Exercise 2

Exercise 1.

1. Compute the singular value decomposition of (some of) the following matrices

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2. With $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, draw the set $\{x \in \mathbb{R}^2 \; ; \; \|Ax\| = 1\}$ (the pre-image of the circle). How can you determine the singular values and right singular vectors of A from this figure?

Exercise 2. Let $A \in \mathbb{R}^{m\times n}$ and $A = U\Sigma V^\top$ its singular value decomposition. We write u_i and v_i the left and right singular vectors respectively, and $\sigma_1 \geqslant \ldots \geqslant \sigma_r > 0$ the non-zero singular values.

1. Check that
$$A = \sum_{i=1}^{r} \sigma_i u_i v_i^\top$$

2. We have seen that $\|A\|_{2,2} := \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = \sigma_1$. Show that
$$\sigma_r = \inf_{x\in(\operatorname{Ker} A)^\perp\setminus\{0\}} \frac{\|Ax\|}{\|x\|}$$

3. For $k < r$, we define
$$A_k := \sum_{i=1}^{k} \sigma_i u_i v_i^\top .$$
Show that $\|A - A_k\|_{2,2} = \sigma_{k=1}$. Show that A_k actually minimizes $\|A - B\|_{2,2}$ among all $B \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank} B \leqslant k$ (this result is known as the Eckart-Young-Mirsky theorem).

Exercise 3 (Pseudo-inverse). Let $A \in \mathbb{R}^{m\times n}$ and $A^\dagger$ its Moore-Penrose pseudo-inverse.

1. Check the identities
$$A^\dagger A A^\dagger = A^\dagger,$$
$$A A^\dagger A = A,$$

2. Show that $A^\dagger A$ and $AA^\dagger$ are orthogonal projections on $(\operatorname{Ker} A)^\perp$ and $\operatorname{Ran} A$ respectively.

Exercise 4. We define the circular convolution of $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ as the vector $(a * x) \in \mathbb{R}^n$ whose entries are given by

$$\forall k \in \{1, \ldots, n\}, \quad (a * x)_k := \sum_{i=1}^{n} a_{[k-i]} x_i$$

where $[i] = i \pmod n$. For $a \in \mathbb{R}^n$, let $A : \mathbb{R}^n \mapsto \mathbb{R}^n, x \mapsto a * x$.

1. Give the matrix of A in the canonical basis.
2. For a complex matrix $M \in \mathbb{C}^{n \times n}$, the singular value decomposition is $M = U \Sigma V^*$ where * denotes the Hermitian transpose, U is unitary ($UU^* = U^*U = I$) and Σ is diagonal with (real) nonnegative entries.

   For $0 \leqslant j \leqslant n-1$, let $u_j = \left(e^{-2\imath\pi kj/n}\right)_{k=0}^{n-1}$, and $U = \left[u_0, \ldots, u_{n-1}\right] \in \mathbb{C}^{n \times n}$. Show that

   $$AU = \mathrm{Diag}(\hat{a}_j)U,$$

   where $\hat{a}$ is the discrete Fourier transform of a. Deduce the singular values of A.
3. Let $a = \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \end{bmatrix}^\top$. Compute the ration $\sigma_{\max}/\sigma_{\min}$ in that case.