Exercises 5

Exercise 1 (Maximum likelihood estimation).

1. Let $x \in \mathbb{R}$, and $y_i = x + w_i$ where $w_i \sim \mathcal{N}(0,1)$. Give the Maximum Likelihood Estimator of x, *i.e.*

$$\hat{x} = \operatorname{argmax}_x p(y|x)$$

2. Same question assuming now a multiplicative Gaussian noise, *i.e.* $y_i \sim x w_i$ with $w_i \sim \mathcal{N}(0,1)$.

Exercise 2. Let $A \in \mathbb{R}^{m\times n}$, $y \in \mathbb{R}^m$, $\eta > 0$ and let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^m$. Show that the solution of the optimization problem

$$x_* = \operatorname{argmin} \|z\|_1 \quad \text{s.t.} \quad \|Az - y\| \leqslant \eta$$

is m-sparse in the case of the uniqueness of the solution. *Hint:* show that the system of columns $\{a_j \ ; \ j \in \operatorname{Supp} x_*\}$ is linearly independent.

Exercise 3 (Null Space Property).

1. Prove the uniform recovery theorem: every k-sparse x_0 is the unique solution of

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax = Ax_0. \tag{BP}$$

if and only if A satisfies the Null Space Property of order k, *i.e.* :

$$\forall I : |I| \leqslant k, \quad \forall h \in \operatorname{Ker} A \setminus 0, \quad \|h_I\|_1 < \|h_{I^c}\|_1$$

2. Show that if NSP(k) holds, then the solution of (BP) is also a solution of

$$\min \|x\|_0 \quad \text{s.t.} \quad Ax = Ax_0$$

3. Let $x_0 \in \mathbb{R}^n$ (not necessarily k-sparse), let $y = Ax_0 + w$, $\|w\| \leqslant \varepsilon$, and let x be a solution of

$$\min \|x\|_1 \quad \text{s.t.} \quad \|Ax - y\| \leqslant \varepsilon \tag{BP-ε}$$

The goal of this question is to prove the uniform robust recovery theorem: if A obeys the robust NSP of order k, *i.e.*

$$\exists 0 < \rho < 1, \quad \exists \tau > 0, \quad \forall I : |I| = k, \quad \forall h \in \operatorname{Ker} A \setminus 0, \quad \|h_I\|_1 \leqslant \rho \|h_{I^c}\|_1 + \tau \|Ah\|_2$$

then for all $x_0 \in \mathbb{R}^n$, any solution of (BP-ε) satisfies

$$\|x - x_0\|_1 \leqslant 2\frac{1+\rho}{1-\rho}\sigma_k(x_0)_1 + 4\frac{\tau}{1-\rho}\varepsilon$$

where $\sigma_k(x_0)_1 := \inf\{\|x_0 - z\|_1 \ ; \ \|z\|_0 \leqslant k\}$ is the best k-sparse approximation with respect to the ℓ^1-norm. We assume that A satisfies the robust NSP of order k.

(a) Let $h = x - x_0$. Show that, for any subset I,

$$\|x_0\|_1 + \|h_{I^c}\| \leqslant 2\|(x_0)_{I^c}\|_1 + \|h_I\|_1 + \|x\|_1$$

(b) Deduce that for a well chosen subset I,

$$\|h_{I^c}\|_1 \leqslant \frac{1}{1-\rho}(2\sigma_k(x_0)_1 + 2\tau\varepsilon)$$

(c) Conclude.