Entanglement detection for spin 1/2 systems

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Abstract

To detect entangled states in many body systems is a fundamental issue in many fields of quantum mechanics such as quantum comunication or quantum cryptography, as well as being helpful to understand the quantum-classical limit. One way of detecting them are the so-called entanglement/separability criteria.

Physical systems usually consist of cold atoms, trapped ions or photons, so addressing their properties individually is of high experimental difficulty. That is why collictive observables (sum of local observables) are more useful to work with. In [1] a family of entanglement criteria expressed via spin squeezing inequalities was presented using first and second moments of collective angular momenta. In this project the states will be characterized in this sense too. We will generate many density matrices and apply the well-known *partial transposition criterion*, using an SDP program for different number of qubits and applying it to all possible partitions. Finally, we will compare our results with the ones found on the mentioned article [1] and characterize the polytope of the PPT states for the case of 2 and 3 qubits.

Our results are a first step towards the total characterization of the fully separable states, since the final goal would be to add many more positive but not completely positive maps to the existing code. We expect that then more entangled states that were missed by the PPT criterion would be detected and even the entire set of fully separable states could be characterized. Another application would be to detect bound entangled states (entangled states from which no entanglement can be distilled) in low dimension by doing the intersection of the PPT polytope with the polytope found in the article [1]. Bound entangled states can be more useful for quantum metrology than separable states, so it is essential to have a way to detect them.

1 Context and motivation

Entanglement is a fundamental aspect of quantum mechanics since many useful propoerties such as quantum correlations or non-locality are directly related to it. That is why having methods to detect it is of high interest in the field of quantum information both experimental and theoretical. Many quantum states, even when their amount of entanglement is really low, can be processed using local operations and classical comunication to condense their collective entanglement content in one strongly entangled state in a process called distillation. However, some states are not distillable: the so-called bound entangled states. Their use could be an advantage in some quantum processing application, as well as in quantum metrology.

The most famous separability criteria is the *partial transposition*. This criterion tells us that after applying the partial transposition to our state (that is, the transposition to only one partition), if the resulting matrix is no longer positive, and therefore not physical, then that partition of the original state is for sure entangled with the rest of it. An entangled state that is positive under partial transpose was shown to be bound entangled [2]. It can be summarized in:

$$\rho \ \text{separable} \Longrightarrow (\mathbb{1} \otimes T)\rho \geq 0 \tag{1}$$

$$(\mathbb{1} \otimes T)\rho \ngeq 0 \Longrightarrow \rho \ \text{entangled} \tag{2}$$

where in the first row we have used that if $\rho \geq 0$ then $\rho^T \geq 0$, being T the transposition operator. If our state is separable then it can be written as $\rho = \sum_k p_k \rho^1 \otimes \rho^2 \otimes ... \otimes \rho^N$, so when we apply the partial transposition to one of the partitions we obtain a positive state in that partition and consequently a positive state as a final state.

2 Project description

The main objective of the project was to apply the PPT criterion (Positive Partial Transpose) to many states using Semi-Definite Programming for low dimensional cases.

For the case of 2 and 3 qubits, characterize the polytope of the PPT states and compare it with the polytope derived in [1].

3 Role of the summer student

First I had to familiarize with the collective measurements and the concepts of separability criteria and entanglement witnesses, specially the ones using positive but not completely positive maps. I retrieved the results of the most relevant paper for the project [1]. The inequalities that are stated there define a polytope in the space of second order momenta such that all the states outside it are entangled, i. e., all separable states lie inside. I also had to learn some basics of matlab and the theory behind the Semi-Definite Programming.

After that, I coded the PPT criteria using an SDP program first for two qubits but then I generalized it to any number of qubits. However, due to computational limitations it was only possible to run the program in a realistical amount of time until 7 qubits. For 2 and 3 qubits the analysis was deeper, doing a characterization of the polytope.

Finally, I started to add another criteria to the code, a positive but not completely positive map called the Breuer map.

4 Discussion of the results

4.1 Retriving previous results

As it has been already said, the project is based on the results of the famous paper by Tóth, Knapp, Gühne and Briegel [1] where they derive a complete set of generalized spin squeezing inequalities based on first and second collective angular momenta, that is, $\langle J_i \rangle$ and $\langle J_i^2 \rangle$ respectively, with $i = x, y, z$ and where $J_i := \frac{1}{2}\sum_{k=1}^{N} \sigma_i^{(k)}$, being $\sigma_i^{(k)}$ the Pauli matrix in the i-direction acting on the k-particle. These inequalities are a separability criteria that can be used for the experimental detection of entanglement in a system of an arbitrary number of spin 1/2 particles which cannot be addressed individually. The inequalities are the following:

$$\langle J_x^2 \rangle + \langle J_y^2 \rangle + \langle J_z^2 \rangle \leq \frac{N(N+2)}{4} \tag{3}$$

$$(\Delta J_x)^2 + (\Delta J_x)^2 + (\Delta J_x)^2 \geq \frac{N}{2} \tag{4}$$

$$\langle J_i^2 \rangle + \langle J_j^2 \rangle - \frac{N}{2} \leq (N-1)(\Delta J_k)^2 \tag{5}$$

$$(N-1)\left[(\Delta J_i)^2 + (\Delta J_j)^2\right] \geq \langle J_k^2 \rangle + \frac{N(N-2)}{4} \tag{6}$$

where i, j, k take all the possible permutations of x, y, z and N is the number of qubits of the system.

This set of inequalities define a polytope in the space of the second angular momenta[1] once the first order momenta are fixed to some value. Then, every density matrix lying outside it, will be entangled.

In the paper they demonstrate that when N is very large, then the volume of the polytope converges to the volume of separable states, meaning that in this limit all the states inside the polytope will be separable. First we will focus in the lowest dimension: 2 qubits.

[1] From now on, every time we say first or second angular momenta we will be referring to the expectation value of it.

4.2 Code PPT criteria for 2 qubits

We coded using matlab an SDP program to implement the PPT criteria when we have just 2 qubits.

In this simple case we already knew which were the vertices of the PPT polytope, so we were able to compare our results with the analytic solution by computing the rate of the respective volumes, as it can be seen in the Fig. 1. In there we can see that the more steps in the code we take, the closer is the numerical volume to the analytical one, showing that the code is running correctly.

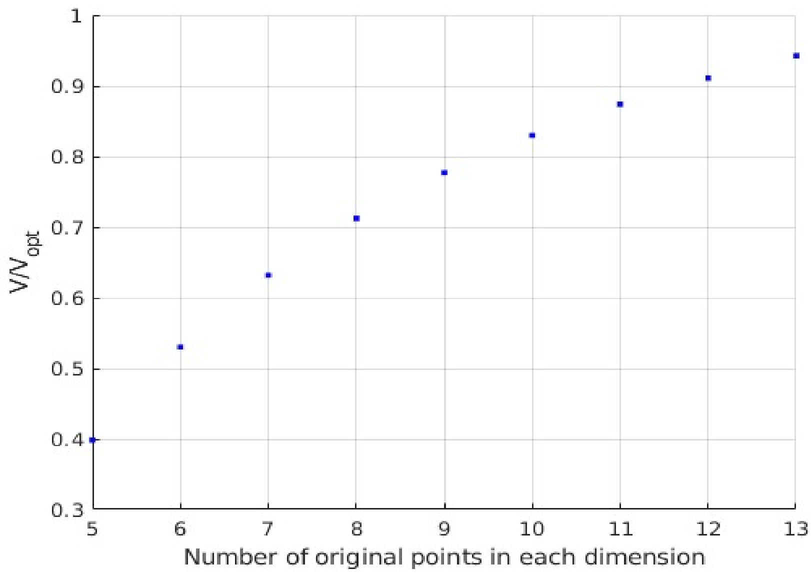


Figure 1: Rate between the volume of the numerical polytope of the PPT states for 2 qubits and the volume of the analytical one with respect to the number of points taken into account in the SDP program in each dimension.

At the same time, in this case the polytope of the paper coincides with the PPT polytope so both criteria are equivalent in dimension 4. In addition, by the Horodecki Theorem [3] we know that if ρ is a state in a 2x2 or 2x3 system, then $\rho^{T_A} \geq 0$ implies that ρ is separable. In other dimensions this is not the case. Therefore, for 2 qubits the polytope of the article is tight to the set of separable states.

4.3 Code PPT criteria in general

The next step was to generalize the code to any number of qubits, even though due to computational limitations we only were able to run it until 7 qubits. In this range we saw several things. First, that in some regions the PPT polytope surpassed the one from the paper. This means that the points that have a positive partial transpose but are outside the polytope of the paper are bound entangled states, since the criteria of the paper ensures that they are entangled. Secondly, we saw that when the number of qubits was odd, there were NPT states (states with a negative partial transpose), i. e., entangled states, inside the polytope of the paper. Looking at it with more detail we also saw that many of them were not even physical. Then we focused on characterizing the set of physical states that was not taken into account in the mentioned paper.

In all the following cases we fix the first order momenta to 0 in each component for simplicity.

4.3.1 Physical region

The physical region turned out to be a cone, which is convex as expected, meaning that all the points outside that cone are not physical states (it does not exist a valid density matrix satisfying the imposed expectation values for the second order momenta).

The equation in the space of second order momenta turned out to be the following:

$$\left(\langle J_x^2\rangle - 0.25\right)^2 + \left(\langle J_y^2\rangle - 0.25\right)^2 + \left(\langle J_z^2\rangle - 0.25\right)^2 =$$
$$= 2\cdot\left[\left(\langle J_x^2\rangle - 0.25\right)\left(\langle J_y^2\rangle - 0.25\right) + \left(\langle J_x^2\rangle - 0.25\right)\left(\langle J_z^2\rangle - 0.25\right) + \left(\langle J_z^2\rangle - 0.25\right)\left(\langle J_y^2\rangle - 0.25\right)\right] \tag{7}$$

which corresponds to a cone with a base radius of $R = 1.225$, a center of the basis at the point $C = (1.25, 1.25, 1.25)$, a vertix at point $P = (0.25, 0.25, 0.25)$ and therefore height of $h = \sqrt{3}$ and angle with the basis of $\theta \approx 0.6155$ rad. For a better visualization see Fig. 2.

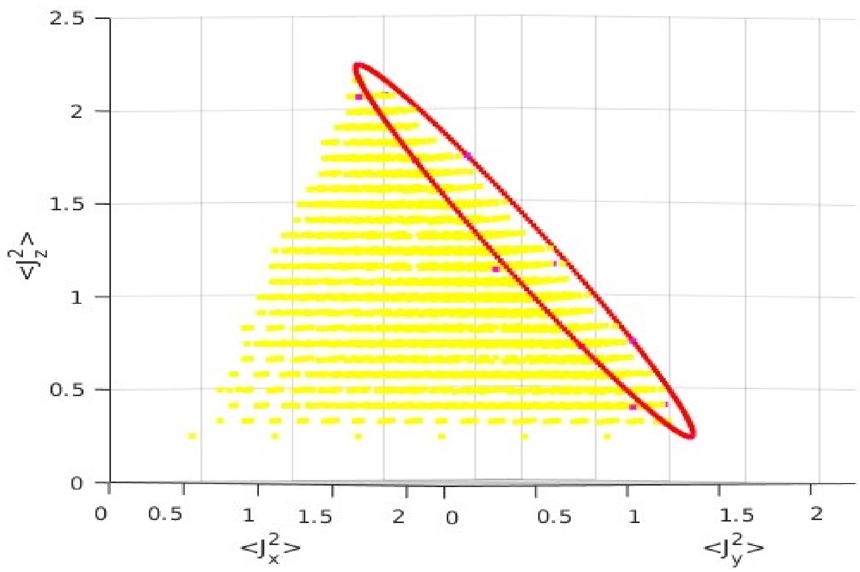


Figure 2: Physical states in the space of second order angular momenta. The yellow points are those values of $\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle$ with $\langle J_i\rangle = 0$ for $i = x, y, z$ for which a valid density matrix that satisfies those constraints exists. The magenta points are just some points that were used to find a good fit for the base of the cone, which is the circle in red in the figure.

After checking the volume of the cone with the dimensions above inferred from the data and the volume given by alphaShape in matlab (which creates the minimum figure that envelops all the points) we obtained a relative error of 0.036, so we attribute the difference to the numerical precision.

We saw that for low dimension and when the number of qubits is odd three of the vertices of the polytope derived in the article by Toht were not physical states. As a consequence, some regions of the polytope are outside the cone of the pysical region, so it is considering states that does not exist, as it can be seen in figure 3. Nonetheless, the PPT polytope is, by definition, always inside the cone of physical states.

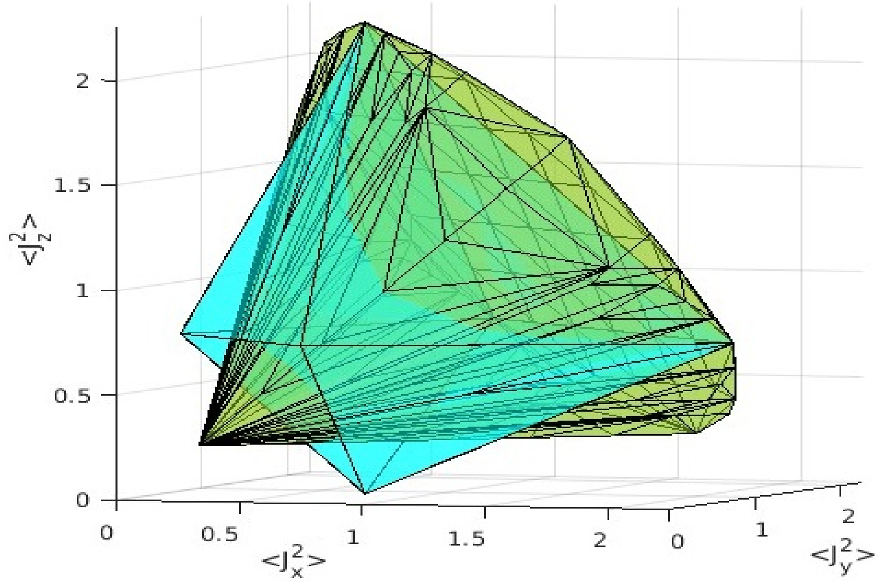


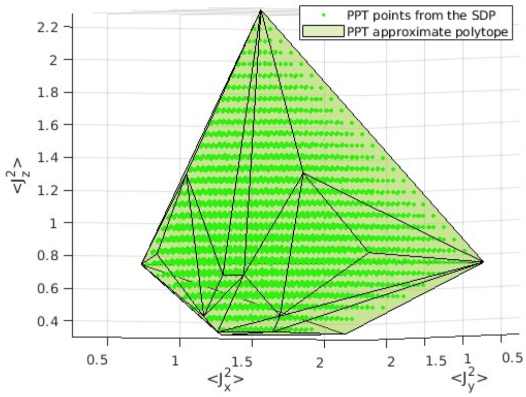
Figure 3: Cone of physical states in the space of second order angular momenta and algebraic polytope found in the paper by Tóth [1]. The green region envelops those values of $\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle$ with $\langle J_i\rangle = 0$ for $i = x, y, z$ for which a valid density matrix that satisfies these constraints exists. The blue polytope is the one defined by the spin squeezing inequalities when the number of qubits is 3 and $\langle J_i\rangle = 0$ for $i = x, y, z$.

4.4 Code PPT criteria for 3 qubits

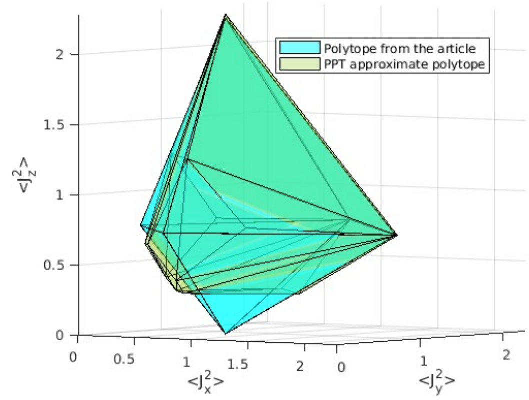
With this knowledge in mind, we studied deeper the case with three qubits. We just passed through the program the states that were physical and saw that even then some NPT states (therefore entangled) lay inside the polytope of the article. This makes us think that adding more maps would get the numerical polytope closer and closer to the set of fully separable states.

We found and approximate polytope of the PPT states, even though it was not tight: there are not PPT states outside it but there are a few NPT states in some regions.

We tried to get a better approximation with matlab by finding the equation of each facet and making the intersection of the corresponding planes in order to get the vertices. To find a more accurate equation of each facet we modified a little the normal vector of the plan and ran the SDP program over that new direction. Finally we chose the direction that was detecting more states. The results were the polytope that can be seen in figures 4a and 4b.



(a) In green points, the states that satisfy the PPT criteria in the space of $\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle$ with $\langle J_i\rangle = 0$ for $i = x, y, z$. In the green region it is plotted the approximate polytope of the PPT points, defined by the vertices above.



(b) The green region is the approximate polytope of the PPT states, defined by the vertices above. The blue region is the algebraic polytope given the spin squeezing inequalities of the article [1]. Both of them are defined in the space of $\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle$ with $\langle J_i\rangle = 0$ for $i = x, y, z$.

4.5 Other criteria

4.5.1 Permutation criteria

4.5.2 Breuer map

The Breuer map is a positive but not completely positive map defined in a space of even dimension $d \geq 4$ as follows:

$$\Lambda_{Breuer}[A] = \mathbb{1}\,\mathrm{Tr}\,A - A - \upsilon[A] \tag{8}$$

where $\upsilon[A] = UA^TU^\dagger$, U is a unitary matrix that can be constructed as $U = RDR^T$, where it is defined $D = \sum_{k=0}^{d/2-1} e^{i\Phi_k}(|2k\rangle\langle 2k+1| - |2k+1\rangle\langle 2k|)$ for any angles Φ_k and R is an arbitrary orthogonal matrix. The criteria using this map is the same as for any positive but not completely positive map, i. e.,

$$\rho \ \text{separable} \Longrightarrow (\mathbb{1}\otimes\Lambda_{Breuer})\rho \geq 0 \tag{9}$$

$$(\mathbb{1}\otimes\Lambda_{Breuer})\rho \ngeq 0 \Longrightarrow \rho \ \text{entangled} \tag{10}$$

We coded it into an SDP program fixing some arbitrary angles in order to see which entangled states are detected by this map. Afterwards, we added both constraints so that the states not detected by the program were PPT and also positive under the Breuer map.

We are still studying this map by modifying the parameters R and Φ_k, so we have not any conclusions yet. However, we have the intuition that the set of undetected states might got smaller. If it was really that case, we would know that the Breuer map can discover PPT entangled states. This we would have a clear sign that the idea to use all PnCP maps is a right way to proceed, and that branching our sdp to an existing algorithm that generates many PnCP maps in not so many amount of time would likely be a very good approximation of the set of separable states. On the contrary, we would have to keep searching for another PnCP map that does the job.

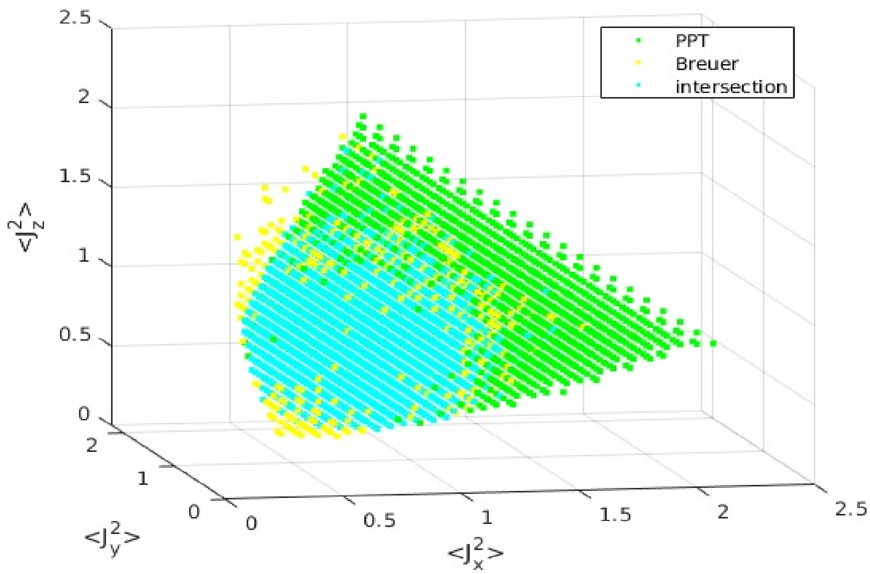


Figure 5: The green points are the PPT states that are not positive under the Breuer map, the yellow points are the states positive under the Breuer map that are not PPT and the blue points are the points that are PPT and positive under the Breuer map. They are represented in the space of $\langle J_x^2\rangle, \langle J_y^2\rangle, \langle J_z^2\rangle$ with $\langle J_i\rangle = 0$ for $i = x, y, z$. The angles fixed in the Breuer map are both 0.

5 Implications of the work

This work is a first step whose applications are twofold:

First, it is a visual method easy to implement to detect bound entangled states, by computing the intersection of the PPT polytope and the one from the article. The states that are PPT but outside the second polytope are bound entangled states.We did it for 3 qubits but the same code could be used for any number of qubits (up to computational time limitations).

Secondly, we could use the existing algorithm that generates positive but not completely positive maps and then select the ones that are useful (in the sense that they detect new entangled states that were not detected by the previous maps) in order to characterize the set of fully separable states. We expect that the volume of the undetected set gets lower and that, after adding many more maps, we end just with separable states. The idea would be to check the action of the selected maps in the limit of large number of qubits, when we know that the polytope of the article is tight to the set of separable states. If both polytopes coincide, we would know that those maps are enough to characterize the set of fully separable states. This could be useful to extend the criteria to higher dimensions, where the spin squeezing inequalities are not valid anymore (there is a modification of them but that does not depend directly on the second momenta and consequently it can not be used by experimentalists) [4].

References

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