Time Series Analysis & Recurrent Neural Networks

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Exercise 4

To be uploaded before the exercise group on May 25th, 2022

Task 1. Granger Causality

1. Create a 2-variate AR(2) time series $x_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \end{pmatrix}$ with $T = 1000$ time steps and the following parameters: $a_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0.2 & 0 \\ -0.2 & 0.1 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 0.1 & 0 \\ -0.1 & 0.1 \end{pmatrix}$, diagonal Gaussian noise matrix $\Sigma = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix}$, and initial condition $x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.
2. Given your knowledge of model parameters above, does x_1 Granger-cause x_2, or does x_2 Granger-cause x_1 according to Granger's definition? Why?
3. In a similar spirit to exercise 1 on the previous sheet, we want to find out if adding parameters to the model increases its explanatory power in a significant way. Given a fixed order of the model ($p = 2$), compute the log-likelihoods of a combined $VAR(2)$ model where the first time series is "causing" the second time series (meaning there are non-zero entries in the respective A matrices), and compare it to the likelihood of fitting an individual $AR(2)$ model on the second time series.
4. Repeat this analysis by setting up a model where the second time series is causing the first one. How do the results for the individual likelihoods compare to your analysis of Granger causality?
5. How could you confirm this result in a statistically meaningful way (hint: compute the log-likelihood-ratio test statistic similar to the last exercise sheet, and eq. 7.35 in the script)?

Task 2. M-Step in a linear Gaussian state space model

Update: As we didn't manage to cover this material yet in the current lecture, this will be treated as a bonus exercise. The theory is also covered in last year's lecture 5 which can be found following `https://cloud.zi-mannheim.de/index.php/s/ng46sWPtNnNDACC/authenticate/showShare`, PW:TSArnn#2021.

Consider a linear Gaussian state space model,

$$\begin{aligned} z_t &= A z_{t-1} + \epsilon, & \epsilon &\sim N(0, \Sigma) \\ x_t &= B z_t + \eta, & \eta &\sim N(0, \Gamma) \end{aligned}$$

In the lecture we derived the M-step to determine the transition matrix A. Derive the M-step for the latent state noise Σ by maximizing the expected log-likelihood $\mathrm{E}[\log p(X, Z)]$, with respect to Σ, where $X = \{x_t \mid t \in 1 \ldots T\}$ and $Z = \{z_t \mid t \in 1 \ldots T\}$ are the sets of all latent states and observations from time 1 to T.
(Hints: First identify the part in the ELBO that explicitly depends on the parameters and ignore the

rest. Multiplying both sides by Σ can be useful at one point in the derivation. Further plugging in A as derived in the lecture will help you simplify the final result. The matrix cookbook by Petersen and Pedersen (2012)
`https://www.math.uwaterloo.ca/~hwolkowi/matrixcookbook.pdf`
provides a helpful summary for matrix algebra).