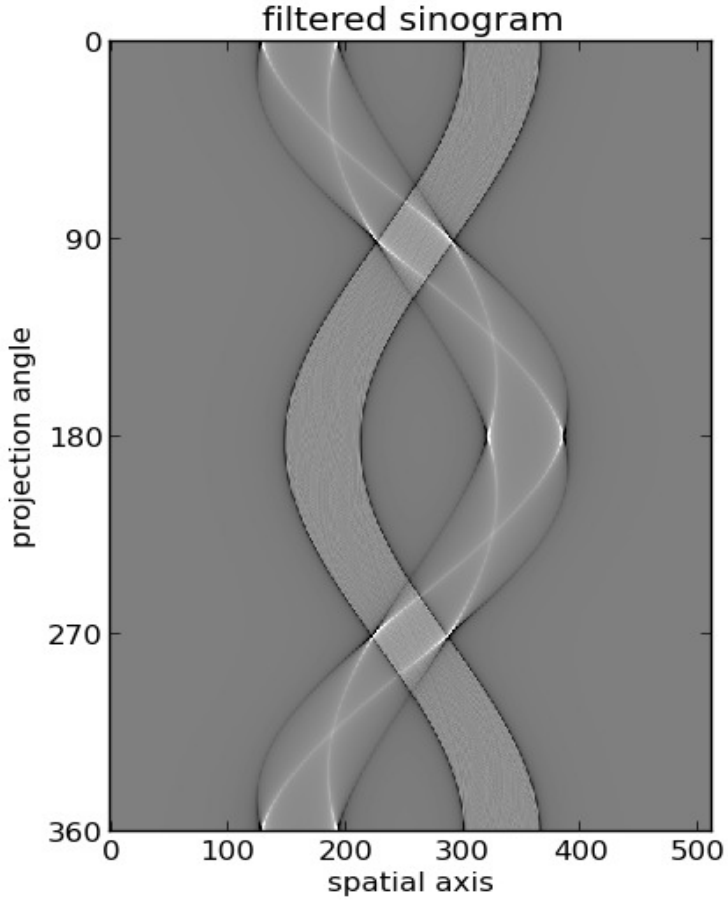
Filered backprojection

Filter sinogram with ramp filter: either multiplication with ramp filter in Fourier space
or convolution with FT(|r|)= … in real space

$P_\theta(r_\perp)$



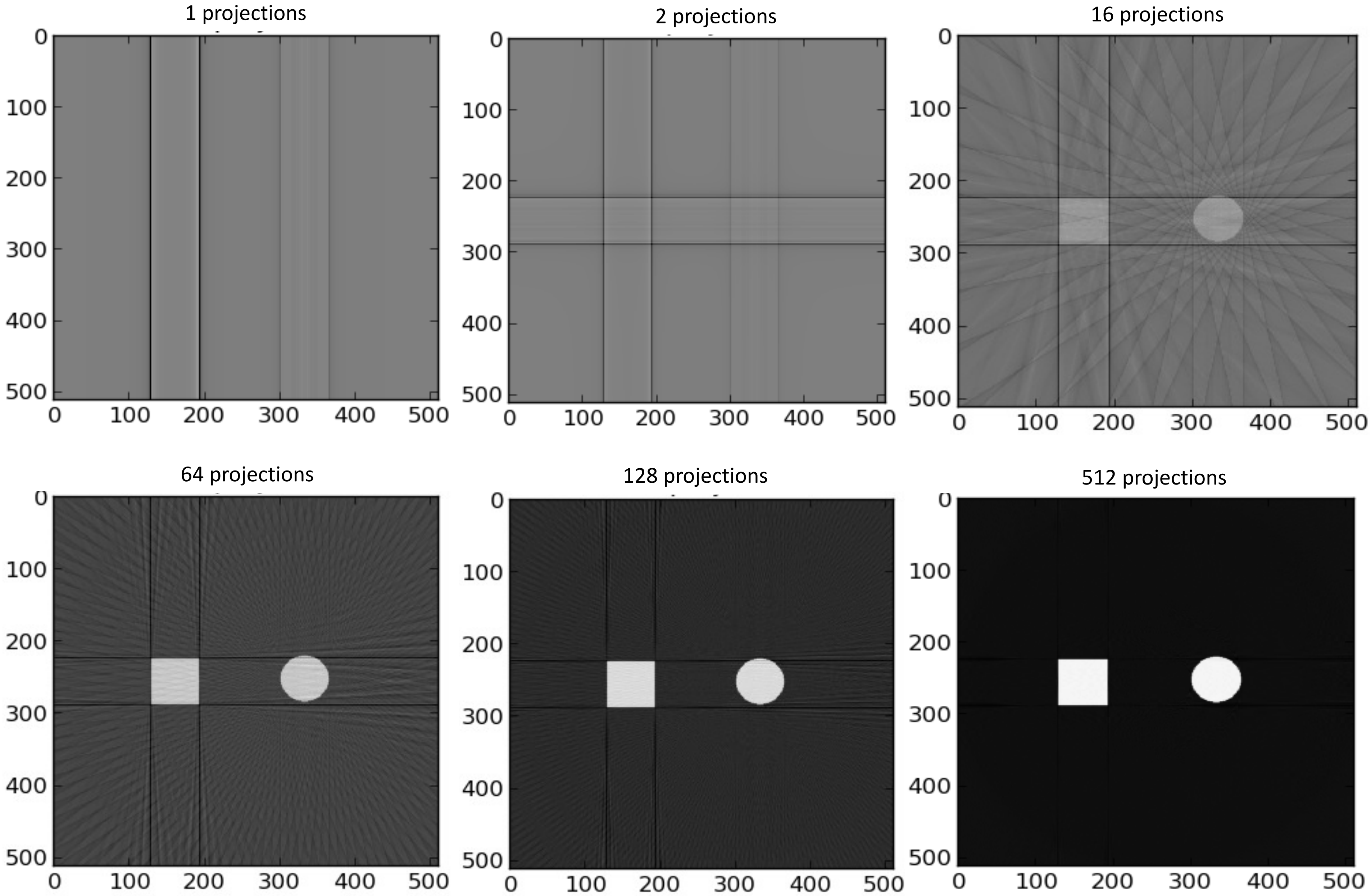
$P_\theta^F(r_\perp)$



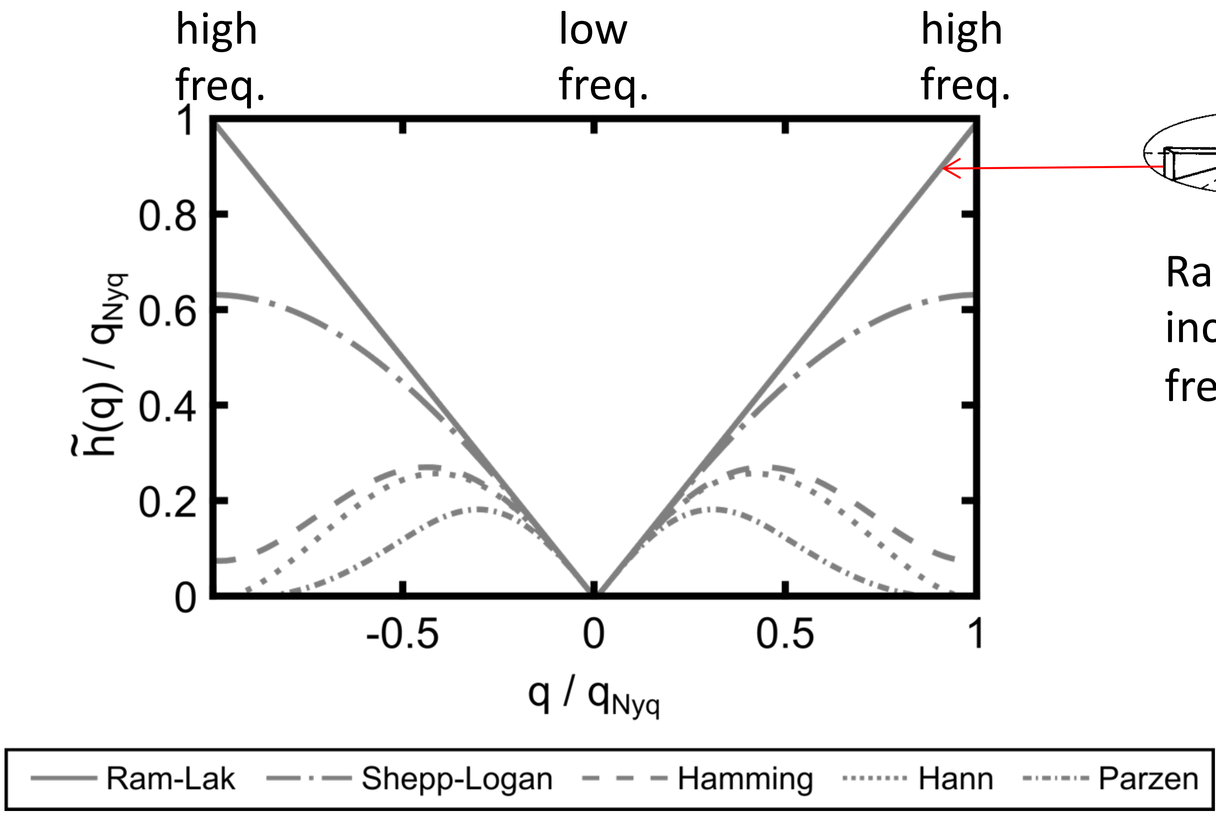
$$P_\theta^F\,(r_\perp) = \; FT^{-1}\,\{\tilde{P}_\theta(q)\;\cdot|q|\,\}$$

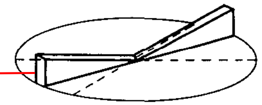
Filtered backprojection

Sum up all backprojections $f(x,y) = \int_0^{\pi} P_\theta^F(r_\perp)\, d\theta$



Other filters for the filtered backprojection

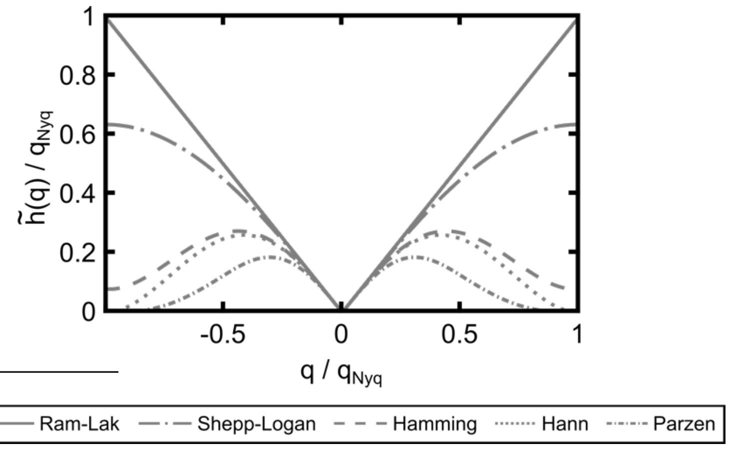




Ram-Lak filter increases high frequencies

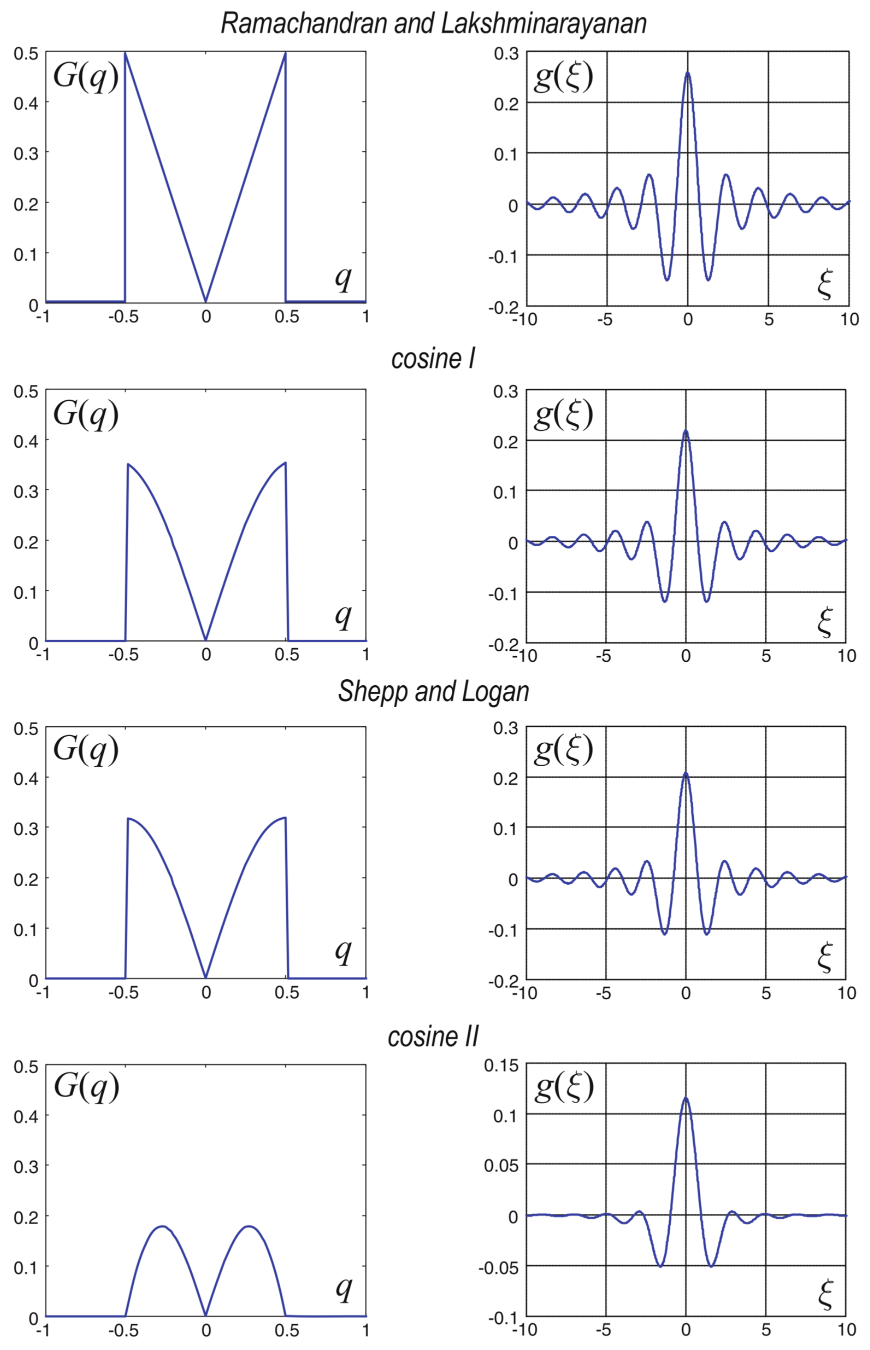
Ramachandran-Lakshminarayan filter is sensitive to noise in projections

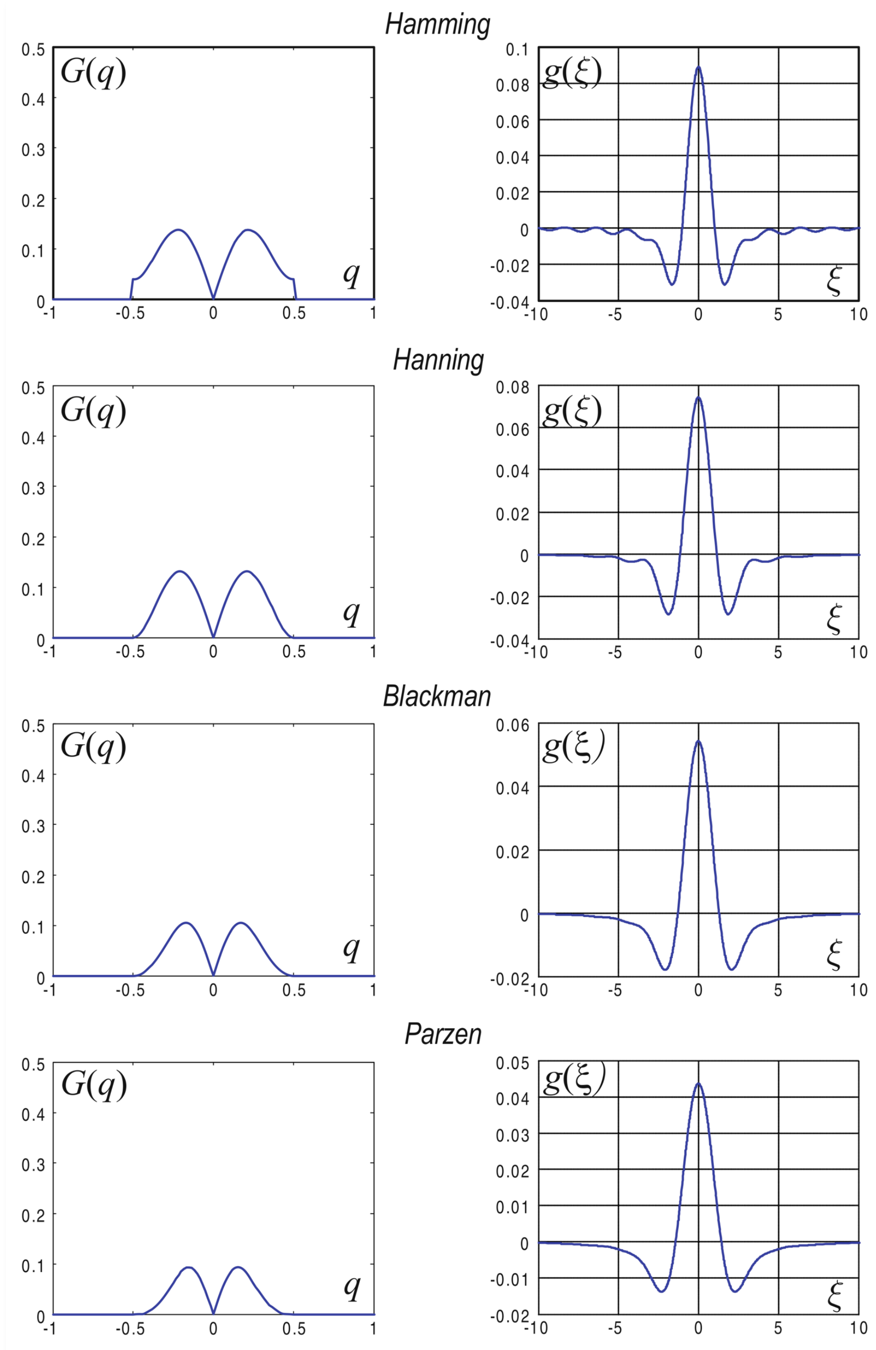
Other filters for the filtered backprojection



filter name	filter function $\tilde{h}(q)$ in Fourier space
Ram-Lak	$\lvert q\rvert$
Shepp-Logan	$\lvert q\rvert \frac{\sin\left(\frac{\pi}{2}\lvert q\rvert\right)}{\frac{\pi}{2}\lvert q\rvert}$
Hann	$\lvert q\rvert \cdot \frac{1}{2}\left[1+\cos\left(\pi \frac{q}{q_{\mathrm{Nyq}}}\right)\right]$
Hamming	$\lvert q\rvert \cdot \left[0.54+0.46\cos\left(\pi \frac{q}{q_{\mathrm{Nyq}}}\right)\right]$
Parzen	$\lvert q\rvert \cdot \begin{cases} 1-6\left\lvert\frac{q}{q_{\mathrm{Nyq}}}\right\rvert^2+6\left\lvert\frac{q}{q_{\mathrm{Nyq}}}\right\rvert^3 & , 0 \leq \left\lvert\frac{q}{q_{\mathrm{Nyq}}}\right\rvert \leq \frac{1}{2} \\ 2\left(1-\left\lvert\frac{q}{q_{\mathrm{Nyq}}}\right\rvert^3\right) & , \frac{1}{2} \leq \left\lvert\frac{q}{q_{\mathrm{Nyq}}}\right\rvert \leq 1 \end{cases}$

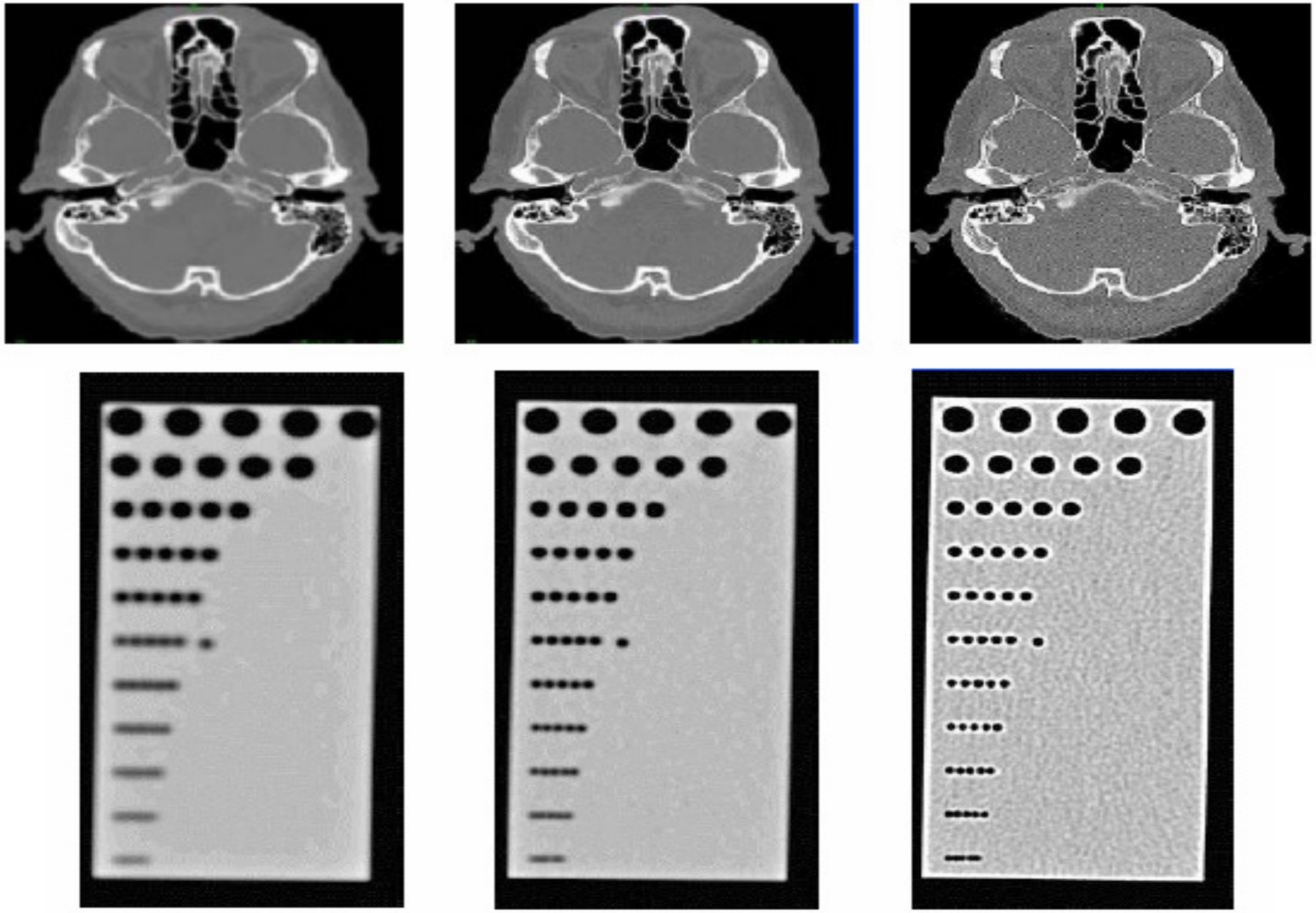
Other filters for the filtered backprojection





Choice of filter strength

Filter can be tuned to achieve image enhancement
Tradeoff between noise and sharpness



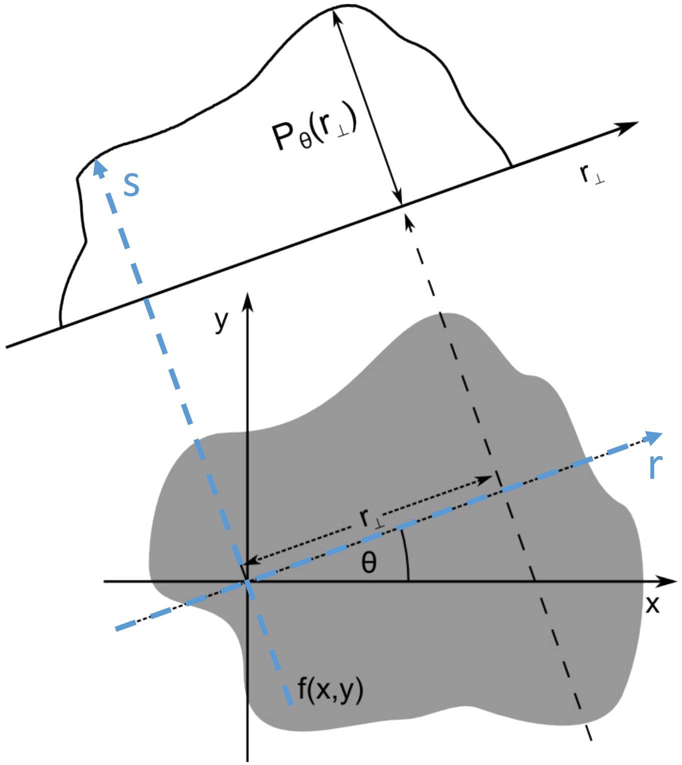
smoothing | standard | edge enhanced

Filter for grating based phase contrast

Projection of the absorption (Radon transform)

$$P_\theta(r_\perp) = R_\theta\{f(x,y)\} = \int_{-\infty}^{\infty} f(r_\perp, s)\, ds$$

$$f(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} FT\big(P_\theta(r_\perp)\big) \cdot |q| \cdot e^{2\pi i q r_\perp}\, dq \right] d\theta$$



Projection of the differential phase

$$d(r_\perp, \theta) = \int_{-\infty}^{\infty} \frac{\partial f(r_\perp, s)}{\partial r_\perp}\, ds$$

$$= \frac{\partial}{\partial r_\perp} \int_{-\infty}^{\infty} f(r_\perp, s)\, ds$$

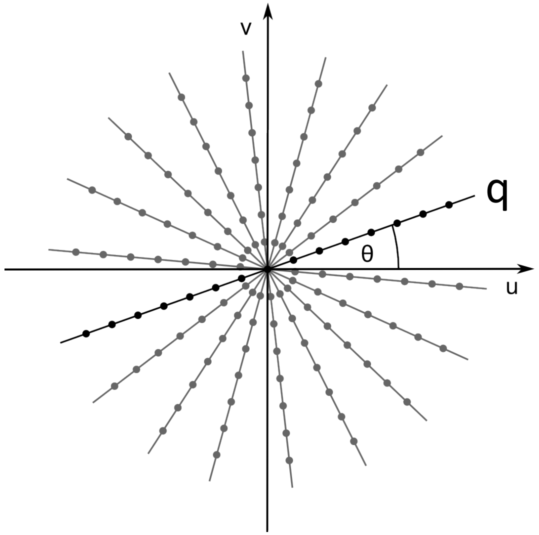
- grating direction along rotation axis z
- beam direction in s
- $r_\perp$ is perpendicular to grating direction

as for the projection of the absorption, we take the Fourier transform of the differential phase

$$FT\{d(r_\perp,\theta)\} = FT\left\{\frac{\partial}{\partial r_\perp}\int_{-\infty}^{\infty} f(r_\perp,s)\,ds\right\} = 2\pi i\cdot q\cdot FT\left\{\int_{-\infty}^{\infty} f(r_\perp,s)\,ds\right\}$$

Fourier derivative theorem

$$= 2\pi i\cdot q\cdot FT\{\,P_\theta\ (r_\perp)\,\} = 2\pi i\cdot q\cdot \tilde{P}_\theta(q)$$



$$\tilde{P}_\theta(q) = \frac{FT\{d(r_\perp,\theta)\}}{2\pi i\cdot q}$$

and instead of the FBP of the absorption signal

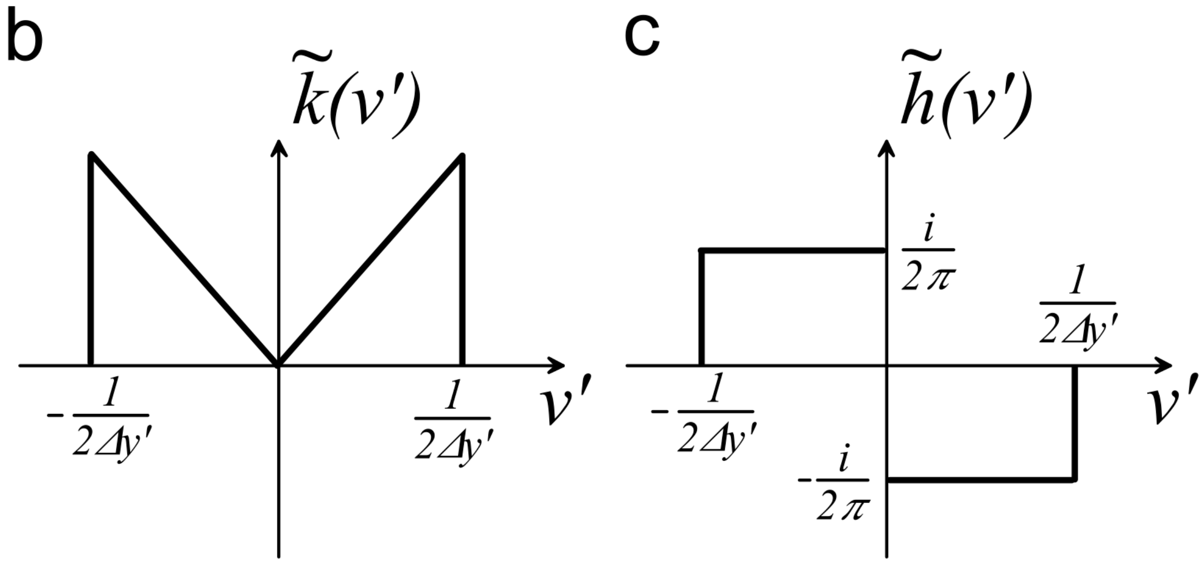
$$f(x,y) = \int_0^{\pi}\left[\int_{-\infty}^{\infty}\tilde{P}_\theta(q)\cdot|q|\cdot e^{2\pi i q r_\perp}\,dq\right]d\theta$$

we have for the FBP
of the differential phase signal

$$f_{ph}(x,y) = \int_0^{\pi}\left[\int_{-\infty}^{\infty}\frac{FT\{d(r_\perp,\theta)\}}{2\pi i\cdot q}\cdot|q|\cdot e^{2\pi i q r_\perp}\,dq\right]d\theta$$

$$f_{ph}(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} \frac{FT\{d(r_\perp,\theta)\}}{2\pi i \cdot q} \cdot |q| \cdot e^{2\pi i q r_\perp} \, dq \right] d\theta$$

$$= -\frac{i}{2\pi} \int_0^{\pi} \left[\int_{-\infty}^{\infty} FT\{d(r_\perp,\theta)\} \cdot \frac{|q|}{q} \cdot e^{2\pi i q r_\perp} \, dq \right] d\theta \qquad \frac{|q|}{q} \begin{cases} 1 & for\ q > 0 \\ -1 & for\ q < 0 \end{cases}$$



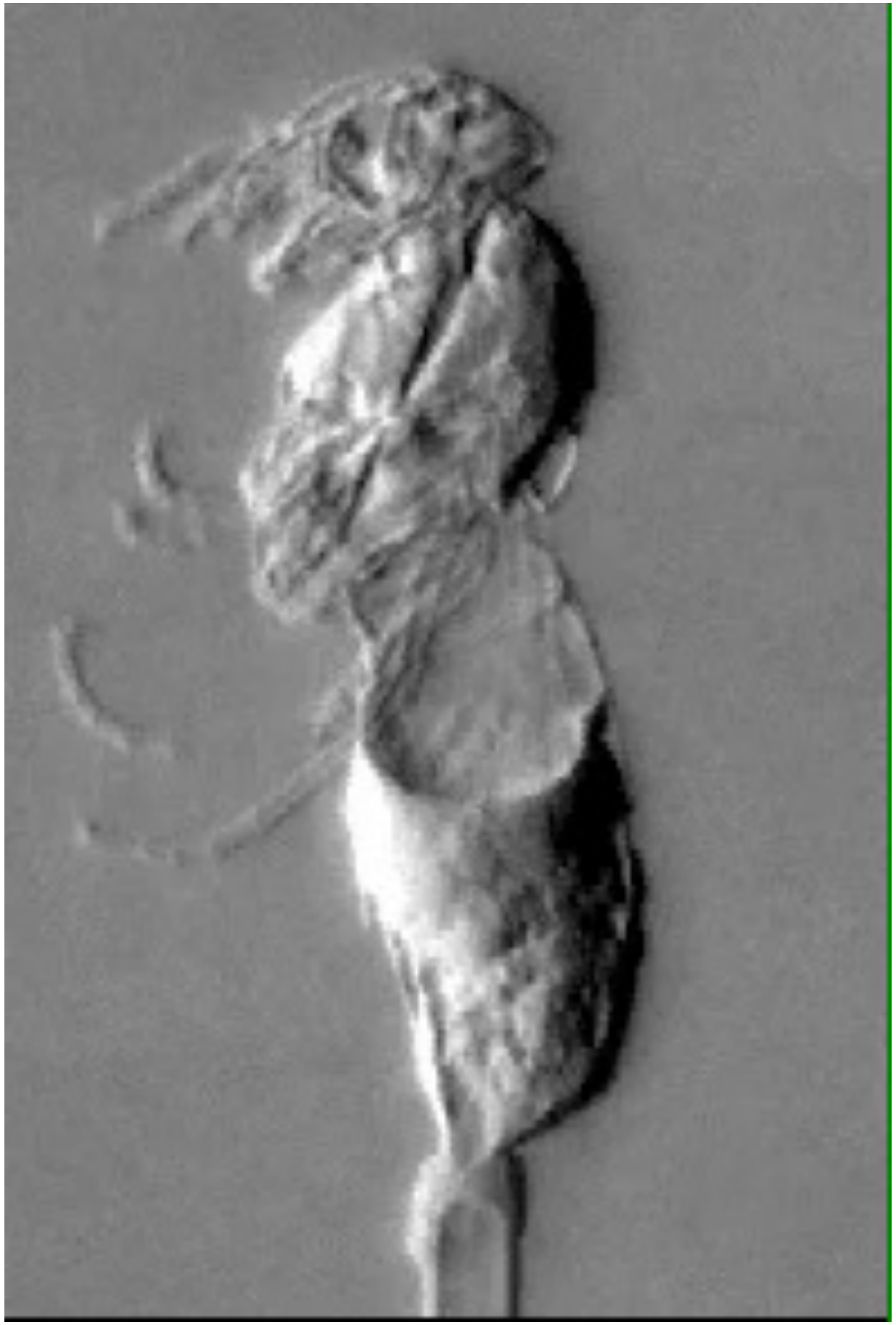
Ram-Lak filter
of absorption

Hilbert filter for differential phase

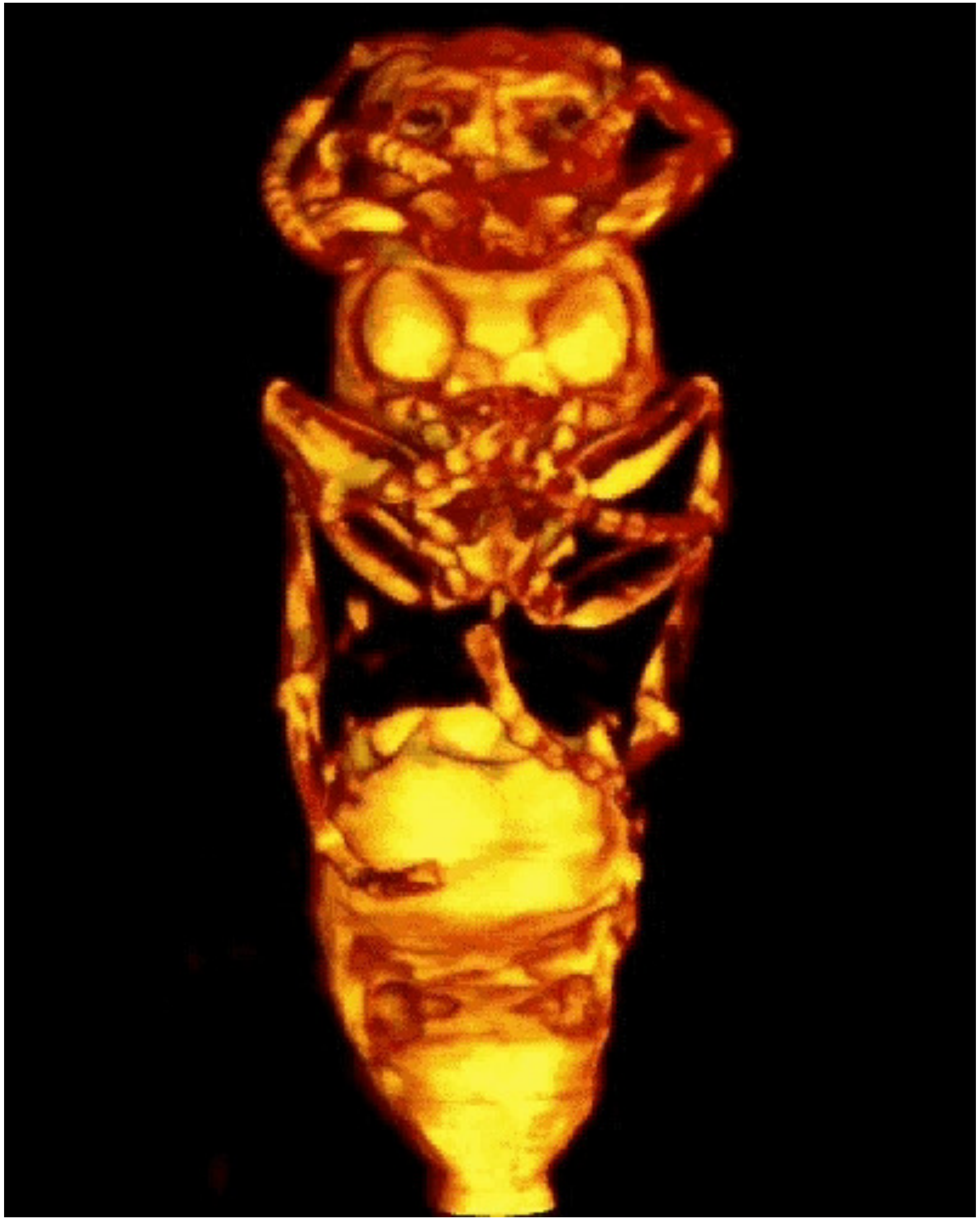
F. Pfeiffer et al., Physical Review Letters 98 (2007) 108105-1

Phase of a rat brain with tumor

Differential phase contrast projection



Real part of the refractive index $\delta(x,y,z)$ of a hornet



Aliasing

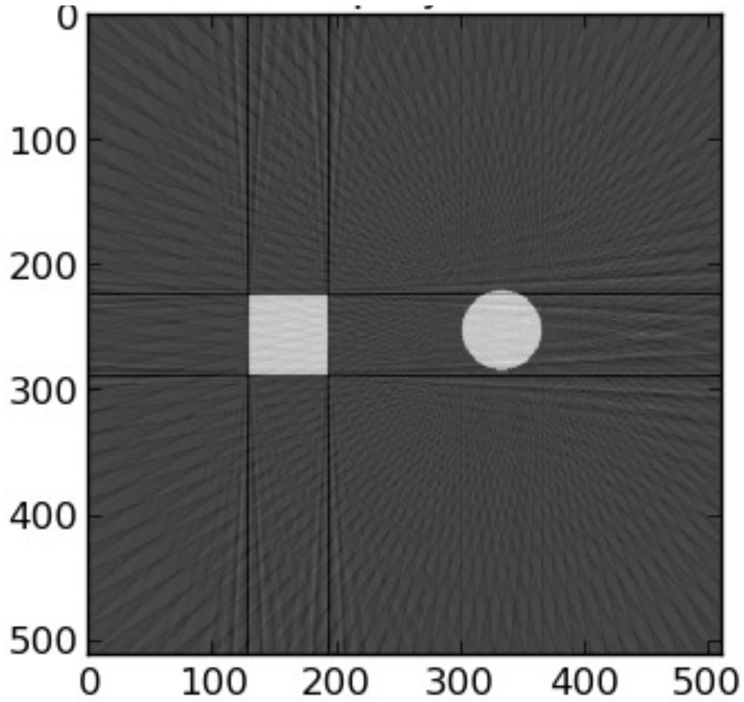
How many pixels per projection are necessary?

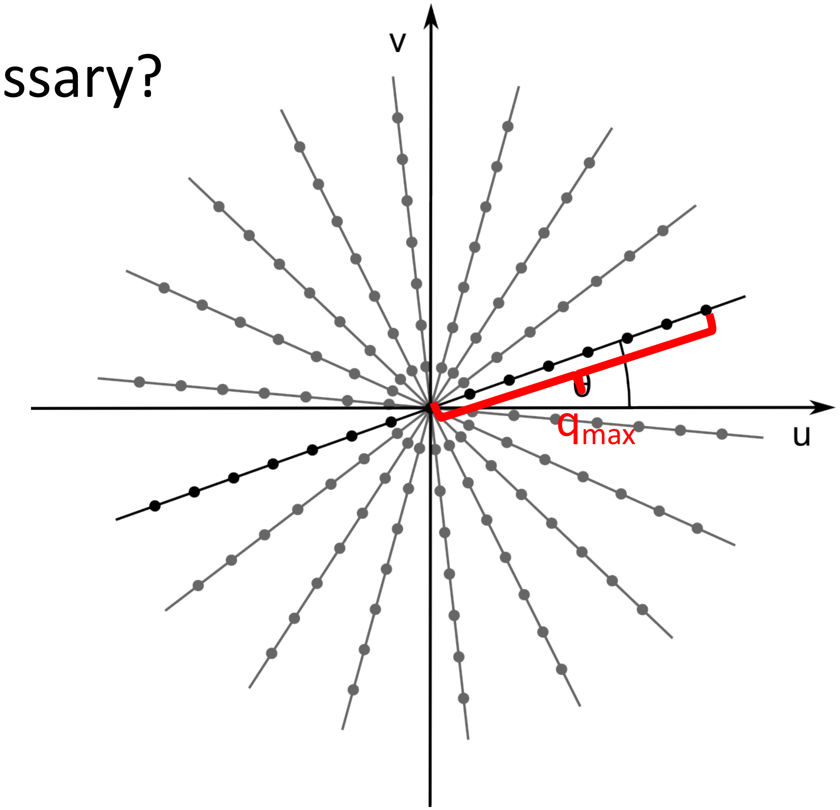
Nyquist criterion

$$p < \frac{1}{2\, q_{max}}$$

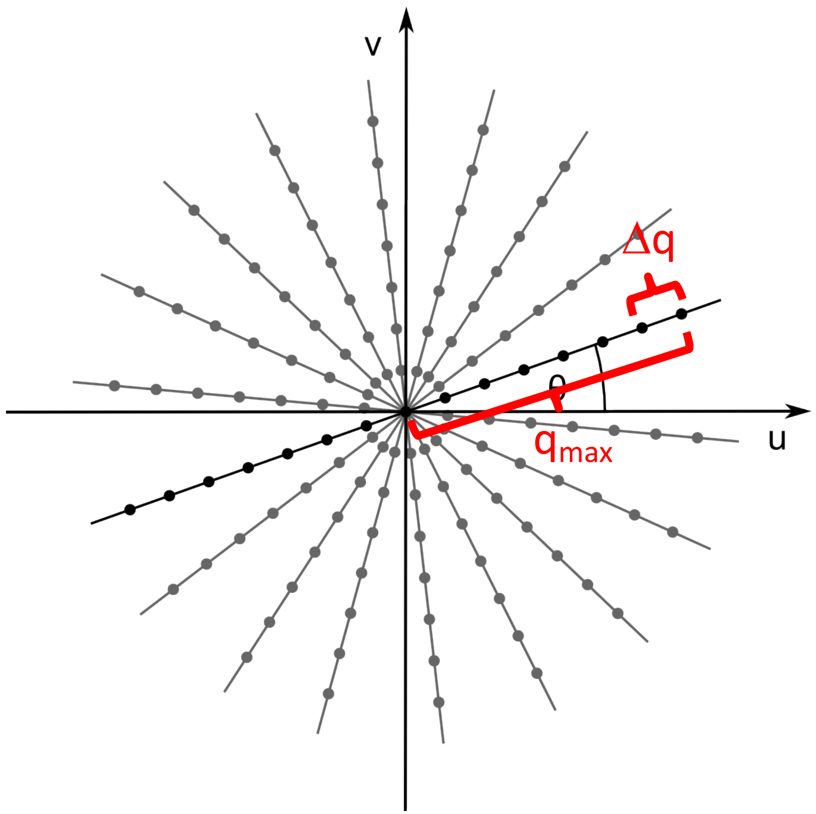
with pixel pitch p

Reconstruction with aliasing





How many projections are necessary?

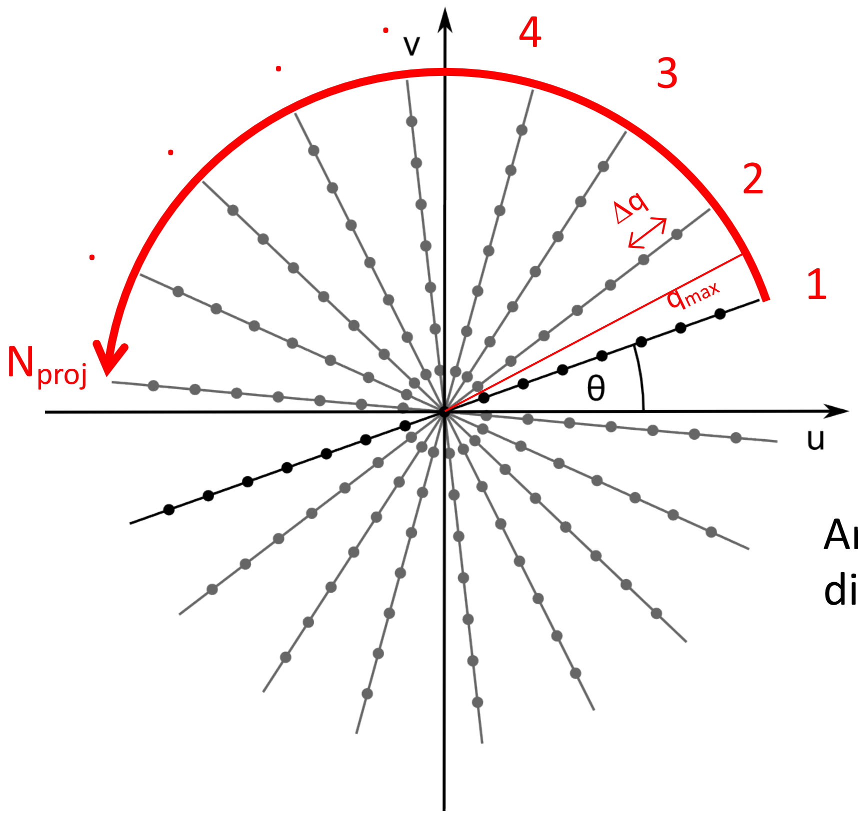


Fourier slice theorem → full 2D-FT of object can be obtained if projection lines are measured over at least $180° = \pi$

maximum (spatial) frequency present in discrete Fourier transform:

$$q_{max} = \Delta q \cdot \frac{N_{pix}}{2}$$

How many projections are necessary?



Fourier slice theorem → full 2D-FT of object can be obtained if projection lines are measured over at least $180° = \pi$

maximum (spatial) frequency present in discrete Fourier transform:

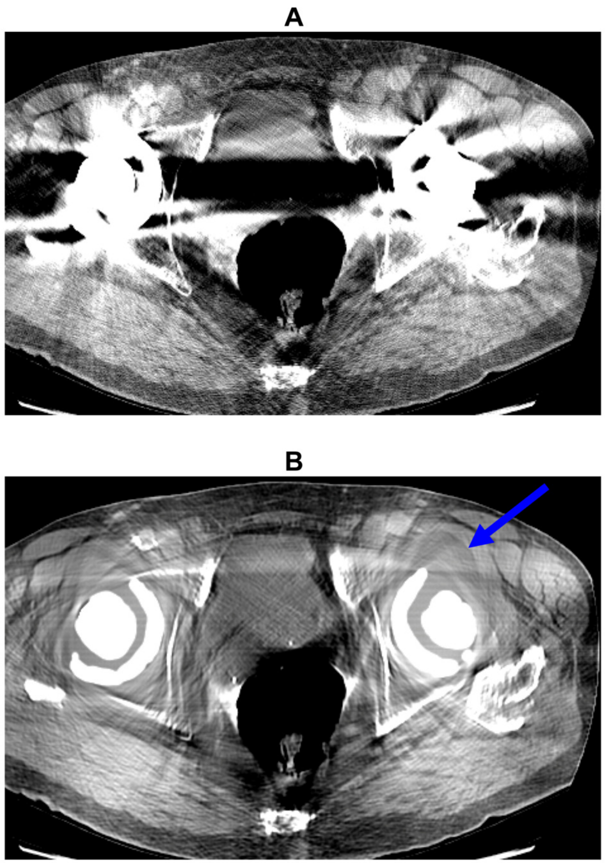
$$q_{max} = \Delta q \cdot \frac{N_{pix}}{2}$$

Angular distance between points = radial distance between points: $q_{\max} \cdot \Delta\theta = \Delta q$

$$\Delta\theta = \frac{\pi}{N_{proj}} \overset{!}{\approx} \frac{\Delta q}{q_{max}} = \frac{2}{N_{pix}}$$

$$\Rightarrow \quad N_{proj} \approx N_{pix} \cdot \frac{\pi}{2}$$

Artifacts in reconstructed volumes

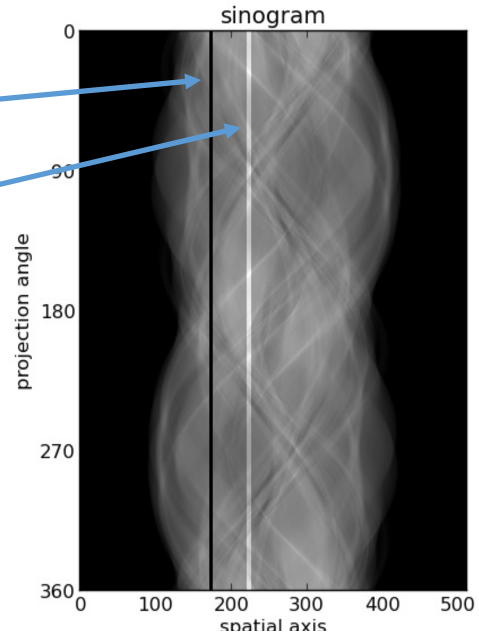


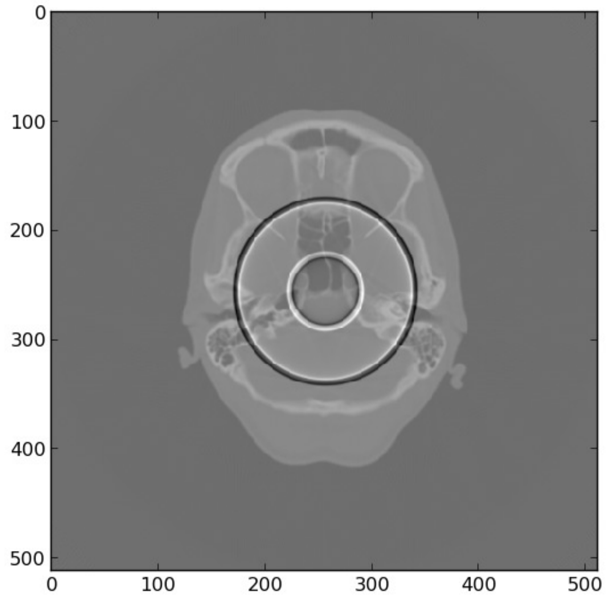
F.E. Boas & D. Fleischmann, Imaging Med. (2012) 4(2), 229-240

Detector inhomogeneities, ring artifact

dead pixel

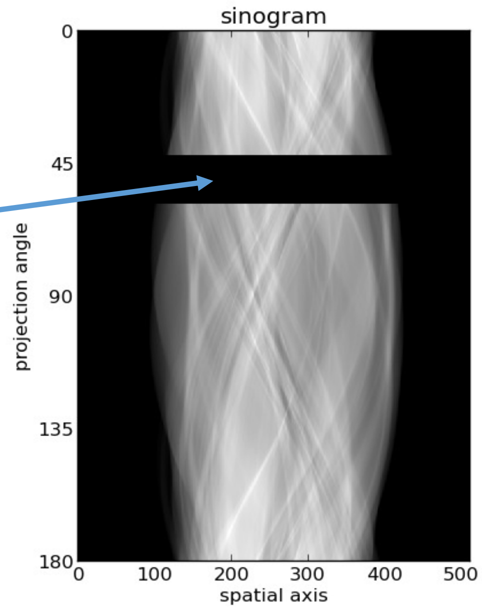
hot pixel

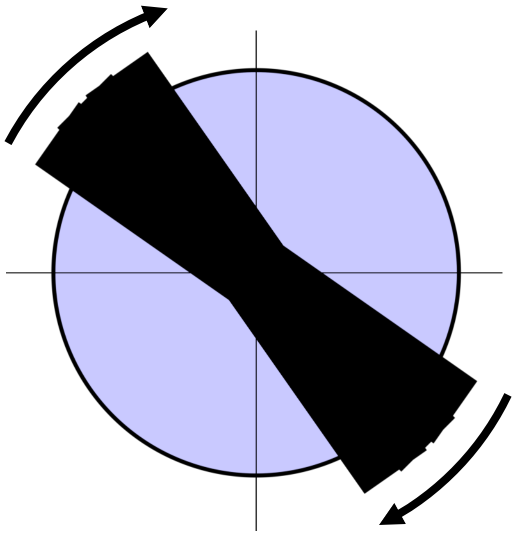




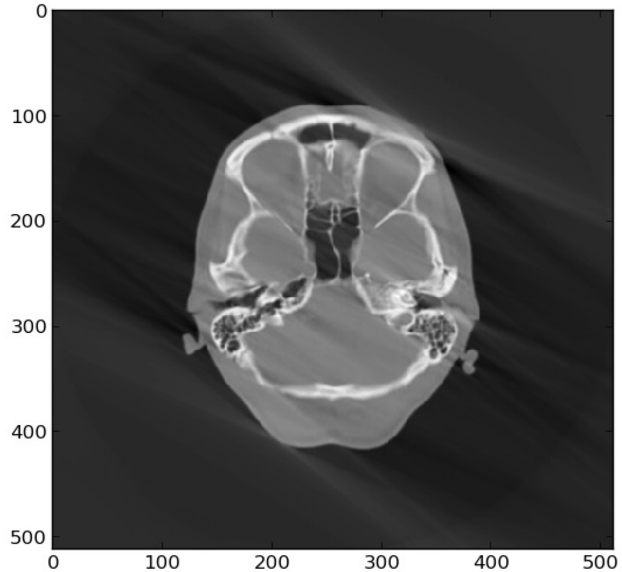
missing/degraded projections

missing angles

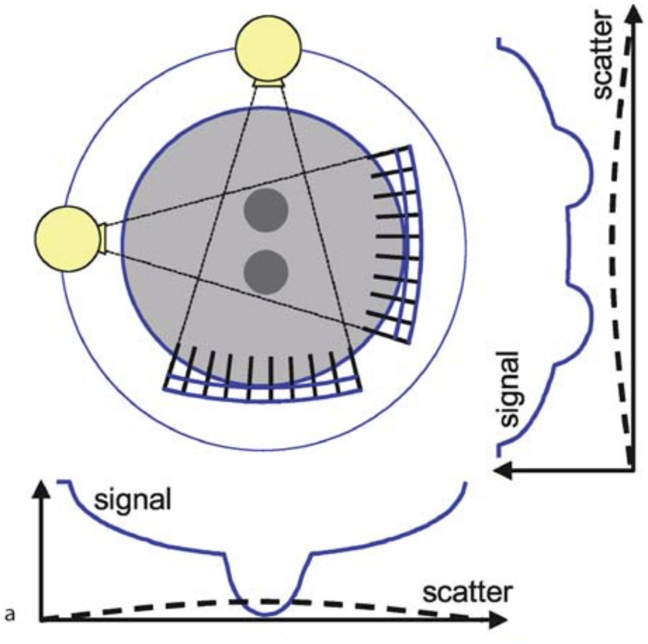




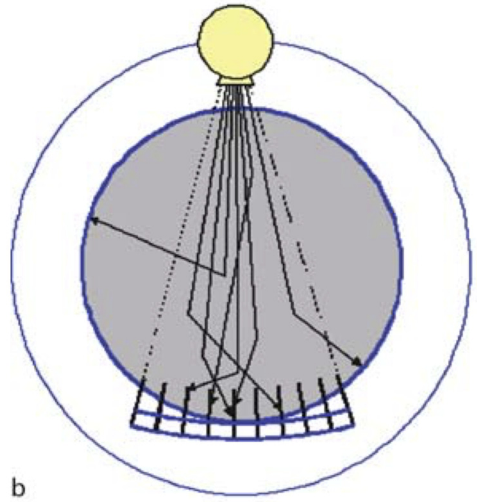
for a parallel beam
with limited width



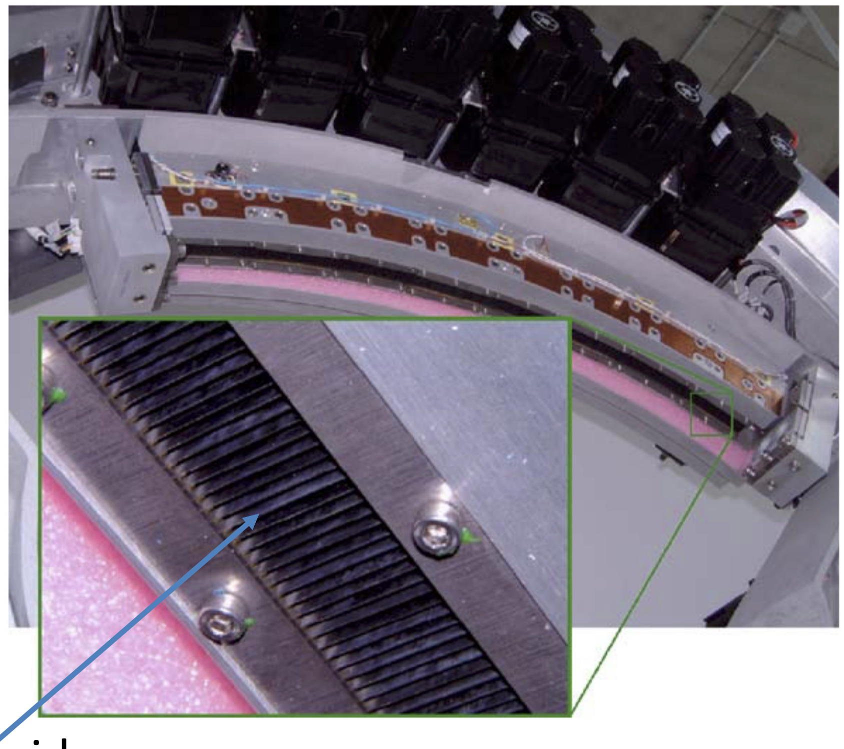
Scatter radiation (Compton)



Compton scattered photons are detected as a broad background signal without structural information



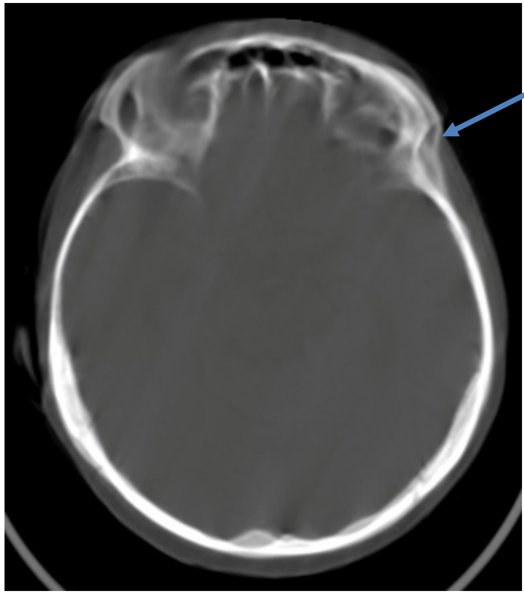
Scattered photons are absorbed in an anti-scatter grid in front of the detector which is curved to a circle segment with center in the X-ray tube



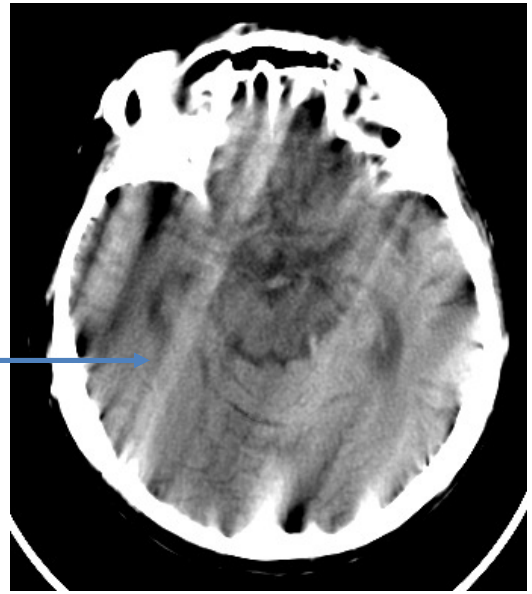
Anti-scatter grid

© Buzug, Springer, 1st ed. 2008

Motion blur



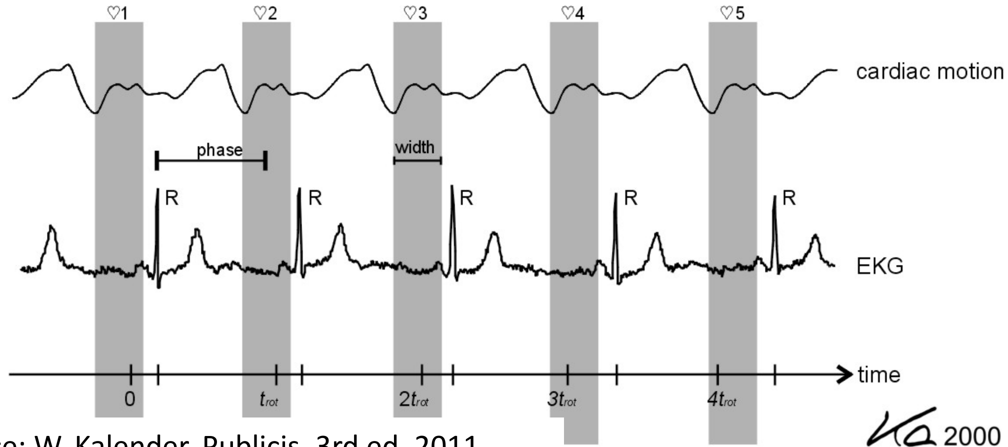
doubling of features



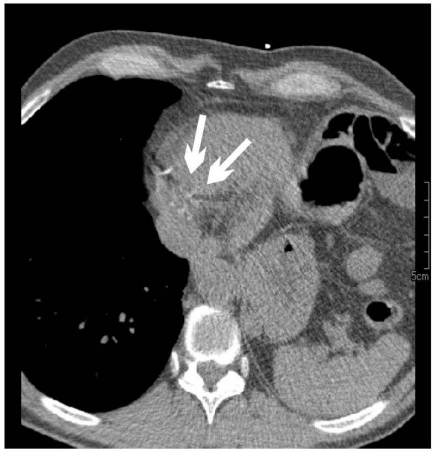
streaks

F.E. Boas & D. Fleischmann, Imaging Med. (2012) 4(2), 229-240

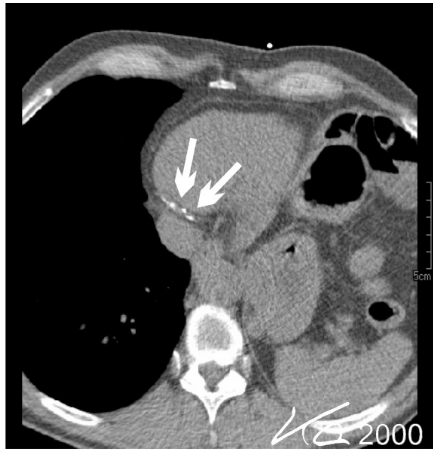
→ ECG-correlated heart CT



source: W. Kalender, Publicis, 3rd ed. 2011



a)



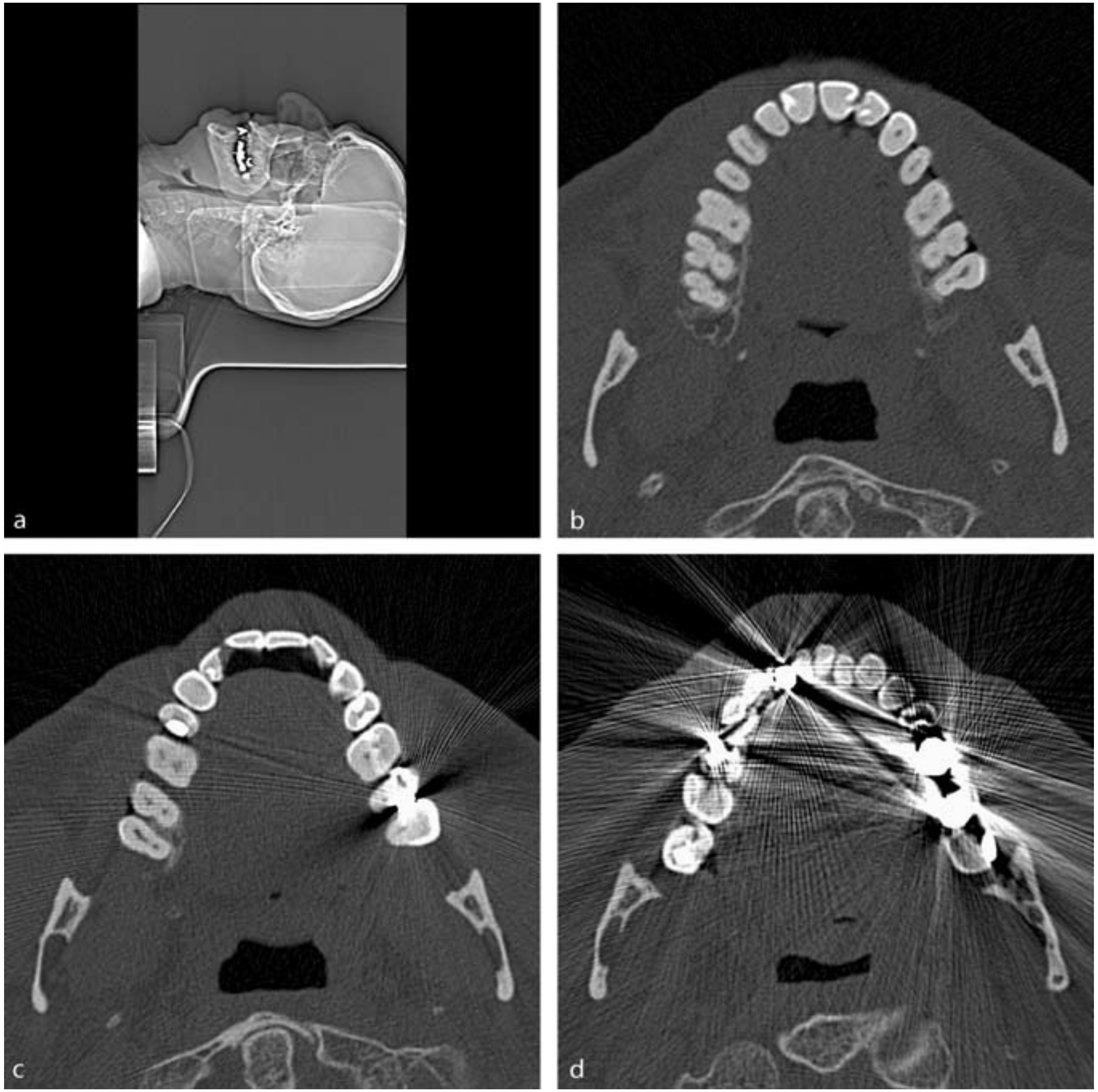
b)

metal artifacts,
beam starvation
(absorption of nearly all photons
→ intensity below noise level)

Low energy photons are strongly absorbed

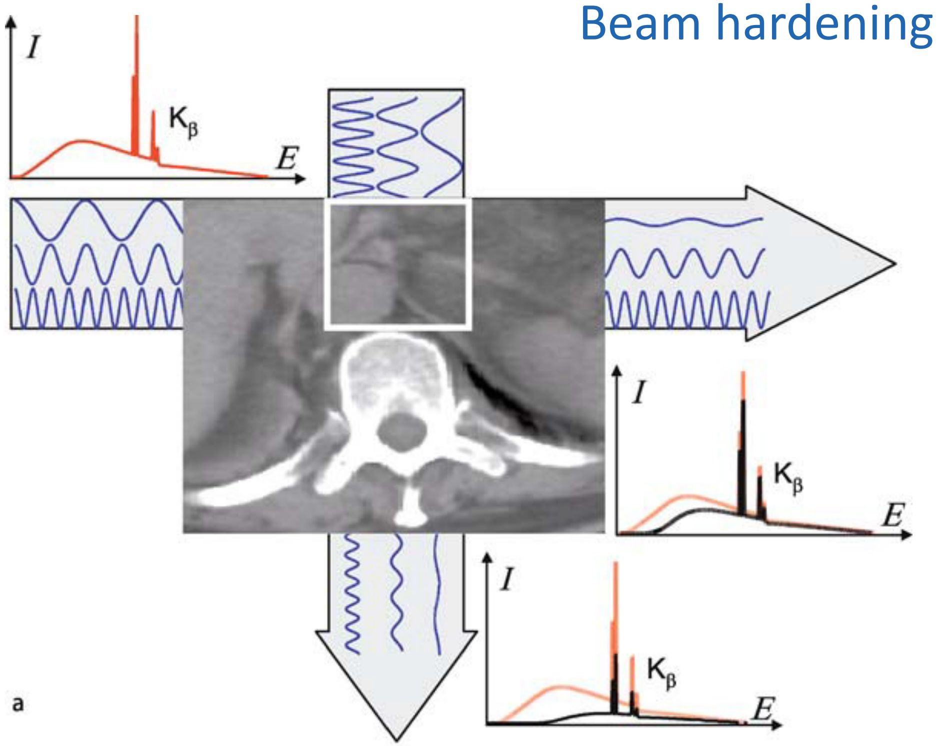
$$\mu_{p.e.} \propto Z^4/E^3$$

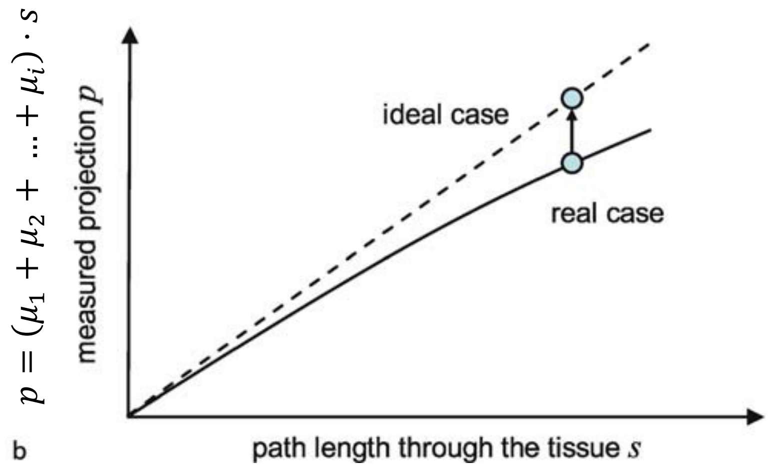
example: amalgam dental fillings

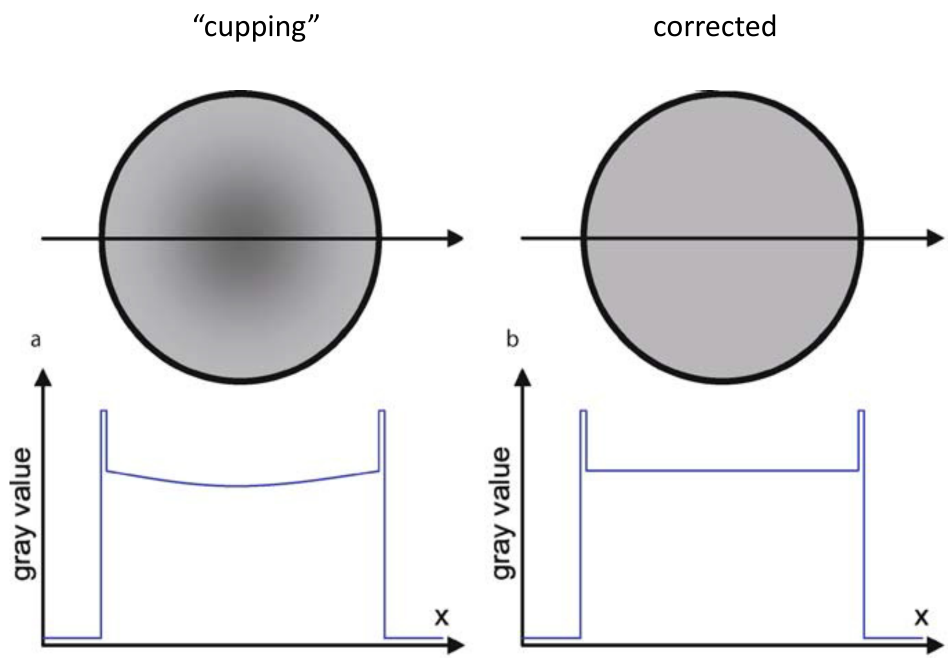


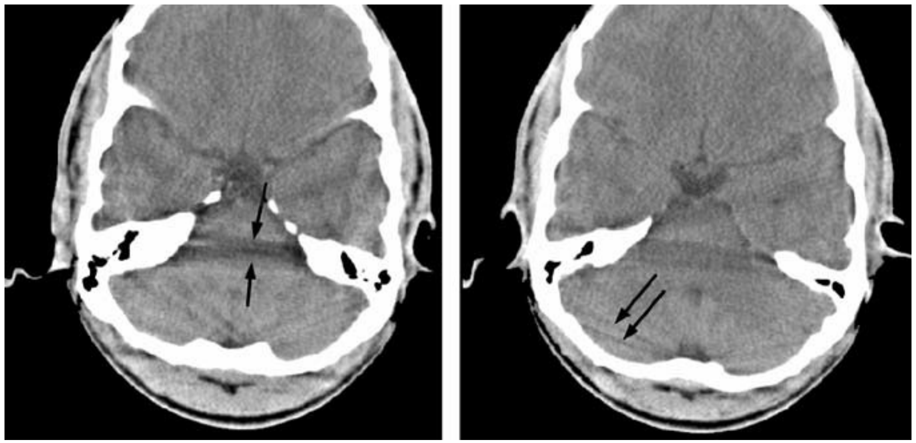
source: Buzug, Springer, 1st ed. 2008

Beam hardening



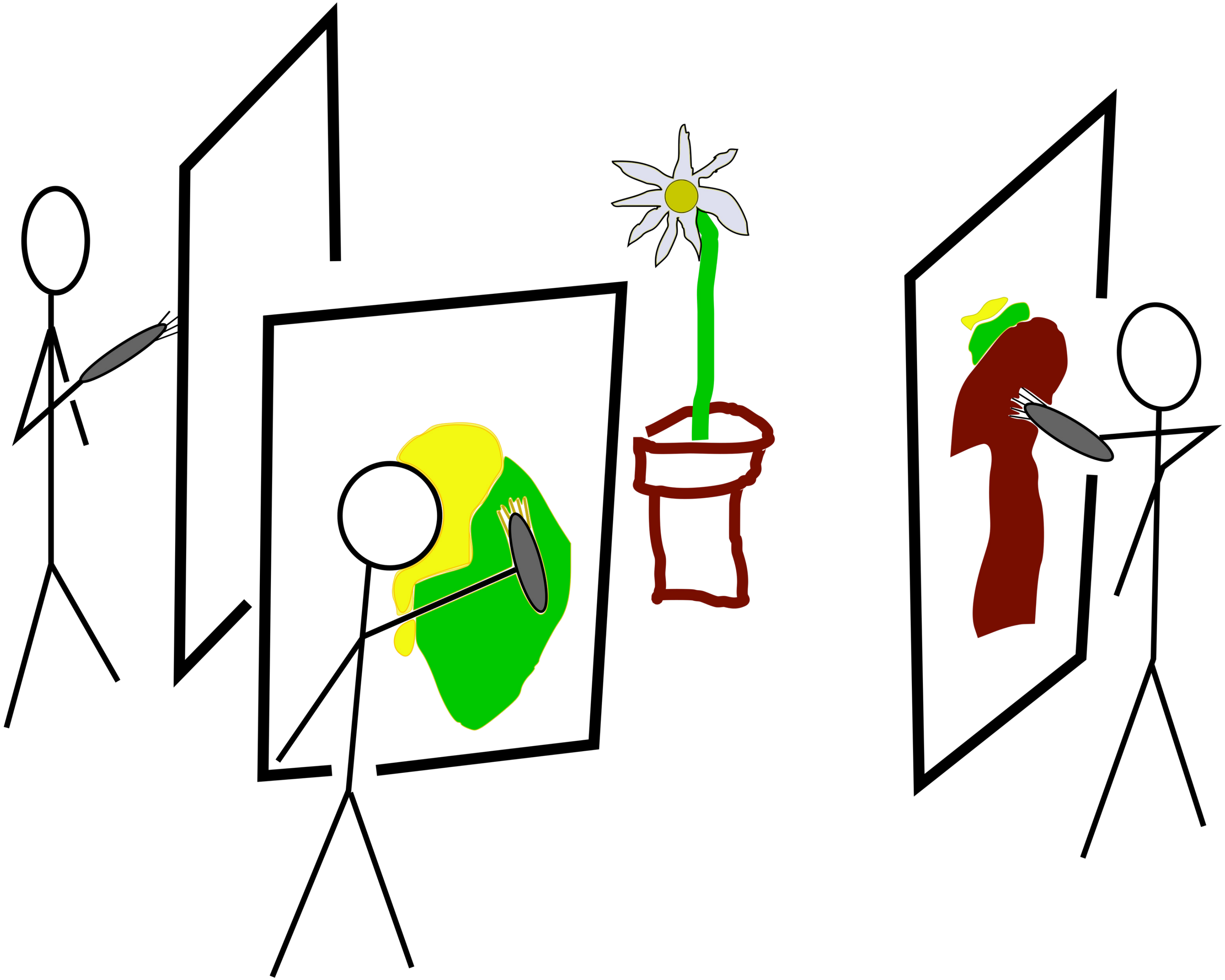






source: Buzug, Springer, 1st ed. 2008

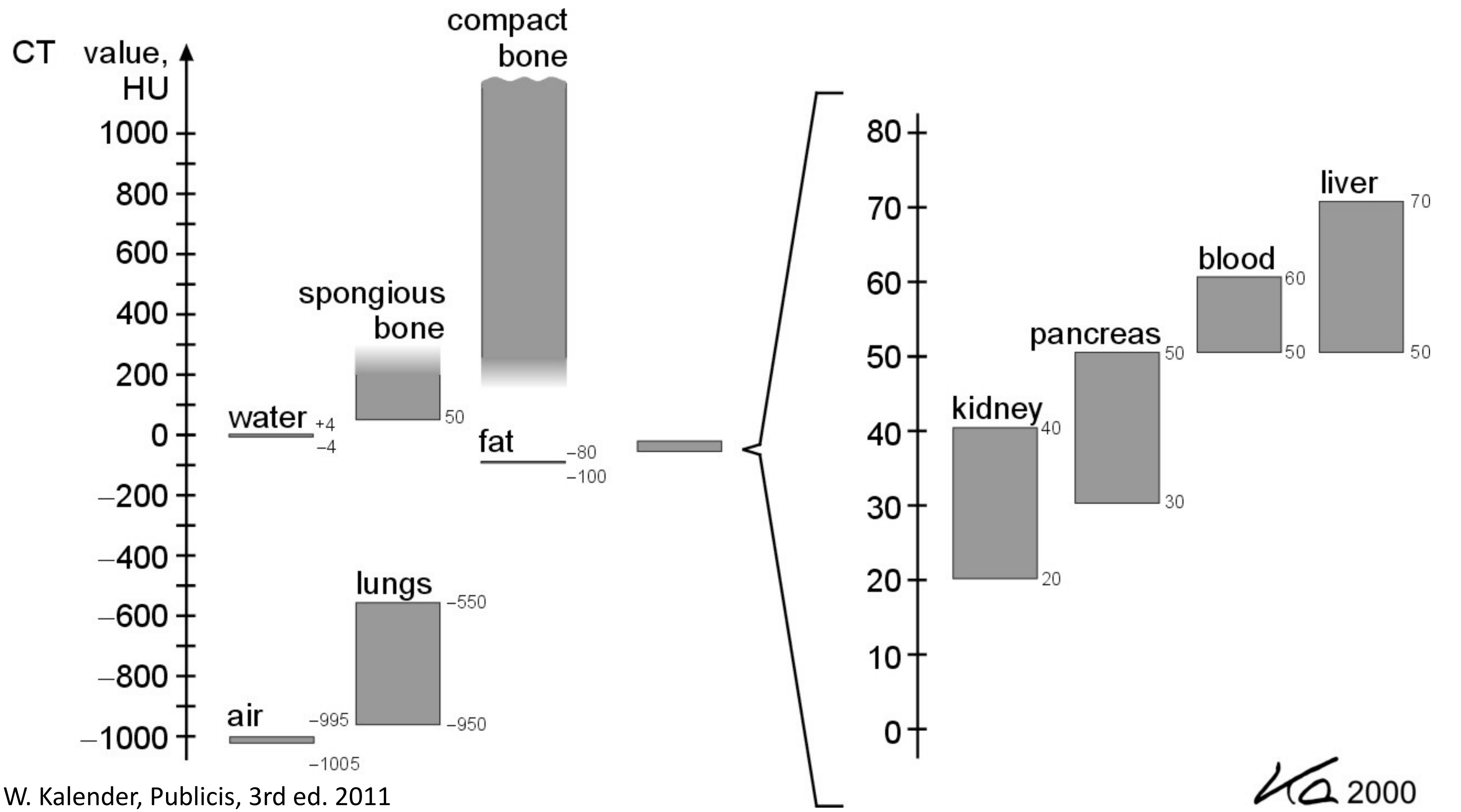
Imaging



Hounsfield units

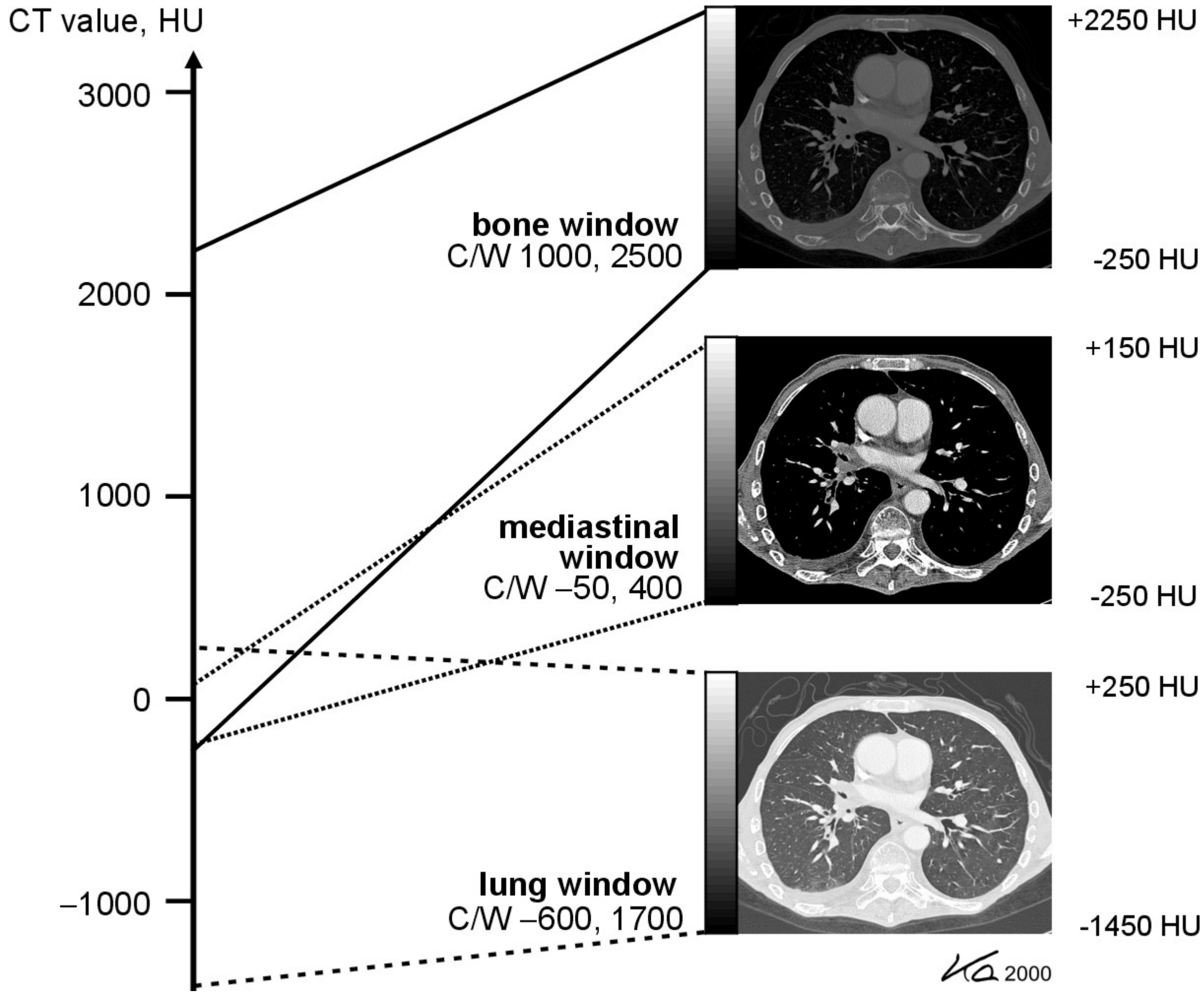
Normalization to water equivalent allows comparison of results from different CT manufacturers

$$\mu[HU] = \frac{\mu - \mu_{water}}{\mu_{water}} \cdot 1000$$



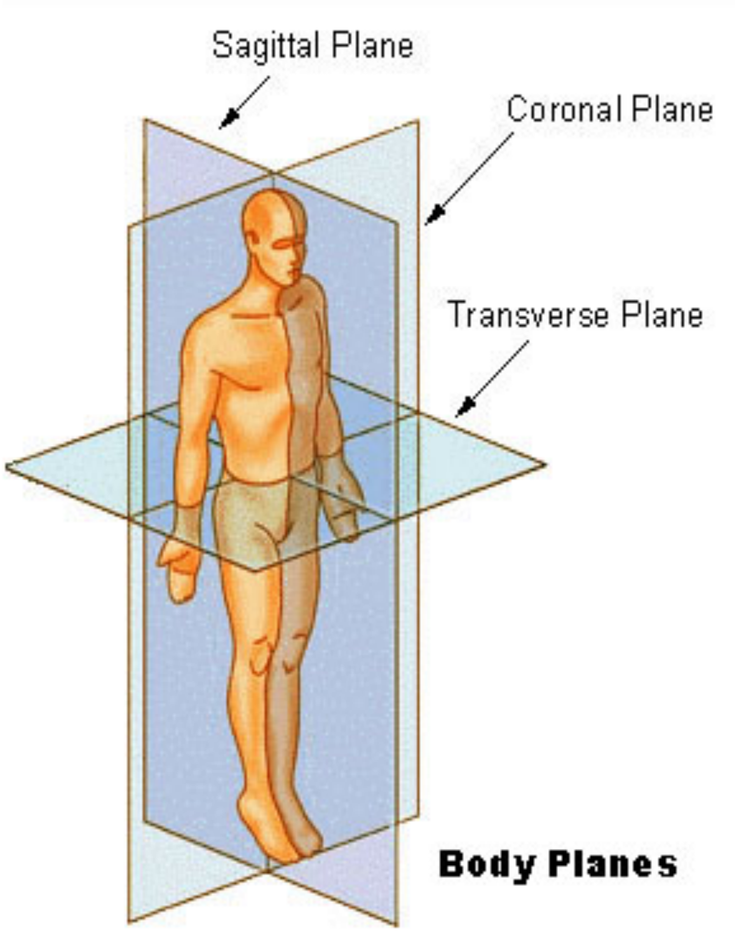
source: W. Kalender, Publicis, 3rd ed. 2011

Windowing, Contrast scaling

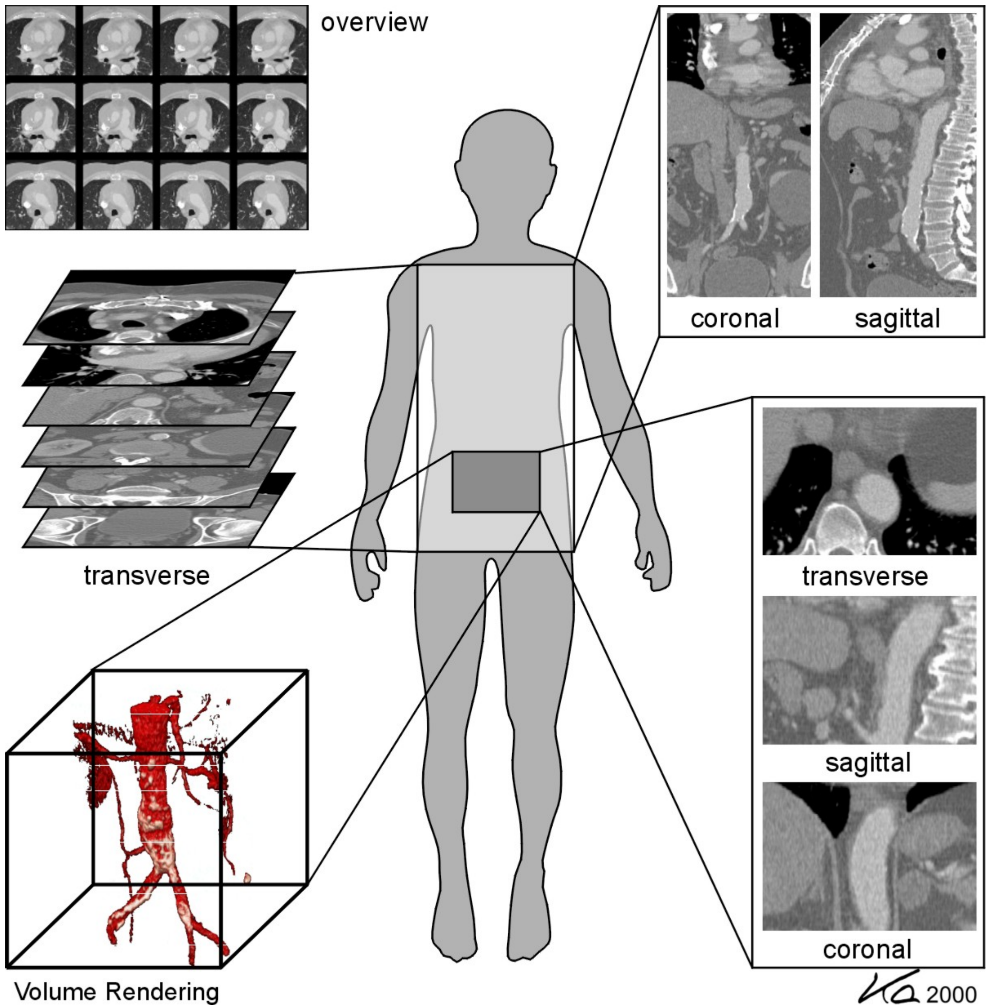


source: Kalender, Publicis, 3rd ed. 2011

Tomographic Display

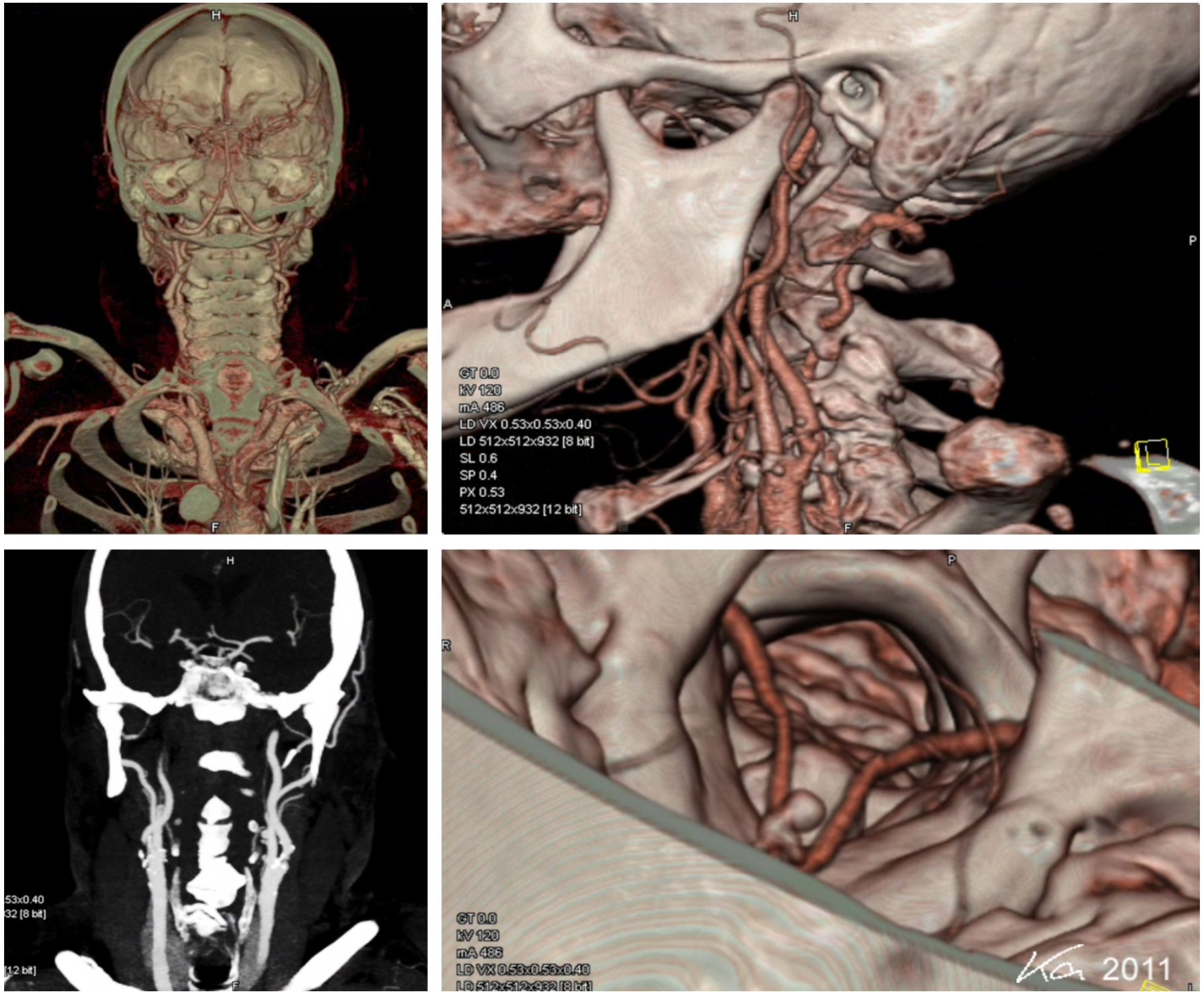


source: wikipedia.org



source: W. Kalender, Publicis, 3rd ed. 2011

Volume rendering display



source: Kalender, Publicis, 3rd ed. 2011

Summary

- Fundamental problem of CT is reconstruction from projections (line integrals)
- Projections and tomographic slices are related by the Fourier slice theorem
- Change from circular to square grid introduces a ramp-like filter, and requires interpolation
- Standard algorithm uses filtered backprojection, FBP
- Different filters can be used to enhance imaging results
- Ram-Lack filter is used for standard projections (attenuation), imaginary Hilbert filter for derivatives (differential phase contrast)

Tomographic reconstruction II

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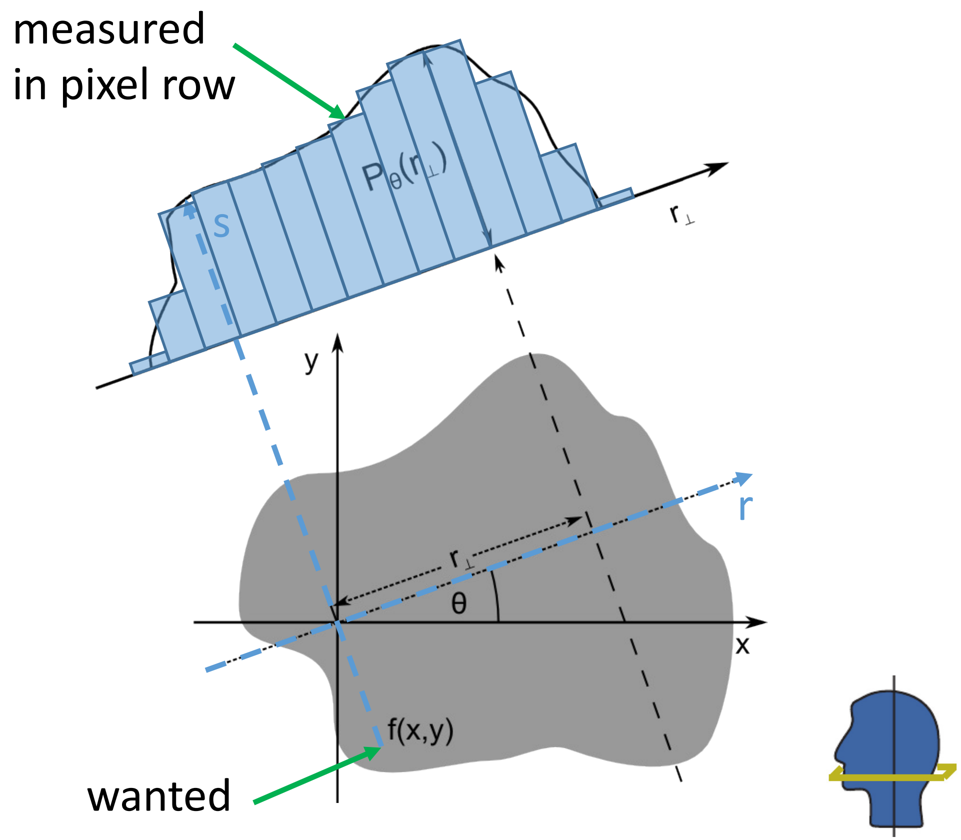
Literature

for Algebraic Reconstruction Technique, ART

- A.C. Kak, M. Slaney
  "Principles of Computerized Tomographic imaging", Chapter 7
  Prentice Hall International, 3rd edition (2008),
  IEEE Press, ISBN 0-87942-198-3
  free electronic copy at http://www.slaney.org

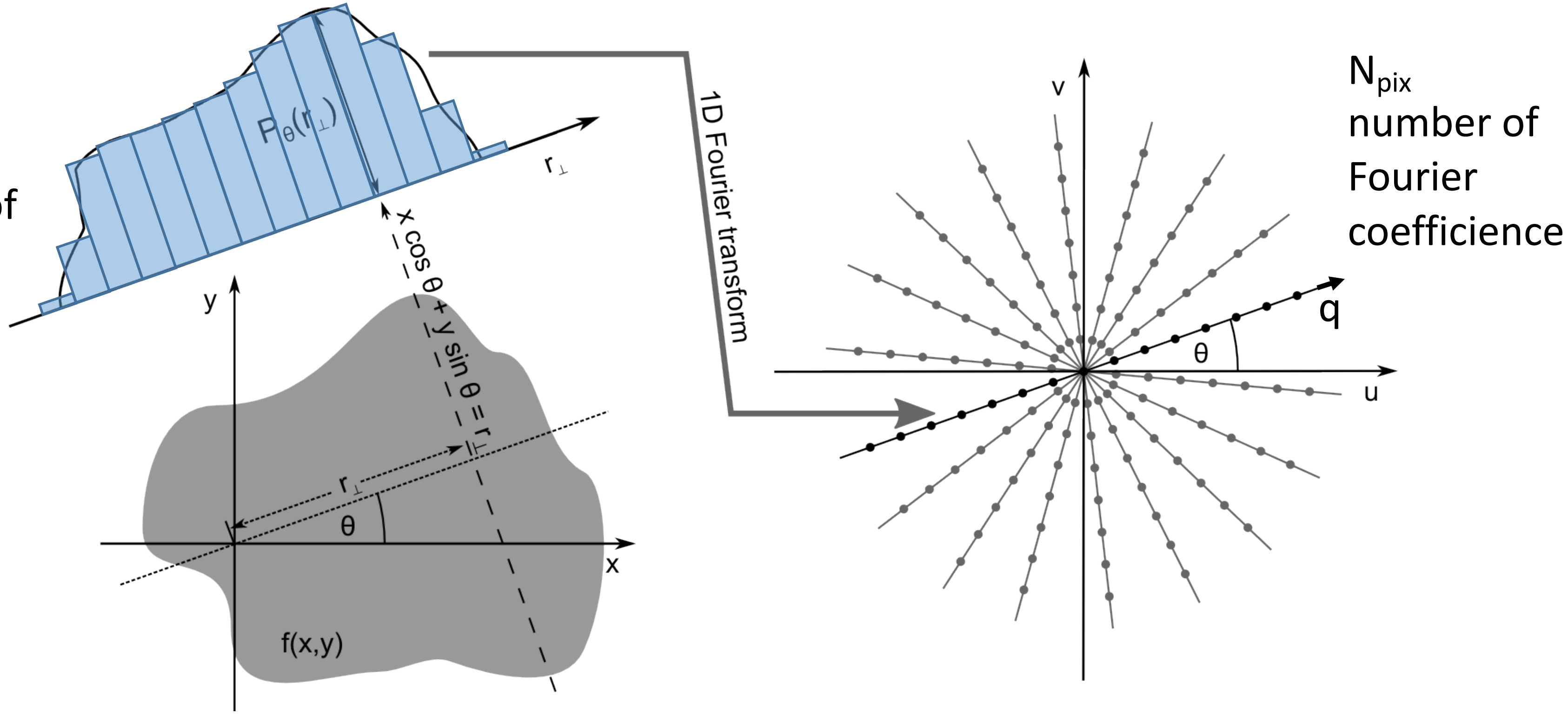
- T.M. Buzug
  "Computed Tomography. From Photon Statistics to Modern Cone-Beam CT" , Chapter 6
  Springer, 1st edition (2008), ISBN-13: 978-354039407
  link.springer.com/book/10.1007/978-3-540-39408-2

Review of Filtered Back Projection, FBP



attenuation contrast	grating based phase contrast
$f_a(x,y) = \mu(x,y)$	$f_{ph}(x,y) = \Phi(x,y)$
$P_\theta(r_\perp) = -\ln\left(\frac{I(\theta,r_\perp)}{I_0}\right) = \int_{-\infty}^{\infty} f_a(r_\perp,s)\,ds$	$P_\theta(r_\perp) = \int_{-\infty}^{\infty} \partial\Phi(r_\perp,s)/\partial r_\perp\,ds = \int_{-\infty}^{\infty} \partial f_{ph}(r_\perp,s)/\partial r_\perp ds$

Review of Filtered Back Projection, FBP

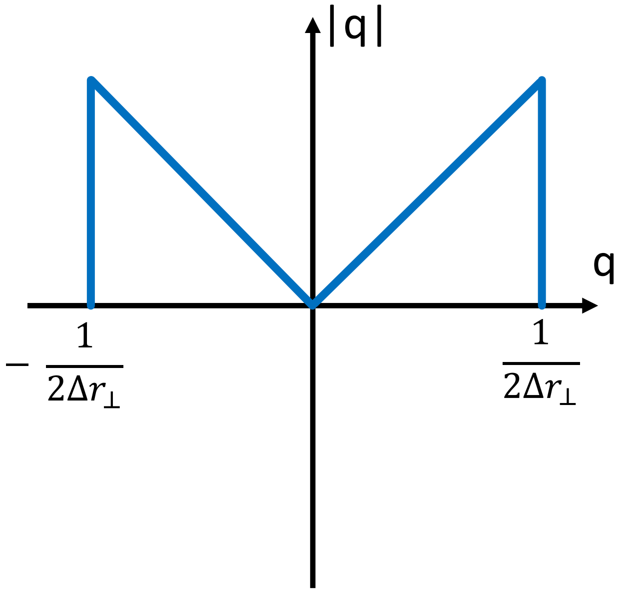


attenuation contrast	grating based phase contrast
$FT(P_\theta(r_\perp)) = \tilde{P}_\theta(q) = \int_{-\infty}^{\infty} P_\theta(r_\perp) \cdot e^{-2\pi i \cdot q r_\perp} \, dr_\perp$	$FT(P_\theta(r_\perp)) = \tilde{P}_\theta(q) = \int_{-\infty}^{\infty} P_\theta(r_\perp) \cdot e^{-2\pi i \cdot q r_\perp} \, dr_\perp$
$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f_a(r_\perp, s) \, ds \right) \cdot e^{-2\pi i \cdot q r_\perp} \, dr_\perp$	$= 2\pi i \cdot q \cdot \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f_{ph}(r_\perp, s) \, ds \right) \cdot e^{-2\pi i \cdot q r_\perp} \, dr_\perp$
$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_a(x, y) \cdot e^{-2\pi i \cdot q (x \cdot cos\theta + y \cdot sin\theta)} \, dxdy$	$= 2\pi i \cdot q \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{ph}(x, y) \cdot e^{-2\pi i \cdot q (x \cdot cos\theta + y \cdot sin\theta)} \, dxdy$

Review of Filtered Back Projection, FBP

attenuation contrast

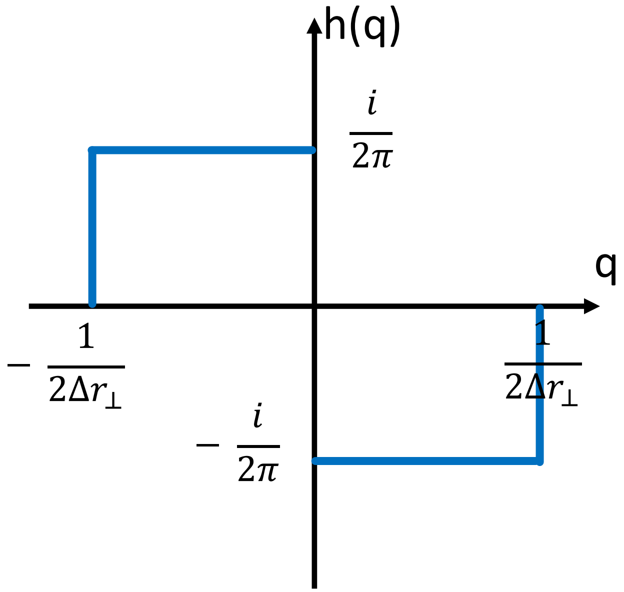
$$f_a(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} FT(P_\theta(r_\perp)) \cdot |q| \cdot e^{2\pi i q r_\perp}\, dq \right] d\theta$$



Ram-Lak filter for absorption

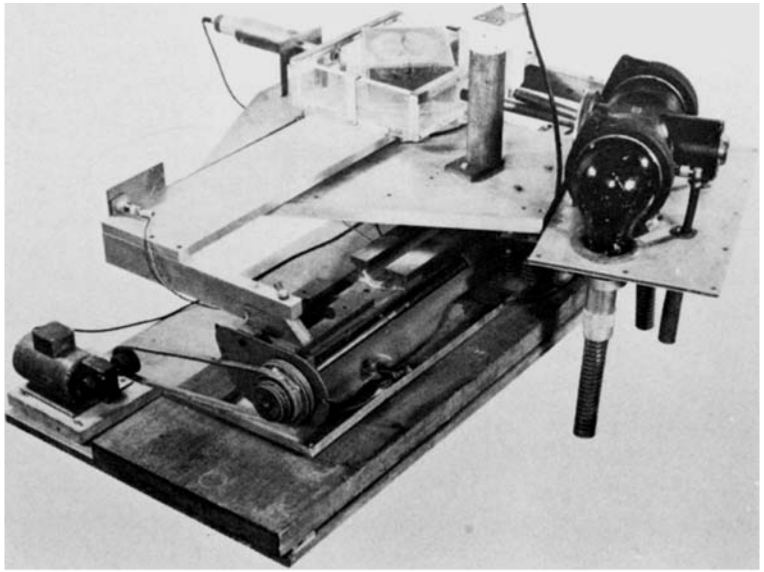
grating based phase contrast

$$f_{ph}(x,y) = \int_0^{\pi} \left[\int_{-\infty}^{\infty} \frac{FT(P_\theta(r_\perp))}{2\pi i \cdot q} \cdot |q| \cdot e^{2\pi i q r_\perp}\, dq \right] d\theta$$



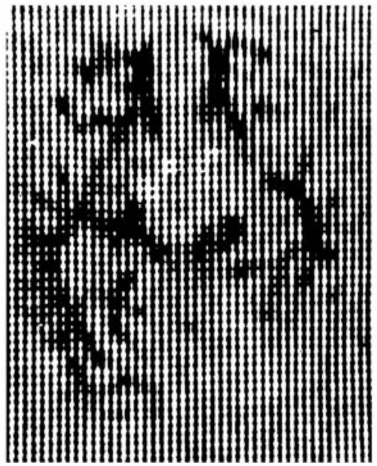
Hilbert filter for differential phase

Algebraic Reconstruction Technique
ART



Hounsfield's first setup

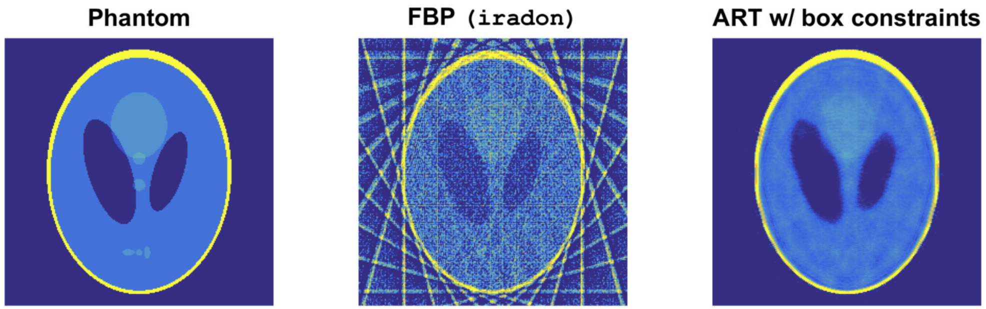
2.5 h reconstruction
time with ART
216 h measurement



Hounsfield's first medical result

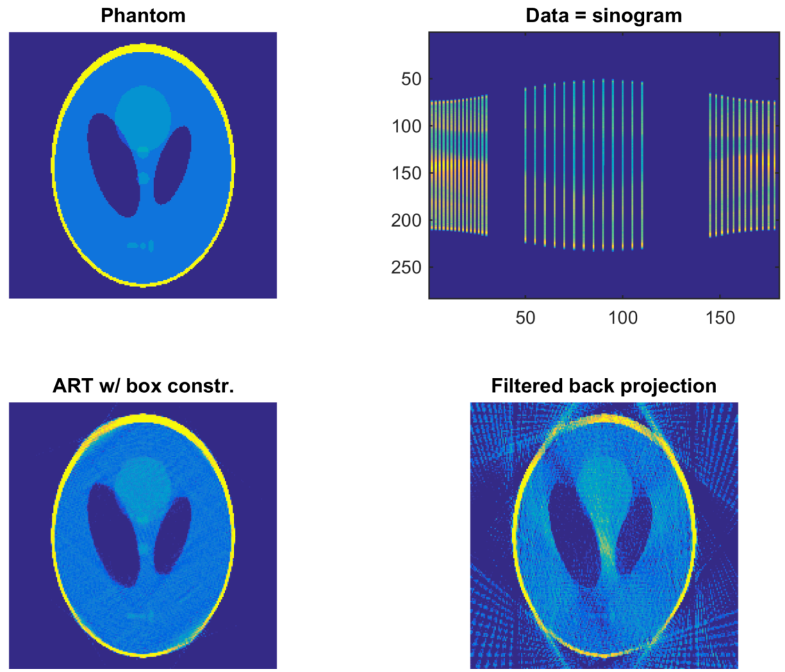
Thorsten M. Buzug, Computed Tomography, Springer 2008

Filtered Back projection	Algebraic Reconstruction Technique
classical method used nearly everywhere	
fast	
low memory requirements	
good results for many and complete projections	good for nonuniform data or if data are missing
difficult to implement constraints	easy to implement constraints
	flexible and adaptive



only 12 projections for 180°

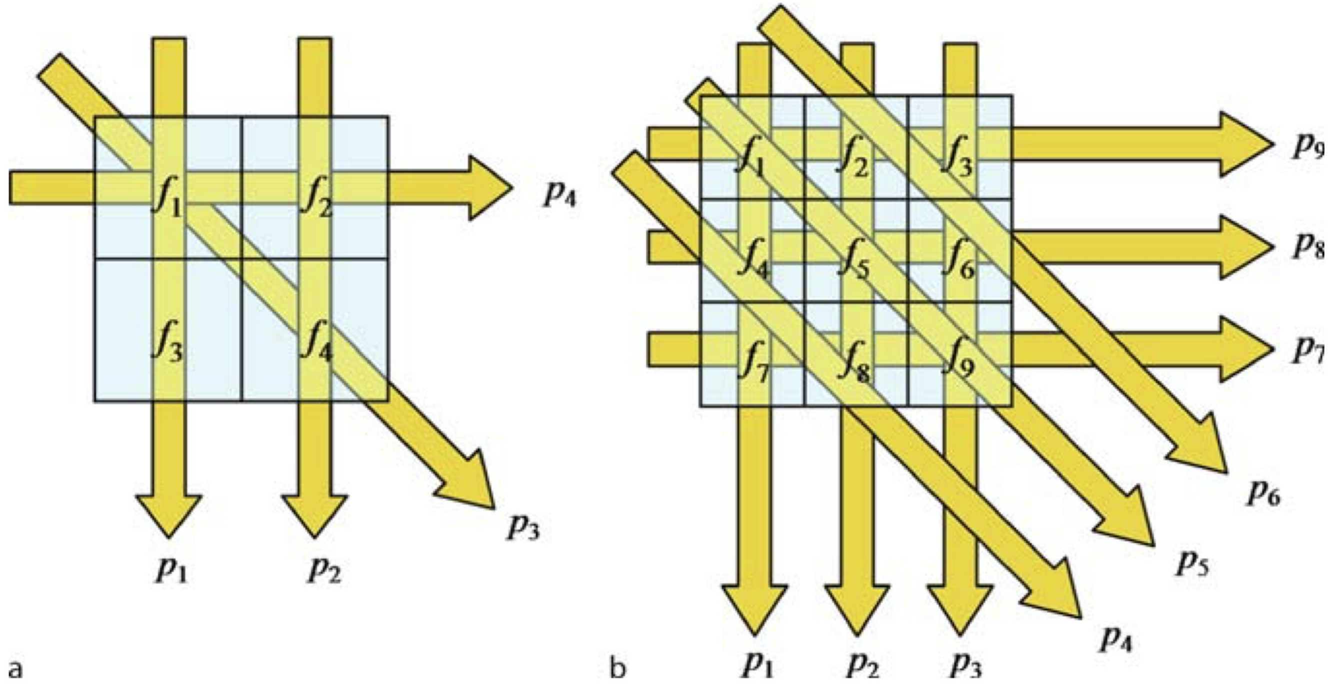
ART with constraint: $pixel\ value > 0$



irregular spaced and missing angles

Algebraic Reconstruction Technique, ART

- Tomography can be formulated as a set of linear equations



$$p_1 = a_1 \cdot f_1 + a_3 \cdot f_3$$

$$p_2 = a_2 \cdot f_2 + a_4 \cdot f_4$$

tomogram f is unknown
projection p is known

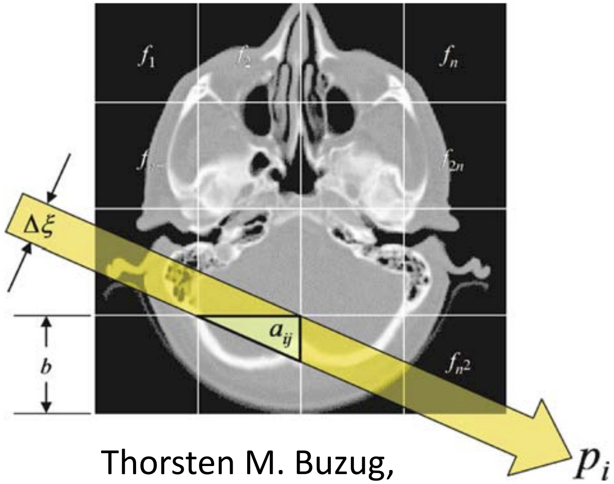
Thorsten M. Buzug, Computed Tomography, Springer 2008

Algebraic Reconstruction Technique, ART

- Tomography can be formulated as a set of linear equations

$$p_i = \sum_{j=1}^{N} a_{ij} \cdot f_j$$

illuminated area of pixel j by ray i / total area of pixel j
depending on beam width $\Delta\xi$ and distance to pixel center



Thorsten M. Buzug, Computed Tomography, Springer 2008

$$\vec{p} = A \cdot \vec{f}$$

unknown tomogram

system matrix

measure projection in detector row (sinogram)

reflects the physics like
a) beam width (see $\Delta\xi$ in the image above)
b) method like Positron Emission Tomography (PET) or CT
c) image given as pixels (rectangular)
or blobs (Kaiser-Bessel window functions)

System Matrix

- Tomography can be formulated as a set of linear equations

$$\vec{p} = A \cdot \vec{f}$$

compare

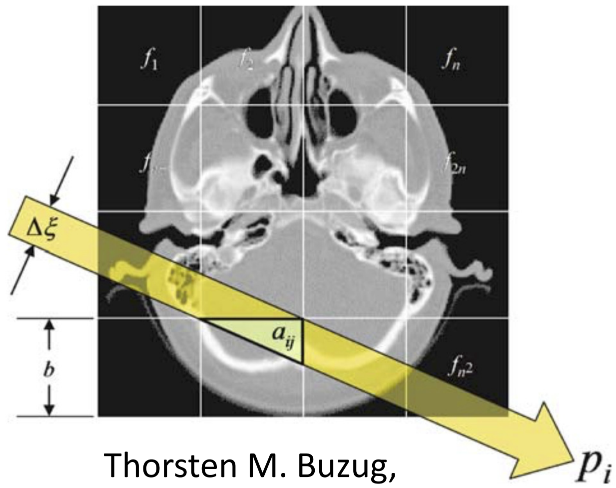
compare with Radon transform

$$P_\theta(r_\perp) = Radon\{f(x,y)\} = \int_{-\infty}^{\infty} f(r_\perp, s)\, ds$$

$$\left.\begin{array}{l} p(\theta, r_\perp) \rightarrow p_i \\ f(x,y) \rightarrow f_i \end{array}\right\} \text{1D vectors}$$

$$p_i = \sum_{j=1}^{N} a_{ij} \cdot f_j \quad \Rightarrow \quad A \cdot \vec{f} = \vec{p}$$

with System Matrix A



Thorsten M. Buzug, Computed Tomography, Springer 2008

System Matrix

- Tomography can be formulated as a set of linear equations

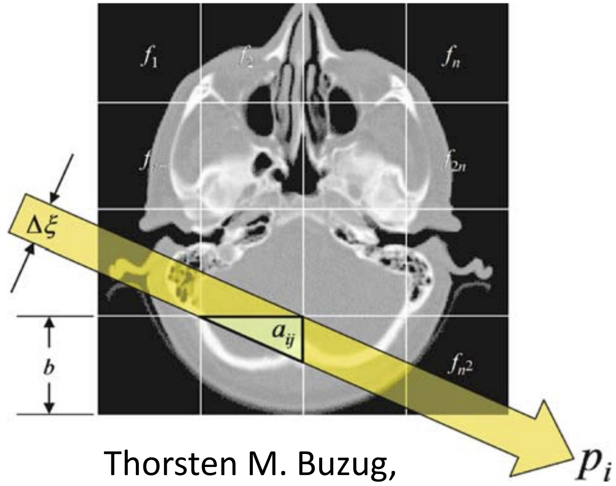
$$p = A \cdot f$$

compare

compare with Radon transform

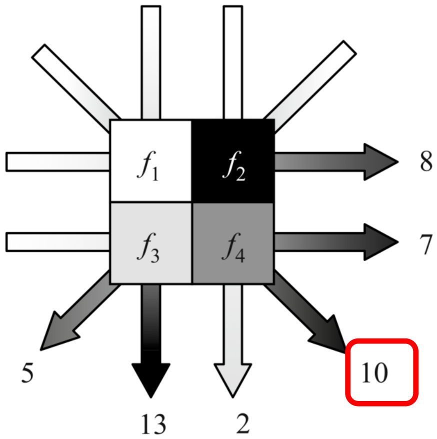
$$P_\theta(r_\perp) = Radon\{f(x,y)\} = \int_{-\infty}^{\infty} f(r_\perp, s)\, ds$$

$$\left.\begin{array}{l} p(\theta, r_\perp) \ \rightarrow p_i \\ f(x,y) \ \rightarrow f_i \end{array}\right\}$$ 1D vectors



Thorsten M. Buzug, Computed Tomography, Springer 2008

$$p_i = \sum_{j=1}^{N} a_{ij} \cdot f_j \quad \Rightarrow \quad A \cdot \vec{f} = \vec{p}$$ with System Matrix A



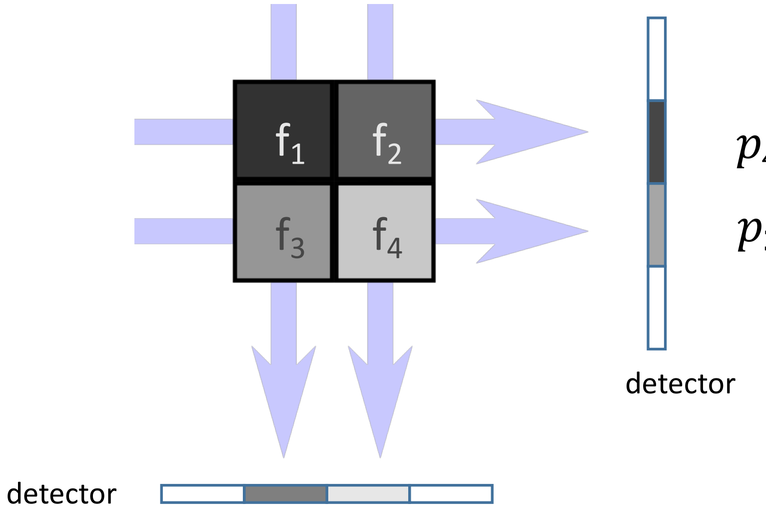
$a_{.j}$

$a_{i.}$

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ \boxed{1 \ \ 0 \ \ 0 \ \ 1} \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 2 \\ \boxed{10} \\ 7 \\ 8 \end{pmatrix}$$

known unknown measured

$$p_4 = 1 \cdot f_1 + 1 \cdot f_4$$



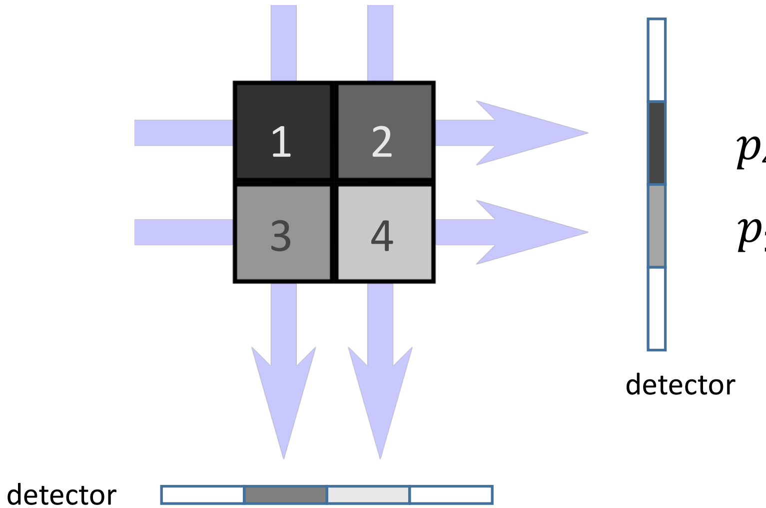
$p_4 = 3$

$p_3 = 7$

$p_1 = 4$

$p_2 = 6$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 7 \\ 3 \end{pmatrix}$$



$p_4 = 3$

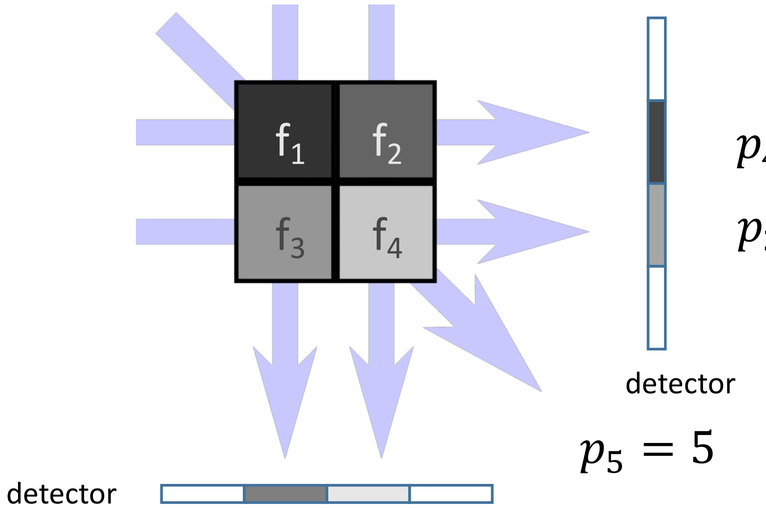
$p_3 = 7$

$p_1 = 4$

$p_2 = 6$

one solution is given here $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

but $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + \begin{pmatrix} x \\ -x \\ -x \\ x \end{pmatrix}$ is also a solution



$p_4 = 3$

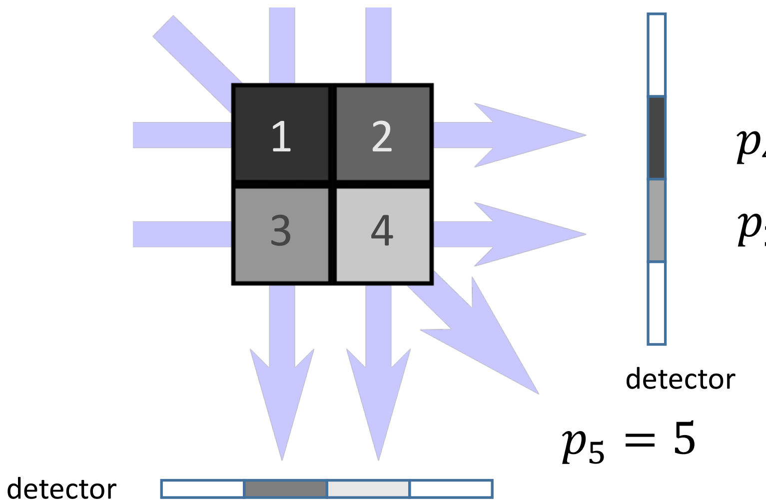
$p_3 = 7$

$p_5 = 5$

$p_1 = 4$

$p_2 = 6$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \\ 7 \\ 3 \\ 5 \end{pmatrix}$$



$p_4 = 3$

$p_3 = 7$

$p_5 = 5$

$p_1 = 4$

$p_2 = 6$

the one solution is given here $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$

Direct Solution?

Solution to the linear equation?

$\vec{p} = A \cdot \vec{f}$ ⟹ $\tilde{\vec{f}} = A^{-1} \cdot \vec{p}$ ⟹ $A^{-1} = ?$

size of the matrix A for
≈ 1000 pixel in a detector row
≈ 1500 projections

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 2 \\ 10 \\ 7 \\ 8 \end{pmatrix}$$

$(sample_{row} \cdot sample_{column}) \approx 10^6$

$(sample_{row} \cdot sample_{column}) \cdot N_{projection} \cdot N_{DetRow} \approx 10^{12}$

$N_{projection} \cdot N_{DetRow} \approx 10^6$

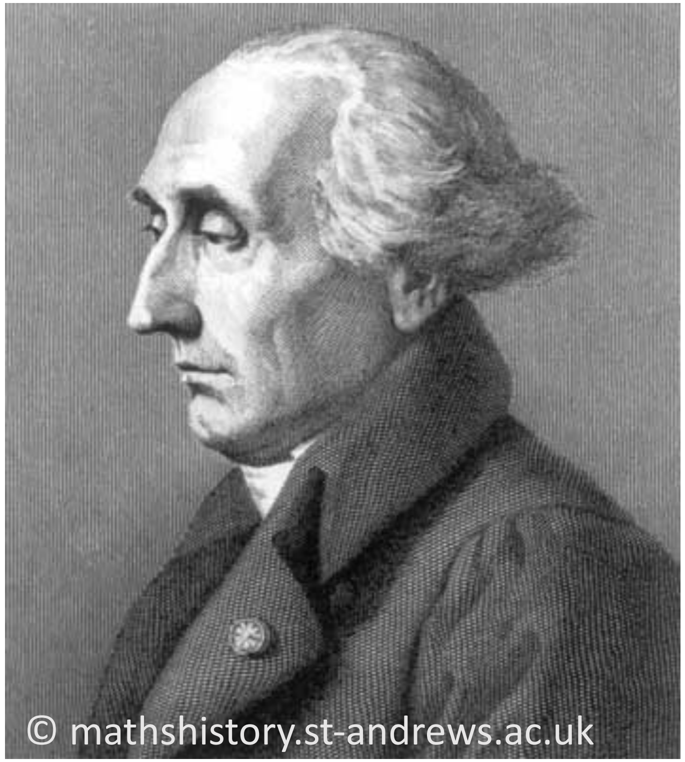
system matrix usually too large for direct inversion $\cancel{A^{-1}}$
need some other solution scheme
→ large pool of numerical solution methods available

Iterative Solution

Guess and iterate to an optimum

Stefan Marian Kaczmarz
20. Mar. 1895 – Sep. 1939

- With Lagrange multipliers
- With Kaczmarz method



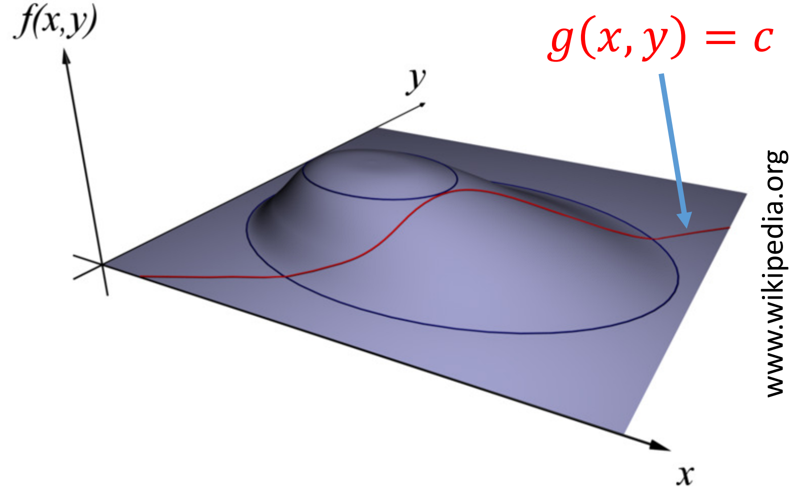
Joseph-Louis Lagrange
25. Jan. 1736 – 10. Apr. 1813

Lagrange multipliers

- Optimization under equality constraints

we search for $\max_{x,y} f(x,y)$ or $\min_{x,y} f(x,y)$ with $g(x,y) = c$

$g(x,y) - c = 0$ is a D-1 dimensional surface in D (D=2 here)



define the Lagrange function L

$$L(x,y,\lambda) = f(x,y) + \lambda(g(x,y) - c)$$

cost function

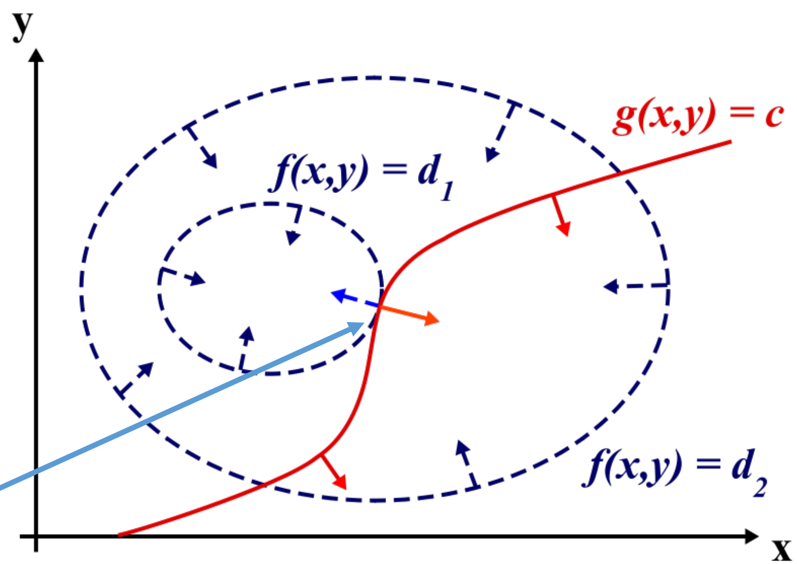
Lagrange multiplier

$\nabla_{x,y,\lambda} L(x,y,\lambda) = 0$ we want to have extremum

check $\dfrac{\partial L}{\partial \lambda} = 0 \implies g(x,y) = c$ constant in x,y

$\nabla_{x,y} = \dfrac{\partial L}{\partial x} + \dfrac{\partial L}{\partial y} = 0 \implies$ extremum of $f(x,y)$

on $g(x,y) = c$



this is the point where the gradients have opposite directions $\nabla_{x,y} f = -\lambda \nabla_{x,y} g$

Lagrange multipliers

- minimize approximation such that it is consistent with the data

Lagrange function
$$L = \underbrace{\sum_j \left(f_j - \hat{f}_j\right)^2}_{\text{cost function}} + \sum_i \lambda_i \left(\sum_j A_{ij} f_j - p_i\right)$$

Lagrange multiplier

minimize difference between guess $\hat{f}_j$ and real value f_j under constraint of measured projection values p_i

we want minima, hence
$$\frac{\partial L}{\partial f_j} \overset{!}{=} 0$$

$$\frac{\partial L}{\partial f_j} = 2\left(f_j - \hat{f}_j\right) + \sum_i \lambda_i A_{ij} = 0$$

$$f_j = \hat{f}_j - \frac{1}{2} \mathrm{A}_{\mathrm{j}}^T \vec{\lambda} \qquad \text{I)}$$

$$\frac{\partial L}{\partial \lambda_i} \overset{!}{=} 0$$

$$\frac{\partial L}{\partial \lambda_i} = \sum_j A_{ij} f_j - p_i = 0$$

now we substitute f_j with I)

Lagrange multipliers

- minimize approximation such that it is consistent with the data

$$\frac{\partial L}{\partial \lambda_i} = \sum_j A_{ij} f_j - p_i = 0$$

now we substitute f_j with I)

$$f_j = \hat{f}_j - \frac{1}{2} \mathrm{A}_{\mathrm{j}}^T \vec{\lambda} = \hat{f}_j - \frac{1}{2} \sum_i \lambda_i A_{ij}$$

$$\sum_j A_{ij} \left(\hat{f}_j - \frac{1}{2} \sum_i \lambda_i A_{ij} \right) - p_i = 0$$

$$\left(\underbrace{\sum_j A_{ij} \hat{f}_j}_{\hat{p}_i} - \frac{1}{2} \underbrace{\sum_j A_{ij} \sum_i \lambda_i A_{ij}}_{(A_i A_i^T)\vec{\lambda}} \right) - p_i = 0$$

$$\Longrightarrow \quad \vec{\lambda} = 2 \left(A_i A_i^T \right)^{-1} (\hat{p}_i - p_i) \qquad \text{II)}$$

we now can put II) into I) to get for the tomogram

$$f_j = \hat{f}_j - \mathrm{A}_{\mathrm{j}}^T \left(A_i A_i^T \right)^{-1} (\hat{p}_i - p_i) = \hat{f}_j - \frac{\mathrm{A}_{\mathrm{j}}^T (\hat{p}_i - p_i)}{\left(A_i A_i^T \right)}$$

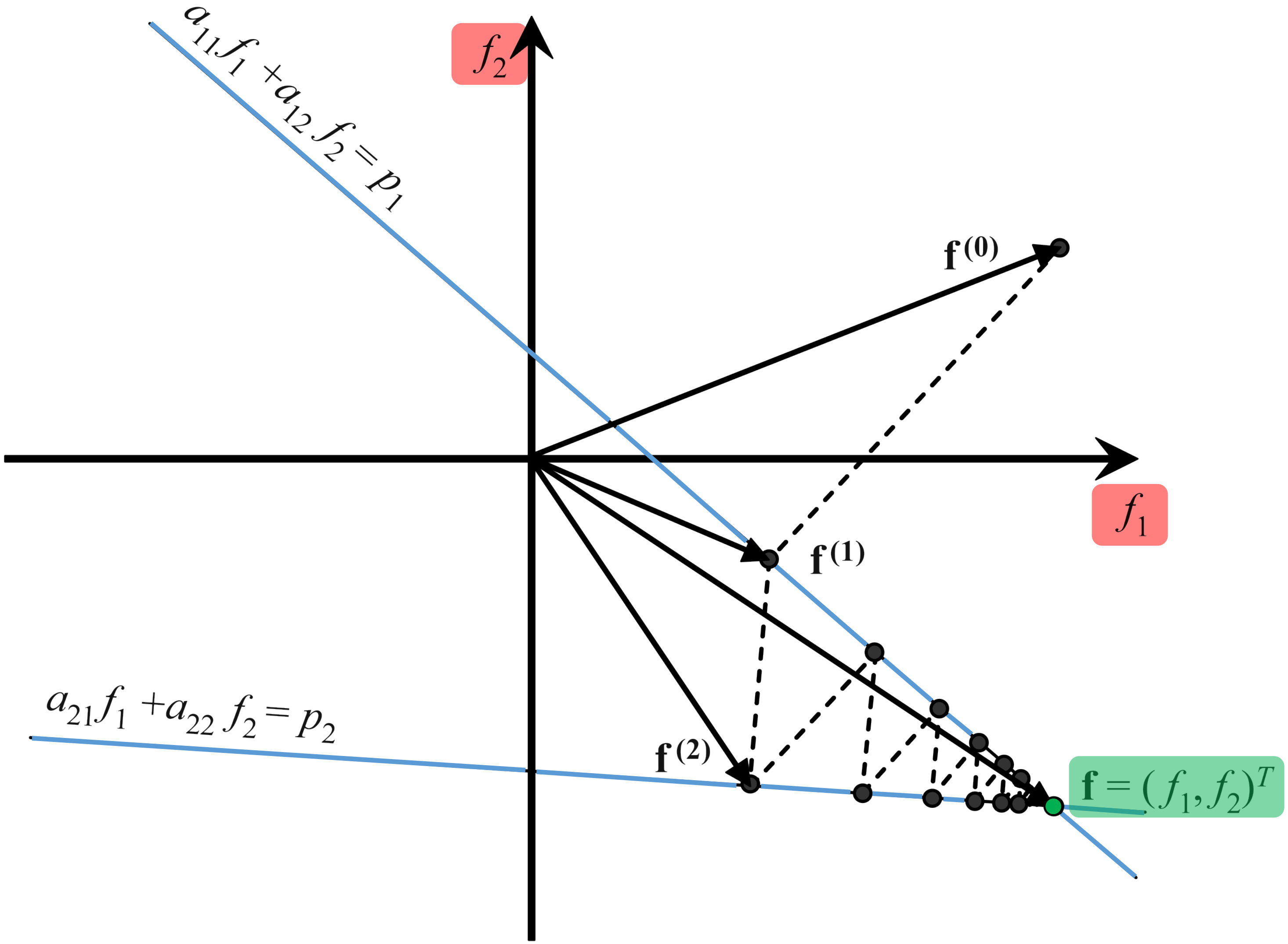
and hence for the n^{th} iteration
and the j components

$$f = f^{(n)} = f^{(n-1)} - \frac{A^T (\hat{p}_i - p_i)}{\left(A_i A_i^T \right)}$$

Also known as
Algebraic Reconstruction Technique (ART)

Algebraic Reconstruction Technique, ART

Step by step example for 2 unknowns f_1, f_2 and 2 measurements (projections) p_1, p_2

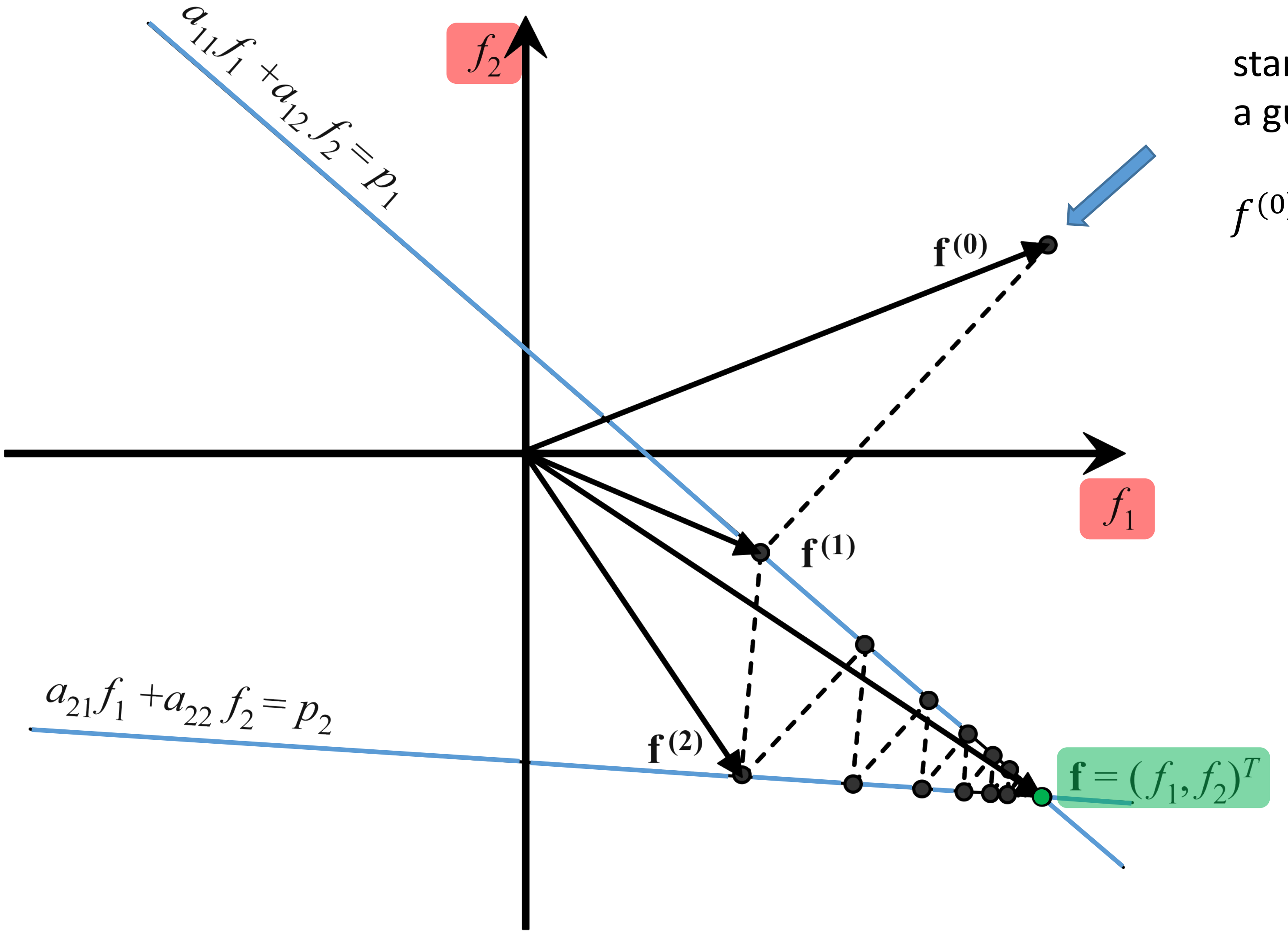


intersection of measurements is solution

Thorsten M. Buzug, Computed Tomography, Springer 2008

Algebraic Reconstruction Technique, ART

Step by step example for 2 unknowns f_1, f_2 and 2 measurements (projections) p_1, p_2



start with
a guess for

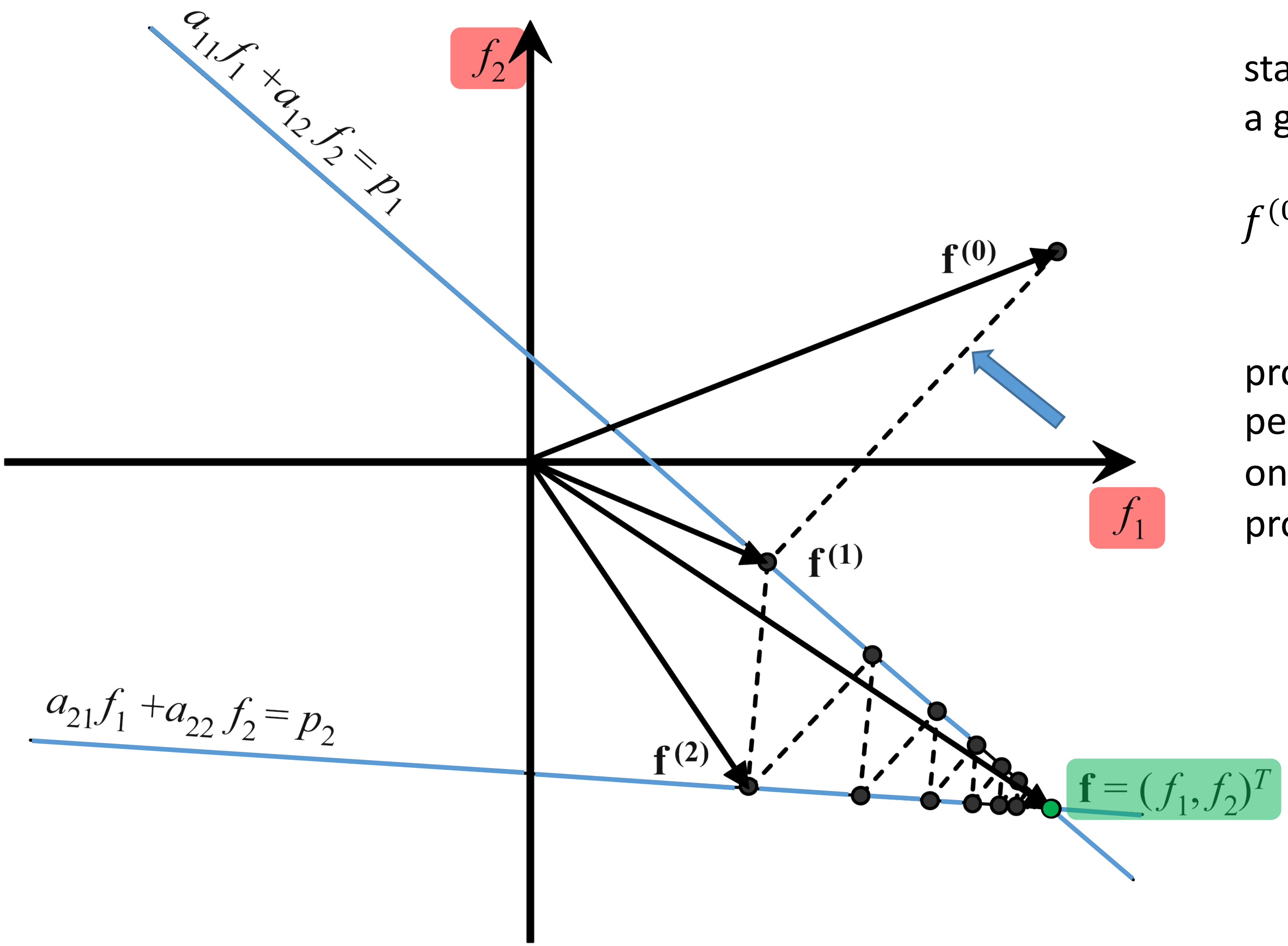
$$f^{(0)} = \begin{pmatrix} f_1^{(0)} \\ f_2^{(0)} \end{pmatrix}$$

intersection of
measurements
is solution

Thorsten M. Buzug, Computed Tomography, Springer 2008

Algebraic Reconstruction Technique, ART

Step by step example for 2 unknowns f_1, f_2 and 2 measurements (projections) p_1, p_2



start with a guess for

$$f^{(0)} = \begin{pmatrix} f_1^{(0)} \\ f_2^{(0)} \end{pmatrix}$$

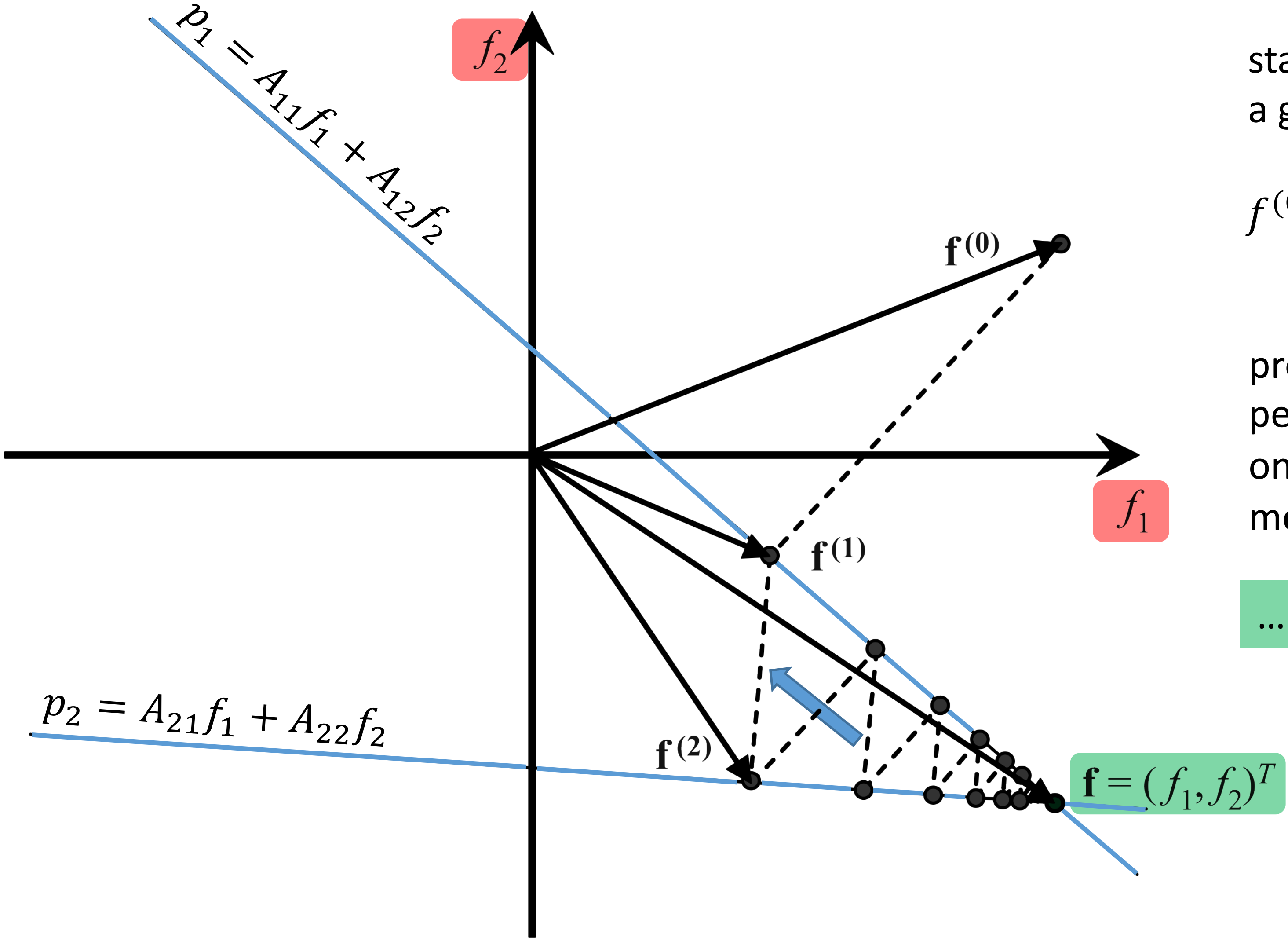
project $f^{(0)}$ perpendicular onto a measured projection

intersection of measurements is solution

Thorsten M. Buzug, Computed Tomography, Springer 2008

Algebraic Reconstruction Technique, ART

Step by step example for 2 unknowns f_1, f_2 and 2 measurements (projections) p_1, p_2



start with
a guess for

$$f^{(0)} = \begin{pmatrix} f_1^{(0)} \\ f_2^{(0)} \end{pmatrix}$$

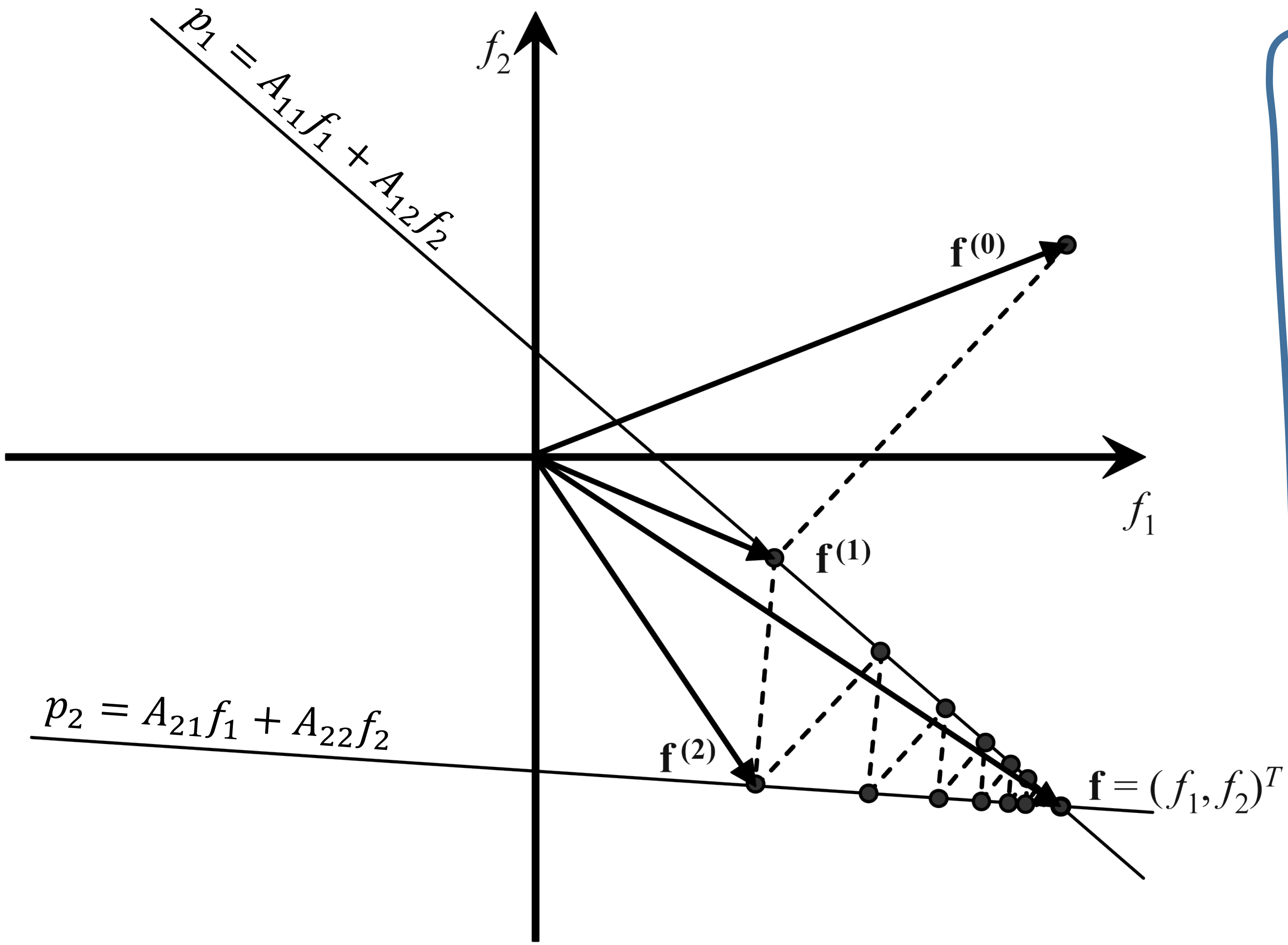
project $f^{(1)}$
perpendicular
onto another
measured projection

... until convergence

intersection of
measurements
is solution

Thorsten M. Buzug, Computed Tomography, Springer 2008

Algebraic Reconstruction Technique, ART



step from (n-1) to n

$$\vec{f}^{(n-1)} = \begin{pmatrix} f_1^{(n-1)} \\ f_2^{(n-1)} \end{pmatrix}$$

i) slope of projection p_i

$$-\begin{pmatrix} A_{i1} \\ A_{i2} \end{pmatrix}$$

ii) slope of all lines perpedicular to p_i

$$\begin{pmatrix} A_{i2} \\ A_{i1} \end{pmatrix}$$

chose the line through $f^{(n-1)}$

iii) calculate intersection point $f^{(n)}$ of projection p_i and line from ii)

$$\vec{f}^{(n)} = \vec{f}^{(n-1)} + (\vec{A}_i)^T \cdot \frac{p_i - \vec{A}_i\,\vec{f}^{(n-1)}}{\vec{A}_i(\vec{A}_i)^T}$$

see appendix

Thorsten M. Buzug, Computed Tomography, Springer 2008

Algebraic Reconstruction Technique, ART

in other words:

- Start with initial guess of f, e.g. zeros, random, or previous FBP

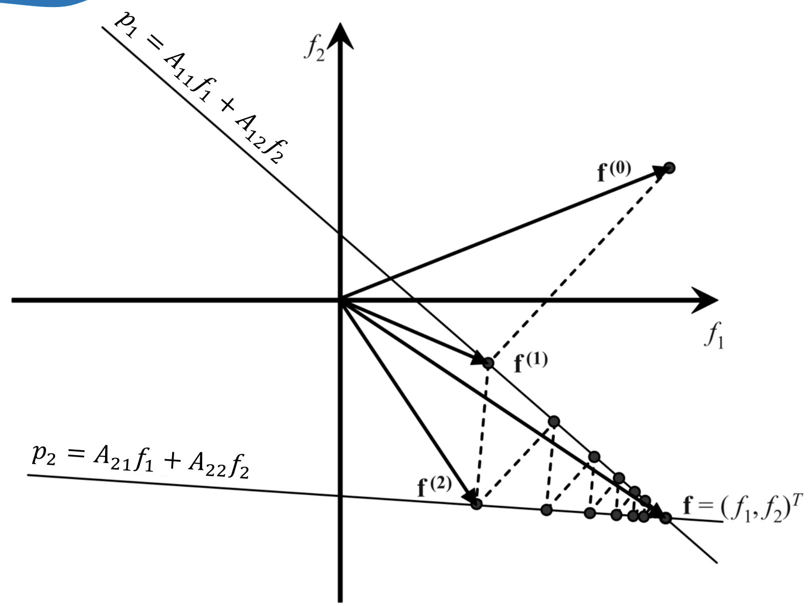
$$\downarrow \vec{f}^0 \; (= \vec{f}^{(n-1)})$$

- Forward-project guess $p_i^{(n-1)} = \vec{A}_i \cdot \vec{f}^{(n-1)}$ for (n-1)th iteration and ith angle that would be the measured projection in case $f^{(n-1)}$ would be correct
- calculate difference to measured projection $\Delta_i^{(n-1)} = p_i - p_i^{(n-1)}$
- Back-project difference along ray

$$\Delta\vec{f}_i^{(n-1)} = \frac{(\vec{A}_i)^T \, \Delta_i^{(n-1)}}{\vec{A}_i (\vec{A}_i)^T}$$

- Update guess with back-projected difference $\vec{f}^{(n)} = \vec{f}^{(n-1)} + \Delta\vec{f}_i^{(n-1)}$

loop over angles i (randomly) and iterations n



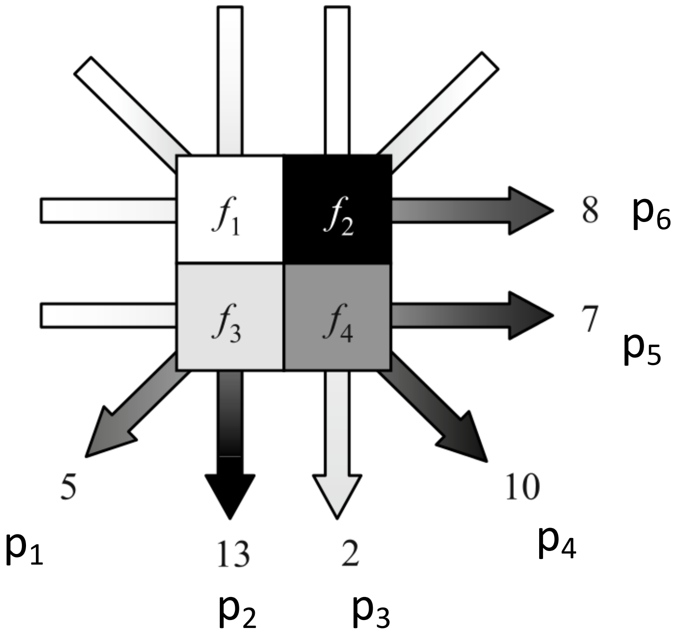
compare with the mathematical result:

$$\vec{f}^{(n)} = \vec{f}^{(n-1)} + (\vec{A}_i)^T \cdot \frac{p_i - \vec{A}_i \vec{f}^{(n-1)}}{\vec{A}_i (\vec{A}_i)^T}$$

for a detailed calculation see the slides in Appendix
(not really part of the lecture)

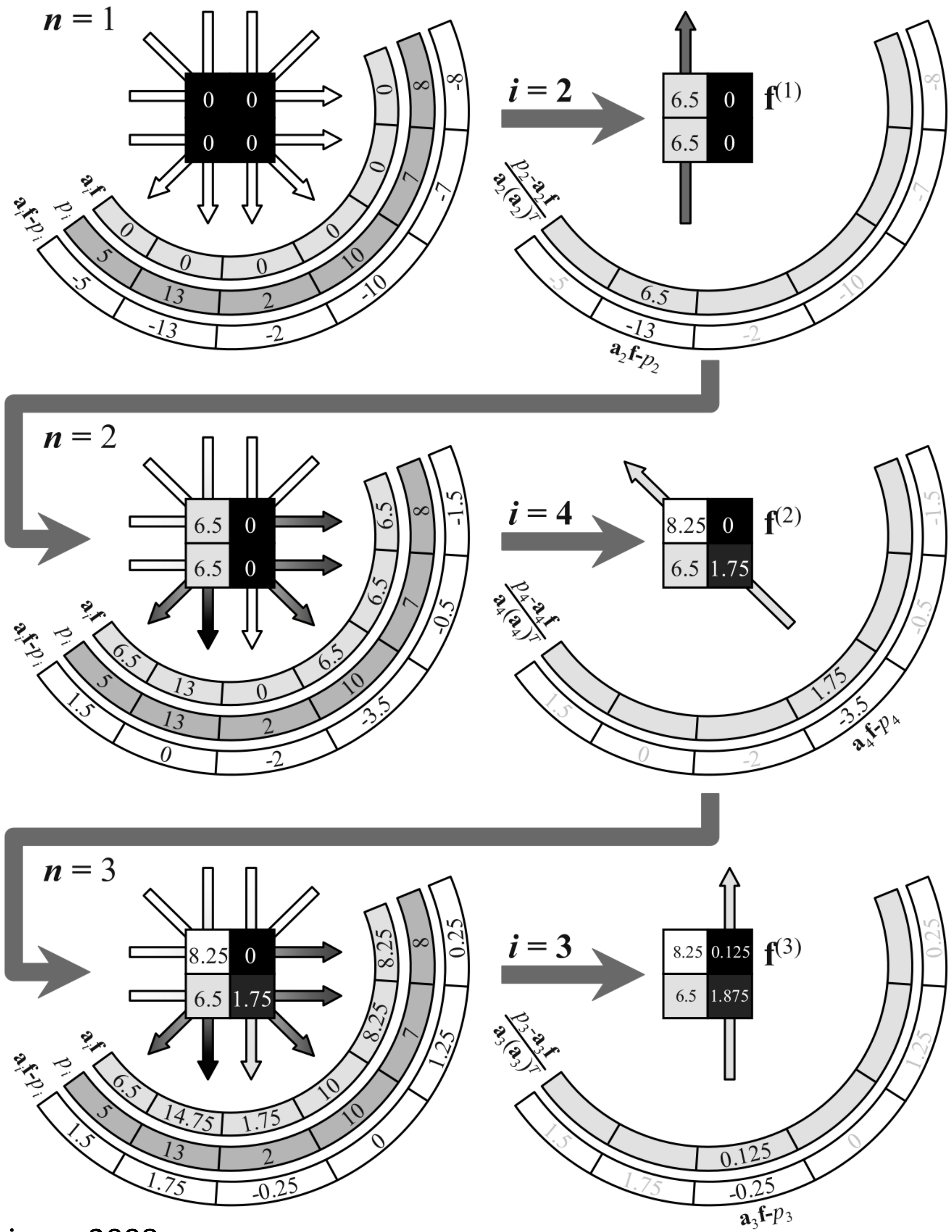
Thorsten M. Buzug, Computed Tomography, Springer 2008

Algebraic Reconstruction Technique, ART



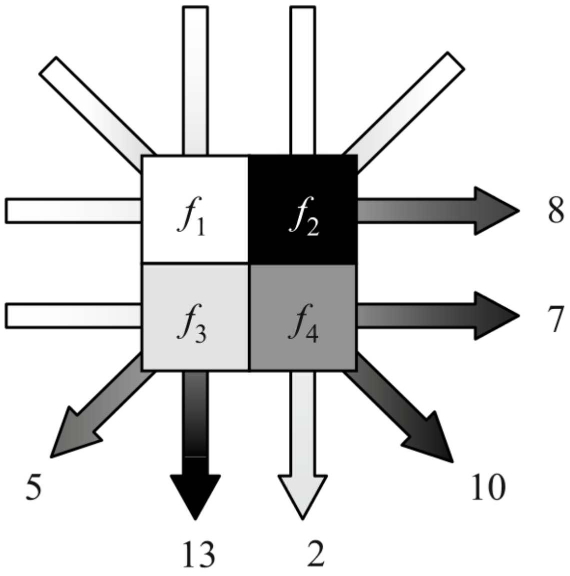
$$
\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 2 \\ 10 \\ 7 \\ 8 \end{pmatrix}
$$

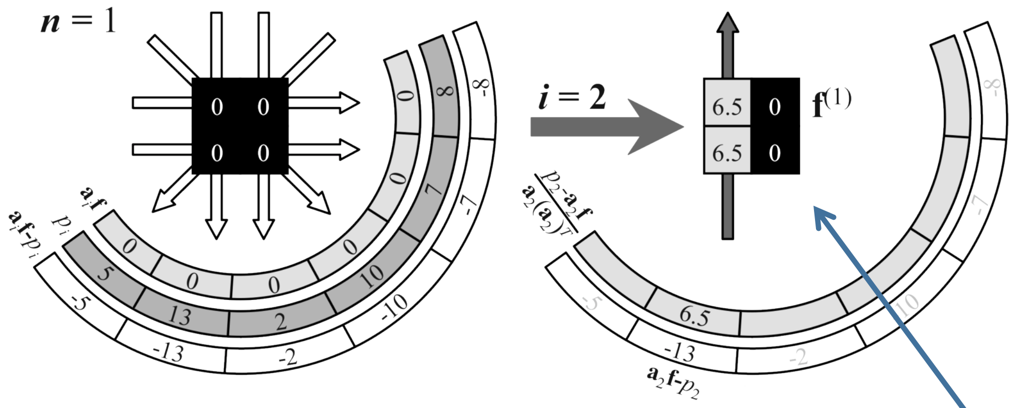
$$
A \cdot \vec{f} = \vec{p}
$$



Thorsten M. Buzug, Computed Tomography, Springer 2008

Algebraic Reconstruction Technique, ART





$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 13 \\ 2 \\ 10 \\ 7 \\ 8 \end{pmatrix}$$

$$A \cdot \vec{f} = \vec{p}$$

iteration: start $\quad f^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

i=2 $\quad A_2 = (1\ 0\ 1\ 0)$

$$p_2^0 = A_2 \cdot f^0 = (1\ 0\ 1\ 0) \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 0$$

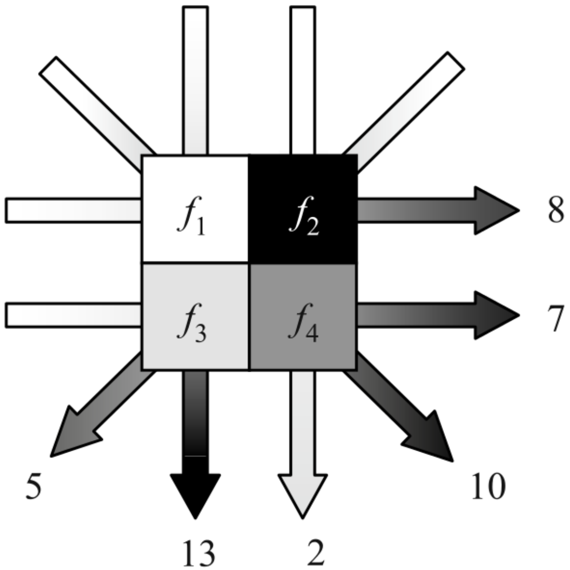
$$\Delta_2^0 = p_2 - p_2^0 = 13 - 0 = 13$$

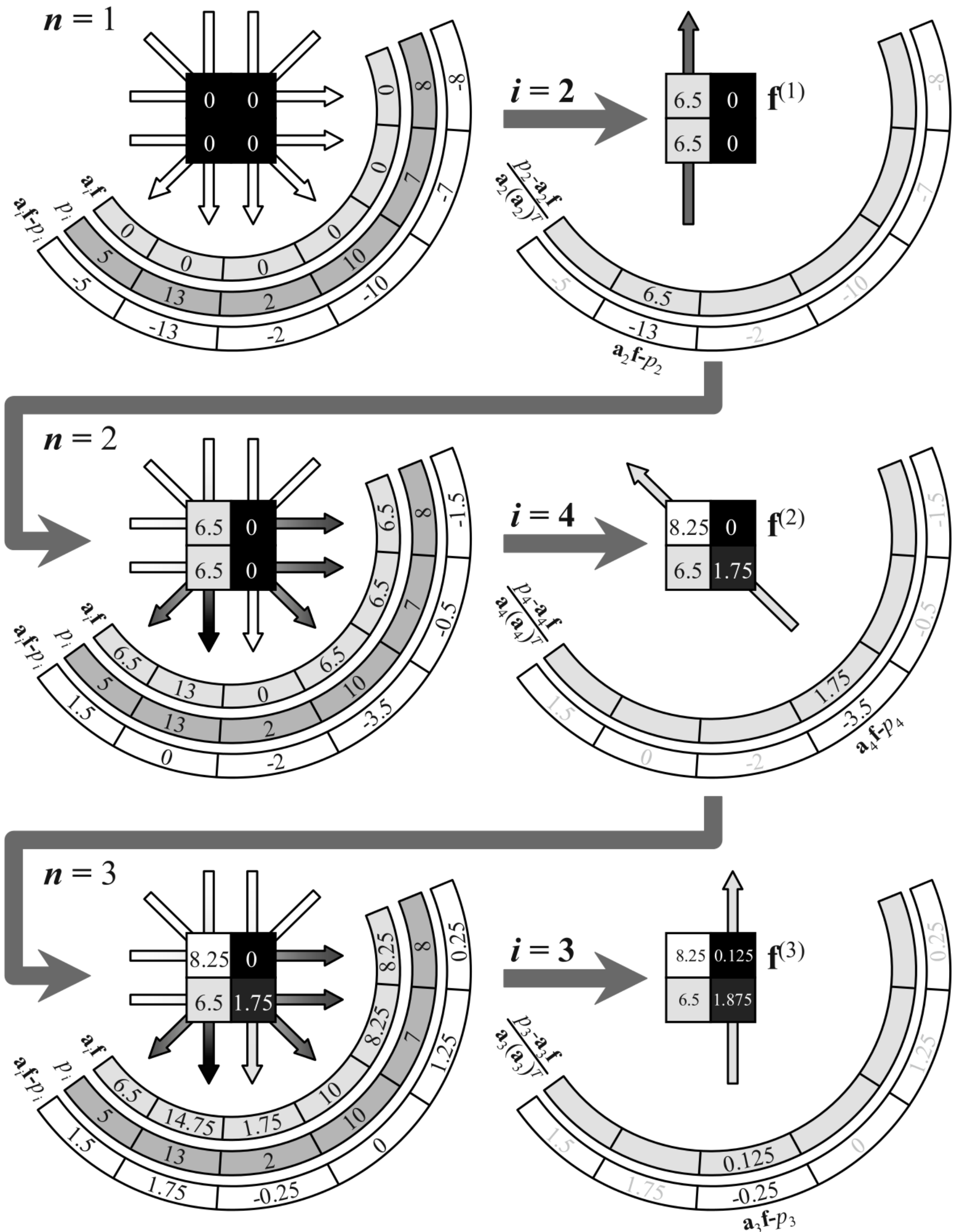
$$\Delta f_2^0 = \frac{A_2^T \Delta_2^0}{|A_2|^2} = \frac{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot 13}{(1\ 0\ 1\ 0) \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}} = \frac{\begin{pmatrix} 13 \\ 0 \\ 13 \\ 0 \end{pmatrix}}{2} = \begin{pmatrix} 6.5 \\ 0 \\ 6.5 \\ 0 \end{pmatrix}$$

$$f^1 = f^0 + \Delta f_2^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 6.5 \\ 0 \\ 6.5 \\ 0 \end{pmatrix} = \begin{pmatrix} 6.5 \\ 0 \\ 6.5 \\ 0 \end{pmatrix}$$

Thorsten M. Buzug, Computed Tomography, Springer 2008

Algebraic Reconstruction Technique, ART





$$
\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix}
=
\begin{pmatrix} 5 \\ 13 \\ 2 \\ 10 \\ 7 \\ 8 \end{pmatrix}
$$

$$A \cdot \vec{f} = \vec{p}$$

correct result

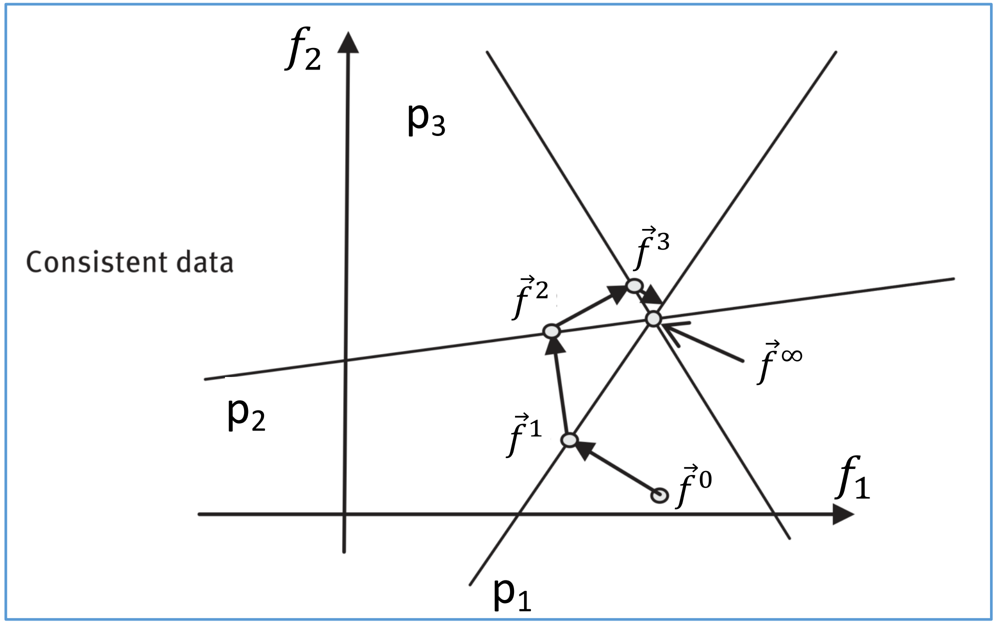
8	0
5	2

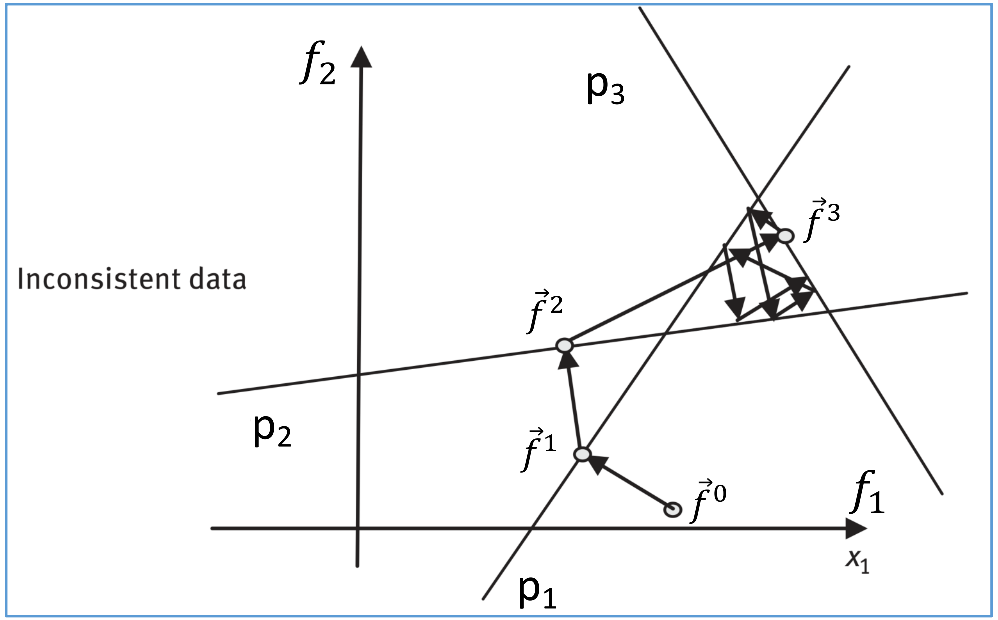
Thorsten M. Buzug, Computed Tomography, Springer 2008

Iterative method: inconsistent data

Kaczmarz method for solving linear equation systems (projection onto convex sets)
Also known as Algebraic Reconstruction Technique (ART)

$$p_i = \sum_{j=1}^{N} A_{ij} \cdot f_j$$





N-dimensional solution space for the N pixels in an image (here N=2)
i=3 projections

G. L. Zeng, Image Reconstruction, De Gruyter Graduate, 2017

ART (dis)advantages over FBP

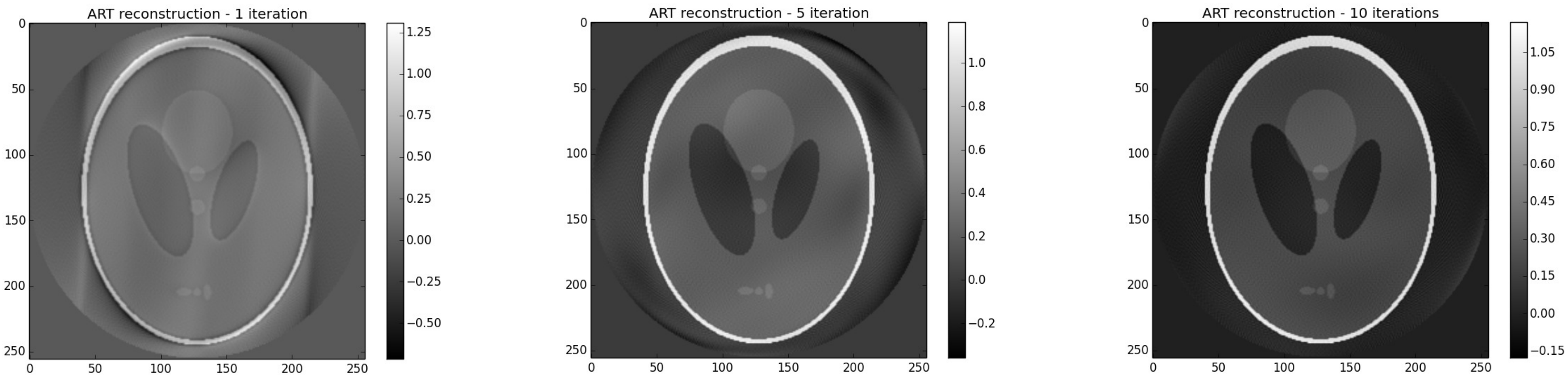
Advantages

- Prior knowledge and restrictions can be implemented into the algorithm (first guess, support, sample shape)
- Artefact reduction/correction can be easily implemented into the algorithm (beam hardening, scatter, etc)
- Missing data can be more easily accounted for (missing views, beam starvation)
- Noise model can be implemented (Poisson, Gaussian, detector readout noise,…), better statistics and SNR
- Result: needs fewer projections/dose

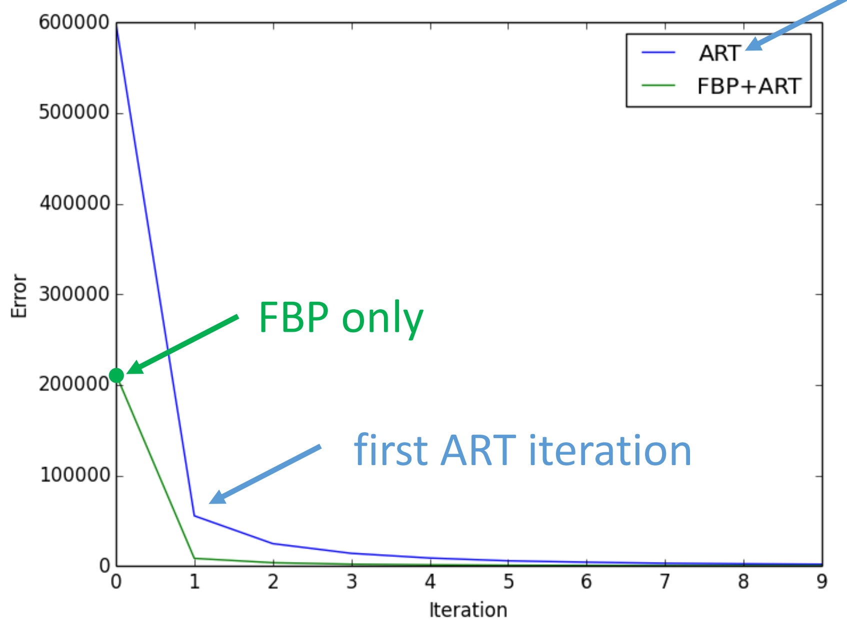
Disadvantages

- Much slower
- More complicated, harder to implement and understand
- Potentially numerically unstable, may need regularization
- New (doctors don’t like change)!

ART example results

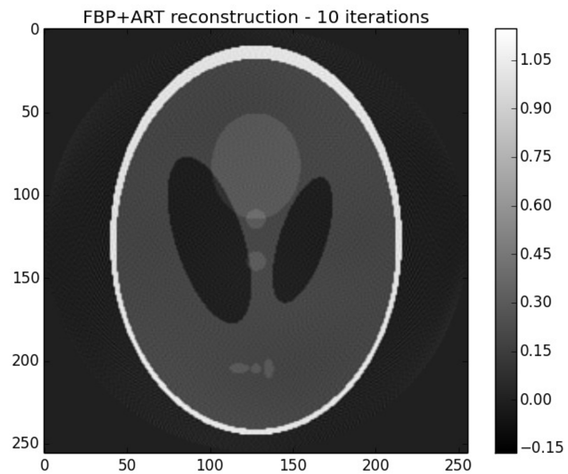


zeros initial condition



FBP only

first ART iteration



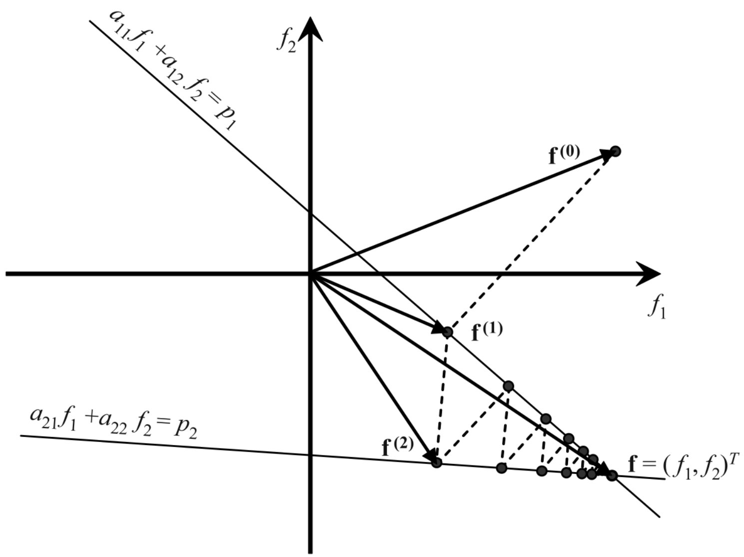
Convergence speed

- Many ways to optimize speed/performance
  - image representation (radial basis functions)

changing the projection angle it is more complicated to calculate the a_{ij} for a square pixel than for a blob

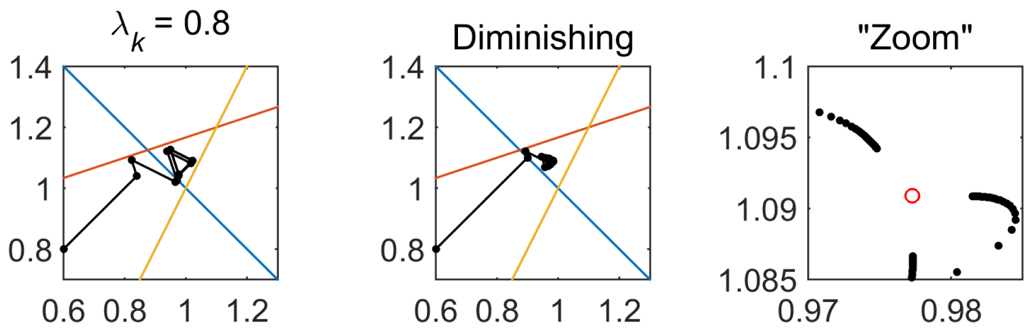
Convergence speed

- Update equation
  - constraint: relaxation parameter λ
  - regularization: a priori knowledge parameter γ



$$\vec{f}^{(n+1)} = \vec{f}^{(n)} + \lambda \cdot \Delta\vec{f} + \gamma \cdot G\left(\vec{f}^{(n)}\right)$$

λ depends on
geometric arrangement of the lines
noise level
number of iterations
iteration number k



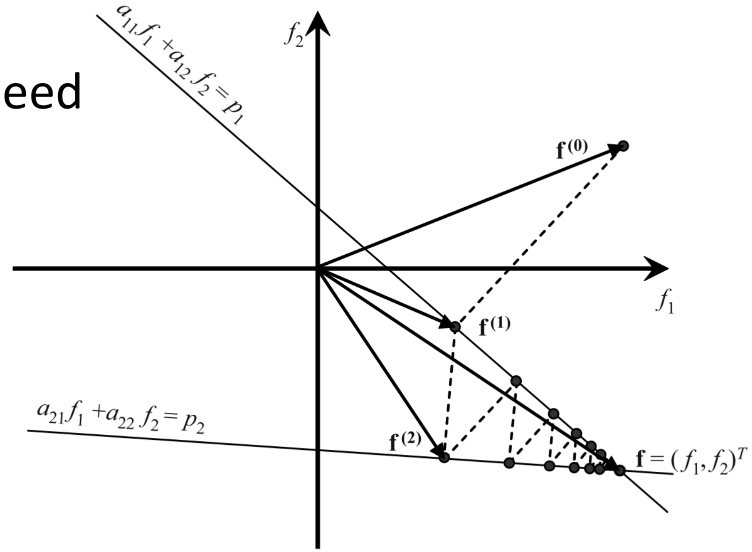
λ_k fixed

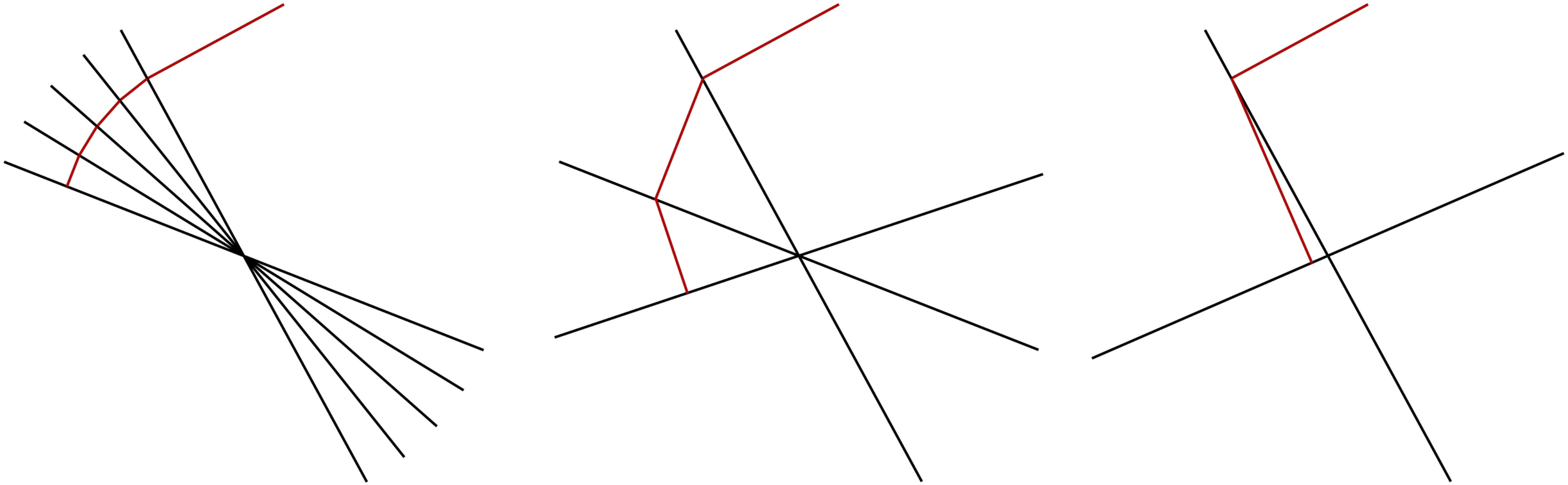
$\lambda_k = 1/\sqrt{k}$

γ can be
values of $f^{(n)}$ are non negative
only a few discrete gray values in sample: Discrete Algebraic Reconstruction Technique (DART)
shape information can be useful if the projections are truncated
how similar have neighbouring pixel values to be without smoothing edges

Convergence speed

- Forward projection
  - angle between rays determines convergence speed
  - slow convergence for neighboring rays
  - choose rays statistically



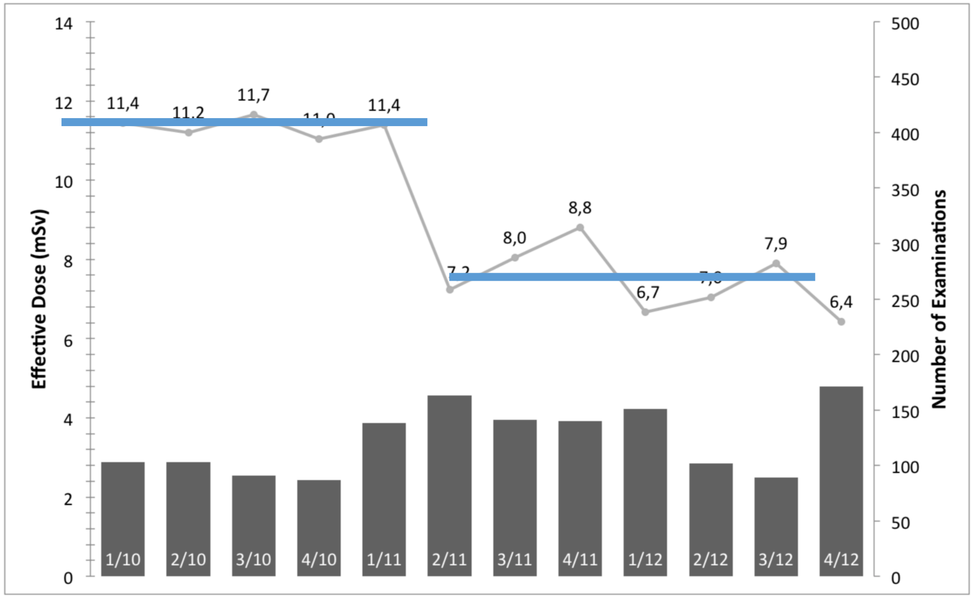


slow | medium | fast convergence

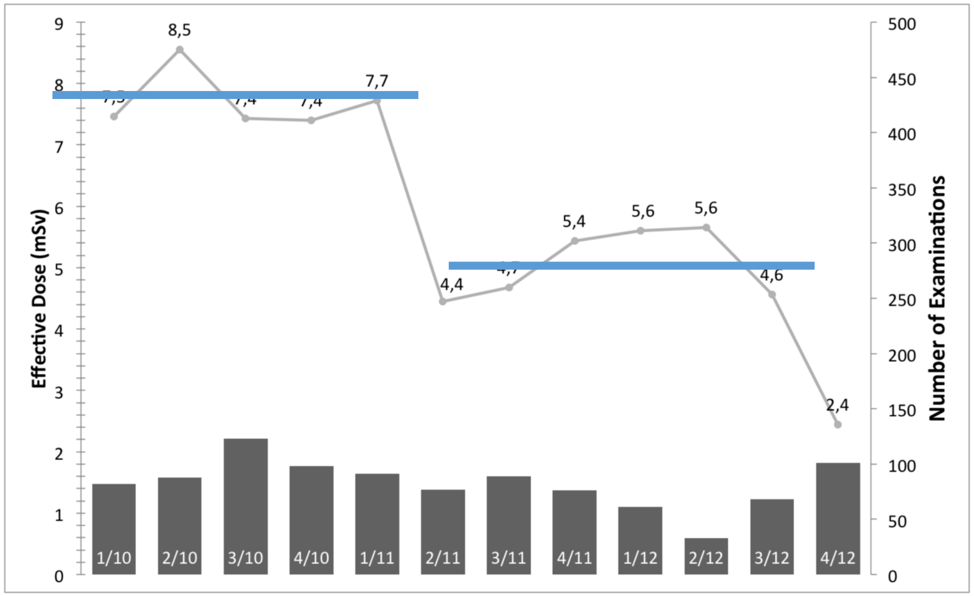
“Real World” results

Median effective dose and number of examinations per quarter year in mSv

aortic CT angiography



pulmonary CT angiography



Introduction of the IR system (between 1/11 and 2/11)

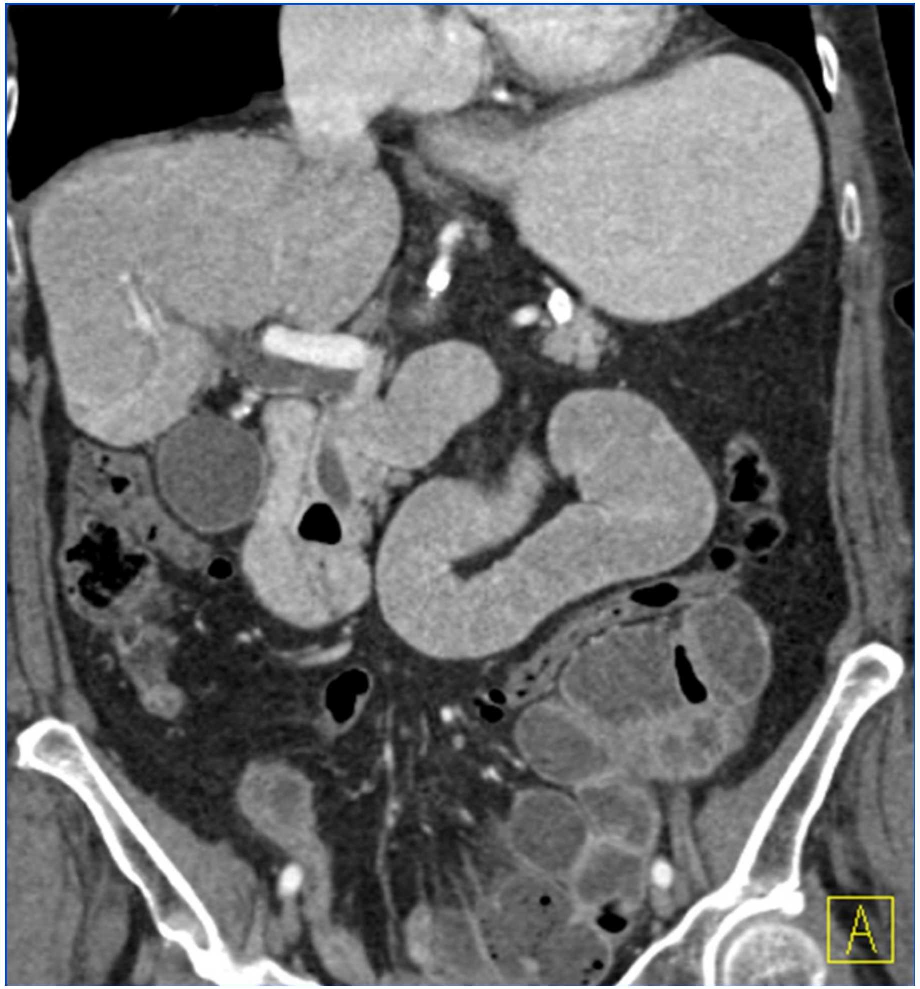
4/12: The IR can compensate the additional noise caused by the reduction of tube-voltage from 120 kVp to 100 kVp.

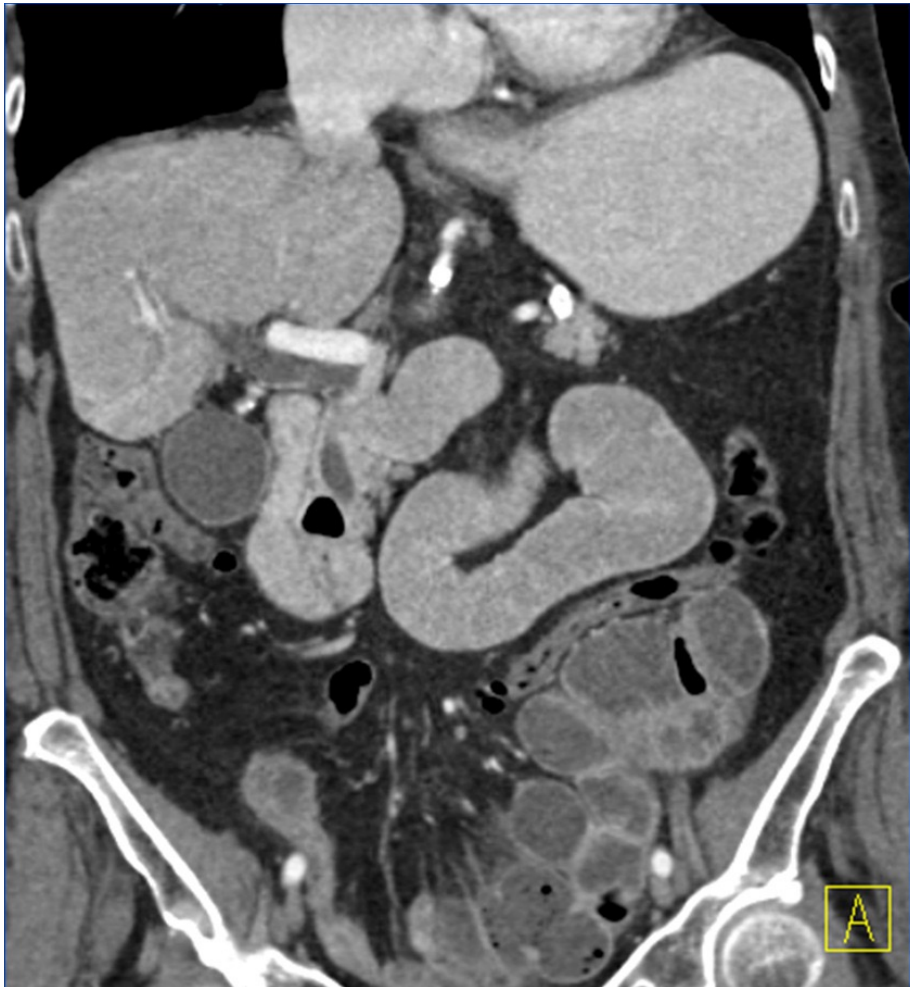
Peter B. Noël et al., PLOS ONE 8 (2013) e81141

“Real World” results

Filtered backprojection 100% dose

Iterative 40% dose

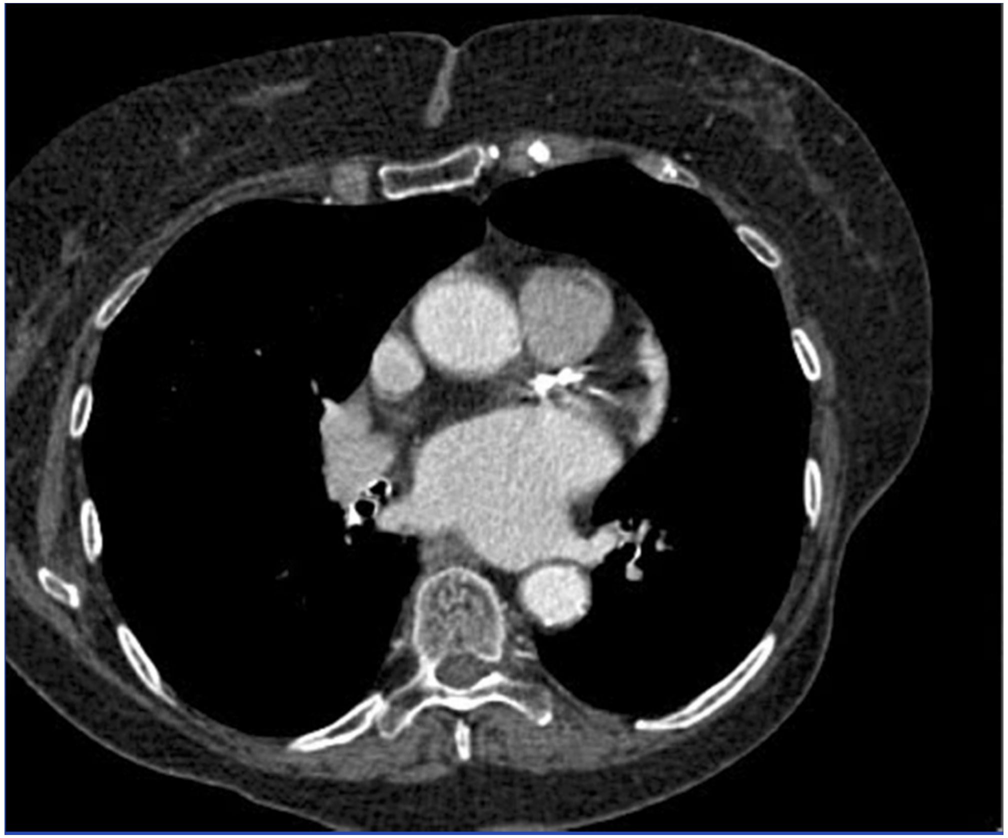




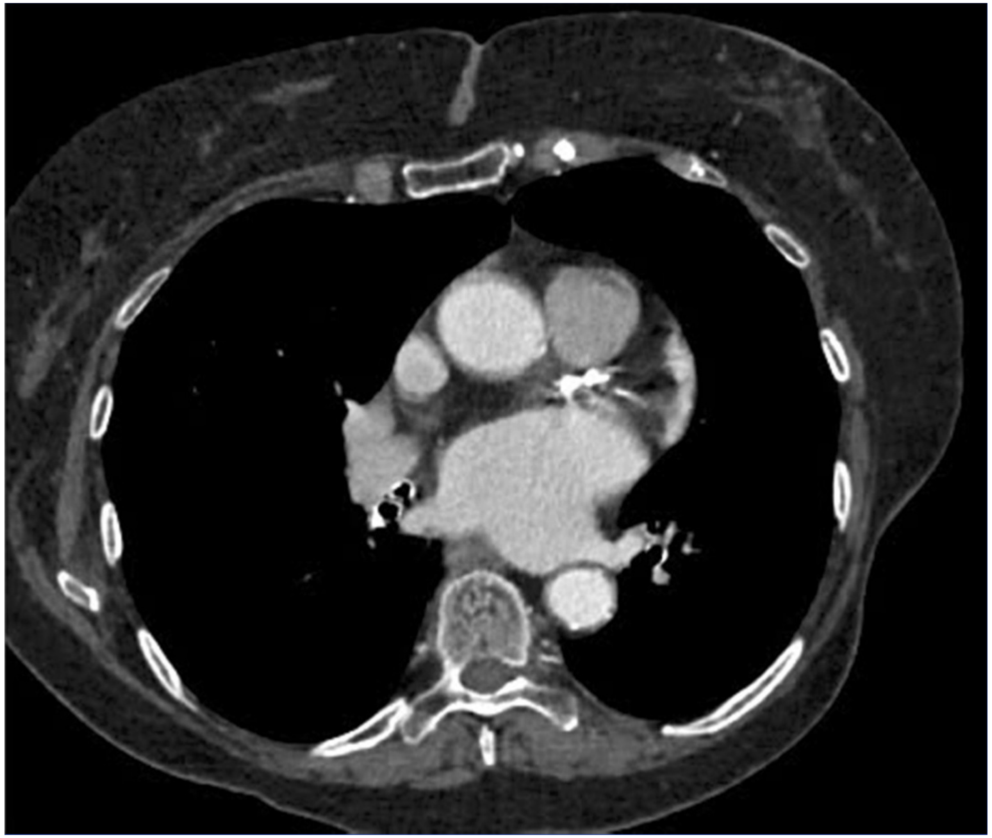
source: Kachelries, http://www.dkfz.de/en/medphysrad/workinggroups/ct/ct_conference_contributions/BasicsOfCTImageReconstruction_Part2.pdf

“Real World” results

Filtered backprojection 100% dose



Iterative 40% dose



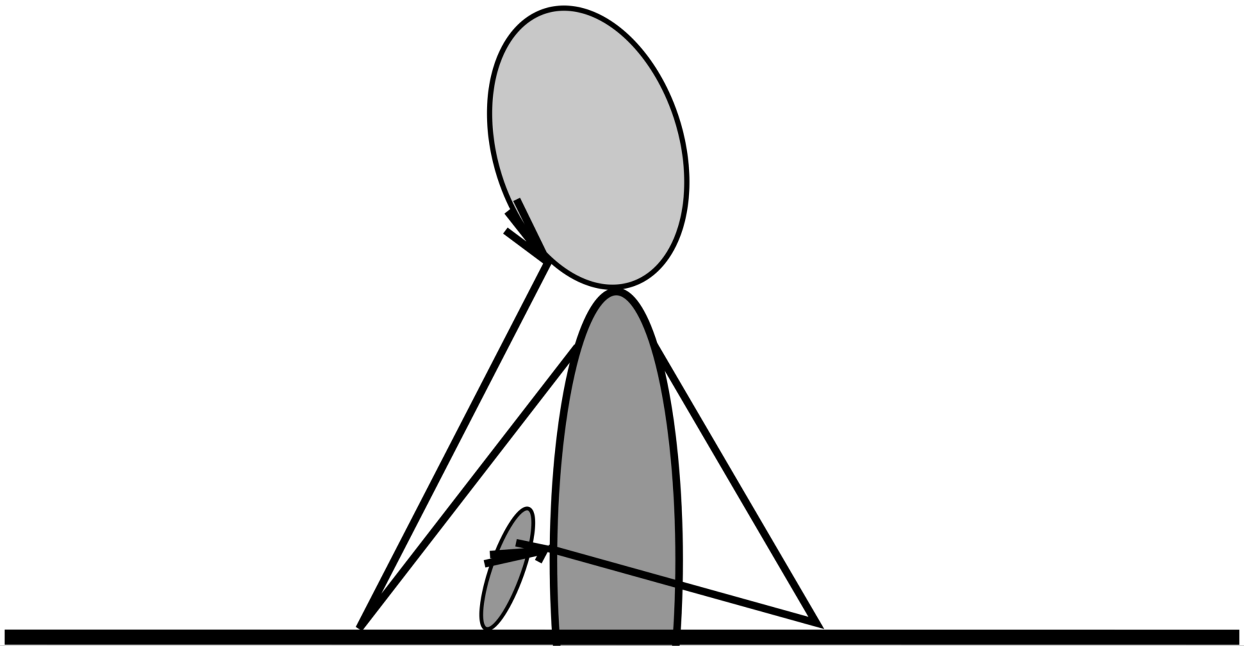
source: Kachelries, http://www.dkfz.de/en/medphysrad/workinggroups/ct/ct_conference_contributions/BasicsOfCTImageReconstruction_Part2.pdf

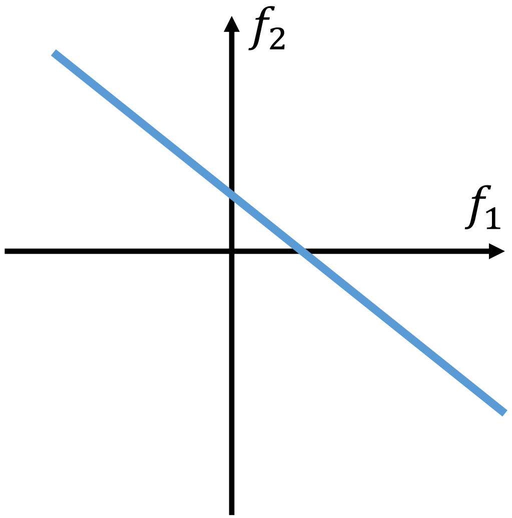
Related iterative procedures

ART	Algebraic reconstruction technique	see discussion above; Kaczmarz' method
SART	Simultaneous ART	updates all pixels of one projection in every iteration
SIRT	Simultaneous iterative reconstruction technique	updates all pixels of all projections in every iteration
DART	Discrete Algebraic Reconstruction Technique	limit the gray values to a few e.g. 2 for black and white
MBIR	Model-based iterative reconstruction	model the physics from source, spectrum, attenuation, scattering, statistics, detector
ML-EM	Maximum likelihood expectation-maximization	two steps: E-step, expectation of the log-likelihood, M-step, next estimate through maximizing the expected log-likelihood
MART	Multiplicative algebraic reconstruction technique	multiplies the update term (whereas ART adds it)
OS-SIRT, OS-EM, OSC,…		

M. Beister, D. Kolditz, W.A. Kalender, Physica Medica (2012) 28, 94-108, doi:10.1016/j.ejmp.2012.01.003

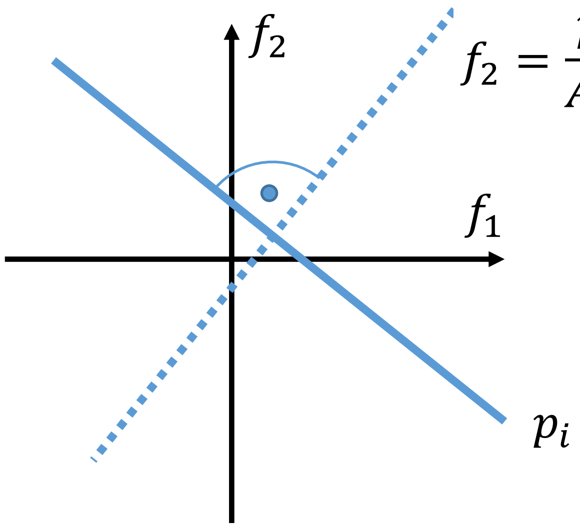
Appendix





$$p_i = A_{i1} f_1 + A_{i2} f_2 \quad \rightarrow \quad f_2 = \frac{p_i - A_{i1} f_1}{A_{i2}} = \frac{p_i}{A_{i2}} - \frac{A_{i1}}{A_{i2}} f_1$$

slope of straight line

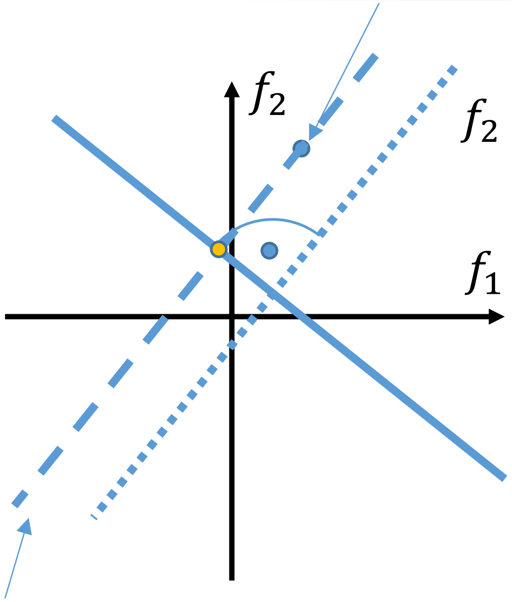


$$f_2 = \frac{p_i}{A_{i2}} + \frac{A_{i2}}{A_{i1}} f_1$$

one of the lines perpendicular to the previous one

notice, that the slope is inverted
and has an opposite sign

(f_1^{n-1}, f_2^{n-1}) guess (n-1)



$$f_2 = \frac{p_i}{A_{i2}} + \frac{A_{i2}}{A_{i1}} f_1$$

$$p_i = A_{i1} f_1 + A_{i2} f_2$$

$$f_2^{n-1} = \frac{p_i}{A_{i2}} + \frac{A_{i2}}{A_{i1}} f_1^{n-1}$$

line through guess (n-1)

1) $$f_2^{n-1} = \frac{p_i}{A_{i2}} + \frac{A_{i2}}{A_{i1}} f_1^{n-1}$$

2) $$f_2 = \frac{p_i}{A_{i2}} + \frac{A_{i2}}{A_{i1}} f_1$$

replace p_i

3) $$f_2 = \frac{A_{i2} f_2^{n-1} - \frac{A_{i2}^2}{A_{i1}} f_1^{n-1}}{A_{i2}} + \frac{A_{i2}}{A_{i1}} f_1$$

$$f_2 = f_2^{n-1} - \frac{A_{i2}}{A_{i1}} f_1^{n-1} + \frac{A_{i2}}{A_{i1}} f_1$$

$$f_2 = f_2^{n-1} - \frac{A_{i2}}{A_{i1}} f_1^{n-1} + \frac{A_{i2}}{A_{i1}} \frac{(p_i - A_{i2} f_2)}{A_{i1}}$$

we look for the point ● on p_i
as a starting point for the next iteration
and replace f_1

$$f_2 = f_2^{n-1} - \frac{A_{i2}}{A_{i1}} f_1^{n-1} + \frac{A_{i2}}{A_{i1}} \ \frac{(p_i - A_{i2} f_2)}{A_{i1}}$$

solve for f_2

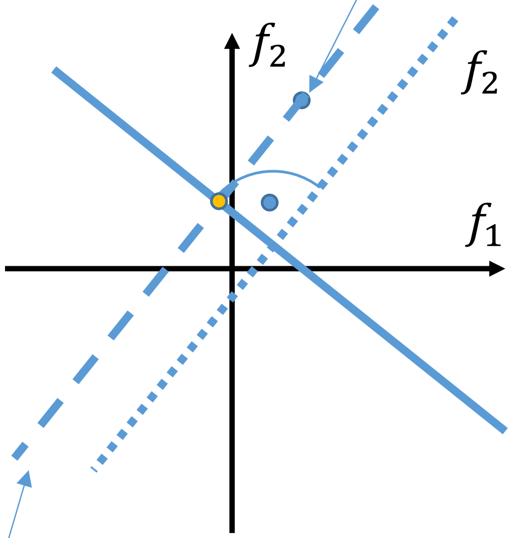
$$f_2 = f_2^{n-1} - \frac{A_{i2}}{A_{i1}} f_1^{n-1} + \frac{A_{i2}}{A_{i1}^2} p_i \ - \ \frac{A_{i2}^2}{A_{i1}^2} f_2$$

$$f_2 \left(1 + \frac{A_{i2}^2}{A_{i1}^2}\right) = f_2^{n-1} - \frac{A_{i2}}{A_{i1}} f_1^{n-1} + \frac{A_{i2}}{A_{i1}^2} p_i$$

$$f_2 \left(A_{i1}^2 + A_{i2}^2\right) = A_{i1}^2 f_2^{n-1} - A_{i1} A_{i2} f_1^{n-1} + A_{i2} p_i \qquad \rightarrow \qquad f_2 = \frac{\left(A_{i1}^2 f_2^{n-1} - A_{i1} A_{i2} f_1^{n-1} + A_{i2} p_i\right)}{\left(A_{i1}^2 + A_{i2}^2\right)} \quad (*)$$

now the same for f_1

(f_1^{n-1}, f_2^{n-1}) guess (n-1)



$$f_2 = \frac{p_i}{A_{i2}} + \frac{A_{i2}}{A_{i1}} f_1$$

$$p_i = A_{i1} f_1 + A_{i2} f_2$$

$$f_2^{n-1} = \frac{p_i}{A_{i2}} + \frac{A_{i2}}{A_{i1}} f_1^{n-1}$$

line through guess (n-1)

1) $$f_2^{n-1} = \frac{p_i}{A_{i2}} + \frac{A_{i2}}{A_{i1}} f_1^{n-1}$$

2) $$f_1 = \frac{A_{i1}}{A_{i2}} f_2 - \frac{A_{i1}}{A_{i2}^2} p_i$$

replace p_i

3) $$f_1 = \frac{A_{i1}}{A_{i2}} f_2 - \frac{A_{i1} f_2^{n-1} - A_{i2} f_1^{n-1}}{A_{i2}}$$

$$f_1 = \frac{A_{i1}}{A_{i2}} f_2 + f_1^{n-1} - \frac{A_{i1}}{A_{i2}} f_2^{n-1}$$

we look for the point on p_i and replace f_2

$$f_1 = \frac{A_{i1}}{A_{i2}} \frac{(p_i - A_{i1} f_1)}{A_{i2}} + f_1^{n-1} - \frac{A_{i1}}{A_{i2}} f_2^{n-1}$$

$$f_1 = \frac{A_{i1}}{A_{i2}} \frac{(p_i - A_{i1} f_1)}{A_{i2}} + f_1^{n-1} - \frac{A_{i1}}{A_{i2}} f_2^{n-1}$$

$$f_1 = \frac{A_{i1}}{A_{i2}} \left(\frac{p_i}{A_{i2}} - \frac{A_{i1}}{A_{i2}} f_1 \right) + f_1^{n-1} - \frac{A_{i1}}{A_{i2}} f_2^{n-1}$$ solve for f_1

$$f_1 = \frac{A_{i1}}{A_{i2}^2} p_i - \frac{A_{i1}^2}{A_{i2}^2} f_1 + f_1^{n-1} - \frac{A_{i1}}{A_{i2}} f_2^{n-1}$$

$$f_1 \left(1 + \frac{A_{i1}^2}{A_{i2}^2} \right) = \frac{A_{i1}}{A_{i2}^2} p_i + f_1^{n-1} - \frac{A_{i1}}{A_{i2}} f_2^{n-1}$$

$$f_1 \left(A_{i2}^2 + A_{i1}^2 \right) = A_{i1} p_i + A_{i2}^2 f_1^{n-1} - A_{i1} A_{i2} f_2^{n-1} \quad \rightarrow \quad f_1 = \frac{\left(A_{i2}^2 f_1^{n-1} - A_{i1} A_{i2} f_2^{n-1} + A_{i1} p_i \right)}{\left(A_{i2}^2 + A_{i1}^2 \right)} \quad (**)$$

(**) and (*) give the coordinates f_1, f_2 on projection p_i where the next step of iteration starts

these coordiantes may be rearranged and put into a vector to summarize

$$f_1^n = \frac{\left(A_{i2}^2 f_1^{n-1} - A_{i1}A_{i2}f_2^{n-1} + A_{i1}p_i\right)}{\left(A_{i2}^2 + A_{i1}^2\right)} \qquad f_2^n = \frac{\left(A_{i1}^2 f_2^{n-1} - A_{i1}A_{i2}f_1^{n-1} + A_{i2}p_i\right)}{\left(A_{i1}^2 + A_{i2}^2\right)}$$

$$\begin{pmatrix} f_1^n \\ f_2^n \end{pmatrix} = \frac{1}{A_{i1}^2 + A_{i2}^2} \cdot \begin{pmatrix} A_{i2}^2 f_1^{n-1} - A_{i1}A_{i2}f_2^{n-1} + A_{i1}p_i \\ A_{i1}^2 f_2^{n-1} - A_{i1}A_{i2}f_1^{n-1} + A_{i2}p_i \end{pmatrix}$$

add a zero vector $\begin{pmatrix} A_{i1}^2 f_1^{n-1} - A_{i1}^2 f_1^{n-1} \\ A_{i2}^2 f_2^{n-1} - A_{i2}^2 f_2^{n-1} \end{pmatrix}$ to be able to extract the vector $\begin{pmatrix} f_1^{n-1} \\ f_2^{n-1} \end{pmatrix}$

$$\begin{pmatrix} f_1^n \\ f_2^n \end{pmatrix} = \begin{pmatrix} f_1^{n-1} \\ f_2^{n-1} \end{pmatrix} + \frac{1}{A_{i1}^2 + A_{i2}^2} \cdot \begin{pmatrix} -A_{i2}^2 f_1^{n-1} - A_{i1}A_{i2}f_2^{n-1} + A_{i1}p_i \\ -A_{i1}^2 f_2^{n-1} - A_{i1}A_{i2}f_1^{n-1} + A_{i2}p_i \end{pmatrix}$$

$$= \begin{pmatrix} f_1^{n-1} \\ f_2^{n-1} \end{pmatrix} + \frac{1}{\begin{pmatrix} A_{i1} \\ A_{i2} \end{pmatrix} \cdot \begin{pmatrix} A_{i1} \\ A_{i2} \end{pmatrix}^T} \cdot \left(p_i \cdot \begin{pmatrix} A_{i1} \\ A_{i2} \end{pmatrix} - \begin{pmatrix} A_{i2}^2 f_1^{n-1} + A_{i1}A_{i2}f_2^{n-1} \\ A_{i1}^2 f_2^{n-1} + A_{i1}A_{i2}f_1^{n-1} \end{pmatrix} \right)$$

$$= \begin{pmatrix} f_1^{n-1} \\ f_2^{n-1} \end{pmatrix} + \frac{1}{(A_{i1}, A_{i2}) \cdot \begin{pmatrix} A_{i1} \\ A_{i2} \end{pmatrix}} \cdot \left(p_i \cdot \begin{pmatrix} A_{i1} \\ A_{i2} \end{pmatrix} - \left[(A_{i1}, A_{i2}) \cdot \begin{pmatrix} f_1^{n-1} \\ f_2^{n-1} \end{pmatrix} \right] \cdot \begin{pmatrix} A_{i1} \\ A_{i2} \end{pmatrix} \right)$$

$$= \begin{pmatrix} f_1^{n-1} \\ f_2^{n-1} \end{pmatrix} + \frac{\left(p_i - (A_{i1}, A_{i2}) \cdot \begin{pmatrix} f_1^{n-1} \\ f_2^{n-1} \end{pmatrix} \right)}{(A_{i1}, A_{i2}) \cdot \begin{pmatrix} A_{i1} \\ A_{i2} \end{pmatrix}} \cdot \begin{pmatrix} A_{i1} \\ A_{i2} \end{pmatrix} = \mathbf{f}^{n-1} + \frac{(p_i - \boldsymbol{A}_i \cdot \mathbf{f}^{n-1})}{\boldsymbol{A}_i \cdot \boldsymbol{A}_i^T} \cdot \boldsymbol{A}_i^T$$