ℓ^1-regularization

Fallstudien der mathematischen Modellbildung, Teil 2

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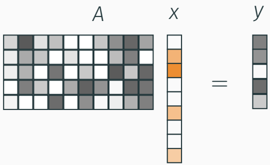
- Enforcing structure helps with ill-conditioning and under-determined systems.
  A popular prior is sparsity, *i.e.* assuming the solution has only a few non-zero entries



- **Rationale**: signals/data are often sparse in some basis / living on low-complexity domain.

- If c_i, $i = 1, \dots, n$ denotes the columns of A, the system rewrites

$$y = \sum_{i=1}^{n} x_i c_i$$

(c_i) is an over-complete basis (or dictionary), and the goal is to select a subset of this basis that is sufficient to express $y \rightarrow$ regressor selection, or variable selection.

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- The corresponding regularized problem is

$$\boxed{\min \|Ax - y\|^2 + \lambda \|x\|_0}$$

and in the noiseless case

$$\boxed{\min \|x\|_0 \quad \text{s.t.} \quad Ax = y}$$

- Remember that the penalized form is always equivalent to a constrained form with adequate parameter, *i.e.*

$$\min_{x \in \mathbb{R}^n} \|Ax - y\|^2 \quad \text{s.t.} \quad \|x\|_0 \leqslant \tau \tag{1}$$

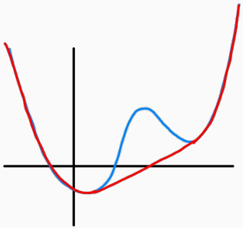
- NP-hard combinatorial, non-convex problem. Direct strategy: check every possible sparsity pattern, *i.e.* fix subsets J of non-zero entries in x and solve the least-squares

$$\min_{\tilde{x} \in \mathbb{R}^n} \|A_J \tilde{x} - y_J\|^2$$

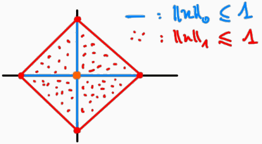
  There are $\binom{n}{k}$ possible supports for each sparsity level $\rightarrow$ infeasible for large n

- Possible approximations of the problem:
  - Greedy algorithms (*e.g.* orthogonal matching pursuit)
  - Convex relaxation

- *Definition (Convex envelope).* The convex envelope of a function $I(x)$ is the largest convex $J(x)$ such that $J(x) \leqslant I(x)$.



- ***Theorem.*** The convex envelope of $\|x\|_0$ for x restricted to $\|x\|_\infty \leqslant \alpha$ is $\|x\|_1/\alpha$



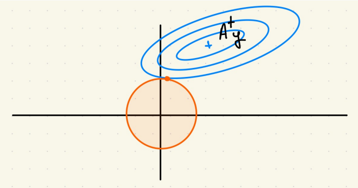
- Relax ℓ^0-penalty into ℓ^1-penalty

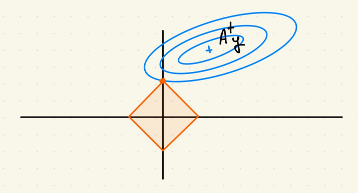
$$\min \|Ax - y\|_2^2 + \lambda \|x\|_1 \qquad \text{(LASSO)}$$

Called LASSO[1] (Least Absolute Shrinkage and Selection Operator) or basis pursuit denoising. When $\lambda = 0$, we obtain the basis pursuit[2] problem

$$\min \|x\|_1 \quad \text{s.t.} \quad Ax = y \qquad \text{(BP)}$$

- Main properties are:
  Shrinkage: like Tikhonov regularization, LASSO penalizes large coefficients
  Selection: unlike Tikhonov, LASSO produces sparse estimates





[1] *Tibshirani, 1996*

[2] *Donoho, early 1990's*

Lagrange Dual Function for Lasso

- We can reformulate the problem under a constrained form

$$\min \frac{1}{2}\|z-y\|^2 + \lambda\|x\|_1 \quad \text{s.t.} \quad z = Ax$$

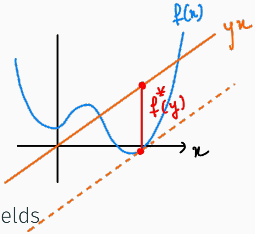
and deduce the Lagrangian:

$$\mathcal{L}(x,z,\nu) = \frac{1}{2}\|z-y\|^2 + \lambda\|x\|_1 + \nu^\top(z-Ax)$$

- Minimization over z yields $\tilde{z} = y - \nu$. Minimization over x on the other hand is less obvious, since we have lost differentiability

$$\inf_x \lambda\|x\|_1 - \langle A^\top\nu,\, x\rangle = -\left(\sup_x \langle A^\top\nu,\, x\rangle - \lambda\|x\|_1\right)$$

> ***Definition (Conjugate function).*** The convex conjugate of $f : \mathbb{R}^n \to \mathbb{R}$ is
>
> $$f^*(y) = \sup_x \langle y,\, x\rangle - f(x)$$



With $J(x) := \lambda\|x\|_1$, the minimization over x and z yields

$$\boxed{\mathcal{L}(\tilde{x},\tilde{z},\nu) = \nu^\top y - \frac{1}{2}\|\nu\|^2 - J^*(A^\top\nu)}$$

- ***Definition (Dual norm).*** Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, the associated dual norm is

$$\|y\|_* = \sup\left\{y^\top x \; ; \; \|x\| \leqslant 1\right\}$$

Example. $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are dual to each other.

- ***Proposition.*** The conjugate function of $\|x\|$ is

$$f^*(y) = \begin{cases} 0 & \text{if} \quad \|y\|_* \leqslant 1 \\ \infty & \text{otherwise} \end{cases}$$

Proof.[1] If $\|y\|_* > 1$, then by definition there exists $w \in \mathbb{R}^n$ such that $\|w\| \leqslant 1$ and $y^\top w > 1$. Taking $x = tw$ and letting $t \to \infty$ we obtain

$$y^\top x - \|x\| = t(y^\top w - \|w\|) \to \infty,$$

hence $f^*(y) = \infty$. If $\|y\|_* \leqslant 1$, since $y^\top x \leqslant \|x\|\|y\|_*$ for all x, then $y^\top x - \|x\| \leqslant 0$, and $x = 0$ is the maximizer.

[1] *Boyd, Vandenberghe, Convex Optimization, Example 3.26*

- If $J(x) = \lambda\|x\|_1$, then $J^*(y)$ is the indicator of $\{\|y\|_\infty \leqslant \lambda\}$.
- Altogether, we obtain

$$\mathcal{L}(\tilde{x}, \tilde{z}, \nu) = \nu^\top y - \frac{1}{2}\|\nu\|^2 - i_{\{\nu : \|\nu\|_\infty \leqslant \lambda\}}(A^\top \nu)$$

where we denote i_C the indicator function of the set C. Hence the LASSO dual problem reads

$$\boxed{\max \nu^\top y - \frac{1}{2}\|\nu\|^2 \quad \text{s.t.} \quad \|A^\top \nu\|_\infty \leqslant \lambda}$$

$\|\cdot\|_1$ is convex but not differentiable at 0. How to derive optimality conditions?

- Recall the standard inequality for convex functions

$$f(y) \geqslant f(x) + \langle \nabla f(x), y - x \rangle$$

- ***Definition (Sub-differential).*** The sub-differential of f at x is

$$\partial f(x) = \left\{ v \in \mathbb{R}^n \; ; \; \forall y \in \mathbb{R}^n, f(y) \geqslant f(x) + \langle v, y - x \rangle \right\}$$

Note that $\partial f(x)$ is convex. If f is differentiable, then $\partial f(x) = \{\nabla f(x)\}$.

- ***Proposition.*** For *any* function f,

$$x_* = \mathrm{argmin}_x f(x) \iff 0 \in \partial f(x)$$

Proof. x_* minimizer of $f \iff \forall x, f(x) \geqslant f(x_*) = f(x_*) + \langle 0, x - x^* \rangle \iff 0 \in \partial f(x)$.

Some basic rules

- $\partial f(x) = \{\nabla f(x)\}$ if f is differentiable at x
- $\partial(\alpha f) = \alpha \partial f$ if $\alpha > 0$
- $\partial(f_1 + f_2)(x) = \partial f_1(x) + \partial f_2(x)$
- if $g(x) = f(Ax + b)$ where f is convex, then $\partial g(x) = A^\top \partial f(Ax + b)$

- $|x|$ is differentiable at any $x \neq 0$ with derivative ± 1. At 0, for any $z \in \mathbb{R}$,

$$|z| \geqslant yz \iff y \in [-1, 1]$$

  so $\partial|0| = [-1, 1]$.

- Generalization:

$$v \in \partial\|x\|_1 \iff \begin{cases} v_i = \textbf{sign}(x_i) & \text{if} \quad x_i \neq 0 \\ v_i \in [-1, 1] & \text{if} \quad x_i = 0 \end{cases}$$