

Exercise 2

Exercise 1. 1. Compute the singular value decomposition of (some of) the following matrices

$$\begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

2. With $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, draw the set $\{x \in \mathbb{R}^2; \|Ax\| = 1\}$ (the pre-image of the circle). How can you determine the singular values and right singular vectors of A from this figure?

Exercise 2. Let $A \in \mathbb{R}^{m \times n}$ and $A = U\Sigma V^\top$ its singular value decomposition. We write u_i and v_i the left and right singular vectors respectively, and $\sigma_1 \geq \dots \geq \sigma_r > 0$ the non-zero singular values.

1. Check that

$$A = \sum_{i=1}^r \sigma_i u_i v_i^\top$$

2. We have seen that $\|A\|_{2,2} := \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sigma_1$. Show that

$$\sigma_r = \inf_{x \in (\text{Ker } A)^\perp \setminus \{0\}} \frac{\|Ax\|}{\|x\|}$$

3. For $k < r$, we define

$$A_k := \sum_{i=1}^k \sigma_i u_i v_i^\top.$$

Show that $\|A - A_k\|_{2,2} = \sigma_{k+1}$. Show that A_k actually minimizes $\|A - B\|_{2,2}$ among all $B \in \mathbb{R}^{m \times n}$ such that $\text{rank } B \leq k$ (this result is known as the Eckart-Young-Mirsky theorem).

Exercise 3 (Pseudo-inverse). Let $A \in \mathbb{R}^{m \times n}$ and A^\dagger its Moore-Penrose pseudo-inverse.

1. Check the identities

$$\begin{aligned} A^\dagger A A^\dagger &= A^\dagger, \\ A A^\dagger A &= A, \end{aligned}$$

2. Show that $A^\dagger A$ and $A A^\dagger$ are orthogonal projections on $(\text{Ker } A)^\perp$ and $\text{Ran } A$ respectively.

Exercise 4. We define the circular convolution of $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$ as the vector $(a * x) \in \mathbb{R}^n$ whose entries are given by

$$\forall k \in \{1, \dots, n\}, \quad (a * x)_k := \sum_{i=1}^n a_{[k-i]} x_i$$

where $[i] = i \pmod{n}$. For $a \in \mathbb{R}^n$, let $A : \mathbb{R}^n \mapsto \mathbb{R}^n, x \mapsto a * x$.

1. Give the matrix of A in the canonical basis.
2. For $0 \leq j \leq n-1$, let $u_j = (e^{-2i\pi kj/n})_{k=0}^{n-1}$, and $U = [u_0, \dots, u_{n-1}] \in \mathbb{C}^{n \times n}$. Show that

$$AU = \text{Diag}(\hat{a}_j)U,$$

where \hat{a} is the discrete Fourier transform of a . Deduce the singular values (σ_j) of A .

3. Let $a = [1 \quad -1 \quad 0 \quad \dots \quad 0]^\top$. Compute the ration $\sigma_{\max}/\sigma_{\min}$ in that case.