Déconvolution Parcimonieuse Sans Grille: une Méthode de Faible Rang

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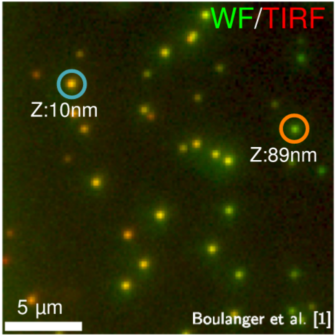
Gretsi 2017, Juan-les-Pins

Sparse Super-Resolution

Recover pointwise sources from low-resolution and noisy observations.

Astrophysics (2D)



Molecule fluorescence (3D)

Also neural spikes (1D), seismic imaging (1.5D), ...

Overview

Introduction

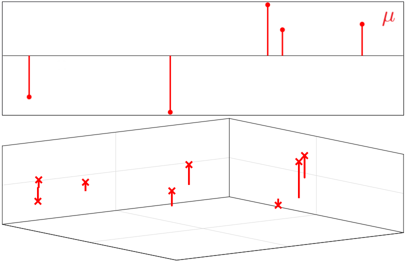
Low-Rank Semidefinite Primal

Toeplitz Relaxation

FFT-Based Conditional Gradient

Conclusion

Degradation Model



Radon measure on $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$

Initial measure:

$\mu_{a,x} = \sum_{i=1}^{r} a_i \delta_{x_i}, a_i \in \mathbb{R}, x_i \in \mathbb{T}^d$

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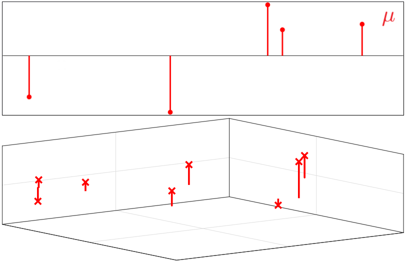


Linear measurements:

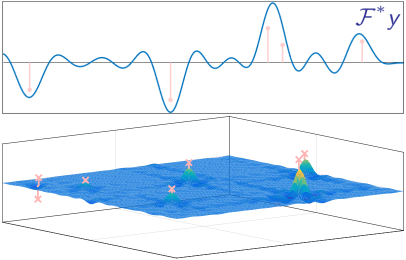
$$y = \mathcal{F}\mu_{a,x} + w$$

$\rightarrow$ Fourier measurements (cutoff frequency f_c)

$$(\mathcal{F}\mu)_k \stackrel{\text{def.}}{=} \int_{\mathbb{T}^d} e^{2i\pi\langle k,x\rangle} \mathrm{d}\mu(x),$$

for $k \in [\![-f_c; f_c]\!]^d$

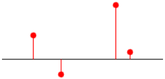
$\rightarrow$ Noise $w \in \mathbb{C}^{(2f_c+1)^d}$



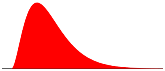
Sparse Recovery

Grid-free regularization: total variation of measures

$$|\mu|(\mathbb{T}^d) \overset{\text{def.}}{=} \sup \left\{ \int \eta \mathrm{d}\mu \;;\; \eta \in \mathcal{C}(\mathbb{T}^d), \|\eta\|_\infty \leqslant 1 \right\}$$



$|\mu|(\mathbb{T}^d) = \|a\|_{\ell^1}$



$|\mu|(\mathbb{T}^d) = \|f\|_{\mathrm{L}^1}$

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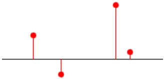

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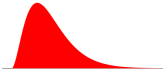

$|\mu|(\mathbb{T}^d) = \|f\|_{L^1}$

BLASSO (de Castro and Gamboa [2012])

$$\min_{\mu \in \mathcal{M}(\mathbb{T}^d)} \frac{1}{2}\|\mathcal{F}\mu - y\|^2 + \lambda|\mu|(X) \qquad (\mathcal{P}_\lambda)$$

$$\max_{p \in \mathbb{C}^{n^d}} \left\{ \langle y, \, p \rangle - \frac{\lambda}{2}\|p\|^2 \; ; \; \|\mathcal{F}^* p\|_\infty \leqslant 1 \right\} \qquad (\mathcal{D}_\lambda)$$

Dual Certificate

$$\max\left\{\langle y,\ p\rangle - \tfrac{\lambda}{2}\|p\|^2\ ;\ \|\mathcal{F}^* p\|_\infty \leqslant 1\right\} \qquad (\mathcal{D}_\lambda)$$

$\Longrightarrow \boldsymbol{\eta_\lambda \overset{\text{def.}}{=} \mathcal{F}^* p_\lambda}$ **dual certificate**.

$\eta_\lambda(x) \;=\; \sum_{k\in[\![-f_c;f_c]\!]^d} p_k e^{-2i\pi\langle k,x\rangle},$
with $p_k = p^*_{-k}$.

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Optimality relation: $\eta_\lambda \in \partial|\mu_\lambda|(X)$

$$\mu_{a,x} \text{ solves } (\mathcal{P}_\lambda) \Leftrightarrow \begin{cases} \eta_\lambda(x_i) = \operatorname{sign}(a_i) \\ \|\eta_\lambda\|_\infty \leqslant 1 \end{cases}$$

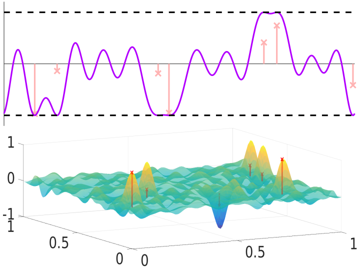

Figure: Dual polynomial

Related Works

- Support discretization $\longrightarrow$ LASSO - **Basis pursuit** (Donoho [1992])
  $\longrightarrow$ fast, inaccurate
- **Greedy support retrieval** (Bredies and Pikkarainen [2013])
  $\longrightarrow$ continuous setting, slow convergence
- **SDP relaxation** (Candès and Fernandez-Granda [2014])
  $\longrightarrow$ simple, stable, not scalable
- Non-variational schemes: **MUSIC** (Schmidt [1986])
  $\longrightarrow$ robust, scalable
  and **Prony's method** (Kunis et al. [2016]), **FRI-based methods** (Wei and Dragotti [2016]), ...

grid-free

Contributions

- SDP approach combined with conditional gradient algorithm
- efficient FFT-based computations

Overview

Semidefinite Hierarchies

- Encode measure in terms of moments
- Moment matrix satisfies **PSD** **+** **Toeplitz** **constraints**
- Keep only moments up to order m

Moment-Relaxations

$$\min_{u,z,\tau} \quad u_0 + \tau + \frac{1}{2}\left\|\frac{y}{\lambda} + 2z\right\|^2$$

$$\text{s.t.} \begin{cases} (a) \quad \mathcal{R} = \begin{bmatrix} R & z \\ z^* & \tau \end{bmatrix} \succeq 0, \\ (b) \quad R = \sum_{\boldsymbol{k} \in \rrbracket -m;m \llbracket^d} u_{\boldsymbol{k}} \Theta_{\boldsymbol{k}} \end{cases} \qquad (\widetilde{\mathcal{P}}_\lambda)$$

$$\Theta_{\boldsymbol{k}} = \theta_{k_d} \otimes \ldots \otimes \theta_{k_1}, \text{ with } \begin{cases} \otimes \quad \text{Kronecker product,} \\ \theta_k = (\delta_{i,i+k})_{i=1,\ldots,m} = \begin{bmatrix} 0 & \ldots & 1 & 0 & \ldots \\ 0 & \ldots & 0 & 1 & \ldots \\ & \ddots & & \ddots & \ddots \end{bmatrix} \end{cases}$$

When Is This Relaxation Tight?

Proposition (Dumitrescu [2017])

When $d = 1$, the relaxation is tight for $m \geqslant 2f_c + 1$.

Proof is based on either

- Fejér-Riesz theorem (Dumitrescu [2017])
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When $d = 2$, there exists an m s.t. the relaxation is tight.
When $d > 2$, we do not know in general.

Support Recovery

Proposition

Let $\mathcal{R}_\lambda = \begin{bmatrix} R_\lambda & z_\lambda \\ z_\lambda^* & \tau \end{bmatrix}$ *be a solution of* $(\widetilde{\mathcal{P}}_\lambda)$ *(suppose the relaxation is tight). The coefficients of the dual polynomial are given by*

$$p_\lambda = \frac{y}{\lambda} + 2z_\lambda$$

Proof.

Comes from primal-dual optimality relations $\square$

$\Longrightarrow$ Support recovered via root-finding on the dual certificate $\mathcal{F}^* p_\lambda$.

Low-Rank Structure

Proposition

When $d = 1$, $(\widetilde{\mathcal{P}}_\lambda)$ *admits a solution* $\mathcal{R}_\lambda$ *such that* $\operatorname{rank} \mathcal{R}_\lambda \leqslant s$, *where* s *is the number of spikes in* μ_λ *(solution of* $(\mathcal{P}_\lambda)$*).*

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In dimension $d \geqslant 2$, we conjecture that this results holds

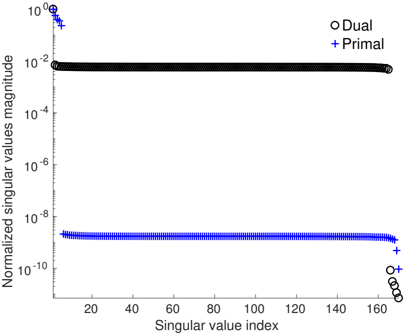

Figure: ($r = 5$ spikes, $f_c = 5$, $d = 2$). Singular values of primal and dual matrices

Overview

Penalized Primal

Numerical challenges:

- Handling both constraints simultaneously
- Toeplitz $\cap$ Low-Rank: analysis difficult

$$\boxed{\begin{aligned} &\min_{u,z,\tau} \; u_0 + \tau + \frac{1}{2}\Big\|\frac{y}{\lambda} + 2z\Big\|^2 \\ &\text{s.t.} \begin{cases} (a) & \mathcal{R} = \begin{bmatrix} R & z \\ z^* & \tau \end{bmatrix} \succeq 0 \\ (b) & R = \sum_{\boldsymbol{k}} u_{\boldsymbol{k}} \Theta_{\boldsymbol{k}} \end{cases} \end{aligned}} \quad (\widetilde{\mathcal{P}}_\lambda)$$

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$\Longrightarrow$ **Penalize Toeplitz constraint (b)**

$$\mathcal{R}_{\lambda,\rho} \in \operatorname*{argmin}_{R,u,z,\tau} \; u_0 + \tau + \frac{1}{2}\|\frac{y}{\lambda} + 2z\|^2 + \frac{1}{2\rho}\|R - \sum_{\boldsymbol{k}\in]\!]-m,m[\![^d} u_{\boldsymbol{k}}\Theta_{\boldsymbol{k}}\|^2 \qquad (\widetilde{\mathcal{P}}_{\lambda,\rho})$$
$$\text{s.t.} \quad \begin{bmatrix} R & z \\ z^* & \tau \end{bmatrix} \succeq 0.$$

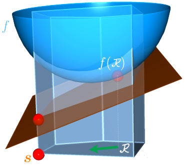
Overview

Alternating Descent Conditional Gradient Method

$$\left.\begin{array}{l} f \text{ convex, smooth} \\ K \text{ convex, bounded} \end{array}\right\} \min_{\mathcal{R}\in K} f(\mathcal{R})$$

Frank-Wolfe steps:

1. $\mathcal{S}^\star = \min_{\mathcal{S}\in\mathcal{D}} \langle \nabla f(\mathcal{R}_r), \mathcal{S} \rangle$
2. $\mathcal{R}_{r+1} = \mathcal{R}_r + c(\mathcal{S}^\star - \mathcal{R}_r)$, with $c \in [0, 1]$



Jaggi [2013]

Sparse iterates + simple LM

Slow convergence:

$f(x_r) - f(x^\star) \leqslant O(1/t)$

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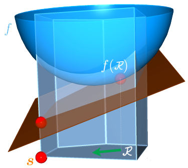

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Slow convergence:

$f(x_r) - f(x^\star) \leqslant O(1/t)$

- $K = \{X \succeq 0\,;\ \mathrm{tr}\, X \leqslant 1\}$
- Step 1. amounts to compute a leading eigenvector of ∇f.
- Low-cost storage: $\mathcal{R}_t = \mathcal{U}_t\mathcal{U}_t^*$
- Fixed-rank **BFGS** step (Boyd et al. [2015]) on $F : U \mapsto f(UU^*)$.

Algo: Recovering Dual Polynomial

Set: $U_0 = [0...0]^\top$, D_0: $\mathrm{tr}\,\mathcal{R}^\star \leqslant D_0$

For $r = 1, \ldots, N$ **do**

1. $v_r = D_0 \operatorname*{argmin}_{\|v\|\leqslant 1} v^\top \cdot \nabla f\,[U_r U_r^*] \cdot v$
2. $\widehat{U}_{r+1} = [\alpha_r U_r, \beta_r v_r]$, where $\alpha_r, \beta_r = \operatorname*{argmin}_{\alpha+\beta\leqslant 1} f(\alpha U_r U_r^* + \beta v_r v_r^*)$
3. $U_{r+1} \leftarrow$ **BFGS**$(F(U)$, init. at $\widehat{U}_{r+1})$

Fast-Fourier-Tranforms-Based Computations

- Leading eigenvector is computed using **Power Iteration**.
- Requires only computing $\nabla f \cdot v$, with

$$\nabla f(UU^*) = \begin{bmatrix} \frac{1}{n} I_n & \frac{y}{\lambda} + 2z \\ \frac{y}{\lambda} + 2z & 1 \end{bmatrix} + \frac{1}{\rho}(UU^* - P_{\mathcal{T}}(UU^*))$$

- Main costly operation: $P_{\mathcal{T}}(UU^*) \cdot v$

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Key Ingredient: $O(n^d \log(n))$ FFT-Based Computations

Toeplitz-Vector Multiplication

Let $x \in \mathbb{C}^{(n+1)^d}$, $t \in \mathbb{C}^{(2n+1)^d}$, and $T = \mathrm{Toep}_t$. Then

$$Tx = \mathrm{Pad}^{-1} \circ \mathcal{F}^{-1}\Big(\langle \mathcal{F} \circ \mathrm{Pad}(x), \mathcal{F}(t) \rangle\Big)$$

Toeplitz Projection

Let $U = [U_1, \ldots, U_r] \in \mathbb{C}^{(n+1)^d \times r}$. Then $P_{\mathcal{T}}[UU^*] = \mathrm{Toep}_t$, with

$$t_i \propto \left[\sum_k \mathcal{F}^{-1}(|\mathcal{F} \circ \mathrm{Pad}(U_k)|^2)\right]_i$$

Numerics

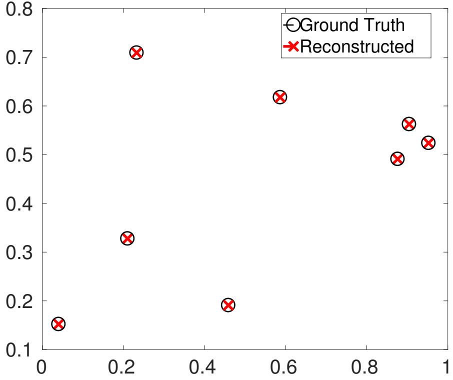

Figure: $r = 8$, $f_c = 13$, $\|w\| = 0.001\|y\|$, $\lambda = 1$, $\rho = 10$, total time: 315s, error (flat-norm): $\|\mu_{\lambda,\rho} - \mu_0\| = 4.57 \times 10^{-3}$

Numerics

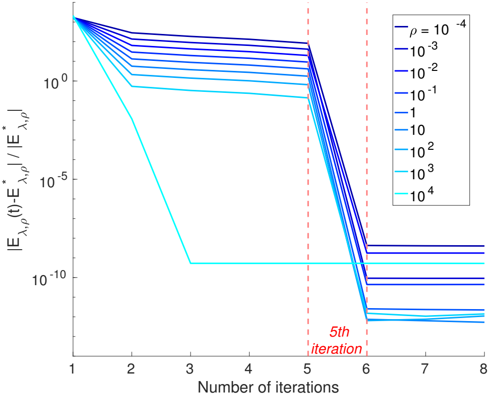

Figure: Relative error between $E_{\lambda,\rho}$ and E_λ. Initial measure has 5 spikes.

Conclusion

- SDP formulation for problem the problem of spikes superresolution...
- ... which admits low-rank solutions
- Scalable method in 2D, based on a conditional gradient approach
- Generalizable framework, to problems of the form

$$\min f(x) \quad \text{s.t.} \quad x \in A \cap B$$

with f smooth, A some convex hull, B an affine space on which we can project easily.

Merci pour votre attention!

Support Recovery via Root-Finding

Dual polynomial $\eta_\lambda = \sum p_k e^{2i\pi\langle k, x\rangle}$

Root-finding:

- $P(X) = \sum_k p_k X^k$, $X \in \mathbb{C}^d$
- Solve $|P(X)|^2 - 1 = 0$
- Select roots s.t. $|X| = 1$

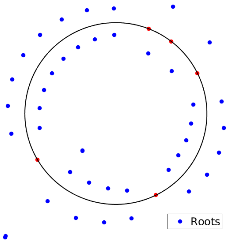

Figure: Roots of $1 - |P|^2$, with $P = \sum p_k X^k$

Sensitivity Analysis

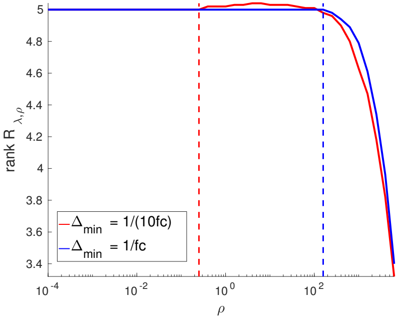

Figure: Rank of $\mathcal{R}_{\lambda,\rho}$ w.r.t. ρ

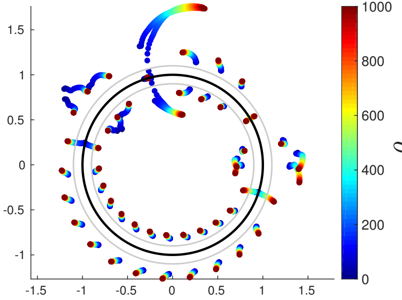

Figure: Roots trajectory w.r.t ρ.

Numerics

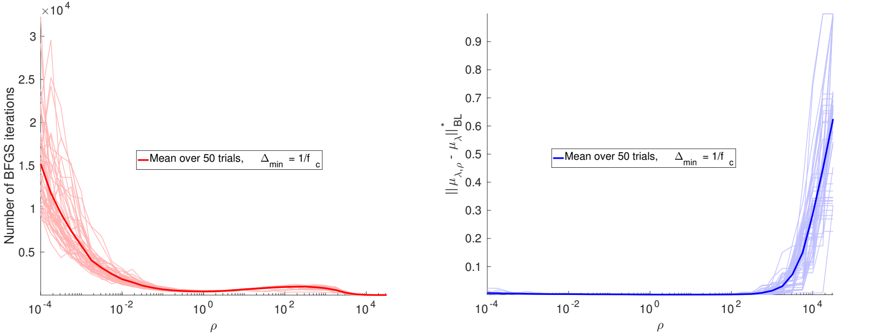

Figure: Left: Number of BFGS iterations, summed over all Frank-Wolfe steps. Right: Error between $\mu_{\lambda,\rho}$ and μ_{λ}, measured with the dual bounded Lipschitz norm.

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